Perinormality in Polynomial and Module-Finite Ring Extensions

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by Andrew McCrady, a candidate for the degree of Doctor of Philosophy of Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

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ABSTRACT

In this dissertation we investigate some open questions posed by Epstein and Shapiro in [9] regarding perinormal domains. More specifically, we focus on the ascent/descent property of perinormality between "canonical" integral domain extensions, in particular, $R \subset R[X]$ and $R \subset \widehat{R}$. We give special conditions under which perinormality ascends from R to the polynomial ring $R[X]$ in the case that R is a universally catenary domain. Whereas we have a characterizing result for when perinormality descends from $R[X]$ to R, the sufficient condition for the descent is cumbersome to check. For this reason, we turn to special cases for which perinormality descends from $R[X]$ to R. In the case of an analytically irreducible local domain $(R, \mathfrak{m})$ and its $\mathfrak{m}$-adic completion $(\widehat{R}, \mathfrak{m}\widehat{R})$, we refer to a technique for generating examples in which perinormality fails to ascend. When $\widehat{R}$ is perinormal, we explore hypotheses under which R must be normal, perinormal, or weakly normal.

Chapter 1

Introduction

Throughout this work a ring is assumed to be a commutative ring with unity, however most of our work will be in the setting of domains. By *overring* we mean a ring extension $R \subseteq S \subseteq K$, where $K := R_{(0)}$ is the quotient field of R. It is well known that a flat ring extension $R \subset S$ satisfies the going down property. A result by Richmond [25, Theorem 2] states that a local overring $(S, \mathfrak{m})$ of R is flat if and only if $R_{\mathfrak{m} \cap R} = S$. In [9], Epstein and Shapiro consider domains R for which the following holds: any overring extension $R \subset S$, satisfying the going down property and with $(S, \mathfrak{m})$ local, must be a flat extension. Epstein and Shapiro name such domains R *perinormal* domains. The class of perinormal domains includes weakly normal, seminormal, Krull, and generalized Krull domains. The main result in [9] is a characterization of perinormality among the universally catenary domains [9, Theorem 4.7]. Locally, the structure of the prime spectrum of such a domain is unique among its overrings. The proof of this result hinges on contracting height one primes from the integral closure of R to height one primes of R which is where the universally catenary assumption is used. The universally catenary assumption on a ring is not too restrictive in the sense that such rings abound in algebraic geometry and algebraic number theory.

In Chapter 2 we search for a convenient paraphrasing of [9, Theorem 4.7]. To use [9, Theorem 4.7], it is necessary that, for *every* $\mathfrak{p} \in \operatorname{Spec}(R)$, no proper, integral overring of $R_{\mathfrak{p}}$ have its spectrum in a one-to-one correspondence with $\operatorname{Spec}(R_{\mathfrak{p}})$. (Let A' stand for the integral closure of the domain A in $A_{(0)}$, and let the domain extension $A \subset B \subset A'$ have the canonical map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be bijective. By Cohen-Seidenberg theorems, the extension $A \subset B$ satisfies the going down property.) McAdam in [17], [18], and [19] explored conditions for a ring extension to satisfy going down. We combine the work of McAdam with Epstein and Shapiro's [9, Theorem 4.7] to offer, in Corollary 2.9, an alternative characterization of perinormality among universally catenary domains. Given a prime $\mathfrak{p} \in \operatorname{Spec}(R)$, rather than accounting for *all* of the integral overrings of $R_{\mathfrak{p}}$ whose spectrum is bijective to $\operatorname{Spec}(R_{\mathfrak{p}})$, we only need look at the module-finite extensions $R_{\mathfrak{p}}[a/b]$ where $a/b \in K$ is integral over $R_{\mathfrak{p}}$.

When the integral closure R' of the noetherian domain R is a finitely generated R-module, then the conductor ideal $R :_R R' := \{r \in R \mid rR' \subseteq R\}$ is a non-zero ideal of both R and R'. The non-normal locus of R is $V_R(R :_R R') := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid R :_R R' \subset \mathfrak{p}\}$, meaning that $R_{\mathfrak{q}}$ is normal for each $\mathfrak{q} \in \operatorname{Spec}(R) - V_R(R :_R R')$. For each such $\mathfrak{q}$ it follows that $R_{\mathfrak{q}}$, being a Krull domain, is perinormal. In the scenario that $V_R(R :_R R') = \{\mathfrak{m}\}$ for some $\mathfrak{m} \in \operatorname{Max}(R)$, we explore when R is perinormal relative to an integral overring (see Definition 1.15 and Proposition 2.13). Such a domain R is normal for all $\mathfrak{p} \neq \mathfrak{m}$, and we show that whenever $R/\mathfrak{m}$ is, for example, algebraically closed, the concepts of seminormality, weak normality and perinormality coincide.

Chapter 3 deals with the question when certain domains and their ideal transforms are perinormal. If $I \subset R$ is an ideal, the ideal transform of I is the overring of R

defined by $T(I) := \{a \in K \mid aI^n \subset R$ for some $n \in \mathbb{N}\}$. Ideal transforms were first introduced by Nagata [22]. Placing mild hypotheses on a domain R affords the conclusion that $T(xR) = R_x = R[1/x]$ for each non-zero $x \in R$ [3, Lemma 2.2]. Proposition 3.6 shows that non-local integral domains R are perinormal if and only if for each non-unit $x \in R$ the ideal transform $T(xR)$ is perinormal. In the setting of a, possibly non-domain, commutative ring R with unity, Matijevic [15] defines the *global transform* T of R as the set of elements in the total quotient ring of R whose conductor into R contains a power of a finite product of maximal ideals in R. Without any restriction on the Krull dimension of R, Matijevic obtains the result that whenever $R \subset A \subset T$ is a ring extension, then A/xA is a finitely generated R-module for each nonzero divisor $x \in R$ [15, Theorem]. In some sense, allowing R to have *any* Krull dimension comes at the expense of restricting the conclusion of the Krull-Akizuki Theorem to subrings of T. Later, Nishimura [23] used [15, Theorem] to explore the relationship between a local domain and its integral closure in the ideal transform of the maximal ideal. Proposition 3.12 shows that, under mild hypothesis, a perinormal local domain $(R, \mathfrak{m})$ is equal to the ideal transform $T(\mathfrak{m})$.

Chapter 4 addresses a question in [9] regarding the ascent/descent of perinormality in the ring extension $R \subset R[X]$. The theory developed by McAdam in [17] concerning the transfer of the going down property from $R \subset S$ to $R[X] \subset S[X]$ will turn out to be vital. For reference, we include the descriptions in [17] regarding the relationships between the prime spectra in extensions $R \subset T$ and $R[X] \subset T[X]$. We show conditions under which, when R is a universally catenary perinormal domain, the extended ring $R[X]$ is also perinormal [Proposition 4.9]. The opposite direction, also, has not yielded unfettered results in the positive. Whereas we do have a characterizing result (Proposition 4.11) for when perinormality descends from $R[X]$ to R, the sufficient condition for the descent is cumbersome to check. For this reason, we turn to special cases for which perinormality descends from $R[X]$ to R. With some restrictions (see [Theorem 4.12], and [Proposition 4.15]), we can establish the descent of perinormality from $R[X]$ to R. We conclude the chapter by showing that $R[X]$ *globally perinormal* [Definition 1.1] implies R weakly normal [Proposition 4.15].

Chapter 5 explores another question posed in [9], namely, whether perinormality ascends or descends between an analytically irreducible local domain $(R, \mathfrak{m})$ and its $\mathfrak{m}$-adic completion. Complete, noetherian, local domains enjoy properties, important in considerations of perinormality. In particular, the integral closure, A', of a complete, noetherian, local domain A (in its quotient field) is a finitely generated A-module, and is, in its own right, a complete local domain. Furthermore, any domain between A and A' is also a complete, local domain. Finally, excellence of A allows us much wiggle-room. It is also known that if a local, noetherian domain R is analytically irreducible, then its integral closure R' (in the quotient field of R) is also local, and R' is a finitely generated R-module. Since $\sqrt{\mathfrak{m}R'} = \mathfrak{m}'$, the $\mathfrak{m}$-adic and $\mathfrak{m}'$-adic completions of R' are isomorphic. Zariski and Samuel in [30] and Dieudonne in [6] give conditions for the integral closure of $\widehat{R}$ to be isomorphic to the $\mathfrak{m}$-adic completion of R'. We show that perinormality need not ascend from R to $\widehat{R}$. To accomplish this, we only need find a suitable example of a complete, non-perinormal, local domain, and, then, apply Ray Heitmann's machinery from [13] to obtain a unique factorization domain (and hence a perinormal domain) whose completion is non-perinormal. After this, we turn to the hypothesis that $\widehat{R}$ is perinormal. With varying hypotheses on R, we show R may be normal, perinormal, or weakly normal.

1.1 Preliminary Results, Definitions, and Notations

Throughout, the ring R will stand for a commutative ring with identity. A (quasi-) local ring R, with $\mathfrak{m}$ its unique maximal ideal, will be denoted by $(R, \mathfrak{m})$. (We reserve the designation *local* for noetherian rings, while *quasi-local* encompasses the possibility that the ring is not noetherian.) As is customary, $\operatorname{Spec}(R)$ and $\operatorname{Max}(R)$ will connote the set of all prime ideals and the set of all maximal ideals of R, respectively. Less conventionally, we shall write $X^1(R)$ for the set of all height 1 prime ideals in R. For an ideal I of R, the set $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subset \mathfrak{p}\}$ is denoted by $V_R(I)$.

In what follows, we fix the terminology for a ring extension $R \subset S$: If $\mathfrak{p} \in \operatorname{Spec}(R)$, then $R_\mathfrak{p} \subset S_\mathfrak{p}$ stands for the fraction ring extension $W^{-1}R \subset W^{-1}S$ where $W := R - \mathfrak{p}$. In the parlance of the Cohen-Seidenberg theorems, a prime $P \in \operatorname{Spec}(S)$ *lies over* a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ if $P \cap R = \mathfrak{p}$; an extension of rings $R \subset S$ satisfies the *going up property* if every chain $\mathfrak{p} \subset \mathfrak{q}$ in $\operatorname{Spec}(R)$, with $P \in \operatorname{Spec}(S)$ lying over $\mathfrak{p}$, forces the existence of $Q \in \operatorname{Spec}(S)$ lying over $\mathfrak{q}$ with $P \subset Q$; an extension of rings $R \subset S$ satisfies the *going down property* if every chain $\mathfrak{p} \subset \mathfrak{q}$ in $\operatorname{Spec}(R)$, with $Q \in \operatorname{Spec}(S)$ lying over $\mathfrak{q}$, forces the existence of $P \in \operatorname{Spec}(S)$ lying over $\mathfrak{p}$ with $P \subset Q$. When R is a domain, its quotient field $R_{(0)}$ will be designated by K. Further, if $R \subset S \subset K$ is a ring extension sitting between the domain R and its quotient field K, then S is called an *overring* of R.

Definition 1.1. Let R be an integral domain. We say that

(i) R is *perinormal* if whenever a (quasi-)local overring extension $R \subseteq (S, \mathfrak{M})$ satisfies the going down property, then $S = R_{\mathfrak{p}}$ for some $\mathfrak{p} \in \operatorname{Spec}(R)$.

(ii) R is *globally perinormal* if whenever an overring extension $R \subseteq S$ (with S not necessarily (quasi-)local) satisfies the going down property, then $S = W^{-1}R$ for some multiplicatively closed subset $W \subset R$.

We now list a result by Richman which will serve us in good stead as we compile basic results about perinormality.

Proposition 1.2 ([25, Theorem 2])**.** *Let S be an overring of the integral domain R. Then S is flat over R if and only if $R_{\mathfrak{M} \cap R} = S_{\mathfrak{M}}$ for all $\mathfrak{M} \in \operatorname{Max}(S)$.*

Suppose that the overring extension $R \subset (S, \mathfrak{M})$ satisfies the going down property. If R is perinormal, then S is flat over R, and by Proposition 1.2 $R_{\mathfrak{M} \cap R} = S$. Hence in Definition 1.1 the prime $\mathfrak{p}$ such that $S = R_{\mathfrak{p}}$ is unambiguously the ideal $\mathfrak{M} \cap R$. This observation yields the following.

Corollary 1.3. *A domain R is perinormal if and only if whenever a local overring extension $R \subset (S, \mathfrak{M})$ satisfies the going down property, then $S = R_{\mathfrak{M} \cap R}$.*

Much of our results will implicitly use the following work-horse.

Proposition 1.4 ([9, Proposition 2.5])**.** *If R is perinormal, so is R_W for every multiplicative set $W \subset R$. Conversely, if $R_{\mathfrak{m}}$ is perinormal for all maximal ideals $\mathfrak{m}$ of R, then so is R.*

In the following architecture, we build up to the definitions of seminormality and weak normality as in [27].

Definition 1.5. Let $R \subset S \subset R'$ be a ring extension with R' the integral closure of the domain R (in its quotient field K).

(i) The extension $R \subset S$ is said to be *subintegral* if the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a bijection, and if for every $\mathfrak{P} \in \mathrm{Spec}(S)$, with $\mathfrak{P} \cap R = \mathfrak{p} \in \mathrm{Spec}(R)$, the corresponding field extensions $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{P}}/\mathfrak{P}_{\mathfrak{P}}$ are isomorphisms.

(ii) Let $R \subset T$ be an extension of domains. Define ${}^{+}_{T}R$ to be the unique largest subextension of R in T such that $R \subset^{+}_{T} R$ is subintegral. We say R is *seminormal in* T if $R =^{+}_{T} R$.

(iii) The domain R is said to be *seminormal* if R is seminormal in R'.

(iv) The extension $R \subset S$ is said to be *weakly subintegral* if the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a bijection, and if for every $\mathfrak{P} \in \mathrm{Spec}(S)$ with $\mathfrak{P} \cap R = \mathfrak{p} \in \mathrm{Spec}(R)$ the corresponding field extensions $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{P}}/\mathfrak{P}_{\mathfrak{P}}$ are purely inseparable.

(v) Let $R \subset T$ be any extension of domains. Define ${}^{+}_{*}R$ to be the unique largest subextension of R in T such that $R \subset^{+}_{*} R$ is weakly subintegral. We say R is *weakly normal in* T if $R =^{+}_{*} R$.

(vi) The domain R is said to be *weakly normal* if R is weakly normal in R'.

From the definition we see that a weakly normal domain is seminormal. This leads us to the following result.

Proposition 1.6 ([9, Proposition 3.4])**.** *A perinormal domain is weakly normal (therefore seminormal).*

In the following definition, we do *not* require that the ring R be noetherian.

Definition 1.7. A domain R is said to be (R_1) if $R_{\mathfrak{p}}$ is a valuation ring for all $\mathfrak{p} \in X^1(R)$.

The above relaxation of the definition of (R_1) allows us to introduce the concept of a generalized Krull domain.

Definition 1.8. The integral domain R is called a *generalized Krull domain* if the following properties are satisfied:

(a) $R = \bigcap_{\mathfrak{p} \in X^1(R)} R_{\mathfrak{p}}$

(b) Any non-zero $r \in R$ is contained in only finitely many height 1 prime ideals of R.

(c) R is (R_1).

If in (c) $R_{\mathfrak{p}}$ is required to be a DVR for each $\mathfrak{p} \in X^1(R)$, then we say that R is a *Krull domain.*

Clearly, then, a Krull domain is also a generalized Krull domain. Somewhat less obvious is the following ring class containment result.

Proposition 1.9 ([9, Proposition 3.10])**.** *A generalized Krull domain is perinormal.*

Following the convention in [9] that a *normal* domain stands for a domain, integrally closed in its quotient field, we note that a noetherian, normal domain R is perinormal, since, by the Mori-Nagata Theorem [11, Theorem 4.3], R must be a Krull domain.

A striking result in [9] is a characterization of perinormality in the setting of universally catenary domains (Theorem 1.12). It states that a universally catenary domain is perinormal if and only if, locally, it has a uniquely structured spectrum among its overrings. This theorem will be the focus of Section 2.1. We recall the definition of universally catenary rings, and then state the aforementioned theorem.

Definition 1.10. A noetherian ring R is called *universally catenary* if every $R-$ algebra A of finite type has the following property: for any $P, Q \in \operatorname{Spec}(A)$ with $P \subset Q$, all maximal chains of prime ideals in $\operatorname{Spec}(A)$ between P and Q have the same length, namely $ht_A(Q/P)$.

It is easy to check the following statement.

Proposition 1.11. *If R is universally catenary, then so is $W^{-1}R$ and R/I for any multiplicatively closed subset $W \subset R$ and any ideal $I \subset R$.*

Finally, we are in the position to reveal the pivotal Epstein and Shapiro theorem.

Theorem 1.12 ([9, Theorem 4.7])**.** *Suppose that R is a universally catenary domain with quotient field K. The following are equivalent.*

(a) R is perinormal.

(b) For each $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_\mathfrak{p}$ is the only ring S such that $R_\mathfrak{p} \subset S \subset K$ and $\operatorname{Spec}(S) \to \operatorname{Spec}(R_\mathfrak{p})$ is an order-reflecting bijection.

(c) R is (R_1), and for each $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_\mathfrak{p}$ is the only ring S such that $R_\mathfrak{p} \subset S \subset R'_\mathfrak{p}$ and $\operatorname{Spec}(S) \to \operatorname{Spec}(R_\mathfrak{p})$ is an order-reflecting bijection.

Remark 1.13. We make two observations regarding Theorem 1.12 (c), above. For both, we use the Cohen-Seidenberg Theorems together with the bijectivity of the map $\theta(\mathfrak{p}) : \operatorname{Spec}(S) \to \operatorname{Spec}(R_\mathfrak{p})$. Firstly, the ring S must be local since $R_\mathfrak{p}$ is local.

Secondly, the "order-reflecting" condition on $\theta(\mathfrak{p})$ is moot since the extension $R_{\mathfrak{p}} \subset S$ satisfies the going-up property, and since $\theta(\mathfrak{p})$ is bijective, and, therefore, inherits the order-reflecting condition automatically.

