# UNIFORM BOUNDS IN F-FINITE RINGS AND THEIR APPLICATIONS

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By

**Thomas Marion Polstra**

Dissertation Advisor:
Professor Ian Aberbach

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

UNIFORM BOUNDS IN PRIME CHARACTERISTIC RINGS AND THEIR APPLICATIONS

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Thomas Marion Polstra

a candidate for the degree of Doctor of Philosophy and hereby certify that in their opinion it is worthy of acceptance.

Professor Ian Aberbach

Professor Steven Dale Cutkosky

Professor Dan Edidin

Professor Jay Dow

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# Contents

## Abstract

This dissertation establishes uniform bounds in characteristic $p$ rings which are either F-finite or essentially of finite type over an excellent local ring. These uniform bounds are then used to show that the Hilbert-Kunz length functions and the normalized Frobenius splitting numbers defined on the spectrum of a ring converge uniformly to their limits, namely the Hilbert-Kunz multiplicity function and the F-signature function. From this we establish that the F-signature function is lower semi-continuous. Lower semi-continuity of the F-signature of a pair is also established. We also give a new proof of the upper semi-continuity of Hilbert-Kunz multiplicity, a result originally proven by Ilya Smirnov.

CHAPTER 1

# Introduction

Throughout this dissertation, unless otherwise stated, all rings are assumed to be commutative, Noetherian, with identity, and of prime characteristic $p$. If $M$ is an $R$-module then we let $\ell(M)$ denote its length, provided $M$ is a finite length $R$-module, and $\mu(M)$ denotes the minimal number of generators needed to generate $M$ as an $R$-module, provided $M$ is a finitely generated $R$-module.

We let $F^e : R \to R$ denote the $e$th iterate of the Frobenius endomorphism on $R$, that is $F^e(r) = r^{p^e}$ for each $r \in R$. It has long been known that behavior of the Frobenius endomorphisms govern the singularities of $R$. Most notable is Kunz's theorem, [**Kun69**], equating flatness of $F^e$ with the property that $R$ is regular. Given that failure of flatness of $F^e$ is equivalent to a ring having singularities, it then becomes natural to introduce asymptotic measurements of the Frobenius endomorphisms which can be used to detect "how much" the Frobenius endomorphisms fail to be flat in hopes to measure "how bad" the singularities of the ring are. The two asymptotic measurements this dissertation is concerned with are Hilbert-Kunz multiplicity and F-signature, which we introduce below.

For the sake of simplicity, we assume for the remainder of the introduction that $R$ is a domain of prime characteristic $p$. If $I \subseteq R$ is an ideal of $R$ then let $I^{[p^e]} = F^e(I)R = (i^{p^e} \mid i \in I)$ be the expansion of $I$ along $F^e$. If $(R, \mathfrak{m}, k)$ is a local ring, then of particular interest is expansion of the maximal ideal along the Frobenius endomorphisms. The sequence of ideals $\mathfrak{m}^{[p^e]}$ will be a descending chain of $\mathfrak{m}$-primary ideals and the numbers $\ell(R/\mathfrak{m}^{[p^e]}R)$ will be bounded below by $p^{ed}$ where $d$ is the Krull dimension of the local ring $R$. In fact, Kunz showed that $R$ being regular is equivalent to $\ell(R/\mathfrak{m}^{[p^e]}R) = p^{ed}$. We then measure the failure of Frobenius to be flat by studying the sequence of ratios of $\ell(R/\mathfrak{m}^{[p^e]}R)$ against the expected value of $p^{ed}$ if $R$ was regular, that is study the sequence of numbers $\frac{1}{p^{ed}}\ell(R/\mathfrak{m}^{[p^e]}R)$. The *Hilbert-Kunz multiplcity* of the local ring $(R, \mathfrak{m}, k)$ is defined to by the limit $\mathrm{e}_{\mathrm{HK}}(R) = \lim_{e\to\infty} \frac{1}{p^{ed}}\ell(R/\mathfrak{m}^{[p^e]}R)$, a limit shown to always exist by Monsky in [**Mon83**]. If $(R, \mathfrak{m}, k)$ is regular, then sequence of numbers $\frac{1}{p^{ed}}\ell(R/\mathfrak{m}^{[p^e]}R)$ is the constant sequence 1 and $\mathrm{e}_{\mathrm{HK}}(R) = 1$. The reverse implication holds as well, that is if $\mathrm{e}_{\mathrm{HK}}(R) = 1$ then $R$ is a regular local ring by work of Watanabe and Yoshida in [**WY00**]. Furthermore, by work stemming from [**BE04**], small enough values of $\mathrm{e}_{\mathrm{HK}}(R)$ imply that $R$ has decent singularities, see Theorem 2.13 below for a precise statement.

To introduce F-signature we impose a further restriction on $R$ for the sake of simplicity. For each $e \in \mathbb{N}$ we let $F^e_*R$ denote the $R$-module which is the $R$-module $R$ obtained via restriction of scalars under $F^e$. Thus given $r \in R$ and $F^e_*s \in F^e_*R$ ($F^e_*s$ denotes the element $s \in R$ but viewed as element of $F^e_*R$) we have $rF^e_*s = F^e_*r^{p^e}s$. The restriction we impose on $R$ is that $F^e_*R$ is a finitely generated $R$-module for each $e \in \mathbb{N}$. Characteristic $p$ rings with this property are called *F-finite*. See Chapter 2 for more information on F-finite rings and how to extend the notion of F-signature to all local rings of prime characteristic.

Let $(R, \mathfrak{m}, k)$ be a local F-finite domain of prime characteristic $p$. By Kunz's theorem discussed above, $R$ being regular is then equivalent to $F^e_*R$ being a finitely generated free $R$-module. Given a finitely generated $R$-module $M$ let $\mathrm{frk}(M)$, the *free rank* of $M$, be the largest rank of a free $R$-module appearing in various direct sum decompositions of $M$. Of particular interest are the *eth Frobenius splitting numbers* $a_e(R) = \mathrm{frk}(F^e_*R)$. Kunz's theorem equates $R$ having singularities with $a_e(R)$ being strictly less than $\mathrm{rank}(F^e_*R)$. We once again can measure the failure of Frobenius being flat by studying the sequence of ratios $\frac{1}{\mathrm{rank}(F^e_*R)}a_e(R)$. The *F-signature* of $R$ is the limit $s(R) = \lim_{e\to\infty} \frac{1}{\mathrm{rank}(F^e_*R)}a_e(R)$. The F-signature of a local ring was first defined by Huneke and Leuschke in [**HL02**], the existence of the limit F-signature was established in several partial cases by various authors, see for example [**Abe08**], [**AE05**], [**HL02**], [**Sin05**], [**WY04**], and was shown to exist in full generality by Tucker in [**Tuc12**]. If $R$ is regular then the sequence $\frac{1}{\operatorname{rank}(F^e*R)}a_e(R)$ is the constant sequence 1 and therefore $s(R) = 1$. Huneke and Leuschke show in [**HL02**] that $s(R)$ being 1 is in fact equivalent to $R$ being regular.

As with Hilbert-Kunz multiplicity, the values of $s(R)$ can determine the severity of the singularities of $R$. The sequence of numbers $\frac{1}{\operatorname{rank}(F^e_*R)}a_e(R)$ are between 0 and 1. Hence the limit value $s(R)$ lies between 0 and 1 with the maximum value being obtained only when $R$ is regular. Aberbach and Leuschke show in [**AL03**] (along with Tucker's proof of the existence of F-signature in [**Tuc12**]) that $s(R) > 0$ is equivalent to $R$ being strongly F-regular, a condition we discuss in Chapter 2.

Suppose that $R$ is an F-finite domain of prime characteristic $p$, not necessarily local. Then each localization of $R$ at a prime ideal is an F-finite local domain. Hence it is natural to study the Hilbert-Kunz multiplicity function which maps a prime $P \mapsto \mathrm{e}_{\mathrm{HK}}(R_P)$, the Hilbert-Kunz multiplicity of $R_P$, and the F-signature function which maps a prime $P \mapsto s(R_P)$, the F-signature of $R_P$. Let us first consider the F-signature function.

The F-signature function is a real-valued function on $X = \operatorname{Spec}(R)$ which takes its values in the unit interval $[0, 1]$. The locus of primes $P \in X$ for which $s(R_P) = 1$ is the regular locus of $X$. Moreover, if $P, Q \in X$ and $s(R_P) > s(R_Q)$, then we interpret this inequality to mean that the Frobenius endomorphism is "more flat" at the prime $P$ than it is at $Q$. Thus we expect the singularity at $P$ to be nicer than the singularity at $Q$. Moreover, it is natural to expect the singularity at a prime $P$ to control the severity of the singularities of $X$ in a small enough open neighborhood around $P$. Thus it is reasonable to conjecture that for any $\varepsilon > 0$, if $P \in X$, then there is a small enough open neighborhood $U \subseteq \operatorname{Spec}(R)$ of $P$ such that $s(R_P) - s(R_Q) < \varepsilon$ for all $Q \in U$, i.e., the F-signature function is lower semi-continuous. The main contribution of this dissertation is an affirmative answer to whether or not the F-signature function is lower semi-continuous. We remark that in the following theorem we do not assume $R$ is a domain.

THEOREM A. *Let $R$ be a ring of prime characteristic $p$ which is either F-finite or essentially of finite type over an excellent local ring. Then the F-signature function on* $\operatorname{Spec}(R)$ *is lower semi-continuous.*

To quote [**BST13**], "in the absolute setting, a difficult and important open problem is to show that the F-signature is lower semi-continuous as a function on the prime spectrum of a ring." The authors of [**BST13**] extend the notion of F-signature to pairs $(R, \mathscr{D})$ in [**BST12**] where $\mathscr{D}$ is a Cartier subalgebra (see Chapter 2) and prove lower semi-continuity of the F-signature in the limited case where $R$ is a regular ring and $\mathscr{D}$ is a Cartier subalgebra associated to a hyperplane and a positive real number, see [**BST13**] for details. See Theorem 4.9 below for a proof that the F-signature function is lower semi-continuous for any F-finite ring $R$ and Cartier subalgebra $\mathscr{D}$ on $R$.

Before the work of Blickle, Schwede, and Tucker in [**BST13**], Enescu and Yao were able to show in [**EY11**] that the F-signature function is naturally the limit of lower semi-continuous functions. They show that for each $e \in \mathbb{N}$ the function which maps a prime $P \mapsto \frac{1}{\operatorname{rank}(F^e_* R_P)} a_e(R_P)$ is lower semi-continuous. We shall establish lower semi-continuity of the F-signature by showing the F-signature function is the *uniform* limit of these functions.

We now shift our attention back to Hilbert-Kunz multiplicity. Once again, assume for sake of simplicity that $R$ is an F-finite domain, not necessarily local, and let $X = \operatorname{Spec}(R)$. The Hilbert-Kunz multiplicity function is a real-valued function on $X$ which takes its values in the interval $[1, \infty)$. The locus of primes for which $\mathrm{e}_{\mathrm{HK}}(R_P) = 1$ is the regular locus of $X$. It is then reasonable to expect the Hilbert-Kunz multiplicity function to be upper semi-continuous on $X$. In fact, upper semi-continuity of Hilbert-Kunz multiplicity was established by Smirnov in [**Smi16**]. Kunz showed that for each

$e \in \mathbb{N}$ the function $\ell_e$ which maps a prime $P \mapsto \frac{1}{p^{e\,\mathrm{ht}(P)}}\ell(R_P/P^{[p^e]}R_P)$ is upper semi-continuous. Hence, the Hilbert-Kunz multiplicity function is naturally the limit of upper semi-continuous functions. Smirnov does not show the sequence of functions $\ell_e$ converge uniformly to its limit functions. However, we are able to establish the uniform convergence of the sequence functions $\ell_e$ and therefore recover Smirnov's theorem.

THEOREM B. *Let $R$ be a ring of prime characteristic $p$ which is either F-finite or essentially of finite type over an excellent local ring. Then the sequence of functions which map a prime $P \mapsto \frac{1}{p^{e\,\mathrm{ht}(P)}}\ell(R_P/P^{[p^e]}R_P)$ converge uniformly to their limit function, namely the Hilbert-Kunz multiplicity function. In particular, we recover Smirnov's theorem that the Hilbert-Kunz multiplicity function is upper semi-continuous at all primes $P \in \mathrm{Spec}(R)$ for which $R_P$ is equidimensional.*

In order to establish lower semi-continuity of the F-signature and to recover Smirnov's result that the Hilbert-Kunz multiplicity function is upper semi-continuous, we will need to establish uniform length bounds of Hilbert-Kunz type functions. It has long been understood that bounds of Hilbert-Kunz lengths in local rings are key for establishing the existence of the limits of Hilbert-Kunz multiplicity and F-signature. We describe the key bound that is used to establish the existence of these limits. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p$ and $M$ a finitely generated $R$-module of dimension $d$. Then there exists a constant $C \in \mathbb{R}$ so that for all $e \in \mathbb{N}$, $\ell(M/\mathfrak{m}^{[p^e]}M) \leqslant Cp^{ed}$. This length bound is used to establish the existence of the limits Hilbert-Kunz multiplicity and F-signature in Chapter 2. The key uniform bound of this dissertation, which is easily derived from Theorem 3.7 and Theorem 3.13, is the following:

THEOREM C. *Let $R$ be a ring of prime characteristic $p$ which is either F-finite or essentially of finite type over an excellent local ring. Let $M$ be a finitely generated*

*$R$-module. There exists a constant $C \in \mathbb{R}$ so that for all $e \in \mathbb{N}$ and all $P \in \operatorname{Spec}(R)$,*

$$\ell\left(\frac{M_P}{P^{[p^e]}M_P}\right) \leqslant Cp^{e\dim(M_P)}.$$

Chapter 2 is used to recall the basics of Frobenius and to discuss in detail the numerical invariants Hilbert-Kunz multiplicity and F-signature. Beyond recalling previously proven results in Chapter 2, we provide an elementary proof of the existence of the limit F-signature. Chapter 3 is the heart of the dissertation. In it are several technical theorems which provide strong numerical bounds which are in the spirit of Theorem C. Chapter 4 is the payoff of the technical results of Chapter 3. In particular, Theorem A is given by Theorem 4.7 and Theorem B is a special case of Theorem 4.1.

CHAPTER 2

# Limits in positive characteristic commutative algebra

## 1. Basics of Frobenius

Suppose $R$ is a commutative Noetherian ring of prime characteristic $p$. We denote by $F : R \to R$ the Frobenius endomorphism, that is $F(r) = r^p$ for all $r \in R$. For each $e \in \mathbb{N}$ we let $F^e : R \to R$ denote the Frobenius endomorphism composed with itself $e$ times. Observe that $F^e(r) = r^{p^e}$ for each $r \in R$. Given an $R$-module $M$ let $F^e_* M$ be the induced $R$-module obtained via restriction of scalars under $F^e$. If $R$ is reduced then $F^e_* R$ is naturally isomorphic to the $R$-module $R \subseteq R^{1/p^e}$ as $R$ sits in a algebraic closure of its total ring of fractions. However, we will refrain from using this notation and henceforth use $F^e_* \_$.

As discussed in the introduction, we are interested in the asymptotic behavior of iterating the Frobenius endomorphism. For example, we will be interested in studying properties of ideals under the image of Frobenius. If $I \subseteq R$ is an ideal, then $F^e(I)R$, which is the expansion of $I$ along $F^e$, is $I^{[p^e]}R := F^e(I)R = (i^{p^e} \mid i \in I)$. If $I$ is generated by $i_1, \ldots, i_s$ then $I^{[p^e]}R$ is seen to be generated by $i_1^{p^e}, \ldots, i_s^{p^e}$. Also observe that if $M$ is an $R$-module then $IF^e_* M = F^e_* I^{[p^e]} M$. Moreover, as restricting scalars is exact, $F^e_* M / IF^e_* M \cong F^e_*(M/I^{[p^e]}M)$.

We say that $R$ is *F-finite* if $F^e_* R$ is a finitely generated $R$-module for all $e \in \mathbb{N}$. The following proposition is a list of well-known, useful, and easy to prove properties concerning F-finite rings and restricting scalars under iterates of the Frobenius endomorphism.

PROPOSITION 2.1. *Let $R$ be a Noetherian ring of prime characteristic $p$.*

(1) *$R$ is F-finite if and only if $F^e_* R$ is finitely generated for some $e \in \mathbb{N}$.*

(2) *If $R$ is F-finite and $M$ a finitely generated $R$-module then $F^e_*M$ is a finitely generated $R$-module.*

(3) *If $M$ is an $R$-module then $F^e_*F^{e'}_*M \cong F^{e+e'}_*M$ for all $e, e' \in \mathbb{N}$.*

(4) *$R$ is F-finite if and only if $R/\sqrt{0}$ is F-finite.*

(5) *If $R$ is F-finite then so is any homomorphic image of $R$.*

(6) *If $R$ is F-finite then so is any ring essentially of finite type over $R$.*

(7) *If $R$ is F-finite then so is the formal power series ring $R[[x_1, \dots, x_n]]$.*

(8) *If $k$ is a field of characteristic $p$, then $k$ is F-finite if and only if $[F^e_*k : k] < \infty$, i.e., $F^e_*k$ is a finitely generated vector space, for each $e \in \mathbb{N}$.*

(9) *If $(R, \mathfrak{m}, k)$ is a complete local ring then $R$ is F-finite if and only if $k$ is an F-finite field.*

(10) *If $P \in \operatorname{Spec}(R)$ and $M$ an $R$-module, then $(F^e_*M)_P \cong F^e_*M_P$.*

We will be using many of the properties listed in Proposition 2.1 without reference. Kunz proved every F-finite ring is excellent in [**Kun76**]. Another fundamental result of Kunz is equating flatness of the modules $F^e_*R$ with $R$ being regular. We outline a proof of this theorem.

THEOREM 2.2 ([**Kun69**, Theorem 2.1]). *Let $R$ be a Noetherian ring of prime characteristic $p$. The following are equivalent:*

(1) *$R$ is regular;*

(2) *$F^e_*R$ is a flat $R$-module for each $e \in \mathbb{N}$;*

(3) *$F^e_*R$ is a flat $R$-module for some $e \in \mathbb{N}$.*

PROOF. Flatness and the property of being regular can be checked locally. If $(R, \mathfrak{m}, k)$ is a regular local ring of prime characteristic $p$, then the minimal free resolution of $k$ is the Koszul complex on a minimal set of generators $x_1, ..., x_d$ of the maximal ideal. Tensoring this complex with $F^e_*R$ yields a new complex, which as a complex of Abelian groups is isomorphic to Koszul complex on the regular sequence $x_1^{p^e}, \dots, x_d^{p^e}$. Hence the first and all higher degree homology groups of this complex are 0. In particular, $\mathrm{Tor}_1^R(k, F^e_* R) = 0$ and $F^e_* R$ is a flat $R$-module.

The implication (2) $\Rightarrow$ (3) is trivial. For the implication (3) $\Rightarrow$ (1), consider the short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}^2 \to R/\mathfrak{m} \to 0.$$

By tensoring the above short exact sequence with the flat $R$-module $F^e_* R$ we obtain the short exact sequence

$$0 \to (\mathfrak{m}/\mathfrak{m}^2) \otimes_R F^e_* R \to F^e_*(R/(\mathfrak{m}^{p^{[e]}})^2) \to F^e_*(R/\mathfrak{m}^{[p^e]}) \to 0.$$

Hence $F^e_*(\mathfrak{m}^{[p^e]}/(\mathfrak{m}^{[p^e]})^2) \cong (\mathfrak{m}/\mathfrak{m}^2) \otimes_R F^e_* R \cong (\mathfrak{m}/\mathfrak{m}^2) \otimes_k F^e_* R/\mathfrak{m} F^e_* R \cong \mathfrak{m}/\mathfrak{m}^2 \otimes_k F^e_*(R/\mathfrak{m}^{[p^e]})$. In particular, $F^e_*(\mathfrak{m}^{[p^e]}/(\mathfrak{m}^{[p^e]})^2)$ is a free $F^e_*(R/\mathfrak{m}^{[p^e]})$-module. Equivalently, $\mathfrak{m}^{[p^e]}/(\mathfrak{m}^{[p^e]})^2$ is a free $R/\mathfrak{m}^{[p^e]}$-module. Suppose that $x_1, \ldots, x_n$ is a set of minimal generators of $\mathfrak{m}$. As $F^e_*(\mathfrak{m}^{[p^e]}/(\mathfrak{m}^{[p^e]})^2) \cong (\mathfrak{m}/\mathfrak{m}^2) \otimes_R F^e_* R$, the elements $x_1^{p^e}, ..., x_n^{p^e}$ minimally generate $\mathfrak{m}^{[p^e]}$. One can then use [**Lec64**, Lemma 3] and [**Lec64**, Lemma 4] to inductively show that $\ell(R/(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})) = \alpha_1 \cdots \alpha_n$ for all $1 \leqslant \alpha_i \leqslant p^e$. In particular, $\ell(R/\mathfrak{m}^{[p^e]}) = p^{en}$. By the Cohen structure theorem, $\widehat{R}$ has a coefficient field $k \subseteq R$ and there is an onto map of rings $\pi : k[[X_1, \ldots, X_n]] \to \widehat{R}$ by mapping $X_i \mapsto x_i$. But an explicit computation shows $\ell(k[[X_1, \ldots, X_n]]/(X_1, \ldots, X_n)^{[p^e]}) = p^{en}$ which is the same as $\ell(R/(x_1, \ldots, x_n)^{[p^e]}) = \ell(\widehat{R}/(x_1, \ldots, x_n)^{[p^e]}\widehat{R})$ and therefore $\ker(\pi) \subseteq (X_1, \ldots, X_n)^{[p^e]}$. However, flatness of $F^e_* R$ implies flatness of $F^{ef}_* R$ for all $f \in \mathbb{N}$. Repeating the above, one sees that $\ker(\pi) \subseteq \cap_{f=1}^{\infty}(X_1, \ldots, X_n)^{[p^{ef}]} = 0$. Hence $\pi$ is an isomorphism and $R$ is regular. $\square$

Another fundamental contribution of Kunz to the study of prime characteristic rings is the computation of $\mathrm{rank}(F^e_* R)$. The key observation is that if $R = k[[x_1, \ldots, x_d]]$ is a regular local ring of dimension $d$ over the F-finite field $k$ and $\Gamma$ a basis for the $k$-vector space $F^e_* k$, then $F^e_* R$ is a free $R$-module with basis $\{\gamma F^e_* x_1^{i_1} \cdots x_d^{i_d} \mid$

$0 \leqslant i_j \leqslant p^e - 1$ and $\gamma \in \Gamma\}$. Hence, $\operatorname{rank}(F^e_* R) = |\Gamma| p^{ed} = [F^e_* k : k] p^{ed}$. Kunz extends this computation to the non-regular case in [**Kun76**].

