Large-amplitude solitary water waves with discontinuous vorticity

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

Large-amplitude solitary water waves with discontinuous vorticity

presented by Adelaide Akers, a candidate for the degree of Doctor of Philosophy of

Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

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Professor Yuri Latushkin

Professor Stephen Montgomery-Smith

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Dedicated to Luke and Zella Akers

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ABSTRACT

Consider a two-dimensional body of water with constant density which lies below a vacuum. The ocean bed is assumed to be impenetrable, while the boundary which separates the fluid and the vacuum is assumed to be a free boundary. Under the assumption that the vorticity is only bounded and measurable, we prove that for any upstream velocity field, there exists a continuous curve of large-amplitude solitary wave solutions. This is achieved via a local and global bifurcation construction of weak solutions to the elliptic equations which constitute the steady water wave problem. We also show that such solutions possess a number of qualitative features; most significantly that each solitary wave is a symmetric, monotone wave of elevation.

CHAPTER 1

Preliminaries

1.1. The governing equations

The governing equations for a fluid in motion were initially developed by Euler in the 1750s and, due to progress made in subsequent centuries by Cauchy [**10**] [**11**], Laplace [**18**] [**41**], Lagrange [**35**] [**18**] [**19**], Poisson [**44**], and Stokes [**50**] [**49**] [**51**] [**52**], the study of water waves has contributed to the historical development of mathematics as a whole. Euler's equations continue to be the basis for models used by present day mathematicians and engineers, but the high nonlinearity of the equations and the difficulties posed by this classical free boundary problem (when the exact shape of the fluid domain is unknown) inspire the development of new methodologies and sharper computational approaches.

1.1.1. Derivation of the governing equations. In this section, we derive the basic equations for water waves, including the boundary conditions which are a key feature of their analytical study. This section is meant to be heuristic in nature, so at the outset we make the overarching assumption of appropriate regularity that is necessary to perform these calculations. We make these more precise in the next section.

Our derivation of the governing equations for inviscid fluid motion is based on two principles:

(P1) **Conservation of mass**. Mass is neither created nor destroyed;

(P2) **Conservation of momentum** (Newton's second law of motion). The rate of change of the fluid momentum equals the amount of force applied to it.

To simplify our analysis, we first make the following underlying assumptions: we will think of the body of water as a continuum via a *macroscopic viewpoint* rather than a microscopic lens in which the fluid is made up of discrete molecules. Mathematically, this means that the fluid is taken to be comprised of a continuum of infinitesimal particles. We may therefore identify a body of water at a fixed time with a region in space. Additionally, we will consider only *two-dimensional flow* and *constant density* ρ throughout the fluid domain.

Given the above assumptions, we fix a Cartesian coordinate system (x, y) and suppose the fluid is contained within a region $\Omega \subset \mathbb{R}^2$. The trajectory of a particle of fluid is the curve traced out by the particle as time progresses. The position of a particle of fluid at time t is given by $\mathbf{x}(t) := (x(t), y(t))^{\mathrm{T}}$, and its motion is prescribed by the velocity field $\mathbf{u} : \mathbb{R} \times \Omega \to \mathbb{R}^2$, where $\mathbf{u}(t, \mathbf{x}) := (u, v)^{\mathrm{T}}$. If the particle starts at position $\mathbf{x_0} := (x(0), y(0))^{\mathrm{T}}$, then the trajectory of the particle at time t is given by the solution to the system of differential equations

$$(1.1) \qquad \frac{d\mathbf{x}}{dt} = \mathbf{u}(t, \mathbf{x}(t)), \qquad \mathbf{x}(0) = (x_0, y_0)^{\mathrm{T}}.$$

We begin by considering an arbitrary subregion Ω' of Ω, whose boundary is the C^1- surface $\partial\Omega'$. The total mass of the fluid contained within Ω' at time t is

$$m(t) := \int_{\Omega'} \rho \, d\mathbf{x}.$$

Next, we examine the rate of change of mass $m(t)$. By the principle of conservation of mass (P1), m increases or decreases depending only on the normal flux of particles through $\partial\Omega'$. To compute the total mass of fluid entering or leaving $\partial\Omega'$ at time t, we consider a small area patch dS on $\partial\Omega'$, which has outward unit normal $\mathbf{n}$. Then the total mass of fluid flowing out of Ω' through dS per unit time is

$$m'(t) := \int_{\partial\Omega'} \rho\, \mathbf{u} \cdot \mathbf{n}\, dS.$$

The conservation of mass is now equivalent to (in integral form)

$$\frac{d}{dt} \int_{\Omega'} \rho\, d\mathbf{x} = -\int_{\partial\Omega'} \rho(\mathbf{u} \cdot \mathbf{n})\, dS. \tag{1.2}$$

The rate of change of the total mass inside Ω' equals the total rate of change of mass density inside Ω', which gives

$$\frac{d}{dt} \int_{\Omega'} \rho\, d\mathbf{x} = \int_{\Omega'} \frac{\partial \rho}{\partial t}\, d\mathbf{x} = 0, \tag{1.3}$$

due to the homogeneity assumption (constant density). Additionally, the divergence theorem yields

$$\int_{\Omega'} \nabla \cdot (\rho \mathbf{u})\, d\mathbf{x} = \int_{\partial\Omega'} (\rho \mathbf{u}) \cdot \mathbf{n}\, dS. \tag{1.4}$$

Using (1.3) and (1.4), we know that this rate of change is zero, hence

$$0 = \int_{\Omega'} \nabla \cdot (\rho \mathbf{u})\, d\mathbf{x}. \tag{1.5}$$

Lastly, we use the fact that Ω' was arbitrary to deduce the *differential form of the law of conservation of mass*, which applies pointwise:

$$\rho(\nabla \cdot \mathbf{u}) = 0. \tag{1.6}$$

This is the first of our two conservation laws. It is important to note that this is equivalent to incompressibility of the flow.

To formulate the conservation of momentum (P2), we must first analyze the forces acting on this body of water. There are two categories of forces acting on a fluid parcel: body forces and stress forces. Gravity is the primary example of a body force, and it takes the form of a constant vector field $\mathbf{g} := (0, -\rho g)^{\mathrm{T}}$, where ρ is the constant

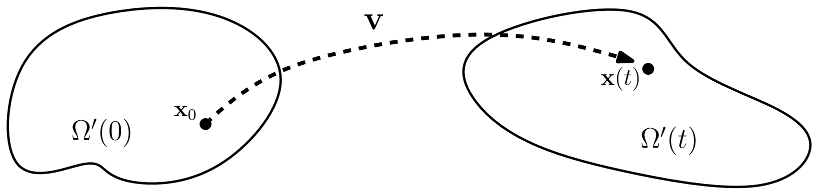

FIGURE 1. Lagrangian flow map $\mathbf{v}$

density and $g > 0$ is the gravitational constant. Stress forces include the pressure field $P(t, \mathbf{x})$, given by the pressure differential $-\nabla P$. For simplicity, we ignore other forces such as viscocity, which is often negligible for water waves.

Suppose $\mathbf{u}$ and P are of class C^1 throughout the fluid and consider an arbitrary subregion $\Omega' \subset \Omega$ at time $t = 0$ as before (with C^1-boundary $\partial\Omega'$). As the fluid flow evolves to some time $t > 0$, let $\Omega'_t := \Omega'(t)$ denote the region of the fluid occupied by particles that originally made up Ω'. The set Ω'_t is the image of Ω' under the *Lagrangian flow map*

$$\mathbf{v} : [0, T) \times \Omega' \to \mathbb{R}^2,$$

solving the ODE

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{u}(t, v(t, \mathbf{x})), \qquad \text{with } v\Big|_{t=0} = \mathrm{id}_{\Omega'},$$

where $\mathbf{v}$ is the location of the particle initially inhabiting position $\mathbf{x}_0$ (see 1). Using the divergence theorem, we calculate that the total force exerted on the fluid inside Ω'_t through the normal stresses exerted across the boundary $\partial\Omega'_t$ is

$$\int_{\partial\Omega'_t} (-P\mathbf{n})\, dS = \int_{\Omega'_t} (-\nabla P)\, d\mathbf{x},$$

where $\mathbf{n}$ is the outward unit normal on $\partial\Omega'_t$. It follows that the total body of force (both body and surface forces) acting on the fluid within Ω'_t is given by

$$\mathbf{F} = \int_{\Omega'_t} (-\nabla P + \mathbf{g})\, d\mathbf{x}. \tag{1.7}$$

Next, we consider the conservation of momentum (density). Letting the position of a fluid particle in Ω at some time t be given by $\mathbf{x}(t) = (x(t), y(t))^{\mathrm{T}} \in \Omega$, we compute that

$$\partial_t \mathbf{u}(t, \mathbf{x}) = \mathbf{u}_t(t, \mathbf{x}(t)) + \mathbf{x}'(t) \cdot (\nabla_{\mathbf{x}} \mathbf{u})(t, \mathbf{x}(t)) =: D_t \mathbf{u},$$

where $\nabla_{\mathbf{x}} := \mathbf{x} \cdot \nabla$ is the directional derivative in the direction of $\mathbf{x}$ and $D_t := \partial_t + \mathbf{u} \cdot \nabla$ is called the *material derivative* or *advective derivative* corresponding to $\mathbf{u}$. The material derivative measures the change of a quantity as it evolves with the flow. Thus, the conservation of momentum (P2) and (1.7) imply that

$$\int_{\Omega_t'} (-\nabla P + \mathbf{g}) \, d\mathbf{x} = \int_{\Omega_t'} \rho D_t \mathbf{u} \, d\mathbf{x}.$$

Dividing by constant density ρ and once again recalling that the volume Ω_t' is arbitrary, we arrive at the Euler equation of motion:

$$D_t \mathbf{u} = -\frac{1}{\rho} \nabla P + (0, -g)^{\mathrm{T}}. \tag{1.8}$$

This completes the second of our two conservation laws.

Combining both conservation laws (P1) and (P2), we have the *incompressible Euler equations*

$$\begin{cases} \nabla \mathbf{u} = 0, \\ D_t \mathbf{u} = -\dfrac{1}{\rho} \nabla P + (0, -g)^{\mathrm{T}}. \end{cases} \tag{1.9}$$

Notice that the pressure P is an additional unknown apart from the velocity $\mathbf{u}$; this together with the nonlinearity of the material derivative of $D_t \mathbf{u}$ poses a great number of difficulties in the analysis of the problem.

1.1.2. Boundary conditions. Next, let us consider the equations governing the motion of the boundary of a fluid domain. As mentioned above, this is one of the most distinctive features of the analysis of water waves. In view of the two-dimensionality

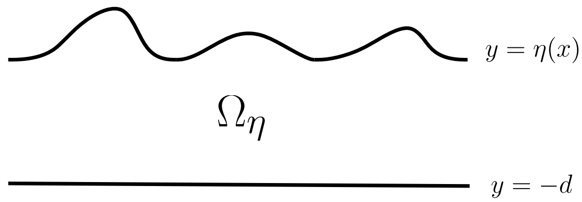

FIGURE 2. Fluid domain Ω_η

assumption, there are two boundaries of the fluid region: the rigid, flat bed and the surface of the water. Since the behavior of the waves at the surface are unknown, the exact shape of the interface between the air and the water is also unknown; we call this boundary a *free surface*.

Due to the two-dimensional assumption, we can describe both the bed and free surface as curves in the plane where x is the horizontal variable and y is the vertical variable. The bed is given by the line $\{y = -d\}$, where $d > 0$ is the depth of the water in its quiescent state; we denote the free surface as the set $S := \{y = \eta(t, x) : \eta > -d\}$, where $\eta : \mathbb{R}^2 \to \mathbb{R}$ is assumed to be smooth and lies strictly above the bed. At time t, the fluid occupies the region (see Figure 2)

$$\Omega_t := \{(x, y) : -d < y < \eta(t, x)\}. \tag{1.10}$$

There are two boundary conditions.

(BC1) **Kinematic boundary condition** Any particle on the boundary remains on the boundary for all time $t > 0$. Specifically, this boundary condition dictates that the boundaries are impermeable—no water particles permeate the air, and no air particles permeate the water.

(BC2) **Dynamic boundary condition** The normal stresses exerted on either side of the surface must be equal.

First, we formulate the kinematic boundary conditions. The impermeability condition indicates that the fluid cannot move out of the fluid domain and into the bed or the air. More precisely, when the water at the interface of a region with which it does not mix, both the water and the surface must have the same velocity normal to the contact surface. We see, then, that a necessary and sufficient condition for impermeability on the bed is

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on} \quad y = -d,$$

where $\mathbf{n}$ is the unit normal vector for the bed. As we are assuming that the bed is perfectly horizontal, the above equation simplifies to

$$v = 0 \quad \text{on} \quad y = -d. \tag{1.11}$$

The kinematic boundary condition on the surface is slightly more complicated. Once again, a necessary and sufficient condition for impermeability is

$$V = \mathbf{u} \cdot \mathbf{n}, \quad \text{on} \quad y = \eta(t, x), \tag{1.12}$$

where V is the normal velocity of S. In this setting, the normal vector to the free surface S is

$$\mathbf{n} = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1)^{\mathrm{T}},$$

and since the normal velocity on the surface is given by $\mathbf{u}(t, x, \eta(t, x)) \cdot \mathbf{n} = \eta_t / \sqrt{1+\eta_x^2}$, it follows that (1.12) becomes

$$-\eta_x u + v = \eta_t. \tag{1.13}$$

Therefore, the kinematic boundary conditions are

$$\begin{cases} v = 0 \quad \text{on} \quad y = -d, \\ -\eta_x u + v = \eta_t \ \text{on} \quad y = \eta. \end{cases} \tag{1.14}$$

The dynamic boundary condition requires that the pressure at the free surface must be equal to the pressure in the air. Assuming the fluid lies below a vacuum, we may approximate the air pressure P by the constant atmospheric pressure P_{atm}; this yields the dynamic boundary condition

$$(1.15) \quad P = P_{\text{atm}} \quad \text{on} \quad y = \eta(t, x).$$

It is important to note that no such condition exists on the bed since there are no stress force interactions there.

Together, the incompressible Euler equations (1.9) and the boundary conditions (1.14) and (1.15), finally furnish the *governing equations for water waves*. The fact that the free surface is an unknown and must be determined as part of the solution presents a distinctive difficulty in the analysis of the problem.

Well-posedness. In order for the water wave problem to be an accurate mathematical model for physical phenomena, it is important to establish the following properties on (at least) a small time interval $[0, T)$ with $T > 0$:

(i) the existence of a solution;

(ii) the uniqueness of solutions;

(iii) the (continuous) dependence of solutions on initial data.

These three conditions are more commonly referred to as *well-posedness* (in the sense of Hadamard) for the Cauchy problem. The issue of well-posedness is nearly a mathematical field in and of itself and is outside the scope of this paper; however we will note that local well-posedness of the water wave problem has been established under certain assumptions on the initial conditions (see, for example, [**62**], [**36**], [**16**], and [**48**]).

1.2. Steady water waves

So far, we have formulated the water wave problem in the following fashion: we consider a two-dimensional, gravity-driven wave in water of constant density ρ. For further simplicity, we will assume that $\rho = 1$. At time t, the fluid occupies the region $\{(x, y) : -d < y < \eta(x, t)\}$, where $\{y = -d\}$ is an impermeable flat bed and the a priori unknown function η is the free surface profile. While there are many types of solutions to the water wave problem, we will focus only on *traveling wave* solutions. These are more commonly known as *steady water waves*, which move at a constant speed $c > 0$ and appear to be stationary when viewed within a moving coordinate frame. More precisely, we begin with the governing equations (1.9), lettting $\mathbf{u}$ and P be functions of $(x - ct, y)$ and η be a function of $x - ct$. We then make the change of coordinates

$$(x - ct, y) \mapsto (x, y),$$

which effectively eliminates the dependence of $\mathbf{u}, P$, and η on time. Thus, in this coordinate system, the wave occupies the (steady) fluid domain (see Figure 2)

$$\Omega_\eta = \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\},$$

and is described mathematically by the incompressible steady Euler equations

$$\text{(1.16a)} \quad \begin{cases} (u - c)u_x + vu_y = -P_x \\ (u - c)v_x + vv_y = -P_y - g \quad \text{in } \Omega_\eta \\ u_x + v_y = 0 \end{cases}$$

where, once again, $(u, v) : \Omega_\eta \to \mathbb{R}^2$ is the velocity vector field, $P : \Omega_\eta \to \mathbb{R}$ is the pressure and g is the gravitational constant. The system is also equipped with the kinematic boundary conditions

$$\text{(1.16b)} \quad v = 0 \text{ on } y = -d, \qquad v = (u - c)\eta_x \text{ on } y = \eta(x)$$

FIGURE 3. Periodic wave

FIGURE 4. Solitary wave

and dynamic boundary condition

(1.16c) $P = P_{\text{atm}}$ on $y = \eta(x)$

with P_{atm} being constant atmospheric pressure. Lastly, we impose a *no (horizontal) stagnation condition*

(1.17) $u < c$,

throughout the fluid, which, in the context of the moving frame, precludes the existence of "stagnant" fluid particles carried with the wave.

The main objective of this thesis is to prove that solutions of (1.16) exist. We distinguish two types of traveling wave solutions: periodic and solitary (see Figures 3 and 4). Periodic solutions are such that the components of the velocity vector field (u, v) and the free surface profile η have period L in the horizontal variable x. A solitary wave, however, decays far away from the crest and are modeled by the following asymptotic conditions: $(u - c, v) \to (U, 0), \eta \to 0$, where U is a (given) smooth background current. The solitary wave phenomenon appears in both nature and experimental simulations, which strongly suggests that it should be possible to rigorously construct mathematically.

1.3. Vorticity for steady waves

An important property of fluid flow is *vorticity*, which measures the local rotation of the fluid and is given by the curl of the velocity,

$$\omega := \nabla \times \mathbf{u}.$$

A water flow which is curl free (i.e., vorticity equal to zero) is called an *irrotational wave*; we call a wave with nontrivial vorticity a *rotational wave*. The vast majority of studies of water waves assume irrotationality. Throughout this work, however, we will concentrate solely on waves with nontrivial vorticity.

Rotational waves occur frequently in nature; in particular, the effects of vorticity are significant for wind-driven waves, shear currents, waves near a ship, and tsunamis approaching a shore. Given the two-dimensional setting, we may define the scalar vorticity as $\omega := v_x - u_y$. The incompressibility (1.6) and the kinematic boundary condition (1.16b), permit us to introduce the *streamline function* ψ defined by

$$\psi_x = -v, \quad \psi_y = u - c, \text{ and } \psi(x, \eta(x)) = 0. \tag{1.18}$$

In this context, the equation (1.16a) in the interior becomes

$$\Delta\psi = -\omega \quad \text{in } \Omega_\eta. \tag{1.19}$$

Furthermore, we prove in Section 2.3.4 that ω is a function of ψ provided there is no stagnation (1.17), and hence

$$\Delta\psi = -\gamma(\psi),$$

for some *vorticity function* γ. The vorticity function γ is highlighted throughout this work—its regularity (or rather lack thereof) in particular, will play a significant role in the analysis.

CHAPTER 2

Solitary water waves with discontinuous vorticity

As outlined in the previous chapter, we know that the classical steady water wave problem concerns solutions of the incompressible Euler equations (1.16). The influence of discontinuous vorticity on these solutions is the main focus of this work.

As in Chapter 1, we consider a two-dimensional inviscid, incompressible fluid of unit density in a gravity-driven flow $\mathbf{u} := (u, v)$ over a horizontal, flat, impermeable bed. Switching to the moving coordinate frame, let

$$\Omega_\eta = \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\}$$

denote the fluid domain. We define solitary waves as traveling waves satisfying the asymptotic conditions

$$\eta \to 0, \qquad v \to 0, \qquad u \to U(y) \qquad \text{as } x \to \pm\infty$$

uniformly in y. Here, $U(y) \in W^{1,\infty}([-d, 0])$ is given arbitrary function, and describes the shear flow at $x = \pm\infty$.

Let us work with a one parameter family of shear flows

$$U(y) = c - FU^*(y)$$

where F a dimensionless parameter called the *Froude number* and U^* is a fixed positive function normalized so that

$$gd^3 = \left(\int_{-d}^{0} U^*(y)\, dy\right)^2 \qquad \text{or} \qquad \frac{1}{F^2} = gd^3 \left(\int_{-d}^{0} (c - U(y))\, dy\right)^{-2}.$$

The function U^* is a rescaling of the relative shear flow at $x = \pm\infty$. The Froude number F is the ratio between inertial and gravitational forces; later, we will uncover the existence of a critical Froude number $F_{\rm cr}$, which will play an important role in identifying the structure of solutions. We will call a solution with $F > F_{\rm cr}$ *supercritical*, and a solution with $F < F_{\rm cr}$ will be called *subcritical*.

Lastly, we define the terms which will be used to describe the qualitative features of the solutions we will construct. A solitary wave is *trivial* if $\eta \equiv 0, v \equiv 0$, and $u \equiv U(y)$. A *laminar* or *shear* flow is a traveling wave whose streamlines are all parallel to the bed. We call a solitary wave *symmetric* if wave has an even axis of symmetry; more precisely, the wave is symmetric if u and η are even and v is odd in the horizontal variable x. A symmetric wave is *monotone* if the height of each streamline (except the one corresponding to the bed) is strictly decreasing from crest to trough. Lastly, we say that a solitary wave is a *wave of elevation* if the height of every streamline (except the one corresponding to the bed) lies above its asymptotic height.

2.1. History

The study of water waves began over two centuries ago. Although many mathematicians had attempted to develop a theory for water waves, it was not until Euler established his governing equations for the fluid that the mathematical community began to form an intense interest. The theory was influenced by many famous mathematicians throughout the 19th century, mainly in the irrotational case. For example, Airy [**1**] formulated the dispersion relation for the wave velocity for irrotational periodic waves with finite depth. Stokes, too, contributed to the theory of irrotational periodic water waves; most significant was his conjecture that steady water waves limit to an extreme wave with a crest angle of 120° [**51**] (see [**17**] for a more comprehensive history).

The early 20th century gave rise to the first constructions of steady periodic irrotational water waves. The first of these results were due to Nekrasov [**43**] in 1921 and Levi-Civita [**38**] in 1924, who independently established the existence of small-amplitude irrotational periodic waves of infinite depth using conformal mappings. Struik (a student of Levi-Civita) extended these results to waves of finite depth [**54**]. It was not until the 1960s and 1970s that analogous results for large-amplitude irrotational periodic waves were obtained by Krasovskiĭ [**33**] and Keady and Norbury [**28**]. Amick, Fraenkel, and Toland [**2**], constructed large-amplitude periodic waves and finally proved Stokes' extreme wave conjecture in 1980 (for a more detailed history, see, for example [**53**]). Each of these results were achieved in the absence of vorticity. In this context, (1.19) implies that the stream function is harmonic, which allows one to use powerful techniques from complex analysis such as conformal mappings to fix the domain.

The presence of vorticity, however, prevents ψ from being harmonic and hence prohibits reduction of the problem to an integral equation on the boundary, as was done in the irrotational cases. Even so, the existence of small-amplitude periodic water waves with non-zero vorticity was first established via the use of a non-conformal change of coordinates formulated by Dubreil-Jacotin in 1934 [**20**]. Extending the Dubreil-Jacotin results, much later Constantin and Strauss [**14**] constructed large-amplitude periodic rotational waves, with $\gamma \in C^1$, using degree-theoretic global bifurcation (due to Healey and Simpson [**25**]). For each of the above works, the periodic solutions are subcritical.

Solitary waves were first observed by Scottish engineer and shipbuilder John Scott Russell in 1834 [**46**]. Riding along the Edinburgh canal, he came upon a peculiar wave comprised of a single crest and exhibiting decay far away from the crest. This wave propagated without changing in shape or size (when viewed in the moving frame of reference of his horse riding along the shore). The mathematical construction of these waves, however, was beyond the scope of Russel's work. In comparison to periodic waves, solitary waves are much more difficult to construct due to issues of compactness. For these waves, the fluid domain is unbounded, which in fact causes the linearized operator not to be Fredholm. This is a serious obstruction that rules out the use of bifurcation theoretic techniques as in the periodic cases.

Nearly a century after Russell's observation, the first constructions of small-amplitude irrotational solitary waves came in the form of long wavelength limits of periodic solutions (see [**37**], [**22**], [**55**]). Beale [**56**] used a generalized implicit function theorem of Nash–Moser type, and later Mielke [**42**] used spatial dynamics techniques. The first large-amplitude irrotational solitary waves were due to Amick and Toland [**3**], who used global bifurcation methods on approximate problems to obtain a connected set of solitary waves. Similar results were proved by Benjamin, Bona, and Bose [**5**]. It is important to note that large-amplitude irrotational solitary waves were constructed using conformal mappings, which transformed problem into an integral equation on the free surface.

The same obstruction persists in the presence of vorticity. Despite this complication, small-amplitude rotational solitary waves were approximated by Burns [**9**] in 1953, Benjamin [**4**] in 1962, and Freeman and Johnson [**21**] in 1970. The first rigorous constructions, however, came in 1960 due to Ter-Krikorov [**55**], who used periodic waves which degenerated into solitary waves. In 2008, Hur [**27**] proved the existence of small-amplitude solitary water waves with arbitrary vorticity function $\gamma \in C^0$, extending Beale's Nash-Moser implicit function method to rotational solitary waves. That same year, a similar small-amplitude existence result was proved independently by Groves and Wahlen [**24**] using spatial dynamics and a center manifold reduction method in the spirit of Mielke [**42**]. They allowed for a general vorticity function $\gamma \in H^1$. Five years later, Wheeler [**59**] proved the existence of large-amplitude solitary waves with γ in the Hölder space $C^{1+\alpha}$ (for $\alpha \in (0,1)$). The solitary water waves in each of the above references (both rotational and irrotational) are supercritical.

It is important to note that the previous works regarding rotational waves each assume some degree of continuity on the vorticity. However, recent numerical simulations in [**31**] and [**32**] indicate that the presence of discontinuous vorticity may give rise to flow patterns which vary drastically from those in the continuous vorticity case. This motivated Constantin and Strauss [**15**] to examine the existence periodic solutions with an arbitrary bounded and potentially discontinuous vorticity. In this work, Constantin and Strauss used Schauder estimates and maximum principles for weak solutions to generalize the global bifurcation techniques from [**14**] and construct periodic solutions for $\gamma \in L^\infty$. To the best of our knowledge, there has been no existence theory for large-amplitude solitary water waves with discontinuous vorticity.

2.2. Statement of results

The purpose of this thesis is to establish the existence and qualitative properties of large-amplitude solitary waves assuming only that the vorticity function γ is bounded and measurable.

Our main result is the following:

THEOREM 2.1 (Existence of large-amplitude solitary waves). *Fix $p \in (2,4)$, wave speed $c > 0$, gravitational constant $g > 0$, asymptotic depth $d > 0$, and positive asymptotic relative velocity $U^* \in W^{1,\infty}([-d,0];\mathbb{R}_+)$. There exists a continuous curve*

$$\mathscr{C} = \{(u(z), v(z), \eta(z), F(z)) : z \in (0,\infty)\}$$

of solitary waves with regularity

$$(u(z), v(z), \eta(z)) \in W^{1,p}(\overline{\Omega(z)}) \times W^{1,p}(\overline{\Omega(z)}) \times W^{2,p}(\mathbb{R}), \tag{2.1}$$

where $\Omega(z)$ denotes the fluid domain corresponding to $\eta(z)$. The solution curve $\mathscr{C}$ has the following properties.

(a) (Extreme wave or infinite mass limit) *Following $\mathscr{C}$, either (i) we encounter waves that are arbitrarily close to having points of (horizontal) stagnation,*

$$\lim_{z\to\infty} \inf_{\Omega(z)} |c - u(z)| = 0, \tag{2.2}$$

or (ii) the excess mass tends to infinity,

$$\lim_{z\to\infty} \int_{\mathbb{R}} \eta(z)(x)\, dx = \infty. \tag{2.3}$$

(b) (Critical laminar flow) *The left endpoint of $\mathscr{C}$ is a critical laminar flow,*

$$\lim_{z\to 0}(u(z), v(z), \eta(z), F(z)) = (c - \frac{1}{F_{cr}^2}U^*, 0, 0, \frac{1}{F_{cr}^2}).$$

(c) (Symmetry and monotonicity) *Every solution in $\mathscr{C}$ is a wave of elevation that is symmetric, monotone, and supercritical.*

The argument leading to Theorem 2.1 is quite long, so let us now discuss briefly the main challenges we must overcome, the machinery we will use, and the overarching structure of the thesis.

Observe that, when γ is merely bounded and measurable, elliptic regularity theory suggests that the corresponding velocity field will, in general, fail to solve the Euler equations (1.16) in the classical sense. We are therefore forced to consider weak solutions of the system and work in the Sobolev spaces (2.1). This fact complicates, to varying degrees, every step of the analysis.

As this is a free boundary problem, we begin by changing coordinates in order to fix the domain. For rotational steady waves without stagnation, the traditional method for doing this is to use the Dubreil-Jacotin transformation (also called semi-Lagrangian variables). However, it is not at all obvious at this level of regularity that this is a valid change of variables. Indeed, confirming the equivalence of the three main formulations of the problem is our first major contribution; see Section 2.3.4.

With this result in hand, we are permitted to apply the Dubreil-Jacotin transformation, which recasts the steady incompressible Euler system as a scalar quasilinear elliptic PDE with fully nonlinear boundary conditions posed on a fixed domain. More precisely, the fluid domain Ω_η is mapped to a fixed infinite strip $R = \mathbb{R} \times (-1, 0)$. For the purposes of this discussion, we can represent the problem as an abstract operator equation of the form $\mathscr{F}(\phi, F) = 0$, where ϕ is a new unknown that describes the deviation of the streamlines from their far-field heights.

At this point, we encounter a second obstacle: the unboundedness of R has serious implications for the compactness properties of the linearized operator $\mathscr{F}_\phi(\phi, F)$. In particular, it is well-known in the literature of solitary waves that $\mathscr{F}_\phi(0, F_{\mathrm{cr}})$ fails to be Fredholm. This means we cannot use a Lyapunov–Schmidt reduction approach to construct small-amplitude waves as was done in the periodic case (see, for example, [**14, 15**]). A similar issue was faced by Wheeler [**59, 61**] and Chen, Walsh, and Wheeler [**13**], who were able to prove that the linearized operator at a *supercritical* wave is in fact Fredholm index 0. However, these authors studied classical solutions with Hölder regularity, and their methods do not directly generalize to the Sobolev space regime. Instead, we obtain an analogous result using an entirely new argument based on the principle of concentration-compactness.

In place of Lyanpunov–Schmidt, we devote Section 2.6 to constructing a family $\mathscr{C}_{\mathrm{loc}}$ of small-amplitude solitary waves using a spatial dynamics approach similar to that of Groves and Wahlén [**24**], as well as Wheeler [**59**]. First, we rewrite the problem once more as an infinite-dimensional Hamiltonian system where the horizontal variable x acts as the time-like variable. At the critical value of the Froude number, 0 is an eigenvalue of algebraic multiplicity 2 for $\mathscr{F}(0, F_{\mathrm{cr}})$ and the rest of the spectrum is bounded away from the imaginary axis. We are therefore able to invoke a variant of the center manifold theorem for quasilinear elliptic equations pioneered by Mielke [**42**] and Kirchgässner [**29, 30**]. This reduces the infinite-dimensional problem to a planar Hamiltonian system that, in fact, is equivalent to the Korteweg–de Vries equation modulo a rescaling. We prove that for every slightly supercritical Froude number, the reduced equation has a homoclinic orbit, and these lift up to give solitary wave solutions of the original Euler problem.