Definition 1.14. If $R \subset (A, \mathfrak{n})$ is a ring extension and $\mathfrak{p} \in \mathrm{Spec}(R)$, we say A is *centered at* $\mathfrak{p}$ if $\mathfrak{n} \cap R = \mathfrak{p}$.

The following is a definition of *relative perinormality* introduced in [10].

Definition 1.15. Let $R \subseteq T$ be a ring extension. We say the R is *perinormal in*, or *relative to* T if for every $\mathfrak{p} \in \mathrm{Spec}(R)$, whenever $(A, \mathfrak{n})$ is a local ring with $R_{\mathfrak{p}} \subset A \subset T_{\mathfrak{p}}$ such that

(i) A is centered at $\mathfrak{p}R_{\mathfrak{p}}$, and

(ii) $R_{\mathfrak{p}} \subset A$ satisfies the going down property,

then $A = R_{\mathfrak{p}}$.

What follows is an easy consequence of this definition.

Proposition 1.16. *Let R, S, T be integral domains with $R \subseteq S \subseteq T$. If R is perinormal in T, then so is R perinormal in S.*

Proof. Suppose that R is perinormal in T. Then for $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_{\mathfrak{p}}$ is the only local ring between $R_{\mathfrak{p}}$ and $T_{\mathfrak{p}}$, and, hence, the only local ring between $R_{\mathfrak{p}}$ and $S_{\mathfrak{p}}$, centered on $\mathfrak{p}R_{\mathfrak{p}}$, and satisfying the going down property over $R_{\mathfrak{p}}$. Thus R is perinormal in S. ∎

The following is a corollary to Theorem 1.12.

Corollary 1.17. *Let R be a universally catenary domain with quotient field $K := R_{(0)}$ and integral closure R'. Then R is perinormal if and only if R is (R_1) and R is perinormal relative to R'.*

Proof. ($\Longrightarrow$) Suppose that R is perinormal. Then by [9, Proposition 3.2] R is (R_1). Let $\mathfrak{p} \in Spec(R)$, then, by [9, Proposition 2.5], $R_{\mathfrak{p}}$ is perinormal. Hence R is perinormal in K. By, Proposition 1.16, R is perinormal in R'.

($\Longleftarrow$) Now suppose that R is (R_1) and that R is perinormal in R' is perinormal. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and suppose that $R_{\mathfrak{p}} \subseteq (S, \mathfrak{m}) \subseteq R'_{\mathfrak{p}}$ is such that $\operatorname{Spec}(S) \to \operatorname{Spec}(R_{\mathfrak{p}})$ is an order-reflecting bijection. In particular, S is centered on $\mathfrak{p}R_{\mathfrak{p}}$ and the extension $R_{\mathfrak{p}} \subseteq S$ has going down. Then, by the relative perinormality hypothesis, $R_{\mathfrak{p}} = S$, and the result follows from Theorem 1.12 (c). ∎

Chapter 2

Module-Finite Extensions

Unless otherwise stated, R is a noetherian domain. By *module-finite* extension we mean an extension $R \subseteq S \subseteq K$ such that S is a finitely generated R-module. Consequently, S is then noetherian as well. The integral closure of R in K is denoted by R'.

2.1 More on Theorem 1.12

The goal of this section is to derive another characterization of perinormality for universally catenary domains. For the sake of convenience, we repeat Theorem 1.12 with a slight modification due to Remark 1.13.

Theorem 2.1 ([9, Theorem 4.7]). *Let R be a universally catenary integral domain with quotient field K. The following are equivalent.*

(a) R is perinormal.

(b) For each $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_\mathfrak{p}$ is the only ring S between $R_\mathfrak{p}$ and K such that the induced map $\operatorname{Spec}(S) \to \operatorname{Spec}(R_\mathfrak{p})$ is an order-reflecting bijection.

(c) R has (R_1), and for each $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_\mathfrak{p}$ is the only ring S between $R_\mathfrak{p}$ and its integral closure such that the induced map $\operatorname{Spec}(S) \to \operatorname{Spec}(R_\mathfrak{p})$ is a bijection.

To ensure the perinormality of R, Theorem 1.12 (c) says that one needs to check that for each $\mathfrak{p} \in \operatorname{Spec}(R)$, *any* local, integral overring of $R_{\mathfrak{p}}$ with identically structured spectrum coincides with $R_{\mathfrak{p}}$. The following results culminate in Corollary 2.9, which reduces (c) in Theorem 1.12 to showing that only local, module-finite extensions of $R_{\mathfrak{p}}$ with identically structured spectrum need be considered.

Definition 2.2. Suppose $R \subset S$ is a ring extension. A prime $\mathfrak{p} \in \operatorname{Spec}(R)$ is *unibranched* if there exists a unique prime $P \in \operatorname{Spec}(S)$ lying over $\mathfrak{p}$. We say the extension $R \subset S$ is *unibranched* if each $\mathfrak{p} \in \operatorname{Spec}(R)$ is unibranched. In other words, the induced map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is bijective.

The result, below, details the relationship between the height-1-spectra in an extension with the going-down property.

Proposition 2.3 ([9, Proposition 3.9])**.** *Let $R \subseteq S$ be an overring extension of the (R_1) domain R, satisfying the going-down property. Then S is (R_1), and the map $X^1(S) \to \{\mathfrak{p} \in X^1(R) : \mathfrak{p}S \neq S\}$, given by $P \mapsto P \cap R$, is bijective.*

We list another result addressing the going-down property.

Proposition 2.4 ([18, Theorem 2])**.** *Let R be a noetherian domain and S a domain lying between R and R'. Then the extension $R \subset S$ satisfies the going down property if and only if $\mathfrak{p} \in \operatorname{Spec}(R)$ is unibranched in S for all primes such that $ht_R(\mathfrak{p}) > 1$.*

With the next result, we underscore the exceptional behaviour in dimension 1.

Lemma 2.5 ([9, Lemma 3.7])**.** *Let R be an integral domain, S an overring of R, and $P \in \operatorname{Spec}(S)$ such that $V := R_{P \cap R}$ is a valuation ring of dimension 1. Then $R_{P \cap R} = S_P$ as subrings of the field K, and $ht_S(P) = 1$.*

Propositions 2.3 , 2.4 and Lemma 2.5 combine to give Corollary 2.6. In particular, the going down property of $R \subset S$, together with some additional hypotheses, imparts, on the extension, the unibranched property.

Corollary 2.6. *Let R be an (R_1) noetherian domain. Let S be a ring lying between R and R'. Then $R \subset S$ has the going down property if and only if the extension is unibranched.*

Proof. ($\Longrightarrow$) By Propositions 2.3 and 2.4, it suffices to show that $X^1(S) \to X^1(R)$ is surjective. Since $R \subset S$ is integral, then by the Cohen-Seidenberg Theorem, $Spec(S) \to Spec(R)$ is onto. Then for every $\mathfrak{p} \in X^1(R)$, there is a $P \in \operatorname{Spec}(S)$ lying above $\mathfrak{p}$. Since R is (R_1), by Lemma 2.5, $R_{\mathfrak{p}} = R_{P \cap R} = S_P$ and $P \in X^1(S)$.

($\Longleftarrow$) This is immediate from Proposition 2.4. ∎

We continue to aggregate results on the going down property.

Proposition 2.7 ([19, Proposition 2])**.** *The integral extension $R \subseteq S$ satisfies the going down property if and only if $R \subseteq R[s]$ satisfies the going down property for all $s \in S$.*

We are now in the position to pull the above results together.

Corollary 2.8. *Let R be an (R_1) noetherian domain and S a ring lying between R and R'. Then the following are equivalent:*

(1) $R \subset S$ has going down.

(2) $R \subset S$ is unibranched.

(3) $R \subset R[s]$ has going down for each $s \in S$.

(4) $R \subset R[s]$ *is unibranched for each* $s \in S$.

(5) For each $s \in S$, *the inclusions* $R \subset R[s]$ *and* $R[s] \subset S$ *naturally induce bijections*

$$\theta_1(s) : \operatorname{Spec}(S) \to \operatorname{Spec}(R[s]) \text{ and } \theta_2(s) : \operatorname{Spec}(R[s]) \to \operatorname{Spec}(R), \textit{respectively}$$

Proof. The results from Proposition 2.3 through Proposition 2.7 make clear that (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3)$\Longleftrightarrow$(4).

Next, we establish (4)$\Longrightarrow$ (5). Assuming (4), clearly, $\theta_2(s)$ is bijective for all $s \in S$ by the hypothesis of (4). Since (4) and (2) are equivalent, we have that $\theta : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is bijective, as well. Then, $[\theta_2(s)]^{-1} \circ \theta = \theta_1(s)$ is bijective for all $s \in S$. So, $\theta_1(s)$ and $\theta_2(s)$ are both bijective for all $s \in S$, and, hence, (5) follows.

It suffices to prove (5)$\Longrightarrow$(1). We assume that (5) holds. Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$ and $Q \in \operatorname{Spec}(S)$ with $\mathfrak{p} \subset \mathfrak{q}$ and Q lying over $\mathfrak{q}$. Since $R \subset S$ is an integral extension, there exists a $P \in \operatorname{Spec}(S)$ lying over $\mathfrak{p}$, and $R \subset S$ satisfies the going up property. Therefore, there exists a $Q' \in \operatorname{Spec}(S)$ with $P \subset Q'$ and $Q' \cap R = \mathfrak{q} = Q \cap R$. By the hypotheses of (5), $\theta_2(s) \circ \theta_1(s) = \theta : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is bijective, and hence $Q = Q'$. Thus (1) follows. ∎

If R is universally catenary and (R_1), then Theorem 1.12 and Corollary 2.8 produce the following characterization of perinormality.

Corollary 2.9. *Let* R *be a universally catenary domain with integral closure* R' *in* $K := R_{(0)}$. *Then* R *is perinormal if and only if* R *is* (R_1) *and for each* $\mathfrak{p} \in \operatorname{Spec}(R)$, *if* $s \in R'_{\mathfrak{p}}$ *is such that* $R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}[s]$ *satisfies the going down property, then* $s \in R_{\mathfrak{p}}$.

Proof. ($\Longrightarrow$) Suppose that R is perinormal. Then R is (R_1) by [9] Theorem 4.7. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and suppose $s \in R'_{\mathfrak{p}}$ is such that $R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}[s]$ satisfies the going down property. Then $R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}[s]$ is unibranched by Corollary 2.6. So, $\mathrm{Spec}(R_{\mathfrak{p}}[s]) \to \mathrm{Spec}(R_{\mathfrak{p}})$ is a bijection which, by necessity, is order-relecting. (This is so since $R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}[s]$ is an integral extension, and thus must satisfy the going up property, and since $\theta_{\mathfrak{p}}(s) : \mathrm{Spec}(R_{\mathfrak{p}}[s]) \to \mathrm{Spec}(R_{\mathfrak{p}})$ is injective. So if $P_1, P_2 \in \mathrm{Spec}(R_{\mathfrak{p}})$, and $Q_1, Q_2 \in \mathrm{Spec}(R_{\mathfrak{p}}[s])$ with $P_1 \subset P_2$, and $Q_i \cap R_{\mathfrak{p}} = P_i$ for $i = 1, 2$, then, by the going up property, there is a $Q_2' \in \mathrm{Spec}(R_{\mathfrak{p}}[s])$ such that $Q_2' \cap R_{\mathfrak{p}} = P_2$ and $Q_1 \subset Q_2'$. Since $\theta_{\mathfrak{p}}(s)$ is injective, then $Q_1 \subset Q_2' = Q_2$, and the order-reflection is established.) In particular $R_{\mathfrak{p}}[s]$ is local. Since R is perinormal, Theorem 1.12 implies that $R_{\mathfrak{p}}[s] = R_{\mathfrak{p}}$, and, therefore, $s \in R_{\mathfrak{p}}$.

($\Longleftarrow$) Now assume that R is (R_1) and for each $\mathfrak{p} \in \mathrm{Spec}(R)$, if $s \in R_{\mathfrak{p}}'$ is such that $R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}[s]$ satisfies the going down property, then $s \in R_{\mathfrak{p}}$. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and suppose that $R_{\mathfrak{p}} \subseteq (S, \mathfrak{m}) \subseteq R_{\mathfrak{p}}'$ is such that $\mathrm{Spec}(S) \to \mathrm{Spec}(R_{\mathfrak{p}})$ is an order-reflecting bijection. Then $R_{\mathfrak{p}} \subseteq S$ is unibranched. By Corollary 2.8, this is equivalent to $R_{\mathfrak{p}} \subset R_{\mathfrak{p}}[s]$ satisfying the going down property for each $s \in S$. By the hypothesis, $s \in R_{\mathfrak{p}}$ for each $s \in S$. Thus $S = R_{\mathfrak{p}}$. ∎

2.2 Sufficient Conditions for Perinormality

In this section, unless otherwise specified, R will stand for an integral domain and R' for the integral closure of R (in $K := R_{(0)}$). Furthermore, we shall assume that R' **is a finitely generated** R**-module**.

Lemma 2.10. *For each* $\mathfrak{p} \in \operatorname{Spec}(R)$ *there are only finitely many primes in* $\operatorname{Spec}(R')$ *lying over* $\mathfrak{p}$.

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Since $R_\mathfrak{p} \subseteq R'_\mathfrak{p}$ is an integral extension, the set $\{\mathfrak{P} \in \operatorname{Spec}(R'_\mathfrak{p}) |\ \mathfrak{P} \cap R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}\}$ is the maximal spectrum, $\operatorname{Max}(R'_\mathfrak{p})$, of $R'_\mathfrak{p}$. Additionally, if $\mathfrak{P} \in \{\mathfrak{P} \in \operatorname{Spec}(R'_\mathfrak{p}) |\ \mathfrak{P} \cap R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}\}$, then $\mathfrak{p}R'_\mathfrak{p} \subseteq \mathfrak{P}$. Thus, $\operatorname{Max}(R'_\mathfrak{p}) = \{\mathfrak{P} \in \operatorname{Spec}(R'_\mathfrak{p}) |\ \mathfrak{P} \cap R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}\} = \{\mathfrak{P} \in \operatorname{Spec}(R'_\mathfrak{p}) |\ \mathfrak{p}R'_\mathfrak{p} \subseteq \mathfrak{P}\}$. Since R' is finitely generated over R, then $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ is finitely generated over the noetherian ring $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ (given that $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a field). Hence, $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ is a noetherian ring in its own right. So, $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ has finitely many minimal prime ideals. By the above tri-set-equality, $\operatorname{Max}(R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}) = \operatorname{Spec}(R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p})$. It follows that $\operatorname{Max}(R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p})$ is the set of all minimal prime ideals in $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$, and so is finite. But, $\operatorname{Max}(R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p})$ is in 1-1 correspondence with $\operatorname{Max}(R'_\mathfrak{p}) = \{\mathfrak{P} \in \operatorname{Spec}(R'_\mathfrak{p}) |\ \mathfrak{P} \cap R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}\}$ which, in turn, is in 1-1 correspondence with $\{P \in \operatorname{Spec}(R') |\ P \cap R = \mathfrak{p}\}$. ∎

The Lemma, below, gives us a glimpse of ideal sharing between R and its integral closure.

Lemma 2.11. *The conductor*

$$I = R :_R R' = \{a \in R |\ aR' \subseteq R\}$$

is a non-zero ideal of R *and of* R'.

Proof. Since R' is finitely generated over R, say, by $b_1/r_1, \ldots, b_l/r_l$ where $b_i, r_i \in R\backslash\{0\}$, then $r := r_1 \cdots r_l$ is a non-zero element of I. Let $c, d \in I$, and $r', r'' \in R'$. Then $(c-d)r', (r''c)r' \in R$ since $(c-d)r' = cr' - dr'$, $(r''c)r' = c(r''r')$, and $cr', dr' \in R$, and $r''r' \in R'$. Thus I is an ideal common to R and R'. ∎

In the following, we see the power that shared ideals have over special localizations of ring extensions.

Lemma 2.12 ([9, Corollary 4.5])**.** *Let $R \subseteq S$ be an arbitrary ring extension and I an ideal of both R and S. Then, for any $P \in \operatorname{Spec}(S)$ with $I \not\subset P$, the inclusion map $R_{P\cap R} \to S_P$ is a ring isomorphism.*

The above lemma and specific information about the conductor ideal I forces equality of classes of rings under discussion.

Proposition 2.13. *Let R be a noetherian domain with integral closure R' finitely generated as an R-module. Let $I = R :_R R'$. Let*

(i) $V_R(I) = \{\mathfrak{m}_0\}$ for some $\mathfrak{m}_0 \in \operatorname{Max}(R)$, and

(ii) $R/\mathfrak{m}_0 = R'/\mathfrak{n}_i$ for all $\mathfrak{n}_i \in \operatorname{Max}(R')$ lying over $\mathfrak{m}_0$.

Then the following are equivalent.

1. *R is perinormal in R',*

2. *R is weakly normal,*

3. *R is seminormal.*

Proof. (1)$\Longrightarrow$(2) Let R be perinormal in R'. Let S be an integral overring of R with $\theta : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ a bijection such that, whenever $P \in \mathrm{Spec}(S)$ and $\mathfrak{p} := P \cap R$, the field extension $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \subseteq S_P/PS_P$ is purely inseparable. It suffices to show that $S = R$. Let $\mathfrak{m} \in \mathrm{Max}(R)$. Since θ is a bijection, there is a unique $M \in \mathrm{Max}(S)$ with $M \cap R = \mathfrak{m}$. Note that $(S_\mathfrak{m}, M_\mathfrak{m})$ is a local ring and that $R_\mathfrak{m} \subseteq S_\mathfrak{m} \subseteq R'_\mathfrak{m}$. Also, $M_\mathfrak{m} \cap R_\mathfrak{m} = \mathfrak{m}R_\mathfrak{m}$, and $R_\mathfrak{m} \subseteq S_\mathfrak{m}$ satisfies the going down property since θ is bijective and $R \subseteq S$ satisfies the going up property, because the ring extension $R \subseteq S$ is integral. Since R be perinormal in R', then $R_\mathfrak{m} = S_\mathfrak{m} = S_M$. Because R and S are integral domains, then

$$R = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} R_\mathfrak{m}, S = \bigcap_{M \in \mathrm{Max}(S)} S_M.$$

Since $\mathrm{Max}(R)$ and $\mathrm{Max}(S)$ are in 1-1 correspondence, and $R_\mathfrak{m} = R_{\theta(M)} = S_M$ then $R = S$.