Proposition 2.3. *If $(R, \mathfrak{m}, k)$ is an F-finite local domain of dimension $d$, then $\operatorname{rank}_R(F^e_* R) = [F^e_* k : k] \cdot p^{ed}$.*

We now discuss a sequence of corollaries of Proposition 2.3. First observe that if $R$ is a local F-finite domain with fraction field $K$, then $\operatorname{rank}_R(F^e_* R) = \operatorname{rank}_K(F^e_* K)$. Thus Proposition 2.3 implies the following.

Corollary 2.4. *Let $R$ be an F-finite ring. Given a prime ideal $P \subseteq R$ let $k(P) = R_P/PR_P$ denote the residue field of the localization $R_P$. For any two prime ideals $P \subseteq Q$ of $R$,*

$$[F^e_* k(P) : k(P)] = [F^e_* k(Q) : k(Q)] \cdot p^{e \dim(R_Q/PR_Q)}.$$

*In particular, if $R$ is locally equidimensional then*

$$[F^e_* k(P) : k(P)] \cdot p^{e \dim(R_P)} = [F^e_* k(Q) : k(Q)] \cdot p^{e \dim(R_Q)}$$

*and the function $\operatorname{Spec}(R) \to \mathbb{N}$ given by $P \mapsto [F^e_* k(P) : k(P)] \cdot p^{e \dim(R_P)}$ is constant on the connected components of $\operatorname{Spec}(R)$.*

If $R$ is an F-finite locally equidimensional ring with connected spectrum, we denote by $\gamma(R)$ the value which $\log_{p^e}(\operatorname{rank}_{R_P}(F^e_* R_P))$ takes on for each $P \in \operatorname{Spec}(R)$. Observe that if $R$ is an F-finite domain, then $R$ is reduced with connected spectrum and $\gamma(R)$ is the unique integer for which $\operatorname{rank}_R(F^e_* R) = p^{e\gamma(R)}$. If $R$ is F-finite, but the function $P \mapsto [F^e_* k(P) : k(P)] \cdot p^{e \dim(R_P)}$ is not necessarily constant, then we let $\gamma(R)$ be the maximum of the function that maps $P \mapsto \log_{p^e}([F^e_* k(P) : k(P)] \cdot p^{e \dim(R_P)})$. It easily follows that $\gamma(R) = \gamma(R_P)$ for some minimal prime of $P$. In fact, if $P$ is any minimal prime ideal of $R$ so that $\dim(R/P) = \dim(R)$, then $\gamma(R) = \gamma(R_P)$. In particular, $p^{e\gamma(R)} = [F^e_* k : k] p^{e \dim(R)}$ for any F-finite local ring.

Corollary 2.5. *If $R$ is a locally equidimensional $F$-finite reduced ring, of dimension $d$, with connected spectrum then there exist short exact sequences*

$$0 \longrightarrow R^{\oplus p^{\gamma(R)}} \longrightarrow F_*R \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow F_*R \longrightarrow R^{\oplus p^{\gamma(R)}} \longrightarrow N \longrightarrow 0$$

*such that* $\dim(M) < d$ *and* $\dim(N) < d$.

Proof. The finitely generated $R$-modules $R^{\oplus p^{\gamma(R)}}$ and $F_*R$ have the same rank by Corollary 2.4. Hence there exists $R$-linear maps $R^{\oplus p^{\gamma(R)}} \to F_*R$ and $F_*R \to R^{\oplus p^{\gamma(R)}}$ whose cokernels, $M$ and $N$ respectively, are torsion. As $R$ is reduced, it must be the case that $R^{\oplus p^{\gamma(R)}} \to F_*R$ and $F_*R \to R^{\oplus p^{\gamma(R)}}$ are injective. □

The following corollary allows us to effectively relate the length of a finitely generated module $M$ over an F-finite local ring $R$ with the length of the $R$-module $F^e_*M$ obtained via restriction of scalars under $F^e$.

Corollary 2.6. *Let $(R, \mathfrak{m}, k)$ be an F-finite ring of prime characteristic $p$ and of dimension $d$. Let $M$ be a finite length $R$-module, then for each $e \in \mathbb{N}$, $\frac{1}{p^{ed}}\ell(M) = \frac{1}{p^{e\gamma(R)}}\ell(F^e_*M)$.*

Proof. Consider a prime filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{\ell(M)} = M$ of $M$. Each of the factors $M_i/M_{i-1}$ is isomorphic to $k$. As restricting scalars is exact, we obtain the filtration $0 = F^e_*M_0 \subseteq F^e_*M_1 \subseteq \cdots \subseteq F^e_*M_{\ell(M)} = F^e_*M$. Moreover, $F^e_*M_i/F^e_*M_{i-1} \cong F^e_*(M_i/M_{i-1}) \cong F^e_*k$. Hence, $\ell(F^e_*M) = \ell(F^e_*k)\ell(M) = [F^e_*k : k]\ell(M)$. The corollary follows by Corollary 2.4. □

## 2. Uniform bounds in local rings

In this section we establish the existence of uniform bounds found in local rings of prime characteristic. The techniques used in this section are elementary but are key to proving the existence of the numerical invariants Hilbert-Kunz multiplicity and

F-signature. In Chapter 3 we establish global versions of the results in this section for large classes rings which are not assumed to be local.

Proposition 2.7. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p$ and of dimension $d$. There exists a constant $C \in \mathbb{R}$ such that for every $\mathfrak{m}$-primary ideal $I \subseteq R$ and for every $e \in \mathbb{N}$, $\ell(R/I^{[p^e]}) \leqslant Cp^{ed}\ell(R/I)$.*

Proof. As $I$ is an $\mathfrak{m}$-primary ideal, there is a filtration of ideals

$$I = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{\ell(R/I)} = R$$

such that $I_{i+1} = (I_i, u_i)$ and $u_i \in (I_i : \mathfrak{m})$. Hence, for each $e \in \mathbb{N}$, there is a filtration of ideals

$$I^{[p^e]} = I_0^{[p^e]} \subseteq I_1^{[p^e]} \subseteq \cdots \subseteq I_{\ell(R/I)}^{[p^e]} = R$$

such that $I_{i+1}^{[p^e]} = (I_i^{[p^e]}, u_i^{p^e})$ and $u_i^{p^e} \in (I_i^{[p^e]} : \mathfrak{m}^{[p^e]})$. In particular, $\ell(R/I^{[p^e]}) \leqslant \ell(R/\mathfrak{m}^{[p^e]})\ell(R/I)$. Thus to prove the proposition, it is enough to show there exists $C \in \mathbb{R}$ such that for every $e \in \mathbb{N}$, $\ell(R/\mathfrak{m}^{[p^e]}) \leqslant Cp^{ed}$. This follows from the observation that if $\mathfrak{m}$ is generated by $n$ elements, then $\mathfrak{m}^{np^e} \subseteq \mathfrak{m}^{[p^e]}$. Hence, $\ell(R/\mathfrak{m}^{[p^e]}) \leqslant \ell(R/\mathfrak{m}^{np^e})$, the latter of which is eventually polynomial in the variable $p^e$ of degree $d$. $\square$

If $M$ is a finitely generated $R$-module and $J = \operatorname{Ann}_R(M)$, then there exists an $N \in \mathbb{N}$ and onto $R$-linear map $R^{\oplus N}/JR^{\oplus N} \to M$. In particular, if $I \subseteq R$ is an $\mathfrak{m}$-primary ideal of $R$, then $\ell(M/I^{[p^e]}M) \leqslant N\ell(R/(J + I^{[p^e]}))$. Thus we obtain the following corollary.

Corollary 2.8. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p$ and $M$ a finitely generated $R$-module of dimension $d$. There exists a constant $C \in \mathbb{R}$ such that for every $\mathfrak{m}$-primary ideal $I \subseteq R$ and for every $e \in \mathbb{N}$, $\ell(M/I^{[p^e]}M) \leqslant Cp^{ed}\ell(R/I)$.*

## 3. Hilbert-Kunz Multiplicity

Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p$ and $M$ a finitely generated $R$-module of dimension $d$. Given an $\mathfrak{m}$-primary ideal $I \subseteq R$, we define the *Hilbert-Kunz multiplicity of $I$ in $M$* to be $\mathrm{e_{HK}}(I; M) = \lim_{e\to\infty} \frac{1}{p^{ed}}\ell(M/I^{[p^e]}M)$, provided the limit exists. If $I = \mathfrak{m}$, we write $\mathrm{e_{HK}}(M)$ to denote $\mathrm{e_{HK}}(I; M)$ and call $\mathrm{e_{HK}}(M)$ the *Hilbert-Kunz multiplicity of $M$*. If $M = R$ we write $\mathrm{e_{HK}}(I)$ to denote $\mathrm{e_{HK}}(I; R)$ and call $\mathrm{e_{HK}}(I)$ the *Hilbert-Kunz multiplicity of $I$*. Lastly, if $I = \mathfrak{m}$ and $M = R$, we write $\mathrm{e_{HK}}(R)$ to denote $\mathrm{e_{HK}}(\mathfrak{m}; R)$ and call $\mathrm{e_{HK}}(R)$ the *Hilbert-Kunz multiplicity of $R$*. Monsky proved the existence of Hilbert-Kunz multiplicity in [**Mon83**].

The purpose of this section is to outline the proof of the existence of $\mathrm{e_{HK}}(I; M)$ for any $\mathfrak{m}$-primary ideal $I$ and finitely generated $R$-module $M$. The techniques of Chapter 3, which are the heart of this dissertation, are inspired by the techniques of this section.

With some work, the proof that the Hilbert-Kunz multiplicity of an ideal with respect to a finitely generated module exists reduces to proving the Hilbert-Kunz multiplicity of an ideal in an F-finite domain exists.

Theorem 2.9. *Let $(R, \mathfrak{m}, k)$ be a local F-finite domain of dimension $d$ and $I \subseteq R$ an $\mathfrak{m}$-primary ideal, then $\mathrm{e_{HK}}(I)$ exists, that is the limit $\lim_{e\to\infty} \frac{1}{p^{ed}}\ell(R/I^{[p^e]}R)$ exists.*

Proof. Denote respectively by ${\mathrm{e_{HK}}}^-(I)$ and ${\mathrm{e_{HK}}}^+(I)$ the limit infimum and limit supremum of the sequence $\frac{1}{p^{ed}}\ell(R/I^{[p^e]}R)$. By Lemma 2.5 there is a short exact sequence

$$0 \to F_*R \to R^{\oplus p^{\gamma(R)}} \to N \to 0$$

and $N$ is an $R$-module of dimension less than $d$. Restricting scalars is exact, hence we obtain for each $e \in \mathbb{N}$ the short exact sequence

$$0 \to F_*^{e+1}R \to F_*^e R^{\oplus p^{\gamma(R)}} \to F_*^e N \to 0.$$

Tensoring with $R/I$ we obtain right exact sequences

$$F_*^{e+1}\left(\frac{R}{I^{[p^{e+1}]}R}\right) \to F_*^e\left(\frac{R}{I^{[p^e]}R}\right)^{\oplus p^{\gamma(R)}} \to F_*^e\left(\frac{N}{I^{[p^e]}N}\right) \to 0.$$

Therefore for each $e \in \mathbb{N}$,

$$p^{\gamma(R)}\ell\left(F_*^e\left(\frac{R}{I^{[p^e]}R}\right)\right) \leqslant \ell\left(F_*^{e+1}\left(\frac{R}{I^{[p^{e+1}]}R}\right)\right) + \ell\left(F_*^e\left(\frac{N}{I^{[p^e]}N}\right)\right).$$

Dividing this inequality by $p^{(e+1)\gamma(R)}$ and applying Corollary 2.6 we see that for each $e \in \mathbb{N}$,

$$\frac{1}{p^{ed}}\ell\left(\frac{R}{I^{[p^e]}R}\right) \leqslant \frac{1}{p^{(e+1)d}}\ell\left(\frac{R}{I^{[p^{e+1}]}R}\right) + \frac{1}{p^{ed}}\ell\left(\frac{N}{I^{[p^e]}N}\right).$$

Applying Corollary 2.8 to the finitely generated $R$-module there is a constant $C$ such that for each $e \in \mathbb{N}$

$$\frac{1}{p^{ed}}\ell\left(\frac{R}{I^{[p^e]}R}\right) \leqslant \frac{1}{p^{(e+1)d}}\ell\left(\frac{R}{I^{[p^{e+1}]}R}\right) + \frac{C}{p^e},$$

i.e.,

$$\frac{1}{p^{ed}}\ell\left(\frac{R}{I^{[p^e]}R}\right) - \frac{1}{p^{(e+1)d}}\ell\left(\frac{R}{I^{[p^{e+1}]}R}\right) \leqslant \frac{C}{p^e}.$$

Therefore for each pair of integers $e, e'$,

$$\begin{aligned}\frac{1}{p^{ed}}\ell\left(\frac{R}{I^{[p^e]}R}\right) - \frac{1}{p^{(e+e')d}}\ell\left(\frac{R}{I^{[p^{e+e'}]}R}\right) &= \sum_{i=e}^{e+e'-1}\frac{1}{p^{ed}}\ell\left(\frac{R}{I^{[p^i]}R}\right) - \frac{1}{p^{(i+1)d}}\ell\left(\frac{R}{I^{[p^{i+1}]}R}\right)\\ &\leqslant \sum_{i=e}^{e+e'-1}\frac{C}{p^i} \leqslant \frac{2C}{p^e}.\end{aligned}$$

Hence for each pair of integers $e, e'$,

$$\frac{1}{p^{ed}}\ell\left(\frac{R}{I^{[p^e]}R}\right) \leqslant \frac{1}{p^{(e+e')d}}\ell\left(\frac{R}{I^{[p^{e+e'}]}R}\right) + \frac{2C}{p^e}.$$

Taking limit infimum as $e' \to \infty$ we see that for each $e \in \mathbb{N}$

$$\frac{1}{p^{ed}}\ell\left(\frac{R}{I^{[p^e]}R}\right) \leqslant \mathrm{e_{HK}}^-(I) + \frac{2C}{p^e}.$$

Taking the limit supremum as $e \to \infty$ shows

$$\mathrm{e_{HK}}^+(I) \leqslant \mathrm{e_{HK}}^-(I),$$

thus $\mathrm{e_{HK}}^-(I) = \mathrm{e_{HK}}^+(I)$ and the limit $\lim_{e\to\infty} \frac{1}{p^{ed}}\ell(R/I^{[p^e]})$ exists. $\square$

The following lemma allows us to reduce the proof of the existence of Hilbert-Kunz multiplicity of an $\mathfrak{m}$-primary ideal $I$ in a finitely generated module $M$ to the scenario of Theorem 2.9.

LEMMA 2.10. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p$ and of dimension $d$. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated $R$-modules. Then $\ell(M/I^{[p^e]}M) = \ell(M'/I^{[p^e]}M') + \ell(M''/I^{[p^e]}M'') + O(p^{e(d-1)})$.*

PROOF. If $N$ is a finitely generated $R$-module, then $\ell_R(N/I^{[p^e]}N) = \ell_{\widehat{R}}(\widehat{N}/I^{[p^e]}\widehat{N})$. Thus we may assume $R$ is complete, in which case, $R$ is module finite over a power series ring $k[[x_1,\ldots,x_d]]$. If $\overline{k}$ denotes the algebraic closure of $k$, then $k[[x_1,\ldots,x_d]] \subseteq \overline{k}[[x_1,\ldots,x_d]]$ is a flat extension of rings such that the maximal ideal of $k[[x_1,\ldots,x_d]]$ extends to the maximal ideal of $\overline{k}[[x_1,\ldots,x_d]]$. Hence $S := R \otimes_{k[[x_1,\ldots,x_d]]} \overline{k}[[x_1,\ldots,x_d]]$ is a faithfully flat extension of $R$ such that the maximal ideal of $R$ extends to the maximal ideal of $S$. In particular, for each finitely generated $R$-module $N$, $\ell_R(N/I^{[p^e]}N) = \ell_S((N \otimes_R S)/I^{[p^e]}(N \otimes_R S))$. Thus we have reduced to the scenario that $R$ is complete with perfect residue field. In particular, $R$ is F-finite.

Let $M_1 = M$ and $M_2 = M' \oplus M''$. The lemma is equivalent to showing

$$\ell(M_1/I^{[p^e]}M_1) = \ell(M_2/I^{[p^e]}M_2) + O(p^{e(d-1)}).$$

First suppose that $R$ is reduced. Then $M_1$ and $M_2$ are finitely generated $R$-modules isomorphic at the minimal primes of $R$. Therefore there are right exact sequences

$$M_1 \to M_2 \to N_1 \to 0$$

and

$$M_2 \to M_1 \to N_2 \to 0$$

such that $N_1, N_2$ are of dimension less than $d$. It follows that

$$\ell(M_2/I^{[p^e]}M_2) \leqslant \ell(M_1/I^{[p^e]}M_1) + \ell(N_1/I^{[p^e]}N_1)$$

and

$$\ell(M_1/I^{[p^e]}M_1) \leqslant \ell(M_2/I^{[p^e]}M_2) + \ell(N_2/I^{[p^e]}N_2).$$

By Proposition 2.8, $\ell(N_1/I^{[p^e]}N_1) = O(p^{e(d-1)})$ and $\ell(N_2/I^{[p^e]}N_2) = O(p^{e(d-1)})$, therefore $\ell(M_1/I^{[p^e]}M_1) = \ell(M_2/I^{[p^e]}M_2) + O(p^{e(d-1)})$, under the assumption $R$ is reduced.

If $R$ is not reduced, we can choose $e_0$ large enough so that $\sqrt{0}^{[p^{e_0}]} = 0$. Then $F^{e_0}(R)$, the image of $R$ under Frobenius, is a reduced ring abstractly isomorphic to $R/\sqrt{0}$. If $N$ is a finite length $R$-module then $N$ is a finite length $F^{e_0}(R)$-module and $\ell_{F^{e_0}(R)}(N) = \ell_R(N)$ since $R$ has a perfect residue field. Observe that for each integer $e$, $(I^{[p^{e_0}]} \cap F^{e_0}(R))^{[p^e]}R = I^{[p^{e+e_0}]}R$. Therefore for each $e \in \mathbb{N}$, and $i = 1, 2$, $\ell_{F^{e_0}(R)}(M_i/(I^{[p^{e_0}]} \cap F^{e_0}(R))^{[p^e]}M_i) = \ell_R(M_i/(I^{[p^{e+e_0}]}M_i)$, thus we are reduced to the reduced case by replacing $R$ with $F^{e_0}(R)$ and $I$ with $I^{[p^{e_0}]} \cap F^{e_0}(R)$. $\square$

THEOREM 2.11 ([**Mon83**, Theorem 1.1]). *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic $p$ and $M$ a finitely generated $R$-module of dimension $d$, and $I \subseteq R$ an $\mathfrak{m}$-primary ideal. Then the Hilbert-Kunz multiplicity of $I$ in $M$ exists, i.e., the limit $\mathrm{e}_{\mathrm{HK}}(I; M) = \lim_{e\to\infty} \frac{1}{p^{ed}}\ell(M/I^{[p^e]}M)$ exists.*

PROOF. As in Lemma 2.10, we may first complete $R$ and extend the residue field to its algebraic closure without changing length. Moreover, all computations are unaffected by replacing $R$ with $R/\operatorname{Ann}_R(M)$, thus we may assume the dimension of $R$ is $d$. Let $\operatorname{Assh}(R) = \{P \in \operatorname{Spec}(R) \mid \dim(R/P) = d\}$. If $0 = M_0 \subseteq \cdots \subseteq M_n = M$ is a prime filtration of $M$ and $P \in \operatorname{Assh}(R)$, then $\ell_{R_P}(M_P)$ is equal to the number of factors isomorphic to $R/P$ in this filtration. Repeated application of Lemma 2.10

and Corollary 2.8 implies

$$\ell(M/I^{[p^e]}M) = \left( \sum_{P \in \mathrm{Assh}(R)} \ell(R/(P+I^{[p^e]})R)\ell_{R_P}(M_P) \right) + O(p^{e(d-1)}).$$

By Theorem 2.9 we may divide by $\frac{1}{p^{ed}}$, take a limit as $e \to \infty$, and get that $\mathrm{e}_{\mathrm{HK}}(M)$ exists and is equal to $\sum_{P \in \mathrm{Assh}(R)} \mathrm{e}_{\mathrm{HK}}(R/P)\ell_{R_P}(M_P)$. $\square$

Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p$ and of dimension $d$. We claim that $\ell(R/\mathfrak{m}^{[p^e]}R) \geqslant p^{ed}$ for each integer $e$. As observed in the proofs of Lemma 2.10 and Theorem 2.11, we may assume $R$ is complete and has perfect residue field when computing the numbers $\ell(R/\mathfrak{m}^{[p^e]}R)$. Choose a minimal prime of $R$ so that $\dim(R) = \dim(R/P)$, then $\ell(R/\mathfrak{m}^{[p^e]}R) \geqslant \ell(R/(P+\mathfrak{m}^{[p^e]})R)$. Replacing $R$ by $R/P$ we are reduced to showing that if $R$ is a complete local domain with perfect residue field, then $\ell(R/\mathfrak{m}^{[p^e]}R) \geqslant p^{ed}$. Since $R$ has a perfect residue field, we have by Lemma 2.3 and Corollary 2.6 that $\ell(R/\mathfrak{m}^{[p^e]}R) = \ell(F^e_*(R/\mathfrak{m}^{[p^e]}R)) = \ell(F^e_*R/\mathfrak{m}F^e_*R) = \mu(F^e_*R) \geqslant \mathrm{rank}(F^e_*R) = p^{ed}$, as desired.