The weakened regularity of our solutions also presents difficulties when attempting to apply traditional maximum principle arguments, which are crucial to proving the existence of large-amplitude waves. For this reason, we follow [**15**] and assume some additional smoothness for U^* near the bed and free surface. Using an elliptic regularity argument that exploits the translation invariance of the domain, we can then infer that the solutions likewise enjoy enough regularity near the boundary so that the Hopf edge-point lemma and Serrin corner point lemma can be applied. Using a moving planes method in the spirit of Maia [**40**], we then prove that solitary wave solutions in $\mathscr{C}$ possess even symmetry, are monotone and (necessarily) waves of elevation. We also show that these qualitative properties persist away from the point of bifurcation.

Finally, we prove Theorem 2.1 by extending $\mathscr{C}_{\mathrm{loc}}$ to a global curve $\mathscr{C}$ using the abstract global bifurcation theory recently developed by Chen, Walsh, Wheeler [**13**]. This result allows for a more general class of operators which may not be locally compact and whose linearized operators are singular at the point of bifurcation. Although the theory in [**13**] was developed for higher regularity solutions, we may apply it to our weaker setting following straightforward generalizations. This section was done in collaboration with Samuel Walsh; the remainder of the thesis is entirely independent work.

2.3. Formulation

As is well known in the previous literature, the governing equations for two-dimensional steady solitary water waves may be formulated in three different ways: the velocity formulation (in terms of the velocity field (u, v) as in (1.16)), the stream-line formulation (in terms of the stream function (1.18)), and the height formulation (in terms of the height function introduced by the Dubreil-Jacotin transformation). However, due to the discontinuous vorticity we introduce to the problem, we must express all three of these formulations in their weak form and consider solutions in the sense of distributions. In section 2.3.4, we prove the equivalence of the three weak formulations.

2.3.1. Weak velocity formulation. We begin with the weak formulation of the governing equations; (1.16) becomes

$$\text{(2.4a)} \qquad \begin{cases} -cu_x + (u^2)_x + (uv)_y = -P_x \\ -cv_x + (uv)_x + (v^2)_y = -P_y - g \qquad \text{in } \Omega_\eta \\ u_x + v_y = 0 \end{cases}$$

with boundary conditions

$$\text{(2.4b)} \qquad \begin{cases} v = 0 & \text{on } y = -d \\ v = (u - c)\eta_x & \text{on } y = \eta(x) \\ P = P_{\text{atm}} & \text{on } y = \eta(x) \end{cases}$$

and asymptotic conditions

$$\text{(2.4c)} \qquad \eta \to 0, \qquad v \to 0, \qquad u \to U(y) = c - FU^*(y), \qquad \text{as } x \to \pm\infty$$

uniformly in y. Here again, P_{atm} is the (constant) atmospheric pressure and g is the gravitational constant of acceleration.

2.3.2. Stream function formulation. We will eliminate the pressure by introducing the relative stream function ψ defined by (1.18). The "no stagnation" condition (1.17) dictates that

$$(2.5) \qquad u - c = \psi_y < 0,$$

throughout the fluid. In particular, this guarantees the existence of a *vorticity function* $\gamma \in L^\infty$ which satisfies

$$-\Delta\psi = \gamma(\psi).$$

Note that this condition is consistent with the requirement that $U^* > 0$. We may then transform the weak velocity formulation (2.4) into the free boundary problem

$$(2.6a) \qquad \begin{cases} \Delta\psi = -\gamma(\psi) & \text{in } \Omega_\eta \\ \psi = 0 & \text{on } y = \eta(x) \\ \psi = m & \text{on } y = -d \\ |\nabla\psi|^2 + 2g(y+d) = Q & \text{on } y = \eta(x) \end{cases}$$

together with the asymptotic conditions

$$(2.6b) \qquad \eta \to 0, \qquad \psi_x \to 0, \qquad \psi_y \to -FU^*(y), \qquad \text{as } x \to \pm\infty$$

uniformly in y. Here, $m > 0$ is the flux

$$(2.7) \qquad m := F \int_{-d}^{0} U^*(y)\, dy,$$

Q is a constant to be determined later, and the vorticity function γ is given implicitly in terms of U^* and F by

$$\gamma(-s) = FU^*_y(y), \qquad \text{where } s = F\int_{-d}^{y} U^* dy'.$$

Here we are making use of the fact that s is strictly increasing as a function from $y = -d$ to $y = -m$.

We continue by writing (2.6a) together with (2.6b) in terms of the dimensionless variables

$$(\tilde{x}, \tilde{y}) = \frac{1}{d}(x, y), \qquad \tilde{\eta}(\tilde{x}) = \frac{1}{d}\eta(x),$$

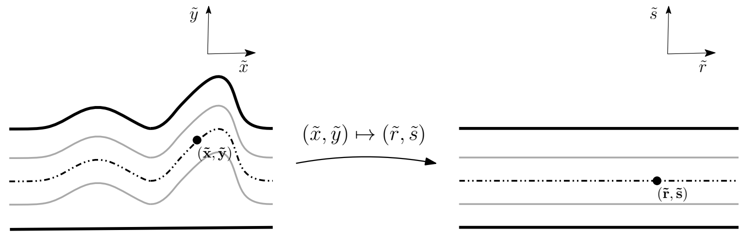

FIGURE 5. Dubreil-Jacotin transformation

$$\tilde{\psi}(\tilde{x},\tilde{y}) = \frac{1}{m}\psi(x,y), \qquad \tilde{\gamma}(\tilde{\psi}) = \frac{d^2}{m}\gamma(\psi)$$

which result from rescaling lengths by d and velocities by m/d. In these variables, the streamline problem (2.6) becomes

$$\text{(2.8a)} \quad \begin{cases} \Delta\tilde{\psi} = -\tilde{\gamma}(\tilde{\psi}) & \text{in } \tilde{\Omega}_{\tilde{\eta}} \\ \tilde{\psi} = 1 & \text{on } \tilde{y} = -1 \\ \tilde{\psi} = 0 & \text{on } \tilde{y} = \tilde{\eta}(\tilde{x}) \\ |\nabla\tilde{\psi}|^2 + \dfrac{2}{F^2}(\tilde{\eta}+1) = \dfrac{Q}{2} & \text{on } \tilde{y} = \tilde{\eta}(\tilde{x}) \end{cases}$$

with asymptotic conditions

$$\text{(2.8b)} \quad \tilde{\eta} \to 0, \qquad \tilde{\psi}_x \to 0, \qquad \tilde{\psi}_y \to \tilde{\Psi}_{\tilde{y}}(\tilde{y}) := \frac{U^*(\tilde{y}d)d}{\int_{-d}^{0} U^*(y)\,dy} \qquad \text{as } x \to \pm\infty.$$

where

$$\tilde{\psi}(\tilde{x},\tilde{y}) \to \tilde{\Psi}(\tilde{y}) := \frac{\int_{-1}^{\tilde{y}d} U^*(y)\,dy}{\int_{-d}^{0} U^*(y)\,dy}$$

as $\tilde{x} \to \pm\infty$ uniformly in $\tilde{y}$.

Lastly, the dimensionless vorticity function $\tilde{\gamma}$ is given in terms of U^* at infinity

$$\tilde{\gamma}(-\tilde{s}) = \frac{d^2}{m}\gamma(-s) = \frac{d^2 U^*_{\tilde{y}}(\tilde{y}d)}{\int_{-d}^{0} U^*\,dy} \qquad \text{where } \tilde{s} = \frac{\int_{-d}^{y} U^*\,dy}{\int_{-d}^{0} U^*\,dy}.$$

2.3.3. Height function formulation. We apply the Dubreil-Jacotin change of variables

$$\tilde{s} = \tilde{\psi}(\tilde{x}, \tilde{y}), \quad \tilde{r} = \tilde{x}, \quad \tilde{h} = \tilde{y} + 1 \quad \text{for } (\tilde{r}, \tilde{s}) \in R := \mathbb{R} \times (0,1)$$

to (2.8) to yield the height formulation

$$\text{(2.9a)} \quad \begin{cases} \left(\dfrac{\tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}}\right)_{\tilde{r}} - \left(\dfrac{1+\tilde{h}_{\tilde{r}}^2}{2\tilde{h}_{\tilde{s}}^2}\right)_{\tilde{s}} + \tilde{\gamma}(-\tilde{s}) = 0 & \text{on } -1 < \tilde{s} < 0 \\ \tilde{h} = 0 & \text{on } \tilde{s} = -1 \\ \dfrac{1+\tilde{h}_{\tilde{r}}^2}{2\tilde{h}_{\tilde{s}}^2} + \dfrac{1}{F^2}\tilde{h} = \dfrac{Q}{2} & \text{on } \tilde{s} = 0 \end{cases}$$

with asymptotic conditions

$$\text{(2.9b)} \quad \tilde{h} \to \tilde{H}(\tilde{s}), \qquad \tilde{h}_{\tilde{r}} \to 0, \qquad \tilde{h}_{\tilde{s}} \to \tilde{H}_{\tilde{s}} \qquad \text{as } \tilde{r} \to \pm\infty$$

uniformly in $\tilde{s}$. The asymptotic height function $\tilde{H}$ is the solution of the differential equation

$$\begin{cases} \tilde{H}_{\tilde{s}}(\tilde{s}) = -\dfrac{1}{\tilde{\Psi}_{\tilde{y}}\left(\tilde{H}(\tilde{s}) - 1\right)} = -\dfrac{\int_{-d}^{0} U^*(y)\, dy}{U^*\left((\tilde{H}(\tilde{s}) - 1)d\right) d} \\ \tilde{H}(-1) = 0, \quad \tilde{H}(0) = 1. \end{cases}$$

2.3.4. Equivalence of formulations. In this section, we prove that the three formulations of the water wave problem are equivalent. Equivalence of weak formulations have been established in the past; for slightly smoother solutions, Varvaruca and Zarnescu [**58**] proved that the weak velocity and weak stream formulations are equivalent in $C^{0,\alpha}$ for $\alpha > 1/3$. Constantin and Strauss [**15**] proved equivalence of the weak formulations for $\gamma \in L^\infty$ and periodic solutions in Sobolev spaces $W^{2,p}$ with $p \in (2,4)$. Most recently Sastre-Gomez [**47**] established that the weak velocity and weak stream function forms are equivalent to the modified-height formulation (due to Henry [**26**]), again for periodic solutions in Hölder spaces. The following theorem states the equivalence of the three formulations for solitary water waves with regularity as in [**15**].

THEOREM 2.2. *Let* $0 < \beta < \dfrac{1}{2}$ *and* $p := \dfrac{2}{1-\beta}$. *Then the following statements are equivalent:*

(i) *there exists a solution of the velocity formulation (*1.16a*)–(*2.4c*) with the regularity* $P, u-U, v \in W^{1,p}(\Omega_\eta) \subset C^\beta(\Omega_\eta)$, $U \in W^{1,\infty}([-d,0])$, *and* $\eta \in C^{1+\beta}(\Omega_\eta)$;

(ii) *there exists a solution of the stream function formulation (*2.8a*)–(*2.8b*) with the regularity* $\tilde{\psi} - \tilde{\Psi} \in W^{2,p}(\Omega_\eta), \Psi_{yy} \in L^\infty(\Omega_\eta), \gamma \in L^\infty\left([0,m]\right)$, *and* $\eta \in C^{1+\beta}(\Omega_\eta)$;

(iii) *there exists a solution of the height formulation (*2.9a*)–(*2.9b*) with the regularity* $h - H \in W^{2,p}(\overline{R}), H_{ss} \in L^\infty([-1,0])$, *and* $\Gamma \in W^{1,\infty}\left([m,0]\right)$.

PROOF. We proceed as in [**15**]. First, let us show that (i) implies (ii). Let $w = u - U$ and $\tilde{\phi} = \tilde{\psi} - \tilde{\Psi}$. By the Sobolev Extension Theorem, we can extend $w, v \in W^{1,p}(\Omega_\eta)$ to functions in $W^{1,p}(\mathbb{R}^2)$. Morrey's inequality yields $w, v \in C^\beta(\overline{\Omega}_\eta)$.

Clearly $\phi \in W^{2,r}(\Omega_\eta) \subset C^{1+\beta}(\overline{\Omega_\eta})$. It follows that

$$\tilde{\phi}_x = \tilde{\psi}_x = -\frac{d}{m}v \to 0 \text{ and } \tilde{\phi}_y = \tilde{\psi}_y - \tilde{\Psi}_y = \frac{d}{m}\left(u - c + FU^*(y)\right) \to 0 \text{ as } x \to \pm\infty.$$

The kinematic boundary conditions

$$\begin{cases} -\psi_x = \psi_y \eta_x & \text{on } y = \eta(x) \\ \psi_x = 0 & \text{on } y = 0 \end{cases}$$

imply that ψ must be constant on the free surface and the flat bed. We set $\psi = 0$ on $\{y = 0\}$ and $\psi = m$ on $\tilde{y} = \eta(x)$ where m is defined as in (2.7).

We claim, and prove below, that $\omega = \gamma(\psi)$ for some $\gamma \in L^\infty([0, m])$. Assuming the claim momentarily, since $\psi_x = -v, \psi_y = u - c$, it follows that (2.4a)–(2.4c) becomes

$$\begin{cases} \Delta\psi = -\gamma(\psi) & \text{in } \Omega_\eta \\ \psi = 0 & \text{on } y = -d \\ \psi = m & \text{on } y = \eta(x) \end{cases}$$

with asymptotic conditions (2.6b). To check the remaining nonlinear boundary condition, define $\Gamma \in W^{1,\infty}([m, 0])$ by

$$\Gamma(s) = \int_{-1}^{s} \gamma(-z)\, dz \tag{2.10}$$

and define

$$E = \frac{(c-u)^2 + v^2}{2} + g(y+d) + P - \Gamma(-\psi).$$

A straightforward calculation shows that the gradient of E vanishes throughout Ω_η, which verifies Bernoulli's law. Indeed, this is equivalent to the first two equations in (2.4a). Furthermore, evaluating E on the free surface and noticing that

$$Q := 2\left(\Gamma(-\psi) - P\right)\Big|_{y=\eta(x)} = 2(\Gamma(0) - P_{\text{atm}}),$$

we obtain the missing part of (2.6a).

Now we wish to show that (ii) implies (i). Given $\tilde{\psi} - \tilde{\Psi} \in W^{2,p}(\tilde{\Omega}_{\tilde{\eta}})$ and $\tilde{\eta} \in C^{1+\beta}(\tilde{\Omega}_{\tilde{\eta}})$, define $w, v \in W^{1,p}(\Omega_\eta)$ by means of

$$w = \psi_y, \qquad v = -\psi_x,$$

which is equivalent to (1.18). Taking the gradient of

$$-P = \frac{|\nabla\psi|^2}{2} + g(y+d) - \Gamma(-\psi) - \frac{Q}{2} - P_{\text{atm}}$$

yields (2.4a), and the boundary conditions follow immediately.

In the process of showing (ii) implies (iii), we will also prove the previous claim that $\tilde{\omega} = \tilde{\gamma}(\tilde{\psi})$ for some $\tilde{\gamma} \in L^\infty([0,m])$. We apply the change of variables

$$\tilde{s} = -\tilde{\psi}(\tilde{x},\tilde{y}), \qquad \tilde{r} = \tilde{x}, \qquad \tilde{h}(\tilde{r},\tilde{s}) = \tilde{y}+1, \qquad \text{for } (\tilde{r},\tilde{s}) \in R, \tag{2.11}$$

so that

$$\tilde{h}_{\tilde{r}} = \frac{v}{u-c}, \qquad \tilde{h}_{\tilde{s}} = \frac{m}{d(c-u)}, \qquad v = \frac{-m\tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}d}, \qquad \text{and } u = c - \frac{m}{d\tilde{h}_{\tilde{s}}}, \tag{2.12}$$

and

$$\partial_{\tilde{x}} = \partial_{\tilde{r}} - \frac{\tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}}\partial_{\tilde{s}}, \qquad \partial_{\tilde{y}} = \frac{1}{\tilde{h}_{\tilde{s}}}\partial_{\tilde{s}}, \tag{2.13}$$

hence

$$\partial_x = \frac{1}{d}\left(\partial_{\tilde{r}} - \frac{\tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}}\partial_{\tilde{s}}\right), \text{ and } \qquad \partial_y = \frac{1}{d}\left(\frac{1}{\tilde{h}_{\tilde{s}}}\partial_{\tilde{s}}\right). \tag{2.14}$$

In addition, for $\tilde{\omega} \in L^p(\Omega_\eta)$, we see that

$$\partial_{\tilde{r}}\tilde{\omega} = \left(\partial_x - \frac{v}{c-u}\partial_y\right)\tilde{\omega}.$$

Taking the curl of the Euler equation (1.16a) yields

$$\begin{aligned} 0 &= (u-c)\tilde{\omega}_x + v\tilde{\omega}_y \\ &= (c-u)\partial_{\tilde{r}}\tilde{\omega} \end{aligned}$$

It follows that $\partial_{\tilde{r}}\tilde{\omega} = 0$, so $\tilde{\omega}$ is a function only of $\tilde{s}$ throughout R. We write $\tilde{\omega} = \tilde{\gamma}(-\tilde{s})$ for some vorticity function $\tilde{\gamma} \in L^p([0,m])$. Evaluating at $r = \pm\infty$, we know that

$$\tilde{\omega} = U_y \in L^\infty([0,m]),$$

and hence $\tilde{\gamma} \in L^\infty([0,m])$.

Then applying the change of variables (2.13) to (2.8a) on the interior of $\tilde{\Omega}_{\tilde{\eta}}$, we see that

$$\begin{aligned}
0 &= \Delta\tilde{\psi} + \tilde{\gamma}(\tilde{\psi}) \\
&= \left(\partial_{\tilde{r}} - \frac{\tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}}\partial_{\tilde{s}}\right)\left(\frac{\tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}}\right) + \left(\frac{1+\tilde{\gamma}(-s)}{\tilde{h}_{\tilde{s}}}\partial_{\tilde{s}}\right)\left(\frac{1}{\tilde{h}_{\tilde{s}}}\right) + \tilde{\gamma}(-s) \\
&= \left(\frac{\tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}}\right)_{\tilde{r}} + \left(\frac{1+\tilde{h}_{\tilde{r}}^2}{2\tilde{h}_{\tilde{s}}^2}\right)_{\tilde{s}} + \tilde{\gamma}(-s).
\end{aligned}$$

The boundary conditions follow immediately. In addition, the asymptotic conditions require that

$$\tilde{h}_{\tilde{s}} = -\frac{1}{\tilde{\Psi}_{\tilde{y}}(\tilde{h}-1)} \to \tilde{H}_{\tilde{s}}(\tilde{s}) = -\frac{1}{\tilde{\Psi}_{\tilde{y}}(\tilde{H}(\tilde{s})-1)}$$

and satisfies $\tilde{H}(-1) = 0$ and $\tilde{H}(0) = 1$. We need to show that $\tilde{h} - \tilde{H} \in W^{2,p}(\overline{R})$. Notice first that $\tilde{\psi} - \tilde{\Psi} \in W^{2,p}(\tilde{\Omega}_{\tilde{\eta}})$, $(\tilde{\psi} - \tilde{\Psi})_{\tilde{y}} \in W^{1,p}(\tilde{\Omega}_{\tilde{\eta}}) \subset C^{0,\beta}(\tilde{\Omega}_{\tilde{\eta}}) \subset L^\infty(\tilde{\Omega}_{\tilde{\eta}})$, $\tilde{\Psi}_{\tilde{y}} \in L^\infty(\tilde{\Omega}_{\tilde{\eta}})$, and hence $\tilde{\psi}_{\tilde{y}} \in L^\infty(\tilde{\Omega}_{\tilde{\eta}})$. This implies

$$\|(\tilde{h}-\tilde{H})_{\tilde{r}}\|_{L^p(R)}^p = \int_R |\tilde{h}_{\tilde{r}}(\tilde{r},\tilde{s})|^p \, d\tilde{r}d\tilde{s} = \int_{\tilde{\Omega}_{\tilde{\eta}}} \left|\frac{\tilde{\psi}_{\tilde{x}}}{\tilde{\psi}_{\tilde{y}}}\right|^p |\tilde{\psi}_{\tilde{y}}| \, d\tilde{x}d\tilde{y}$$

$$\leq \left\|\frac{1}{\tilde{\psi}_{\tilde{y}}^{p-1}}\right\|_{L^\infty(\tilde{\Omega}_{\tilde{\eta}})} \int_{\tilde{\Omega}_{\tilde{\eta}}} |\tilde{\psi}_{\tilde{x}}|^p \, d\tilde{x}d\tilde{y} < \infty.$$

In a similar fashion,

$$\|(\tilde{h}-\tilde{H})_{\tilde{s}}\|_{L^p(R)}^p \leq \left\|\frac{1}{\tilde{\Psi}_{\tilde{y}}^p \tilde{\psi}_{\tilde{y}}^{p-1}}\right\|_{L^\infty(\tilde{\Omega}_{\tilde{\eta}})} \int_{\tilde{\Omega}_{\tilde{\eta}}} |\tilde{\psi}_{\tilde{y}} - \tilde{\Psi}_{\tilde{y}}|^p \, d\tilde{x}d\tilde{y} < \infty,$$

since $\tilde{\psi}_{\tilde{y}} - \tilde{\Psi}_{\tilde{y}} \in L^p(\tilde{\Omega}_{\tilde{\eta}})$. Examining the mixed derivative of $\tilde{h} - \tilde{H}$, we see that

$$\|(\tilde{h}-\tilde{H})_{\tilde{r}\tilde{s}}\|_{L^p(R)}^p \leq \left\|\frac{1}{\tilde{\psi}_{\tilde{y}}^{p-1}}\right\|_{L^\infty(\tilde{\Omega}_{\tilde{\eta}})} \int_{\tilde{\Omega}_{\tilde{\eta}}} |\tilde{\psi}_{\tilde{x}}\tilde{\psi}_{\tilde{y}\tilde{y}} - \tilde{\psi}_{\tilde{y}}\tilde{\psi}_{\tilde{x}\tilde{y}}|^p \, d\tilde{x}d\tilde{y}.$$

We may further bound the L^p norm of $\tilde{\psi}_{\tilde{x}}\tilde{\psi}_{\tilde{y}\tilde{y}} - \tilde{\psi}_{\tilde{y}}\tilde{\psi}_{\tilde{x}\tilde{y}}$ by utilizing Minkowski's inequality twice as follows:

$$\begin{aligned}\|\tilde{\psi}_{\tilde{x}}\tilde{\psi}_{\tilde{y}\tilde{y}} - \tilde{\psi}_{\tilde{y}}\tilde{\psi}_{\tilde{x}\tilde{y}}\|_{L^p(\tilde{\Omega}_{\tilde{\eta}})} &\le \|\tilde{\psi}_{\tilde{x}}\tilde{\psi}_{\tilde{y}\tilde{y}}\|_{L^p(\tilde{\Omega}_{\tilde{\eta}})} + \|\tilde{\psi}_{\tilde{y}}\tilde{\psi}_{\tilde{x}\tilde{y}}\|_{L^p(\tilde{\Omega}_{\tilde{\eta}})} \\ &\le \|\tilde{\psi}_{\tilde{x}}\|_{L^\infty(\tilde{\Omega}_{\tilde{\eta}})}\|\tilde{\psi}_{\tilde{y}\tilde{y}} - \tilde{\Psi}_{\tilde{y}\tilde{y}}\|_{L^p(\tilde{\Omega}_{\tilde{\eta}})} \\ &\quad + \|\tilde{\psi}_{\tilde{x}}\|_{L^p(\tilde{\Omega}_{\tilde{\eta}})}\|\tilde{\Psi}_{\tilde{y}\tilde{y}}\|_{L^\infty([-1,0])} + \|\tilde{\psi}_{\tilde{y}}\|_{L^\infty(\tilde{\Omega}_{\tilde{\eta}})}\|\tilde{\psi}_{\tilde{x}\tilde{y}}\|_{L^p(\tilde{\Omega}_{\tilde{\eta}})} \\ &< \infty.\end{aligned}$$

It follows that $(\tilde{h} - \tilde{H})_{\tilde{r}\tilde{s}} \in L^p(\tilde{\Omega}_{\tilde{\eta}})$. Lastly,

$$\|(\tilde{h} - \tilde{H})_{\tilde{s}\tilde{s}}\|^p_{L^p(R)} = \int_{\tilde{\Omega}_{\tilde{\eta}}} \frac{|\tilde{\Psi}_{\tilde{y}\tilde{y}}\tilde{\psi}^3_{\tilde{y}} - \tilde{\psi}_{\tilde{y}\tilde{y}}\tilde{\Psi}^3_{\tilde{y}}|^p}{|\tilde{\Psi}_{\tilde{y}}|^{3p}|\tilde{\psi}_{\tilde{y}}|^{3p-1}}\, d\tilde{x}d\tilde{y}.$$

Adding and subtracting $\tilde{\Psi}^3_{\tilde{y}}\tilde{\Psi}_{\tilde{y}\tilde{y}}$ yields

$$\|(\tilde{h} - \tilde{H})_{\tilde{s}\tilde{s}}\|^p_{L^p(R)} = \int_{\tilde{\Omega}_{\tilde{\eta}}} \frac{|\tilde{\Psi}^3_{\tilde{y}}(\tilde{\Psi}_{\tilde{y}\tilde{y}} - \tilde{\psi}_{\tilde{y}\tilde{y}}) + \tilde{\Psi}_{\tilde{y}\tilde{y}}(\tilde{\psi}^3_{\tilde{y}} - \tilde{\Psi}^3_{\tilde{y}})|^p}{|\tilde{\psi}_{\tilde{y}}|^{3p-1}|\tilde{\Psi}_{\tilde{y}}|^{3p}}\, d\tilde{x}\, d\tilde{y}$$

By once again making use of Minkowski's inequality, we see that

$$\begin{aligned}\|(\tilde{h} - \tilde{H})_{\tilde{s}\tilde{s}}\|^p_{L^p(R)} \le& \left\|\frac{1}{\tilde{\psi}^{3-1/p}_{\tilde{y}}}\right\|_{L^\infty(\tilde{\Omega}_{\tilde{\eta}})} \|\tilde{\Psi}_{\tilde{y}\tilde{y}} - \tilde{\psi}_{\tilde{y}\tilde{y}}\|_{L^p(\tilde{\Omega}_{\tilde{\eta}})} \\ &+ \left\|\frac{\tilde{\Psi}_{\tilde{y}\tilde{y}}(\tilde{\psi}_{\tilde{y}} - \tilde{\Psi}_{\tilde{y}})}{\tilde{\Psi}^3_{\tilde{y}}}\right\|_{L^p(\tilde{\Omega}_{\tilde{\eta}})} \left\|\frac{(\tilde{\psi}^2_{\tilde{y}} + \tilde{\psi}_{\tilde{y}}\tilde{\Psi}_{\tilde{y}} + \tilde{\Psi}^2_{\tilde{y}})}{\tilde{\psi}^{3-1/p}_{\tilde{y}}}\right\|_{L^\infty(\tilde{\Omega}_{\tilde{\eta}})} \\ <& \infty.\end{aligned}$$

This shows that $(\tilde{h} - \tilde{H})_{\tilde{s}\tilde{s}} \in L^p(\tilde{\Omega}_{\tilde{\eta}})$ and ultimately proves that $\tilde{h} - \tilde{H} \in W^{2,p}(R)$.

Let us now verify that $\tilde{\Gamma}(-\tilde{\psi}) \in W^{1,\infty}(\tilde{\Omega}_{\tilde{\eta}})$ and the chain rule holds for $\tilde{\Gamma} \in W^{1,\infty}([m,0])$. First, notice that $\tilde{\Gamma}(-\tilde{\psi}) \in C^\beta(\overline{\tilde{\Omega}_{\tilde{\eta}}})$ since $\Gamma \in C^\beta([m,0])$ and $\tilde{\psi} - \tilde{\Psi} \in W^{2,\infty}(\tilde{\Omega}_{\tilde{\eta}}) \subset C^{1+\beta}(\overline{\tilde{\Omega}_{\tilde{\eta}}})$. We aim to justify the following:

$$\partial_{\tilde{x}}\tilde{\Gamma}(-\tilde{\psi}) = -\tilde{\gamma}(\tilde{\psi})\tilde{\psi}_{\tilde{x}} \in L^r(\tilde{\Omega}_{\tilde{\eta}})$$

Let $\varphi \in \mathcal{D}(\tilde{\Omega}_{\tilde{\eta}})$ be a test function and observe the action of the distribution $\partial_{\tilde{x}}\tilde{\Gamma}(-\tilde{\psi})$ on φ is given by

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-1}^{\tilde{\eta}(\tilde{x})} \partial_{\tilde{x}}\tilde{\Gamma}(-\tilde{\psi})\varphi \, d\tilde{y}\, d\tilde{x} &= -\int\int_R \tilde{h}_{\tilde{s}}\tilde{\Gamma}(\tilde{s})\left(\partial_{\tilde{r}} - \frac{\tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}}\partial_{\tilde{s}}\right)\varphi \, d\tilde{r}\, d\tilde{s} \\
&= -\int\int_R \frac{\tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}}\tilde{\gamma}(-\tilde{s})\tilde{h}_{\tilde{s}}\varphi \, d\tilde{r}\, d\tilde{s} \\
&= \int_{-\infty}^{\infty}\int_{-1}^{\tilde{\eta}(\tilde{x})} \tilde{\psi}_{\tilde{x}}\tilde{\gamma}(\tilde{\psi})\varphi \, d\tilde{y}\, d\tilde{x}
\end{aligned}$$

by first using the definition of the distributional derivative, applying the change of variables (2.11) and subsequently (2.14), performing the differentiation, utilizing the compact support of the distribution and integrating by parts, and finally changing back to the original variables. Therefore, we have shown that $\partial_{\tilde{x}}\tilde{\Gamma}(-\tilde{\psi}) = -\tilde{\gamma}(\tilde{\psi})\tilde{\psi}_{\tilde{x}}$ in the distributional sense. In a similar fashion, it is easy to prove the fact that

$$\partial_{\tilde{y}}\tilde{\Gamma}(-\tilde{\psi}) = -\tilde{\gamma}(\tilde{\psi})\tilde{\psi}_{\tilde{y}} \in L^{\infty}(\tilde{\Omega}_{\tilde{\eta}})$$

It remains to verify that (iii) implies (ii). Given a solution $\tilde{h}$ of (2.9) with $\tilde{h} - \tilde{H} \in W^{2,p}(R)$, the free surface is identified as

$$\tilde{\eta}(\tilde{x}) = \tilde{h}(\tilde{x}, 1) - 1 \qquad \text{since } \tilde{h}(\tilde{r}, \tilde{s}) = \tilde{y} + 1$$

Notice that if $\tilde{r}$ is fixed, then for $G : (\tilde{r}, \tilde{s}) \mapsto (\tilde{r}, \tilde{h}(\tilde{r}, \tilde{s}))$,

$$\partial_{\tilde{s}}G = \tilde{h}_{\tilde{s}} = \frac{m}{(c-u)d} > 0$$

since $u < c$ throughout Ω_η. Thus G is strictly monotone and hence bijective. Furthermore, by the inverse function theorem, G is a local $C^{1,\beta}$-diffeomorphism which extends to a global $W^{2,p}_{\text{loc}}$-diffeomorphism. As for its inverse, the first component must be given by $(\tilde{x}, \tilde{y}) \mapsto \tilde{x}$. Define $-\tilde{\psi}(\tilde{x}, \tilde{y})$ to be its second component so that

$G^{-1} : (\tilde{x}, \tilde{y}) \mapsto (\tilde{x}, \tilde{\psi}(\tilde{x}, \tilde{y}))$. We know that $\tilde{\psi} = 0$ on $\tilde{y} = 0$, $\tilde{\psi} = -1$ on $\tilde{y} = \tilde{\eta}(\tilde{x})$, and

$$\tag{2.15} \tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{y}) = -\frac{\tilde{h}_{\tilde{r}}(\tilde{x}, -\tilde{\psi}(\tilde{x}, \tilde{y}))}{\tilde{h}_{\tilde{s}}(\tilde{x}, -\tilde{\psi}(\tilde{x}, \tilde{y}))} \quad \text{and} \quad \tilde{\psi}_{\tilde{y}} = -\frac{1}{\tilde{h}_{\tilde{s}}(\tilde{x}, -\tilde{\psi}(\tilde{x}, \tilde{y}))}$$

Additionally, $Y_{\tilde{s}} = (\tilde{H} - 1)_{\tilde{s}} = \tilde{H}_s > 0$, hence Y is monotone. Therefore it has an inverse $-\tilde{\Psi} \in C^{1,\beta}([-1, 0])$. In particular, $\tilde{\Psi}(-1) = 1$ and $\tilde{H}(0) = 1$ implies that $Y(0) = 0$ which in turn implies that $\tilde{\Psi}(0) = 0$. It remains to show that $\tilde{\psi} - \tilde{\Psi} \in W^{2,p}(\tilde{\Omega}_{\tilde{\eta}})$ which may be verified by invoking the same strategy used to show that $\tilde{h} - \tilde{H} \in W^{2,p}(\overline{R})$ in (ii) implies (iii). The nonlinear boundary condition in (2.6a) follows immediately by applying (2.15) to the free surface boundary condition in (2.9a).