(2)$\Longrightarrow$(3) This implication holds by Proposition 1.6.

(3)$\Longrightarrow$(1). Let R be seminormal. Define

$$X = \{\mathfrak{P} \in \mathrm{Spec}(R') |\ I \not\subset \mathfrak{P}\} \qquad \text{and}$$

$$Y = \{\mathfrak{P} \cap R |\ \mathfrak{P} \in X\} \subset \mathrm{Spec}(R)$$

Then $I \not\subset \mathfrak{p}$ for any $\mathfrak{p} \in Y$. The map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ induces a surjective map $\sigma : X \to Y$. If $\mathfrak{P}_1, \mathfrak{P}_2 \in X$ are such that $\mathfrak{P}_1 \cap R = \mathfrak{p} = \mathfrak{P}_2 \cap R$ then, by Lemma 2.12, $R'_{\mathfrak{P}_1} = R_\mathfrak{p} = R'_{\mathfrak{P}_2}$. Thus, $\mathfrak{P}_1 = \mathfrak{P}_2$ and, hence, σ is injective. Thus σ is bijective. Let $X' := \{\mathfrak{P} \in \mathrm{Spec}(R') |\ \mathfrak{P} \cap R \subsetneq \mathfrak{m}_0\} \subseteq X$ and $Y' := \{\mathfrak{p} \in \mathrm{Spec}(R) |\ \mathfrak{p} \subsetneq \mathfrak{m}_0\} = \{\mathfrak{P} \cap R |\ \mathfrak{P} \in X'\} \subseteq Y$. By definition of X' and Y', it is clear that the restriction $\sigma| \ : X' \to Y'$ of σ is well-defined. Since σ is injective, then so is $\sigma|$. The ring extension $R \subseteq R'$ is integral. So, if $\mathfrak{p} \in Y'$, then there is a $\mathfrak{P} \in \operatorname{Spec}(R')$ such that $\mathfrak{p} = \mathfrak{P} \cap R$. If $\mathfrak{P} \notin X'$, then either $\mathfrak{p} = \mathfrak{P} \cap R = \mathfrak{m}_0$, or $\mathfrak{p} = \mathfrak{P} \cap R \not\subseteq \mathfrak{m}_0$, both of which contradict the fact that $\mathfrak{p} \subsetneq \mathfrak{m}_0$. Thus, $\mathfrak{P} \in X'$, making $\sigma|$ surjective, and, hence, bijective.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and suppose that $R_\mathfrak{p} \subseteq (T, M) \subseteq R'_\mathfrak{p}$. Since $R_\mathfrak{p} \subseteq T$ is an integral extension, then $M \cap R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Suppose that $R_\mathfrak{p} \subseteq T$ satisfies the going down property. We want to prove that $R_\mathfrak{p} = T$. Since $\operatorname{Spec}(R) = Y \cup \{\mathfrak{m}_0\}$, we only need analyze two cases.

- Case 1: Suppose $\mathfrak{p} \in Y$. Then there exists a unique $\mathfrak{P} \in X$ lying over $\mathfrak{p}$ and, by Lemma 2.12, we see

$$R'_\mathfrak{P} = R_\mathfrak{p} \subseteq T \subseteq R'_\mathfrak{p} \subseteq R'_\mathfrak{P} \tag{2.1}$$

  hence $R_\mathfrak{p} = T$.

- Case 2: Suppose $\mathfrak{p} = \mathfrak{m}_0$. We shall show that $\operatorname{Spec}(T) \to \operatorname{Spec}(R_{\mathfrak{m}_0})$ is a bijection, and that $(R_{\mathfrak{m}_0})_\mathfrak{q}/\mathfrak{q}(R_{\mathfrak{m}_0})_\mathfrak{q} \cong T_Q/QT_Q$, for all $Q \in \operatorname{Spec}(T)$ lying over $\mathfrak{q}R_{\mathfrak{m}_0} \in \operatorname{Spec}(R_{\mathfrak{m}_0})$. Since R is seminormal, and seminormality is a local property, then $R_{\mathfrak{m}_0}$ is seminormal, also. Hence, $R_{\mathfrak{m}_0} = T$.

  Proof of the bijectivity of $\operatorname{Spec}(T) \to \operatorname{Spec}(R_{\mathfrak{m}_0})$**:**

  Note that I is an ideal of T as well, since $I \subseteq IT \subseteq IR' = I$. Let $P \in \operatorname{Spec}(T)$ with $I \subset P$. Then $P \cap R = \mathfrak{m}_0$. Since the ring extension $R_{\mathfrak{m}_0} \subseteq T$ is integral, $R_{\mathfrak{m}_0}$ is local, and $P \cap R = \mathfrak{m}_0$, it follows that $P = M$. So, $V_T(I) = \{M\}$. Let $V := \{\mathfrak{P}_{\mathfrak{m}_0} | \ \mathfrak{P} \in X'\}$ and $W := \{\mathfrak{p}_{\mathfrak{m}_0} | \ \mathfrak{p} \in Y'\}$. Note that X' and V, and Y' and W are in 1-1 correspondence, respectively. Then, the map $\sigma|_{\ \mathfrak{m}_0} : V \to W$ induced by $\sigma|$ is a bijection, and filters through $\operatorname{Spec}(T) \backslash \{\mathfrak{m}_0\}$ via the map

$\sigma|_{\mathfrak{m}_0,1} : \operatorname{Spec}(T)\backslash\{\mathfrak{m}_0\} \to W$. It follows that $\sigma|_{\mathfrak{m}_0,1}$ is a bijection as well, since $T \subset R'_{\mathfrak{m}_0}$ is an integral ring extension. As $R_{\mathfrak{m}_0}$ and T are both local, $\operatorname{Spec}(T)$ and $W \cup \{\mathfrak{m}_0\} = \operatorname{Spec}(R_{\mathfrak{m}_0})$ are in 1-1 correspondence, and, therefore, $\operatorname{Spec}(T) \to \operatorname{Spec}(R_{\mathfrak{m}_0})$ is a bijection.

Proof of the isomorphism between $(R_{\mathfrak{m}_0})_{\mathfrak{q}}/\mathfrak{q}(R_{\mathfrak{m}_0})_{\mathfrak{q}}$ and T_Q/QT_Q, for all $Q \in \operatorname{Spec}(T)$ lying over $\mathfrak{q}R_{\mathfrak{m}_0} \in \operatorname{Spec}(R_{\mathfrak{m}_0})$:

We observe that the bijection $\operatorname{Spec}(T) \to \operatorname{Spec}(R_{\mathfrak{m}_0})$ forces $T_{\mathfrak{q}}$ to be local for each $\mathfrak{q} \in \operatorname{Spec}(R_{\mathfrak{m}_0})$. Let $Q \in \operatorname{Spec}(T)$ with $Q \cap R_{\mathfrak{m}_0} = \mathfrak{q}$. Then $T_{\mathfrak{q}} = T_Q$. If $\mathfrak{q} \neq \mathfrak{m}_0$, then, $(R_{\mathfrak{m}_0})_{\mathfrak{q}} = T_Q$ by Lemma 2.12. Thus, $(R_{\mathfrak{m}_0})_{\mathfrak{q}}/\mathfrak{q}(R_{\mathfrak{m}_0})_{\mathfrak{q}} \cong T_Q/QT_Q$, for all $Q \in \operatorname{Spec}(T)$ lying over $\mathfrak{q}R_{\mathfrak{m}_0} \in \operatorname{Spec}(R_{\mathfrak{m}_0})\backslash\{\mathfrak{m}_0 R_{\mathfrak{m}_0}\}$.

Hypothesis (ii) implies

$$R_{\mathfrak{m}_0}/\mathfrak{m}_0 R_{\mathfrak{m}} \cong R/\mathfrak{m}_0 \subseteq T/M \subseteq R'/\mathfrak{n}_i = R/\mathfrak{m}_0, \tag{2.2}$$

and so $R_{\mathfrak{m}_0}/\mathfrak{m}_0 R_{\mathfrak{m}} \cong T_M/MT_M$.

Hence, $(R_{\mathfrak{m}_0})_{\mathfrak{q}}/\mathfrak{q}(R_{\mathfrak{m}_0})_{\mathfrak{q}} \cong T_Q/QT_Q$, for all $Q \in \operatorname{Spec}(T)$ lying over $\mathfrak{q}R_{\mathfrak{m}_0} \in \operatorname{Spec}(R_{\mathfrak{m}_0})$.

Cases 1 and 2 show, for all $\mathfrak{p} \in \operatorname{Spec}(R)$, that $R_{\mathfrak{p}}$ is the only local ring between $R_{\mathfrak{p}}$ and $R'_{\mathfrak{p}}$ that is centered on $\mathfrak{p}R_{\mathfrak{p}}$ and satisfies the going down property over $R_{\mathfrak{p}}$. Therefore R is perinormal in R'. ∎

For noetherian domains, the concepts of perinormality and normality coincide whenever $R \subseteq R'$ has going down. Further, we do not require that R' be R-module finite.

Proposition 2.14. *Let R be a noetherian domain with integral closure R' in $K :=$*

$R_{(0)}$. *We suppose that the ring extension* $R \subseteq R'$ *satisfies the going down property. Then* R *is perinormal if and only if* R *is normal.*

Proof. ($\Longrightarrow$) Suppose that R is perinormal. Then so is $R_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. Let $\mathcal{N}(\mathfrak{m}) := \{\mathfrak{n} \in \mathrm{Spec}(R') |\ \mathfrak{n} \cap R = \mathfrak{m}\}$. Since $R \subseteq R'$ is an integral extension, then $\emptyset \neq \mathcal{N}(\mathfrak{m}) \subseteq \mathrm{Max}(R')$. Again, because $R \subseteq R'$ is integral, and, therefore, satisfies the going up property, it follows that

$$\mathrm{Max}(R') = \bigcup_{\mathfrak{m} \in \mathrm{Max}(R)} \mathcal{N}(\mathfrak{m}).$$

The ring extension $R_{\mathfrak{m}} \subset R'_{\mathfrak{n}}$ must, also, satisfy the going down property since $R \subseteq R'$ does. Proposition 1.3 implies that

$$R'_{\mathfrak{n}} = (R_{\mathfrak{m}})_{\mathfrak{n}R'_{\mathfrak{n}} \cap R_{\mathfrak{m}}}. \tag{2.3}$$

But $\mathfrak{n}R'_{\mathfrak{n}} \cap R_{\mathfrak{m}} = \mathfrak{m}R_{\mathfrak{m}}$, and, thus,

$$R'_{\mathfrak{n}} = R_{\mathfrak{m}}. \tag{2.4}$$

Since R and R' are domains, we obtain the equalities

$$R = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} R_{\mathfrak{m}} = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} \Big(\bigcap_{\mathfrak{n} \in \mathcal{N}(\mathfrak{m})} R'_{\mathfrak{n}} \Big) = \bigcap_{\mathfrak{n} \in \mathrm{Max}(R')} R'_{\mathfrak{n}} = R'.$$

So $R = R'$, and thus R is normal.

($\Longleftarrow$) Since R is noetherian, $R = R'$ is a Krull domain. Hence R is perinormal by Proposition 1.9 . ∎

Chapter 3

Ideal Transforms

3.1 Background and and transforms $T(xR)$

We begin with the definition of an ideal transform, first introduced by Nagata in [20].

Definition 3.1. Let R be an integral domain with quotient field K and let I be an ideal in R. The ideal transform $T(I)$ is an overring of R defined as

$$T(I) = \{a \in K \mid aI^n \subset R \text{ for some } n \in \mathbb{N}\} = \bigcup_{n=1}^{\infty} (R :_K I^n).$$

The prime spectrum of $T(I)$ is partially characterized by the following lemma.

Lemma 3.2 ([20, Lemma 3])**.** *There is a one-to-one correspondence between prime ideals of $T(I)$ not containing I and prime ideals of R not containing I, given by $P \mapsto P \cap R$. In fact, $T(I)_P = R_{P \cap R}$ for all such $P \in \operatorname{Spec}(T(I))$.*

In [3], Brewer described explicitly the structure of ideal transforms of finitely generated ideals I. Namely, $T(I) = \bigcap_{\mathfrak{p} \notin V_R(I)} R_{\mathfrak{p}}$. Moreover, for a non-local domain R, Brewer showed that R is the intersection of all the ideal transforms of proper principal ideals.

Proposition 3.3 ([3, Theorem 2.1])**.** *Suppose that R is an integral domain with more than one maximal ideal. Let $\{x_\alpha\}_\alpha$ be the collection of non-units of $R - \{0\}$. Then*

$R = \bigcap_{\alpha} T(x_\alpha R)$.

In addition, Brewer established that the ideal transform of a proper principal ideal is a localization of R.

Lemma 3.4 ([3, Lemma 2.2])**.** *Suppose that R is an integral domain with more than one maximal ideal. Let x be a nonzero element of R. Then $T(xR) = R[x^{-1}] = R_x$.*

These facts provide the background to showing that a non-local domain R is perinormal if and only if $T(xR)$ is perinormal for each non-unit $x \in R\backslash\{0\}$.

Lemma 3.5. *Suppose that R is a domain with quotient field $K := R_{(0)}$, such that R has more than one maximal ideal. Let $R \subset (S, \mathfrak{n}) \subsetneq K$ be ring extensions. Then there exists a non-unit $x \in R\backslash\{0\}$ such that $R_x \subset (S, \mathfrak{n})$. Further, $x \notin \mathfrak{n}$.*

Proof. By way of contradiction suppose that $x^{-1} \notin (S, \mathfrak{n})$ for all non-units $x \in R\backslash\{0\}$. Then $x \in \mathfrak{n}$ for all such x, yielding that $\mathfrak{n} \cap R$ consists of all the non-units of R, and is, thus, the unique maximal ideal of R, a contradiction. Since x is a unit in R_x, it is also a unit in $(S, \mathfrak{n})$, so $x \notin \mathfrak{n}$. ∎

We are, now, in the position to see that perinormality of a ring is reciprocally bound to of perinormality of its principal ideal transforms.

Proposition 3.6. *Suppose that R is an integral domain with more than one maximal ideal. Then R is perinormal if and only if $T(xR)$ is perinormal for each non-unit $x \in R\backslash\{0\}$.*

Proof. Suppose that R is perinormal. Since perinormality localizes, then, by Lemma 3.4, $T(xR) = R_x$ is perinormal for each non-unit $x \in R\backslash\{(0)\}$.

We now prove the converse. Let $(S, \mathfrak{n})$ be an overring of R such that $R \subset (S, \mathfrak{n})$ satisfies the going down property. We want to show that $S = R_{\mathfrak{n} \cap R}$.

By Lemma 3.5 there exists a non-zero, non-unit $y \in R \backslash \mathfrak{n}$ such that $R_y \subseteq (S, \mathfrak{n})$. We consider the the ring extensions $R \subset T(yR) = R_y \subset (S, \mathfrak{n})$. By hypothesis $T(yR) = R_y$ is perinormal. If we show that the going down property holds for $R_y \subset (S, \mathfrak{n})$, then $S = (R_y)_{\mathfrak{n} \cap R_y}$ by Proposition 1.3. Let $\mathfrak{p} = \mathfrak{n} \cap R$. Then $y \notin \mathfrak{p}$, and $\mathfrak{n} \cap R_y = \mathfrak{p}_y$. So, $(R_y)_{\mathfrak{n} \cap R_y} = (R_y)_{\mathfrak{p}_y} = R_{\mathfrak{p}} = R_{\mathfrak{n} \cap}$. Hence $S = R_{\mathfrak{n} \cap R}$.

Proof that $R_y \subset (S, \mathfrak{n})$ satisfies the going down property:

Suppose $P_1 \subsetneq P_2$ in R_y and that there exists $\mathfrak{P}_2 \in \operatorname{Spec}(S)$ with $\mathfrak{P}_2 \cap R_y = P_2$. Let $\mathfrak{p}_i = P_i \cap R = \mathfrak{P}_i \cap R$ for $i = 1, 2$. Then $P_i = (\mathfrak{p}_i)_y$ and $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. Since $R \subseteq S$ satisfies the going down property, then there is a $\mathfrak{P}_1 \in \operatorname{Spec}(S)$ such that $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$ and $\mathfrak{P}_1 \cap R = \mathfrak{p}_1$. So, $\mathfrak{P}_1 \cap R_y = (\mathfrak{P}_1)_y \cap R_y = (\mathfrak{P}_1 \cap R)_y = (\mathfrak{p}_1)_y = P_1$. Hence $R_y \subset S$ satisfies the going down property. ∎

3.2 Integrality and Perinormality of $T(\mathfrak{m})$

The following result by Nishimura allows us to generalize previous statements tied to principal ideal transforms in the setting of noetherian rings.

Proposition 3.7 ([23, Lemma 1.1, eq. (1.1.1)])**.** *Let $I = (a_1, \ldots, a_n)$ be an ideal in a noetherian ring R where each a_i is not a zero-divisor. Then $T(I) = \bigcap_{i=1}^{n} R_{a_i}$.*

Remark 3.8. Lemma 3.4 is true for local noetherian domains by Proposition 3.7.

Remark 3.9. Let $(R, \mathfrak{m})$ be a local, noetherian domain and $\mathfrak{m} = (a_1, \ldots, a_n)R$, so that $T(\mathfrak{m}) = \bigcap_{i=1}^{n} R_{a_i}$. Then none of the domains $T(a_i R) = R_{a_i}$ is integral over R, for otherwise a_i would be a unit in R and could not be a generator for $\mathfrak{m}$.

Let R' denote the integral closure of the local domain $(R, \mathfrak{m})$ in its quotient field K. The following describes, among other properties of $T(\mathfrak{m})$, a sufficient condition for the $R \subset T(\mathfrak{m})$ to be an integral extension.