As $\ell(R/\mathfrak{m}^{[p^e]}R) \geqslant p^{ed}$ for any local ring $(R, \mathfrak{m}, k)$ of prime characteristic $p$ and of dimension $d$, we immediately obtain that $\mathrm{e}_{\mathrm{HK}}(R) \geqslant 1$ for any prime characteristic local ring $R$. An application of Kunz's theorem, Theorem 2.2, is that $\ell(R/\mathfrak{m}^{[p^e]}R) = p^{ed}$ if $R$ is a regular local ring. In particular, if $R$ is a regular local ring, then $\mathrm{e}_{\mathrm{HK}}(R) = 1$. Amazingly enough, the reverse implication holds, as long as we assume the completion of $R$ is unmixed, i.e., $(R, \mathfrak{m}, k)$ is assumed to be *formally unmixed*. Recall that a local ring $(R, \mathfrak{m}, k)$ is said to be unmixed if $\dim(R) = \dim(R/P)$ for each $P \in \mathrm{Ass}(R)$.

THEOREM 2.12 ([**WY00**]). *Let $(R, \mathfrak{m}, k)$ be a formally unmixed local ring of prime characteristic $p$. Then $\mathrm{e}_{\mathrm{HK}}(R) = 1$ if and only if $R$ is a regular local ring.*

Given that the Hilbert-Kunz multiplicity of a local ring being minimal implies the local ring in question is regular, it is reasonable to expect that small values of Hilbert-Kunz multiplicity imply the ring in question has decent singularities. Theorems of

Blickle and Enescu in [**BE04**], and improvements of their Theorems later given by Aberbach and Enescu in [**AE08**], confirm this suspicion.

THEOREM 2.13 (Blickle-Enescu, Aberbach-Enescu). *Let $(R, \mathfrak{m}, k)$ be a formally equidimensional local ring of characteristic $p$. Let $e$ be the Hilbert-Samuel multiplicity of $R$. If $\mathrm{e}_{\mathrm{HK}}(R) \leqslant 1 + \max\{1/\dim(R)!, 1/e\}$, then $R$ is strongly F-regular and Gorenstein. In particular, $R$ is a normal domain.*

Simple computations show that the Hilbert-Kunz multiplicity of a local ring need not be an integer. In fact, Brenner has shown that the Hilbert-Kunz multiplicity of a local ring can even be irrational, see [**Bre13**]. As the Hilbert-Kunz multiplicity of a local ring $(R, \mathfrak{m}, k)$ is at least 1, it is natural to ask "how close" can $\mathrm{e}_{\mathrm{HK}}(R)$ can be to 1. Initial investigations of such a question were originated by Blickle and Enescu in [**BE04**] and were significantly improved by Aberbach and Enescu in [**AE08**] and Celikbas, Dao, Huneke, and Zhang in [**CDHZ12**].

THEOREM 2.14 (Blickle-Enescu, Aberbach-Enescu, Celikbas-Dao-Huneke-Zhang). *Fix $d \in \mathbb{N}$. There is a number $\delta > 0$ such that if $(R, \mathfrak{m}, k)$ is formally unmixed of dimension $d$, of any prime characteristic, and $\mathrm{e}_{\mathrm{HK}}(R) \leqslant 1 + \delta$, then $R$ is regular.*

We shall need the following well-know lemma, whose proofs is given for the sake of completion.

LEMMA 2.15. *Let $(R, \mathfrak{m}, k)$ be a local characteristic $p$ ring, $I$ be an $\mathfrak{m}$-primary ideal, and $M$ a finitely generated $R$-module. Then*

$$\lim_{e_2 \to \infty} \frac{1}{p^{e_2 \dim(M)}} \ell(M/I^{[p^{e_1+e_2}]}M) = p^{e_1 \dim(M)} \mathrm{e}_{\mathrm{HK}}(I, M).$$

PROOF. We only have to observe that

$$\begin{aligned}
\lim_{e_2\to\infty} \frac{1}{p^{e_2 \dim(M)}} \ell(M/I^{[p^{e_1+e_2}]}M) &= \lim_{e_2\to\infty} \frac{p^{e_1 \dim(M)}}{p^{(e_1+e_2)\dim(M)}} \ell(M/I^{[p^{e_1+e_2}]}M) \\
&= p^{e_1 \dim(M)} \lim_{e\to\infty} \frac{1}{p^{e \dim(M)}} \ell(M/I^{[p^e]}M) \\
&= p^{e_1 \dim(M)} \, \mathrm{e}_{\mathrm{HK}}(I, M). \quad \square
\end{aligned}$$

## 4. F-signature

We now turn our attention to the study of Frobenius splittings in local rings. A *Frobenius splitting* of a local ring $(R, \mathfrak{m}, k)$ will mean an onto $R$-linear map $F^e_* R \to R$. We will be interested in "how many" Frobenius splittings a local ring admits.

DEFINITION 2.16. Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of prime characteristic $p$. Then the *eth Frobenius splitting number of* $R$ is denoted $a_e(R)$ and is the largest rank of a free $R$-module $F$ for which there exists a surjective $R$-linear map $F^e_* R \to F$.

As free $R$-modules are projective, the $e$th Frobenius splitting number of an F-finite local ring $R$ is, equivalently, the largest rank of a free $R$-module appearing in various direct sum decomposition of $F^e_* R$.

LEMMA 2.17. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. If $a_e(R) \geqslant 1$ for some $e \in \mathbb{N}$ then $a_e(R) \geqslant 1$ for all $e \in \mathbb{N}$. Moreover, if $a_e(R) \geqslant 1$ for some $e \in \mathbb{N}$ then $R$ is reduced.*

PROOF. Suppose that $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$ and $r \in R$ is such that $\varphi(F^e_* r) = 1$. Let $\psi \in \mathrm{Hom}_R(F^e_* R, R)$ be the composition of maps $F^e_* R \xrightarrow{\cdot F^e_* r} F^e_* R \xrightarrow{\varphi} R$. Then $\psi(F^e_* 1) = 1$. Let $\eta \in \mathrm{Hom}_R(F_* R, R)$ be the composition of maps

$$F_* R \xrightarrow{F} F^2_* R \xrightarrow{F} \cdots \xrightarrow{F} F^e_* R \xrightarrow{\psi} R.$$

Then $\eta(F_*1) = 1$, thus $a_1(R) \geqslant 1$. Let $e' \in \mathbb{N}$, then $F_*^{e'}\eta \in \operatorname{Hom}_{F_*^{e'}R}(F_*^{e'+1}R, F_*^{e'}R) \subseteq \operatorname{Hom}_R(F_*^{e'+1}R, F_*^{e'}R)$ is such that $F_*^{e'}\eta(F_*^{e'+1}1) = F_*^{e'}1$. By induction, $a_e(R) \geqslant 1$ for all $e \in \mathbb{N}$.

To see that $R$ is reduced let $r \in R$ and suppose $r^n = 0$. Then $r^{p^e} = 0$ for $p^e \geqslant n$. The above shows that there exists $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$ so that $\varphi(F_*^e 1) = 1$. Then $0 = \varphi(F_*^e(r^{p^e})) = \varphi(rF_*^e 1) = r\varphi(F_*^e 1) = r$. $\square$

Lemma 2.18. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p$ and $F_*^e R \cong R^{\oplus n} \oplus M_e$ a choice of decomposition of $F_*^e R$ such that $\operatorname{Hom}_R(M_e, R) = \operatorname{Hom}_R(M_e, \mathfrak{m})$, i.e., $M_e$ does not contain a free $R$-summand. Let $I_e = \{r \in R \mid \varphi(F_*^e r) \in \mathfrak{m} \,\forall \varphi \in \operatorname{Hom}_R(F_*^e R, R)\}$. Then $I_e$ is an ideal of $R$ containing $\mathfrak{m}^{[p^e]}R$ and $F_*^e I_e \cong \mathfrak{m}R^{\oplus n} \oplus M_e$. In particular, $n = \ell(F_*^e(R/I_e))$ and therefore $n$ is equal to $a_e(R)$, as $I_e$ did not depend on the choice of decomposition of $F_*^e R$.*

Proof. Let $r, s \in I_e$, and $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$. Then $\varphi(F_*^e r + F_*^e s) = \varphi(F_*^e r) + \varphi(F_*^e s) \in \mathfrak{m} + \mathfrak{m} = \mathfrak{m}$, thus $r + s \in I_e$. Let $t \in R$, $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$, and let $\psi$ be the composition $F_*^e R \xrightarrow{F_*^e r} F_*^e R \xrightarrow{\varphi} R$. Then $\psi \in \operatorname{Hom}_R(F_*^e R, R)$, hence $\varphi(F_*^e tr) = \psi(F_*^e r) \in \mathfrak{m}$, thus $tr \in I_e$ and $I_e$ is indeed an ideal.

To show $\mathfrak{m}^{[p^e]}R \subseteq I_e$ it is enough to show each generator of $\mathfrak{m}^{[p^e]}R$ is an element of $I_e$. Let $r \in \mathfrak{m}$, then $(F_*^e r^{p^e}) = rF_*^e 1 \in \mathfrak{m}F_*^e R$ and therefore $\varphi((F_*^e r^{p^e})) \in \mathfrak{m}$ for all $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$, i.e., $r^{p^e} \in I_e$.

By definition of $I_e$, $F_*^e I_e$ are the elements of $F_*^e R$ which cannot be mapped to a unit of $R$ via an element of $\operatorname{Hom}_R(F_*^e R, R)$. Under the decomposition $F_*^e R \cong R^{\oplus n} \oplus M_e$ this is precisely the elements of $\mathfrak{m}R^{\oplus n} \oplus M_e$. $\square$

Suppose that $(R, \mathfrak{m}, k)$ is an F-finite domain of dimension $d$. Then for each $e \in \mathbb{N}$, $a_e(R)$ is naturally bounded from above by $\operatorname{rank}(F_*^e R) = p^{e\gamma(R)}$. Moreover, $\frac{1}{p^{e\gamma(R)}} a_e(R) = \frac{1}{p^{e\gamma(R)}} \ell(F_*^e(R/I_e)) = \frac{1}{p^{ed}} \ell(R/I_e)$. These observations inspire the definition of *F-signature*. Let $(R, \mathfrak{m}, k)$ be a local F-finite ring, not necessarily a domain. The F-signature of $R$ is defined to be the limit $s(R) = \lim_{e\to\infty} \frac{1}{p^{ed}} \ell(R/I_e)$, provided the limit exists. Tucker gave the first proof the existence of the F-signature in [**Tuc12**]. In this section we prove the existence of this numerical invariant, but the techniques we use are simpler than those used in [**Tuc12**] and follow more closely the methods used by Polstra and Tucker in [**PT16**].

Before continuing, we first observe that study of F-signature typically reduces to the study of rings which are *strongly F-regular*. A local F-finite ring $(R, \mathfrak{m}, k)$ is called strongly F-regular if for each nonzero element $c \in R$, there exists $e \in \mathbb{N}$ and $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$ so that $\varphi(F_*^e r) = 1$.

LEMMA 2.19. *Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of prime characteristic $p$. If $R$ is not strongly F-regular then $s(R)$ exists and is equal to* 0.

PROOF. To say that $R$ is not strongly F-regular is equivalent to there existing a non-zero element $c \in R$ which is an element of the ideal $I_e$ for each $e \in \mathbb{N}$. In this case $(\mathfrak{m}^{[p^e]}, c)R \subseteq I_e$ for all $e \in \mathbb{N}$ and $\ell(R/I_e) \leqslant \ell(R/(\mathfrak{m}^{[p^e]}, c)R) = O(p^{e(d-1)})$ by Corollary 2.8. Hence $\lim_{e\to\infty} \frac{1}{p^{ed}}\ell(R/I_e)$ exists and is equal to 0. $\square$

Every strongly F-regular local ring is a domain, hence the study of F-signature can typically be reduced to studying local domains.

Length is an additive function on exact sequences and this fact is crucial to proving the existence of Hilbert-Kunz multiplicity. Our next lemma is an analogue of this fact for counting free summands in finitely generated modules and will allow us to prove the existence of F-signature. Given a finitely generated $R$-module we let $\operatorname{frk}(M)$ denote the largest rank of a free module $F$ for which there exists an onto $R$-linear map $M \to F$. In particular, if $(R, \mathfrak{m}, k)$ is a local F-finite ring, then $a_e(R) = \operatorname{frk}(F_*^e R)$.

LEMMA 2.20. *Let $(R, \mathfrak{m}, k)$ be a local ring and let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated $R$-modules. Then $\operatorname{frk}(M) \leqslant \operatorname{frk}(M') + \mu(M'')$.*

PROOF. Let $n = \operatorname{frk}(M)$ and $n' = \operatorname{frk}(M')$. Then one can find decompositions $M \cong R^n \oplus N$ and $M' \cong R^{n'} \oplus N'$ so that $N$ and $N'$ do not have a free $R$-summand.

Let $f$ be the map $M' \to M$ and $g$ be the map $M \to M''$. Then $f(N') \subseteq mR^n \oplus N$, or else the free rank of $M'$ will be strictly larger then $n'$. Hence there is an induced map

$$(R^{n'} \oplus N)/N \cong R^{n'} \xrightarrow{\overline{f}} (R^n \oplus N)/(\mathfrak{m}R^n \oplus N) \cong k^n.$$

Let $\overline{M''}$ be the cokernel of $\overline{f}$. Then $\overline{M''} \cong M''/g(\mathfrak{m}R^n \oplus N)$. There is a right exact sequence

$$R^{n'} \xrightarrow{\overline{f}} k^n \to \overline{M''} \to 0.$$

The desired result now follows, because if $A \to B \to C \to 0$ is a right exact sequence of finitely generated $R$-modules, then $\mu(B) \leqslant \mu(A) + \mu(C)$. $\square$

THEOREM 2.21 ([**Tuc12**, Main Result]). *Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of prime characteristic $p$ and of dimension $d$. Then the F-signature of $R$ exists, i.e., the limit $s(R) = \lim_{e\to\infty} \frac{1}{p^{ed}}\ell(R/I_e)$ exists.*

PROOF. By Lemma 2.19 we may assume $R$ is an F-finite domain. Let $s_e(R) = \frac{1}{p^{ed}}\ell(R/I_e) = \frac{1}{p^{e\gamma(R)}}a_e(R)$ and denote by $s^-(R)$ and $s^+(R)$ respectively the limit infimum and limit supremum as $e \to \infty$ of the sequence $s_e(R)$. By Lemma 2.5 there is a short exact sequence

$$0 \to F_*R \to R^{p^{\gamma(R)}} \to N \to 0$$

with $N$ an $R$-module of dimension less than $d$. As restricting scalars is exact, we have for each $e \in \mathbb{N}$ a short exact sequence

$$0 \to F_*^{e+1}R \to (F_*^eR)^{p^{\gamma(R)}} \to F_*^eN \to 0.$$

By Lemma 2.20, $p^{\gamma(R)}a_e(R) \leqslant a_{e+1}(R) + \mu(F_*^eN)$. Observe

$$\mu(F_*^eN) = \ell(F_*^eN/\mathfrak{m}F_*^eN) = \ell(F_*^e(N/\mathfrak{m}^{[p^e]}N)).$$

Thus by Corollary 2.6 there is a constant $C$ such that for every $e \in \mathbb{N}$, $\mu(F_*^eT) \leqslant Cp^{e(\gamma-1)}$. Dividing by $p^{(e+1)\gamma(R)}$ we find that the constant $C$ is such that for all $e \in \mathbb{N}$,

$s_e(R) \leqslant s_{e+1}(R) + \frac{C}{p^e}$, since $\frac{C}{p^{\gamma(R)+e}} \leqslant \frac{C}{p^e}$. Observe that for all $e, e' \in \mathbb{N}$,

$$s_e(R) - s_{e+e'}(R) = \sum_{i=0}^{e'-1} s_{e+i} - s_{e+i+1} \leqslant \sum_{i=0}^{e'-1} \frac{C}{p^{e+i}} = \frac{C}{p^e} \sum_{i=0}^{e'-1} \frac{1}{p^i} \leqslant \frac{2C}{p^e}.$$

This shows that for all $e, e' \in \mathbb{N}$ that $s_e(R) \leqslant s_{e+e'}(R) + \frac{2C}{p^e}$. Taking the limit infimum as $e' \to \infty$ shows that for all $e \in \mathbb{N}$ that $s_e(R) \leqslant s^-(R) + \frac{2C}{p^e}$. Taking the limit supremum as $e \to \infty$ shows $s^+(R) \leqslant s^-(R)$. $\square$

Similar to Hilbert-Kunz multiplicity, F-signature can be used to measure the singularities of a local ring of prime characteristic. We first discuss how various Frobenius splitting numbers of $(R, \mathfrak{m}, k)$ can be used to determine if $R$ is a regular local ring. But before doing so, we remark that the following theorem is Theorem 2.2 in disguise and it therefore should be attributed to Kunz.

Theorem 2.22. *Let $(R, \mathfrak{m})$ be a local F-finite ring of prime characteristic $p$. Then $\ell(R/I_e) \leqslant p^{ed}$ for each $e \in \mathbb{N}$. Furthermore, the following are equivalent.*

(1) *$\ell(R/I_e) = p^{ed}$ for all $e \in \mathbb{N}$.*
(2) *$\ell(R/I_e) = p^{ed}$ for some $e \in \mathbb{N}$.*
(3) *$R$ is a regular local ring.*

Proof. We begin by remarking that if $a_e(R) > 0$ then $R$ is reduced. Hence we may assume $R$ is reduced. Observe that $\frac{1}{p^{ed}}\ell(R/I_e) = \frac{1}{[F^e_* k:k]p^{ed}} a_e(R) = \frac{1}{p^{e\gamma(R)}} a_e(R)$. If $F^e_* R \cong R^{\oplus a_e(R)} \oplus M_e$ is a choice of decomposition of $R$ then for each $P \in \operatorname{Spec}(R)$, $F^e_* R_P \cong R_P^{\oplus a_e(R)} \oplus (M_e)_P$ and we clearly have $a_e(R_P) \geqslant a_e(R)$. Choose a minimal prime $P \in \operatorname{Spec}(R)$ so that $\gamma(R_P) = \gamma(R)$. Since $R$ is reduced, $R_P$ is a field, and $a_e(R) \leqslant a_e(R_P) = p^{e\gamma(R)}$. Therefore $\ell(R/I_e) \leqslant p^{ed}$ for all $e \in \mathbb{N}$.