Notice that by differentiating $\tilde{\psi}_{\tilde{y}}$ in (2.15) with respect to $\tilde{y}$, we obtain

$$\tilde{\psi}_{\tilde{y}\tilde{y}} = \left(\frac{\tilde{h}_{\tilde{s}\tilde{s}}}{\tilde{h}_{\tilde{s}}^3} (\tilde{x}, -\tilde{\psi}(\tilde{x}, \tilde{y})) \right)$$

Similarly, if we reformulate $\tilde{\psi}_{\tilde{x}}$ in (2.15) to be

$$\tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{y}) \tilde{h}_{\tilde{s}}(\tilde{x}, -\tilde{\psi}(\tilde{x}, \tilde{y})) = \tilde{h}_{\tilde{r}}(\tilde{x}, -\tilde{\psi}(\tilde{x}, \tilde{y}))$$

and differentiate with respect to $\tilde{x}$, we obtain

$$\tilde{\psi}_{\tilde{x}\tilde{x}} = \frac{\tilde{h}_{\tilde{r}\tilde{r}}}{\tilde{h}_{\tilde{s}}} - 2\frac{\tilde{h}_{\tilde{r}\tilde{s}} \tilde{h}_{\tilde{r}}}{\tilde{h}_{\tilde{s}}^2} + \frac{\tilde{h}_{\tilde{s}\tilde{s}} \tilde{h}_{\tilde{r}}^2}{\tilde{h}_{\tilde{s}}^3}$$

Finally,

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + \tilde{\psi}_{\tilde{y}\tilde{y}} = -\gamma(\psi) \qquad \text{in } \tilde{\Omega}_{\tilde{\eta}}$$

follows from (2.9a). ∎

2.4. Linearized Operators

2.4.1. Function spaces and the operator equation. Here and in the sequel, we drop the $\sim$ notation in both the stream function and the height function formulations. Furthermore, we notice that the upstream and downstream conditions on h allow us to write the function Γ in terms of the asymptotic height function H in the following way:

$$\Gamma(s) = \frac{1}{2H_s^2}.$$

With this in hand, the height function formulation (2.9a) may be written exclusively in terms of h and H

$$\text{(2.16)} \quad \begin{cases} \left(\dfrac{h_r}{h_s}\right)_r - \left(\dfrac{1+h_r^2}{2h_s^2} - \dfrac{1}{2H_s^2}\right)_s = 0 & \text{for } -1 < s < 0, \\ h = 0 & \text{for } s = -1, \\ \dfrac{1+h_r^2}{2h_s^2} - \dfrac{1}{2H_s^2} + \dfrac{1}{F^2}(h - H) = 0 & \text{for } s = 0. \end{cases}$$

Introducing the difference $\phi(r,s) := h(r,s) - H(s)$, the height equation (2.16) becomes

$$\text{(2.17)} \quad \begin{cases} \left(\dfrac{\phi_r}{\phi_s + H_s}\right)_r - \left(\dfrac{1+\phi_r^2}{2(\phi_s + H_s)^2} - \dfrac{1}{2H_s^2}\right)_s = 0 & \text{in } R \\ \phi = 0 & \text{on } B \\ \dfrac{1+\phi_r^2}{2(\phi_s + H_s)^2} - \dfrac{1}{2H_s^2} + \dfrac{1}{F^2}\phi = 0 & \text{on } T \end{cases},$$

together with the asymptotic conditions

$$\text{(2.18)} \quad \phi \to 0 \qquad \text{and} \qquad D\phi \to 0 \qquad \text{as } r \to \pm\infty$$

uniformly in s, where B and T are the boundaries

$$B := \{(r,s) : s = -1\} \qquad \text{and} \qquad T := \{(r,s) : s = 0\}.$$

We will (almost) exclusively use this version of the height function formulation for the remainder of the work. Define the Banach spaces

$$X := \left\{ \phi \in W^{2,p}_{\mathrm{e}}(R) : \phi = 0 \text{ on } B \right\},$$

$$Y := L^p_{\mathrm{e}}(R),$$

where the subscript "e" indicates evenness in the horizontal variable r. Finally, we can introduce the operator equation

$$\mathscr{F}(\phi, F) = 0, \tag{2.19}$$

where $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) : \mathcal{U} \subset X \times \mathbb{R} \to Y$ is given by

$$\mathscr{F}_1(\phi, F) := \left(\frac{\phi_r}{\phi_s + H_s} \right)_r - \left(\frac{1 + \phi_r^2}{2(\phi_s + H_s)^2} - \frac{1}{2H_s^2} \right)_s, \tag{2.20}$$

$$\mathscr{F}_2(\phi, F) := \left. \frac{1 + \phi_r^2}{2(\phi_s + H_s)^2} - \frac{1}{2H_s^2} + \frac{1}{F^2}\phi \right|_T, \tag{2.21}$$

and the set $U \subset X$ is defined to be

$$U := \left\{ (\phi, F) \in X \times \mathbb{R} : \inf_R (\phi_s + H_s) > 0, F > F_{\mathrm{cr}} \right\} \subset \mathcal{X} \times \mathbb{R}. \tag{2.22}$$

Here, $\phi_s + H_s > 0$ is equivalent to the no stagnation condition (2.5).

2.4.2. Sturm-Liouville Problem. Let us begin by examining the eigenvalue problem for the linearized operator $\mathscr{F}_\phi(0, F)$ restricted to variations $\dot{\phi}$ that are laminar in the sense that $\dot{\phi} = \dot{\phi}(s)$. If we consider r-independent functions only, we have

$$\begin{cases} -\left(\dfrac{\dot{\phi}_s}{H_s^3} \right)_s = \nu \dfrac{\dot{\phi}}{H_s} & -1 < s < 0 \\ \dot{\phi} = 0 & s = -1 \\ -\dfrac{\dot{\phi}_s}{H_s^3} + \mu \dot{\phi} = 0 & s = 0 \end{cases} \tag{2.23}$$

where ν is the eigenvalue, and

$$\mu := \frac{1}{F^2}, \tag{2.24}$$

is introduced for notational convenience throughout this section. We begin by analyzing the case when $\nu = 0$, and we wish to find the smallest value μ for which

$$\begin{cases} -\left(\frac{\dot{\phi}_s}{H_s^3}\right)_s = 0 & -1 < s < 0 \\ \dot{\phi} = 0 & s = -1 \\ -\frac{\dot{\phi}_s}{H_s^3} + \mu\dot{\phi} = 0 & s = 0 \end{cases} \tag{2.25}$$

has nontrivial solution $\dot{\phi}$. Consider the unique solution Φ to the initial value problem

$$\begin{cases} -\left(\frac{\Phi_s}{H_s^3}\right)_s = 0 & -1 < s < 0 \\ \Phi = 0 & s = -1 \\ \Phi_s = 1 & s = -1 \end{cases} \tag{2.26}$$

along with the affine function

$$A(\mu) := -\frac{\Phi_s(0)}{H_s^3(0)} + \mu\Phi(0).$$

Notice that for fixed μ, Φ solves (2.25) if and only if $A(\mu) = 0$. We arrive at the following lemma:

Lemma 2.1 (Existence of the critical Froude number). *There exists a unique $\mu_{cr} > 0$ such that*

(i) *The Sturm-Liouville problem* (2.25) *has a nontrivial solution $\dot{\phi} = \Phi$ for $\mu = \mu_{cr}$;*

(ii) *$A(\mu) < 0$ for $\mu < \mu_{cr}$ and $A(\mu) > 0$ for $\mu > \mu_{cr}$.*

Proof. To prove (i), first notice that we can solve (2.26) and write Φ explicitly:

$$\Phi(s) = \frac{1}{H_s^3(-1)} \int_{-1}^{s} H_s^3(t)\, dt.$$

Using our explicit solution Φ to solve $A(\mu) = 0$ for μ, find that

$$\frac{\Phi_s(0)}{H_s^3(0)} = \mu\Phi(0),$$

if and only if

$$\mu = \left(\int_{-1}^{0} H_s^3(t)\, dt\right)^{-1},$$

i.e., there exists a unique $\mu = \mu_{\text{cr}} > 0$ such that (2.25) has a nontrivial solution $\dot{\phi} = \Phi$, where

$$\mu_{\text{cr}} := \left(\int_{-1}^{0} H_s^3(t)\, dt\right)^{-1},$$

proving (i). Finally, (ii) follows immediately from the fact that $A(\mu)$ is affine with $A(\mu_{\text{cr}}) = 0$. ∎

With this in hand, we may now analyze the full eigenvalue problem (2.23) by first fixing $\mu = \mu_{\text{cr}}$.

LEMMA 2.2 (Spectrum of the Sturm-Liouville operator). *Let Σ denote the set of eigenvalues ν for the Sturm-Liouville problem* (2.23) *with $\mu = \mu_{cr}$ fixed. Then the following holds:*

(i) $\Sigma = \{\nu_j\}_{j=0}^{\infty}$, *where $\nu_j \to \infty$ as $j \to \infty$, and $\nu_j < \nu_{j+1}$ for all $j \geq 0$;*

(ii) $\nu_0 = 0$*; and*

(iii) $\nu_j \in \Sigma$ *has both geometric and algebraic multiplicity 1 for all $j \geq 0$.*

PROOF. Let us begin by introducing the solution $\Phi := \Phi(s; \nu)$ to the initial value problem

$$\begin{cases} -\left(\dfrac{\Phi_s}{H_s^3}\right)_s = \nu\dfrac{\Phi}{H_s} & -1 < s < 0 \\ \Phi = 0 & \text{for } s = -1 \\ \Phi_s = 1 & \text{for } s = -1 \end{cases} \tag{2.27}$$

and the associated function

$$B(\nu) := \frac{\Phi_s(0;\nu)}{\Phi(0;\nu)}.$$

Notice first that $\dot\phi = \Phi(\cdot;\nu)$ solves (2.23) for ν provided

$$B(\nu) = \mu_{\rm cr} H_s(0)^3.$$

Also, B will have poles at each Dirichlet eigenvalue, i.e., the eigenvalues of the Dirichlet problem

$$\text{(2.28)} \quad -\left(\frac{\dot\phi_s}{H_s^3}\right)_s = \nu_D \frac{\dot\phi}{H_s} \text{ for } -1 < s < 0, \quad \dot\phi(-1) = 0, \quad \dot\phi(0) = 0,$$

provided that $\dot\phi \not\equiv 0$. We wish to show that $\nu_D > 0$, i.e., $\Sigma_D := \{\nu_D^{(j)}\}_{j=1}^\infty \subset (0,\infty)$.

First, notice that (2.28) is a self-adjoint Sturm-Liouville problem, and hence has a countable set of simple eigenvalues accumulating only at $+\infty$. By way of contradiction, suppose that there exists some ν_D with $\nu_D \leq 0$ and let $\dot\phi$ be a nontrivial solution of (2.28) for such fixed ν_D. Applying the maximum principle to the above initial value problem and recalling that $\dot v(-1) = \dot v(0) = 0$, we find that $\dot v \equiv 0$, which implies that $\dot\phi \equiv 0$. This contradicts the assumption that $\dot\phi$ was a nontrivial solution to the Dirichlet problem. Hence $\nu_D > 0$, i.e., $\Sigma_D := \{\nu_D^{(j)}\}_{j=1}^\infty \subset (0,\infty)$, as desired.

In order to proceed with our argument, we wish to differentiate with respect to ν. But we must proceed with caution, for we do not yet know that B (or Φ) depends smoothly on ν. To show that this is indeed the case, we first make the $C^{1,1}$ change of variables

$$s \mapsto y := \int_{-1}^{s} \frac{1}{H_s}\, ds$$

which allows us to rewrite (2.27)

$$\begin{cases} -\left(\dfrac{\Phi_y}{H_s^4}\right)_s = \nu\Phi, & 0 < y < 1 \\ \Phi = 0 & \text{for } y = 0 \\ \Phi_y = H_s(-1) & \text{for } y = 0 \end{cases} \tag{2.29}$$

Now define

$$w(y;\nu) := \frac{\Phi_y}{H_s^4(y)},$$

so that we may consider (2.27) as the linear system

$$\begin{cases} \Phi_y = H_s^4(y)w \\ w_y = -\nu\Phi \end{cases} \tag{2.30}$$

for $y \in (0,1)$ together with the new associated function in y

$$B(\nu) := \frac{w(1;\nu)}{\Phi(1;\nu)}. \tag{2.31}$$

With these in hand, we define the following difference quotients in ν

$$\begin{aligned} N(y) &:= \frac{\Phi(y;\nu_1) - \Phi(y;\nu_2)}{\nu_1 - \nu_2}, \text{ and} \\ v(y) &:= \frac{w(y;\nu_1) - w(y;\nu_2)}{\nu_1 - \nu_2}, \end{aligned}$$

which transforms (2.30) to

$$\begin{cases} N_y = H_s^4(y)v \\ v_y = -\nu_1 N + \Phi(y;\nu_2) \end{cases} \tag{2.32}$$

This is equivalent to the elliptic equation $LN = \Phi(y;\nu_2)$, defined by

$$\left(\frac{N_y}{H_s^4(y)}\right)_y + \nu_1 N = \Phi(y;\nu_2),$$

whose coefficients lie in $C^{0,\alpha}([0,1])$. Then classical Schauder estimates show that

$$\|N\|_{C^{2,\alpha}} \lesssim \|LN\|_{C^{0,\alpha}} + \|N\|_{L^\infty}, \tag{2.33}$$

and we see that N is uniformly bounded as $\nu_2 \to \nu_1$. Therefore, the difference quotient is well-defined and hence B is differentiable with respect to ν.

We may proceed by differentiating (2.25) with respect to ν, which yields

$$\begin{cases} -\left(\dfrac{\Phi_{s\nu}}{H_s^3}\right)_s = \dfrac{\Phi}{H_s} + \nu\dfrac{\Phi_\nu}{H_s} & -1 < s < 0 \\ \Phi_\nu = 0 & \text{for } s = -1 \\ \Phi_{s\nu} = 0 & \text{for } s = -1 \end{cases}$$

Notice, then, that multiplying by Φ on both sides and integrating in s, we have

$$\int_{-1}^{0} \frac{\Phi_s \Phi_{s\nu}}{H_s^3}\, ds - \frac{\Phi\Phi_{s\nu}}{H_s^3}\bigg|_{s=-1}^{s=0} = \int_{-1}^{0} \frac{\Phi^2}{H_s}\, ds + \int_{-1}^{0} \nu\frac{\Phi\Phi_\nu}{H_s}\, ds.$$

Using (2.25), we know that $\nu\dfrac{\Phi}{H_s} = -\left(\dfrac{\Phi_s}{H_s^3}\right)_s$, and hence integrating by parts yields

$$\frac{\Phi_s\Phi_\nu - \Phi\Phi_{s\nu}}{H_s^3}\bigg|_{s=-1}^{s=0} = \int_{-1}^{0} \frac{\Phi^2}{H_s^3}\, ds.$$

Therefore,

$$B'(\nu) = \frac{\Phi\Phi_{s\nu} - \nu_s\Phi_\nu}{\Phi^2}\bigg|_{s=-1}^{s=0} = -\frac{H_s^3(0)}{\Phi^2(0)}\int_{-1}^{0} \frac{\Phi^2}{H_s}\, ds < 0,$$

for $\Phi \neq \Phi(0; \frac{1}{F_{\text{cr}}^2})$. Hence, B is a strictly decreasing function of ν. Now, since B has poles at each Dirichlet eigenvalue $\nu_D^{(j)}$ for $j \geq 0$, it follows that $B(\nu)$ decreases from $+\infty$ to $-\infty$ on the intervals $(-\infty, \nu_D^{(0)})$ and $(\nu_D^{(j)}, \nu_D^{(j+1)})$ for all $j \in \mathbb{N} \cup \{0\}$. Thus, for each j, there exists a unique $\nu_j \in (\nu_D^{(j)}, \nu_D^{(j+1)})$ for which

$$B(\nu_j) = \frac{1}{F_{\text{cr}}^2} H_s(0)^3.$$

Additionally, there exists at most one $\nu_0 \in (-\infty, \nu_0^D)$ for which the same holds. We have shown, then, that $\nu_0 = 0$. By definition of B, we see that $\{\nu_j\}_{j=0}^{\infty}$ are exactly the eigenvalues of (2.23), completing the proofs for (i) and (ii).

To prove (iii), we notice once again that (2.23) is a self-adjoint Sturm-Liouville problem, and hence has a countable set of simple eigenvalues $\{\nu_j\}_{j=0}^{\infty}$ accumulating only at $+\infty$. ∎

2.4.3. Local Properness and Fredholm Index. In this section, we analyze the linearized operator $\mathscr{F}_\phi(\phi, F)$ for $(\phi, F) \in U$. Specifically, we identify important properties possessed by the the operators $\mathscr{F}_\phi(0, F)$ obtained by linearizing the about the trivial solution $\phi = 0$; these properties will play a signature role in the subsequent analysis. The assumption of supercriticality $F > F_{\mathrm{cr}}$ is necessary throughout this section.

Now taking the Frechét derivative of $\mathscr{F}$ about ϕ in the $\dot\phi$ direction, we have the linearized operator $\mathcal{L} := \mathscr{F}_\phi(\phi, F)\dot\phi$:

$$\mathcal{A}\dot\phi := \left(\frac{1}{\phi_s + H_s}\dot\phi_r - \frac{\phi_r}{(\phi_s + H_s)^2}\dot\phi_s\right)_r - \left(\frac{\phi_r}{(\phi_s + H_s)^2}\dot\phi_r - \frac{1 + \phi_r^2}{(\phi_s + H_s)^3}\dot\phi_s\right)_s$$

$$B\dot\phi := \frac{\phi_r}{(\phi_s + H_s)^2}\dot\phi_r - \frac{1 + \phi_r^2}{(\phi_s + H_s)^3}\dot\phi_s + \frac{1}{F^2}\dot\phi\bigg|_T$$

We also define $\tilde{\mathcal{L}} := (\tilde{\mathcal{A}}, \tilde{B})$ to be the "limiting operator" of $\mathcal{L}$ at infinity by evaluating its coefficients as $|r| \to \infty$:

$$\tilde{\mathcal{A}}\dot\phi := \left\{\frac{1}{H_s}\dot\phi_r\right\}_r + \left\{\frac{1}{H_s^3}\dot\phi_s\right\}_s, \qquad \tilde{B}\dot\phi := -\frac{1}{H_s^3}\dot\phi + \frac{1}{F^2}\dot\phi\bigg|_T \tag{2.34}$$

Lastly, we define the family of shifted operators $\mathcal{L}^{n\pm} := (\mathcal{A}^{n\pm}, B^{n\pm})$ for $\{r_n\} \subset \mathbb{R}$ such that $|r_n| \to +\infty$ to be

$$\mathcal{A}^{n\pm}\dot\phi := \left(\frac{1}{\phi_s^{n\pm} + H_s}\dot\phi_r - \frac{\phi_r^{n\pm}}{(\phi_s^{n\pm} + H_s)^2}\dot\phi_s\right)_r - \left(\frac{\phi_r^{n\pm}}{(\phi_s^{n\pm} + H_s)^2}\dot\phi_r - \frac{1 + (\phi_r^{n\pm})^2}{(\phi_s^{n\pm} + H_s)^3}\dot\phi_s\right)_s$$

$$B^{n\pm}\dot\phi := \frac{\phi_r^{n\pm}}{(\phi_s^{n\pm} + H_s)^2}\dot\phi_r - \frac{1 + (\phi_r^{n\pm})^2}{(\phi_s^{n\pm} + H_s)^3}\dot\phi_s + \frac{1}{F^2}\dot\phi\bigg|_T$$

where $\phi^{n\pm}(r,s) := \phi(r \pm r_n, s)$. Our goal now is to prove that $\mathcal{L}$ is locally proper. As a preliminary step, we first show that $\tilde{\mathcal{L}}\dot{\phi} = f$ is uniquely solvable in the Sobolev space $W^{2,p}(R)$.

LEMMA 2.3 (Solvability in $W^{2,p}(R)$). *If $F > F_{cr}$, then for any $\lambda \geq 0$ and $(f,g) \in L^p(R) \times L^p(T)$, there exists a unique $\dot{\phi} \in W^{2,p}(R)$ solving $\tilde{\mathcal{L}}\dot{\phi} = (f,g)$.*

A key argument throughout the remainder of this paper is the use of the maximum principle. As such, it is important to note that the coefficient last in B has a sign which is not suitable for maximum principle arguments. Luckily, we can overcome this difficulty by modifying the problem using the following "dividing trick"—a slight modification of the function Φ introduced in (2.26).

LEMMA 2.4. *Fix $0 < \epsilon \ll 1$ and let $\tilde{\Phi} := \tilde{\Phi}(s; F, \epsilon)$ be the solution to the initial value problem*

$$\left(\frac{\tilde{\Phi}_s}{H_s^3}\right)_s + \frac{1}{F^2}\epsilon\tilde{\Phi} = 0 \quad \text{in } (-1,0), \quad \tilde{\Phi}(-1) = \epsilon, \quad \tilde{\Phi}_s(-1) = 1. \tag{2.35}$$

For supercritical waves, and for $\epsilon > 0$ sufficiently small,

$$\tilde{\Phi} > 0 \text{ for } -1 < s \leq 0, \quad \tilde{\Phi}_s > 0 \text{ for } -1 \leq s \leq 0, \tag{2.36}$$

and

$$-\frac{\tilde{\Phi}_s}{H_s^3} + \frac{1}{F^2}\tilde{\Phi} < 0 \quad \text{on } s = 0. \tag{2.37}$$

PROOF. It is easy to see that $\tilde{\Phi} = \Phi$ whenever $\epsilon = 0$. It follows that the second inequality in (2.36) and the inequality in (2.37) hold for sufficiently small ϵ. Indeed, $\tilde{\Phi}_s$ is uniformly bounded away from zero for ϵ sufficiently small, and since $\tilde{\Phi} = 0$, integrating the second inequality in (2.36) yields the first inequality. ∎

PROOF OF LEMMA 2.3. Fix $F > F_{\mathrm{cr}}$ and let $\tilde{\Phi}$ be defined as in Lemma 2.4. Making the change of variables

$$\dot{\phi} =: \tilde{\Phi} v,$$

we see that the operator at infinity $\tilde{\mathcal{L}}$ becomes

$$\begin{cases} \left(\frac{v_r}{H_s}\right)_r + \left(\frac{v_s}{H_s^3}\right)_s + \frac{2\tilde{\Phi}_s}{\tilde{\Phi} H_s^3} v_s - \frac{1}{F^2}\epsilon v = \frac{f}{\tilde{\Phi}}, & \text{in } R \\ -\frac{v_s}{H_s^3} + \frac{1}{\tilde{\Phi}}\left(-\frac{\phi_s}{H_s^3} + \frac{1}{F^2}\tilde{\Phi}\right) v = \frac{g}{\tilde{\Phi}} & \text{on } T \\ v = 0, & \text{on } B. \end{cases} \tag{2.38}$$

Notice that the zeroth order term in (2.38) now has the correct sign in order to apply the maximum principle; the result follows from a straightforward generalization of [**34**, Theorem 11.6.2]. ∎

LEMMA 2.5 (Local properness). *If the limiting operator $\tilde{\mathcal{L}}$ has a trivial kernel in X, then $\mathcal{L}$ is locally proper (i.e., for any compact set $K \subset Y$ and any closed and bounded set $D \subset X, \mathcal{L}^{-1}(K) \cap D$ is compact in X).*

PROOF. Consider the bounded sequence $\{\phi_n\} \subset X$ and notice that

$$\mathcal{L}\phi_n \to 0 \text{ in } Y.$$

By reflexivity of $W^{2,p}(R)$, we can extract a subsequence, still denoted ϕ_n such that $\phi_n \rightharpoonup \phi$ weakly in $W^{2,p}(R)$, and hence, $\mathcal{L}\phi = 0$. Now let $\mu_n := \phi_n - \phi$. Since $\mathcal{L}\phi_n = 0$, we know from a priori Schauder estimates that

$$\|\mu_n\|_{W^{2,p}} \lesssim \|\mu_n\|_{L^p},$$

so it suffices to show that $\mu_n \to 0$ strongly in $L^p(R)$.

By way of contradiction, suppose that $\mu_n \nrightarrow 0$ in $L^p(R)$. Then $\|\mu_n\|_{L^p} \neq 0$ and define

$$v_n = \frac{\mu_n}{\|\mu_n\|_{L^p}}.$$

Thus,

$$\|v_n\|_{L^p} = 1, \quad \mathcal{L}v_n \to 0 \text{ in } Y \subset L^p(R), \qquad \text{and } v_n \rightharpoonup 0 \text{ in } W^{2,p}(R). \tag{2.39}$$

By redefining $\nu_n := |v_n|^p$ so that

$$\int_R \nu_n \, drds = 1,$$

we satisfy the hypotheses of the concentration-compactness principle, developed by Lions [**39**]. Specifically, we modify the principle to accommodate the two-dimensional infinite strip R. By the concentration-compactness principle for the sequence ν_n, there are three possibilities:

(i) (*vanishing*) For any $K > 0$,

$$\lim_{n\to\infty} \sup_{r_0 \in \mathbb{R}} \int_{-1}^{0} \int_{r_0-K}^{r_0+K} \nu_n \, drds = 0; \tag{2.40}$$

(ii) (*compactness*) There exists a sequence $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that for any $\epsilon > 0$, there exists $K > 0$ depending on ϵ and there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$S = \int_{-1}^{0} \int_{r_n-K}^{r_n+K} \nu_n \, drds \geq 1 - \epsilon; \tag{2.41}$$

(iii) (*dichotomy*) There exists $l \in (0, 1)$ such that

$$\lim_{K\to\infty} Q(K), \tag{2.42}$$

where

$$Q(K) := \lim_{n\to\infty} \sup_{r_0 \in \mathbb{R}} \int_{-1}^{0} \int_{r_0-K}^{r_0+K} \nu_n \, drds. \tag{2.43}$$

The goal is to achieve a contradiction by proving that none of these three occur.

Suppose *vanishing* occurs. Then for any $(r,s) \in R$,

$$\left(\int_{-1}^{0}\int_{r}^{r+1} \nu_n \, drds\right) \le \sup_{r_0\in\mathbb{R}} \int_{-1}^{0}\int_{r_0}^{r_0+1} \nu_n \, drds,$$

so that

$$\left(\int_{-1}^{0}\int_{r}^{r+1} \nu_n \, drds\right)^2 \le \left(\sup_{r_0\in\mathbb{R}} \int_{-1}^{0}\int_{r_0}^{r_0+1} \nu_n \, drds\right)\left(\int_{-1}^{0}\int_{r}^{r+1} \nu_n \, drds\right). \tag{2.44}$$

Now since

$$\left(\sum_{k=0}^{\infty} a_k\right)^2 \le 2\sum_{k=0}^{\infty} a_k^2,$$

we have

$$\begin{aligned}\sum_{r\in\mathbb{Z}}\left(\int_{-1}^{0}\int_{r}^{r+1} \nu_n \, drds\right)^2 &\gtrsim \left(\sum_{r\in\mathbb{Z}}\int_{-1}^{0}\int_{r}^{r+1} \nu_n \, drds\right)^2\\ &= \left(\int_{R} \nu_n \, drds\right)^2\end{aligned}$$

so that, after summing over $\mathbb{Z}$, the inequality (2.44) becomes

$$1 = \|v_n\|_{L^p(R}^{2p} = \left(\int_R \nu_n \, drds\right)^2 \lesssim \left(\sup_{r_0\in\mathbb{R}} \int_{-1}^{0}\int_{r_0}^{r_0+1} \nu_n \, drds\right)\|v_n\|_{L^p(R)}^p \to 0,$$

a contradiction.