Proposition 3.10 ([24, Remarqué]). *Suppose that all maximal ideals of R' have height at least 2. Then*

1. *$T(\mathfrak{m})$ is integral over R,*

2. *$(T(\mathfrak{m}), \mathfrak{n}_1, \ldots, \mathfrak{n}_l)$ is a semi-local domain.*

3. *$\operatorname{depth} T(\mathfrak{m})_{n_i} > 1$ for each maximal ideal $\mathfrak{n}_i$ of $T(\mathfrak{m})$.*

This result by Querré was later improved upon by Nishimura, as seen below.

Proposition 3.11 ([23, Corollary 1.7]). *Let $(R, \mathfrak{m})$ be a local noetherian domain. Then $T(\mathfrak{m})$ is integral over R if and only if R' has no maximal ideals of height one.*

Let $(R, \mathfrak{m}) \subset T(\mathfrak{m})$ be an integral extension with $\operatorname{Max}(T(\mathfrak{m})) = \{\mathfrak{n}_1, \ldots, \mathfrak{n}_l\}$. Let $X = \operatorname{Spec}(T(\mathfrak{m})) - \operatorname{Max}(T(\mathfrak{m}))$ and $Y = \operatorname{Spec}(R) - \{\mathfrak{m}\}$. The map $\operatorname{Spec}(T(\mathfrak{m})) \to \operatorname{Spec}(R)$ is surjective, and $\mathfrak{n}_i \mapsto \mathfrak{m}$ for each $i = 1, \ldots, l$. Lemma 3.2 implies that the induced map $X \to Y$ is a bijection and that $T(\mathfrak{m})_Q = R_{Q \cap R}$ for all $Q \in X$. Since perinormality localizes, it follows that if $T(\mathfrak{m})$ is perinormal, then $R_{\mathfrak{q}}$ is perinormal for all $\mathfrak{q} \neq \mathfrak{m}$.

Proposition 3.12. *Let $(R, \mathfrak{m})$ be a noetherian local domain of dimension greater than 1, and suppose that R is analytically irreducible. If R is perinormal then $R = T(\mathfrak{m})$.*

Proof. By Proposition 3.11 the extension $R \subset T(\mathfrak{m})$ is integral. As in the preamble to this proposition, letting $X = \operatorname{Spec}(T(\mathfrak{m})) - \operatorname{Max}(T(\mathfrak{m}))$ and $Y = \operatorname{Spec}(R) - \{\mathfrak{m}\}$, the map $X \to Y$ is bijective, with $T(\mathfrak{m})_P = R_{P \cap R}$ for each $P \in X$. The integral closure R' of R is a local domain by the analytic irreducibility of R (see Proposition 5.5 and Exercise 5.7). Since R' is integral over $T(\mathfrak{m})$, it follows that $T(\mathfrak{m})$ is local, also. Thus $X \to Y$ extends to a bijection $\operatorname{Spec}(T(\mathfrak{m})) \to \operatorname{Spec}(R)$. The map is order-reflecting by the going up property of $R \subset T(\mathfrak{m})$, all of this, in turn, yielding the going down property for $R \subset T(\mathfrak{m})$. In summary, $T(\mathfrak{m})$ is a local overring of $(R, \mathfrak{m})$ with the going down property. By Corollary 1.3, $R = T(\mathfrak{m})$. ∎

Remark 3.13. It is known that $R = T(I)$ if and only of $\operatorname{grade}(I) > 1$. (See the discussion following [28, Lemma 2.8].) So a local, perinormal domain $(R, \mathfrak{m})$ satisfying the conditions of Corollary 3.12 has $\operatorname{depth}(R) > 1$.

Example 3.14. It is easy to see that the 2-dimensional domain $R = \mathbb{C}[x^4, x^3y, xy^3, y^4]$ is not perinormal. Namely, R is not seminormal (hence, not perinormal) since $b = x^2y^2$ is such that $b^2, b^3 \in R$ with $b \notin R$. Still, we shall demonstrate the failure of R to be perinormal using Proposition 3.12 and Remark 3.13, as it points to an obstruction for perinormality in rings of low depth.

Let $\mathfrak{m} := (x^4, x^3y, xy^3, y^4) \in \operatorname{Max}(R)$. We shall show that $R_\mathfrak{m}$ is not perinormal. Since perinormality localizes, it will follow that R is not perinormal.

First we establish that $\operatorname{depth}(R_\mathfrak{m}) = 1$**:**

We shall do so by showing that a system of parameters for $R_\mathfrak{m}$ is not a regular $R_\mathfrak{m}$-sequence. Thereby, the 2-dimensional $R_\mathfrak{m}$ is not Cohen-Macaulay, and hence $\operatorname{depth}(R_\mathfrak{m}) \leqslant 1$. Note that $\mathfrak{m}_\mathfrak{m}^4 \subset (x^4, y^4)_\mathfrak{m}$. Abusing notation, $\{x^4, y^4\}$ is a system of parameters for $R_\mathfrak{m}$. But $\{x^4, y^4\}$ is not an $R_\mathfrak{m}$-regular sequence since y^4 is a zero-divisor on $R_\mathfrak{m}/(x^4)_\mathfrak{m}$. This is so because the product $y^4(x^3y)^2 = x^4(xy^3)^2 \in (x^4)_\mathfrak{m}$, but

$(x^3y)^2 \notin (x^4)_{\mathfrak{m}}$ since $(x^3y)^2 = x^4(x^2y^2)$ and $x^2y^2 \notin R_{\mathfrak{m}}$. So, the system of parameters $\{x^4, y^4\}$ is not a regular $R_{\mathfrak{m}}$-sequence. Thus $\text{depth}(R_{\mathfrak{m}}) = 1$.

Next we show that the integral closure $R'_{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ in its fraction field is a local ring:

It suffices to show that the fiber ring $R'/\mathfrak{m}R'$ is local, i.e., that there is only one maximal ideal $\mathfrak{m}'$ in R' lying over $\mathfrak{m}$, thereby confirming that $R'_{\mathfrak{m}}$ is local.

Notice $x^2y^2 = [(x^3y)^2]/x^4$ is in the quotient field of R and is a root of $T^2 - x^4y^4 \in R[T]$, thus the integral closure of R in its quotient field is $R' = \mathbb{C}[x^4, x^3y, x^2y^2, xy^3, y^4] = R[x^2y^2]$. (See [5, Example 1.3.9].)

Notice that any element of R' has the form $a+bx^2y^2+m$ where $m \in (x^4, x^3y, xy^3, x^4)R' = \mathfrak{m}R' = \mathfrak{m}$ (in particular all powers of x^2y^2 are achievable by elements of $\mathfrak{m}R'$). Consider the epimorphism $R' \to \mathbb{C}[T]/(T^2)$, given by $a + bx^2y^2 + m \mapsto a + bT + T^2\mathbb{C}[T]$. Its kernel is $\{a + bx^2y^2 + m \mid a = b = 0\} = \mathfrak{m}R'$. Thus $R'/\mathfrak{m}R'$ is isomorphic to the local ring $\mathbb{C}[T]/(T^2)$. It follows that $\mathfrak{m}' = (x^4, x^3y, x^2y^2, xy^3, y^4)R'$ is the only maximal ideal lying over $\mathfrak{m}$.

Finally, we show that the $\mathfrak{m}_{\mathfrak{m}}$-adic completion $\widehat{R_{\mathfrak{m}}}$ is a domain:

Since the $\mathfrak{m}_{\mathfrak{m}}$-adic completion $\widehat{R_{\mathfrak{m}}}$ of $R_{\mathfrak{m}}$ is a subring of the $\mathfrak{m}_{\mathfrak{m}}$-adic completion $\widehat{R'_{\mathfrak{m}}}$ of $R'_{\mathfrak{m}}$, it suffices to show that $\widehat{R'_{\mathfrak{m}}}$ is a domain. Now, $R'_{\mathfrak{m}} = R'_{\mathfrak{m}'}$. Since the $\mathfrak{m}$-adic and $\mathfrak{m}'$-adic topologies of $R'_{\mathfrak{m}'}$ are the same, we only need show that the $\mathfrak{m}'_{\mathfrak{m}'}$-adic completion $\widehat{R'_{\mathfrak{m}'}}$ of $R'_{\mathfrak{m}'}$ is a domain. We, next, note that

$$R'_{\mathfrak{m}'} = \mathbb{C}[x^4, x^3y, x^2y^2, xy^3, y^4]_{(x^4,x^3y,x^2y^2,xy^3,y^4)}$$

is an excellent normal domain. But, excellent normal domains are analytically so. Thus, $\widehat{R'_{\mathfrak{m}'}}$ is a domain.

Conclusion:

So $R_{\mathfrak{m}}$ is a noetherian, local, analytically irreducible domain of Krull dimension greater than 1. By Proposition 3.12 and Remark 3.13 the domain $R_{\mathfrak{m}}$ is not perinormal.

Chapter 4

Polynomial Rings

4.1 Preliminary Results

Let R be a domain and let $R[X]$ denote the polynomial ring over R in the indeterminate X. As a parting gesture in [9], the authors suggest exploring whether the perinormality of one of the domains R, or $R[X]$ implies the perinormality of the other. We will assume that R is universally catenary, so that we are afforded the characterization of perinormality given in Theorem 1.12. First, we describe the ascent and descent of the (R_1)-property between R and $R[X]$.

Lemma 4.1. *If R is an (R_1), universally catenary ring, then so is $R[X]$.*

Proof. By the definition of the universally catenary property, $R[X]$ inherits this property from R. Let $P \in X^1(R[X])$ and $\mathfrak{p} := P \cap R$. Then $\mathrm{ht}_R(\mathfrak{p}) \leqslant 1$. Since R is (R_1), then $R_\mathfrak{p}$ is regular, and, hence, $R_\mathfrak{p}[X]$ is regular. Now, $R[X]_P = (R_\mathfrak{p}[X])_{P_\mathfrak{p}}$ is regular since the localization of the regular ring $R_\mathfrak{p}[X]$ is regular. Thus $R[X]$ is also (R_1). ∎

In a more general setting, we have the following lemma.

Lemma 4.2 ([14, Theorem 4.5.5])**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local map of noetherian rings, and let $k \in \mathbb{N}$.*

(a) If S satisfies the regularity condition (R_k), then so does R. If S satisfies the Serre condition (S_k), then so does R.

(b) If R and $\kappa(\mathfrak{p}) \otimes_R S$, for all $\mathfrak{p} \in \operatorname{Spec}(R)$, satisfy the regularity condition (R_k), then so does S. If R and $\kappa(\mathfrak{p}) \otimes_R S$, for all $\mathfrak{p} \in \operatorname{Spec}(R)$, satisfy the Serre condition (S_k), then so does S.

The following lemma, together with the proof of Lemma 4.1, shows the ascent/descent quality of R_1 between the ring and its polynomial ring extension.

Lemma 4.3. *Let R be a noetherian domain. If $R[X]$ satisfies (R_1), then R satisfies (R_1).*

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $R_\mathfrak{p} \subseteq R[X]_{\mathfrak{p}R[X]}$ is a flat local homomorphism. Since $R[X]_{\mathfrak{p}R[X]}$ is (R_1), by Lemma 4.2 so is $R_\mathfrak{p}$. So, for any prime $\mathfrak{q}R_\mathfrak{p} \in X^1(R_\mathfrak{p})$, it follows that $(R_\mathfrak{p})_{\mathfrak{q}R_\mathfrak{p}} = R_\mathfrak{q}$ is a regular. Since $\mathfrak{p}$ was arbitrary we have $R_\mathfrak{q}$ is regular for any $\mathfrak{q} \in X^1(R)$. ∎

Next we present some definitions and results from [17] which illustrate the behavior of primes in extensions $R \subset S$ and $R[X] \subset S[X]$, with S an overring of R.

Definition 4.4. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. We shall denote its residue field $(R/\mathfrak{p})_\mathfrak{p}$ by $\kappa_R(\mathfrak{p})$, its extended ideal $\mathfrak{p}R[X]$ by $\mathfrak{p}^*$, and a monic, irreducible polynomial in $\kappa_R(\mathfrak{p})[X]$ by $\alpha(X)$. Finally, we define

$$\langle \mathfrak{p}, \alpha(X) \rangle := \{g(X) \in R[X] \mid \alpha(X) \text{ divides } \overline{g}(X)\}$$

where $\overline{g}(X)$ is the reduction of $g(X)$ modulo $\mathfrak{p}$.

Below we provide information about all the prime ideals of $R[X]$ lying over the prime ideal $\mathfrak{p}$ of R.

Theorem 4.5 ([17, Theorem 1])**.** *Let $\mathfrak{p}$ be a prime of R and let $\mathfrak{P}$ be a prime of $R[X]$ with $\mathfrak{P} \cap R = \mathfrak{p}$. Then either $\mathfrak{P} = \mathfrak{p}^*$ or $\mathfrak{P}$ is of the form $\langle \mathfrak{p}, \alpha(X) \rangle$. If $\langle \mathfrak{p}, \alpha(X) \rangle \subset \langle \mathfrak{p}, \alpha'(X) \rangle$ then $\langle \mathfrak{p}, \alpha(X) \rangle = \langle \mathfrak{p}, \alpha'(X) \rangle$, which in turn implies that $\alpha(X) = \alpha'(X)$. Furthermore, $\mathfrak{p}^* \subsetneq \langle \mathfrak{p}, \alpha(X) \rangle$, with $\mathfrak{p}^*, \langle \mathfrak{p}, \alpha(X) \rangle \in \operatorname{Spec}(R[X])$.*

Let $R \subset S$ be a ring extension. The following theorem presents information about prime ideals in $S[X]$ lying over the two types of prime ideals in $R[X]$.

Theorem 4.6 ([17, Theorem 2])**.** *Let $R \subset S$ be a ring extension and $\mathfrak{p}$ be a prime ideal of R.*

(i) The set of all prime ideals in $S[X]$ lying over $\langle \mathfrak{p}, \alpha(X) \rangle$ is the set

$$\{\langle \mathfrak{q}, \beta(X) \rangle |\ \mathfrak{q} \text{ is a prime ideal of } S \text{ lying over } \mathfrak{p}, \text{ and } \beta(X) \text{ divides } \alpha(X) \text{ in } \kappa_S(\mathfrak{q})[X]\},$$

where $\kappa_S(\mathfrak{q}) := (S/\mathfrak{q})_{\mathfrak{q}}$.

(ii) The set of all prime ideals in $S[X]$ lying over $\mathfrak{p}^*$ is the set

$$\begin{aligned} &\{\mathfrak{q}^* |\ \mathfrak{q} \text{ is a prime of } S \text{ lying over } \mathfrak{p}\} \\ &\cup \{\langle \mathfrak{q}, \beta(X) \rangle |\ \mathfrak{q} \text{ is a prime of } S \text{ lying over } \mathfrak{p}, \text{ and} \\ &\quad \beta(X) \in \kappa_S(\mathfrak{q})[X] \text{ does not divide any polynomial of } \kappa_R(\mathfrak{p})[X]\}. \end{aligned}$$

The definition that follows is a refinement of the concept of a unibranched extension.

Definition 4.7. The extension $R \subset S$ is a *U-extension* if:

(i) $R \subset S$ is unibranched (i.e., $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is bijective), and

(ii) for every prime $\mathfrak{q}$ of S with $\mathfrak{q} \cap R = \mathfrak{p}$, the field extension $\kappa_R(\mathfrak{p}) \subseteq \kappa_S(\mathfrak{q})$ is algebraic, and purely inseparable.

The theorem, below, will allow us to formulate a characterization for the descent of perinormality from $R[X]$ to R.

Theorem 4.8 ([17, Theorem 4])**.** *Let $R \subset S$ be domains. The ring extension $R[X] \subset S[X]$ is a U-extension if and only if the ring extension $R \subset S$ is a U-extension.*

4.2 Ascending Perinormality

This section is devoted solely to proving Proposition 4.9. In this preamble, we shall fix the notation that will permeate the theorem.

- The symbol B' will, exclusively, stand for the integral closure of the integral domain B in its field of fractions.
- Let R be a domain with $K := R_{(0)}$.
- Let $\mathfrak{P}_0 \in \mathrm{Spec}(R[X])$, and $\mathfrak{p}_0 := \mathfrak{P}_0 \cap R$.
- Let $(R[X])_{\mathfrak{P}_0} \subseteq T \subseteq (R'[X])_{\mathfrak{P}_0}$ be a tower of integral domains (and, hence, a tower of integral ring extensions since $R'[X]_{\mathfrak{P}_0}$ is the integral closure of $R[X]_{\mathfrak{P}_0}$).
- Let $\mathcal{T} := T \cap R'[X]$. Then $\mathcal{T}_{\mathfrak{P}_0} = T$ since $\mathcal{T}_{\mathfrak{P}_0} = (T \cap R'[X])_{\mathfrak{P}_0} = T_{\mathfrak{P}_0} \cap (R'[X])_{\mathfrak{P}_0} = T \cap (R'[X])_{\mathfrak{P}_0} = T$, as $T_{\mathfrak{P}_0} = T$ and $T \subseteq R'[X]_{\mathfrak{P}_0}$. (We may assume that R is local with maximal ideal $\mathfrak{p}_0$. By the Cohen-Seidenberg Theorems, any integral overring of R is local, too.)

- Let $\mathcal{T}_i := \{a_i \in R' | \ \exists$ a polynomial $a_0 + a_1X + \ldots + a_iX^i + \ldots + a_sX^s \in \mathcal{T}\}$. Each $\mathcal{T}_i$ contains R and is an R-module. Of special import is $\mathcal{T}_0$ which is a local R-algebra.

- Let $v \in R' \backslash R$, let $f(X) \in R[X]$ be the irreducible, monic polynomial such that $f(v) = 0$, and let $\mathfrak{P}_v \in \operatorname{Spec}(R[X])$ be such that $R[u] \cong R[X]/\mathfrak{P}_v$. Since $\dim R = \dim R'$, then $\mathfrak{P}_v \in X^1(R[X])$. Note that $f(X) \notin \mathfrak{p}^*$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ as $f(X)$ is monic. Thus $\mathfrak{P}_v = \langle (0), \alpha_v(X) \rangle$ for some $\alpha_v(X) \in K[X]$, monic and irreducible.