The implication (1) $\Rightarrow$ (2) is trivial and (3) $\Rightarrow$ (1) follows by Theorem 2.2 and Lemma 2.3. Suppose that for some $e$, $\ell(R/I_e) = p^{ed}$, equivalently $a_e(R) = p^{e\gamma(R)}$. To show $R$ is a regular local ring it suffices to show $F^e_* R$ is a free $R$-module by Theorem 2.2. Suppose we have decomposition $F^e_* R \cong R^{\oplus p^{e\gamma(R)}} \oplus M_e$. Since $a_e(R) \geqslant 1$, $R$ is a reduced ring and therefore $R_Q$ is a field for each $Q \in \operatorname{Min}(R)$. By the definition of $\gamma(R)$, $(M_e)_Q = 0$ for each $Q \in \mathrm{Min}(R)$, or else, $\gamma(R_Q) > \gamma(R)$ for some $Q \in \mathrm{Min}(R)$. Let $F^e_* r$ be an element of $M_e$. There exists $x \in R - Q$ such that $xF^e_* r = F^e_* x^{p^e} r = 0$. Then $xrF^e_* 1 = F^e_*(xr)^{p^e} = 0$, but $R \xrightarrow{F^e} R$ is injective as $R$ is reduced. Thus $xr = 0$, $r \in Q$ for each $Q \in \mathrm{Min}(R)$, and $R$ being reduced implies $r = 0$ and $M_e = 0$, i.e., $F^e_* R \cong R^{\oplus p^{e\gamma(R)}}$. $\square$

Hence a natural upper bound of the F-signature of a local ring $(R, \mathfrak{m}, k)$ is 1 and this upper bound is achieved if $R$ is regular. The reverse implication holds for all F-finite rings of prime characteristic.

THEOREM 2.23 ([**HL02**, Corollary 16]). *Let* $(R, \mathfrak{m}, k)$ *be an F-finite local of prime characteristic. Then* $\mathrm{s}(R) = 1$ *if and only if* $R$ *is regular.*

As well as proving Theorem 2.23, Huneke and Leuschke prove that, if $(R, \mathfrak{m}, k)$ is Gorenstein, then the positivity of the F-signature is equivalent to $R$ being F-rational, which is equivalent to $R$ being strongly F-regular under the Gorenstein hypothesis by [**HH94**, Corollary 4.7 (a) and Theorem 5.5 (f)]. Aberbach and Leuschke extend the equivalence of positivity of the F-signature and strong F-regularity to all local F-finite rings in [**AL03**].

THEOREM 2.24 ([**AL03**, Main Result]). *Let* $(R, \mathfrak{m}, k)$ *be an F-finite local ring of prime characteristic* $p$. *Then* $\mathrm{s}(R) > 0$ *if and only if* $R$ *is strongly F-regular.*

Observations of Yao in [**Yao06**] allow us to extend the notion of F-signature to local rings $(R, \mathfrak{m}, k)$ of prime characteristic $p$ which are not assumed to be F-finite. Let $E = E_R(k)$ denote the injective hull of the residue field and let $u \in E$ generate the socle $(0 :_E \mathfrak{m})$. Yao shows that if $R$ is F-finite then $I_e = \{r \in R \mid u \otimes F^e_* r = 0 \in E \otimes_R F^e_* R\}$. In particular, a local F-finite ring is strongly F-regular if for each nonzero element $r \in R$ there exists $e \in \mathbb{N}$ so that $u \otimes F^e_* r \neq 0$ as an element of $E \otimes_R F^e_* R$. Thus the notion of strong F-regularity can be naturally extended to non-F-finite rings. Moreover, we can extend the definition of the ideals $I_e$, and therefore the definition of F-signature, to rings which are not assumed to be F-finite. Specifically, if $(R, \mathfrak{m}, k)$ is a local ring of prime characteristic $p$ and of dimension $d$ then we define $I_e = \{r \in R \mid u \otimes F^e_* r = 0 \in E \otimes_R F^e_* R\}$ and the F-signature of $R$ to be $s(R) = \lim_{e\to\infty} \frac{1}{p^{ed}} \ell(R/I_e)$, provided the limit exists. Yao proves that the existence of the limit $s(R)$ follows from the F-finite case and that Theorem 2.23 and Theorem 2.24 hold without the F-finite hypothesis as well.

We shall also show that under mild hypotheses, for each $e \in \mathbb{N}$ there exists an $\mathfrak{m}$-primary $I$ and socle generator $u \in (I :_R \mathfrak{m})$ so that $I_e = (I^{[p^e]} :_R u^{p^e})$. In particular, $\ell(R/I_e) = \ell(R/(I^{[p^e]} :_R u^{p^e})) = \ell((I, u)^{[p^e]}R/I^{[p^e]}R)$, thus the normalized Frobenius splitting numbers $\frac{1}{p^{ed}} \ell(R/I_e)$ can be measured as a relative length of Hilbert-Kunz type lengths. Expressing Frobenius splitting numbers as relative Hilbert-Kunz type lengths will be useful in Chapter 4 to establish Theorem B. But first, we outline a proof of the well-known fact that if $(R, \mathfrak{m}, k)$ is an F-finite local ring of prime characteristic $p$ then $I_e = \{r \in R \mid u \otimes F^e_* r = 0 \in E \otimes_R F^e_* R\}$. The result needed to make this identification of ideals is the following criterion of Hochster.

Proposition 2.25 ([**Hoc77**]). *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, not necessarily of prime characteristic, $M$ a finitely generated $R$-module, and $E = E_R(k)$ the injective hull of the residue field. Then an $R$-linear map $R \to M$ splits if and only if the map $E \otimes_R R \to E \otimes_R M$ is injective.*

The proof of Proposition 2.25 implicitly appears in the proof of [**Hoc77**, Theorem 2.6]. For a direct statement and proof of Proposition 2.25, see [**Hoc**, Page 155].

Corollary 2.26. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p$. Let $E = E_R(k)$ denote the injective hull of the residue field and let $u \in E$ generate the socle $(0 :_E \mathfrak{m})$. Then for each $e \in \mathbb{N}$, $I_e = \{r \in R \mid u \otimes F^e_* r = 0 \in E \otimes_R F^e_* R\}$.*

Proof. An element $r \in R$ is an element of $I_e$ if and only if the map $R \xrightarrow{\varphi} F^e_* R$ defined by $\varphi(1) = F^e_* r$ does not split. By Proposition 2.25, $\varphi$ does not split if and only if the map $E \cong E \otimes_R R \xrightarrow{1\otimes\varphi} E \otimes F^e_* R$ is not injective. But $E$ is an essential extension of $k \cong R \cdot u$, hence $\ker(1 \otimes \varphi)$ is non-zero if and only if $u$ is in $\ker(1 \otimes \varphi)$ if and only if $u \otimes F_*^e r = 0$ as an element of $E \otimes_R F_*^e R$. $\square$

We now sketch a proof that the Frobenius splitting numbers of a local ring can be measured as relative Hilbert-Kunz type lengths, as long as the local ring we are studying is reduced and excellent. Every F-finite ring is excellent and so is a localization of a ring which is essentially of finite type over an excellent ring. Recall that F-finite rings and rings essentially of finite type are the two classes of rings we will be interested in in Chapter 3 and Chapter 4. Recall, if $(R, \mathfrak{m}, k)$ is of prime characteristic $p$ and $I_e \neq R$, i.e., $\ell(R/I_e) \neq 0$, then $R$ must be reduced.

COROLLARY 2.27. *Let $(R, \mathfrak{m}, k)$ be an excellent reduced local ring of dimension $d$. For each $e \in \mathbb{N}$ there is an irreducible $\mathfrak{m}$-primary ideal $I$ and $u \in (I : \mathfrak{m})$ such that $I_e = (I^{[p^e]} : u^{p^e})$. Moreover, if $e_1, e_2 \in \mathbb{N}$, then there is irreducible $\mathfrak{m}$-primary ideal and $u \in (I : \mathfrak{m})$ so that $I_{e_1} = (I^{[p^{e_1}]} : u^{p^{e_1}})$ and $I_{e_1+e_2} = (I^{[p^{e_1+e_2}]} : u^{p^{e_1+e_2}})$.*

PROOF. The ring $R$ is reduced and excellent, hence $R$ is approximately Gorenstein [**Hoc77**, Theorem 1.7]. That is, there exists a descending chain of irreducible $\mathfrak{m}$-primary ideals $\{I_t\}_{t\in\mathbb{N}}$ which is cofinal with $\{\mathfrak{m}^t\}_{t\in\mathbb{N}}$. Let $u_t$ generate the socle mod $I_t$. Then $R/I_t$ is a 0-dimensional Gorenstein ring, that is $R/I_t = E_{R/I_t}(k)$. Moreover, since $I_{t+1} \subseteq I_t$ we must have $Ru_{t+1} + I_{t+1} = (I_{t+1} : \mathfrak{m}) \subseteq (I_t : \mathfrak{m}) = Ru_t + I_t$. Thus for each $t$ there exists $x_t \in R$ so that $x_t u_t = u_{t+1} \mod I_{t+1}$. As $R/I_t$ is an injective $R/I_t$-module there exists a map $R/I_t \xrightarrow{\varphi_t} R/I_{t+1}$ so that $\varphi_t(u_t) = x_t u_t = u_{t+1}$. The direct limit system $\varinjlim R/I_t \xrightarrow{\varphi_t} R/I_{t+1}$ produces an injective $R$-module which is an essential extension of the residue field $k$, i.e., $E := E_R(k) \cong \varinjlim R/I_t \xrightarrow{\varphi_t} R/I_{t+1}$ and the image of $u_t$ at the $t$th spot in this system generates the socle element $u$ of $E_R(k)$.

Suppose $r \in I_e$. By Corollary 2.26 this equivalent to $u \otimes F_*^e r$ being 0 in $E \otimes_R F_*^e R \cong \varinjlim R/I_t \otimes_R F_*^e R \cong \varinjlim F_*^e(R/I_t^{[p^e]}) \cong F_*^e \left(\varinjlim R/I_t^{[p^e]}\right)$. Chasing through these isomorphisms, $u \otimes F_*^e r$ being 0 is equivalent to $u_t^{p^e} r^{p^e} \in I_t^{[p^e]}$ for all $t$ large enough.

Hence, $I_e$ is the union of the ascending chain of ideals $(I_t^{[p^e]} : u_t^{p^e})$ and therefore for each $e \in \mathbb{N}$ there is a $t_e$ such that for all $t \geqslant t_e$, $I_e = (I_t^{[p^e]} : u_t^{p^e})$. $\square$

## 5. Cartier subalgebras and F-signature of pairs

In [**BST12**], Blickle, Schwede, and Tucker use the language of Cartier subalgebras to greatly generalize the notion of the F-signature. Their generatlization of the F-signature provides the correct framework to answer a question of Aberbach and Enescu, see ([**AE05**], Question 4.9) and ([**BST12**], Remark 4.6).

We assume $R$ is F-finite. One can make $\mathscr{C}^R := \oplus_{e\in\mathbb{N}} \operatorname{Hom}_R(F^e_* R, R)$ a graded $\mathbb{F}_p$-algebra in a natural way. The 0th graded piece of $\mathscr{C}^R$ is $\operatorname{Hom}_R(R, R) \cong R$. If $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ and $\psi \in \operatorname{Hom}_R(F^{e'}_* R, R)$, then we let $\varphi \bullet \psi := \varphi \circ F^e_* \psi \in \operatorname{Hom}_R(F^{e+e'}_* R, R)$. One should observe that $\mathscr{C}^R$ is noncommutative and that $R \cong \operatorname{Hom}_R(R, R)$ is not central in $\mathscr{C}^R$. If $r \in R$, $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$, and $F^e_* s \in F^e_* R$, then $r \bullet \varphi(F^e_* s) = r\varphi(F^e_* s) = \varphi(rF^e_* s) = \varphi(F^e_* r^{p^e} s) \neq \varphi(F^e_* rs) = \varphi(F^e_* s) \bullet r$.

A Cartier subalgebra $\mathscr{D}$ is a graded $\mathbb{F}_p$-subalgebra of $\mathscr{C}^R$ such that the 0th graded piece of $\mathscr{D}$ is $\operatorname{Hom}_R(R, R)$, which is all of the 0th graded piece of $\mathscr{C}^R$. Cartier subalgebras can be seen as a natural generalization of a number of commonly studied settings in positive characteristic commutative algebra. We briefly discuss two classes of Cartier subalgebras which recover the framework of [**HY03**] and [**HW02, Tak04**].

If $R$ is an $F$-finite normal domain and $D$ a Weil divisor on $X = \operatorname{Spec}(R)$, we use $R(D)$ to denote $\Gamma(X, \mathcal{O}_X(D))$. If $\Delta$ is an effective $\mathbb{Q}$-divisor then one may may recover the setting of [**HW02, Tak04**] by studying the Cartier subalgebra $\mathscr{C}^{(R,\Delta)} = \bigoplus_{e\geqslant 0} \mathscr{C}^{(R,\Delta)}_e$ where

$$\mathscr{C}^{(R,\Delta)}_e = \operatorname{im}\left(\operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R) \to \operatorname{Hom}_R(F^e_* R, R)\right).$$

That is the Cartier subalgebra $\mathscr{C}^{\Delta}$ may be used to study the pair $(R, \Delta)$.

Similarly, if $\mathfrak{a} \subseteq R$ is an ideal and $t \in \mathbb{R}_{\geqslant 0}$ then we may use a Cartier algebra to study the pair $(R, \mathfrak{a}^t)$. Specifically, if $R$ is an F-finite domain and $0 \neq \mathfrak{a} \subseteq R$ is an ideal and $t \in \mathbb{R}_{\geqslant 0}$, the Cartier subalgebra $\mathscr{C}^{\mathfrak{a}^t} = \bigoplus_{e \geqslant 0} \mathscr{C}_e^{\mathfrak{a}^t}$ where

$$\begin{aligned} \mathscr{C}_e^{\mathfrak{a}^t} &= F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil} \operatorname{Hom}_R(F_*^e R, R) \\ &= \{\phi(F_*^e x \cdot \_\_) \mid x \in \mathfrak{a}^{\lceil t(p^e-1) \rceil} \text{ and } \phi \in \operatorname{Hom}_R(F_*^e R, R)\} \end{aligned}$$

recovers the framework of [**HY03**]. For more information on Cartier subalgebras, see [**BS13**].

Given a Cartier subalgebra $\mathscr{D}$ we call a summand $M$ of $F_*^e R$ a $\mathscr{D}$-summand if $M \cong R$ and the map $F_*^e R \to M \cong R$ is an element of $\mathscr{D}$. The assumption that $\mathscr{D}_0 = \operatorname{Hom}_R(R, R)$ implies that the chosen isomorphism of $M \cong R$ does not change whether $M$ is a $\mathscr{D}$-summand or not. The $e$th Frobenius splitting number of $(R, \mathscr{D})$ is definied to be the maximal number of $\mathscr{D}$-summands appearing in various direct sum decompositions of $F_*^e R$ and is denoted $a_e^{\mathscr{D}}$. Observe that if $\mathscr{D} = \mathscr{C}^R$ then $a_e^{\mathscr{D}} = a_e(R)$ is the usual $e$th Frobenius splitting number of $R$.

Given a Cartier subalgebra $\mathscr{D}$ we let $\Gamma_{\mathscr{D}} := \{e \in \mathbb{N} \mid \mathscr{D}_e \neq 0\}$. One can easily check that $\Gamma_{\mathscr{D}}$ is actually a subsemigroup of $\mathbb{N}$. If $R$ is local we let $s_e(R, \mathscr{D}) := a_e^{\mathscr{D}}/p^{e\gamma}$ and define the F-signature of $(R, \mathscr{D})$ to be $s(R, \mathscr{D}) := \lim_{e \in \Gamma_{\mathscr{D}} \to \infty} s_e(R, \mathscr{D})$, provided the limit exists. Blickle, Schwede, and Tucker are indeed able to show that the $s(R, \mathscr{D})$ exists in [**BST12**]. We provide an alternative proof of the existence of the limit $s(R, \mathscr{D})$. The techniques used here to establish the existence of $s(R, \mathscr{D})$ will be useful for establishing lower semi-continuity of the F-signature of a pair, Theorem 4.9.

Lemma 2.28. *Let $R$ be an F-finite domain and $\mathscr{D}$ a Cartier subalgebra and suppose $\mathscr{D}_e \neq 0$. Then there is a short exact sequence $0 \to F_*^e R \to R^{\oplus p^{e\gamma(R)}} \to T \to 0$ such that the inclusion $F_*^e R \to R^{p^{e\gamma(R)}}$ is a direct sum of $p^{e\gamma(R)}$ elements of $\mathscr{D}_e$ and $T$ is a torsion $R$-module.*

Proof. Let $W = R - \{0\}$ and let $K = R_W$ be the field of fractions of $R$. Then $\operatorname{Hom}_R(F_*^e R, R)_W \cong \operatorname{Hom}_{R_W}(F_*^e R_W, R_W) \cong \operatorname{Hom}_K(F_*^e K, K)$, which as a $F_*^e K$-vector space is isomorphic to $F_*^e K$ which is isomorphic to $K^{\oplus p^{e\gamma(R)}}$ as a $K$-vector space. Let $\varphi \in \mathscr{D}_e$ be any nonzero element, then $\varphi/1$ is a generator for $\operatorname{Hom}_K(F_*^e K, K)$ as a

$F^e_*K$-vector space. Since we are assuming that $\mathscr{D}_0 = \mathrm{Hom}_R(R,R)$, any $F^e_*K$-multiple of $\varphi/1$, which is premultiplication by an element of $F^e_*K$, is still an element of $(\mathscr{D}_e)_W$. It follows that any choice of isomorphism $F^e_*K \to K^{p^{e\gamma(R)}}$ is a direct sum of $p^{e\gamma(R)}$ elements of $(\mathscr{D}_e)_W$. Hence the assumption that $\mathscr{D}_e \neq 0$ allows us to find an inclusion of $R$-modules $F^e_*R \to R^{p^{e\gamma(R)}}$, which is a direct sum of $p^{e\gamma(R)}$ elements of $\mathscr{D}_e$, and so that its cokernel is torsion. □

THEOREM 2.29 ([**BST12**], Theorem 3.11). *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain and let $\mathscr{D}$ be a Cartier subalgebra. Then the F-signature of $(R, \mathscr{D})$ exists, i.e., the limit $\lim\limits_{e\in\Gamma_{\mathscr{D}}\to\infty} s_e(R,\mathscr{D})$ exists.*

PROOF. Denote by $s^-(R,\mathscr{D})$ and $s^+(R,\mathscr{D})$ the limit infimum and limit supremum as $e \in \Gamma_{\mathscr{D}} \to \infty$ of the sequence $s_e(R,\mathscr{D})$ respectively. It is a well known fact that every subsemigroup of $\mathbb{N}$ is finitely generated, ([**Gri01**], Proposition 4.1). So suppose that $\Gamma_{\mathscr{D}}$ is generated by $\{e_1, ..., e_N\} \subseteq \mathbb{N}$ as a semigroup. By Lemma 2.28, for each $1 \leqslant i \leqslant N$ we can find a short exact sequence

$$0 \to F^{e_i}_*R \xrightarrow{\Phi_i} R^{\oplus p^{e_i\gamma(R)}} \xrightarrow{\Psi_i} T_i \to 0,$$

so that the map $F^{e_i}_*R \to R^{\oplus p^{e\gamma(R)}}$ is direct sum of elements from $\mathscr{D}_{e_i}$, and $T_i$ is a torsion $R$-module.

Let $e \in \Gamma_{\mathscr{D}} - \{0\}$ and write $e = a_1e_1 + \cdots + a_Ne_N$ and suppose that $a_i \neq 0$. Let $f \in \mathbb{N}$ be such that $e = e_i + f$. Consider the induced short exact sequence

$$0 \to F^e_*R \xrightarrow{F^f_*\Phi_i} F^f_*R^{\oplus p^{e_i\gamma(R)}} \xrightarrow{F^f_*\Psi_i} F^f_*T_i \to 0.$$

There are decompositions $F^e_*R \cong R^{\oplus a^{\mathscr{D}}_e} \oplus M_e$ with each projection $R^{\oplus a^{\mathscr{D}}_e} \to R$ an element of $\mathscr{D}_e$ and $F^f_*R^{\oplus p^{e_i\gamma(R)}} \cong (R^{\oplus a^{\mathscr{D}}_f} \oplus M_f)^{p^{\oplus e_i\gamma(R)}}$ with each projection $R^{\oplus a^{\mathscr{D}}_f} \to R$ an element of $\mathscr{D}_f$. Observe that $F^f_*\Phi_i$ maps every element of $M_e$ to an element of $(\mathfrak{m}R^{\oplus a^{\mathscr{D}}_f} \oplus M_f)^{\oplus p^{e_i\gamma(R)}}$. To see this, suppose that $\Psi : F^f_*R^{\oplus p^{e_i\gamma(R)}} \to R$ is a map of the form $F^f_*R^{\oplus p^{e_i\gamma(R)}} \xrightarrow{\pi} F^f_*R \xrightarrow{\psi} R$ where $\pi$ is one of the natural $p^{e_i\gamma(R)}$ projections and

$\psi \in \mathscr{D}_f$. Then the fact that $\Phi_i$ is a direct sum of elements from $\mathscr{D}_{e_i}$ and $\mathscr{D}$ a Cartier subalgebra implies $\Psi \circ F^f_* \Phi_i$ is an element of $\mathscr{D}_e$. Hence if $F^f_*\Phi$ mapped an element of $M_e$ to an element not in $(\mathfrak{m}R^{\oplus a^{\mathscr{D}}_f} \oplus M_f)^{\oplus p^{e_i\gamma(R)}}$, then the $e$th Frobenius splitting number of $(R, \mathscr{D})$ would be strictly larger than $a^{\mathscr{D}}_e$.