Now suppose *compactness* occurs. Then there exists a sequence $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that for any $\epsilon > 0$, there exists $K > 0$ depending on ϵ and there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$,

$$S = \int_{-1}^{0}\int_{r_n-K}^{r_n+K} \nu_n \, drds \ge 1-\epsilon;$$

Define the shifted function $w_n(r,s) := v_n(r-r_n,s)$. It follows that

$$\|w_n\|_{L^p(R)} = \|v_n\|_{L^p(R)} = 1, \qquad \|w_n\|_{W^{2,p}(R)} = \|v_n\|_{W^{2,p}(R)}.$$

Then $w_n \rightharpoonup w$ in $W^{2,p}(R)$, and from embedding theorems, we have that $w_n \to w$ in $L^p_{\text{loc}}(R)$. From (2.41), we know that

$$\int_{-1}^{0}\int_{-K}^{K} |w_n|^p \, drds \geq 1-\epsilon.$$

Taking the limit as $n \to \infty$, we see that

$$1-\epsilon \leq \lim_{n\to\infty} \int_{-1}^{0}\int_{-K}^{K} |w_n|^p \, drds = \int_{-1}^{0}\int_{-K}^{K} |w|^p \, drds.$$

and hence $\|w\|_{L^p(R)} = 1$ and $w_n \to w$ in $L^p(R)$. Additionally, we claim that $\mathcal{L}^{n-}w_n \rightharpoonup \tilde{\mathcal{L}}w$ in Y. Indeed, for test functions $\psi \in C_c^\infty(\overline{R})$, consider

$$\langle \mathcal{L}^{n-}w_n - \tilde{\mathcal{L}}w, \psi\rangle = \int_R (\mathcal{L}^{n-} - \mathcal{L})w_n\psi \, drds + \int_R \mathcal{L}(w_n - w)\psi \, drds.$$

Let us turn our attention to the first integral:

$$\begin{aligned}\int_R (\mathcal{L}^{n-} - \mathcal{L})w_n\psi \, drds = &\int_R \left[\left(\frac{1}{\phi_s^{n-}+H_s} - \frac{1}{H_s}\right)(w_n)_r - \frac{\phi_r^{n-}}{(\phi_s^{n-}+H_s)^2}(w_n)_s\right]_r \psi \, drds \\ &- \int_R \left[\frac{\phi_r^{n-}}{(\phi_s^{n-}+H_s)^2}(w_n)_r - \left(\frac{1+(\phi_r^{n-})^2}{(\phi_s^{n-}+H_s)^3} - \frac{1}{H_s^3}\right)(w_n)_s\right]_s \psi \, drds.\end{aligned}$$

Integrating by parts and recalling that $Dw \in W^{1,r}(R) \subset C^{0,\alpha}(R) \subset L^\infty(R)$, we may apply Lebesgue Dominated Convergence Theorem to see that, as $n \to \infty$:

$$\frac{1}{\phi_s^{n-}+H_s} - \frac{1}{H_s} \to 0, \quad \frac{\phi_r^{n-}}{(\phi_s^{n-}+H_s)^2} \to 0, \quad \text{and} \quad \frac{1+(\phi_r^{n-})^2}{(\phi_s^{n-}+H_s)^3} - \frac{1}{H_s^3} \to 0,$$

which means that $(\mathcal{L}^{n-} - \tilde{\mathcal{L}})w_n \rightharpoonup 0$ in Y. Now we turn our attention to the second integral

$$\int_R \mathcal{L}(w_n - w)\psi \, drds.$$

It is easy to see that $\tilde{\mathcal{L}}(w_n - w) \rightharpoonup 0$ in Y due to the assumption that $w_n \rightharpoonup w$. Therefore, we have shown that $\mathcal{L}^{n-}w_n \rightharpoonup \tilde{\mathcal{L}}w$ in Y. But from (2.39), we see that

$$\mathcal{L}^{n-}w_n = \mathcal{L}^{n-}v_n(r - r_n, s) \to 0 \quad \text{in } Y,$$

implying that $\tilde{\mathcal{L}}w = 0$, and by assumption, $w = 0$. This contradicts the fact that $\|w\|_{L^p(R)} = 1$.

Lastly, assume that *dichotomy* occurs. Then for any $\epsilon > 0$, there exists $K_0 > 0$ such that for all $K > K_0$,

$$|Q(K) - l| < \frac{\epsilon}{2}, \qquad \text{and} \qquad l - \frac{\epsilon}{2} < Q(K) \leq Q(2K) \leq l.$$

Now let

$$Q_n(K) := \sup_{r_0 \in R} \int_{-1}^{0} \int_{r_0-K}^{r_0+K} \nu_n \, drds,$$

and notice that by fixing K from above, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$l - \frac{2\epsilon}{3} \leq Q_n(K) \leq Q_n(2K) \leq l + \frac{2\epsilon}{3}.$$

It is possible to find $r_n \in \mathbb{R}$ such that

$$l - \epsilon < \int_{-1}^{0} \int_{r_n-K}^{r_n+K} \nu_n \, drds \leq \int_{-1}^{0} \int_{r_n-K}^{r_n+K} \nu_n \, drds < l + \epsilon.$$

Now set $\Phi \in C_c^\infty([-2, 2])$ and $\Psi \in C_b^\infty(\mathbb{R})$ such that

$$\Phi, \Psi \geq 0, \qquad \Phi \equiv 1 \text{ on } [-1, 1], \qquad \text{and } \Phi^p + \Psi^p = 1.$$

Moreover, define

$$\Phi_q(\cdot) := \Phi\left(\frac{\cdot}{q}\right), \qquad \Psi_q(\cdot) := \Psi\left(\frac{\cdot}{q}\right) \qquad \text{for } q > 0.$$

It follows that

$$|D^k \Phi_q| = \mathcal{O}\left(\frac{1}{q^k}\right), \qquad \text{and } |D^k \Psi_q| = \mathcal{O}\left(\frac{1}{q^k}\right) \qquad \text{for } k \in \mathbb{N} \cup \{0\}. \tag{2.45}$$

Lastly, define

$$v_n^1(r, s) := \Phi_{q_n}(r - r_n) v_n(r, s), \quad v_n^2 := \Psi_{q_n}(r - r_n) v_n(r, s),$$

where $q_n > K$ and $|q_n| \to \infty$ as $n \to \infty$. By this construction, we have the following:

$$\|v_n^1\|_{L^p(R)} + \|v_n^2\|_{L^p(R)} = \|v_n\|_{L^p(R)};$$

$$\|v_n^1\|^p_{W^{2,p}(R)} + \|v_n^2\|^p_{W^{2,p}(R)} = \|v_n\|^p_{W^{2,p}(R)} + \mathcal{O}\left(\frac{1}{q_n^p}\right); \text{ and}$$

$$\|v_n^1\|^p_{L^p(R)} \geq \|v_n^1\|_{L^p((r_n-K,r_n+K)\times(-1,0))} > 1-\epsilon.$$

Now let $w_n^1 := v_n^1(r+r_n,s) = \Phi_{q_n}(r)v_n(r+r_n,s)$. It follows that $\|w_n^1\|^P_{W^{2,p}(R)} = \|v_n^1\|^p_{W^{2,p}(R)}$ is bounded. Furthermore, $w_n^1 \rightharpoonup w^1$ in $W^{2,p}(R)$ and hence $w_n^1 \to w^1$ in $L^p_{\text{loc}}(R)$. Also

$$l - e < \|v_n^1\|^p_{L^p((r_n-K,r_n+K)\times(-1,0))} = \|w_n^1\|^p_{L^p((-K,K)\times(-1,0))} \to \|w^1\|_{L^p(-K,K)\times(-1,0)}.$$

which implies that

$$0 < l - \epsilon < \|w^1\|_{L^p((-K,K)\times(-1,0))} < \|w^1\|_{L^p(R)}. \tag{2.46}$$

A direct computation shows that

$$\mathcal{A}^{n+}w_n^1 = \Phi_{q_n}(\mathcal{A}^{n+}v_n) + \frac{1}{q_n^2}\frac{\Phi_{rr}\left(\frac{\cdot}{q_n}\right)}{\phi_s^{n+}+H_s}v_n,$$

$$B^{n+}w_n^1 = \left(\frac{1}{q_n}\frac{\phi_r^{n+}}{(\phi_s^{n+}+H_s)^2}\Phi_r\left(\frac{\cdot}{q_n}\right) + \frac{1}{F^2}\Phi_{q_n}\right)v_n + (B^{n+}v_n)\Phi_{q_n}.$$

The weak convergence of w_n^1 in $W^{2,p}(R)$ together with the bounds (2.45), we see that $\mathcal{L}^{n+}w_n^1 \rightharpoonup \tilde{\mathcal{L}}w^1$ in Y. On the other hand, since $\mathcal{L}v_n \to 0$ in Y, we see that

$$\mathcal{L}^{n+}w_n^1 = \Phi_{q_n}\mathcal{L}^{n+}v_n(r+r_n,s) + \mathcal{O}\left(\frac{1}{q_n}\right) \to 0 \text{ in } Y.$$

Thus, $\tilde{\mathcal{L}}w^1 = 0$, which implies that $w^1 = 0$, contradicting (2.46).

We have shown that none of the three possibilities can occur, and arrive at a contradiction. Therefore, we must have that $u_n \to 0$ strongly in $L^p(R)$. ∎

THEOREM 2.3 (semi-Fredholm operators). *Assume that $\tilde{\mathcal{L}}$ is semi-Fredholm with index 0. Then $\mathcal{L}$ is also semi-Fredholm with index 0.*

Proof of Theorem 2.3. Consider the family of operators $\mathcal{L}_t = \tilde{\mathcal{L}} + t(\mathcal{L} - \tilde{\mathcal{L}})$ for $t \in [0,1]$. Notice that for each t, $\mathcal{L}_t$ is locally proper. By Yood's criterion [**45**], $\mathcal{L}_t$ is also semi-Fredholm for each t, and by continuity of index, $\mathcal{L}_1 = \mathcal{L}$ and $\mathcal{L}_0 = \tilde{\mathcal{L}}$ have the same Fredholm index. ∎

2.5. Qualitative properties

2.5.1. Elevation. In this subsection, we show that the waves $(\phi^\epsilon, F^\epsilon)$ constructed in Lemma 2.16 which are supercritical must also be waves of elevation. Particularly, we are interested in solutions h of (2.16) which are independent of the horizontal variable r. As mentioned in Section 2.6.1, these solutions represent horizontal laminar flows with constant depth, and are solutions of (2.47). Any solution of this form satisfies

$$-\left(\frac{1}{2h_s^2} - \frac{1}{2H_s^2}\right)_s = 0, \quad h(-1) = 0. \tag{2.47}$$

Notice that the asymptotic solution $(H(s), 0)$ is such a solution. Recalling the subsequent relationship between H and Γ

$$\Gamma(s) = \frac{1}{2H_s^2},$$

we can solve for $H(s)$ to reveal a family of horizontal laminar solutions

$$H(s;\kappa) := \int_{-1}^{s} \frac{1}{\sqrt{\kappa + 2\Gamma(t)}}\, dt, \tag{2.48}$$

where κ is a constant of integration. The functions $H(s;\kappa)$ generalize the function $H(s)$ from Section 2.3; notice, in particular, that there exists a unique value λ such that

$$1 = H(0;\lambda) := \int_{-1}^{0} \frac{1}{\sqrt{\lambda + 2\Gamma(t)}}\, dt,$$

and $H(s;\lambda) = H(s)$. Let $d^* \in (d, \infty]$ be the maximum depth of these trivial flows,

$$d^* = d \cdot \sup_{\kappa} H(0;\kappa) = d \cdot H(0; -2\Gamma_{\min}),$$

where

$$\Gamma_{\min} = \min_{s\in[-1,0]} \Gamma(s).$$

We will need the following lemma concerning the properties of the flows $H(s;\kappa)$.

Lemma 2.6. *Define* $A : (-2\Gamma_{min}, \infty) \to \mathbb{R}$ *by*

$$A(\kappa) := \begin{cases} \dfrac{1}{2}\left(\dfrac{\kappa - \lambda}{1 - H(0;\kappa)}\right), & \kappa \neq \lambda, \\ \\ \mu_{cr} & \kappa = \lambda. \end{cases}$$

Then A *is* C^1 *and strictly increasing. Moreover, if* $d^* = \infty$*, then* $\lim_{\kappa\downarrow -2\Gamma_{min}} A(\kappa) = 0$*, and if* $d^* < \infty$*, then* $\lim_{\kappa\downarrow -2\Gamma_{min}} A(\kappa) > 0$*.*

Proof. By differentiating under the integral, we see that

$$\frac{\partial}{\partial\kappa} H(0;\kappa) = -\frac{1}{2\mu_{\text{cr}}}.$$

Thus $H(0;\kappa)$ is a strictly decreasing function of κ for $\kappa \in (-2\Gamma_{\min}, \infty)$. A similar computation shows that $H(0;\kappa)$ is a strictly convex function of $\kappa \in (-2\Gamma_{\min}, \infty)$. Now notice that for $\kappa \neq \lambda$, A is a difference quotient in κ due to the fact that $H(0;\lambda) = 1$. A straightforward calculation shows that

$$\lim_{\kappa\to\lambda} A(\kappa) = \mu_{\text{cr}} > 0,$$

so we may conclude that A is C^1 and strictly increasing for $\kappa \neq \lambda$. Finally, $\lambda > -2\Gamma_{\min}$ and $d^* > d$ imply

$$\lim_{\kappa\downarrow -2\Gamma_{\min}} A(\kappa) = \frac{1}{2}\left(\frac{2\Gamma_{\min} + \lambda}{d^*/d - 1}\right),$$

which is clearly positive if $d^* < +\infty$ and 0 if $d^* = +\infty$. ∎

In order to prove that supercriticality is equivalent to waves of elevation, we require maximum principle arguments; however, the regularity of our solutions currently do not satisfy the hypothesis for such arguments, particularly at the boundary. To overcome this obstacle, we introduce a slightly modified version of Theorem 7 from [**15**].

LEMMA 2.7. *Assume that $\alpha \in (0,1)$ and $\frac{1}{2H_s^2}$ is of class $C_{loc}^{1+\alpha}$ for s near $s = 0$ and $s = -1$. Then h_{ss} belongs to C_{loc}^{α} near both the top T and the bottom B. The same is true for h_{rs}.*

PROOF. Consider the infinite strip $S_1 := \{s_1 < s < 0\}$ near the top. By assumption, if s_1 is sufficiently close to 0, then

$$\left(\frac{1}{H_s^2}\right)_s \in C_{\text{loc}}^{\alpha}\left([2s_1, 0]\right).$$

Taking the finite difference in s, we have

$$\zeta^{\epsilon}(r,s) := \frac{h(r,s) - h(r,s-\epsilon)}{\epsilon}.$$

Now taking the difference of the height equations for h and $h^{\epsilon}(r,s) := h(r, s-\epsilon)$, we obtain the uniformly elliptic problem

$$(2.49) \qquad \begin{cases} \mathcal{A}\zeta^{\epsilon} = -\dfrac{1}{2\epsilon}\left\{\dfrac{1}{(H_s^{\epsilon})^2} - \dfrac{1}{H_s^2}\right\}_s & \text{in } S_1 \\[2ex] B\zeta^{\epsilon} = 0 & \text{in } T_1 \end{cases}$$

where $\mathcal{A}$ and B are the operators in (2.55) with h^{λ} replaced by h^{ϵ}. Note also that $\zeta^{\epsilon}\Big|_{s=s_1}$ is bounded in $C_{\text{loc}}^{\alpha}(\mathbb{R})$. Since all of the coefficients of (2.49) are C_{loc}^{α} on the strip S_1 and its top boundary T_1, we may apply the Schauder-type estimate from Theorem 3 of [**15**] on the strip $\overline{S}_1$:

$$(2.50) \qquad |\zeta^{\epsilon}|_{1+\alpha;\text{loc}} \lesssim \left|\frac{1}{H_s^2}\right|_{\alpha;\text{loc}} + |\zeta^{\epsilon}|_{L^{\infty}}.$$

This implies that $\{\zeta^\epsilon\}$ is bounded in $C^{1+\alpha}_{\rm loc}(\overline{S}_1)$.

Since $C^{1+\beta}_{\rm loc}$ is a Banach space for all $\beta \in (0, \alpha)$, it follows that $\{\zeta^\epsilon\}$ has a limit in $C^{1+\beta}_{\rm loc}$ as $\epsilon \to 0$ for all $\beta \in (0, \alpha)$, and hence, $h_s \in C^{1+\beta}_{\rm loc}(\overline{S}_1)$. Then $h_s \in C^{1+\beta}_{\rm loc}(\overline{S}_1) \subset C^{1}_{\rm loc}(\overline{S}_1)$ satisfies an elliptic PDE with oblique boundary conditions and coefficients in $C^{1+\beta}_{\rm loc}(\overline{S}_1) \subset C^{\alpha}_{\rm loc}(\overline{S}_1)$. Applying a similar Schauder-estimate argument as that above, we conclude that $h_{ss} \in C^{\alpha}_{\rm loc}(\overline{S}_1)$.

A similar argument shows that $h_{rs} \in C^{\alpha}_{\rm loc}(\overline{S}_1)$. ∎

We arrive at the main theorem for this subsection.

THEOREM 2.4 (Elevation). *Every nontrivial solitary wave solution* $(\phi, F) \in W^{2,p}(\overline{R}) \times \mathbb{R}$ *of* (2.17) *is supercritical if and only if it is a wave of elevation in the sense that*

$$\phi > 0 \text{ in } R \cup T.$$

PROOF OF THEOREM 2.4. That elevation implies supercriticality is due to a direct application of Theorem 1.1 in [**60**]. The converse follows a similar proof as [**59**, Theorem 2.1], but requires the additional regularity at the boundary as in Lemma 2.7 in order to apply the Hopf lemma.

Now suppose that (ϕ, F) is a nontrivial solitary wave with $\epsilon \geq 0$. We wish to show that $h(r, 0) \geq 0$ for all $r \in \mathbb{R}$, where $h := H(s; \lambda) + \phi$. By way of contradiction, assume $h(r, 0) < 0$. Since $h(r, 0) \to 0$ as $r \to \pm\infty$, we know that $h(r, 0)$ must achieve its minimum at some $r = r_0 \in \mathbb{R}$. Recalling that $H(0; \kappa)$ is decreasing with $H(0; \lambda) = 1$ and $H(0, \kappa) \to 0$ as $\kappa \to \infty$, there exists a unique $\kappa_* > \lambda$ such that

$$h(r_0, 0) = H(0; \kappa_*) < 0.$$

Define

$$\varphi(r,s) := h(r,s) - H(s;\kappa_*).$$

Taking the difference of the height equations solved by h and $H^{\kappa_*} := H(s;\kappa_*)$, we see that φ solves the elliptic equation

$$\begin{cases} \mathscr{L}\varphi = 0, & \text{in } R \\ \mathscr{B}\varphi = 0 & \text{on } T \\ \varphi = 0 & \text{on } B \end{cases} \tag{2.51}$$

where

$$\mathscr{L}\varphi := \left(\frac{\varphi_r}{h_s}\right)_r - \left(\frac{h_r}{2h_s^2}\varphi_r - \frac{h_s + H_s^{\kappa_*}}{2h_s^2(H_s^{\kappa_*})^2}\right)_s,$$

$$\mathscr{B}\varphi := \frac{h_r}{2h_s}\partial_r + \frac{h_s + H_s^{\kappa_*}}{2h_s^2(H_s^{\kappa_*})^2}\varphi_s + \frac{1}{F^2}\varphi.$$

By design, $\varphi \geq 0$ on $s = 0$ and $\varphi = 0$ on $s = -1$. Since $\kappa > \lambda$, the asymptotic height condition (2.9b) indicates that

$$\lim_{r\pm\infty} \varphi(r,s) = H(s;\lambda) - H(s;\kappa) \geq 0,$$

uniformly in s. An application of the maximum principle (Theorem A.1.1) allows us to conclude that $\varphi \geq 0$ in $\overline{R}$.

Since $\phi(r_0, 0) = 0$ and $\phi \not\equiv =$, we may conclude that

$$\varphi_s(r_0, 0) = h_s(r_0, 0) - H_s(0;\kappa_*) < 0, \tag{2.52}$$

by the Hopf lemma. Recalling that $H_s(0;\kappa_*) = (\kappa_*)^{-1/2}$, it follows that $h_s(r_0, 0) < (\kappa_*)^{-1/2}$. However, the nonlinear boundary condition (2.9a) at $(r_0, 0)$ together with the fact that

$$h_r(r_0,0) = 0, \quad h(r_0,0) = H(0;\kappa_*), \text{ and } \quad h_s(r_0,0) < (\kappa_*)^{-1/2},$$

implies that

$$\mu > \frac{1}{2} \cdot \frac{\kappa_* - \lambda}{1 - H(0;\kappa_*)} =: A(\kappa_*).$$

With $\kappa_* > \lambda$, Lemma 2.6 indicates that

$$\mu > A(\kappa_*) \geq \mu_{\mathrm{cr}},$$

contradicting our assumption that $\mu \leq \mu_{\mathrm{cr}}$. Therefore, $h(r,0) \geq 0$ for all $r \in \mathbb{R}$.

With this result in hand, we know immediately that $\phi(r,s) := h(r,s) - H(s;\lambda)$ satisfies $\phi \geq 0$ on $s = 0$ and $\phi = 0$ on $s = -1$. Applying the maximum principle as before, we conclude that $\phi \geq 0$ in R, and hence $\phi > 0$ by the strong maximum principle.

We wish to show that $\phi(r,0) \geq 0$ for all $r \in \mathbb{R}$. By way of contradiction, assume $\phi(r_0,0) = 0$ for some $r_0 \in \mathbb{R}$. Since $\phi \geq 0$ in $\overline{R}$ and $\phi \not\equiv 0$, we apply the Hopf lemma to obtain

$$\phi_s(r_0,0) = h_s(r_0,0) - \lambda^{-1/2} < 0. \tag{2.53}$$

On the other hand, the nonlinear boundary condition of (2.17) at $(r_0,0)$ yields

$$\lambda = h_s(r_0,0)^2,$$

contradicting the strict inequality (2.53). Therefore, $\phi > 0 \in R \cup T$. ∎

2.5.2. Symmetry and monotonicity. In order to eventually establish existence of large-amplitude solitary water waves with discontinuous vorticity using the method originally developed in [**14**], we will need the operator to satisfy a few specific properties. Recall the height equation (2.17) satisfied by

$$\phi(r,s) := h(r,s) - H(s)$$

with the asymptotic conditions

$$\phi \to 0, \quad \phi_r \to 0, \quad \phi_s \to 0 \qquad \text{as } r \to \pm\infty$$

uniformly in s.

We now prove even symmetry for solitary water waves with discontinuous vorticity by using a moving planes method introduced by Maia in [**40**].

THEOREM 2.5 (Symmetry and monotonicity). *Let $(\phi, F) \in W^{2,p}(\overline{R}) \times \mathbb{R}$ be a supercritical solution of* (2.17) *which satisfies the upstream (downstream) condition*

$$\phi \to 0, \quad D\phi \to 0. \tag{2.54}$$

Then, after a translation, ϕ is a symmetric and monotone solitary wave; that is, there exists $r_* \in \mathbb{R}$ such that $r \mapsto \phi(r, \cdot)$ is symmetric about $\{r = r_*\}$ and

$$\pm\phi_r > 0 \qquad \textit{for } \pm(r_* - r) > 0, -1 < s \leq 0.$$

We aim to apply the moving planes method as in [**12**]. To do so, we start by considering the reflected functions

$$h^\lambda(r, s) := h(2\lambda - r, s)$$

where $\{r = \lambda\}$ is the axis of reflection. We also introduce the difference

$$v^\lambda(r, s) := h^\lambda(r, s) - h(r, s).$$

In this setting, it suffices to prove that $\{r = r_*\}$ is an axis of even symmetry if and only if $v^{r_*} \equiv 0$.

Define the λ-dependent sets

$$T^\lambda := \{(r, 0) : r < \lambda\}; \quad R^\lambda := \{(r, s) \in R : r < \lambda\}; \quad B^\lambda := \{(r, -1) : r < \lambda\}.$$

If h solves the height equation (2.16), then for each λ, v^λ solves

$$\tag{2.55} \begin{cases} \mathcal{A}v^\lambda = 0 & \text{in } R^\lambda, \\ Bv^\lambda = 0 & \text{on } T^\lambda, \\ v^\lambda = 0 & \text{on } B^\lambda, \end{cases}$$

where, in self-adjoint form,

$$\mathcal{A}v^\lambda := \left\{ \frac{1}{h_s^\lambda} v_r^\lambda - \frac{h_r}{h_s^\lambda h_s} v_s^\lambda \right\}_r - \left\{ \frac{h_r^\lambda + h_r}{2(h_s^\lambda)^2} v_r^\lambda - \frac{(1+h_r^2)(h_s^\lambda + h_s)}{2(h_s^\lambda)^2 h_s^2} v_s^\lambda \right\}_s ,$$

$$Bv^\lambda := \frac{h_r^\lambda + h_r}{2(h_s^\lambda)^2} v_r^\lambda + \frac{(1+h_r^2)(h_s^\lambda + h_s)}{2(h_s^\lambda)^2 h_s^2} v_s^\lambda + \frac{1}{F^2} v^\lambda.$$

Lemma 2.8. *Under the hypotheses of Theorem 2.5, there exists $K > 0$ such that*

$$\tag{2.56} v^\lambda \geq 0 \quad \text{in } R^\lambda \qquad \text{for all } \lambda < -K, \text{ and}$$

$$\tag{2.57} h_r \geq 0 \quad \text{in } R^\lambda \qquad \text{for all } \lambda < -K.$$

To proceed, we intend to use maximum principle arguments; however, the regularity of our solutions currently do not satisfy the hypothesis for such arguments. Once again, we must invoke Lemma 2.7 in order to apply the maximum principle.

Proof of Lemma 2.8. It is important to note that, once again, the coefficient last in B has a sign which is not suitable for maximum principle arguments. Luckily, we can overcome this difficulty by modifying the problem using the same "dividing trick" used in Section 2.4.3. Let Φ be defined as in Lemma 2.4; in particular, we recall that

$$\tag{2.58} \left(-\frac{1}{H_s^3} \Phi_s + \frac{1}{F^2} \Phi \right) \Big|_{s=0} < 0 \qquad \text{for } 0 < \frac{1}{F^2} < \frac{1}{F_{\text{cr}}^2}.$$

We also choose ϵ sufficiently small so that $\Phi > \epsilon$ on $(-1, 0)$ and we may now define u^λ by $v^\lambda = u^\lambda \Phi$. Recalling that Φ is only a function of s, a straightforward calculation shows that

$$
\begin{aligned}
0 &= \mathcal{A}v^\lambda \\
&= (\mathcal{A}'u^\lambda)\Phi - \left[\frac{2h_r^\lambda}{(h_s^\lambda)^2}\Phi_s\right] u_r^\lambda + \left[\frac{2(1+(h_r^\lambda)^2)}{(h_s^\lambda)^3}\Phi_s\right] u_s^\lambda + \mathcal{Z}u^\lambda \\
&=: \tilde{\mathcal{A}}u^\lambda,
\end{aligned}
$$

where $\mathcal{A}'$ is the principal part of $\mathcal{A}$:

$$
\mathcal{A}' := \frac{1}{h_s^\lambda}\partial_r^2 - \frac{2h_r^\lambda}{(h_s^\lambda)^2}\partial_r\partial_s + \frac{1+(h_r^\lambda)^2}{(h_s^\lambda)^2}\partial_s^2,
$$

and

$$
\mathcal{Z} := \mathcal{A}\Phi = \frac{1+(h_r^\lambda)^2}{(h_s^\lambda)^3}\Phi_{ss} + \left(\frac{1}{2H_s^2}\right)_s \left[\frac{(h_s^\lambda)^2 + h_s^\lambda h_s + h_s^2}{(h_s^\lambda)^3}\right]\Phi_s.
$$

A similar computation reveals that

$$
0 = Bv^\lambda = (B'u^\lambda)\Phi + (B'\Phi)u^\lambda =: \tilde{B}u^\lambda,
$$

where B' is the principal part of B:

$$
B' := \frac{h_r^\lambda + h_r}{2h_s^2}\partial_r - \frac{(h_s^\lambda + h_s)(1+(h_r^\lambda)^2)}{2(h_s^\lambda)^2 h_s^2}\partial_s.
$$

Therefore, u^λ solves the PDE

$$
\begin{cases}
\tilde{\mathcal{A}}u^\lambda = 0 & \text{in } R^\lambda, \\
\tilde{B}u^\lambda = 0 & \text{on } T^\lambda, \\
u^\lambda = 0 & \text{on } B^\lambda.
\end{cases} \tag{2.59}
$$

We claim that there exists $K > 0$ large enough that $u^\lambda \geq 0$ in R^λ for all $\lambda \leq -K$. By way of contradiction, assume that for every K, there exists some $\lambda_0 \leq -K$ such that u^{λ_0} takes on a negative value in R^{λ_0}. Since h is a wave of elevation, h^λ must also be a wave of elevation for any λ. Now we know that $u^\lambda = 0$ on $\{r = \lambda\}$, and

$$
u^\lambda = \frac{h^\lambda - h}{\Phi} > \frac{H - h}{\Phi}.
$$

Furthermore, as $r \to -\infty$, we must have that

$$\frac{H-h}{\Phi} \to 0.$$

Therefore, u^{λ_0} taking on a negative value in R^{λ_0} implies that there exists a point $(r_0, s_0) \in R^{\lambda_0} \cup T^{\lambda_0}$ such that

$$u^{\lambda_0}(r_0, s_0) = \inf_{R^{\lambda_0}} u^{\lambda_0} < 0. \tag{2.60}$$

Case I ($(\mathbf{r_0}, \mathbf{s_0}) \in \mathbf{R}^{\lambda_0}$). If $(r_0, s_0) \in R^{\lambda_0}$, then u^{λ_0} has a global maximum, and

$$0 = \nabla u^{\lambda_0}(r_0, s_0) \tag{2.61}$$

$$= \left[\frac{\nabla v^{\lambda_0}}{\Phi} + v^{\lambda_0}\left(0, \frac{-\Phi_s}{\Phi^2}\right)\right](r_0, s_0) \tag{2.62}$$

$$= \left[\frac{\nabla v^{\lambda_0}}{\Phi} - v^{\lambda_0}\left(0, \frac{\Phi_s}{\Phi^2}\right)\right](r_0, s_0). \tag{2.63}$$

Recalling that $\phi := h - H$ satisfies the upstream condition (2.54), we know that for each $\delta > 0$, there exists a K sufficiently large enough so that

$$|h(r_0, s) - H(s)| < \delta \quad \text{ on } R^{-K}.$$

Additionally, from the chain of inequalities

$$H(s_0) < h^{\lambda_0}(r_0, s_0) < h(r_0, s_0) < H(s_0) + \delta$$

we conclude that $|v^{\lambda_0}(r_0, s_0)| < \delta$. Furthermore, (2.61) allows us to obtain bounds on $\nabla v^{\lambda_0}(r_0, s_0)$:

$$|\nabla v^{\lambda_0}(r_0, s_0)| = \left|\frac{\Phi_s(s_0)}{\Phi(s_0)} v^{\lambda_0}(r_0, s_0)\right| < C\delta,$$

with C depending on ϵ. From this we may conclude that (for K large enough)

$$\mathcal{Z}(s_0) = \left(\frac{\Phi_s}{H_s^3}\right)_s \Big|_{s_0} + \mathcal{O}(\delta).$$

But conditions (2.35) and (2.36) guarantee that $\mathcal{Z}(s_0) < 0$ for K sufficiently large. Hence $\mathcal{Z}$ has the correct sign, and an application of the maximum principle to (2.59) at (r_0, s_0) leads to a contradiction.