- We call an element $v \in \mathcal{T}_0 \backslash R$ *obedient* if for every $\mathfrak{P} \in \operatorname{Spec}(R[X])$ with $\langle (0), \alpha_v(X) \rangle \subset \mathfrak{P} \subseteq \mathfrak{P}_0$, there is an $\alpha(X) \in \kappa_R(\mathfrak{p})[X]$ such that for every $g(X) \in \langle (0), \alpha_v(X) \rangle$, there is an $n \in \mathbb{N}^+$ with $g(X) + \mathfrak{p}_{\mathfrak{p}}[X] = (\alpha(X))^n$. (Here, $\mathfrak{p} := \mathfrak{P} \cap R$, and $\alpha(X)$ depends on $\mathfrak{p}$ and v, while n depends on $\mathfrak{p}, v$, and $g(X)$.)

- We call the domain R *obedient* if for every prime ideal $\mathfrak{P}_0$ of $R[X]$ and every integral overring T of $R[X]_{\mathfrak{P}_0}$ such that $\operatorname{Spec}(T) \to \operatorname{Spec}((R[X])_{\mathfrak{P}_0})$ is bijective, the domain $\mathcal{T}_0$ has an obedient element.

The notation, developed above, is handed down to the proof of the Proposition.

Proposition 4.9. *Let R be a perinormal, obedient ring. Then $R[X]$ is perinormal.*

Proof. For some $\mathfrak{P}_0 \in \operatorname{Spec}(R[X])$, let $(R[X])_{\mathfrak{P}_0} \subseteq T \subseteq (R'[X])_{\mathfrak{P}_0}$ be a tower of integral domains with $\operatorname{Spec}(T) \to \operatorname{Spec}((R[X])_{\mathfrak{P}_0})$ bijective. Let $v \in \mathcal{T}_0 \backslash R$ be an obedient element. Then $R \subseteq R[v]$ is an integral extension of *local* domains. (We assume that $\mathfrak{p}_0$ is the maximal ideal of R. Let P_0 be the maximal ideal of $R[v]$. Then $\mathfrak{p}_0 = P_0 \cap R$, and there exists an $\alpha_0(X) \in \kappa_R(\mathfrak{p}_0)$ monic and irreducible such that $P_0 = \langle \mathfrak{p}_0, \alpha_0(X)\rangle / \langle (0), \alpha_v(X)\rangle$.) We will show that $\mathrm{Spec}(R[v]) \to \mathrm{Spec}(R)$ is a bijection. Note that $\mathrm{Spec}(R[v])$ is in 1-1 correspondence with the set

$$\{\langle \mathfrak{p}, \alpha(X)\rangle \subseteq \langle \mathfrak{p}_0, \alpha_0(X)\rangle |$$

$$\langle (0), \alpha_v(X)\rangle \subseteq \langle \mathfrak{p}, \alpha(X)\rangle, \text{ and } \mathfrak{p} \in \mathrm{Spec}(R), \alpha(X) \in \kappa_R(\mathfrak{p})[X], \text{ monic and irreducible}\}.$$

Since v is obedient, it follows that for each $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathfrak{p}^* \subsetneq P_0$, there is a unique $\alpha(X) \in \kappa_R(\mathfrak{p})[X]$ with $\langle (0), \alpha_v(X)\rangle \subseteq \langle \mathfrak{p}, \alpha(X)\rangle$. Hence $\mathrm{Spec}((R[v]) \to \mathrm{Spec}(R)$ is bijective. By the perinormality of R, it follows that $R = R[v]$, contradicting that $v \neq R$. Thus $\mathcal{T}_0 = R$.

We split our consideration into two cases.

(i) Suppose $X \notin \mathfrak{P}_0$.

Then $(R[X])_{\mathfrak{P}_0} = \mathcal{T}_{\mathfrak{P}_0} = T$. (This is so by the following. Let $a_0 + a_1X + \ldots + a_sX^s \in \mathcal{T}$. Then $a_0 \in \mathcal{T}_0 = R \subset R[X] \subset \mathcal{T}$ implies that $X(a_1 + a_2X + \ldots + a_sX^{s-1}) = a_1X + \ldots + a_sX^s = (a_0 + a_1X + \ldots + a_sX^s) - a_0 \in \mathcal{T}$. Since X is a unit in $(\mathcal{T})_{P_0}$, it follows that $a_1 + a_2X + \ldots + a_sX^{s-1} \in \mathcal{T}_{\mathfrak{P}_0} = T$. But, $a_1 + a_2X + \ldots + a_sX^{s-1} \in R'[X]$, also. Thus $a_1 + a_2X + \ldots + a_sX^{s-1} \in T \cap R'[X] = \mathcal{T}$. Thus $a_1 \in \mathcal{T}_0 = R$. We repeat the above reasoning until we get that $a_0, a_1, \ldots, a_s \in \mathcal{T}_0 = R$, and, hence, $a_0 + a_1X + \ldots + a_sX^s \in R[X]$, allowing us to conclude that $R[X] = \mathcal{T}$. Upon localization, we get the desired result that $(R[X])_{\mathfrak{P}_0} = \mathcal{T}_{\mathfrak{P}_0} = T$.)

(ii) Suppose $X \in \mathfrak{P}_0$.

Then $\mathfrak{P}_0 = \langle \mathfrak{p}_0, X\rangle$ by Theorem 4.5, and we see $\langle \mathfrak{p}_0, X\rangle = (\mathfrak{p}_0, X)R[X]$. Let $\mathcal{T}' := T_{\mathfrak{p}_0 R[X]} \cap R'[X]$. Consider the extension $R[X]_{\mathfrak{p}_0 R[X]} \subset T_{\mathfrak{p}_0 R[X]} \subset R'[X]_{\mathfrak{p}_0 R[X]}$.

The natural map $\psi_{\mathfrak{p}_0 R[X]} : \operatorname{Spec}(T_{\mathfrak{p}_0 R[X]}) \to \operatorname{Spec}(R[X]_{\mathfrak{p}_0 R[X]})$ is bijective. (It is routine to check that $\psi_{\mathfrak{p}_0 R[X]}$ is well-defined. The surjectivity of this map follows from the fact that $R[X]_{\mathfrak{p}_0 R[X]} \subset T_{\mathfrak{p}_0 R[X]}$ is an integral extension. The injectivity of this map follows from the fact that $\psi : \operatorname{Spec}(T) \to \operatorname{Spec}(R[X]_{\mathfrak{P}_0})$ is injective.)

Now $\operatorname{Spec}(T_{\mathfrak{p}_0 R[X]}) \to \operatorname{Spec}(R[X]_{\mathfrak{p}_0 R[X]})$ is bijective and $X \notin \mathfrak{p}_0 R[X]$, thus, by case (ii), we conclude that $\mathcal{T}' = R[X]$, since $\mathcal{T}' = T_{\mathfrak{p}_0 R[X]} \cap R'[X] = R[X]$. Then

$$R[X] \subset \mathcal{T} \subset \mathcal{T}' = R[X],$$

and, thus, $\mathcal{T} = R[X]$. Localizing at $\mathfrak{P}_0$ yields that $T = R[X]_{\mathfrak{P}_0}$.

Thus for all $\mathfrak{P} \in \operatorname{Spec}(R[X])$ we have satisfied Theorem 2.1 (c), so $R[X]$ is perinormal. ∎

The above proof establishes the fact that the only integral overring S of R such that $S[X] \subseteq \mathcal{T}$ is the ring $R[X]$.

4.3 Descending Perinormality in Certain Cases

We now turn our attention to the possibility that perinormality descends from $R[X]$ to R. We introduce the following definition.

Definition 4.10. Let R be a domain such that whenever $\mathfrak{p} \in \operatorname{Spec}(R)$, and T is an *integral* overring of R with $\operatorname{Spec}(T) \to \operatorname{Spec}(R_\mathfrak{p})$ bijective, then $R_\mathfrak{p} \subseteq T$ is a U-extension (i.e., for every $P' \in \operatorname{Spec}(R_\mathfrak{p})$, and $Q' \in \operatorname{Spec}(T)$ lying over P', the field extension $\kappa_{R_\mathfrak{p}}(P') \subseteq \kappa_T(Q')$ is purely inseparable). We call such a ring R an *U-ring.*

Proposition 4.11. *Let $R[X]$ be a universally catenary, perinormal domain. Then R is a U-ring if and only if R is perinormal.*

Proof. If $R[X]$ is perinormal, then $R[X]$ is (R_1), and, therefore, R is (R_1). (See Lemma 4.3.) If $R[X]$ is universally catenary, then so is R.

($\Longrightarrow$) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and T be an integral overring of $R_{\mathfrak{p}}$ with $\operatorname{Spec}(T) \to \operatorname{Spec}(R_{\mathfrak{p}})$ bijective. Since R is a U-ring, then $R_{\mathfrak{p}} \subseteq T$ is a U-extension. By Theorem 4.8, $R_{\mathfrak{p}}[X] \subseteq T[X]$ is a U-extension, and hence $\operatorname{Spec}(T[X]) \to \operatorname{Spec}(R_{\mathfrak{p}}[X])$ is bijective. Then $\operatorname{Spec}(T[X]_{\mathfrak{p}^*}) \to \operatorname{Spec}(R_{\mathfrak{p}}[X]_{\mathfrak{p}^*})$ is bijective also. Since $R[X]$ is perinormal, then $R_{\mathfrak{p}}[X]_{\mathfrak{p}^*} = T[X]_{\mathfrak{p}^*}$. To obtain the desired result $T = R_{\mathfrak{p}}$, we only need show that $T \subseteq R_{\mathfrak{p}}$. Let $t \in T$. Then, by the above equality, there are $f(X), s(X) \in R_{\mathfrak{p}}[X]$ with $s(X) \notin \mathfrak{p}^*$ such that $s(X)t = f(X)$. Let s_i and a_i be the i^{th} coefficients of $s(X)$ and $f(X)$, respectively. Then, there exists an $i \in \{0, 1, 2, \ldots, \deg s(X)\}$ with s_i a unit in $R_{\mathfrak{p}}$. Then, $s_i t = a_i$ implies that $t \in R_{\mathfrak{p}}$. Hence $T \subseteq R_{\mathfrak{p}}$, and, therefore $T = R_{\mathfrak{p}}$. By Theorem 2.1 (c), R is perinormal.

($\Longleftarrow$) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and T be an integral overring of $R_{\mathfrak{p}}$ with $\operatorname{Spec}(T) \to \operatorname{Spec}(R_{\mathfrak{p}}$ bijective. Let Q' be the unique prime ideal lying over $P \in \operatorname{Spec}(R_{\mathfrak{p}})$. By perinormality of R, $T = R_{\mathfrak{p}}$, and hence $R_{\mathfrak{p}} \subseteq T$ is trivially a U-extension, and, hence, R is a U-ring. ∎

Even though the above theorem gives a characterization for the descent of perinormality from $R[X]$ to R, it is, in some sense, unsatisfying in that the U-ring quality is hard to check. The following two results seem to be more accessible in this regard.

Proposition 4.12. *Let R be an (R_1), universally catenary domain with $\dim R < 3$. If $R[X]$ is perinormal then R is perinormal.*

Proof. If $\dim R = 0$ then R is perinormal since it is a field. Let $\dim R \gneq 0$. By Proposition 1.4, R is perinormal if and only if $R_{\mathfrak{m}}$ is perinormal for all $\mathfrak{m} \in \operatorname{Spec}(R)$.

Since R satisfies the (R_1) property, then $R_\mathfrak{p}$ is a valuation ring for all $\mathfrak{p} \in X^1(R)$. If $\dim R = 1$, then $R_\mathfrak{m}$ is perinormal for all $\mathfrak{m} \in \operatorname{Spec}(R)$, and hence R is perinormal.

It remains to tackle the case when $\dim R = 2$. We shall show that $R_\mathfrak{m}$ is perinormal for every $\mathfrak{m} \in \operatorname{Max}(R)$. Let $R_\mathfrak{p} \subset T \subset R'_\mathfrak{p}$, where $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\mathfrak{p} \subset \mathfrak{m}$, be a tower of integral domains such that $\theta : \operatorname{Spec}(T) \to \operatorname{Spec}(R_\mathfrak{p})$ is a bijection. By Theorem 2.1 (c), it suffices to show that $T = R_\mathfrak{p}$.

Since $R \subset T$ is integral, the Cohen-Seidenberg Theorems apply to both $R \subset T$ and $R[X] \subset T[X]$. We fix the notation $\mathfrak{P} := \mathfrak{p}R_\mathfrak{p}$. We may assume that $\operatorname{ht}_{R_\mathfrak{p}} \mathfrak{P} = 2$, since the smaller heights were handled above. Let $\mathfrak{P}' \in \operatorname{Spec}(R_\mathfrak{p})$ and $\mathfrak{Q}'$ be the unique (by the bijectivity of the prime spectra) prime ideal of T such that $\mathfrak{Q}' \cap R_\mathfrak{p} = \mathfrak{P}'$. (We shall use the accustomed notation $\mathfrak{P}'^* := \mathfrak{P}'R[X]$ and $\mathfrak{Q}'^* := \mathfrak{Q}'T[X]$ and $\{\langle \mathfrak{P}', \alpha(X)\rangle = \{g(X) \in R_\mathfrak{p}[X] |\ \alpha(X) \text{ divides } \overline{g(X)}\}$, and $\{\langle \mathfrak{Q}', \beta(X)\rangle = \{h(X) \in T[X] |\ \beta(X) \text{ divides } \overline{h(X)}\}$.)

If we establish that $\operatorname{Spec}(T[X]_{\mathfrak{P}^*}) \to \operatorname{Spec}(R[X]_{\mathfrak{P}^*})$ is bijective, then, by the perinormality of $R[X]$, it will follow that $T[X]_{\mathfrak{P}^*} = R[X]_{\mathfrak{P}^*} = R_\mathfrak{p}[X]_{\mathfrak{P}^*}$. From this equality, we get the desired equality $T = R_\mathfrak{p}$. (To see this, we note the containment $R_\mathfrak{p} \subseteq T$. To show the reverse containment $T \subseteq R_\mathfrak{p}$, we let $t \in T$. By the previous equality, we can find $f(X), s(X) \in R_\mathfrak{p}[X]$, with $s(X) \in R_\mathfrak{p}[X] \backslash \mathfrak{P}^*$, such that $s(X)t = f(X)$. Let s_i and a_i be the the i^{th} coefficients of $s(X)$, and $f(X)$, respectively. Since $s(X) \notin \mathfrak{P}^*$, there is a $j \in \{0, 1, 2, \ldots, \deg s(X)\}$ with s_i a unit in $R_\mathfrak{p}$. Thus $ts_i = a_i$, and therefore $t \in R_\mathfrak{p}$. Hence, $T \subseteq R_\mathfrak{p}$.) Then

Theorem 4.5, gives us insight into $\operatorname{Spec}(R_\mathfrak{p}[X]_{\mathfrak{P}^*})$; namely, $\operatorname{Spec}(R_\mathfrak{p}[X]_{\mathfrak{P}^*}) = \{\mathfrak{P}'^*_{\mathfrak{P}^*} |\ \mathfrak{P}' \in \operatorname{Spec}(R_\mathfrak{p})\} \cup \{\langle \mathfrak{P}', \alpha(X)\rangle_{\mathfrak{P}^*} |\ \mathfrak{P}' \in \operatorname{Spec}(R_\mathfrak{p})$ and $\alpha(X) \in \kappa_{R_\mathfrak{p}}(\mathfrak{P}')[X]$ is monic and irreducible$\}$. Now, $\mathrm{ht}_{R_{\mathfrak{p}}[X]}\langle \mathfrak{P}', \alpha(X)\rangle = \mathrm{ht}_{R_{\mathfrak{p}}[X]}\mathfrak{P}' + 1 \leqslant 2$. (This can be seen by noting, firstly, that $R_{\mathfrak{p}}[X]$ is catenary, and, secondly, that $\mathfrak{P}'^* \subsetneq \langle \mathfrak{P}', \alpha(X)\rangle$, and by using the last part of Theorem 4.5 to conclude that any prime ideal of $R_{\mathfrak{p}}[X]$, lying between $\mathfrak{P}'^*$ and $\langle \mathfrak{P}'^*, \alpha(X)\rangle$, must equal one or the other of these two prime ideals.) So, $\mathfrak{P}'^*_{\mathfrak{P}^*} \in \mathrm{Spec}(R_{\mathfrak{p}}[X]_{\mathfrak{P}^*})$ implies that $\mathrm{ht}_{R_{\mathfrak{p}}}\mathfrak{P}' \leqslant 2$, while $\langle \mathfrak{P}', \alpha(X)\rangle_{\mathfrak{P}^*} \in \mathrm{Spec}(R_{\mathfrak{p}}[X]_{\mathfrak{P}^*})$ implies that $\mathrm{ht}_{R_{\mathfrak{p}}}\mathfrak{P}' \leqslant 1$. Further, since $\langle \mathfrak{P}', \alpha(X)\rangle \subseteq \mathfrak{P}^*$, then $\mathfrak{P}' = (0)$. (By way of contradiction, suppose $\mathrm{ht}\,\mathfrak{P}' = 1$. Then $\mathrm{ht}\langle \mathfrak{P}', \alpha(X)\rangle = 2 = \mathfrak{P}^*$, and, so, $\langle \mathfrak{P}', \alpha(X)\rangle = \mathfrak{P}^*$. This, in turn, forces the equality $\langle \mathfrak{P}', \alpha(X)\rangle \cap R_{\mathfrak{p}} = \mathfrak{P}^* \cap R_{\mathfrak{p}}$. But, $\langle \mathfrak{P}', \alpha(X)\rangle \cap R_{\mathfrak{p}} = \mathfrak{P}'$, a height one prime ideal, and $\mathfrak{P}^* \cap R_{\mathfrak{p}} = \mathfrak{P}$, a height two prime ideal. This contradiction allows us to conclude that $\mathfrak{P}' = (0)$.) So, $\langle \mathfrak{P}', \alpha(X)\rangle = \langle (0), \alpha(X)\rangle$. Hence, $\mathrm{Spec}(R_{\mathfrak{p}}[X]_{\mathfrak{P}^*}) = \{\mathfrak{P}'^*_{\mathfrak{P}^*} \mid \mathfrak{P}' \in \mathrm{Spec}(R_{\mathfrak{p}})\} \cup \{\langle (0), \alpha(X)\rangle_{\mathfrak{P}^*} \mid \alpha(X) \in K[X]$ is monic and irreducible$\}$.