It now follows that there is an induced map

$$F^e_* R/M_e \cong R^{\oplus a^{\mathscr{D}}_e} \xrightarrow{\overline{F^f_*\Phi_i}} F^f_* R^{\oplus p^{e_i\gamma(R)}}/(\mathfrak{m}R^{\oplus a^{\mathscr{D}}_f} \oplus M_f)^{\oplus p^{e_i\gamma(R)}} \cong (R/\mathfrak{m})^{\oplus a^{\mathscr{D}}_f p^{e_i\gamma(R)}}.$$

The cokernel of $\overline{F^f_*\Phi_i}$ is $\overline{F^f_*T_i} := F^f_*T_i/F^f_*\Psi_i((\mathfrak{m}R^{\oplus a^{\mathscr{D}}_f} \oplus M_f)^{\oplus p^{e_i\gamma(R)}})$. Therefore there is a right exact sequence

$$R^{\oplus a^{\mathscr{D}}_e} \xrightarrow{\overline{F^f_*\Phi_i}} (R/\mathfrak{m})^{\oplus a^{\mathscr{D}}_f p^{e_i\gamma(R)}} \to \overline{F^f_*T_i} \to 0.$$

Counting the number of minimal generators is subadditive on right exact sequences, hence $p^{e_i\gamma(R)} a^{\mathscr{D}}_f \leqslant a^{\mathscr{D}}_e + \mu(F^f_*T_i)$. As the modules $T_i$ are torsion, we can apply Corollary 2.8 to know that there is a constant $C$ so that for all $f \in \mathbb{N}$ and for all $1 \leqslant i \leqslant N$, $\mu(F^f_*T) \leqslant Cp^{f(\gamma(R)-1)}$. Dividing the inequality $p^{e_i\gamma(R)} a^{\mathscr{D}}_f \leqslant a^{\mathscr{D}}_e + \mu(F^f_*T_i) \leqslant p^{e_i\gamma(R)} a^{\mathscr{D}}_f \leqslant a^{\mathscr{D}}_e + Cp^{f(\gamma(R)-1)}$ by $p^{e\gamma(R)}$ shows that $s_f(R,\mathscr{D}) = s_{e-e_i}(R,\mathscr{D}) \leqslant s_e(R,\mathscr{D}) + \frac{C}{p^e}$, since $\frac{Cp^{f(\gamma(R)-1)}}{p^{e\gamma(R)}} = \frac{C}{p^{e_i\gamma(R)+e-e_i}} \leqslant \frac{C}{p^e}$. This allows us to conclude that for all $e \in \Gamma_{\mathscr{D}}$ and for all $1 \leqslant i \leqslant N$,

$$s_e(R,\mathscr{D}) \leqslant s_{e+e_i}(R,\mathscr{D}) + \frac{C}{p^{e+e_i}} \leqslant s_{e+e_i}(R,\mathscr{D}) + \frac{C}{p^e}.$$

It follows, after using a very similar trick to that used in the proofs of Theorem 2.11 and Theorem 2.21, that for all $e, e' \in \Gamma_{\mathscr{D}}$ that

$$s_e(R,\mathscr{D}) \leqslant s_{e+e'}(R,\mathscr{D}) + \frac{2C}{p^e}.$$

Taking the limit infimum as $e' \in \Gamma_{\mathscr{D}} \to \infty$ shows that for all $e \in \Gamma_{\mathscr{D}}$, $s_e(R,\mathscr{D}) \leqslant s^-(R,\mathscr{D}) + \frac{2C}{p^e}$. Taking limit supremum as $e \in \Gamma_{\mathscr{D}} \to \infty$ shows $s^+(R,\mathscr{D}) \leqslant s^-(R,\mathscr{D})$.

$\square$

Recall that we were able to measure the $e$th normalized Frobenius splitting number of a local ring $(R, \mathfrak{m}, k)$ as $\frac{1}{p^{ed}}\ell(R/I_e)$ where $I_e = \{r \in R \mid \varphi(F^e_* r) \in \mathfrak{m}\, \forall \varphi \in \mathrm{Hom}_R(F^e_* R, R)\}$. Similarly, given a Cartier subalgebra $\mathscr{D}$ over $(R, \mathfrak{m}, k)$ we define $I_e^{\mathscr{D}} = \{r \in R \mid \varphi(F^e_* r) \in \mathfrak{m}$ for all $\varphi \in \mathscr{D}_e\}$. The following Lemma is a list of basic properties about the sets $I_e^{\mathscr{D}}$ which can all be found in Section 3 of [**BST12**].

Lemma 2.30. *Let $(R, \mathfrak{m}, k)$ be a local F-finite ring and $\mathscr{D}$ a Cartier subalgebra and let $e, e_1, e_2 \in \mathbb{N}$ and $\varphi \in \mathscr{D}_{e_1}$. Then*

(1) *$I_e^{\mathscr{D}} \subseteq R$ is an ideal,*

(2) *$\mathfrak{m}^{[p^e]} \subseteq I_e^{\mathscr{D}}$,*

(3) *$\varphi(F^{e_1}_* I^{\mathscr{D}}_{e_1+e_2}) \subseteq I^{\mathscr{D}}_{e_2}$,*

(4) *$\frac{1}{p^{ed}}\ell(R/I_e^{\mathscr{D}}) = \frac{1}{p^{e\gamma(R)}} a_e^{\mathscr{D}}$.*

Proof. The proofs of (1), (2), and (4) follow exactly as in the proof of Lemma 2.18. Suppose for a contradiction that $r \in I^{\mathscr{D}}_{e_1+e_2}$, $\varphi \in \mathscr{D}_{e_1}$, and $\varphi(F^{e_1}_* r) \notin I^{\mathscr{D}}_{e_2}$. Then there exists $\psi \in \mathscr{D}_{e_2}$ so that $\psi(F^{e_2}_*(\varphi(F^{e_1}_* r))) = \psi \circ F^{e_2}_* \varphi(F^{e_1+e_2}_* r) = 1$. However, $\psi \circ F^{e_2}_* \in \mathscr{D}_{e_1+e_2}$ and we have therefore contradicted the assumption that $r \in I^{\mathscr{D}}_{e_1+e_2}$. $\square$

CHAPTER 3

# Uniform Bounds

This chapter is the heart of the dissertation. We establish uniform bounds for all rings which are either F-finite or essentially of finite type over an excellent local ring. In Chapter 4 we use the uniform bounds of this chapter to prove the Hilbert-Kunz multiplicity and F-signature functions are uniform limits of upper semi-continuous, respectively lower semi-continuous, functions.

## 1. F-finite rings

In [**Dut83**], Sankar Dutta showed that if $(R, \mathfrak{m}, k)$ is an F-finite local domain of dimension $d$ then there exists a finite set of nonzero primes $\mathcal{S}(R)$ and a constant $C$ such that for all $e \in \mathbb{N}$ there is a containment of $R$-modules $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_* R$ which has a prime filtration whose prime factors are isomorphic to $R/P$, where $P \in \mathcal{S}(R)$, and such a prime factor appears no more than $Cp^{e\gamma(R)}$ times in the filtration. In particular, the length of the prime filtration of $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_* R$ has length no more than $C|\mathcal{S}(R)|p^{e\gamma(R)}$. This result, for local domains whose residue field is perfect, is exercise 10.4 in [**Hun96**], whose proof is given in the second appendix by Karen Smith, and this result is explicitly stated and proved in [**Hun13**] as Lemma 4. Our first goal of the chapter will be to show Dutta's lemma holds without the local hypothesis. First we note the following.

REMARK 3.1. *If $R$ is an F-finite domain and $P \in \operatorname{Spec}(R)$ a nonzero prime, then $\gamma(R/P) = \log_p[F^e_* k(P) : k(p)] < [F^e_* k(P) : k(P)] + \operatorname{ht}(P) = \gamma(R)$.*

LEMMA 3.2. *Let $R$ be an F-finite domain. Then there exists a finite set of nonzero primes $\mathcal{S}(R)$, and a constant $C$, such that for every $e \in \mathbb{N}$,*

(1) *there is a containment of $R$-modules $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_* R$,*

(2) *which has a prime filtration whose prime factors are isomorphic to $R/P$, where $P \in \mathcal{S}(R)$,*

(3) *and for each $P \in \mathcal{S}(R)$, the prime factor $R/P$ appears no more than $Cp^{e\gamma(R)}$ times in the prime filtration of the containment $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_* R$.*

Proof. We shall prove the statement by induction on the Krull dimension of $R$. If the dimension $R$ is 0, then $R$ is a field and the lemma is trivial.

Now suppose that $\dim(R) > 0$. Then $F_* R$ is a torsion-free $R$-module of rank $p^{\gamma(R)}$. Hence there is an injection of $R$-modules $R^{\oplus p^{\gamma(R)}} \subseteq F_* R$ so that the support of the cokernel $F_* R / R^{\oplus p^{\gamma(R)}}$ consists of nonzero primes. Therefore $R^{\oplus p^{\gamma(R)}} \subseteq F_* R$ has a prime filtration of the following form with the quotients $M_i / M_{i-1} \cong R/P_i$ where $P_i$ is a nonzero prime of $R$,

$$R^{\oplus p^{\gamma(R)}} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_h = F_* R.$$

The quotients $R/P_i$ are F-finite domains of smaller Krull dimension than $R$, and so we may assume by induction that the result holds for each $R/P_i$, with finite collection of primes $\mathcal{S}(R/P_i)$ and constant $C_i$. Let $C' = \sum C_i$ and $\mathcal{S}(R) = \bigcup(\mathcal{S}(R/P_i) \cup \{P_i\})$. Observe that the above filtration shows that $R^{\oplus p^{\gamma(R)}} \subseteq F_* R$ has a prime filtration consisting of no more than $C'$ quotients isomorphic to $R/P$ for each $P \in \mathcal{S}(R)$ and all prime factors are isomorphic to $R/P$ for some $P \in \mathcal{S}(R)$. We shall show by induction that $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_* R$ has a prime filtration whose prime factors are isomorphic to $R/P$ with $P \in \mathcal{S}(R)$ with no more than $C' p^{e\gamma(R)}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^e})$ quotients isomorphic to $R/P$ for each $P \in \mathcal{S}(R)$.

Suppose that $R^{\oplus p^{e\gamma(R)}} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = F^e_* R$ is a prime filtration of $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_* R$ whose prime factors are isomorphic to $R/P$ with $P \in \mathcal{S}(R)$, and with no more than $C' p^{e\gamma(R)}(1 + \frac{1}{p} + \cdots + \frac{1}{p^e})$ quotients isomorphic to $R/P$ for each $P \in \mathcal{S}(R)$. Restrict scalars under $F^e$ of the inclusion $R^{\oplus p^{\gamma(R)}} \subseteq F_* R$ to get the following new filtration,

$$(F^e_* R)^{\oplus p^{\gamma(R)}} = F^e_* M_0 \subseteq F^e_* M_1 \subseteq \cdots \subseteq F^e_* M_h \cong F^{e+1}_* R.$$

Each of the quotients $F^e_* M_i / F^e_* M_{i+1} \cong F^e_*(R/P_i)$. By induction there exists a prime filtration of $F^e_* M_{i-1} \subseteq F^e_* M_i$ with precisely $p^{e\gamma(R)}$ prime factors isomorphic to $R/P_i$ and each other prime factor is isomorphic to $R/P$ for some $P \in \mathcal{S}(R/P_i)$ and such a prime factor appears no more than $C_i p^{e\gamma(R/P_i)}$ times in the filtration. Furthermore, the prime filtration $R^{\oplus p^{e\gamma(R)}} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = F^e_* R$ gives the following filtration of $(R^{\oplus p^{e\gamma(R)}})^{\oplus p^{\gamma(R)}} = R^{\oplus p^{(e+1)\gamma(R)}} \subseteq (F^e_* R)^{\oplus p^{\gamma(R)}}$,

$$R^{p^{(e+1)\gamma(R)}} = N_0^{\oplus p^{\gamma(R)}} \subseteq N_1^{\oplus p^{\gamma(R)}} \subseteq \cdots \subseteq N_m^{\oplus p^{\gamma(R)}} = (F^e_* R)^{\oplus p^{\gamma(R)}}.$$

Hence $R^{\oplus p^{(e+1)\gamma(R)}} \subseteq (F^e_* R)^{\oplus p^{\gamma(R)}}$ has a prime filtration with prime factors isomorphic to $R/P$ with $P \in \mathcal{S}(R)$ and such a prime factor appears no more than $C' p^{(e+1)\gamma(R)}(1 + \frac{1}{p} + \cdots + \frac{1}{p^e})$ times in the filtration.

Putting the above information together we get that there is an embedding of $R^{\oplus p^{(e+1)\gamma(R)}} \subseteq F^{e+1}_* R$ with prime filtration whose prime factors are isomorphic to $R/P$ with $P \in \mathcal{S}(R)$ and there are no more than the following number of quotients isomorphic to $R/P$ for each $P \in \mathcal{S}(R)$:

$$C' p^{(e+1)\gamma(R)} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^e}\right) + \sum_{i=1}^{h} C_i p^{e\gamma(R/P_i)}.$$

By Remark 3.1 we know that each $\gamma(R/P_i) \leqslant \gamma(R) - 1$, and so we have the following estimates,

$$
\begin{aligned}
C'p^{(e+1)\gamma(R)}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^e}\right) \quad + \quad \sum_{i=1}^{h} C_i p^{e\gamma(R/P_i)} & \\
&\leqslant C'p^{(e+1)\gamma(R)}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^e}\right)+\sum_{i=1}^{h} C_i p^{e(\gamma(R)-1)} \\
&= C'p^{(e+1)\gamma(R)}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^e}\right)+C'p^{e(\gamma(R)-1)} \\
&= C'p^{(e+1)\gamma(R)}\left(1+\frac{1}{p}+\cdots\frac{1}{p^e}+\frac{1}{p^{\gamma(R)+e}}\right) \\
&\leqslant C'p^{(e+1)\gamma(R)}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^e}+\frac{1}{p^{e+1}}\right).
\end{aligned}
$$

Observe $1+\frac{1}{p}+\cdots+\frac{1}{p^e} \leqslant 1+\frac{1}{2}+\cdots+\frac{1}{2^e} \leqslant 2$. It now follows by induction that for every $p^e$, the containment $R^{\oplus p^{e\gamma(R)}} \subseteq F_*^e R$ will have a prime filtration whose factors are isomorphic to $R/P$ for some $P \in \mathcal{S}(R)$ with no more than $C'(1+\frac{1}{p}+\cdots+\frac{1}{p^e})p^{e\gamma(R)} \leqslant 2C'p^{e\gamma(R)}$ quotients isomorphic to $R/P$ for each $P \in \mathcal{S}(R)$. $\square$

We will find it useful to have a version of Lemma 3.2 in which the inclusions $R^{\oplus p^{e\gamma(R)}} \subseteq F_*^e R$ are reversed.

Corollary 3.3. *Let $R$ be an F-finite domain. Then there exists a finite set of nonzero primes $\mathcal{S}(R)$, and a constant $C$, such that for every $e \in \mathbb{N}$,*

1. *there is a containment of $R$-modules, $F_*^e R \subseteq R^{\oplus p^{e\gamma(R)}}$,*
2. *which has a prime filtration whose prime factors are isomorphic to $R/P$, where $P \in \mathcal{S}(R)$,*
3. *and for each $P \in \mathcal{S}(R)$, the prime factor $R/P$ appears no more than $Cp^{e\gamma(R)}$ times in the prime filtration of the containment $F_*^e R \subseteq R^{\oplus p^{e\gamma(R)}}$.*

Proof. As $F_*R$ is torsion-free of rank $p^{\gamma(R)}$, there is an embedding $F_*R \subseteq R^{\oplus p^{\gamma(R)}}$ such that the support of the cokernel of $R^{p^{\oplus\gamma(R)}}/F_*R$ consists of nonzero primes. Therefore there is prime filtration $F_*R = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = R^{\oplus p^{\gamma(R)}}$ with $M_i/M_{i-1} \cong R/P_i$ with $P_i$ a nonzero prime ideal of $R$. By Remark 3.1, $\gamma(R/P_i) < \gamma(R)$. Let $\mathcal{S}(R/P_i)$ and constant $C_i$ be the collection of primes and constant as described in Lemma 3.2 for the F-finite domain $R/P_i$. As in the proof of Lemma 3.2 we let $C' = \sum C_i$ and $\mathcal{S}(R) = \bigcup(\mathcal{S}(R/P_i) \cup \{P_i\})$. We show by induction on $e \in \mathbb{N}$ that there is an embedding $F^e_*R \subseteq R^{\oplus p^{e\gamma(R)}}$ with a prime filtration whose prime factors are isomorphic to $R/P$ with $P \in \mathcal{S}(R)$ with no more than $C'p^{e\gamma(R)}(1 + \frac{1}{p} + \cdots + \frac{1}{p^e})$ quotients isomorphic to $R/P$ for each $P \in \mathcal{S}(R)$. The above filtration of $F_*R \subseteq R^{\oplus p^{\gamma(R)}}$ shows the induction step when $e = 1$.

Suppose that $F^e_*R = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = R^{\oplus p^{e\gamma(R)}}$ is a prime filtration of an embedding $F^e_*R \subseteq R^{\oplus p^{e\gamma(R)}}$ such that $N_j/N_{j-1} \cong R/P_j$ for some $P_j \in \mathcal{S}(R)$ and such a prime factor appears no more than $C'p^{e\gamma(R)}(1 + \frac{1}{p} + \cdots + \frac{1}{p^e})$ times in the filtration. Then $F^e_*(R^{\oplus p^{\gamma(R)}}) \cong (F^e_*R)^{\oplus p^{\gamma(R)}} \subseteq (R^{\oplus p^{e\gamma(R)}})^{\oplus p^{\gamma(R)}} \cong R^{\oplus p^{(e+1)\gamma(R)}}$ has a prime filtration with prime factors $R/P_j$ with $P_j \in \mathcal{S}(R)$ and such a prime factor appears no more than $C'p^{(e+1)\gamma(R)}(1+\frac{1}{p}+\cdots+\frac{1}{p^e})$ times in the filtration. Furthermore, the prime filtration $F_*R = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = R^{\oplus p^{\gamma(R)}}$ gives the following filtration of $F^{e+1}_*R \cong F^e_*(F_*R) \subseteq F^e_*(R^{\oplus p^{\gamma(R)}})$,

$$F^e_*(F_*R) = F^e_*M_0 \subseteq F^e_*M_1 \subseteq \cdots \subseteq F^e_*M_n = F^e_*(R^{\oplus p^{\gamma(R)}}).$$

As $F^e_*M_i/F^e_*M_{i-1} \cong F^e_*(R/P_i)$, we apply Lemma 3.2 to know there is a prime filtration of $F^e_*M_{i-1} \subseteq F^e_*M_i$ whose prime factors come from $\mathcal{S}(R/P_i)$ and such a prime factor appears no more than $C_ip^{e\gamma(R/P_i)} \leqslant C'p^{(\gamma(R)-1)e}$ times in the filtration. Putting all of this information together we get an embedding $F^{e+1}_*R \subseteq R^{\oplus p^{(e+1)\gamma(R)}}$ with a prime filtration whose prime factors come from $\mathcal{S}(R)$ and such a prime factor appears no more than the following number of times in the filtration,

$$
\begin{aligned}
p^{\gamma(R)}C'p^e\gamma(R)\left(1+\frac{1}{p}+\cdots+\frac{1}{p^e}\right)+\sum_{i=1}^{n}C_ip^{e\gamma(R/P_i)}&\\
&\leqslant C'p^{(e+1)\gamma(R)}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{(e+1)}}\right)\\
&\leqslant 2C'(p^{(e+1)\gamma(R)}).
\end{aligned}
$$

□

We combine of Lemma 3.2 and Corollary 3.3 into a single statement for convenience.

COROLLARY 3.4. *Let $R$ be an F-finite domain. There exists a finite set of nonzero primes $\mathcal{S}(R)$ and a constant $C$ such that for every $e\in\mathbb{N}$, there is a containment of $R$-modules $F_*^eR\subseteq R^{\oplus p^{e\gamma(R)}}$ and $R^{\oplus p^{e\gamma(R)}}\subseteq F_*^eR$ which each has a prime filtration whose prime factors are isomorphic to $R/P$, where $P\in\mathcal{S}(R)$, and such a prime factor appears no more than $Cp^{e\gamma(R)}$ times in the filtration.*

DISCUSSION 3.5. If $(R,\mathfrak{m})$ is a local ring, an $\mathfrak{m}$-*primary pair of ideals* will be a containment of ideals of $R$, $I\subseteq J$, such that $I$ is $\mathfrak{m}$-primary. Observe that either $J$ is also $\mathfrak{m}$-primary or is $R$ itself. If $I\subseteq J$ is an $\mathfrak{m}$-primary pair of ideals, then there is an ascending chain of ideals $I\subseteq(I,u_1)\subseteq(I,u_1,u_2)\subseteq\cdots\subseteq(I,u_1,u_2,...,u_{\ell(J/I)})=J$ where $u_{i+1}\in(I,u_1,...,u_i):\mathfrak{m}$. Given an $\mathfrak{m}$-primary pair of ideals $I\subseteq J$ and choosing elements $u_1,\dots,u_{\ell(J/I)}$ as above, let $I_0=I$ and $I_i=(I,u_1,...,u_i)$ for $1\leqslant i\leqslant\ell(J/I)$.