Case II ($(\mathbf{r_0}, \mathbf{s_0}) = (\mathbf{r_0}, \mathbf{0}) \in \mathbf{T}^{\lambda_0}$). Suppose $(r_0, s_0) \in T^{\lambda_0}$. By the Hopf lemma,

(2.64) $\quad \nu \cdot \nabla u^{\lambda_0}(r_0, 0) > 0$

implies that

(2.65) $\quad u_r^{\lambda_0}(r_0, s_0) = 0 \;\text{ and }\; u_s^{\lambda_0}(r_0, 0) > 0.$

This, of course, further implies that

(2.66) $\quad h_r^{\lambda_0}(r_0, 0) = h_r(r_0, 0)$

and a reiteration of the argument in the previous case guarantees that for K sufficiently large,

(2.67) $\quad |h(r_0, 0) - H(0)| < \delta,$

(2.68) $\quad |h_s(r_0, 0) - H(0)| < \delta, \text{ and}$

(2.69) $\quad |h^{\lambda_0}(r_0, 0) - H(0)| < \delta.$

From the boundary condition of (2.9a) evaluated at $(r_0, 0)$, we see that

$$\frac{1 + (h_r^{\lambda_0})^2}{2(h_s^{\lambda_0})^2} + \frac{1}{F^2} h^{\lambda_0} = \frac{1 + h_r^2}{2h_s^2} + \frac{1}{F^2} h.$$

Then (2.66) and (2.67) yield the estimate

$$|v_r^{\lambda_0}(r_0, 0)| = |h_r^{\lambda_0}(r_0, 0) - h(r_0, 0)| < C\delta.$$

Together, the boundary condition of (2.59) becomes

$$\begin{aligned}\tilde{B}u^{\lambda_0} &= -\left(\frac{(h_s^{\lambda_0} + h_s)(1 + (h_r^{\lambda_0})^2)}{2(h_s^{\lambda_0})^2 h_s^2} u_s^{\lambda_0}\right)\Phi + \left(-\frac{1}{H^3}\Phi_{ss} + \frac{1}{F^2}\Phi + \mathcal{O}(\delta)\right)u^{\lambda_0} \\ &= 0,\end{aligned}$$

on T^{λ_0}. But in the context of (2.37), this means that the coefficient of u^{λ_0} above is negative, and hence (2.60) implies that $u_s^{\lambda_0}(r_0, 0) > 0$, a contradiction. ∎

With Lemma 2.8 in hand, we may finally turn to the proof of Theorem 2.5.

Proof. Consider the set

$$\Lambda := \{\lambda_0 : v^\lambda > 0 \text{ in } R^\lambda, \forall \lambda < \lambda_0\},$$

which is nonempty in light of Lemma 2.8. Then we may define

$$\lambda_\star := \sup \Lambda,$$

and consider two cases:

Case I ($\lambda_\star < +\infty$) Suppose $\lambda_\star < +\infty$ and notice that, by continuity of v^λ,

$$v^{\lambda_\star} \geq 0 \text{ on } R^{\lambda_\star}$$

Recall that $v^{\lambda_\star}$ satisfies the elliptic equation (2.55) with $\lambda = \lambda_\star$; an application of the strong maximum principle implies either of the following possibilities:

(i) $v^{\lambda_\star} > 0$, or

(ii) $v^{\lambda_\star} \equiv 0$ in $R^{\lambda_\star}$.

We claim that $v^{\lambda_\star} \equiv 0$ in $R^{\lambda_\star}$. By way of contradiction, assume $v^{\lambda_\star} > 0$ in $R^{\lambda_\star}$; then there exists a sequence $\{\lambda_k\}_{k=1}^\infty$ with $\lambda_k \searrow \lambda_\star$ and a sequence of points $\{(r_k, s_k)\}_{k=1}^\infty \subset \overline{R}^{\lambda_k}$ such that

$$v^{\lambda_k}(r_k, s_k) = \inf_{R^{\lambda_k}} v^{\lambda_k} < 0.$$

Since $v^{\lambda_k} = 0$ on B^{λ_k} and $\{r = \lambda_k\}$, the strong maximum principle guarantees that $(r_k, s_k) \in T^{\lambda_k}$, implying that

$$v_s^{\lambda_k}(r_k, 0) \leq 0 \quad \text{and} \quad v_r^{\lambda_k}(r_k, 0) = 0.$$

We want to show that $\{r_k\}$ is bounded from below. If $\{r_k\}$ were not bounded from below, then $r_k < -K$ for all K sufficiently large satisfying (2.56). Consider $u^{\lambda_k} := \dfrac{v^{\lambda_k}}{\Phi}$ as in Lemma 2.4. This implies that

$$\tilde{B}u^{\lambda_k}(r_k, 0) > 0,$$

a contradiction.

Hence $\{r_k\}$ is bounded from below by $-K$, and given that $\lambda_k \searrow \lambda_\star$, it follows that $\{r_k\}$ is also bounded above by λ_1. Invoking Bolzano-Weierstrass, there exists a subsequence $\{r_{k_j}\}_{j=1}^{\infty}$ such that

$$(r_{k_j}, 0) \to (r_\star, 0) \text{ in } \overline{T}^{\lambda_\star} \text{ as } j \to \infty,$$

for some $r_\star \in [K, \lambda_\star]$. We know that $v^{\lambda_\star} > 0$ in $R^{\lambda_\star}$, which implies that

$$\lim_{j\to\infty} v^{\lambda_{k_j}}(r_{k_j}, 0) = v^{\lambda_\star}(r_\star, 0).$$

If $r_\star < \lambda_\star$, we have, by continuity, that

$$v^{\lambda_\star}(r_\star, 0) = v_r^{\lambda_\star}(r_\star, 0) = 0.$$

Now, an application of the Hopf lemma leads us to conclude that

$$\nu\dot{\nabla}v^{\lambda_\star}(r_\star, 0) < 0 \iff v_s^{\lambda_\star}(r_\star, 0) < 0,$$

where ν is the outward unit normal to $R^{\lambda_\star}$ at $(r_\star, 0)$. Reconsidering the operator B, we know that $Bv^{\lambda_\star} = 0$, and

$$0 = (Bv^{\lambda_\star})(r_\star, 0) = -\frac{(h_s^{\lambda_\star} + h_s)(1 + (h_r^{\lambda_\star})^2)}{2h_s^2(h_s^{\lambda_\star})^2} v_s^{\lambda_\star}(r_\star, 0) > 0,$$

which is impossible. Therefore, $(r_\star, 0)$ must be a corner point of $R^{\lambda_\star}$, i.e., $r_\star = \lambda_\star$. Since $v_r^{\lambda_\star}(r_\star, 0) = 0 = v^{\lambda_\star}(r_\star, 0)$, we have that $h^{\lambda_\star}(r_\star, 0) = 0$. We may rewrite the top boundary condition by clearing the denominators:

$$0 = Bv^\lambda = (h_s^\lambda)^2(h_r^\lambda + h_r)v_r^\lambda - (h_s^\lambda + h_s)(1 + (h_r^\lambda)^2)v_s^\lambda + 2\frac{1}{F^2}h_s^2(h_s^\lambda)^2 v^\lambda. \tag{2.70}$$

It follows that for $\lambda = \lambda_\star$,

$$0 = Bv^\lambda = (h_s^{\lambda_\star})^2(h_r^{\lambda_\star} + h_r)v_r^{\lambda_\star} - (h_s^{\lambda_\star} + h_s)(1 + (h_r^{\lambda_\star})^2)v_s^{\lambda_\star} + 2\frac{1}{F^2}h_s^2(h_s^{\lambda_\star})^2.$$

Differentiating with respect to r and evaluating at $(r_\star, 0) = (\lambda_\star, 0)$, and using the facts

$$h_r^{\lambda_\star}(\lambda_\star, 0) = -h_r(\lambda_\star, 0), \quad h_s^{\lambda_\star}(\lambda_\star, 0) = h_s(\lambda_\star, 0), \quad \text{and } h_{sr}^{\lambda_\star}(\lambda_\star, 0) = h_{rs}(\lambda_\star, 0),$$

we see that

$$\begin{aligned} 0 = (Bv^{\lambda_\star})_r &= -(h_{sr}^{\lambda_\star} + h_{sr})v_s^{\lambda_\star} - (h_s^{\lambda_\star} + h_s)v_{sr}^{\lambda_\star}\Big|_{(\lambda_\star,0)} \\ &= 2h_s(\lambda_\star, 0)v_{rs}^{\lambda_\star}(\lambda_\star, 0). \end{aligned}$$

Since we assume the absence of stagnation (i.e., $h_s > 0$), we immediately obtain

$$v_{rs}^{\lambda_\star}(\lambda_\star, 0) = 0.$$

Furthermore, since $v^{\lambda_\star}(\lambda_\star, \cdot) \equiv 0$, it follows that

$$v_s^{\lambda_\star}(\lambda_\star, 0) = v_{ss}^{\lambda_\star}(\lambda_\star, 0) = 0.$$

Lastly, notice that by (2.55), we may rewrite $\mathcal{A}v^{\lambda_\star} = 0$ so that for solving for $v_{rr}^{\lambda_\star}$ for $v \in C^{0,\alpha}_{\text{loc}}$ near T,

$$\begin{aligned} v_{rr}^{\lambda_\star} = h_s^{\lambda_\star}\Bigg(&\frac{2h_r^{\lambda_\star}}{(h_s^{\lambda_\star})^2}v_{rs}^{\lambda_\star} - \frac{1 + (h_r^{\lambda_\star})^2}{(h_s^{\lambda_\star})^3}v_{ss}^{\lambda_\star} - \frac{h_s s(h_r^{\lambda_\star} + h_r) - 2h_s^{\lambda_\star}h_{rs}}{(h_s^{\lambda_\star})^3}v_r^{\lambda_\star} \\ &- \left(\frac{1}{H_s^2}\right)_s \frac{(h_s^{\lambda_\star})^2 + h_s^{\lambda_\star}h_s + h_s^2}{(h_s^{\lambda_\star})^2}v_s^{\lambda_\star}\Bigg) \end{aligned}$$

Evaluating at $(\lambda_\star, 0)$, we see that

$$v_{rr}^{\lambda_\star}(\lambda_\star, 0) = 0.$$

Hence $v^{\lambda_\star}$ and all of its derivatives up to second order vanish at $(\lambda_\star, 0)$. Since $v^{\lambda_\star}$ solves (2.55) in $R^{\lambda_\star}$, this violates the Serrin Edgepoint Lemma, which concludes that either the first or second order derivatives (in the direction of the unit normal at the surface) must be negative, a contradiction.

Therefore, we must conclude that $v^{\lambda_\star} \equiv 0$ in $R^{\lambda_\star}$ and hence h and ϕ are symmetric about the axis $\{r = \lambda_\star\}$.

Now we wish to consider the monotonicity of h. Once again, for $\lambda < \lambda_\star$, we have $v^\lambda > 0$ in R^λ. Notice that v^λ vanishes on $r = \lambda$, and it attains its minimum on $\overline{R}^\lambda$ there. Yet another iteration of the Hopf maximum principle determines that

$$\nu \cdot \nabla v^\lambda(\lambda, s) < 0 \iff v_r^\lambda(\lambda_s) < 0,$$

and furthermore,

$$h_r(\lambda, s) = -\frac{1}{2} v_r^\lambda(\lambda, s) > 0 \text{ for } \lambda < \lambda_\star,\ -1 < s < 0, \tag{2.71}$$

where we used the fact that $h_r^\lambda(\lambda, s) = -h_r(\lambda, s)$. Now, on the top boundary $T^{\lambda_\star}$, we know that $h_r \geq 0$ by continuity. By way of contradiction, assume that $h_r(\lambda, 0) = 0$ for some $(\lambda, 0) \in T^{\lambda_\star}$. Then using (2.70), differentiating with respect to r and evaluating at $(\lambda, 0)$, and using the identities

$$h_r(\lambda, 0) = -h_r^\lambda(\lambda, 0), \qquad v^\lambda(\lambda, 0) = v_r^\lambda(\lambda, 0) = 0;$$

$$h_s(\lambda, 0) = h_s^\lambda(\lambda, 0), \qquad h_{rs}(\lambda, 0) = -h_{rs}^\lambda(\lambda, 0),$$

we have

$$0 = 2h_s(\lambda, 0) v_{rs}^\lambda(\lambda, 0),$$

which implies that $v^\lambda_{rs}(\lambda, 0) = 0$. Once again, using the (2.55) to solve for v^λ_{rr}, we find that $v^\lambda_{rr}(\lambda, 0) = 0$. But, once again, the Serrin Edgepoint Lemma is violated since v^λ satisfies (2.55). This implies, of course, that $h_r > 0$ on $T^{\lambda_\star}$. This and (2.71) implies that $h_r > 0$ on all of $R^{\lambda_\star} \cup T^{\lambda_\star}$. The same is true for ϕ_r, and hence, $\{r = \lambda_\star\}$ is an axis of even symmetry, as desired.

Case II ($\lambda_\star = +\infty$) If $\lambda_\star = +\infty$, then $v^\lambda \geq 0$ in R^λ for all λ. Since v^λ solves (2.55) in R^λ, the maximum principle implies that $v^\lambda > 0$ in R^λ for all λ. This implies that as $|r| \to \infty$, $h \to H$ and $h^\lambda \to H^*$, where $H^* > H$. But this contradicts the fact that $h - H \in W^{2,p}(R)$. ∎

2.5.3. Asymptotic monotonicity and nodal properties. The global bifurcation theory will require us to prove that the symmetry and monotonicity of the small-amplitude solutions persist away from the point of bifurcation. The main difficulty of showing this lies in the fact that monotonicity is neither an open nor closed property in the topology of our Banach spaces $\mathcal{X}, \mathcal{Y}$. To overcome this obstruction, we will exploit the structure of our equation; namely a collection of sign conditions, commonly referred to as "nodal properties," which imply monotonicity while simultaneously defining a subset of $\mathscr{F}^{-1}(0)$ that is both open and closed in the appropriate topology.

We begin by dividing R into two subregions: a finite rectangle and two semi-infinite tails in which ϕ is small. On the finite rectangle, we can apply the same analysis for periodic solutions as in [**15**]. On the tails, we apply the strategy of [**59**] by considering solutions of (2.17) in the half strip

$$R^+ := \{(r, s) \in R : r > 0\}$$

with boundary

$$L^+ := \{(0,s) : s \in [-1,0]\}, \qquad T^+ := \mathbb{R}_+ \times \{0\}, \qquad B^+ := \mathbb{R}_+ \times \{-1\}.$$

PROPOSITION 2.1 (Asymptotic monotonicity). *There exists $\delta > 0$ such that if $\phi \in W^{2,p}(\overline{R^+})$, with additional regularity at the boundary as in Lemma 2.7, is a supercritical solution of the height equation (2.17) in R^+, and*

$$\|\phi\|_{W^{2,p}(R^+)} < \delta,$$

then ϕ_r exhibits the following monotonicity property:

$$\text{If } \pm\phi_r \le 0 \text{ on } L^+, \text{ then } \pm\phi_r < 0 \text{ in } R^+ \cup T^+. \tag{2.72}$$

Differentiating the height equation (2.17) with respect to r, we obtain a linear elliptic PDE for $v := \phi_r$ with coefficients depending on ϕ:

$$\begin{cases} \left(\dfrac{1}{\phi_s + H_s} v_r - \dfrac{\phi_r}{(\phi_s+H_s)^2} v_s\right)_r - \\ \qquad\qquad \left(\dfrac{\phi_r}{(\phi_s+H_s)^2} v_r - \dfrac{1+\phi_r^2}{(\phi_s+H_s)^3} v_s\right)_s = 0 & \text{in } R^+ \\ v = 0 & \text{in } B^+ \\ \dfrac{\phi_r}{(\phi_s+H_s)^2} v_r - \dfrac{1+\phi_r^2}{(\phi_s+H_s)^3} v_s + \dfrac{1}{F^2} v = 0 & \text{on } T^+. \end{cases} \tag{2.73}$$

Once again, we notice that the coefficient of the zeroth-order term has the “bad sign” in order to apply maximum principles. To avoid this complication, we require the following lemma.

LEMMA 2.9. *There exists $\epsilon > 0$ such that the function*

$$\hat{\Phi}(s) := \left(\epsilon + \int_{-1}^{s} H_s^3(t)\, dt\right)^{1-\epsilon} \tag{2.74}$$

satisfies

$$
\text{(2.75)} \qquad \begin{cases} \left(\dfrac{\hat{\Phi}_s}{H_s^3}\right)_s < 0 & \textit{in } [-1,0], \\[2ex] \hat{\Phi} > 0 & \textit{in } [-1,0], \\[2ex] \dfrac{\hat{\Phi}_s}{H_s^3} - \dfrac{1}{F^2}\hat{\Phi} > 0 & \textit{for } s = 0. \end{cases}
$$

Proof of Lemma 2.9. That $\hat{\Phi} > 0$ on $[-1, 0]$ for any $\epsilon > 0$ is clear by definition of $\hat{\Phi}$. Also, since

$$
\text{(2.76)} \qquad \hat{\Phi}_s = (1-\epsilon)\left(\epsilon + \int_{-1}^{s} H_s^3\,dt\right)^{-\epsilon} \cdot H_s^3,
$$

it follows that for $0 < \epsilon < 1$,

$$
\left(\frac{\hat{\Phi}}{H_s^3}\right) = -\epsilon(1-\epsilon)\left(\epsilon + \int_{-1}^{s} H_s^3\,dt\right)^{-1-\epsilon} \cdot H_s^3 < 0,
$$

for $s \in [-1, 0]$. Lastly,

$$
\left(\frac{\hat{\Phi}_s}{H_s^3} - \frac{1}{F^2}\hat{\Phi}\right)\Bigg|_{s=0} = (1-\epsilon)\left(\epsilon + \int_{-1}^{s} H_s^3\,dt\right)^{-\epsilon} - \frac{1}{F^2}\left(\epsilon \int_{-1}^{0} H_s^3\,dt\right)^{1-\epsilon},
$$

depends continuously on ϵ when ϵ is in a sufficiently small neighborhood of the origin. Recalling the definition of the critical Froude number F_{cr} from Lemma 2.1, we see that when $\epsilon = 0$,

$$
\begin{aligned}
\frac{\hat{\Phi}_s}{H_s^3}(0) - \frac{1}{F^2}\hat{\Phi}(0) &= 1 - \frac{1}{F^2}\int_{-1}^{0} H_s^3\,dt \\
&= 1 - \frac{F_{\text{cr}}^2}{F^2} > 0,
\end{aligned}
$$

since we are supercritical $(F > F_{\text{cr}})$. ∎

Proof of Proposition 2.1. Writing $u =: v\hat{\Phi}$, where $\hat{\Phi}$ is chosen according to Lemma 2.9, we can rewrite (2.73) as an equation solved by u. Specifically, the equation in the interior becomes

$$0 = \left(\frac{\hat{\Phi}}{\phi_s + H_s} u_r - \frac{\hat{\Phi}\phi_r}{(\phi_s + H_s)^2} u_s \right)_r - \left(\frac{\hat{\Phi}\phi_r}{(\phi_s + H_s)^2} u_r - \frac{\hat{\Phi}(1+\phi_r^2)}{(\phi_s + H_s)^3} u_s \right)_s \tag{2.77}$$

$$- \frac{\hat{\Phi}_s\phi_r}{(\phi_s + H_s)^2} u_r + \frac{\hat{\Phi}_s(1+\phi_r^2)}{(\phi_s + H_s)^3} u_s + \mathcal{Z}_1 u, \tag{2.78}$$

where $\mathcal{Z}_1$ is the zeroth-order coefficient in the interior given by

$$\mathcal{Z}_1 := -\left(\frac{\hat{\Phi}_s\phi_r}{(\phi_s + H_s)^2} \right)_r + \left(\frac{\hat{\Phi}_s(1+\phi_r^2)}{(\phi_s + H_s)^3} \right)_s . \tag{2.79}$$

On the boundary T^+, u satisfies

$$0 = \frac{\hat{\Phi}\phi_r}{(\phi_s + H_s)^2} u_r - \frac{\hat{\Phi}(1+\phi_r^2)}{(\phi_s + H_s)^3} u_s + \mathcal{Z}_2 u, \tag{2.80}$$

where $\mathcal{Z}_2$ is the zeroth order coefficient on the boundary given by

$$\mathcal{Z}_2 := \frac{1}{F^2}\hat{\Phi} - \frac{\hat{\Phi}_s(1+\phi_r^2)}{(\phi_s + H_s)^3}. \tag{2.81}$$

We aim to show that both $\mathcal{Z}_1$ and $\mathcal{Z}_2$ have the appropriate sign for maximum principle arguments. Previously, we have done so simply by verifying that the zeroth order coefficient is Hólder continuous and negative in the pointwise sense; however, the coefficients in the PDE satisfied by ϕ_r (and hence u) are only bounded and measurable. In light of this, we must use another version of the maximum principle (Theorem A.2) which states the non-positivity condition in a generalized sense. To this end, let $\varphi \in C_0^1(R^+)$ with $\varphi \geq 0$. Then integrating by parts, we have

$$\begin{aligned}
\int_{R^+} \mathcal{Z}_1 \varphi \, dr \, ds &= \int_{R^+} \hat{\Phi} \frac{\phi_r}{(\phi_s + H_s)^2} \varphi_r \, dr \, ds - \int_R^+ \hat{\Phi} \frac{1+\phi_r^2}{(\phi_s + H_s)^3} \varphi_s \, drds \\
&= -\int_{R^+} \hat{\Phi} \left(\frac{\phi_r}{(\phi_s + H_s)^2} \varphi_r - \frac{1+\phi_r^2}{(\phi_s + H_s)^3} \varphi_s \right)_s drds \\
&= -\int_{R^+} \hat{\Phi} \left(\frac{\phi_{rs}\varphi_r + \phi_r\varphi_{rs}}{(\phi_s + H_s)^2} - \frac{2\phi_r\varphi_r(\phi_{ss} + H_{ss})}{(\phi_s + H_s)^3} \right. \\
&\qquad \left. - \frac{2\phi_r\phi_{rs}\varphi_s + (1+\phi_r^2)\varphi_{ss}}{(\phi_s + H_s)^3} + \frac{3(1+\phi_r^2)(\phi_{ss} + H_{ss})\varphi_s}{(\phi_s + H_s)^4} \right) drds.
\end{aligned}$$

Since $\hat{\Phi} > 0$ in R^+, and the remaining terms above can be controlled in $L^\infty(R^+)$, the above term is negative as long as $\|\phi\|_{W^{2,p}(R^+)} < \delta$, for δ sufficiently small. In a similar fashion, we see that $\int_R^+ \mathcal{Z}_2 \varphi \, dr \, ds$ also has the correct sign for $\|\phi\|_{W^{2,p}}$ small enough. It follows that u satisfies an equation of the form

$$\sum_{i,j} \partial_i (a_{ij} \partial_j u) + \sum_i b_i \partial_i u + cu = 0,$$

with nonpositive zeroth order coefficient and whose coefficients have uniform bounds for $\|\phi\|_{W^{2,p}(R^+)} < \delta$ sufficiently small. Therefore, we may apply the maximum principle to conclude that (2.72) holds. ∎

With Lemma 2.1 in hand, we may address the nodal properties.

LEMMA 2.10. *Let* (ϕ, F) *be a solution of the height equation* (2.17)*, where* $\phi \in W_e^{2,p}(\overline{R})$*, with additional regularity at the boundary as in Lemma* 2.7*, is monotone in the sense that* $\phi_r < 0$ *in* $R^+ \cup T^+$*. Then* ϕ *exhibits the following nodal properties:*

(2.82a) $\phi_r < 0$ *in* $R^+ \cup T^+$,

(2.82b) $\phi_{rr} < 0$ *on* L^+,

(2.82c) $\phi_{rs} < 0$ *on* B^+,

(2.82d) $\phi_{rrs} < 0$ *at* $(0, -1)$,

(2.82e) $\phi_{rr} < 0$ *at* $(0, 0)$.

PROOF OF LEMMA 2.10. As in the proof of Lemma 2.1, $v := \phi_r$ satisfies the uniformly elliptic PDE (2.73), for which the zeroth-order term has an adverse sign. This time, the function $\hat{\Phi}$ is not helpful, since we are not assuming smallness of ϕ in

$W^{2,p}$. However, since ϕ is even in r and vanishes identically on B^+, we know that

$$(2.83) \qquad v = 0 \qquad \text{on } L^+ \cup B^+.$$

By assumption, we also have

$$v := \phi_r < 0 \qquad \text{in } R^+ \cup T^+.$$

Therefore, we know that v achieves its maximum value on $L^+ \cup B^+$, and we can apply the Hopf and Serrin edge point lemmas at the points on $\overline{L^+ \cup B^+}$.

Now, the first nodal property (2.82a) follows by assumption, and an application of the Hopf lemma on $L^+ \cup B^+$ yields

$$v_r < 0 \text{ on } L^+, \qquad \text{and} \qquad v_s < 0 \text{ on } B^+,$$

which are respectively equivalent to (2.82b) and (2.82c).

Now consider the corners. By (2.83), we know that

$$v = v_r = v_s = v_{ss} = v_{rr} = 0 \qquad \text{at } (0, -1).$$

Hence, by the Serrin edge point lemma, (2.82d) holds. Lastly, at the upper left corner point $(0,0)$,

$$(2.84) \qquad v(0,0) = v_s(0,0) = 0.$$

Differentiating the top boundary condition of the height equation (2.17) twice in r and evaluating at $(0,0)$, we find that

$$(2.85) \qquad 0 = \left(\frac{v_r^2}{(\phi_s + H_s)^2} - \frac{v_{sr}}{(\phi_s + H_s)^3} + \frac{1}{F^2} v_r \right) \Bigg|_{(0,0)}.$$

By way of contradiction, assume $v_r(0,0) = \phi_{rr}(0,0) = 0$. Using (2.85) and the fact that ϕ is even in r, we see that, in addition to (2.84),

$$(2.86) \qquad v_{rs}(0,0) = 0.$$

Using the equation satisfied by v (2.73) on the top boundary, specifically at the point $(0,0)$, we find that $v_s(0,0) = 0$ and $v_{ss}(0,0) = 0$. Lastly, using (2.73) for the interior, we may write v_{rr} in terms of the other derivatives of v, which implies that $v_{rr}(0,0) = 0$ as well. This clearly violates the Serrin edge point lemma. Therefore, $v_r(0,0) = 0$, which is equivalent to (2.82e), and the proof is complete. ∎

LEMMA 2.11 (Open property). *Let* $(\phi, F), (\tilde{\phi}, \tilde{F})$ *be two supercritical solutions of* (2.17) *with* $\phi, \tilde{\phi} \in W_e^{2,p}(\overline{R})$ *and additional regularity at the boundary as in Lemma* 2.7. *If* ϕ *satisfies the nodal properties* (2.82), *then there exists* $\epsilon = \epsilon(\phi) > 0$ *such that*

$$\|\phi - \tilde{\phi}\|_{W^{2,p}(R)} + |F - \tilde{F}| < \epsilon$$

implies that $\tilde{\phi}$ *also satisfies* (2.82).

PROOF OF LEMMA 2.11. By Lemma 2.10, we need only to show that $\tilde{\phi}_r < 0$ in $R^+ \cup T^+$. To this end, we first divide R^+ into two components: a finite rectangle

$$R_1^+ := \{(r,s) \in R^+ : r < 2K\},$$

and a "tail" region

$$R_2^+ := \{(r,s) \in R^+ : r > K\},$$

with the top $T_{1,2}^+$, bottom $B_{1,2}^+$, and left $L_{1,2}^+$ boundary components defined likewise. The constant $K > 0$ is to be determined. We begin by analyzing the finite rectangle R_1^+. Arguing as in [**15**, Theorem 8], we may conclude that $\tilde{\phi}$ satisfies the nodal properties (2.82) with R^+, T^+, B^+, L^+ replaced by R_1^+, T_1^+, B_1^+, and L_1^+, respectively.

We are able to invoke [**15**, Theorem 8] since the finite rectangle R_1^+ is analogous to the finite region obtained in the periodic case. In particular, it is important to note that the additional regularity near the boundary assumption is crucial; without it, the weak form of the ∂_s^2 terms in the PDE satisfied by $\tilde{v} := \tilde{\phi}_r$ would prohibit differentiating twice with respect to s, which is necessary near each of the four corners. With the additional regularity assumption near the boundary, it is possible to take two derivatives with respect to s in a neighborhood of the boundary which includes the corners. Additionally, it allows us to use the Serrin edgepoint lemma (Lemma A.4) since we know that $\tilde{v}$ is locally $C^{1,\alpha}$ near the boundary.

Specifically, [**15**, Theorem 8] implies that $\tilde{\phi}_r < 0$ in $R_1^+ \cup T_1^+$. Now turning to the "tail" region R_2^+, we can choose K to be large enough so that

$$\|\tilde{\phi}\|_{W^{2,p}(R_2^+)} < \frac{\delta}{2},$$

where δ is given as in the hypothesis of Lemma 2.1. Additionally, since $L_2^+ \subset R^+1 \cup T_1^+$, $\tilde{\phi}_r < 0$ in $R_1^+ \cup T_1^+$ means that $\tilde{\phi}_r < 0$ on L_2^+. An application of Lemma 2.1 allows us to conclude that $\tilde{\phi}_r < 0$ on $R_2^+ \cup T_2^+$, and recognizing that $(R_1^+ \cup T_1^+) \cup (R_2^+ \cup T_2^+) = R^+ \cup T^+$ completes the proof. ∎

LEMMA 2.12 (Closed condition). *Let* $\{(\phi_n, F_n)\} \subset U$ *be a sequence of solutions to* (2.17) *which converges in* $W^{2,p}(\overline{R})$ *to a solution* $(\phi, F) \in U$*. If each* ϕ_n *satisfies the nodal properties* (2.82)*, then* ϕ *also satisfies* (2.82) *unless* $\phi \equiv 0$*.*

PROOF OF LEMMA 2.12. Once again, we need only to show that $v := \phi_r < 0$ in $R^+ \cup T^+$. By continuity, that $v \leq 0$ in $\overline{R_1^+}$. Furthermore, v satisfies a uniformly elliptic PDE (2.73) in R_1^+. Since v vanishes identically on $L_1^+ \cup B_1^+$, the maximum principle implies that $v < 0$ in $R_1^+ \cup T_1^+$ unless $v = 0$ at some point $(r^*, 0) \in T^+$.

By way of contradiction, suppose that $v(r^*, 0) = 0$. Then the boundary condition in (2.73) becomes

$$0 = \frac{v_s(r^*, 0)}{h_s(r^*, 0)},$$

which further implies that $v_s(r^*, 0) = 0$. But the Hopf lemma implies that $v_s(r^*, 0) > 0$, a contradiction. ∎

2.6. Small-amplitude existence theory

In this chapter, we will establish the existence of small-amplitude solitary waves, which appear as solutions (ϕ, F) of $\mathscr{F}(\phi, F) = 0$. The main result is the following theorem, which states the existence of a one-parameter curve of small-amplitude solutions $(\phi^\epsilon, F^\epsilon)$ for $0 < \epsilon < \epsilon_*$.

THEOREM 2.6. *(Small-amplitude solitary waves) Fix $H \in W^{2,\infty}([-1, 0])$, $p \in (2, 4)$ and let $\phi := h - H$. There exists a curve of small-amplitude solitary waves*

$$\mathscr{C}_{loc} := \{(\phi^\epsilon, F^\epsilon) : \epsilon \in (0, \epsilon_*)\} \subset W^{2,p}(R) \times \mathbb{R},$$

where each $(\phi^\epsilon, F^\epsilon)$ is a solution of (2.17) with the following properties:

(i) (Continuity) The map $\epsilon \mapsto \phi^\epsilon$ is continuous from $(0, \epsilon_*)$ to $\mathcal{X}$, with $\|\phi\|_X \to 0$ as $\epsilon \to 0$.