Recall from Theorem 4.6, that the only prime ideals of $T[X]$ lying over $\mathfrak{P}'^*$ are of the type $\mathfrak{Q}'^*$, or of the type $\langle \mathfrak{Q}', \beta(X)\rangle$ where $\beta(X)$ is a monic, irreducible polynomial in $\kappa_T(\mathfrak{Q}')$ which does not divide any polynomial of $\kappa_{R_{\mathfrak{p}}}(\mathfrak{P}')[X]$. However, the latter prime ideal cannot exist. (Otherwise, both ideals lie over $\mathfrak{P}'^*$, and $\mathfrak{Q}'^* \subsetneq \langle \mathfrak{Q}', \beta(X)\rangle$. The incomparability clause of the Cohen-Seidenberg Theorem for the integral extension $R_{\mathfrak{p}}[X] \subseteq T[X]$ of domains produces a contradiction.) So, the only prime ideal of $T[X]$ lying over $\mathfrak{P}'^*$ is $\mathfrak{Q}'^*$.

Recall from Theorem 4.6, that the only prime ideals of $T[X]$ lying over $\langle (0), \alpha(X)\rangle$ are of the type $\langle (0), \beta(X)\rangle$ where $\beta(X) \in K[X]$ is a monic, irreducible polynomial and divides the monic, irreducible $\alpha(X) \in K[X]$. Hence $\alpha(X) = \beta(X)$. Thus $\langle (0), \beta(X)\rangle = \langle (0), \alpha(X)\rangle$ is the only prime ideal lying over $\langle (0), \alpha(X)\rangle$.

The two cases show us that $\mathrm{Spec}(T[X]_{\mathfrak{P}^*}) \to \mathrm{Spec}(R[X]_{\mathfrak{P}^*})$ is bijective, and the proof is complete. ∎

We next enumerate easily established facts useful to the lemma below.

Let R be a noetherian domain and R' be its integral closure in the fraction field $K := R_{(0)}$. Let R' be a finitely generated R-module. Let T be an integral overring of R. Define $I := R :_R R'$. Then

(i) $R :_R R' = R :_K R'$.

(ii) $I \neq (0)$.

(iii) I is an ideal of both R and R', and, hence, an ideal of T.

(iv) I is maximal among ideals common to both R and R', and, hence, I is maximal among ideals common to both R and T.

In the lemma, below, we shall use Theorem 4.6 without explicitly saying so.

Lemma 4.13. *Let $(R, \mathfrak{m})$ be a noetherian domain with integral closure R' in the fraction field $K := R_{(0)}$, and let $I = R :_R R'$. Assume the following:*

(i) R' is a finitely generated R-module,

(ii) $V_R(I) = \{\mathfrak{m}\}$,

(iii) $R/\mathfrak{m} \cong R'/\mathfrak{m}'$ for every maximal ideal $\mathfrak{m}' \subset R'$.

Let T be an integral overring of R. If the map $\theta : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is bijective, then so is $\psi : \mathrm{Spec}(T[X]) \to \mathrm{Spec}(R[X])$.

Proof. Note that since $\theta : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is injective and $R \subset T$ is integral, then T is local with maximal ideal, say $\mathfrak{n}$. Let

$$\mathcal{B} := \{\mathfrak{n}^*\} \cup \{\langle \mathfrak{n}, \beta(X)\rangle | \ \beta(X) \in (T/\mathfrak{n})[X] \text{ monic and irreducible}\} \text{ and}$$

$$\mathcal{A} := \{\mathfrak{m}^*\} \cup \{\langle \mathfrak{m}, \alpha(X)\rangle | \ \alpha(X) \in (R/\mathfrak{m})[X] \text{ monic and irreducible}\}.$$

By Lemma 2.12, $R_{\mathfrak{p}} = T_{\mathfrak{q}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R) \backslash V_R(I) = \mathrm{Spec}(R) \backslash \{\mathfrak{m}\}$ with $\mathfrak{q}$ the unique prime of T lying over $\mathfrak{p}$. Further, for all such $\mathfrak{p}$ it follows that $\kappa_R(\mathfrak{p}) = \kappa_T(\mathfrak{q})$, and, hence, $\kappa_R(\mathfrak{p})[X] = \kappa_T(\mathfrak{q})[X]$.

We, next, show that $\psi| : (\mathrm{Spec}(T[X]) \backslash \mathcal{B}) \to (\mathrm{Spec}(R[X]) \backslash \mathcal{A})$ is a bijection.

Let $\mathfrak{Q} \in \mathrm{Spec}(T[X]) \backslash \mathcal{B}$. Then there is a $\mathfrak{q} \in \mathrm{Spec}(T[X])$ with $\mathfrak{q} \subsetneq \mathfrak{n}$ such that $\mathfrak{Q} = \mathfrak{q}^*$, or $\mathfrak{Q} = \langle \mathfrak{q}, \beta(X)\rangle$ where $\beta(X) \in \kappa_T(\mathfrak{q})[X]$, is monic and irreducible. Let $\mathfrak{p} = \mathfrak{q} \cap R$. Then $\mathfrak{p} \subsetneq \mathfrak{m}$ since $\mathfrak{q} \subsetneq \mathfrak{n}$, and $\mathfrak{n}$ is the unique prime ideal lying over $\mathfrak{m}$.

In the following, we, repeatedly, invoke Theorem 4.6 without specifying so. If $\mathfrak{Q} = \mathfrak{q}^*$, then $\mathfrak{Q} \cap R[X] = \mathfrak{p}^*$. If $\mathfrak{Q} = \langle \mathfrak{q}, \beta(X)\rangle$, we have two possibilities, ostensibly. Firstly, $\mathfrak{Q} \cap R[X] = \langle \mathfrak{q}, \beta(X)\rangle \cap R[X] = \langle \mathfrak{p}, \alpha(X)\rangle$, where $\beta(X) | \alpha(X)$ in $\kappa_T(\mathfrak{q})[X] = \kappa_R(\mathfrak{p})[X]$, and hence $\alpha(X) = \beta(X)$. Secondly, $\mathfrak{Q} \cap R[X] = \langle \mathfrak{q}, \beta(X)\rangle \cap R[X] = \mathfrak{p}^*$, where $\beta(X)$ does not divide any $\alpha(X) \in \kappa_R(\mathfrak{p})[X]$. But, $\kappa_T(\mathfrak{q})[X] = \kappa_R(\mathfrak{p})[X]$, so such $\beta(X)$ cannot exist. Thus $\mathfrak{Q} \cap R[X] = \langle \mathfrak{q}, \beta(X)\rangle \cap R[X] = \langle \mathfrak{p}, \beta(X)\rangle$ is the only possibility. Therefore, $\psi|$ is well-defined. Additionally, from the above discussion, we glean that $\psi|$ is surjective, and that $\mathfrak{p}^*$ has only one prime in $S[X]$ lying over it, namely $\mathfrak{q}^*$. It remains to show that $\langle \mathfrak{p}, \alpha(X)\rangle$ has a unique preimage under $\psi|$.

Let $\mathfrak{P} = \langle \mathfrak{p}, \alpha(X)\rangle$, then there are monic, irreducible polynomials $\beta_1(X), \beta_2(X), \ldots, \beta_n(X) \in \kappa_S(\mathfrak{q})[X]$, such that $\alpha(X) = \beta_1(X) \cdot \beta_2(X) \cdot \ldots \cdot \beta_n(X)$. Then $\{\langle \mathfrak{q}, \beta_i(X)\rangle | \ i = 1, 2, \ldots, n\}$ is the set of all prime ideals of $T[X]$ lying over $\langle \mathfrak{p}, \alpha(X)\rangle$. However, $\kappa_R(\mathfrak{p})[X] =$

$\kappa_S(\mathfrak{q})[X]$, and, $\alpha(X), \beta_1(X), \beta_2(X), \ldots, \beta_n(X)$ are, all, monic and irreducible in $\kappa_R(\mathfrak{p})[X]$. Thus $n = 1$, and $\langle \mathfrak{p}, \alpha(X) \rangle$ has a unique preimage under $\psi|$, namely $\{\langle \mathfrak{q}, \alpha(X) \rangle$.

So, $\psi| : (\mathrm{Spec}(S[X]) - \mathcal{B}) \to (\mathrm{Spec}(R[X]) - \mathcal{A})$ is a bijection.

Next we show that $\psi|| : \mathcal{B} \to \mathcal{A}$ is a bijection. By Theorems 4.5 and 4.6 we see that the primes lying over $\mathfrak{m}^*$ are either $\mathfrak{n}^*$ or of the form $\langle \mathfrak{n}, \beta(X) \rangle$, and we note that $\mathfrak{n}^* \subsetneq \langle \mathfrak{n}, \beta(X) \rangle$. As $R \subset T$ is integral, so is $R[X] \subset T[X]$. By the incomparability clause of the Cohen-Seidenberg theorems, we see that $\mathfrak{n}^*$ is the unique prime of $S[X]$ lying over $\mathfrak{m}^*$. By Theorem 4.6 a prime of $T[X]$ lying over $\langle \mathfrak{m}, \alpha(X) \rangle$ has the form $\langle \mathfrak{n}, \beta(X) \rangle$ where $\beta(X) | \alpha(X)$ in $(T/\mathfrak{n})[X]$. We consider the field extensions $R/\mathfrak{m} \subseteq T/\mathfrak{n} \subseteq R'/\mathfrak{m}'$ and the canonical field isomorphism $R/\mathfrak{m} \cong R'\mathfrak{m}'$ for all $\mathfrak{m}' \in \mathrm{Max}(R')$. It follows that $R/\mathfrak{m} \cong T/\mathfrak{n}$, so $\beta(X) | \alpha(X)$ if and only if $\beta(X) = \alpha(X)$ since they are both irreducible over the same field. So $\langle \mathfrak{n}, \alpha(X) \rangle$ is the unique prime of $T[X]$ lying over $\langle \mathfrak{m}, \alpha(X) \rangle$. This completes the proof that $\psi|| : \mathcal{B} \to \mathcal{A}$ is a bijection, and completes the proof that $\psi : \mathrm{Spec}(T[X]) \to \mathrm{Spec}(R[X])$ is a bijection. ∎

We can now state another "descent of perinormality" result.

Proposition 4.14. *Let $(R, \mathfrak{m})$ be a universally catenary, local integral domain with its integral closure R' finitely generated as an R-module. Suppose that $V_R(R :_R R') = \{\mathfrak{m}\}$, and $R/\mathfrak{m} \cong R'/\mathfrak{m}'$ for every maximal ideal $\mathfrak{m}' \subset R'$. If $R[X]$ is perinormal, then R is perinormal.*

Proof. By Theorem 2.1 (c), it remains to show that for every $\mathfrak{p} \in \mathrm{Spec}(R)$ whenever T is an integral overring of $R_\mathfrak{p}$ with $\mathrm{Spec}(T) \to \mathrm{Spec}(R_\mathfrak{p})$ bijective, then $T = R_\mathfrak{p}$.

By Lemma 4.13, $\mathrm{Spec}(T[X]) \to \mathrm{Spec}(R_\mathfrak{p}[X])$ is a bijection, also. It is straightforward to note that $\mathrm{Spec}(T[X]_{\mathfrak{p}^*}) \to \mathrm{Spec}(R[X]_{\mathfrak{p}^*})$ is a bijection, as well, for every

$\mathfrak{p} \in \operatorname{Spec}(R)$.

By Theorem 2.1 (c), $T[X]_{\mathfrak{p}^*} = R[X]_{\mathfrak{p}^*}$, and so if $t \in T$, there must exist $f(X), s(X) \in R[X]$ with $s(X) \notin \mathfrak{p}^*$ such that $s(X)t = f(X)$. Let s_i and a_i be the i^{th} coefficients of $s(X)$ and $f(X)$, respectively. Since $s(X) \notin \mathfrak{p}^*$, there is an $i \in \{0, 1, 2, \ldots, \deg s(X)\}$ with s_i a unit in $R_{\mathfrak{p}}$. Then $s_i t = a_i$ implies that $t \in R_{\mathfrak{p}}$, and, hence, $T = R_{\mathfrak{p}}$. ∎

We close this chapter with another descent result from $R[X]$ to R.

Proposition 4.15. *If the integral domain $R[X]$ is globally perinormal, then R is weakly normal (and hence seminormal).*

Proof. Let R' denote the integral closure of R in K and let ${}^*_{R'}R$ denote the weak subintegral closure of R in R', meaning ${}^*_{R'}R$ is the largest subextension of R in R' such that $R \subseteq {}^*_{R'}R$ is a weakly subintegral extension. This means that $R \subseteq {}^*_{R'}R$ is a U-extension. Theorem 4.8 allows us to conclude that the map $\operatorname{Spec}({}^*_{R'}R[X]) \to \operatorname{Spec}(R[X])$ is bijective. But, the integral extension $R[X] \subseteq {}^*_{R'}R[X]$ satisfies the going down property, also. (To see this, suppose that $P_1 \subset P_2 \in \operatorname{Spec}(R[X])$ with $Q_2 \in \operatorname{Spec}({}^*_{R'}R[X])$ such that $Q_2 \cap R[X] = P_2$. Let $Q_1 \in \operatorname{Spec}({}^*_{R'}R[X])$ be such that $Q_1 \cap R[X] = P_1$. Since $R[X] \subseteq {}^*_{R'}R[X]$ satisfies the going up property, there exists a $Q_2' \in \operatorname{Spec}({}^*_{R'}R[X])$ with $Q_1 \subset Q_2'$ and $Q_2' \cap R[X] = P_2$. But the map $\operatorname{Spec}({}^*_{R'}R[X]) \to \operatorname{Spec}(R[X])$ is bijective, and, thus, $Q_2 = Q_2'$. Hence, $R[X] \subseteq {}^*_{R'}R[X]$ satisfies the going down property.)

Given that $R[X]$ is globally perinormal, there must be a multiplicatively closed subset $S \subset R[X]$ such that ${}^*_{R'}R[X] = S^{-1}R[X]$. From the above bijection on the prime spectra, it follows that $\operatorname{Spec}(S^{-1}R[X]) \to \operatorname{Spec}(R[X])$ is bijective, from which we conclude that

$$\{P \in \operatorname{Spec}(R[X]) \mid P \cap S = \emptyset\} = \operatorname{Spec}(R[X]).$$

Therefore, S consists of units of $R[X]$, hence $S^{-1}R[X] = R[X]$. Thus, ${}^{*}_{R'}R[X] = R[X]$, and, so, ${}^{*}_{R'}R = R$. ∎

Chapter 5

Completions

Throughout this section, unless stated otherwise, let $(R, \mathfrak{m})$ be a local domain with $\mathfrak{m}$-adic completion $(\widehat{R}, \widehat{\mathfrak{m}})$. Let R' denote the integral closure of R in its quotient field K.

5.1 Preliminary Results

In this section, we list some well-known results about completions. For some of them, we supply proofs.

Proposition 5.1 ([2, Corollary 2.1.13]). *A local complete noetherian ring $(\widehat{R}, \widehat{\mathfrak{m}})$ is universally catenary.*

Proof. By Cohen's Structure Theorem, $\widehat{R}$ is the homomorphic image of a formal power series ring $k[[X_1, \ldots, X_n]]$, where k is either a field or a DVR. Since $k[[X_1, \ldots, X_n]]$ is Cohen-Macaulay, it is universally catenary. So $\widehat{R}$ is also universally catenary by Proposition 1.11. ∎

If $\widehat{R}$ is (R_1), then R is (R_1), since $R \subset \widehat{R}$ is a faithfully flat ring extension. Conversely, if R and all its formal fibers are (R_1), then Lemma 4.2 ensures $\widehat{R}$ is (R_1).

Proposition 5.2 ([1, III, § 3.5, Corollary 4])**.** *Let* $(R, \mathfrak{m})$ *be a local, noetherian, analytically irreducible domain with* $\mathfrak{m}$*-adic completion* $(\widehat{R}, \widehat{\mathfrak{m}})$*. Then* $\widehat{R} \cap K = R$*.*

Proof. Clearly, $R \subseteq K \cap \widehat{R}$. Conversely, let $x \in K \cap \widehat{R}$. We have the exact sequence

$$0 \to R \to xR + R \to \frac{xR + R}{R} \to 0 \tag{5.1}$$

Tensoring with $\widehat{R}$ yields the exact sequence

$$0 \to \widehat{R} \to \widehat{R} \otimes_R (xR + R) \to \widehat{R} \otimes_R \frac{xR + R}{R} \to 0 \tag{5.2}$$

Now, $\widehat{R} \otimes_R xR \cong xR$ since $\widehat{R}$ is a domain. Also, $\widehat{R} \otimes_R R \cong R$. Since $\widehat{R} \otimes_R (xR + R) = \widehat{R} \otimes_R xR + \widehat{R} \otimes_R R$, then $\widehat{R} \otimes_R (xR + R) \cong x\widehat{R} + \widehat{R}$. But, $x \in \widehat{R}$, and, so, $x\widehat{R} \subseteq \widehat{R}$. Therefore, $\widehat{R} \otimes_R (xR + R) = \widehat{R}$.

Thus the exact sequence 5.2 becomes the exact sequence

$$0 \to \widehat{R} \to \widehat{R} \to \widehat{R} \otimes_R \frac{xR + R}{R} \to 0 \tag{5.3}$$

Here, the map $R \to R$ is an isomorphism. We conclude that $\widehat{R} \otimes_R \dfrac{xR + R}{R} = 0$. But, $\widehat{R}$ is faithfully flat over R, forcing $\dfrac{xR + R}{R} = 0$, whence $x \in R$. ∎

The next two results discuss the integral closure of a complete, local domain.