LEMMA 3.6. *Let $(R,\mathfrak{m})$ be a local ring of characteristic $p$ and $M$ a finitely generated $R$-module. If $I\subseteq J$ is an $\mathfrak{m}$-primary pair of ideals, then $\ell(J^{[p^e]}M/I^{[p^e]}M)\leqslant\ell(M/\mathfrak{m}^{[p^e]}M)\ell(J/I)$.*

PROOF. Using notation as in Discussion 3.5, observe that

$$\ell(J^{[p^e]}M/I^{[p^e]}M)=\sum_{i=1}^{\ell(J/I)}\ell(I_i^{[p^e]}M/I_{i-1}^{[p^e]}M),$$

hence it is enough to show that if $I$ is $\mathfrak{m}$-primary and $u \in (I : \mathfrak{m})$, then $\ell((I,u)^{[p^e]}M/I^{[q]}M) \leqslant \ell(M/\mathfrak{m}^{[q]}M)$. Note that $(I,u)^{[p^e]}M/I^{[p^e]}M \cong M/(I^{[p^e]}M :_M u^{p^e})$. Since $u \in I : \mathfrak{m}$ we have that $\mathfrak{m}^{[p^e]}M \subseteq (I^{[p^e]}M :_M u^{p^e})$, hence $\ell(M/(I^{[p^e]}M :_M u^{p^e})) \leqslant \ell(M/\mathfrak{m}^{[p^e]}M)$. $\square$

THEOREM 3.7. *Let $R$ be an F-finite ring and $M$ a finitely generated $R$-module. There exists a constant $C > 0$ such that for all $P \in \operatorname{Spec}(R)$ and $e \in \mathbb{N}$, if $IR_P \subseteq JR_P$ is a $PR_P$-primary pair of ideals, then*

$$\ell\left(\frac{J^{[p^e]}M_P}{I^{[p^e]}M_P}\right) \leqslant Cp^{e\dim(M_P)}\ell\left(\frac{JR_P}{IR_P}\right).$$

PROOF. By Lemma 3.6 we only need to find a constant $C$ such that for all $P \in \operatorname{Spec}(R)$ and all $e \in \mathbb{N}$, $\ell(M_P/P^{[p^e]}M_P) \leqslant Cp^{e\dim(M_P)}$.

Consider a prime filtration of $M$,

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that $M_{i+1}/M_i \cong R/Q_i$ with $Q_i \in \operatorname{Spec}(R)$. It readily follows that for each $e \in \mathbb{N}$ and $P \in \operatorname{Spec}(R)$ that

$$\ell(M_P/P^{[p^e]}M_P) \leqslant \sum_{i=1}^{n} \ell(R_{Q_i}/(Q_i + P^{[p^e]})R_{Q_i}).$$

Therefore it suffices to prove the theorem in the case that $R$ is an F-finite domain and $M$ is the $R$-module $R$. That is, if $R$ is an F-finite domain, we need to show the existence of a constant $C \in \mathbb{R}$ such that for each $e \in \mathbb{N}$ and $P \in \operatorname{Spec}(R)$, $\ell(R_P/P^{[p^e]}R_P) \leqslant Cp^{e\,\mathrm{ht}(P)}$. Equivalently, we need to show there exists a constant $C \in \mathbb{R}$ such that $\ell(F^e_*R_P/PF^e_*R_P) \leqslant Cp^{e\gamma(R_P)} = Cp^{e\gamma(R)}$. Let $C$ and $\mathcal{S}(R)$ be as in Lemma 3.2. Then there exists an embedding $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_*R$ and prime filtration

$$R^{\oplus p^{e\gamma(R)}} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_h = F^e_*R$$

such that $N_{i+1}/N_i \cong R/Q_i$, each $Q_i \in \mathcal{S}(R)$, and such a prime factor appears no more than $Cp^{e\gamma(R)}$ times in the filtration. In particular, $h \leqslant C|\mathcal{S}(R)|p^{e\gamma(R)}$ and it readily follows that

$$\begin{aligned}\ell(F^e_* R_P/PF^e_* R_P) &\leqslant \sum_{i=1}^{h} \ell(R_P/(Q_i+P)R_P) + \ell(R_P^{\oplus p^{e\gamma(R)}}/PR_P^{\oplus p^{e\gamma(R)}}) \\ &\leqslant h + p^{e\gamma(R)} \leqslant (C|\mathcal{S}(R)|+1)p^{e\gamma(R)}. \qquad \square\end{aligned}$$

COROLLARY 3.8. *Let $R$ be an F-finite ring, $N, M$ two finitely generated $R$-modules which are isomorphic at minimal primes of $R$. Then there is a constant $C$ such that for all $P \in \operatorname{Spec}(R)$ and $e \in \mathbb{N}$, if $IR_P \subseteq JR_P$ is a $PR_P$-primary pair of ideals, then*

$$\left|\ell\left(\frac{J^{[p^e]}M_P}{I^{[p^e]}M_P}\right) - \ell\left(\frac{J^{[p^e]}N_P}{I^{[p^e]}N_P}\right)\right| \leqslant Cp^{e(\operatorname{ht}(P)-1)}\ell\left(\frac{JR_P}{IR_P}\right).$$

PROOF. Using the notation in Discussion 3.5 and applying the triangle inequality

$$\left|\ell\left(\frac{J^{[p^e]}M_P}{I^{[p^e]}M_P}\right) - \ell\left(\frac{J^{[p^e]}N_P}{I^{[p^e]}N_P}\right)\right| \leqslant \sum_{i=1}^{\ell(JR_P/IR_P)} \left|\ell\left(\frac{I_i^{[p^e]}M_P}{I_{i-1}^{[p^e]}M_P}\right) - \ell\left(\frac{I_i^{[p^e]}N_P}{I_{i-1}^{[p^e]}N_P}\right)\right|.$$

Thus we may reduce to the case that $J = (I, u)$ where $u \in (I : P)$. There are right exact sequences $M \xrightarrow{\varphi} N \to T_1 \to 0$ and $N \xrightarrow{\psi} M \to T_2 \to 0$, for which $T_1, T_2$ are 0 when localized at minimal primes of $R$. Observe that $\varphi(I^{[p^e]}M :_M u^{p^e}) \subseteq (I^{[p^e]}N :_N u^{p^e})$ so that there is an induced map $\frac{M_P}{(I^{[p^e]}M_P :_{M_P} u^{p^e})} \to \frac{N_P}{(I^{[p^e]}N_P :_{N_P} u^{p^e})}$ whose cokernel, say $(T_1')_P$ is naturally the homomorphic image of $(T_1)_P$. Thus we have the following commutative diagram.

$$\begin{tikzcd}
M_P \arrow[r, "\varphi"] \arrow[d] & N_P \arrow[r] \arrow[d, "\pi_1"] & (T_1)_P \arrow[r] \arrow[d, "\pi_2"] & 0 \\
\frac{M_P}{(I^{[q]}M_P :_{M_P} u^q)} \arrow[r] & \frac{N_P}{(I^{[q]}N_P :_{N_P} u^q)} \arrow[r] & (T_1')_P \arrow[r] & 0
\end{tikzcd}$$

Therefore $\ell\left(\frac{N_P}{(I^{[p^e]}N_P :_{N_P} u^{p^e})}\right) - \ell\left(\frac{M_P}{(I^{[p^e]}M_P :_{M_P} u^{p^e})}\right) \leqslant \ell((T_1')_P)$. Observe that $P^{[q]}N_P \subseteq (I^{[p^e]}N_P :_{N_P} u^{p^e})$ so that $\pi_1(P^{[p^e]}N_P) = 0$ and therefore $\pi_2(P^{[p^e]}(T_1)_P) = 0$. Hence

$(T_1')_P$ is the homomorphic image of $\frac{(T_1)_P}{P^{[p^e]}(T_1)_P}$. Thus

$$\begin{aligned}
\ell\left(\frac{(I,u)^{[p^e]}N_P}{I^{[p^e]}N_P}\right) &- \ell\left(\frac{(I,u)^{[p^e]}M_P}{I^{[p^e]}M_P}\right) \\
&= \ell\left(\frac{N_P}{(I^{[p^e]}N_P :_{N_P} u^{p^e})}\right) - \ell\left(\frac{M_P}{(I^{[p^e]}M_P :_{M_P} u^{p^e})}\right) \\
&\leqslant \ell\left(\frac{(T_1)_P}{P^{[p^e]}(T_1)_P}\right).
\end{aligned}$$

A similar argument applied to the exact sequence $N \to M \to T_2 \to 0$ implies that

$$\left|\ell\left(\frac{J^{[p^e]}M_P}{I^{[p^e]}M_P}\right) - \ell\left(\frac{J^{[p^e]}N_P}{I^{[p^e]}N_P}\right)\right| \leqslant \max_{i=1,2}\left\{\ell\left(\frac{(T_i)_P}{P^{[p^e]}(T_i)_P}\right)\right\}.$$

The corollary now follows by Theorem 3.7. □

Corollary 3.9. *Let $R$ be an F-finite ring and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of finitely generated $R$-modules. There exists a constant $C$ such that for all $P \in \operatorname{Spec}(R)$ and $e \in \mathbb{N}$, if $IR_P \subseteq JR_P$ is a $PR_P$-primary pair of ideals, then*

$$\left|\ell\left(\frac{J^{[p^e]}M_P}{I^{[p^e]}M_P}\right) - \ell\left(\frac{J^{[p^e]}M'_P}{I^{[p^e]}M'_P}\right) - \ell\left(\frac{J^{[p^e]}M''_P}{I^{[p^e]}M''_P}\right)\right| \leqslant Cp^{e(\dim(M_P)-1)}\ell\left(\frac{JR_P}{IR_P}\right).$$

Proof. Observe that $\ell\left(\frac{(J+\operatorname{Ann}_R M)R_P}{(I+\operatorname{Ann}_R M)R_P}\right) \leqslant \ell\left(\frac{JR_P}{IR_P}\right)$. Therefore we can begin by replacing $R$ with $R/\operatorname{Ann}_R M$ so that $\operatorname{ht}(P) = \dim M_P$ for all $P \in \operatorname{Spec}(R)$. If $R$ is reduced then $M$ is isomorphic to $M' \bigoplus M''$ at minimal primes of $R$ and we can apply Corollary 3.8. Suppose $R$ is not reduced. We will use a trick similar to that used in Lemma 2.10 to reduce ourselves to the reduced case. Let $e_0$ be a large enough integer so that for $\sqrt{0}^{[p^{e_0}]} = 0$. Let $F : R \to R$ be the Frobenius endomorphism. Then $F^{e_0}(R)$ is abstractly isomorphic to the reduced ring $R/\sqrt{0}$ and $R$ is module finite over $F^{e_0}(R)$. Then for all $P \in \operatorname{Spec}(R)$ and $IR_P \subseteq PR_P$ which is $PR_P$-primary,

$$\begin{aligned}
\frac{1}{[F_*^{e_0}k(P) : k(P)]}\ell_{F^{e_0}(R_P)}\left(\frac{M_P}{(I^{[p^{e_0}]} \cap F^{e_0}(R))^{[p^e]}M_P}\right) &= \ell_{R_P}\left(\frac{M_P}{(I^{[p^{e_0}]} \cap F^{e_0}(R))^{[p^e]}M_P}\right) \\
&= \ell_{R_P}\left(\frac{M_P}{I^{[p^{e+e_0}]}M_P}\right).
\end{aligned}$$

Thus by replacing $R$ with $F^{e_0}(R)$ we are reduced to the reduced case and may apply Corollary 3.8. $\square$

THEOREM 3.10. *Let $R$ be an F-finite ring and $M$ a finitely generated $R$-module. There exists a constant $C$ such that for all $P \in \operatorname{Spec}(R)$, for all $e_1, e_2 \in \mathbb{N}$, if $IR_P \subseteq JR_P$ is a $PR_P$-primary pair of ideals, then*

$$\left|\ell\left(\frac{J^{[p^{e_1}]}M_P}{I^{[p^{[e_1]}]}M_P}\right)p^{e_2 \operatorname{ht}(P)} - \ell\left(\frac{J^{[p^{e_1+e_2}]}M_P}{I^{[p^{e_1+e_2}]}M_P}\right)\right| \leqslant Cp^{e_2 \dim(M_P)}p^{e_1(\dim(M_P)-1)}\ell\left(\frac{JR_P}{IR_P}\right).$$

PROOF. As in the proof of Corollary 3.9, we may replace $R$ by $R/\operatorname{Ann}_R M$ so that $\dim(M_P) = \operatorname{ht} P$ for all $P \in \operatorname{Spec}(R)$. If there is a short exact sequence $0 \to M' \to M \to M'' \to 0$, then

$$\left|\ell\left(\frac{J^{[p^{e_1}]}M_P}{I^{[p^{e_1}]}M_P}\right)p^{e_2 \operatorname{ht}(P)} - \ell\left(\frac{J^{[p^{e_1+e_2}]}M_P}{I^{[p^{e_1+e_2}]}M_P}\right)\right| \leqslant A_1 + A_2 + A_3 + A_4.$$

Where

$$\begin{aligned}
A_1 &= \left|\ell\left(\frac{J^{[p^{e_1}]}M_P}{I^{[p^{e_1}]}M_P}\right) - \ell\left(\frac{J^{[p^{e_1}]}(M'_P \bigoplus M''_P)}{I^{[p^{e_1}]}(M'_P \bigoplus M''_P)}\right)\right|p^{e_2 \operatorname{ht}(P)}\\
A_2 &= \left|\ell\left(\frac{J^{[p^{e_1+e_2}]}M_P}{I^{[p^{e_1+e_2}]}M_P}\right) - \ell\left(\frac{J^{[p^{e_1+e_2}]}(M'_P \bigoplus M''_P)}{I^{[p^{e_1+e_2}]}(M'_P \bigoplus M''_P)}\right)\right|\\
A_3 &= \left|\ell\left(\frac{J^{[p^{e_1}]}M'_P}{I^{[p^{e_1}]}M'_P}\right)p^{e_2 \operatorname{ht}(P)} - \ell\left(\frac{J^{[p^{e_1+e_2}]}M'_P}{I^{[p^{e_1+e_2}]}M'_P}\right)\right|\\
A_4 &= \left|\ell\left(\frac{J^{[p^{e_1}]}M''_P}{I^{[p^{e_1}]}M''_P}\right)p^{e_2 \operatorname{ht}(P)} - \ell\left(\frac{J^{[p^{e_1+e_2}]}M''_P}{I^{[p^{e_1+e_2}]}M''_P}\right)\right|.
\end{aligned}$$

By Corollary 3.9 there is a constant $C$ such that $A_1 \leqslant Cp^{e_1(\operatorname{ht}(P)-1)}\ell\left(\frac{JR_P}{IR_P}\right)p^{e_2 \operatorname{ht}(P)}$ and $A_2 \leqslant Cp^{(e_1+e_2)(\operatorname{ht}(P)-1)}\ell\left(\frac{JR_P}{IR_P}\right)$. Therefore, by considering a prime filtration of the module $M$, we can reduce proving the theorem to the scenario that $M = R/P$ for some prime $P \in \operatorname{Spec}(R)$, i.e., we may assume that $M = R$ is an F-finite domain. Observe that by Lemma 2.4 and Corollary 2.6, $\ell\left(\frac{J^{[p^{e_1+e_2}]}R_P}{I^{[p^{e_1+e_2}]}R_P}\right) = \frac{1}{[F_*^{e_2}k(P):k(P)]}\ell\left(\frac{J^{[p^{e_1}]}F_*^{e_2}R_P}{I^{[p^{e_1}]}F_*^{e_2}R_P}\right)$. Therefore the theorem is now reduced to showing that there is a constant $C$ independent of $P, I, J, e_1, e_2$ such that

$$\left|\ell\left(\frac{J^{[p^{e_1}]}R_P}{I^{[p^{e_1}]}R_P}\right)p^{e_2\gamma(R)} - \ell\left(\frac{J^{[p^{e_1}]}F_*^{e_2}R_P}{I^{[p^{e_1}]}F_*^{e_2}R_P}\right)\right| \leqslant Cp^{e_2\gamma(R)}p^{e_1(\operatorname{ht}(P)-1)}\ell\left(\frac{JR_P}{IR_P}\right).$$

As in the proof of Corollary 3.8 we can further reduce to the scenario that $J = (I, u)$ where $u \in (I : P)$.

Let $C$ and $\mathcal{S}(R)$ be as in Lemma 3.4 with corresponding inclusions of $R$-modules $F_*^eR \to R^{\oplus p^{e\gamma(R)}}$ and $R^{\oplus p^{e\gamma(R)}} \to F_*^eR$. So there are exact sequences $0 \to F_*^eR \to R^{\oplus p^{e\gamma(R)}} \to T_1(e) \to 0$ and $0 \to R^{\oplus p^{e\gamma(R)}} \to F_*^eR \to T_2(e) \to 0$ so that both $T_1(e)$ and $T_2(e)$ have a prime filtration whose prime factors are isomorphic to $R/Q$ where $Q \in \mathcal{S}(R)$ and such a prime factor appears no more than $Cp^{e\gamma(R)}$ times in the filtration. As in the proof of Corollary 3.8, there will be the following commutative diagrams with all vertical maps being surjective.

$$\begin{tikzcd}
F_*^{e_2}R_P \arrow[r] \arrow[d] & R_P^{\oplus p^{e_2\gamma(R)}} \arrow[r] \arrow[d] & T_1(e_2)_P \arrow[r] \arrow[d] & 0 \\
\frac{F_*^{e_2}R_P}{(I^{[p^{e_1}]}F_*^{e_2}R_P :_{F_*^{e_2}R_P} u^{p^{e_1}})} \arrow[r] & \frac{R_P^{\oplus p^{e_2\gamma(R)}}}{(I^{[p^{e_1}]} :_R u^{p^{e_1}})R_P^{\oplus p^{e_2\gamma(R)}}} \arrow[r] & T_1'(e_2)_P \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
R_P^{\oplus p^{e_2\gamma(R)}} \arrow[r] \arrow[d] & F_*^{e_2}R_P \arrow[r] \arrow[d] & T_2(e_2)_P \arrow[r] \arrow[d] & 0 \\
\frac{R_P^{\oplus p^{e_1\gamma(R)}}}{(I^{[p^{e_1}]} :_R u^{p^{e_1}})R_P^{\oplus p^{e_1\gamma(R)}}} \arrow[r] & \frac{F_*^{e_2}R_P}{(I^{[p^{e_1}]}R_P :_{F_*^{e_2}R_P} u^{p^{e_1}})} \arrow[r] & T_2'(e_2)_P \arrow[r] & 0
\end{tikzcd}$$

Furthermore, $T_i'(e_2)_P$ will be the homomorphic image of $\frac{T_i(e_2)_P}{P^{[p^{e_1}]}T_i(e_2)_P}$ for $i = 1, 2$. It follows that

$$\left|\ell\left(\frac{J^{[p^{e_1}]}R_P}{I^{[p^{e_1}]}R_P}\right)p^{e_2\gamma(R)} - \ell\left(\frac{J^{[p^{e_1}]}F_*^{e_2}R_P}{I^{[p^{e_1}]}F_*^{e_2}R_P}\right)\right| \leqslant \max_{i=1,2}\left\{\ell\left(\frac{T_i(e_2)_P}{P^{[p^{e_1}]}T_i(e_2)_P}\right)\right\}.$$

For each $i = 1, 2$, $\ell\left(\frac{T_i(e_2)_P}{P^{[p^{e_1}]}T_i(e_2)_P}\right) \leqslant Cp^{e_1\gamma(R)} \max_{Q\in\mathcal{S}(R)}\{\left(\frac{R_P}{(Q+P^{[p^{e_1}]})R_P}\right)\}$. We can now apply Proposition 3.7 to know that the desired bound exists. $\square$

## 2. Rings essentially of finite type over an excellent local ring

**2.1. Gamma Constructions.** If $R$ is essentially of finite type over a complete local ring $A$, then we let $\Lambda$ be a $p$-base of the residue field of $A$. Let $\Gamma$ denote a cofinite subset of $\Lambda$. For each such $\Gamma$ there is an associated $R$-algebra, $R^\Gamma$, which satisfies the following:

Theorem 3.11. [**HH90**, Section 6] *Let $R$ be a characteristic $p$ ring essentially of finite type over a complete local ring. Then for each $\Gamma \leqslant \Lambda$, $R^\Gamma$ is a faithfully flat, purely inseparable, F-finite $R$-algebra.*

To say that $R \to R^\Gamma$ is purely inseparable is to say that for each $s \in R^\Gamma$, there exists an $n \in \mathbb{N}$ such that $s^n \in R$. From this it follows that the induced map $\operatorname{Spec}(R^\Gamma) \to \operatorname{Spec}(R)$ is a homeomorphism. The inverse map sends a prime $P \in \operatorname{Spec}(R)$ to $\sqrt{PR^\Gamma}$. If $P \in \operatorname{Spec}(R)$ we shall let $P_\Gamma = \sqrt{PR^\Gamma}$.

If $R$ is essentially of finite type over a complete local ring, then for each $\Gamma$ we have that $PR^\Gamma_{P_\Gamma}$ is $P_\Gamma R^\Gamma_{P_\Gamma}$-primary. If $R$ is essentially of finite type over an excellent local ring $A$, then $\Gamma$ shall represent a cofinite subset of a $p$-base for a coefficient field of $\hat{A}$. If $R$ is essentially of finite type over a complete local ring and $M$ a finitely generated $R$-module, then we let $M^\Gamma = R^\Gamma \otimes_R M$.