(ii) (Invertibility) The linearized operator $\mathcal{F}_\phi(\phi^\epsilon, F^\epsilon)$ is invertible for each $\epsilon \in (0, \epsilon_*)$.

(iii) (Uniqueness) If $\phi \in X$ satisfies $\phi > 0$ on T, and if $\|\phi\|_X$ is sufficiently small, then for any $\epsilon \in (0, \epsilon_*)$, $\mathcal{F}(\phi, F^\epsilon)$ implies that $\phi = \phi^\epsilon$.

Such waves were constructed by Hur in [**27**] for $\gamma \in C^0$, Groves and Wahlén in [**24**] for $\gamma \in H^1([-1,0])$, and Wheeler in [**59**] for $\gamma \in C^{1,\alpha}$ with $\alpha \in (0, \frac{1}{2}]$. Parts (i) - (iii) were not proven in [**27**] and [**24**], but were proven in [**59**], with the addition of weighted Hölder spaces. The main difficulty in our work lies in showing the solutions exist in $W^{2,p}$—a space whose natural decay at infinity no longer requires a weighted function space. We do this by exploiting the inherent exponential localization of the Sobolev space.

We will prove Theorem 2.6 incrementally; the proof closely follows the spatial dynamics method from [**24**] as well as [**59**, Theorem 4.1]. In section 2.6.1 we formulate the problem $\mathscr{F}(\phi, F) = 0$ as an infinite-dimensional Hamiltonian system. We then perform several changes of variable that transform Hamilton's equations into an evolution equation $u_r = Lu + N^\epsilon(u)$, where r acts as the time variable. The remainder of section 2.6.1 and all of section 2.6.2 are devoted to proving prerequisite spectral properties of the linearized operator L. Even though our current setting involves an infinite Hamiltonian system, our goal is to be able to apply techniques for a finite-dimensional Hamiltonian system. To do so, we will utilize the Center Manifold Reduction theorem, which states that $(\mathcal{M}, \Upsilon, \mathcal{H}^\epsilon)$ is locally equivalent to a finite dimensional Hamiltonian system. More precisely, we will utilize [**8**, Theorem 4.1], which is a version of the Center Manifold Reduction Theorem specifically designed to handle Hamiltonian systems. In sections 2.6.3 and 2.6.4, we invoke center manifold reduction techniques and construct a family of small-amplitude solitary solutions $(\phi^\epsilon, F^\epsilon)$. Finally, we stitch together these results and arrive at the proof of Theorem 2.6 in section 2.7. Throughout, we deviate from the arguments in [**24**] and [**59**] only to verify that our small-amplitude solitary solutions exist in the appropriate Sobolev space $W^{2,p}(R)$.

2.6.1. Formulation as a Hamiltonian system. Using $\phi = \phi(r,s)$ as before, let us introduce the variable

$$w := \frac{\phi_r}{\phi_s + H_s}. \tag{2.87}$$

Throughout this section, we will suppress the dependence of (ϕ, w) on r; specifically, we identify $\phi = \phi(r,s)$ as a C^1 mapping $r \mapsto \phi(r,\cdot)$ taking values in the Hilbert space of s-dependent functions. Specifically, we work with two such spaces

$$\mathcal{X} := \{(\phi, w) \in H^1(-1,0) \times L^2(-1,0) : \phi(-1) = 0\}$$

$$\mathcal{Y} := \{(\phi, w) \in H^2(-1,0) \times H^1(-1,0) : \phi(-1) = 0\}.$$

Let us also define the set

$$\mathcal{M} := \{(\phi, w) \in \mathcal{Y} : h_s(s) > 0 \text{ for each } s \in [-1,0]\}.$$

The set $\mathcal{M}$ is called a *manifold domain* of $\mathcal{Y}$, since $\mathcal{Y}$ is both dense and smoothly embedded in $\mathcal{X}$. Consider the symplectic manifold $(\mathcal{X}, \Upsilon)$, with symplectic form $\Upsilon : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ defined by

$$\Upsilon|_{(\phi,w)}((\phi_1, w_1), (\phi_2, w_2)) = \int_0^1 (w_2\phi_1 - w_1\phi_2)\, ds.$$

Then $(\mathcal{M}, \Upsilon)$ together with the Hamiltonian

$$\mathcal{H}(\phi, w) = \int_{-1}^0 \left\{ \frac{1}{2}\left(w^2 - \frac{1}{(\phi_s + H_s)^2} \right) - \frac{1}{2H_s^2} \right\} (\phi_s + H_s)\, ds \; + \frac{1}{2F^2}\phi(0)^2 \tag{2.88}$$

is a Hamiltonian system. Taking variations yields

$$\mathcal{H}_w(\phi, w)[\dot{w}] = \int_{-1}^0 (\phi_s + H_s) w\dot{w}\, ds, \qquad \mathcal{H}_\phi(\phi, w)[\dot{\phi}] = \int_{-1}^0 (-w_r)\dot{\phi}\, ds$$

so that

$$\phi_r = \frac{\delta \mathcal{H}}{\delta w} \qquad \text{and} \qquad - w_r = \frac{\delta \mathcal{H}}{\delta \phi}.$$

Recall that domain $\mathcal{D}(\mathcal{V}_\mathcal{H})$ of the Hamiltonian vector field $\mathcal{V}_\mathcal{H}$ associated to $(M, \Upsilon, \mathcal{H})$ consists of all $m \in M$ satisfying

$$\Upsilon_m(\mathcal{V}_\mathcal{H}\big|_m, \mathcal{V}_1\big|_m) = d\mathcal{H}\big|_m(\mathcal{V}_1\big|_m)$$

for all tangent vectors $\mathcal{V}_1\big|_m \in TM\big|_m \subset T\mathcal{X}\big|_m$. A straightforward computation shows that

$$\begin{aligned} d\mathcal{H}\big|_m(\mathcal{V}_1\big|_m) = \int_{-1}^{0} -\frac{1}{2}\left(w^2 + \frac{1}{(\phi_s + H_s)^2} - \frac{1}{H_s^2}\right)_s \phi_1 \, ds \\ + \int_{-1}^{0} (\phi_s + H_s) w w_1 \, ds \\ + \frac{1}{2}\left(w^2(0) + \frac{1}{(\phi_s(0) + H_s(0))^2} + \frac{1}{F^2}(\phi(0))\right) \phi_1(0), \end{aligned}$$

for $m = (\phi, w) \in \mathcal{M}$ and $\mathcal{V}_1 = (\phi_1, w_1) \in T\mathcal{M}\big|_m$.

$$\begin{aligned} \mathcal{D}(\mathcal{V}_\mathcal{H}) = \Bigg\{ (\phi, w) \in \mathcal{M} : w(-1) = 0, \\ \frac{1}{2}\left(w^2 + \frac{1}{(\phi_s + H_s)^2}\right) - \frac{1}{2H_s^2} + \frac{1}{F^2}\phi\Bigg|_{s=0} = 0 \Bigg\}. \end{aligned}$$

It follows that Hamilton's equations are explicitly

$$\text{(2.89a)} \quad \begin{cases} \phi_r = w(\phi_s + H_s) \\ w_r = \dfrac{1}{2}\left(w^2 + \dfrac{1}{\phi_s + H_s}\right)_s - \left(\dfrac{1}{2H_s^2}\right)_s, \end{cases}$$

with the addtional conditions on the boundary

$$\text{(2.89b)} \quad \phi(r, -1) = w(r, -1) = 0, \qquad \left(\frac{1}{2}\left(w^2 + \frac{1}{\phi_s + H_s}\right) - \frac{1}{2H_s^2} + \frac{1}{F^2}\phi\right)\Bigg|_{s=0} = 0.$$

In the above formulation, the horizontal variable r operates as the traditional time variable in the Hamiltonian system. Furthermore, the Hamiltonian equations are reversible in the sense that if $(\phi, w)(r)$ is a solution to (2.89), then $(\phi, -w)(-r)$ is also a solution. Due to $\psi_y < 0$ and the asymptotic conditions (2.9b), we are seeking solutions (ϕ, w) of Hamilton's equations for which

$$h_s(r,s) > 0 \qquad \text{and} \qquad \phi(r,0) \to 0 \text{ as } r \to \pm\infty.$$

Such solutions arise from perturbations of equilibrium solutions $(\phi(s), 0)$ of (2.89a) such that $\phi(-1) = 0$, which are r-independent solutions representing horizontal laminar flows.

Since the qualitative behavior of the system changes significantly when $F = F_{\text{cr}}$ as established in Section 2.4, we look at solutions resulting from perturbations of the Froude number; specifically, let $F = F_{\text{cr}}$ and choose ϵ in a neighborhood V of the origin in $\mathbb{R}$. For convenience, we introduce the notation

$$\text{(2.90)} \qquad \frac{1}{(F^\epsilon)^2} := \left(\frac{1}{F_{\text{cr}}^2} - \epsilon\right)^{-1/2}.$$

It is important to notice that, with this convention, $\epsilon > 0$ implies that $\frac{1}{(F^\epsilon)^2}$ is a supercritical Froude number.

We now define the ϵ-dependent Hamiltonian by

$$\mathcal{H}^\epsilon(\phi, w) = \int_{-1}^{0} \left\{\frac{1}{2}\left(w^2 - \frac{1}{(\phi_s + H_s)^2}\right) - \frac{1}{2H_s^2}\right\}(\phi_s + H_s)\, ds + \frac{1}{2}\left(\frac{1}{(F^\epsilon)^2}\right)\phi(0)^2,$$

so that Hamilton's equations (2.89a) become

$$\text{(2.91)} \qquad \begin{cases} \phi_r = w(\phi_s + H_s) \\ w_r = \dfrac{1}{2}\left(w^2 + \dfrac{1}{(\phi_s + H_s)^2} + \dfrac{1}{H_s^2}\right)_s \end{cases},$$

for which the domain of the Hamiltonian vector field is the set

$$\mathcal{D}(\mathcal{V}_{\mathcal{H}^\epsilon}) := \left\{(\phi, w) : w(-1) = 0, \left(\frac{1}{2}\left(w^2 + \frac{1}{(\phi_s + H_s)^2}\right) - \frac{1}{2H_s^2} + \left(\frac{1}{(F^\epsilon)^2}\right)\phi\right)\bigg|_{s=0} = 0\right\}.$$

Note that (2.91) together with the condition that $(\phi, w) \in \mathcal{D}(\mathcal{V}_{\mathcal{H}^\epsilon})$ is equivalent to the height equation (2.17). Linearizing the Hamiltonian system $(\mathcal{M}, \Upsilon, \mathcal{H}^0)$ about the equilibrium solution $(\phi, w) = (0, 0)$ results in the problem

$$(\dot{\phi}, \dot{w})_r = L(\dot{\phi}, \dot{w}),$$

where $L : \mathcal{D}(L) \subset \mathcal{X} \to \mathcal{X}$ is given by

$$L(\dot{\phi}, \dot{w}) := \begin{pmatrix} H_s \dot{w} \\ \\ -\left(\dfrac{\dot{\phi}_s}{H_s^3}\right)_s \end{pmatrix} \tag{2.92}$$

with domain

$$\mathcal{D}(L) := \left\{ (\dot{\phi}, \dot{w}) \in \mathcal{Y} : \dot{w}(-1) = 0, -\frac{\dot{\phi}(0)}{H_s^3(0)} + \frac{1}{F^2}\dot{\phi}(0) = 0 \right\}$$

This operator L is closely related to the Sturm–Liouville problem introduced in Section 2.4.2; as such, we obtain the following lemma regarding the spectrum of L.

LEMMA 2.13 (Spectral Properties of L). *The linearized operator $L : \mathcal{D}(L) \subset \mathcal{X} \to \mathcal{X}$ satisfies the following:*

(i) *The spectrum of L consists of an eigenvalue at 0 with algebraic multiplicity 2, together with simple eigenvalues $\pm\sqrt{\nu_j}$, where ν_j are the nonzero eigenvalues of the corresponding Sturm-Liouville problem* (2.4.2). *The eigenvector and generalized eigenvector associated with the eigenvalue $\nu = 0$ are*

$$u_1 := \left(\int_{-1}^{s} H_s^3 \, dt, 0\right)^T, \qquad u_2 := \left(0, \frac{1}{H_s}\int_{-1}^{s} H_s^3 \, dt\right)^T \tag{2.93}$$

(ii) *There exists $\Theta > 0$ and $C > 0$ such that*

$$\|u\|_{\mathcal{Y}} \leq C\|(L - i\theta I)u\|_{\mathcal{X}}, \quad \textit{and} \quad \|u\|_{\mathcal{X}} \leq \frac{C}{|\theta|}\|(L - i\theta I)u\|_{\mathcal{Y}}$$

for $u \in \mathcal{D}(L)$ and $\theta \in \mathbb{R}$ with $|\theta| > \Theta$.

Proof. We claim that zero is an eigenvalue of L if and only if $\mu = \mu_{\mathrm{cr}}$. Notice that if zero is an eigenvalue, then there exists a solution Φ to

$$\begin{cases} -\left(\dfrac{\Phi_s}{H_s^3}\right)_s = 0, \\ \Phi(-1) = 0 \\ \mu\Phi(0) - \dfrac{\Phi_s(0)}{H_s^3(0)} = 0 \end{cases}$$

This is exactly the Sturm–Liouville operator equation from Lemma 2.2, so it follows that $\mu = \mu_{\mathrm{cr}}$. In addition, for $\nu = \lambda^2$, the zero eigenvalue has algebraic multiplicity 2; a straight forward calculation shows that the eigenvector u_1 and generalized eigenvector $u_2 \in \mathcal{D}(L)$, with $Lu_1 = 0$ and $Lu_2 = u_1$ are exactly (2.93).

To prove (ii), let $u = (\xi, w) \in \mathcal{D}(L)$ and $\theta \in \mathbb{R}$ be given. Write $(f, g) := (L - i\theta I)u$, we have

$$f := H_s w - i\theta\xi, \qquad \text{and} \qquad g := -\left(\frac{\xi_s}{H_s^3}\right)_s - i\theta w, \tag{2.94}$$

keeping in mind that the definition of $\mathcal{D}(L)$ also requires that

$$w(-1) = 0 \quad \text{and} \quad (\mu_{\mathrm{cr}} + \epsilon)\xi(0) - \frac{\xi_s(0)}{H_s^3(0)} = 0.$$

Now notice from (2.94) we have that

$$|f_s|^2 = |(H_s w)_s|^2 + |\theta|^2|\xi_s|^2 + 2\,\mathrm{Im}\,(\theta\xi_s(H_s w)_s),$$

and

$$|g|^2 = \left|\left(\frac{\xi_s}{H_s^3}\right)_s\right|^2 + |\theta|^2|w|^2 - 2\,\mathrm{Im}\left(\theta w \left(\frac{\bar{\xi}_s}{H_s^3}\right)_s\right).$$

It follows that

$$|f_s|^2 + |g|^2 = |(H_s w)_s|^2 + \left|\left(\frac{\xi_s}{H_s^3}\right)_s\right| + |\theta|^2(|w|^2 + |\xi_s|^2)$$
$$+ 2\theta\,\mathrm{Im}\left(\xi_s(H_s\bar{w})_s - w\left(\frac{\bar{\xi}_s}{H_s^3}\right)_s\right).$$

Further simplification yields

$$\frac{|f_s|^2}{H_s^3} + H_s|g|^2 = \left|\frac{(H_s w)_s}{H_s^3}\right|^2 + H_s\left|\left(\frac{\xi_s}{H_s^3}\right)_s\right| + |\theta|^2\left(\frac{|w|^2}{H_s^3} + H_s|\xi_s|^2\right)$$
$$+ 2\theta\,\mathrm{Im}\left(\frac{\xi_s(H_s\bar{w})_s}{H_s^3} - w\left(\frac{\bar{\xi}_s}{H_s^3}\right)_s\right). \tag{2.95}$$

Notice also that

$$\int_{-1}^{0}\frac{\xi_s}{H_s^3}(H_s\bar{w})_s\,ds - \int_{-1}^{0} H_s w\left(\frac{\bar{\xi}_s}{H_s^3}\right)_s\,ds$$
$$= \left(\frac{\xi_s(0)}{H_s^3(0)}\right)H_s(0)\bar{w}(0) - \int_{-1}^{0} 2\,\mathrm{Re}\left(H_s\bar{w}\left(\frac{\xi_s}{H_s^3}\right)_s\right)\,ds,$$

so that integrating by parts and taking the imaginary part of (2.95), we have

$$C(\|f\|_{H^1}^2 + \|g\|_{L^2}^2) \geq \int_{-1}^{0}\left(\left|\frac{(H_s w)_s}{H_s^3}\right|^2 + |\theta|^2\left(\frac{|w|^2}{H_s^3}\right)\right)\,ds$$
$$+ \int_{-1}^{0}\left(H_s\left|\left(\frac{\xi_s}{H_s^3}\right)_s\right|^2 + |\theta|^2 H_s|\xi_s|^2\right)\,ds$$
$$+ \left(\frac{\xi_s(0)}{H_s^3(0)}\right)H_s\bar{w}(0)$$

$$=:\ \mathrm{I} + \mathrm{II} + \mathrm{III}\ .$$

Keeping in mind that H_s and $H_s^{-1} \in W^{1,\infty}$ and both are uniformly positive, let us first consider I:

$$\mathrm{I} = \int_{-1}^{0}\frac{H_{ss}^2|w|^2 + H_s^2|w_s|^2 + 2\,\mathrm{Re}\,(H_{ss}H_s w\bar{w}_s)}{H_s^6}\,ds + \int_{-1}^{0}|\theta|^2\left(\frac{|w|^2}{H_s^3}\right)$$
$$\gtrsim \|w\|_{H^1}^2 + 2\int_{-1}^{0}\mathrm{Re}\,(H_{ss}H_s w\bar{w}_s)\,ds + \int_{-1}^{0}|\theta|^2\left(\frac{|w|^2}{H_s^3}\right)$$
$$\gtrsim \|w\|_{H^1}^2 + |w||w_s| + |\theta|^2\|w\|_{L^2}^2.$$

Similarly, we estimate

$$\mathrm{II} = \int_{-1}^{0} H_s\left(\frac{H_s^6|\xi_s s|^2 + 9H_s^4H_{ss}^2|\xi_s|^2 - 2\,\mathrm{Re}\,(3H_s^5H_{ss}\xi_{ss}\bar{\xi}_s)}{H_s^{12}}\right)ds + \int_{-1}^{0}|\theta\xi|^2 H_s\,ds$$
$$\gtrsim \|\xi\|_{L^2}^2 - |\xi_{ss}||\xi_s| + |\theta|^2\|\xi_s\|_{L^2}^2.$$

Lastly, to estimate III, first notice that $|\xi(0)| \lesssim \|\xi_s\|_{L^2}$. Using the definition of f_s and boundary conditions to find that

$$\begin{aligned} |H_s(0)w(0)|^2 &= 2\int_{-1}^{0} \operatorname{Re}\left(H_s\bar{w}(H_s w)_s\right)\, ds \\ &= 2\operatorname{Re}\left(\int_{-1}^{0} H_s\bar{w}(f_s + i\theta\xi_s)\, ds\right) \\ &\leq \delta^2(\|f_s\|_{L^2}^2 + |\theta|^2\|\xi_s\|_{L^2}^2) + \frac{1}{\delta^2}\|w\|_{L^2}^2. \end{aligned}$$

For $|\theta|$ sufficiently large and δ sufficiently small, we combine the above estimates for I, II, and III to arrive at the estimate

$$\begin{aligned} C(\|f\|_{H^1}^2 + \|g\|_{L^2}^2) &\gtrsim \|w_s\|_{L^2}^2 + |\theta|^2(\|w\|_{L^2}^2 + \|\xi_s\|_{L^2}^2) + \|\xi_{ss}\|_{L^2}^2 \\ &\gtrsim \|\xi\|_{H^2}^2 + \|w\|_{H^1}^2 + |\theta|^2(\|w\|_{L^2}^2 + \|\xi\|_{H^1}^2) \end{aligned}$$

as desired. ∎

2.6.2. Further change of variables. Prior to applying the center manifold reduction theorem, we need to restructure (2.91) to get rid of the nonlinearity in the boundary condition, which effectively flattens $\mathcal{D}(\mathcal{V}_{\mathcal{H}^\epsilon})$. We proceed by making the following change of variables: let $\mathcal{G} : \Lambda \times \mathcal{Z} \subset \mathbb{R} \times \mathcal{Y} \to H^1(-1,0)$ be defined by

$$\mathcal{G}(\phi, w) = -\frac{1}{2}\left(w^2 + \frac{1}{(\phi_s + H_s)^2}\right) + \frac{1}{2H_s^2} - \frac{1}{H_s^3}\phi_s \tag{2.96}$$

so that

$$\mathcal{G}(\phi, w)\Big|_{s=0} = \left(\frac{1}{(F^\epsilon)^2}\right)\phi(0) - \frac{1}{H_s^3(0)}\phi_s(0). \tag{2.97}$$

Now let

$$\zeta = \phi - H_s^3(0)\int_s^0 \mathcal{G}(\phi, w)(t)\, dt$$

and consider the mapping $\mathcal{G}_1 : \mathcal{Z} \to H^2(-1,0) \times H^1(-1,0)$ defined by $\mathcal{G}_1(\phi, w) = (\zeta, w)$. Moreover, ζ satisfies the boundary condition

$$\left(\frac{1}{(F^\epsilon)^2}\right)\zeta(0) - \frac{1}{H_s^3(0)}\zeta_s(0) = 0. \tag{2.98}$$

Lastly, we wish to consider the linear function $\mathcal{G}_2^\epsilon : \mathcal{Y} \to \mathcal{Y}$ defined by $\mathcal{G}_2^\epsilon(\zeta, w) = (\xi, w)$, where

$$\xi = \theta + \epsilon H_s^3(0) s \int_s^0 \theta(t)\, dt.$$

It follows that the boundary conditions (2.98) become

$$\frac{1}{F_{\mathrm{cr}}^2}\zeta(0) - \frac{1}{H_s^3(0)}\zeta_s(0) = 0.$$

Denoting $\mathcal{G}^\epsilon(\phi, w) := \mathcal{G}_2^\epsilon \circ \mathcal{G}_1$, the following lemma verifies that $\mathcal{G}^\epsilon$ is a well-defined change of variables.

LEMMA 2.14. *For $\Lambda \times \mathcal{Z}$, a neighborhood of the origin in $\mathbb{R} \times \mathcal{Y}$, the following hold:*

(i) *For each $\epsilon \in \Lambda$, $\mathcal{G}^\epsilon : \mathcal{Z} \to \mathcal{Y}$ is a diffeomorphism onto its image. The mappings $\mathcal{G}^\epsilon$ and $(\mathcal{G}^\epsilon)^{-1}$ depend smoothly on ϵ.*

(ii) *For each $(\epsilon, \phi, w) \in \Lambda \times \mathcal{Z}$, the derivative $D\mathcal{G}^\epsilon(\phi, w) : \mathcal{Y} \to \mathcal{Y}$ extends to an isomorphism $\widehat{D\mathcal{G}^\epsilon}(\phi, w) : \mathcal{X} \to \mathcal{X}$. The operators $\widehat{D\mathcal{G}^\epsilon}$ and $(\widehat{D\mathcal{G}^\epsilon})^{-1}$ depend smoothly on (ϵ, ϕ, w).*

PROOF. The proof is exactly [**59**, Lemma 4.7], which uses the facts

$$\mathcal{G}^\epsilon(0,0) = 0, \quad D\mathcal{G}^0(\phi, w) = \mathrm{id} : \mathcal{Y} \to \mathcal{Y},$$

and an application of the inverse function theorem. ∎

2.6.3. Center Manifold Reduction. In finite-dimensional dynamical systems, the existence of a homoclinic orbit is frequently proved by performing a center manifold reduction, which requires that the linearized problem has a finite number of purely imaginary eigenvalues, and all other eigenvalues have non-zero real part.

Thus far, we have successfully transformed the original Hamiltonian system into a reversible Hamiltonian system $(\mathcal{M}, \Upsilon, \mathcal{H}^\epsilon)$ with linear boundary conditions. We are currently working with an infinite-dimensional dynamical system, so we aim to utilize [**8**, Theorem 4.1], an infinite-dimensional analog to the center manifold reduction techniques pioneered by Kirchgässner [**30**] and Mielke [**42**]. This theorem states that, under the appropriate conditions on the spectrum of the linearized operator, $(\mathcal{M}, \Upsilon, \mathcal{H}^\epsilon)$ is locally equivalent to a finite dimensional Hamiltonian system.

Let $\mathcal{X}^c \subset \mathcal{X}$ be the two-dimensional (generalized) eigenspace for L associated with the eigenvalue 0. Also let P^c be the associated spectral projection, and write $P^* := I - P^c, \mathcal{X}^* := P^*\mathcal{X}$. We denote $u_c \in P^c\mathcal{D}(L)$; more precisely, $u_c = z_1 u_1 + z_2 u_2$, where $u_1, u_2 \in \mathcal{D}(L)$ are the eigenvector and generalized eigenvector from Lemma 2.13(i).

LEMMA 2.15. *(Center manifold reduction) For any integer $k \geq 2$, there exists a neighborhood $\Lambda \times \mathcal{U}$ of the origin in $\mathbb{R} \times \mathcal{D}(L)$ such that, for each $\epsilon \in \Lambda$, there exists a two-dimensional manifold $\mathcal{W}^\epsilon \subset \mathcal{U}$ together with an invertible coordinate map*

$$\chi^\epsilon := P^c\Big|_{\mathcal{W}^\epsilon} : \mathcal{W}^\epsilon \to \mathcal{U}^c := P^c\mathcal{U},$$

with the following properties:

(i) *Defining* $\Psi^\epsilon : \mathcal{U}^c \to \mathcal{U}^* := P^*\mathcal{U}$ *by*

$$u_c + \Psi^\epsilon(u_c) := (\chi^\epsilon)^{-1}(u_c),$$

the map $(\epsilon, u) \mapsto \Psi^\epsilon(u)$ *is* $C^k(\Lambda \times \mathcal{U}^c, \mathcal{U}^*)$. *Moreover,* $\Psi^\epsilon(0) = 0$ *for all* $\epsilon \in \Lambda$ *and* $D\Psi^0(0) = 0$.

(ii) *Every initial condition* $u_0 \in \mathcal{W}^\epsilon$ *determines a unique solution* u *of* $u_r = Lu + N^\epsilon(u)$, *which remains in* $\mathcal{W}$ *as long as it remains in* $\mathcal{U}$.

(iii) *If* u *solves* $u_r = Lu + N^\epsilon(u)$ *and lies in* $\mathcal{U}$ *for all* r, *then* u *lies entirely in* $\mathcal{W}^\epsilon$.

(iv) *If* $u_c \in C^1((a,b), \mathcal{U}^c)$ *solves the reduced system*

$$(u_c)_r = f^\epsilon(u_c) := Lu_c + P^c N^\epsilon(u_c + \Psi^\epsilon(u_c)), \tag{2.99}$$

then $u = (\chi^\epsilon)^{-1}(u_c)$ *solves the full system* $u_r = Lu + N^\epsilon(u)$.

(v) *With* u_c *and* u *as above, if* $\dot{u}_c \in C^1(\mathbb{R}, \mathcal{U}^c)$ *solves the linearized reduced equation* $(\dot{u}_c)_r = Df^\epsilon(u_c)\dot{u}_c$, *then* $\dot{u} = \dot{u}_c + D_u\Psi^\epsilon(u_c)\dot{u}_c$ *solves the full linearized system* $\dot{u}_r = L\dot{u} + D_uN^\epsilon(u)\dot{u}$.

(vi) *The reduced system* (2.99) *can be transformed into a Hamiltonian system* $(U^c, \upsilon, \mathcal{K}^\epsilon)$ *via a* C^{k-1} *change of variables, where* U^c *is a neighborhood of the origin in* $\mathbb{R}^2$, υ *is the canonical symplectic form*

$$\upsilon((z_1, z_2), (z_1', z_2')) := z_1 z_2' - z_1' z_2, \quad (z_1, z_2), (z_1', z_2') \in \mathbb{R}^2,$$

and the reduced Hamiltonian is given by

$$\mathcal{K}^\epsilon(z_1, z_2) := \mathcal{H}^\epsilon(z_1 u_1 + z_2 u_2 + \Theta^\epsilon(z_1 u_1 + z_2 u + 2)), \tag{2.100}$$

where $(\epsilon, u_c) \mapsto \Theta^\epsilon(u_c)$ *is of class* $C^{k-1}(\Lambda \times \mathcal{U}^c, \mathcal{U})$ *and satisfies* $\Theta^\epsilon(0) = 0$ *for all* $\epsilon \in \Lambda$, *and* $D_{u_c}\Theta^0(0) = 0$. *The system is reversible with reverser* $S(z_1, z_2) = (z_1, -z_2)$.