Proposition 5.3 ([21, Proposition 6])**.** *Let* $(R, \mathfrak{m})$ *be a complete local domain (not necessarily noetherian) and* $R \subset T \subset R'$*. Then* T *is local, and if* T *is a finite* R*-module, then* T *is a complete.*

For noetherian, complete, local domains, we can go further.

Proposition 5.4 ([14, Theorem 4.3.4])**.** *If* $(R, \mathfrak{m})$ *is a complete local noetherian domain, then the integral closure* R' *is a finitely generated* R*-module. More generally, if*

L is a finite field extension of the field of fractions K of R, then the integral closure S of R in L is module finite over R. Furthermore, S is a complete noetherian local domain.

If $(R, \mathfrak{m})$ is a noetherian, analytically irreducible domain, Propositions 5.3 and 5.4 imply that every integral overring T of $\widehat{R}$ is a complete, local domain that is finitely generated as an $\widehat{R}$-module.

Proposition 5.5 ([1, III, § 2, no. 13, Corollary to Proposition 19])**.** *Let T be a semi-local ring with maximal ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_l$ and Jacobson radical $\mathfrak{r} := \bigcap_{i=1}^{l} \mathfrak{n}_i$. Then the completion $\widehat{T}$ with respect to $\mathfrak{r}$ is a semi-local ring that is canonically isomorphic to the product $\prod_{i=1}^{l} \widehat{T_{\mathfrak{n}_i}}$ where $\widehat{T_{\mathfrak{n}_i}}$ is the $\mathfrak{n}_i T_{\mathfrak{n}_i}$-adic completion of $T_{\mathfrak{n}_i}$.*

In the following two results we have replaced the phrase "ring" in the original text by the phrase "domain," as it is in the context of the domain that we shall be using the two results.

Proposition 5.6 ([16, 31.E p. 236])**.** *If $(R, \mathfrak{m})$ is a local, noetherian, analytically unramified domain, then R' is a finitely generated R-module.*

Next, we quote and prove a Bourbaki exercise.

Exercise 5.7 ([1, V, §2, Exercise 8])**.** (a) If $(R, \mathfrak{m})$ is a local integral domain, then the integral closure R' of R in K is local if and only if every overring of R which is finitely generated as an R-module is a local domain.

(b) If $(R, \mathfrak{m})$ is a noetherian, local, analytically irreducible domain, then the integral closure R' of R in K is local.

Proof. (a) Let T be an overring of R such that T is a finitely generated R-module. Then T is integral over R, and $R \subset T \subset R'$.

($\Longrightarrow$) Suppose $(R', \mathfrak{m}')$ is local. Let $\mathfrak{n} \in \operatorname{Max}(T)$. Since $T \subseteq R'$ is an integral extension, then there is a $\mathfrak{q} \in \operatorname{Spec}(R')$ such that $\mathfrak{q} \cap T = \mathfrak{n}$. Note that $\mathfrak{q} \subseteq \mathfrak{m}'$. Hence $\mathfrak{n} \subseteq \mathfrak{q} \cap T \subseteq \mathfrak{m}' \cap T$. Since $\mathfrak{n} \in \operatorname{Max}(T)$, and $\mathfrak{m}' \cap T \in \operatorname{Spec}(T)$, then $\mathfrak{n} = \mathfrak{m}' \cap T$. Hence, both $\mathfrak{q}$ and $\mathfrak{m}'$ lie over $\mathfrak{n}$. By the incomparability clause of the Cohen-Seidenberg Theorem, $\mathfrak{q} = \mathfrak{m}'$. So, every maximal ideal of T equals $\mathfrak{m}' \cap T$. Hence, T is local.

($\Longleftarrow$) Now suppose that every overring of R which is finitely generated as an R-module is a local ring. By way of contradiction suppose that $\mathfrak{m}', \mathfrak{m}'' \in \operatorname{Max}(R')$ with $\mathfrak{m}' \neq \mathfrak{m}''$. Let $x \in \mathfrak{m}' \backslash \mathfrak{m}''$. Since x is integral over R, then $R[x]$ is an integral overring of R and is finitely generated as an R-module. By the hypothesis, $R[x]$ is local, with maximal ideal, say, $\mathfrak{n}$. Note that $x \in \mathfrak{n}$ since otherwise x is a unit of $R[X]$, and thus a unit of R', contradicting $x \in \mathfrak{m}'$.

The Cohen-Seidenberg Theorems imply that $\mathfrak{m}'$ and $\mathfrak{m}''$ both contract to $\mathfrak{n}$. Hence, $x \in \mathfrak{n} = \mathfrak{m}' \cap R[x] = \mathfrak{m}'' \cap R[x]$. But, then $x \in \mathfrak{m}''$, a contradiction. So, R' is local.

(b) We shall apply part (a) to prove part (b). Let T be an overring of R which is a finitely generated R-module, and let $\mathcal{K} := \widehat{R}_{(0_{\widehat{R}})}$ be the quotient field of the domain $\widehat{R}$. Since $R \subset T$ is an integral extension, then $\operatorname{Max}(T)$ is the set of all prime ideals of T lying over $\mathfrak{m}$ in R, i.e., the set of all minimal prime over-ideals of $\mathfrak{m}T$. Since T is noetherian, $\mathfrak{m}T$ has finitely many minimal prime over-ideals. Thus T is semi-local with $\operatorname{Max}(T) = \{\mathfrak{n}_1, \ldots, \mathfrak{n}_l\}$ and Jacobson radical $\mathfrak{r} = \bigcap_{i=1}^{l} \mathfrak{n}_i$; and $\widehat{T}$, the $\mathfrak{m}T$-adic completion of T, is finitely generated as a $\widehat{R}$-module. Further, $\widehat{T}$ is an overring of $\widehat{R}$ since

$$\widehat{T} \cong T \otimes_R \widehat{R} \subset K \otimes_R \widehat{R} \cong (R - \{0_R\})^{-1}\widehat{R} \subset \mathcal{K}.$$

Since $\widehat{T}$ is integral over $\widehat{R}$, we see that $\widehat{T} \subset (\widehat{R})'$. Next, we note that $\mathfrak{m}T$-adic and $\mathfrak{r}$-adic topologies are the same. (To see this, note that $\mathfrak{r}$ is the radical of $\mathfrak{m}T$. Since T is noetherian, there is a positive integer t such that $\mathfrak{r}^t \subseteq \mathfrak{m}T$. Clearly, $\mathfrak{m}^1 T \subseteq \mathfrak{r}$. Therefore, the two topologies are the same.) Hence, $\widehat{T}$ equals the $\mathfrak{r}$-adic completion of T.

Proposition 5.5, then, allows us to conclude that

$$\widehat{T} \cong \prod_{i=1}^{l} \widehat{T_{\mathfrak{n}_i}},$$

where each $\widehat{T_{\mathfrak{n}_i}}$ is complete with respect to the $\mathfrak{n}_i T_{\mathfrak{n}_i}$-adic topology. But $\widehat{T}$ is a domain, so $l = 1$. Thus, $\widehat{T}$ is a local ring, and, therefore, T is a local ring, also. Part (a) implies that R' is local.

∎

Remark 5.8. If R is a noetherian, local, analytically irreducible domain, then, by Proposition 5.6, the integral closure R' (in $K := R_{(0)}$) is both a finitely generated R-module, and a local ring. Since R is a noetherian ring, then R' is a noetherian ring, as well. So R' is a local, noetherian domain, and finitely generated as an R-module.

Let $(R, \mathfrak{m})$ be a noetherian, local, analytically irreducible domain with integral closure $(R', \mathfrak{m}')$ in K. Since $\mathfrak{m}R'$ is $\mathfrak{m}'$-primary, the $\mathfrak{m}$-adic and $\mathfrak{m}'$-adic topologies on R' are equivalent. Now consider the integral closure $(\widehat{R})'$ of $\widehat{R}$ in its quotient field $\mathcal{K}$. This domain is a local, complete, finitely generated $\widehat{R}$ module by Proposition 5.4.

Next, let $\widehat{(R')}$ denote the $\mathfrak{m}'$-adic completion of R', and observe that it equals the $\mathfrak{m}$-adic completion of R'. In section 3, we shall limit our consideration to domains R for which the isomorphism $(\widehat{R})' \cong \widehat{(R')}$ holds. The existence of such rings is guaranteed in the settings of Lemmas 5.9 and 5.11.

Lemma 5.9 ([6, Theorem 6.5, (3)]). *Let R be a reduced noetherian local ring with geometrically regular formal fibers, and let R' be the integral closure of R in its total ring of fractions. Then the completion $\widehat{(R')}$ of R' is isomorphic to the integral closure of $\widehat{R}$ in its total ring of fractions.*

Definition 5.10 ([30, p. 314]). We say a local domain $(R, \mathfrak{m})$ satisfies condition (D) if there exists an element $d \neq 0$ in R such that, if $(\widehat{R})'$ denotes the integral closure of $\widehat{R}$ in its total quotient ring, then $d(\widehat{R})' \subset \widehat{R}$.

Lemma 5.11 ([30, Lemma 3, Ch. VIII]). *Let R be a local domain satisfying condition (D). Then R', the integral closure of R in its total quotient ring, is semilocal. Furthermore, if, for every maximal ideal $\mathfrak{m}'$ of R', the local ring $R'_{\mathfrak{m}'}$ and all its residue class rings $R'_{\mathfrak{m}'}/\mathfrak{p}$ ($\mathfrak{p}$ prime) satisfy condition (D), then the ring $\widehat{R}'$ is canonically isomorphic to $(\widehat{R})'$.*

In [30], Zariski and Samuel show that if R is an "algebro-geometric," local ring (i.e., a domain that is essentially of finite-type over a field), then the list of rings satisfying condition (D) includes R, $R/\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(R)$, and $R'_{\mathfrak{m}'}/\mathfrak{p}'$ with $\mathfrak{p}' \in \mathrm{Spec}(R')$.

5.2 R perinormal need not imply $\widehat{R}$ perinormal

Let $(T, \mathfrak{M})$ be a complete, local ring. In [12] and [13] Heitmann investigated conditions on T which guarantee the existence of a local ring $(R, \mathfrak{m})$ with $\widehat{R} \cong T$ such that R

possesses a "nice" quality, while T does not. (The "nice" quality may be having, at most, an isolated singularity, or being a unique factorization domain.)

Proposition 5.12 ([13, Theorem 8])**.** *Let $(T, \mathfrak{M})$ be a complete local ring such that no integer is a zerodivisor in T and depth $T > 1$. Then there exists a local unique factorization domain $(R, \mathfrak{m})$ such that $\widehat{R} \cong T$ and $|R| = \sup(\aleph_0,\ |T/\mathfrak{M}|)$. If $p \in \mathfrak{M}$ where p is a nonzero prime integer, then pR is a prime ideal.*

In [9, Example 3.6], the authors present Karl Schwede's example of a weakly normal domain that is not perinormal. The domain is $k[X, Y, XZ, YZ, Z^2]$ where k is a field of any characteristic except two. We adapt this example to show that perinormality of R does not guarantee perinormality of $\widehat{R}$.

Example 5.13. Let $T = \mathbb{C}[[X, Y, XZ, YZ, Z^2]]$ and let $\mathcal{K} = \mathbb{C}((X, Y, Z))$ be its quotient field. First, we observe that $T' = \mathbb{C}[[X, Y, Z]]$ is the integral closure of T in $\mathcal{K}$. To see this, we make a few observations. The rings T and T' share the same quotient field, and T' is a normal domain. Thus, it suffices to show that each $u \in \mathbb{C}[[X, Y, Z]]$ is integral over T. Notice that $\mathbb{C}[[X, Y, Z]] = T[[Z]]$, and that $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{i,j,k} X^i Y^j Z^k$ (where $a_{i,j,k} \in \mathbb{C}$) is an element of T whenever k is even, or when $i + j \geqslant 1$ and k is odd. So, to see that T' is the integral closure of T, it is enough to show that $u := \sum_{k=0}^{\infty} a_k Z^{2k+1}$ is integral over T. There exists $l \in \mathbb{N}$ such that $u = Z^{2l+1} \sum_{i=0}^{\infty} b_i Z^{2i}$ where $b_i \in \mathbb{C}$ and $b_0 \neq 0$. Let $s := \sum_{i=0}^{\infty} b_i Z^{2i}$ and notice $s \in T$. Then u is a root of the monic polynomial $f(W) = W^2 - s^2 (Z^2)^{2l+1} \in T[W]$.

Next we show that T is not perinormal. Let

$$I := \left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{i,j,k} X^i Y^j Z^k \mid i + j \geqslant 1;\ a_{i,j,k} \in \mathbb{C} \right\}.$$

Then I is an ideal of T and of T'. Explicitly, $I = (X, Y, XZ, YZ)T = (X, Y)T'$. Furthermore, I is a prime ideal in T and in T'. By Proposition 1.4, perinormality is a local property. Thus, if we show that T_I is not perinormal, then it will follow that T, too, cannot be perinormal. Now, $T_I = \mathbb{C}((Z^2))[[X, Y, XY, YZ]]_{(X,Y,XZ,YZ)T}$ and $(T_I)' = T_I' = \mathbb{C}((Z))[[X, Y]]_{(X,Y)T'}$. We claim that $\mathrm{Spec}((T_I)') \xrightarrow{\psi} \mathrm{Spec}(T_I)$ is a bijection. Surjectivity follows from integrality of the extension. Also $\mathrm{Max}(T_I) = \{I_I\}$ and $\mathrm{Max}((T_I)') = \mathrm{Max}((T')_I) = \{I_I\}$. Let $\mathfrak{P} \in \mathrm{Spec}(T')$ be such that $\mathfrak{P} \subsetneq I$, and let $\mathfrak{p} := \mathfrak{P} \cap T$. Since $\mathfrak{P} \subsetneq (X, Y)T'$, we may assume without loss of generality that $X \notin \mathfrak{P}$. This implies $(T')_{\mathfrak{P}} \cong \mathbb{C}((X, Z))[[Y]]_{\mathfrak{P}}$. By Lemma 2.12, $T_{\mathfrak{p}} \cong \mathbb{C}((X, Z))[[Y]]_{\mathfrak{p}}$. Suppose that $\mathfrak{P}' \in \mathrm{Spec}(T')$ is such that $\mathfrak{P}' \subsetneq I$ and $\mathfrak{P}' \cap T = \mathfrak{p}$. Then $X \notin \mathfrak{P}'$ since $X \notin \mathfrak{p} = \mathfrak{P} \cap T$. By the above, $(T')_{\mathfrak{P}'} = T_{\mathfrak{p}} = (T')_{\mathfrak{P}}$, so $\mathfrak{P} = \mathfrak{P}'$. Thus, $\mathrm{Spec}((T_I)') \xrightarrow{\psi} \mathrm{Spec}(T_I)$ is a bijection.

By [9, Proposition 3.3], if $(R, \mathfrak{m})$ is a local, perinormal domain and S is an integral overring of R with $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ bijective, then $S = R$. Therefore, if T_I were perinormal, then $T_I = (T')_I$. But $Z \in (T')_I \backslash T_I$. Hence, T_I is not perinormal, and, consequentally, T is not perinormal.

Now we apply Proposition 5.12. The ring T is a local, complete domain with depth $T > 1$. So there exists a local unique factorization domain $(R, \mathfrak{m})$ such that $\widehat{R} \cong T$. This R is normal, hence perinormal, while T is not.

5.3 When $\widehat{R}$ is the completion of an isolated singularity

Given a complete local domain T, there could be many domains $(R, \mathfrak{m})$ such that $\widehat{R} = T$. In [12] the author shows that a large class of complete, noetherian, local domains can be realized as the completion of an isolated singularity. In this subsection, we focus on such a pair of local domains $(R, \mathfrak{m})$ and $(T, \mathfrak{M})$ where R is an isolated singularity and $T \cong \widehat{R}$. Since T is a domain, R must be analytically irreducible. By Remark 5.8, the analytic irreducibility of R implies that the integral closure R' of R in $K := R_{(0)}$, is a local, noetherian domain, and finitely generated as an R-module.

Since R is an isolated singularity, then $R_{\mathfrak{p}} = R'_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R) \backslash \{\mathfrak{m}\}$. A consequence to this equality is that $R'_{\mathfrak{p}}$ is a local ring. This implies that $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is a bijection. Since $R \subset R'$ satisfies the going up property, $R \subset R'$ satisfies the going down property also. Thus, Proposition 2.14 applies in this setting, and we conclude that R is perinormal if and only if $R = R'$. We consider the extension $\widehat{R} \subset (\widehat{R})'$ under conditions for which $(\widehat{R})' \cong \widehat{(R')}$. It is known that the going down property, unlike flatness, is not stable under faithfully flat base change. (See [7, Example 3.9], for an example.) So, if $R \subset R'$ satisfies the going down property, the same need not necessarily hold for $\widehat{R} \subset (\widehat{R})' \cong \widehat{(R')} \cong \widehat{R} \otimes R'$. (The latter isomorphism stems from the fact that R is analytically irreducible, and, so, forces R' to be finitely generated as an R-module.) We recall the stronger *universally going down homomorphism* from [8].

Definition 5.14 ([8, Corollary 2.3])**.** Let $f : A \to B$ be a homomorphism of integral domains. Then f is universally going down if and only if $f_n : A[X_1, \ldots, X_n] \to B[X_1, \ldots, X_n]$ satisfies going down for each $n \geqslant 0$.

In [8] the authors remark that when $f : A \to B$ is universally going down then $S \to S \otimes_A B$ has going down for each base change $A \to S$. The following result in [4] characterizes universally going down homomorphisms $f : A \to B$ in terms of the weak normalization of A with respect to f.

Lemma 5.15 ([4, Proposition 6.4])**.** *For an integral extension of domains $f : A \to B$, the map f is universally going down if and only if f is going down and B is the weak normalization of A with respect to f (i.e., B is the weak normalization of A in B).*

We apply this lemma in the following.