**2.2. Uniform bounds via Gamma constructions.**

Lemma 3.12. *Let $R \to S$ be a faithfully flat homomorphism of characteristic $p$ Noetherian rings with regular fibers. Let $M$ be a finitely generated $R$-module, $P \in \operatorname{Spec}(R)$*

*and $IR_P$ an $PR_P$-primary ideal. Then*

$$\frac{1}{p^{e\,\mathrm{ht}(P)}}\ell_{R_P}\left(\frac{M_P}{I^{[p^e]}M_P}\right) = \frac{1}{p^{e\,\mathrm{ht}(Q)}}\ell_{S_Q}\left(\frac{(S\otimes_R M)_Q}{(I,\underline{x})^{[p^e]}(S\otimes_R M)_Q}\right),$$

*where $Q$ is a prime of $S$ lying over $P$ and $\underline{x}$ is a regular system of parameters for $S_Q/PS_Q$.*

Proof. The first thing to observe is that $\frac{(S\otimes_R M)_Q}{(I,\underline{x})^{[p^e]}(S\otimes_R M)_Q} \cong \frac{S_Q}{\underline{x}^{[p^e]}S_Q} \otimes_{R_P} \frac{M_P}{I^{[p^e]}M_P}$. Since $R_P \to S_Q$ is flat and $S_Q/PS_Q$ is regular, we have that

$$\begin{aligned}\ell_{S_Q}\left(\frac{(S\otimes_R M)_Q}{(I,\underline{x})^{[p^e]}(S\otimes_R M)_Q}\right) &= \ell_{S_Q}\left(\frac{S_Q}{\underline{x}^{[p^e]}S_Q}\otimes_{R_P}\frac{M_P}{I^{[p^e]}M_P}\right)\\ &= \ell_{S_Q}\left(\frac{S_Q}{(P+\underline{x}^{[p^e]})S_Q}\right)\ell_{R_P}\left(\frac{M_P}{I^{[p^e]}M_P}\right)\\ &= p^{e(\mathrm{ht}(Q)-\mathrm{ht}(P))}\ell_{R_P}\left(\frac{M_P}{I^{[p^e]}M_P}\right).\end{aligned}$$

Dividing both sides of the equation by $p^{e\,\mathrm{ht}(Q)}$ gives the desired result. $\square$

Suppose that $R$ is essentially of finite type over the excellent local ring $A$. Let $\hat{A}$ denote the completion of $A$ with respect to its maximal ideal. Then $R \to \hat{A}\otimes_A R$ is a faithfully flat homomorphism with regular fibers [**Mat80**, Section 33, Lemma 4]. This observation and Lemma 3.12 allow us to reduce proving statements about rings essentially of finite type over an excellent local ring to rings which are essentially of finite type over a complete local ring.

Theorem 3.13. *Let $R$ be essentially of finite type over an excellent local ring and let $M$ be a finitely generated $R$-module. There exists a constant $C > 0$ such that for all $P \in \mathrm{Spec}(R)$ and $e \in \mathbb{N}$, if $IR_P \subseteq JR_P$ is a $PR_P$-primary pair of ideals, then*

$$\ell\left(\frac{J^{[p^e]}M_P}{I^{[p^e]}M_P}\right) \leqslant Cp^{e\dim(M_P)}\ell\left(\frac{JR_P}{IR_P}\right).$$

Proof. By Lemma 3.12 and the remarks that follow, we may reduce to the scenario that $R$ is essentially of finite type over a complete local ring. Choose any $\Gamma$. Then for each $P \in \operatorname{Spec}(R)$ one sees that by tensoring a prime filtration of $\frac{J^{[p^e]}M_P}{I^{[p^e]}M_P}$ with $R^{\Gamma}_{P_\Gamma}$ that

$$\ell_{R^{\Gamma}_{P_\Gamma}}\left(\frac{J^{[p^e]}M^{\Gamma}_{P_\Gamma}}{I^{[p^e]}M^{\Gamma}_{P_\Gamma}}\right) = \ell_{R_P}\left(\frac{J^{[p^e]}M_P}{I^{[p^e]}M_P}\right)\ell_{R^{\Gamma}_{P_\Gamma}}(R^{\Gamma}_{P_\Gamma}/PR^{\Gamma}_{P_\Gamma}).$$

We can now apply Proposition 3.7 to the F-finite ring $R^{\Gamma}$ so that we know there exists a constant $C$ such that for all $P \in \operatorname{Spec}(R)$ and for all $e \in \mathbb{N}$,

$$\begin{aligned} \ell_{R_P}\left(\frac{J^{[p^e]}M_P}{I^{[p^e]}M_P}\right) &= \frac{\ell_{R^{\Gamma}_{P_\Gamma}}\left(J^{[p^e]}M^{\Gamma}_{P_\Gamma}/I^{[p^e]}M^{\Gamma}_{P_\Gamma}\right)}{\ell_{R^{\Gamma}_{P_\Gamma}}(R^{\Gamma}_{P_\Gamma}/PR^{\Gamma}_{P_\Gamma})} \\ &\leqslant \frac{Cp^{e\,\mathrm{ht}(P_\Gamma)}\ell_{R^{\Gamma}_{P_\Gamma}}\left(JR^{\Gamma}_{P_\Gamma}/IR^{\Gamma}_{P_\Gamma}\right)}{\ell_{R^{\Gamma}_{P_\Gamma}}(R^{\Gamma}_{P_\Gamma}/PR^{\Gamma}_{P_\Gamma})} \\ &= Cp^{e\,\mathrm{ht}(P)}\ell_{R_P}\left(\frac{JR_P}{IR_P}\right). \quad \square \end{aligned}$$

Theorem 3.14. *Let $R$ be essentially of finite type over an excellent local ring and let $M$ be a finitely generated $R$-module. There exists a constant $C$ such that for all $P \in \operatorname{Spec}(R)$, for all $e_1, e_2 \in \mathbb{N}$, if $IR_P \subseteq JR_P$ is a $PR_P$-primary pair of ideals, then*

$$\left|\ell\left(\frac{J^{[p^{e_1}]}M_P}{I^{[p^{e_1}]}M_P}\right)p^{e_2\,\mathrm{ht}(P)} - \ell\left(\frac{J^{[p^{e_1+e_2}]}M_P}{I^{[p^{e_1+e_2}]}M_P}\right)\right| \leqslant Cp^{e_2\,\mathrm{ht}(P)}p^{e_1(\mathrm{ht}(P)-1)}\ell\left(\frac{JR_P}{IR_P}\right).$$

Proof. The proof of this theorem is identical to the proof of Proposition 3.13. Lemma 3.12 allows us to reduce to the scenario that $R$ is essentially of finite type over a complete local ring. Pick a $\Gamma$ and let $C$ be as in Theorem 3.10 for the F-finite ring $R^\Gamma$, then

$$\left|\ell_{R_P}\left(\frac{J^{[p^{e_1}]}M_P}{I^{[p^{e_1}]}M_P}\right)p^{e_2\,\mathrm{ht}(P)} - \ell_{R_P}\left(\frac{J^{[p^{e_1+e_2}]}M_P}{I^{[p^{e_1+e_2}]}M_P}\right)\right|$$

$$= \left|\ell_{R^\Gamma_{P_\Gamma}}\left(\frac{J^{[p^{e_1}]}M^\Gamma_{P_\Gamma}}{I^{[p^{e_1}]}(M^\Gamma_{P_\Gamma})}\right)p^{e_2\,\mathrm{ht}(P)} - \ell_{R^\Gamma_{P_\Gamma}}\left(\frac{J^{[p^{e_1+e_2}]}M^\Gamma_{P_\Gamma}}{I^{[p^{e_1+e_2}]}M^\Gamma_{P_\Gamma}}\right)\right| \Big/ \ell_{R^\Gamma_{P_\Gamma}}\left(\frac{R^\Gamma_{P_\Gamma}}{PR^\Gamma_{P_\Gamma}}\right)$$

$$\leqslant \frac{Cp^{e_2\,\mathrm{ht}(P_\Gamma)}p^{e_1(\mathrm{ht}(P_\Gamma)-1)}\ell_{R_{P_\Gamma}}\left(\frac{JR_{P_\Gamma}}{IR_{P_\Gamma}}\right)}{\ell_{R^\Gamma_{P_\Gamma}}\left(\frac{R^\Gamma_{P_\Gamma}}{PR^\Gamma_{P_\Gamma}}\right)} = Cp^{e_2\,\mathrm{ht}(P)}p^{e_1(\mathrm{ht}(P)-1)}\ell_{R_P}\left(\frac{JR_P}{IR_P}\right). \qquad \square$$

## 3. Cartier subalgebras

We will need a version of Corollary 3.3 for Cartier subalgebras in order to establish the lower semi-continuity of the F-signature of pairs function in Theorem 4.9. We first remark the following.

REMARK 3.15. If $R$ is an F-finite ring, $\mathscr{D}$ a Cartier subalgebra, $N, M \in \mathbb{N}$, $f \in \mathscr{D}_{e_1}^{\oplus N}$, $g \in \mathscr{D}_{e_2}^{\oplus M}$, then the natural map $\underbrace{(g, g, ..., g)}_{\text{N times}} \circ F^{e_2}_* f : F^{e_1+e_2}_* R \to R^{\oplus NM}$ is an element of $\mathscr{D}^{\oplus NM}_{e_1+e_2}$.

The remark follows from the assumption that if $\varphi \in \mathscr{D}_{e_1}$ and $\psi \in \mathscr{D}_{e_2}$, then $\psi \circ F^{e_2}_* \varphi \in \mathscr{D}_{e_1+e_2}$.

PROPOSITION 3.16. *Let $R$ be an F-finite domain and $\mathscr{D}$ a Cartier subalgebra. Then there exists a finite set of nonzero primes $\mathcal{S}(R)$, and a constant $C \in \mathbb{R}$, such that for every $e \in \Gamma_{\mathscr{D}}$,*

(1) *there is a containment of $R$-modules, $F^e_* R \subseteq R^{\oplus p^{e\gamma(R)}}$ which is an element of $\mathscr{D}_e^{\oplus p^{e\gamma(R)}}$,*
(2) *which has a prime filtration whose prime factors are isomorphic to $R/P$, where $P \in \mathcal{S}(R)$,*
(3) *and for each $P \in \mathcal{S}(R)$, the prime factor $R/P$ appears no more than $Cp^{e\gamma(R)}$ times in the prime filtration of the containment $F^e_* R \subseteq R^{\oplus p^{e\gamma(R)}}$.*

Proof. Let $\Gamma_{\mathscr{D}}$ be generated by $\Lambda_{\mathscr{D}} = \{e_1, ..., e_m\}$ as a semigroup. By Lemma 2.28, for each $e_i$ we can fix an embedding $F_*^{e_i} R \subseteq R^{\oplus p^{e_i \gamma(R)}}$ which is an element of $\mathscr{D}_{e_i}^{\oplus p^{e_i \gamma(R)}}$ which is an isomorphism when localized at 0.

For each $e_j \in \Lambda_{\mathscr{D}}$ we consider a prime filtration of $F_*^e R \subseteq R^{\oplus p^{e\gamma(R)}}$, say $F_*^e R = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = R^{\oplus p^{e\gamma(R)}}$ and $N_i/N_{i-1} \cong R/P_{j,i}$. Let $\mathcal{S}(R/P_{j,i})$ and $C_{j,i}$ be as in Lemma 3.2 for the F-finite domain $R/P_{j,i}$ and let $\mathcal{S}_j(R, \mathscr{D}) = \bigcup_{i=1}^n \mathcal{S}(R/P_{j,i}) \cup \{P_{j,i}\}$ and $\mathcal{S}(R, \mathscr{D}) = \bigcup_{e_j \in \Lambda_{\mathscr{D}}} \mathcal{S}_j(R, \mathscr{D})$. We now set $C' = \sum C_{j,i}$. Every $e \in \Gamma_{\mathscr{D}}$ can be expressed as $\sum_{i=1}^m a_i e_i$ where $a_i \in \mathbb{N}$. We show by induction on $\sum a_i$ that for each $e \in \Gamma_{\mathscr{D}}$ there is a containment of $R$-modules $F_*^e R \subseteq R^{\oplus p^{e\gamma(R)}}$ which is an element of $\mathscr{D}_e^{\oplus p^{e\gamma(R)}}$, which has a prime filtration whose prime factors are isomorphic to $R/P$, where $P \in \mathcal{S}(R, \mathscr{D})$, and such a prime factor appears no more than $C' p^{e\gamma(R)} \left(1 + \frac{1}{p} + \cdots + \frac{1}{q}\right)$ times in the filtration. This trivially holds for $\sum a_i = 1$.

Now suppose $e = \sum_{e_i \in \Lambda_{\mathscr{D}}} a_i e_i$ with $\sum a_i > 1$. Without loss of generality we may suppose that $a_1 \geqslant 1$ so that $e' = (a_1 - 1)e_2 + \sum_{i \geqslant 2} a_i e_i \in \Gamma_{\mathscr{D}}$. By induction, we can find filtration $F_*^{e'} R = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = R^{\oplus p^{e'\gamma(R)}}$ of an embedding $F_*^{e'} R \subseteq F_*^{e'} R^{\oplus p^{e'\gamma(R)}}$ in $\mathscr{D}_{e'}^{\oplus p^{e'\gamma(R)}}$, each $N_j/N_{i-j} \cong R/P_j$ for some $P_j \in \mathcal{S}(R, \mathscr{D})$, and such a prime factor appears no more than $C' p^{e'\gamma(R)}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{e'}})$ times in the filtration. Therefore $F_*^{e'}(R^{\oplus p^{e_1\gamma(R)}}) \cong (F_*^{e'} R)^{\oplus p^{e_1\gamma(R)}} \subseteq (R^{\oplus p^{e'\gamma(R)}})^{\oplus p^{e_1\gamma(R)}} \cong R^{\oplus p^{e\gamma(R)}}$ has a prime filtration with prime factors $R/P_j$ with $P_j \in \mathcal{S}(R, \mathscr{D})$ and such a prime factor appears no mare than $C' p^{e\gamma(R)}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^e})$ times in the filtration. Furthermore, the prime filtration $F_*^{e_1} R = N_{1,0} \subseteq N_{1,1} \subseteq \cdots \subseteq N_{1,n} = R^{\oplus p^{e_1\gamma(R)}}$ gives the following filtration of $F_*^e R = F_*^{e'}(F_*^{e_1} R) \subseteq F_*^{e'}(R^{\oplus p^{e_1\gamma(R)}})$,

$$F_*^{e'}(F_*^{e_1} R) = F_*^{e'} N_{1,0} \subseteq F_*^{e'} N_{1,1} \subseteq \cdots \subseteq F_*^{e'} N_{1,n} = F_*^{e'}(R^{\oplus p^{e_1\gamma(R)}}).$$

Since $F_*^{e'} N_{1,i}/F_*^{e'} N_{1,i-1} \cong F_*^{e'}(R/P_{1,i})$, we apply Lemma 3.2 to know there is a prime filtration of each $F_*^{e'} N_{q,i-1} \subseteq F_*^{e'} N_{1,i}$ whose prime factors come from $\mathcal{S}(R/P_{1,i})$ and such a prime factor appears no more than $C_i p^{e\gamma(R/P_{1,i})} \leqslant C' p^{e(\gamma(R)-1)}$ times in the filtration. Putting all of this information together we get an embedding $F_*^e R \subseteq$

$R^{\oplus p^{e\gamma(R)}}$, which is an element of $\mathscr{D}_e^{\oplus p^{e\gamma(R)}}$ by Remark 3.15, with a prime filtration whose prime factors come from $\mathcal{S}(R, \mathscr{D})$, and such a prime factor appears no more than the following number in the filtration,

$$
\begin{aligned}
p^{e_1\gamma(R)}C'p^{e'\gamma(R)}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{e'}}\right)+\sum_{i=1}^{h}C_i q^{\gamma(R/P_{1,i})} \\
\leqslant C'p^{e\gamma(R)}(1+\frac{1}{p}+\cdots+\frac{1}{p^{e'}}+\frac{1}{p^{e}}) \leqslant 2C'p^{e\gamma(R)}.
\end{aligned}
$$

$\square$

CHAPTER 4

# Uniform Convergence and Semi-Continuity Results

## 1. Hilbert-Kunz multiplicity

A map of primary ideals will be an assignment $I(-)$ of each prime ideal $P \in \operatorname{Spec}(R)$ to a $PR_P$-primary ideal $I(P)R_P$. Using the uniform bounds of Chapter 3, we are able to describe convergence rates of functions $\operatorname{Spec}(R) \to \mathbb{R}$ defined by $P \mapsto \ell(M_P/I(P)^{[p^e]}M_P)/p^{e\dim(M_P)}$ where $M$ is a finitely generated $R$-module.

To condense notation, given a finitely generated $R$-module $M$ and map of primary ideals $I(-)$, we let $\ell_e(I(-))$ denote the real-valued function on $\operatorname{Spec}(R)$ defined by $P \mapsto \ell_e^M(I(P)) := \ell(M_P/I(P)^{[p^e]}M_P)/p^{e\dim(M_P)}$. We let $\ell_e(I(-)) = \ell_e^R(I(-))$ and observe that $\ell_0(I(P)) = \ell(R_P/I(P)R_P)$. In particular, if $I(P)$ is the map of primary ideals defined by $I(P) = P$ for each $P \in \operatorname{Spec}(R)$, then $\ell_0(I(P)) = 1$ for each $P \in \operatorname{Spec}(R)$.

Theorem 4.1. *Let $R$ be a ring of prime characteristic $p$ which is either F-finite or essentially of finite type over an excellent local ring and let $M$ be a finitely generated $R$-module. Let $I(-)$ be a map of primary ideals. The sequence of functions $\frac{\ell_e^M(I(-))}{\ell_0(I(-))}$ : $\operatorname{Supp}(M) \to \mathbb{R}$, which sends a prime $P \in \operatorname{Supp}(M)$ to $\frac{\ell(M_P/I(P)^{[p^e]}M_P)}{p^{e\dim(M_P)}\ell(R_P/I(P)R_P)}$, converges uniformly to the scaled Hilbert-Kunz multiplicity function $\frac{\mathrm{e_{HK}}(I(-),M_-)}{\ell_0(I(-))}$, which sends a prime $P \in \operatorname{Supp}(M)$ to $\frac{\mathrm{e_{HK}}(I(P),M_P)}{\ell(R_P/I(P)R_P)}$ as $e \to \infty$.*

Proof. Given $\varepsilon > 0$, our goal is to show that there exists an $e' \in \mathbb{N}$ such that for all $P \in \operatorname{Supp}(M)$ and for all $e \geqslant e'$, $|\frac{1}{\ell(R_P/I(P)R_P)}\ell_e^M(I(P)) - \frac{1}{\ell(R_P/I(P)R_P)}\mathrm{e_{HK}}(I(P),M_P)| < \varepsilon$. After modding out $R$ by $\operatorname{Ann}_R M$, it follows by Theorem 3.10 and Theorem 3.14 that there exists a constant $C > 0$ such that, for all $P \in \operatorname{Supp}(M)$ and for all

$e_1, e_2 \in \mathbb{N}$,

$$\left|\ell\left(\frac{M_P}{I(P)^{[p^e]}M_P}\right)p^{e_2\dim(M_P)} - \ell\left(\frac{M_P}{I(P)^{[p^{e_1+e_2}]}M_P}\right)\right|$$

$$\leqslant Cp^{e_2\dim(M_P)}p^{e_1(\dim(M_P)-1)}\ell\left(\frac{R_P}{(I(P)+\operatorname{Ann}_R(M))R_P}\right)$$

$$\leqslant Cp^{e_2\dim(M_P)}p^{e_1(\dim(M_P)-1)}\ell\left(\frac{R_P}{I(P)R_P}\right).$$

Dividing both sides of the inequality by $p^{e_2\dim(M_P)}$, letting $e_2 \to \infty$, and applying Lemma 2.15, gives that for all $P \in \operatorname{Supp}(M)$ and for all $e = e_1$,

$$\left|\ell\left(\frac{M_P}{I(P)^{[p^e]}M_P}\right) - p^{e\dim(M_P)}\,\mathrm{e}_{\mathrm{HK}}(I(P), M_P)\right| \leqslant Cp^{e(\dim(M_P)-1)}\ell\left(\frac{R_P}{I(P)R_P}\right).$$

Choose $e'$ large enough that $\frac{C}{p^{e'}} < \varepsilon$ and let $e \geqslant e'$. Dividing the above inequality by $p^{e\dim(M_P)}\ell(R_P/I(P)R_P)$ gives that for all $P \in \operatorname{Supp}(M)$ and all $e \geqslant e'$,

$$\left|\frac{\ell_e^M(I(P))}{\ell(R_P/I(P)R_P)} - \frac{\mathrm{e}_{\mathrm{HK}}(I(P), M_P)}{\ell(R_P/I(P)R_P)}\right| \leqslant \frac{C}{p^e} < \varepsilon.$$

$\square$

Let $e \in \mathbb{N}$ and set $f_e(P) = \frac{1}{p^{e\dim(M_P)}}\ell(M_P/I(P)^{[p^e]}M_P)$ and let $f$ be the limit function $f(P) = \mathrm{e}_{\mathrm{HK}}(I(P), M_P)$. Theorem 4.1 implies there exists a strictly positive function $g : \operatorname{Spec}(R) \to \mathbb{R}$, namely $g(P) = \frac{1}{\ell(R_P/I(P)R_P)}$, which does not depend on $e$, such that $gf_n$ converges uniformly to the function $gf$. If there exists a $\delta > 0$ such that for all $P \in \operatorname{Spec}(R)$ $g(P) \geqslant \delta$, then $f_e$ converges uniformly to $f$. To see this, choose $e$ so large so that for all $|gf_e - gf| < \varepsilon\delta$. Then $|f_e - f| < \varepsilon\delta/g \leqslant \varepsilon\delta/\delta = \varepsilon$. Using this observation we obtain the following corollary to Theorem 4.1.