Proof. By Lemma 2.13, it follows that L satisfies (H1) and (H2) of Theorem A.6, and hence the eigenvalue 0 with multiplicity 2 is the only part of $\sigma(L)$ lying on the imaginary axis. Additionally, by Lemma 2.14, a quick calculation verifies that $N^0(0) = 0$ and $D_u N^0(0) = 0$, which satisfies (H3) in Theorem A.6, which in turn satisfies (i) - (iv). Lastly, part (vi) follows by undoing the near-identity transformation $\mathcal{G}^\epsilon$ in favor of working with the original variables, and then employing a parameter-dependent Darboux transformation to obtain the reversible Hamiltonian system $(U^c, \upsilon, \mathcal{K}^\epsilon)$ (see, for example, [**7**, Theorem 4]). ∎

2.6.4. Calculation of the reduced system. In this section, we compute the reduced system necessary for the center manifold reduction arguments. For brevity, we write $u = (\phi, w)$ and $\dot{u} = (\dot{\phi}, \dot{w})$, etc. for variations as appropriate. We begin by calculating first derivative of $\mathcal{H}^\epsilon$ in u:

(2.101a)
$$\mathcal{H}^\epsilon_u(\phi, w)[\dot{u}] = \int_{-1}^{0} \left\{ \frac{1}{2}\dot{\phi}_s w^2 + (H_s + \phi_s) w\dot{w} \frac{\dot{\phi}_s}{2(H_s+\phi_s)^2} - \frac{\dot{\phi}_s}{2H_s^2} \right\} ds + (\mu_{\mathrm{cr}} + \epsilon)\phi(0)\dot{\phi}(0).$$

The second derivative is given by

(2.101b)
$$\mathcal{H}^\epsilon_{uu}(\phi, w)[\dot{u}, \ddot{u}] = \int_{-1}^{0} \left\{ \dot{\phi}_s w \ddot{w} + \ddot{\phi}_s w \dot{w} + (H_s + \phi_s)\ddot{w}\dot{w} - \frac{\dot{\phi}_s \ddot{\phi}_s}{(H_s+\phi_s)^3} \right\} ds + (\mu_{\mathrm{cr}} + \epsilon)\ddot{\phi}(0)\dot{\phi}(0),$$

and likewise, the third is

(2.101c)

$$\mathcal{H}^{\epsilon}_{uuu}(\phi, w)[\dot{u}, \ddot{u}, \dddot{u}] = \int_{-1}^{0} \dot{\phi}_s \ddot{w}\dddot{w} + \dddot{\phi}_s \ddot{w}\dot{w} + \frac{3\dot{\phi}_s\ddot{\phi}_s\dddot{\phi}_s}{(H_s + \phi_s)^4}\, ds.$$

Let $\phi_c = \int_{-1}^{s} H_s^3(t)\, dt$, so that the eigenvector and generalized eigenvector (2.93) may be written $u_1 = (\phi_c, 0)^{\mathrm{T}}, u_2 = (0, \phi_c/H_s^3)^{\mathrm{T}}$. We know from the Center Manifold Theorem that the center manifold is spanned by u_1 and u_2, and hence any vector on the center manifold is of the form

$$u_c = \left(z_1\phi_c, z_2\frac{\phi_c}{H_s} \right)$$

where z_1, z_2 are constants. Evaluating (2.101) at $u = 0$ and

$$\dot{u} = \ddot{u} = \dddot{u} = u_c$$

we see that

$$\begin{aligned} \mathcal{H}^{\epsilon}_{u}(0)[u_c] &= \mathcal{H}^{\epsilon}_{u\epsilon}(0)[u_c] = 0; \\ \mathcal{H}^{0}_{uu}(0)[u_c, u_c] &= \int_{-1}^{0} (z_2\frac{\phi_c^2}{H_s})\, ds \\ \mathcal{H}^{\epsilon}_{uu\epsilon}(0)[u_c, u_c] &= z_1^2\phi_c(0)^2 \\ \mathcal{H}^{\epsilon}_{uuu}(0)[u_c, u_c, u_c] &= \int_{-1}^{0} 3z_1z_2^2H_s^3\phi_c^2 + 3z_1^3H_s^5\, ds. \end{aligned}$$

The goal is to consider the Taylor expansion of the Hamiltonian on the center manifold only; i.e., for u_c in the center space and for each ϵ, we consider

$$\mathcal{K}^{\epsilon}(u_c) := \mathcal{H}^{\epsilon}(v_c),$$

where $v_c := u_c + \Psi^{\epsilon}(u_c)$. Performing the Taylor expansion of $\mathcal{K}^{\epsilon}$ about $u = 0$ and again with $\dot{u} = \ddot{u} = \dddot{u} = 0$, we see that

$$\mathcal{K}^{\epsilon}(u_c) = \mathcal{K}^{\epsilon}(0) + \mathcal{K}^{\epsilon}_{u_c}(0)[u_c] + \frac{1}{2}\mathcal{K}^{\epsilon}_{u_cu_c}(0)[u_c, u_c]$$

$$+\frac{1}{6}\mathcal{K}^{\epsilon}_{u_cu_cu_c}(0)[u_c,u_c,u_c]+O(\|u_c\|^4)$$

Now we compute the variations of $\mathcal{K}^{\epsilon}$. Notice that

$$\mathcal{K}^{\epsilon}_{u_c}(u_c)[u_c]=\mathcal{H}^{\epsilon}_{u}v_c[(1+\Psi^{\epsilon}_{u_c}(u_c))[u_c]]$$

so that

$$\mathcal{K}^{\epsilon}_{u_c}(0)[u_c]=\mathcal{H}^{\epsilon}_{u}(\Psi^{\epsilon}(0))(1+\Psi^{\epsilon}_{u_c}(0))[u_c]=0$$

Additionally, we want to compute the quadratic term by expanding in ϵ near $\epsilon=0$:

$$\mathcal{K}^{\epsilon}_{u_cu_c}(u_c)[u_c,u_c]=\mathcal{H}^{\epsilon}_{uu}v_c\Big[(1+\Psi^{\epsilon}_{u_c}(u_c))[u_c],(1+\Psi^{\epsilon}_{u_c}(u_c))[u_c]\Big]$$
$$+\mathcal{H}^{\epsilon}_{u}v_c[\Psi^{\epsilon}_{u_cu_c}(u_c)[u_c]]$$

so that

$$\mathcal{K}^{\epsilon}_{u_cu_c}(0)[u_c,u_c]=\mathcal{H}^{\epsilon}_{uu}(\Psi^{\epsilon}(0))\Big[(1+\Psi^{\epsilon}_{u_c}(0))[u_c],(1+\Psi^{\epsilon}_{u_c}(0))[u_c]\Big]$$
$$+\mathcal{H}^{\epsilon}_{u}(0)(\Psi^{\epsilon}_{u_cu_c}(0)[u_c,u_c]$$

and particularly

$$\mathcal{K}^{0}_{u_cu_c}(0)[u_c,u_c]=\mathcal{H}_{uu}(0)[u_c,u_c]$$

Similarly computation yields

$$\mathcal{K}^{0}_{u_cu_c\epsilon}(0)[\dot{u}_c,\dot{u}_c]=2\mathcal{H}^{0}_{uu}(0)[\dot{u}_c,\Psi^{0}_{\epsilon u_c}(0)[\dot{u}_c]]+\mathcal{H}^{0}_{uu\epsilon}(0)[\dot{u}_c,\dot{u}_c]$$

However, recalling that $(\mu_{\mathrm{cr}}+\epsilon):=(\phi_c(0))^{-1}$, we also have that for any variation $\dot{u}$,

$$\begin{aligned}\mathcal{H}^{0}_{uu}(0)[(\phi_c,0),\dot{u}]&=-\int_{-1}^{0}\frac{(\phi_c)_s\dot{\phi}_s}{H_s^3}\,ds+(\mu_{\mathrm{cr}}+\epsilon)\phi_c(0)\dot{\phi}(0)\\&=-\int_{-1}^{0}\dot{\phi}_s\,ds+\dot{\phi}(0)=0.\end{aligned}$$

It follows that

$$\mathcal{K}^{0}_{\epsilon u_cu_c}(0)[\dot{u}_c,\dot{u}_c]=2\mathcal{H}^{0}_{uu}(0)\left[(z_1\phi_c,z_2\frac{\phi_c}{H_s}),\Psi^{0}_{\epsilon u_c}(0)[(z_1\phi_c,z_2\frac{\phi_c}{H_s})]\right]+z_1^2\phi_c(0)^2$$

We can therefore write the quadratic terms as

$$\mathcal{K}^{\epsilon}_{u_cu_c}(0)[u_c,u_c] = \int_{-1}^{0} z_2^2 \frac{\phi_c^2}{H_s}\, ds + \epsilon z_1^2\phi_c(0)^2 + \mathcal{O}(|\epsilon||s_2^2||(z_1,z_2)|) + \mathcal{O}(\epsilon^2|(z_1,z_2)|^2).$$

Similarly the cubic terms are given by

$$\begin{aligned}
&\mathcal{K}^{\epsilon}_{u_cu_cu_c}(u_c)[\dot{u}_c,\ddot{u}_c,\dddot{u}_c] \\
&\qquad = \mathcal{H}^{\epsilon}_{uuu}v_c\left[(1+\Psi^{\epsilon}_{u_c}(u_c)[\dot{u}_c],(1+\Psi^{\epsilon}_{u_c}(u_c))[\ddot{u}_c],(1+\Psi^{\epsilon}_{u_c}(u_c))[\dddot{u}_c]\right] \\
&\qquad\quad + \mathcal{H}^{\epsilon}_{uu}v_c\left[(\Psi^{\epsilon}_{u_cu_c}(u_c)[\ddot{u}_c,\dddot{u}_c]),(1+\Psi^{\epsilon}_{u_c}(u_c)[\dot{u}_c]\right] \\
&\qquad\quad + \mathcal{H}^{\epsilon}_{uu}v_c\left[(1+\Psi^{\epsilon}_{u_c}(u_c)[\ddot{u}_c]),(\Psi^{\epsilon}_{u_cu_c}(u_c)[\dot{u}_c,\dddot{u}_c]\right]
\end{aligned}$$

so that at $u=0$ and along the diagonal,

$$\mathcal{K}^{\epsilon}_{u_cu_cu_c}(0)[u_c,u_c,u_c] = \int_{-1}^{0} 3z_1z_2^2H_s^3\phi_c^2 + 3z_1^3H_s^5\, ds + \mathcal{O}(|z_2|^2|(z_1,z_2)|)$$

Finally, combining the above calculations, we arrive at the expansion of $\tilde{\mathcal{K}}^{\epsilon}(z_1,z_2)$:

$$\begin{aligned}
\tilde{\mathcal{K}}^{\epsilon}(z_1,z_2) = \frac{1}{2}\left(\int_{1}^{0}\frac{\phi_c^2}{H_s}\, ds\right)z_2^2 + \frac{1}{2}\phi_c(0)^2\epsilon z_1^2 + \frac{1}{2}\left(\int_{-1}^{0} H_s^3\phi_c^2\, ds\right)z_1z_2^2 \\
+ \left(\frac{1}{2}\int_{-1}^{0} H_s^5\, ds\right)s_1^3 + R(z_1,z_2,\epsilon)
\end{aligned}$$

where $R(z_1,z_2,\epsilon)$ is the remainder terms. From this we may compute

$$\begin{aligned}
\tilde{\mathcal{K}}^{\epsilon}_{z_1}(z_1,z_2) &= \phi_c(0)^2\epsilon z_1 + \frac{1}{2}\left(\int_{-1}^{0} H_s^3\phi_c^2\, ds\right)z_2^2 + \frac{3}{2}\left(\int_{-1}^{0} H_s^5\, ds\right)z_1^2, \\
\tilde{\mathcal{K}}^{\epsilon}_{z_2}(z_1,z_2) &= \left(\int_{-1}^{0}\frac{\phi_c^2}{H_s}\, ds\right)z_2 + \left(\int_{-1}^{0} H_s^3\phi_c^2\, ds\right)z_1z_2.
\end{aligned}$$

These calculations furnish the reduced Hamiltonian system

$$\begin{cases}
(z_1)_r = \tilde{\mathcal{K}}^{\epsilon}_{z_2}(z_1,z_2) + R_1(z_1,z_2,\epsilon) \\
(z_2)_r = -\tilde{\mathcal{K}}^{\epsilon}_{z_1}(z_1,z_2) + R_2(z_1,z_2,\epsilon)
\end{cases}$$

where the higher order remainder terms are given by

$$R_1 = \mathcal{O}(|(z_1,z_2)|^2 + |z_2||(\epsilon,z_1,z_2)|^2 + \epsilon|z_2||(z_1,z_2)|),$$

$$R_2 = \mathcal{O}(|z_1||(\epsilon,z_2)|^2 + |z_2||(\epsilon,z_1)|).$$

We can simplify the system further by introducing the scaled variables (Z_1, Z_2) and R defined as

(2.102a)

$$z_1 := |\epsilon|\phi_c(0)^2 \left(\int_{-1}^{0} H_s^5 \, ds\right)^{-1} Z_1$$

(2.102b)

$$z_2 := |\epsilon|^{3/2} \left(\frac{\phi_c(0)^2}{(\int_{-1}^{0} \frac{\phi_c^2}{H_s})^{1/2}(\int_{-1}^{0} H_s^5)} \right) Z_2$$

(2.102c)

$$r := |\epsilon|^{-1/2} \left(\phi_c(0)^2 \int_{-1}^{0} \frac{\phi_c^2}{H_s} \, ds\right)^{-1/2} R$$

which at last yields the reduced system

$$\text{(2.103)} \quad \begin{cases} (Z_1)_R = Z_2 + R_3(Z_1, Z_2, \epsilon) \\ (Z_2)_R = -(\text{sgn}\epsilon) Z_1 - \dfrac{3}{2} Z_1^2 + R_4(Z_1, Z_2, \epsilon) \end{cases}$$

where $R_3 = \mathcal{O}(\epsilon^{1/2})$ and $R_4 = \mathcal{O}(\epsilon^{1/2})$ are the new remainder terms. These calculations furnish the following existence result.

Lemma 2.16 (Existence of ϕ^ϵ). *There exists $\epsilon_* > 0$ such that for each $\epsilon \in (0, \epsilon^*)$, there is a corresponding solution $(\phi^\epsilon, F^\epsilon)$ to the height equation* (2.17).

Proof of Lemma 2.16. Passing to the limit as $\epsilon \searrow 0$, the system (2.103) becomes

$$\text{(2.104)} \quad (Z_1)_{RR} = Z_1 - \frac{3}{2} Z_1^2$$

which is exactly the equation satisfied by the KdV soliton $Z_1^0(R) = \text{sech}^2 (R/2)$ and whose phase portrait depicts a symmetric homoclinic orbit through the origin. Due to [**30**, Proposition 5.1], we may exploit reversibility to conclude that the phase portrait of (2.103) is qualitatively the same for $\epsilon > 0$ sufficiently small. More precisely, there exists a reversible homoclinic orbit $(Z_1^\epsilon, Z_2^\epsilon)$ for $0 < \epsilon < \epsilon_*$, with $Z_1^\epsilon > 0$. Since $(Z_1^\epsilon, Z_2^\epsilon)(0)$ depends continuously on ϵ, we have uniform bounds

$$(2.105)\quad \sum_{k=0}^{2} |D_R^k(Z_1^\epsilon, Z_2^\epsilon)| \leq Ce^{-|R|/2}.$$

Using the change of variables defined in (2.102a), we can write $(z_1^\epsilon, z_2^\epsilon)$ in terms of $(Z_1^\epsilon, Z_2^\epsilon)$ to obtain a reversible homoclinic orbit $u_c^\epsilon := z_1^\epsilon u_1 + z_2^\epsilon u_2$ of the reduced system (2.103). Notice that

$$\sum_{k=0}^{2} \|D_r^k(u_c^\epsilon)\|_{H^2([-1,0])\times H^1([-1,0])} \leq \sum_{k=0}^{2} |(D_r^k(z_1^\epsilon(r), z_2^\epsilon(r))) \cdot (u_1(s), u_2(s)|,$$

so that applying the change of variables (2.102a) and the uniform bound (2.105) yields

$$(2.106)\quad \sum_{k=0}^{2} |D_r^k(u_c^\epsilon)| \leq C_1 \epsilon e^{-C_2|\epsilon|^{1/2}|r|},$$

for positive constants C_1, C_2.

Using the pullback $(\chi^\epsilon)^{-1}$, define

$$(2.107)\quad u^\epsilon := (\chi^\epsilon)^{-1}(u_c^\epsilon) = u_c^\epsilon + \Psi^\epsilon(u_c^\epsilon).$$

According to Lemma 2.15(iv), u^ϵ is a reversible homoclinic orbit of the full system $u_r = Lu + N^\epsilon(u)$. Furthermore,

$$(2.108)\quad u^\epsilon := (\phi, w) = u_c + \Psi^\epsilon(u_c) \in \mathcal{U} \subset \mathcal{D}(L).$$

At the beginning of this section, we suppressed dependence on r for solutions in $\mathcal{D}(L)$; we wish additionally to show that $u^\epsilon \in W^{2,p}(R) \times W^{1,p}(R)$ for $p \in (2, 4)$. Recalling the definition of w in (2.87), it suffices to show that $(u^\epsilon)_1 =: \phi^\epsilon$ is exponentially localized. To this end, we notice that, by Minkowski's inequality

$$\|u_1^\epsilon\|_{H^2(R)} \leq \|(u_c^\epsilon)_1\|_{H^2(R)} + \|\Psi_1^\epsilon(u_c^\epsilon)\|_{H^2(R)}.$$

By (2.106), we already know that $(u_c^\epsilon)_1(r)$ is exponentially localized, so we need only to examine $\Psi_1^\epsilon(u_c^\epsilon)$. Thanks to the properties of Ψ^ϵ from Lemma 2.15(i), we see that for $k = 0, 1, 2$,

$$\begin{aligned} \int_{\mathbb{R}} \int_{-1}^{0} |\partial_s^k \Psi_1^\epsilon(u_c^\epsilon(r))(s)|^2 \, ds \, dr &\leq \int_{\mathbb{R}} \|\partial_s^k \Psi_1^\epsilon\|^2_{C^{0,1}(\mathcal{U}^c;\mathcal{U}^*)} \|u_c^\epsilon(r,s)\|^2_{H^2 \times H^1} \, dr \\ &\leq \int_{\mathbb{R}} C_1 \epsilon e^{-C_2 |\epsilon|^{1/2} |r|} \, dr \\ &< \infty. \end{aligned}$$

Therefore, $\phi^\epsilon \in H^2(R)$. Lastly, we know that ϕ_r^ϵ solves a linear equation with $C^{0,\alpha})$ coefficients, and by a standard elliptic regularity argument have that $\phi^\epsilon \in W^{2,p}(R)$ for $p \in (2,4)$, as desired. ∎

2.7. Proof of small-amplitude existence

Combining the results from all of our previous sections, we finally have all of the necessary components to prove the main result of this chapter.

Proof of Theorem 2.6. We already constructed the family of small-amplitude solitary wave solutions $(\phi^\epsilon, F^\epsilon)$ in Lemma 2.16. Recalling that $\epsilon > 0$ implies $F^\epsilon > F_{\text{cr}}$, the invertibility of $\mathcal{F}_\phi(\phi^\epsilon, F^\epsilon)$ in part (ii) follows immediately from Lemma 2.3.

It remains to show parts (i) and (iii). Recalling ϵ_* and u_c^ϵ as in Lemma 2.16, we see that, after possibly shrinking ϵ_*, the exponential estimates also hold for $u_1^\epsilon = \phi^\epsilon$, and part (i) follows.

Now suppose we have a solution (ϕ, F^ϵ) of the height equation (2.17) with $\epsilon + \|\phi\|_X < \delta$, where δ is to be determined. By way of contradiction, assume $\phi \not\equiv \phi^\epsilon$. We wish to show that ϕ is not a supercritical solution.

By the properties of the center manifold reduction theorem, we know that ϕ is determined by a homoclinic orbit (z_1, z_2) of (2.99). Since we know that this equation already has a homoclinic orbit $(z_1^\epsilon, z_2^\epsilon)$, and that ϕ is not a translation of ϕ^ϵ, it is impossible for (z_1, z_2) to be a translate of $(z_1^\epsilon, z_2^\epsilon)$. By the properties of the phase portrait at the origin, we conclude that $z_1 < 0$ for $|r|$ sufficiently large and

$$(2.109)\quad \lim_{|r|\to\infty} \frac{z_2(r)}{z_1(r)} = \pm\epsilon^{1/2} + \mathcal{O}(\epsilon).$$

Tracing back the various changes of variable, we see that

$$(2.110)\quad \phi(r,s) = z_1(r)u_1(s) + \mathscr{R}(r,s),$$

where the remainder term satisfies

$$(2.111)\quad \|\mathscr{R}(r,\cdot)\|_{H^2(-1,0)} \le C(|\epsilon| + |z_1| + |z_2|)(|z_1| + |z_2|),$$

with constant C independent of ϵ. Now taking δ small enough, we have

$$(2.112)\quad \|\mathscr{R}(r,0)\|_X \le \frac{u_1(0)}{2}(|z_1(r)| + |z_2(r)|).$$

Shrinking δ further, (2.109) and (2.111) yield

$$(2.113)\quad \|\mathscr{R}(r,0)\|_X < u_1(0)|z_1(r)|,$$

for $|r|$ sufficiently large. Finally, for $|r|$ large enough that $r > 0$ and $\mathscr{R}(r,0)$ has the above upper bound, we know that

$$(2.114)\quad \phi(r,0) = z_1(r)u_1(0) + \mathscr{R}(r,0) < 0.$$

Since Lemma (2.4) proved that supercritical solutions are equivalent to waves of elevation, (2.114) contradicts the fact that ϕ is a supercritical solution. Hence $\phi = \phi^\epsilon$, and the theorem is proved. ∎

2.8. Large-amplitude existence theory

In this section, we establish the last few details needed to prove Theorem 2.1. Using analytic global bifurcation theory, we extend the curve of small-amplitude solutions $\mathscr{C}_{\text{loc}}$ to a curve $\mathscr{C}$ consisting of large-amplitude solutions. An abstract global-bifurcation-theoretic result due to Chen, Walsh, Wheeler [**13**] (see Theorem A.7) applies to our problem following some slight modifications to accommodate the weakened regularity of our constructed solutions.

2.8.1. Continuation.

Consider the set U of supercritical waves satisfying the no stagnation condition (1.17), as defined in (2.22). In Section 2.7, we established the existence of a local curve $\mathscr{C}_{\text{loc}} \subset U$ of solutions to the height equation (2.17). Applying the abstract global bifurcation result from Theorems A.7 and A.1 together with the nodal properties described in Section 2.5.3, we are now able to show that $\mathscr{C}_{\text{loc}}$ persists far away from the bifurcation point.

Theorem 2.7 (Global continuation). *The local curve $\mathscr{C}_{loc}$ is contained in a continuous curve of solutions, parametrized as*

$$\mathscr{C} = \{(\phi(z), F(z)) : 0 < z < \infty\} \subset U$$

with the following properties.

(a) *Near each point $(\phi(z_0), F(z_0)) \in \mathscr{C}$, we can reparametrize $\mathscr{C}$ so that the mapping $z \mapsto (\phi(z), F(z))$ is real analytic.*

(b) $(\phi(z), F(z)) \notin \mathscr{C}_{loc}$ *for z sufficiently large.*

(c) *Each wave in $\mathscr{C}$ is a monotonic wave of elevation in the sense that*

$$\phi(z) > 0 \quad on\ R \cup T \qquad and \qquad \partial_r \phi_n(z) \leq 0 \quad on\ \{r \geq 0\}.$$

(d) *As* $z \to \infty$,

$$N(z) := \|\phi(z)\|_X + \frac{1}{\inf_R(\phi_s(z) + H_s)} + F(z) + \frac{1}{F(z) - F_{cr}} \to \infty. \tag{2.115}$$

Proof. In the construction of the local curve $\mathscr{C}_{\text{loc}}$, it was already confirmed that $\lim_{\lambda \searrow 0} \phi(\lambda) = 0 \in \partial U$, where the parameter ϵ from the definition of $\dfrac{1}{(F^\epsilon)^2}$ in (2.90) plays the role of the parameter λ, and ϵ_* that of λ_*. From Section 2.4.3, we know that when (ϕ, F) is supercritical, the linearized operator $F_w(w, \lambda) : X \to Y$ is indeed Fredholm index 0. An application of Theorem A.7 furnishes the existence of a global curve of solutions $\mathscr{C} \subset U$ with the stated parameterization, and exhibiting properties (a) and (b) above. Additionally, the monotonicity from part (c) follows from the nodal properties in Section 2.5.3.

Finally, we wish to show that (d) occurs. By way of contradiction, suppose $N(z)$ remains bounded along the continuum. This is exactly Alternative (a)(ii) from Theorem A.7, from which we know that there exists a sequence $\{z_n\} \subset (0, \infty)$ such that $\sup_n N(z_n) < \infty$ but the sequence of solutions $\{\phi(z_n), F(z_n)\}$ has no convergent subsequence in $X \times \mathbb{R}$. Since $N(z_n)$ is bounded, we know that the sequence $\{\phi(z_n), F(z_n)\}$ is uniformly supercritical, and by Lemma A.1 it is also precompact in $X \times \mathbb{R}$, which is impossible. ∎

2.8.2. Bounds. We next explore the physical meaning for the blowup in Theorem 2.7(d). This entails deriving uniform bounds for the various quantities represented by terms in the function $N(z)$.

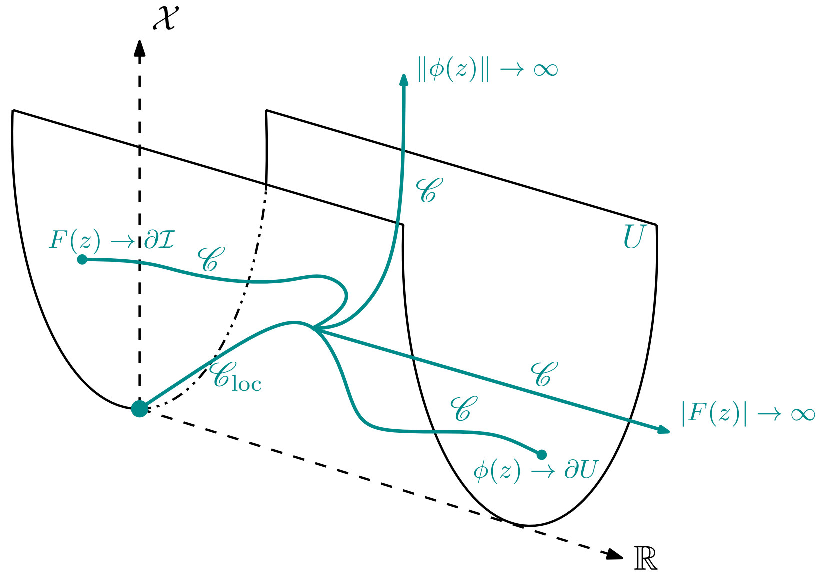

FIGURE 6. Blowup alternatives for $\mathscr{C}$ (Theorem 2.7(d))

In doing this, we will make use of some qualitative theory developed by Chen, Walsh, and Wheeler [**13**] in their study of solitary stratified water waves. It is important to mention that the results of [**13**] assume a higher level of regularity that is not present in our regime. We overcome this via a straightforward generalization of their proofs.

The first of these results is an upper bound on the Froude number (see [**13**, Theorem 4.7])

THEOREM 2.8 (Upper bound on F). *Let* $(\phi, F) \in X \times \mathbb{R}$ *be a solution of the height equation. Then the Froude number* F *satisfies the bound*

$$F^2 \leq \frac{2}{\pi} \|H_s\|_{L^\infty}^2 \|\phi_s(0,\cdot) + H_s(0)\|_{L^\infty}. \tag{2.116}$$

On the other hand, we have a form of lower bound on the Froude number along $\mathscr{C}$ provided that certain norms remain bounded. Let $A(1/F^2)$ be the affine function

$$A\left(\frac{1}{F^2}\right) := -\frac{1}{H_s(-1)^3} + \frac{1}{F^2} \int_{-1}^{0} \frac{H_s^3}{H_s(-1)^3}\, ds.$$

In Lemma 2.1(ii), we established that $A(1/F^2) < 0$ whenever $F > F_{\mathrm{cr}}$, while $A(1/F_{\mathrm{cr}}) = 0$. But, arguing as in [**13**, Lemma 4.3] we can prove that any solution $(\phi, F) \in X \times \mathbb{R}$ of the height equation satisfies

$$(2.117)\quad \frac{1}{H_s(-1)^3}\int_R \frac{H_s^3\phi_r^2 + (H_s + 2h_s)\phi_s^2}{2h_s^2 H_s^3}\, ds\, dr + A\Big(\frac{1}{F^2}\Big)\int_T (\phi + H)\, dr = 0.$$

Notice that the first integral above is equivalent to the H^1 norm of ϕ, provided that there exists uniform upper and lower bounds on $\|h_s\|_{L^\infty}$. Exploiting this fact, we obtain the following lemma.

Lemma 2.17 (Asymptotic supercriticality). *If $\|\phi(z)\|_X$ and $\|\phi_s(z) + H_s\|_{L^\infty}$ are uniformly bounded along $\mathscr{C}$, then*

$$(2.118)\quad \liminf_{z\to\infty} F(z) > F_{\mathrm{cr}}.$$

Proof. Arguing by contradiction, suppose that there exists a sequence $z_n \to \infty$ with

$$(2.119)\quad \limsup_{n\to\infty}\left(\|\phi(z_n)\|_X + \|\phi_s(z_n) + H_s\|_{L^\infty}\right) < \infty \qquad \text{and} \qquad F(z_n) \to F_{\mathrm{cr}}.$$

As each $F(z_n) > F_{\mathrm{cr}}$, Theorem 2.8 implies that $\phi_s(z_n) + H_s$ is uniformly bounded away from 0. As $\phi(z_n)$ is uniformly bounded in X and $A(1/F(z_n)) \to 0$, the identity (2.117) implies that $\|\phi(z_n)\|_{H^1} \to 0$. A simple elliptic regularity argument allows us to improve this to $\|\phi(z_n)\|_{H^2} \to 0$, as each $\phi(z_n)$ is a solution of the height equation. But now, local uniqueness ensures that $(\phi(z_n), F(z_n)) \in \mathscr{C}_{\mathrm{loc}}$ for n sufficiently large, and this contradicts Theorem 2.7(b). ∎

LEMMA 2.18 (Bounds on the pressure and velocity). *For any solitary wave with Froude number $F > F_0$, the pressure and velocity field satisfy the bounds*

$$P - P_{\text{atm}} + \|\gamma\|_{L^\infty}\psi > 0 \qquad \text{in } \Omega, \tag{2.120}$$

and

$$(u-c)^2 + v^2 \le 2\|\gamma_+\|_{L^\infty} + \frac{2}{F_0} + 2\|E\|_{L^\infty} \qquad \text{in } \Omega. \tag{2.121}$$

PROOF. The estimate (2.120) essentially comes from the pressure bounds for periodic waves with continuous vorticity obtained by Varvaruca [**57**, Theorem 3.1]. Here, we can adapt his argument to the solitary wave setting following [**13**, Proposition 4.1] and relax the assumptions on the regularity as in the proof of Theorem 1 in [**15**, Section 7]. As this is rather straightforward, we omit the details.

For the bound on the velocity field (2.121), we may likewise use [**13**, Proposition 4.1] setting the density to be constant. Bernoulli's law and the above pressure estimate then give (2.121). As $\nabla\psi$ and E are in $C^{0,\alpha}$, no modification of the argument is required. ∎

LEMMA 2.19 (Uniform boundedness). *For each $K > 0$, there exists a constant $C = C(K) > 0$ such that, if $(\phi, F) \in \mathscr{C}$ and*

$$\|\phi_s\|_{L^\infty} + \int_{\mathbb{R}} (\phi(r,0) + H(0))\, dr < C, \tag{2.122}$$

then $\|\phi\|_X < K$.

PROOF. Let $K > 0$ be given. Throughout this proof, we will use C to denote a generic positive quantity depending only on the norms occurring on the left-hand side of (2.122). Returning to the (nondimensionalized) Eulerlian variables, we have

$$\frac{1+\phi_r^2}{2\left(\phi_s+H_s\right)^2} = \frac{1}{2}\left((u-c)^2+v^2\right) < \|\gamma_+\|_{L^\infty} + \frac{1}{F_{\mathrm{cr}}} + \|E\|_{L^\infty},$$

where the last inequality comes from (2.121). It follows that $\|\phi_r\|_{L^\infty} < C$. On the other hand, from (2.117), we have in fact that $\|\phi\|_{H^1} < C$, and so via interpolation $\|\phi\|_{W^{1,p}} < C$.

At this point the argument becomes quite standard in the sense that analogous results have been proved by many authors investigating large-amplitude steady water waves. Indeed, we can simply proceed as in [**59**, Section 5.5], substituting a priori L^p estimates for Schauder estimates where necessary. The only modification that is required is to deal with the lower regularity, but this can be carried out in precisely the same manner as in [**15**, Lemma 5]. We therefore omit the details. ∎

2.8.3. Proof of the main result. At long last, we arrive at the proof of Theorem 2.1.

Proof of Theorem 2.1. Due to the formulation equivalence, it suffices to prove this result working entirely with the height equation. In Theorem 2.7, we already constructed the global curve $\mathscr{C}$ and concluded that $N(z) \to \infty$ as $z \to \infty$, where $N(z)$ is the quantity defined in (2.115). In view of Lemma 2.17, we know that if $F(z) \to F_{\mathrm{cr}}$, then it must also be the case that one of $\|\phi(z)\|_X$ and $\|\phi_p(z)\|_{L^\infty}$ also blowup in the limit. Likewise, the upper bound on the Froude number (2.116) implies that if $F(z) \to \infty$, then $\|\phi_s(z)\|_{L^\infty} \to \infty$. Finally, Lemma 2.19 tells us that the blowup of $\|\phi(z)\|_X$ forces $\|\phi_s(z)\|_{L^\infty}$ or $\int \eta(z)\,dx$ to likewise diverge to infinity. This proves part (i).