Proposition 5.16. *Let $(R, \mathfrak{m})$ be a noetherian, local, analytically irreducible domain with an isolated singularity such that $\widehat{(R')} \cong (\widehat{R})'$. If $\widehat{R}$ is perinormal, then R is weakly normal.*

Proof. Following the introduction to this section, the inclusion $f : R \hookrightarrow R'$ satisfies going down. If S is the weak normalization of R with respect to f (meaning the weak normalization of R in R') then $R \subseteq S \subseteq R'$, and the induced map $g : R \hookrightarrow S$ satisfies the going down property. By Lemma 5.15, g is universally going down. Since R is analytically irreducible and noetherian, R' is a finitely generated R-module, forcing S to be so, as well. Furthermore, we have the induced integral domain extensions $\widehat{R} \subseteq \widehat{S} \subseteq \widehat{(R')} \cong (\widehat{R})'$, with each combination of the ring extensions being an integral extension. By Proposition 5.4, $(\widehat{R})'$ is local and a finitely generated $\widehat{R}$-module. Since $\operatorname{Spec}((\widehat{R})') \to \operatorname{Spec}(\widehat{S})$ is surjective, and since $\widehat{R}$ is noetherian, then $\widehat{S}$ is local, and finitely generated as an $\widehat{R}$-module.

Since g is universally going down, the base change $\widehat{R} \subset \widehat{R} \otimes_R S$ has the going down property. If $\mathfrak{M}$ is the maximal ideal of $\widehat{R} \otimes_R S \cong \widehat{S}$, then, by integrality, $\mathfrak{M} \cap \widehat{R} = \widehat{\mathfrak{m}}$. By Corollary 1.3, since $\widehat{R}$ is perinormal, it follows that $\widehat{R} = \widehat{R}_{\mathfrak{M} \cap \widehat{R}} = \widehat{S}_{\mathfrak{M}} = \widehat{S}$. Thus $S = \widehat{S} \cap S = \widehat{R} \cap S \subseteq \widehat{R} \cap K = R$ by Proposition 5.2. Hence $S = R$. ∎

If we add the hypothesis that $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ is a bijection, we can get that R is normal.

Proposition 5.17. *Let $(R, \mathfrak{m})$ be a noetherian, local, analytically irreducible domain with an isolated singularity. Suppose that the extension $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ is a bijection, and that $\widehat{R}$ is perinormal. Then R is normal.*

Proof. Let R' be the integral closure of R in its quotient field K. By Remark 5.8, R' is finitely generated as an R-module, and is a noetherian, local domain with maximal ideal $\mathfrak{m}'$, say. By the Cohen-Seidenberg Theorems, any integral overring of R must be local, as well. Let $v \in R'$. Then $R[v]$ is a finitely generated R-module, and a noetherian, local domain with maximal ideal $\mathfrak{n}$. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. Then $\widehat{R}$ is a local domain with maximal ideal $\widehat{\mathfrak{m}} =: \mathfrak{M}$. We let $(\widehat{R})'$ be the integral closure of $\widehat{R}$ in its field of fractions $\mathcal{K}$. By Proposition 5.4, the ring $(\widehat{R})'$ is a finitely generated $\widehat{R}$-module, and a noetherian, local domain with maximal ideal $\mathfrak{M}'$, say. By the Cohen-Seidenberg Theorems, $\widehat{R}[v]$ is a finitely generated $\widehat{R}$-module, and a noetherian, local domain with maximal ideal $\mathfrak{N}$, say. By Proposition 5.3, $\widehat{R}[v]$ is complete with respect to the $\mathfrak{N}$-adic topology. We next note that the $\mathfrak{m}$-adic and the $\mathfrak{n}$-adic topologies on $R[v]$ are the same since $\mathfrak{m}R[v]$ is $\mathfrak{n}$-primary, and that the $\mathfrak{N}$-adic, the $\mathfrak{n}\widehat{R}[v]$-adic, and the $\mathfrak{m}\widehat{R}[v]$-adic topologies on $\widehat{R}[v]$ are the same since $\mathfrak{n}\widehat{R}[v]$ and $\mathfrak{m}\widehat{R}[v]$ are $\mathfrak{N}$-primary. Thus the $\mathfrak{m}R[v]$-adic completion $\widehat{R[v]}$ of $R[v]$ is isomorphic to the $\mathfrak{m}\widehat{R}[v]$-adic completion $\widehat{\widehat{R}[v]}$ of $\widehat{R}[v]$. Since $\widehat{R}[v]$ is complete with respect to the $\mathfrak{N}$-adic, and thus with respect to the $\mathfrak{m}\widehat{R}[v]$-adic topology, then $\widehat{R[v]} \cong \widehat{R}[v]$. Thus the extension $R[v] \subset \widehat{R}[v]$ is faithfully flat, and, therefore, satisfies the going down property.

As seen by the discussion following this section's introduction, the ring extension $R \subseteq R'$ satisfies the going down propery. Thus, the extension $R \subseteq R[v]$ satisfies the going down property, also.

If the ring extensions $R \subseteq R[v]$ and $R[v] \subset \widehat{R}[v]$, both, satisfy the going down property, then the ring extension $R \subset \widehat{R}[v]$ satisfies the going down property, as well.

We, next, claim that $\widehat{R} \subset \widehat{R}[v]$ satisfies the going down property. Suppose that $P \subsetneq Q$ are primes in $\widehat{R}$ with $\mathfrak{Q} \in \operatorname{Spec}(\widehat{R}[v])$ lying over Q. Since $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ is bijective, then $P \cap R \neq Q \cap R$. So, $P \cap R \subsetneq Q \cap R$ is a chain in $\operatorname{Spec}(R)$ with $\mathfrak{Q}$ lying over $Q \cap R$.

Given that $R \subset \widehat{R}[v]$ satisfies the going down property, there exists $\mathfrak{P} \in \operatorname{Spec}(\widehat{R}[v])$ such that $\mathfrak{P} \subset \mathfrak{Q}$ and $\mathfrak{P} \cap R = P \cap R$. We see that $\mathfrak{P} \cap \widehat{R}, P \in \operatorname{Spec}(\widehat{R})$ both lie over $P \cap R$, so since $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ is bijective we must have $\mathfrak{P} \cap \widehat{R} = P$. Thus $\widehat{R} \subset \widehat{R}[v]$ has going down, and, as $\widehat{R}$ is perinormal, this forces $\widehat{R} = \widehat{R}_{\widehat{\mathfrak{m}}} = \widehat{R}_{\mathfrak{N} \cap \widehat{R}} = \widehat{R}[v]_{\mathfrak{N}} = \widehat{R}[v]$ by Corollary 1.3. Now $v \in \widehat{R} \cap R' \subseteq R$ by Proposition 5.2. Thus $R' \subseteq R$, and, so, $R = R'$. Therefore R is normal. ∎

5.4 A Result for Certain Approximation Domains

We can replace the isolated singularity hypothesis by the assumption that R is an approximation domain. The following notation will be utilized in the definition of an approximation domain:

- $\underline{X}$ is a sequence of variables $X_1, \ldots, X_s$;
- $\underline{f}$ is a sequence of polynomials $f_1, \ldots f_t \in R[\underline{X}]$;
- $\underline{x}$ is a sequence of elements $x_1, \ldots, x_s \in R$;

- $\underline{\tilde{x}}$ is a sequence of elements $\underline{\tilde{x_1}}, \ldots, \underline{\tilde{x_s}} \in \widehat{R}$;

- $\underline{x} \equiv \underline{\tilde{x}} \bmod \widehat{\mathfrak{m}}^n$ is a system of congruences

$$\begin{cases} x_1 & \equiv \tilde{x_1} \bmod \widehat{\mathfrak{m}}^n \\ & \vdots \\ x_s & \equiv \tilde{x_s} \bmod \widehat{\mathfrak{m}}^n \end{cases}$$

Definition 5.18. [29] Let $(R, \mathfrak{m})$ be a local, noetherian ring with completion $(\widehat{R}, \widehat{\mathfrak{m}})$. Suppose that given any sequence $\underline{f} \in R[\underline{X}]$ having a solution $\underline{\tilde{x}} \in \widehat{R}^s$, and given any $n \in \mathbb{N}^+$, there exists a solution $\underline{x} \in R^s$ for $\underline{f}$ such that $\underline{x} \equiv \underline{\tilde{x}} \bmod \widehat{\mathfrak{m}}^n$. Then R is called an *approximation ring.*

Remark 5.19. Here are some useful facts about approximation domains.

1. Approximation rings are excellent and Henselian. (See [26].)

2. A module-finite ring extension of an approximation ring is again an approximation ring.

In local approximation rings, prime ideals extend to prime ideals in the completion.

Proposition 5.20 ([29, Corollary 3])**.** *Let* $(R, \mathfrak{m})$ *be an approximation ring and* $\mathfrak{p} \in \operatorname{Spec}(R)$. *Then* $\widehat{\mathfrak{p}} \in \operatorname{Spec}(\widehat{R})$.

The above will come in handy in the following descent-of-perinormality result.

Proposition 5.21. *Let* $(R, \mathfrak{m})$ *be an analytically irreducible approximation domain such that* $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ *is bijective. If* $\widehat{R}$ *is perinormal, then* R *is perinormal.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and let S be an integral overring of $R_\mathfrak{p}$ is such that $\operatorname{Spec}(S) \to \operatorname{Spec}(R_\mathfrak{p})$ is bijective. Let $T := S \cap R'$ and notice that $T_\mathfrak{p} = S$ since $T_\mathfrak{p} = (S \cap R')_\mathfrak{p} = S \cap R'_\mathfrak{p} = S$.

By Remark 5.8, since R is a noetherian, local, and analytically irreducible domain, $(R', \mathfrak{m}')$ must be a local, noetherian domain, finitely generated as an R-module. The domain T, lying between R and R', must also be a local, noetherian domain, finitely generated as an R-module. Let $\mathfrak{n}$ be the unique maximal ideal of T. We note that $\mathfrak{m}T$ and $\mathfrak{n}$ induce the same topology on T, and that $\mathfrak{m}R', \mathfrak{n}R'$, and $\mathfrak{m}'$ induce the same topology on R'. Thus, we have the induced ring extensions $\widehat{R} \subset \widehat{T} \subset \widehat{(R')}$.

With remark 5.19, we have R to be excellent. We apply Lemma 5.9 to conclude that the $\mathfrak{m}$-adic completion of R' is the integral closure of the domain $\widehat{R}$ in its field of fractions $\mathcal{K}$. So $\widehat{(R')} \cong (\widehat{R})'$, and by Propositions 5.3 and 5.4, both $\widehat{(R')}$ and $\widehat{T}$ are noetherian, complete, local, domains, finitely generated as $\widehat{R}$-modules.

The bijection $\operatorname{Spec}(T_{\mathfrak{p}}) = \operatorname{Spec}(S) \to \operatorname{Spec}(R_{\mathfrak{p}})$ allows us to conclude that

$$\{Q \in \operatorname{Spec}(T) \mid Q \cap R \subset \mathfrak{p}\} \to \{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{q} \subset \mathfrak{p}\},$$

defined by $Q \mapsto Q \cap R$, is a bijection. Since R is an approximation ring, $\mathfrak{p}\widehat{R} \in \operatorname{Spec}(\widehat{R})$. Additionally, since $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ is bijective, $\mathfrak{p}\widehat{R}$ must be the unique prime of $\widehat{R}$ lying over $\mathfrak{p}$.

Claim 5.22. The map $\operatorname{Spec}(\widehat{T}_{\mathfrak{p}\widehat{R}}) \to \operatorname{Spec}(\widehat{R}_{\mathfrak{p}\widehat{R}})$ is a bijection.

Proof of Claim 5.22*:* Let $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ be distinct prime ideals of $\widehat{T}$ such that $\mathfrak{Q}_i \cap \widehat{R} = \mathfrak{q}\widehat{R} \in \operatorname{Spec}(\widehat{R})$ and $\mathfrak{Q}_i \cap R \subseteq \mathfrak{p}$ for $i = 1, 2$. By the bijectivity of the map $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$, we must have $\mathfrak{q} = \mathfrak{q}\widehat{R} \cap R$. So, $\mathfrak{q} = \mathfrak{Q}_i \cap R$. The domains R' and T are finitely generated as modules over the approximation ring R, and, as such, are approximation rings in their own right. It follows that $Q\widehat{T} \in \operatorname{Spec}(\widehat{T})$ for all

$Q \in \mathrm{Spec}(T)$. Then

$$\mathfrak{q} = \mathfrak{Q}_i \cap R = (\mathfrak{Q}_i \cap T) \cap R = (\mathfrak{Q}_i \cap \widehat{R}) \cap R. \tag{5.4}$$

Let $Q_i := \mathfrak{Q}_i \cap T \in \mathrm{Spec}(T)$ for each $i = 1, 2$. Thus $Q_1 \cap R = \mathfrak{q} = Q_2 \cap R$ and $Q_i \cap R \subseteq \mathfrak{p}$. The bijective map $\mathrm{Spec}(S) \to \mathrm{Spec}(R_{\mathfrak{p}})$ forces $Q_1 = Q_2$. Let us call $\mathfrak{Q}_i \cap T = Q$. Then $Q\widehat{T} \in \mathrm{Spec}(\widehat{T})$ and $Q\widehat{T} \subseteq \mathfrak{Q}_i$ for each $i = 1, 2$. Since $\mathfrak{Q}_1 \neq \mathfrak{Q}_2$, we may assume that $Q\widehat{T} \subsetneq \mathfrak{Q}_1$. Note that $Q\widehat{T}$ and $\mathfrak{Q}_1$, both, lie over $\mathfrak{q}\widehat{R} \in \mathrm{Spec}(\widehat{R})$. (Namely, $\mathfrak{q}\widehat{R} \subseteq Q\widehat{T} \cap \widehat{R} \subseteq \mathfrak{Q}_i \cap \widehat{R} = \mathfrak{q}\widehat{R}$.) But, that contradicts the incomparability clause of the Cohen-Seidenberg Theorem for the integral ring extension $\widehat{R} \subseteq \widehat{T}$.

Thus $\mathfrak{Q}_1 = \mathfrak{Q}_2$ and $\mathrm{Spec}(\widehat{T}_{\mathfrak{p}\widehat{R}}) \to \mathrm{Spec}(\widehat{R}_{\mathfrak{p}\widehat{R}})$ is bijective and the claim is proven.

The perinormality of $\widehat{R}$ and the claim imply $\widehat{T}_{\mathfrak{p}\widehat{R}} = \widehat{R}_{\mathfrak{p}\widehat{R}}$.

Claim 5.23. $\widehat{R}_{\mathfrak{p}\widehat{R}} \cap T_{\mathfrak{p}} = R_{\mathfrak{p}}$.

Proof of Claim 5.23 By the bijectivity of the map $\mathrm{Spec}(\widehat{R}) \to \mathrm{Spec}(R)$, we must have $\mathfrak{p} = \mathfrak{p}\widehat{R} \cap R$. Thus $R_{\mathfrak{p}} \subseteq \widehat{R}_{\mathfrak{p}\widehat{R}}$, and, so, $R_{\mathfrak{p}} \subset \widehat{R}_{\mathfrak{p}\widehat{R}} \cap T_{\mathfrak{p}}$. Let $x \in \widehat{R}_{\mathfrak{p}\widehat{R}} \cap T_{\mathfrak{p}}$. We have the exact sequence

$$0 \to R_{\mathfrak{p}} \to xR_{\mathfrak{p}} + R_{\mathfrak{p}} \to \frac{xR_{\mathfrak{p}} + R_{\mathfrak{p}}}{R_{\mathfrak{p}}} \to 0$$

Since $\widehat{R}_{\mathfrak{p}\widehat{R}}$ is faithfully flat over $R_{\mathfrak{p}}$ we have the exact sequence

$$0 \to \widehat{R}_{\mathfrak{p}\widehat{R}} \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}} \to \widehat{R}_{\mathfrak{p}\widehat{R}} \otimes_{R_{\mathfrak{p}}} (xR_{\mathfrak{p}} + R_{\mathfrak{p}}) \to \widehat{R}_{\mathfrak{p}\widehat{R}} \otimes_{R_{\mathfrak{p}}} \frac{xR_{\mathfrak{p}} + R_{\mathfrak{p}}}{R_{\mathfrak{p}}} \to 0 \tag{5.5}$$

Since $x \in \widehat{R}_{\mathfrak{p}\widehat{R}}$, it follows that

$$\widehat{R}_{\mathfrak{p}\widehat{R}} \otimes_{R_{\mathfrak{p}}} (xR_{\mathfrak{p}} + R_{\mathfrak{p}}) \cong x\widehat{R}_{\mathfrak{p}\widehat{R}} + \widehat{R}_{\mathfrak{p}\widehat{R}} = \widehat{R}_{\mathfrak{p}\widehat{R}}.$$

Thus equation 5.5 becomes

$$0 \to \widehat{R}_{\mathfrak{p}\widehat{R}} \to \widehat{R}_{\mathfrak{p}\widehat{R}} \to \widehat{R}_{\mathfrak{p}\widehat{R}} \otimes_{R_\mathfrak{p}} \frac{xR_\mathfrak{p} + R_\mathfrak{p}}{R_\mathfrak{p}} \to 0$$

namely $\widehat{R}_{\mathfrak{p}\widehat{R}} \otimes_{R_\mathfrak{p}} \dfrac{xR_\mathfrak{p} + R_\mathfrak{p}}{R_\mathfrak{p}} = 0$. By faithful flatness of $\widehat{R}_{\mathfrak{p}\widehat{R}}$ over $R_\mathfrak{p}$ we have $\dfrac{xR_\mathfrak{p} + R_\mathfrak{p}}{R_\mathfrak{p}} = 0$, hence $x \in R_\mathfrak{p}$ and the claim is finished.

Finally, Claim 5.23 and $\widehat{T}_{\mathfrak{p}\widehat{R}} \cap T_\mathfrak{p} = T_\mathfrak{p}$ imply that $R_\mathfrak{p} = T_\mathfrak{p} = S$. Hence, R is perinormal. ∎

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Vita

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