Corollary 4.2. *Let $R$ be a ring of prime characteristic $p > 0$ which is either F-finite or essentially of finite type over an excellent local ring and let $M$ be a finitely generated $R$-module. Let $I(-)$ be a map of primary ideals. Suppose that there exists an $e_0$ such* *that* $P^{[p^{e_0}]} \subseteq I(P)$ *for all* $P \in \operatorname{Supp}(M)$*, or more generally there exists a constant* $D$ *such that* $\ell(R_P/I(P)R_P) \leqslant D$ *for all* $P \in \operatorname{Supp}(M)$*. Then the sequence of functions* $\ell_e(I(-)) : \operatorname{Supp}(M) \to \mathbb{R}$*, which sends a prime* $P$ *to* $\frac{1}{p^{e \dim(M_P)}}\ell(M_P/I(P)^{[p^e]}M_P)$*, converges uniformly to the Hilbert-Kunz multiplicity function* $\mathrm{e}_{\mathrm{HK}}(I(-), M_{-})$*, which sends a prime* $P$ *to* $\mathrm{e}_{\mathrm{HK}}(I(P), M_P)$*.*

Proof. By the above remarks we only need to find $\delta > 0$ such that for all $P \in \operatorname{Supp}(M)$, $\frac{1}{\ell(R_P/I(P)R_P)} \geqslant \delta$, or equivalently that there exists a $D$ such that for all $P \in \operatorname{Supp}(M)$, $\ell(R_P/I(P)R_P) \leqslant D$. We are assuming that for each $P \in \operatorname{Spec}(R)$ that $P^{[p^{e_0}]} \subseteq I(P)$. Hence by Lemma 3.7 and Lemma 3.13 there exists a constant $C$ such that for all $P \in \operatorname{Supp}(M)$, $\ell(R_P/I(P)R_P) \leqslant \ell(R_P/P^{[p^{e_0}]}R_P) \leqslant Cp^{e_0 \operatorname{ht}(P)} \leqslant Cp^{e_0 \dim(R)}$. Therefore $D = Cp^{e_0 \dim(R)}$ works. $\square$

Corollary 4.2 gives an alternative proof of Smirnov's result that if $R$ is F-finite or essentially of finite type over an excellent local ring, then $\mathrm{e}_{\mathrm{HK}}(-)$ is upper semi-continuous at primes $P$ such that $R_P$ is equidimensional.

Corollary 4.3. *Let* $R$ *be either F-finite or essentially of finite type over an excellent local ring. Then the Hilbert-Kunz function* $\mathrm{e}_{\mathrm{HK}}(-) : \operatorname{Spec}(R) \to \mathbb{R}_{\geqslant 1}$ *which sends a prime* $P \mapsto \mathrm{e}_{\mathrm{HK}}(R_P)$*, is upper semi-continuous at all* $P \in \operatorname{Spec}(R)$ *such that* $R_P$ *is equidimensional.*

Proof. Consider the map of primary ideals $I(-)$ which sends a prime $P$ to $P$. Corollary 4.2 says that $\ell_e(-)$ converges uniformly to $\mathrm{e}_{\mathrm{HK}}(-)$. E. Kunz originally showed in [**Kun76**] that for each $e \in \mathbb{N}$ the function $\ell_e(-)$ which sends a prime $P \mapsto \frac{1}{p^{e \operatorname{ht}(P)}}\ell(R_P/P^{[p^e]}R_P)$, is upper semi-continuous on all rings which are locally equidimensional. If $R_P$ is equidimensional, then $R$ being catenary implies that there is an $s \in R - P$ such that $R_s$ is locally equidimensional. The $s$ which works is 1 if $\min(R_P) = \min(R)$. If $\min(R_P) \subsetneq \min(R)$, then just choose $s \in \cap_{Q \in \min(R) - \min(R_P)} Q \setminus P$. Therefore, if $R_P$ is equidimensional, then in an open neighborhood of $P$, $\mathrm{e}_{\mathrm{HK}}(-)$

is the uniform limit of upper semi-continuous functions, hence $\mathrm{e_{HK}}(-)$ is upper semi-continuous as well. $\square$

## 2. F-signature

Let $R$ be a ring of prime characteristic $p$. For each $P \in \operatorname{Spec}(R)$ let $E_{R_P}(k(P))$ be the injective hull of the residue field of the local ring $(R_P, PR_P, k(P))$ and let $u_P \in E_{R_P}(k(P))$ generate the socle $0 :_{E_{R_P}(k(P))} PR_P$. For each $e \in \mathbb{N}$ and $P \in \operatorname{Spec}(R)$ let $I_e(P) = \{r \in R_P \mid u_P \otimes F^e_* r = 0 \in E_{R_P}(k(P)) \otimes_{R_P} F^e_* R_P\}$. Let $s_e : \operatorname{Spec}(R) \to \mathbb{R}$ be the function defined by $P \mapsto s_e(R_P) := \frac{1}{p^{e\,\mathrm{ht}(P)}}\ell(R_P/I_e(P)R_P)$ and let $b_e : \operatorname{Spec}(R) \to \mathbb{R}$ be the function defined by $P \mapsto b_e(R_P) := p^{e\,\mathrm{ht}(P)} s_e(R_P)$. Observe that the functions $s_e$ converge to the F-signature function $s : \operatorname{Spec}(R) \to \mathbb{R}$ defined by $P \mapsto s(R_P)$, the F-signature of $R_P$, as $e \to \infty$. As with Hilbert-Kunz multiplicity, we will show that if $R$ is F-finite or essentially of finite type over an excellent local ring, then the functions $s_e$ converge uniformly to their limit as $e \to \infty$.

THEOREM 4.4. *Let $R$ be either F-finite or essentially of finite type over an excellent local ring. There exists a constant $C$ such that, for all $P \in \operatorname{Spec}(R)$, and for all $e_1, e_2 \in \mathbb{N}$,*

$$|b_{e_1}(R_P)p^{e_2\,\mathrm{ht}(P)} - b_{e_1+e_2}(R_P)| \leqslant C p^{e_2\,\mathrm{ht}(P)} p^{e_1(\mathrm{ht}(P)-1)}.$$

PROOF. It is well known that if $b_e(P) > 0$ for some, equivalently for all, $e$, then $R_P$ is a reduced ring. Therefore $C = 0$ is a constant which works for all $P \in \operatorname{Spec}(R)$ such that $R_P$ is not reduced. If $R_P$ is reduced there exists an $s \in R - P$ such that $R_s$ is reduced. Therefore by quasi-compactness of $\operatorname{Spec}(R)$, we may reduce our considerations to when $R$ is a reduced ring. The theorem follows by Lemma 2.27, Theorem 3.10, and Theorem 3.14. $\square$

THEOREM 4.5. *Let $R$ be either F-finite or essentially of finite type over an excellent local ring. The eth normalized Frobenius splitting number function, which maps a prime $P \mapsto s_e(R_P)$, converges uniformly to the F-signature function, which maps a prime $P \mapsto s(R_P)$ as $e \to \infty$.*

Proof. Let $\varepsilon > 0$, let $C$ be as in Theorem 4.4, and choose $e$ so large that $\frac{C}{e} < \varepsilon$. Then for all $P \in \operatorname{Spec}(R)$ we have that

$$|b_{e_1}(R_P)p^{e_2 \operatorname{ht}(P)} - b_{e_1+e_2}(R_P)| \leqslant Cp^{e_2 \operatorname{ht}(P)}p^{e_1(\operatorname{ht}(P)-1)}.$$

Therefore

$$\left| b_{e_1}(R_P) - p^{e_1 \operatorname{ht}(P)} \frac{b_{e_1+e_2}(R_P)}{(p)^{(e_1+e_2)\operatorname{ht}(P)}} \right| \leqslant Cp^{e_1(\operatorname{ht}(P)-1)}.$$

Letting $e_2 \to \infty$ we have that for all $P \in \operatorname{Spec}(R)$ that

$$|b_{e_1}(R_P) - p^{e_1 \operatorname{ht}(P)} s(R_P)| \leqslant Cp^{e_1(\operatorname{ht}(P)-1)}.$$

Hence for all $e_1 \geqslant e$ and all $P \in \operatorname{Spec}(R)$,

$$\left| \frac{b_{e_1}(R_P)}{p^{e_1 \operatorname{ht}(P)}} - s(R_P) \right| \leqslant \frac{C}{e_1} \leqslant \frac{C}{e} < \varepsilon.$$

This verifies that $\frac{b_e(R_P)}{e^{\operatorname{ht}(P)}} = s_e(R_P)$ converges uniformly to $s(R_P)$ as $e \to \infty$. $\square$

Enescu and Yao proved that the functions $s_e : \operatorname{Spec}(R) \to \mathbb{R}$ are lower semi-continuous, provided $R$ is locally equidimensional and is either F-finite, essentially of finite type over an excellent local ring, is excellent and Gorenstein, or is the homomorphic image of an excellent ring. We explicitly state their theorem in the F-finite and and essentially of finite type over an excellent local ring cases.

Theorem 4.6 ([**EY11**]). *Let $R$ be a locally equidimensional ring of prime characteristic $p$ which is either F-finite or essentially of finite type over an excellent local ring. Then the functions $s_e : \operatorname{Spec}(R) \to [0,1]$ are lower semi-continuous.*

Theorem 4.7. *Let $R$ be either F-finite or essentially of finite type over an excellent local ring. The F-signature function on $\operatorname{Spec}(R)$ is lower semi-continuous.*

Proof. Let $\varepsilon > 0$ and let $P \in \operatorname{Spec}(R)$. If $s(R_P) = 0$ then it is the case that for all $Q \in \operatorname{Spec}(R)$, that $s(R_P) - s(R_Q) \leqslant 0 < \varepsilon$. If $P \in \operatorname{Spec}(R)$, then $s(R_P) > 0$ if and only if $R_P$ is strongly F-regular by Theorem 2.24. In particular we have that $R_P$

is a domain. By Theorem 2.27 and Theorem 4.7, we have that in a neighborhood of $P$, the F-signature function is the uniform limit of lower semi-continuous functions, hence itself is lower semi-continuous at $P$. $\square$

## 3. F-signature of Pairs

We establish lower semi-continuity for F-signature of pairs. Throughout this section $R$ will denote an F-finite ring and $\mathscr{D}$ will represent a Cartier subalgebra. If $W \subseteq R$ is a multiplicatively closed set then we let $\mathscr{D}_W$ be the induced Cartier subalgebra on $R_W$ whose $e$th graded piece is $(\mathscr{D}_W)_e := (\mathscr{D}_e)_W$. Our goal is to show that the function $s(-, \mathscr{D}) : \operatorname{Spec}(R) \to \mathbb{R}$ defined by $P \mapsto s(R_P, \mathscr{D}_P)$ is lower semi-continuous. This result is a generalization of Theorem 4.7 in the F-finite case as the F-signature function is the F-signature function of the pair $(R, \mathscr{C}^R)$ where $\mathscr{C}^R$ denotes the total Cartier algebra.

Recall that for each $P \in \operatorname{Spec}(R)$, $s(R_P, \mathscr{D}_P) \in [0, 1]$. Hence the F-signature function is trivially lower semi-continuous at all points $P \in \operatorname{Spec}(R)$ for which $s(R_P, \mathscr{D}_P) = 0$. Moreover, if $s(R_P, \mathscr{D}_P) > 0$ then $s(R_P) > 0$ and $R_P$ is strongly F-regular by Theorem 2.24. In particular, $R_P$ is a domain. Lower semi-continuity is a local condition. Thus when establishing lower semi-continuity for the F-signature of pairs at $P \in \operatorname{Spec}(R)$ we may assume the ambient F-finite ring is a domain. Therefore throughout the rest of this section we assume $R$ is an F-finite domain.

For each $e \in \mathbb{N}$ we let $a_e(-, \mathscr{D}) : \operatorname{Spec}(R) \to \mathbb{R}$ be the function defined by $P \mapsto a_e(R_P, \mathscr{D}_P)$. Let $s_e(-, \mathscr{D}) = \frac{1}{p^{e\gamma(R)}} a_e(-, \mathscr{D})$. The sequence of functions $s_e(-, \mathscr{D})$ converge to $s(-, \mathscr{D})$ as $e \in \Gamma_{\mathscr{D}} \to \infty$. To establish lower semi-continuity of the function $s(-, \mathscr{D})$, we first show that the functions $s_e(-, \mathscr{D})$ are lower semi-continuous. We will not show the functions $s_e(-, \mathscr{D})$ converge uniformly to $s(-, \mathscr{D})$. Instead, we show for each $\varepsilon > 0$, there exists $e_0 \in \mathbb{N}$ so that for each $e \geqslant e_0$ and $P \in \operatorname{Spec}(R)$, $s_e(R_P, \mathscr{D}_P) - s(R_P, \mathscr{D}_P) < \varepsilon$. It is then an easy exercise to verify that the lower semi-continuity of the functions $s_e(-, \mathscr{D})$ imply the lower semi-continuity of the function

$s(-,\mathscr{D})$. One should compare this technique of establishing lower semi-continuity of $s(-,\mathscr{D})$ with those used in Theorem 4.7.

We begin by establishing lower semi-continuity of the functions $a_e(-,\mathscr{D})$ and $s_e(-,\mathscr{D})$. The proof technique we use is similar to the technique used by Enescu and Yao in [**EY11**] to establish Theorem 4.6 in the F-finite case.

LEMMA 4.8. *Let $R$ be an F-finite domain and $\mathscr{D}$ a Cartier subalgebra. Then for each $e \in \mathbb{N}$ the functions $a_e(-,\mathscr{D})$ and $s_e(-,\mathscr{D})$ are lower semi-continuous.*

PROOF. The function $s_e(-,\mathscr{D})$ is obtained by dividing the function $a_e(-,\mathscr{D})$ by a constant. Hence lower semi-continuity of $a_e(-,\mathscr{D})$ is equivalent lower semi-continuity of $s_e(-,\mathscr{D})$. To establish lower semi-continuity of $a_e(-,\mathscr{D})$ it is enough to show that for each $r \in \mathbb{R}$, $\{Q \in \operatorname{Spec}(R) \mid a_e(R_Q,\mathscr{D}_Q) > r\}$ is an open set.

Let $r \in \mathbb{R}$ and $P \in \{Q \in \operatorname{Spec}(R) \mid a_e(R_Q,\mathscr{D}_Q) > r\}$. Then $F^e_* R_P \cong R_P^{\oplus a_e(R_P,\mathscr{D}_P)} \oplus M_P$ is such that each of the $a_e(R_P,\mathscr{D}_P)$ projections $F^e_* R_P \to R_P$ is an element of $(\mathscr{D}_P)_e$. It follows that there is an $s \in R - P$ such that $F^e_* R_s \cong R_s^{\oplus a_e(R_P,\mathscr{D}_P)} \oplus M_s$ and each of the $a_e(R_P,\mathscr{D}_P)$-projections $F^e_* R_s \to R_s$ is an element of $(\mathscr{D}_s)_e$. Hence for all $P' \in D(s)$, $a_e(R_{P'},\mathscr{D}_{P'}) \geqslant a_e(R_P,\mathscr{D}_P)) > r$ and $\{Q \in \operatorname{Spec}(R) \mid a_e(R_Q,\mathscr{D}_Q) > r\}$ is indeed an open set. $\square$

THEOREM 4.9. *Let $R$ be an F-finite domain and $\mathscr{D}$ a Cartier subalgebra of $R$. Then the F-signature function which sends $P \in \operatorname{Spec}(R)$ to $s(R_P,\mathscr{D}_P)$ is lower semi-continuous.*

PROOF. Let $C, \mathcal{S}(R)$ be as in Proposition 3.16. Let $e_1 \in \Gamma_{\mathscr{D}}$ and let $S(e_1)$ be the cokernel of the inclusion $F^{e_1}_* R \xrightarrow{\varphi_{e_1}} R^{\oplus p^{e_1\gamma(R)}}$ described in Proposition 3.16. Then we have the following short exact sequences

$$0 \to F^{e_1}_* R \to R^{\oplus p^{e_1\gamma(R)}} \to S(e_1) \to 0.$$

By part 3 of Lemma 2.30, if $e_2 \in \Gamma_{\mathscr{D}}$, $\varphi_{e_1}$ must map $F_*^{e_1} I_{e_1+e_2}^{\mathscr{D}}(P)$ into $I_{e_2}^{\mathscr{D}}(P) R_P^{\oplus p^{e_1\gamma(R)}}$. Hence we have right exact sequences

$$\frac{F_*^{e_1} R_P}{F_*^{e_1} I_{e_1+e_2}^{\mathscr{D}}(P)} \to \frac{R_P^{\oplus p^{e_1\gamma(R)}}}{I_{e_2}^{\mathscr{D}}(P) R_P^{\oplus p^{e_1\gamma(R)}}} \to \tilde{S}(e_1) \to 0,$$

and $\tilde{S}(e_1)$ is the homomorphic image of $S(e_1)_P / I_{e_1}^{\mathscr{D}}(P) S(e_1)_P$. Length is additive on exact sequences. Thus by parts (2) and (4) of Lemma 2.30 and Corollary 2.4,

$$\begin{aligned}\frac{a_{e_2}(R_P, \mathscr{D}_P)}{p^{e_2(\gamma(R)-\mathrm{ht}(P))}} p^{e_1\gamma(R)} - \frac{a_{e_1+e_2}(R_P, \mathscr{D}_P)}{p^{(e_1+e_2)(\gamma(R)-\mathrm{ht}(P))}} p^{e_1(\gamma(R)-\mathrm{ht}(P))} \\ \leqslant \ell(\tilde{S}(e_1)_P) \leqslant \ell\left(\frac{S(e_1)_P}{I_{e_2}^{\mathscr{D}}(P) S(e_1)_P}\right) \\ \leqslant \ell\left(\frac{S(e_1)_P}{P^{[p^{e_2}]} S(e_1)_P}\right).\end{aligned}$$

By Proposition 3.7 there is a constant $C_1$, independent of $P, e_2$, such that

$$\max_{Q \in \mathcal{S}(R)} \ell\left(R_P/(Q + P^{[p^{e_2}]})R_P\right) \leqslant C_1 p^{e_2(\mathrm{ht}(P)-1)}.$$

Therefore

$$\begin{aligned}\frac{a_{e_2}(R_P, \mathscr{D}_P)}{p^{e_2(\gamma(R)-\mathrm{ht}(P))}} p^{e_1\gamma(R)} - \frac{a_{e_1+e_2}(R_P, \mathscr{D}_P)}{p^{e_2(\gamma(R)-\mathrm{ht}(P))}} \leqslant \ell\left(\frac{S(e)_P}{P^{[p^{e_2}]} S(e_1)_P}\right) \\ \leqslant C C_1 |\mathcal{S}(R)| p^{e_1\gamma(R)} p^{e_2(\mathrm{ht}(P)-1)}.\end{aligned}$$

Dividing through the inequality by $p^{e_1\gamma(R)} p^{e_2 \,\mathrm{ht}(P)}$ shows that

$$s_{e_2}(R_P, \mathscr{D}_P) - s_{e_1+e_2}(R_P, \mathscr{D}_P) \leqslant \frac{C C_1 |\mathcal{S}(R)|}{p^{e_2}}.$$

Letting $e_1 \in \Gamma_{\mathscr{D}} \to \infty$, and relabeling constants, shows that there is a constant $C$, independent of $P \in \mathrm{Spec}(R)$ and $e \in \Gamma_{\mathscr{D}}$ such that

$$s_e(R_P, \mathscr{D}_P) - s(R_P, \mathscr{D}_P) < \frac{C}{p^e}.$$

Given $\varepsilon > 0$ we may choose $e_0 \in \Gamma_{\mathscr{D}}$ so that for all $e \geqslant e_0$, $\frac{C}{p^e} < \varepsilon$. Lower semi-continuity of the function $s(-, \mathscr{D})$ follows from Lemma 4.8. $\square$

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## Vita

Thomas Polstra is from the Atlanta, Georgia area. He received a Bachelor of Science in Mathematics at Georgia State University under the direction of Florian Enescu. Thomas Polstra is married to Jennifer Polstra. Thomas and Jennifer are soon to be parents and are expecting their first child June of 2017. Besides mathematics, Thomas enjoys cycling, running, playing guitar, and spending time with his three dogs, Marla, Brando, and Vito.