The fact that $\mathscr{C}$ begins at the critical laminar flow is a consequence of the construction of $\mathscr{C}_{\text{loc}}$, thus part (ii) is obvious. Finally, the assertion in part (iii) that the solutions in $\mathscr{C}$ are monotone waves of elevation was already established in Theorem 2.7(c) using the nodal properties. The proof is therefore complete. ∎

APPENDIX A

Quoted results

A.1. Maximum principle arguments

Let $\Omega \subset \mathbb{R}^n$ be a connected, open set, and consider the second-order operator L in divergence form given by

$$\text{(A.1)} \qquad L := \sum_{i,j=1}^{n} \partial_j (a_{ij}(x)\partial_i) + \sum_{i=1}^{n} b_i(x)\partial_i + c(x),$$

where $\partial_i := \partial_{x_i}$ and the coefficients a_{ij}, b_i, c are of class $C^{0,\alpha}(\overline{\Omega})$. We assume that L is uniformly elliptic in the sense that there exists $\lambda > 0$ with

$$\text{(A.2)} \qquad \sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda |\xi|^2, \qquad \text{for all } \xi \in \mathbb{R}^n, x \in \overline{\Omega},$$

and that a_{ij} is symmetric. Let $u \in W^{2,p}(\overline{\Omega})$ be a weak solution of $Lu = 0$ in Ω.

We now state some bedrock results from elliptic theory; namely, we recall the maximum prinicple, the Hopf boundary lemma, and the Serrin edge point lemma, with appropriate modifications for generalized solutions and reduced regularity.

THEOREM A.1 (Maximum principle, [**34**]). *Suppose $\Omega \in C^2$. Let the operator L be such that $c < -\delta$ for some constant $\delta > 0$ and suppose it satisfies the ellipticity condition* (A.2). *Then there exists a constant N (independent of Ω) such that for any $u \in W^{2,p}(\Omega)$ satisfying $u\big|_{\partial\Omega} \leq 0$ and $\lambda \geq 0$, we have*

$$\|u_+\|_{L^p(\Omega)} \leq N \|(\lambda u - Lu)_+\|_{L^p(\Omega)}.$$

In particular, if $u \in W^{2,p}(\Omega)$, $u\big|_{\partial\Omega} \leq 0$, and $Lu \geq 0$ in Ω, then $u \leq 0$ in Ω.

In some cases, the coefficients of L need not exist as functions, in which case the hypotheses for the maximum principle must be interpreted in a generalized sense.

THEOREM A.2 (Strong maximum principle, [**23**]). *Let the operator L satisfy* (A.2), *and suppose, L has the following additional properties:*

(i) (Bounded coefficients) For some constants Λ and $\nu \geq 0$, and for all $x \in \Omega$,

$$\sum_{i,j} |a^{ij}(x)|^2 \leq \Lambda^2, \tag{A.3a}$$

(ii) (Non-positive zeroth order coefficient) For all test functions $\varphi \in C_0^1(\Omega)$ such that $\varphi \geq 0$, the zeroth order coefficient c satisfies

$$\int_\Omega c\varphi \, dx \leq 0. \tag{A.3b}$$

Assume $u \in W^{1,2}(\Omega)$ satisfies $Lu \geq 0$ in Ω. Then, if for some ball $B \subset\subset \Omega$ we have

$$\sup_B u = \sup_\Omega u \geq 0, \tag{A.4}$$

the function u must be constant in Ω.

COROLLARY A.1.1 (Consequence of maximum principle, [**59**]). *Let $\mathscr{D} := \{(x,y) \in \mathbb{R}^2 : 0 < y < f(x)\}$, where f is a continuous function with limits as $x \to \pm\infty$, and suppose that $c = 0$. If $u \in C^2(\overline{\Omega})$ satisfies $u \geq 0$ on $\partial\mathscr{D}$, and*

$$\limsup_{|x|\to\infty} \sup_{0<y<f(x)} u(x,y) \geq 0,$$

then $u \geq 0$ in $\mathscr{D}$.

The following statement of the Hopf boundary lemma is a formal statement of the maximum principle outlined in [**15**, Section 3.2].

Theorem A.3 (Hopf boundary lemma). *Suppose u attains its maximum value on $\partial\Omega$ at a point $x_0 \in \partial\Omega$ for which there exists an open ball $B \subset \Omega$ with $\overline{B} \cap \partial\Omega = \{x_0\}$. Assume that $c \leq 0$. Then u is a constant function or*

$$\nu \cdot \nabla u(x_0) > 0, \tag{A.5}$$

where ν is the (unit) exterior normal.

Theorem A.4 (Serrin edge point lemma). *Let x_0 be an "edge point" in the sense that near x_0, the boundary $\partial\Omega$ consists of two transversally intersecting C^2 hypersurfaces $\{\beta(x) = 0\}$ and $\{\sigma(x) = 0\}$. Suppose that $\beta, \sigma < 0$ in Ω. If $u \in C^2(\overline{\Omega})$, then $u > 0$ in Ω, and $u(x_0) = 0$. Assume further that $a_{ij} \in C^{2,\alpha}_{loc}$ in a neighborhood of x_0,*

$$B(x_0) = 0, \qquad \text{and} \qquad \partial_\tau B(x_0) = 0, \tag{A.6}$$

for every differential operator ∂_τ tangential to $\{\beta = 0\} \cap \{\sigma = 0\}$ at x_0. Then for any unit vector v outward from Ω at x_0, either

$$\partial_v u(x_0) < 0 \qquad \text{or} \qquad \partial_v^2 u(x_0) < 0.$$

The last theorem of this section addresses solvability of elliptic operators in Sobolev spaces.

Theorem A.5 (Krylov [**34**]). *Let $c < -\delta$, where $\delta > 0$ is a constant. Then for any $f \in L^p$, there exists a unique $u \in W^{2,p}$ such that $Lu = f$. Moreover, for any $u \in W^{2,p}$ and $\lambda \geq 0$, we have*

$$\|u\|_{W^{2,p}} \lesssim \|\lambda u - Lu\|_{L^p}.$$

A.2. Center manifold reduction

THEOREM A.6 (Buffoni, Groves, Toland [**8**]). *Suppose that* $(\mathcal{X}, \Upsilon^\epsilon, \mathcal{H}^\epsilon)$ *is a one-parameter family of reversible Hamiltonian systems, where* $\mathcal{X}$ *is a Hilbert space,* Υ^ϵ *is a symplectic form on* $\mathcal{X}$*, and* $\mathcal{H}^\epsilon$ *the Hamiltonian. Write the corresponding Hamilton equation in the form*

$$u_r = Lu + N^\epsilon(u), \tag{A.7}$$

where $u(r)$ *is assumed to lie in* $\mathcal{X}$ *for each* r*. We assume that* $L : \mathcal{D}(L) \subset \mathcal{X} \to \mathcal{X}$ *is a densely defined, closed linear operator. Suppose that* 0 *is an equilibrium for* (A.7) *at* $\epsilon = 0$ *and that the following conditions hold:*

(H1) The spectrum $\sigma(L)$ *of* L *contains at most finitely-many eigenvalues on the imaginary axis; each of which has finite multiplicity. Moreover,* $\sigma(L) \cap i\mathbb{R}$ *is separated from* $\sigma(L)\backslash i\mathbb{R}$ *in the sense of Kato. Let* P^c *denote the spectral projection corresponding to* $\sigma(L) \cap i\mathbb{R}$ *and put* $\mathcal{X}^c := P^c\mathcal{X}, \mathcal{X}^* := (1 - P^c)\mathcal{X}$*. We let* n *be the finite dimension of* $\mathcal{X}^c$*.*

(H2) There exists $C > 0$ *such that the operator* L *satisfies the resolvent estimate*

$$\|u\|_{\mathcal{X}} \leq \frac{C}{1 + |\xi|}\|(L - i\xi I)u\|_{\mathcal{X}}, \tag{A.8}$$

for all $\xi \in \mathbb{R}$ *and* $u \in \mathcal{X}^*$*.*

(H3) There exists a natural number k*, and interval* $\Lambda \subset \mathbb{R}$ *containing* 0*, and a neighborhood* $\mathcal{U}$ *of* 0 *in* $\mathcal{D}(L)$ *such that* N *is* C^{k+1} *in its dependence on* (ϵ, u) *on* $\Lambda \times \mathcal{U}$*. Moreover,* $N^0(0) = 0$ *and* $D_u N^0(0) = 0$*.*

Then after possibly shrinking the interval Λ *and neighborhood* $\mathcal{U}$*, we have that, for each* $\epsilon \in \Lambda$*, there exists an* n*-dimensional local center manifold* $\mathcal{W}^\epsilon \subset \mathcal{U}$ *together with* *an invertible coordinate map*

$$\chi^\epsilon := P^c\Big|_{\mathcal{W}^\epsilon} : \mathcal{W}^\epsilon \to \mathcal{U}^c := P^c\mathcal{U},$$

with the following properties:

(i) *Defining* $\Psi^\epsilon : \mathcal{U}^c \to \mathcal{U}^* := P^*\mathcal{U}$ *by* $u_c + \Psi^\epsilon(u_c) := (\chi^\epsilon)^{-1}(u_c)$, *the map* $(\epsilon, u) \mapsto \Psi^\epsilon(u)$ *is* $C^k(\Lambda \times \mathcal{U}^c, \mathcal{U}^*)$. *Moreover,* $\Psi^\epsilon(0) = 0$ *for all* $\epsilon \in \Lambda$ *and* $D\Psi^0(0) = 0$.

(ii) *Every initial condition* $u_0 \in \mathcal{W}^\epsilon$ *determines a unique solution* u *of* (A.7), *which remains in* $\mathcal{W}$ *as long as it remains in* $\mathcal{U}$.

(iii) *If* u *solves* (A.7) *and lies in* $\mathcal{U}$ *for all* r, *then* u *lies entirely in* $\mathcal{W}^\epsilon$.

(iv) *If* $u_c \in C^1((a,b), \mathcal{U}^c)$ *solves the reduced system*

$$(u_c)_r = f^\epsilon(u_c) := Lu_c + P^cN^\epsilon(u_c + \Psi^\epsilon(u_c)), \tag{A.9}$$

then $u = (\chi^\epsilon)^{-1}(u_c)$ *solves the full system* (A.7).

(v) $\mathcal{M}^\epsilon$ *is a symplectic submanifold of* $\mathcal{X}$ *when equipped with the symplectic form* $\Upsilon^\epsilon\Big|_{\mathcal{M}^\epsilon}$ *and Hamiltonian* $\mathcal{K}^\epsilon(u_c) = \mathcal{H}^\epsilon(u_c + \Psi^\epsilon(u_c))$. *The reduced system* (A.9) *corresponds to the Hamiltonian flow for* $(\mathcal{M}^\epsilon, \Upsilon\Big|_{\mathcal{M}^\epsilon}, \mathcal{K}^\epsilon)$. *In fact, it is reversible and coincides with the restriction of the full Hamiltonian to the center manifold.*

A.3. Abstract global bifurcation theory

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $\mathcal{I}$ an open interval (possibly unbounded) with $0 \in \overline{\mathcal{I}}$, and $\mathcal{U} \subset \mathcal{X}$ an open set with $0 \in \partial\mathcal{U}$. Consider the abstract operator equation

$$\mathcal{F}(x, \lambda) = 0,$$

where $\mathcal{F} \mapsto \mathcal{U} \times \mathcal{I} \to \mathcal{Y}$ is an analytic mapping. Assume that for any $(x, \lambda) \in \mathcal{U} \times \mathcal{I}$ with $\mathcal{F}(x, \lambda) = 0$, the Fréchet derivative $\mathcal{F}_x(x, \lambda) \mapsto \mathcal{X} \to \mathcal{Y}$ is Fredholm with index 0.

THEOREM A.7 (Chen, Walsh, and Wheeler [**12, 13**]). *Suppose that there exists a continuous curve $\mathscr{C}_{loc}$ of solutions to $\mathcal{F}(x, \lambda) = 0$, parametrized as*

$$\mathscr{C}_{loc} := \{(\tilde{x}(\lambda), \lambda) : 0 < \lambda < \lambda_*\} \subset \mathcal{F}^{-1}(0)$$

for some $\lambda_* > 0$ and continuous $\tilde{x} \mapsto (0, \lambda_*) \to \mathcal{U}$. If

$$\lim_{\lambda \searrow 0} \tilde{x}(\lambda) = 0 \in \partial\mathcal{U}, \qquad \mathcal{F}_x(\tilde{x}(\lambda), \lambda) \mapsto \mathcal{X} \to \mathcal{Y} \text{ is invertible for all } \lambda, \tag{A.10}$$

then $\mathscr{C}_{loc}$ is contained in a curve of solutions $\mathscr{C}$, parametrized as

$$\mathscr{C} := \{(x(s), \lambda(s)) : 0 < s < \infty\} \subset \mathcal{F}^{-1}(0)$$

for some continuous $(0, \infty) \ni s \mapsto (x(s), \lambda(s)) \in \mathcal{U} \times \mathcal{I}$, with the following properties.

(a) *One of the following alternatives holds:*

(i) (Blowup) *As $s \to \infty$,*

$$N(s) := \|x(s)\|_{\mathcal{X}} + \frac{1}{\operatorname{dist}(x(s), \partial\mathcal{U})} + \lambda(s) + \frac{1}{\operatorname{dist}(\lambda(s), \partial\mathcal{I})} \to \infty. \tag{A.11}$$

(ii) (Loss of compactness) *There exists a sequence $s_n \to \infty$ such that $\sup_n N(s_n) < \infty$ but $\{x(s_n)\}$ has no subsequences converging in $\mathcal{X}$.*

(b) *Near each point $(x(s_0), \lambda(s_0)) \in \mathscr{C}$, we can reparametrize $\mathscr{C}$ so that $s \mapsto (x(s), \lambda(s))$ is real analytic.*

(c) *$(x(s), \lambda(s)) \notin \mathscr{C}_{loc}$ for s sufficiently large.*

Alternative (a)(i) is viewed as desirable because, in many applications, the blow up of each of the quantities appearing in (A.11) can be interpreted physically. On the other hand, Alternative (a)(ii) at this level of generality tells us very little about the behavior of the solutions along the global branch. For the specific case where the zero-set of $\mathcal{F}$ represented classical solutions to a fully nonlinear elliptic PDE set on an infinite cylinder, it was shown that the loss of compactness occurs precisely when (up to translations) there exists a subsequence that convergences locally to a front; see [**13**, Lemma 6.3]. This result unfortunately cannot be applied directly to the curve of solitary waves with discontinuous vorticity as they are weak solutions lying in Sobolev spaces. However, in the next lemma we rule out Alternative (a)(ii) entirely using a concentration-compactness argument that exploits the monotonicity and symmetry of the solutions along with the good behavior of the operator at infinity.

In fact, we can do this in a much more general setting. Let $\Omega := \mathbb{R} \times B$ be an infinite cylinder whose base $B \subset \mathbb{R}^{n-1}$ is a bounded C^2 domain. We will denote points in Ω as (x, y) where $x \in \mathbb{R}$ and $y \in B$. Partition the components of ∂B as $\partial B = \Gamma_0 \cup \Gamma_1$ (either may be empty).

For $p > n$, we wish to consider weak solutions $u \in W^{2,p}(\Omega)$ to conormal derivative problem for the quasilinear elliptic equation

$$\text{(A.12)} \qquad \begin{cases} \partial_i \mathcal{A}^i(y, \nabla u, \lambda) + \mathcal{B}(y, u, \nabla u, \lambda) = 0 \text{ in } \Omega, \\ -\nu^i \mathcal{A}^i(y, \nabla u, \lambda) + \mathcal{G}(y, u, \lambda) = 0 \text{ on } \mathbb{R} \times \Gamma_1, \\ u = 0 \text{ on } \mathbb{R} \times \Gamma_0, \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$, the parameter $\lambda \in \mathbb{R}^m$, and the coefficients have the regularity

$$\mathcal{A}^i(y, \cdot, \cdot) \in C^1_{\mathrm{b}}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m), \qquad \mathcal{A}^i(\cdot, \xi, \lambda) \in L^\infty(B)$$

$$\mathcal{B}(y, \cdot, \cdot, \cdot) \in C^1_{\mathrm{b}}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m), \qquad \mathcal{B}(\cdot, z, \xi, \lambda) \in L^\infty(B)$$

$$\mathcal{G}(y,\cdot,\cdot) \in C^1_{\mathrm{b}}(\mathbb{R}\times\mathbb{R}^m), \qquad\qquad \mathcal{G}(\cdot,z,\lambda) \in L^\infty(\Gamma_1),$$

for all $(y,z,\xi,\lambda) \in B\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$. Moreover, assume that the interior equation is uniformly elliptic in the sense that there exists $c>0$ such that

$$\mathcal{A}^i_{\xi^j}(y,\xi,\lambda)\zeta_i\zeta_j \geq c|\zeta|^2, \qquad \text{for all } y\in B,\ \xi,\zeta\in\mathbb{R}^n,\ \lambda\in\mathbb{R}^m. \tag{A.13}$$

We will also assume that the so-called limiting operator is injective in the following sense. Let

$$a^{ij} := \mathcal{A}^j_{\xi^i}(y,0,\lambda), \quad b^i := \mathcal{B}_{\xi^i}(y,0,0,\lambda), \quad c_n := \mathcal{B}_z(y,0,0,\lambda), \quad \gamma := \mathcal{G}_z(y,0,\lambda), \tag{A.14}$$

and consider the linear problem

$$\begin{cases} \partial_i\left(a^{ij}(y,\lambda)\partial_j u\right) + b^i(y,\lambda)\partial_i u + c(y,\lambda)u = 0 \text{ in } \Omega, \\ \qquad\qquad -\nu^i a^{ij}\partial_j u + \gamma(y,\lambda)u = 0 \text{ on } \mathbb{R}\times\Gamma_1, \\ \qquad\qquad\qquad u = 0 \text{ on } \mathbb{R}\times\Gamma_0. \end{cases} \tag{A.15}$$

In view of the situation for the solitary waves with discontinuous vorticity, we will suppose that there exist a (possibly unbounded) open set $\Lambda\subset\mathbb{R}^m$ such that there are no nontrivial $u\in W^{2,p}(\Omega)$ for which (u,λ) solves (A.15) in a weak sense and $\lambda\in\Lambda$.

Lemma A.1 (Compactness). *Suppose that $\{(u_n,\lambda_n)\}\subset W^{2,p}(\Omega)\times\Lambda$ is a sequence of solutions to* (A.12) *that is uniformly bounded in $W^{2,p}(\Omega)\times\mathbb{R}^m$, with the additional monotonicity property*

$$u_n(x,y) \text{ is even in } x \text{ and } \partial_x u_n \leq 0 \text{ for } x\geq 0 \tag{A.16}$$

for each n, then either

(i) *$\{(u_n,\lambda_n)\}$ is precompact $W^{2,p}(\Omega)\times\mathbb{R}$, or*

(ii) *$\{\lambda_n\}$ has a limit point on $\partial\Lambda$.*

Proof. We will assume that (ii) does not occur, and then prove (i). Consider the function

$$M(x) := \sup_n \sup_{y\in B} |u_n(x,y)| \qquad \text{for all } x \in \mathbb{R}.$$

Suppose first that $M(x)$ does not vanish in the limit $|x| \to \infty$. We can then find $\epsilon > 0$ and a sequence $\{(x_n, y_n)\} \subset \Omega$ with $x_n \to \infty$ such that

$$|u_n(x_n, y_n)| > \epsilon.$$

As $\{u_n\}$ is uniformly bounded in $W^{2,p}(\Omega) \subset C_{\mathrm{b}}^{1,\alpha}(\Omega)$, for $\alpha := 1 - n/p$, we know that there exists $\delta > 0$ independent of n such that

$$\sup_n \sup_{y \in B\cap B_\delta(y_n)} |u_n(x_n, y)| > \frac{\epsilon}{2}.$$

Because u_n is even and monotonic in x, this further gives

$$|u_n(x,y)| > \frac{\epsilon}{2} \qquad \text{for all } x \in [-x_n, x_n],\ y \in B \cap B_\delta(y_n).$$

Applying Chebyshev's inequality, we arrive at the estimate

$$\delta^{n-1}|x_n| \lesssim \operatorname{meas}\left\{(x,y) \in \Omega : |u_n(x,y)| > \frac{\epsilon}{2}\right\} \lesssim \frac{1}{\epsilon^p}\|u_n\|_{L^p(\Omega)}^p,$$

which contradicts the uniform boundedness of $\{u_n\}$ in $W^{2,p}(\Omega)$. We therefore conclude that $M(x) \to 0$ as $|x| \to \infty$.

In fact, the sequence of gradients $\{\nabla u_n\}$ is also equidecaying. This is a simple consequence of the Gagliardo–Nirenberg interpolation inequality: letting $\Omega_N := \{(x,y) \in \Omega : |x| > N\}$, we find that

$$\|u_n\|_{\dot{W}^{1,\infty}(\Omega_N)} \le C\|u_n\|_{L^\infty(\Omega_N)}^{\theta}\|u_n\|_{\dot{W}^{2,p}(\Omega_N)}^{1-\theta}, \qquad \text{for } \theta := \frac{1}{2-n/p} \in (1/2, 1),$$

where $C > 0$ is independent of n and N. As $\{u_n\}$ is uniformly bounded in $W^{2,p}(\Omega)$, taking the supremum over n and using the equidecay of $\{u_n\}$ yields the equidecay of

$\{\nabla u_n\}$, since

$$\sup_n \|u_n\|_{\dot{W}^{1,\infty}(\Omega_N)} \leq C|M(N)|^\theta \to 0 \qquad \text{as } N \to \infty.$$

Now, by the reflexivity of $W^{2,p}(\Omega)$, there exists $(u,\lambda) \in W^{2,p}(\Omega) \times \mathbb{R}^m$ such that up to a subsequence $\lambda_n \to \lambda$ and $u_n \rightharpoonup u$ weakly in $W^{2,p}(\Omega)$. Using imbedding theorems and a diagonalization argument, we can further conclude that $u_n \to u$ in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$. Our goal is now to show strong convergence in L^p, which can then be used in conjunction with a priori estimates to deduce the full convergence in $W^{2,p}$.

Seeking a contradiction, suppose that $u_n - u$ does *not* converge to 0 in $L^p(\Omega)$. Then consider the normalized sequence of differences $v_n := (u_n - u)/\|u_n - u\|_{L^p}$, and set $\mu_n := |v_n|^p$. Clearly $\|\mu_n\|_{L^1(\Omega)} = 1$ for all n, and hence from the concentration-compactness principle of Lions [**39**, Lemma 1.1] we conclude that there must be a subsequence along which one of the following three alternatives occurs.

(i) *Compactness.* There exists $\{x_n\} \subset \mathbb{R}$ such that, for all $\epsilon \in (0,1)$, there exists $R > 0$ satisfying

$$\liminf_{n\to\infty} \int_{x_n - R}^{x_n + R} \int_B \mu_n \, dy \, dx \geq 1 - \epsilon.$$

(ii) *Dichotomy.* $\lim_{R\to\infty} Q(R) = \delta$, for some $\delta \in (0,1)$, where Q is the concentration function:

$$Q(R) := \lim_{n\to\infty} \sup_{x_0 \in \mathbb{R}} \int_{x_0 - R}^{x_0 + R} \int_B \mu_n \, dy \, dx. \tag{A.17}$$

(iii) *Vanishing.* For all $R > 0$, $Q(R) = 0$.

We will eliminate each of these possibilities one at a time

Compactness. Suppose first that the we are in the compactness case and let $\{x_n\}$ be the sequence of translations. If $\{x_n\}$ is bounded, then there exists $R > 0$

such that

$$\liminf_{n\to\infty} \int_{-R}^{R} \int_{B} \mu_n \, dy \, dx \geq \frac{1}{2(1+\sup_n \|u_n\|_{L^p})},$$

and hence

$$\liminf_{n\to\infty} \int_{-R}^{R} \int_{B} |u_n - u|^p \, dy \, dx \geq \frac{\|u_n - u\|_{L^p}}{2(1+\sup_n \|u_n\|_{L^p})} \geq \frac{1}{2}.$$

This contradicts the fact that $u_n \to u$ in $W^{1,p}_{\mathrm{loc}}$, and hence we may assume instead that $\limsup |x_n| = \infty$. Without loss of generality, then, say that $x_n \to +\infty$. We work with the translated functions $w_n := v_n(\cdot - x_n, \cdot)$, which naturally have the same L^p and $W^{2,p}$ norms as v_n. This ensures that, possibly passing to a subsequence, $w \rightharpoonup w$ in $W^{2,p}(\Omega)$, for some $w \in W^{2,p}(\Omega)$. We note again that $w_n \to w$ in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$ as well.

Now, for any $\epsilon \in (0,1)$, we may let $R = R(\epsilon)$ be given such that

$$1 - \epsilon \leq \liminf_{n\to\infty} \int_{-R}^{R} \int_{B} |w_n|^p \, dy \, dx = \int_{-R}^{R} \int_{B} |w|^p \, dy \, dy \leq 1,$$

where the first equality comes from the local convergence and the second from the weak lower semicontinuity of the L^p norm. The above chain of inequalities implies that $\|w\|_{L^p} = 1$, and hence by applying the Brezis–Lieb Lemma [**6**], we conclude that $w_n \to w$ in $L^p(\Omega)$.

To go further we must exploit the specific structure of the equation (A.12). Observe that as the coefficients are independent of x, the translates $u_n(\cdot - x_n, \cdot)$ are also solutions. Moreover, each v_n solves the linear problem

$$\text{(A.18)} \quad \begin{cases} \partial_i \left(a_n^{ij} \partial_j v_n \right) + b_n^i \partial_i v_n + c_n v_n = 0 & \text{in } \Omega, \\ -\nu^i a_n^{ij} \partial_j v_n + \gamma_n v_n = 0 & \text{on } \mathbb{R} \times \Gamma_1, \\ v_n = 0 & \text{on } \mathbb{R} \times \Gamma_0, \end{cases}$$

where the coefficients $a_n^{ij}, b_n^i, c_n, \beta_n^i, \gamma_n$ are defined in terms of the convex combinations $u_n^{(s)} := su_n + (1-s)u$ and $\lambda^{(s)} := s\lambda_n + (1-s)\lambda$ by

$$a_n^{ij} := \int_0^1 \mathcal{A}^j_{\xi^i}(y, \nabla u_n^{(s)}, \lambda^{(s)})\, ds, \qquad b_n^i := \int_0^1 \mathcal{B}_{\xi^i}(y, u_n^{(s)}, \nabla u_n^{(s)}, \lambda^{(s)})\, ds,$$
$$c_n := \int_0^1 \mathcal{B}_z(y, u_n^{(s)}, \nabla u_n^{(s)}, \lambda^{(s)})\, ds, \qquad \gamma_n := \int_0^1 \mathcal{G}_z(y, u_n^{(s)}, \lambda^{(s)})\, ds.$$

Finally, this means that w_n solves a translated problem

$$\begin{cases} \partial_i \left(\tilde{a}_n^{ij} \partial_j w_n\right) + \tilde{b}_n^i \partial_i w_n + \tilde{c}_n w_n = 0 & \text{in } \Omega, \\ -\nu^i \tilde{a}_n^{ij} \partial_j w_n + \tilde{\gamma}_n w_n = 0 & \text{on } \mathbb{R} \times \Gamma_1, \\ w_n = 0 & \text{on } \mathbb{R} \times \Gamma_0, \end{cases} \tag{A.19}$$

for $\tilde{a}_n^{ij} := a_n^{ij}(\cdot - x_n, \cdot)$ and so on.

We will write (A.19) more concisely as $\mathcal{L}_n w_n = 0$, where $\mathcal{L}_n$. We claim that $\mathcal{L}_n w_n \rightharpoonup \mathcal{L} w$ in $L^p(\Omega)$, with $\mathcal{L}$ being the operator at infinity given by (A.15). Let a test function $\varphi \in C_c^\infty(\overline{\Omega})$ be given and consider the pairing

$$\langle (\mathcal{L}_n - \mathcal{L}) w_n, \varphi \rangle = \int_\Omega \Big(- \left(\tilde{a}_n^{ij} - a^{ij}\right) \partial_i w_n \partial_j \varphi + \left(\tilde{b}_n^i - b^i\right) \partial_i w_n \varphi + \left(\tilde{c}_n - c\right) w_n \varphi \Big)\, dy\, dx$$
$$+ \int_{\mathbb{R} \times \Gamma_1} \left(\tilde{\gamma}_n - \gamma\right) w_n \varphi \, dS.$$

As M vanishes at infinity, and in light of the regularity of the coefficients for the nonlinear problem (A.12), we see that for all $\epsilon > 0$, there exists $R > 0$ such that

$$\sup_n \left(|a_n^{ij} - a^{ij}| + |b_n^i - b^i| + |c_n - c| + |\gamma_n - \gamma| \right) < \epsilon \qquad \text{on } (R, \infty) \times B.$$

It follows that the integrand above is converging to 0 almost everywhere as $n \to \infty$. As w_n is locally convergent to w in $W^{1,p}(\Omega)$, we can then apply Lebesgue dominated convergence to conclude that $\mathcal{L}_n w_n - \mathcal{L} w_n \rightharpoonup 0$ in $L^p(\Omega)$. On the other hand, because $w_n \rightharpoonup w$ in $W^{2,p}(\Omega)$, clearly $\mathcal{L}(w_n - w) \rightharpoonup 0$ in $L^p(\Omega)$. In total, then, we have shown that w is a weak solution of $\mathcal{L}w = 0$, which by assumption implies $w = 0$. However, $\|w\|_{L^p} = 1$, and so this is impossible.

Dichotomy. Next, consider the situation where dichotomy occurs. Here the argument will be similar to the compactness case, but we will allow some portion of the mass to leak out to infinity. Let $\delta \in (0, 1)$ be the limiting concentration. Letting

$$Q_n(R) := \sup_{x_0 \in \mathbb{R}} \int_{x_0 - R}^{x_0 + R} \int_B \mu_n \, dy \, dx,$$

for any $\epsilon > 0$, may may choose $R = R(\epsilon) > 0$ so that

$$\delta - \epsilon < Q_n(R) \le Q_n(2R) < \delta + \epsilon \qquad \text{for all } n \text{ sufficiently large.}$$

This implies that there exists a sequence $\{x_n\} \subset \mathbb{R}$ such that

$$\frac{\delta}{2} < \int_{x_n - R}^{x_n + R} \int_B \mu_n \, dy \, dx \le \int_{x_n - 2R}^{x_n + 2R} \int_B \mu_n \, dy \, dx < \frac{3\delta}{2}.$$

Intuitively, we have found that an arbitrarily large proportion of the δ mass is concentrated in compact neighborhoods that shifts with the sequence $\{x_n\}$.

Now, we introduce a cut-off function $\Phi \in C_c^\infty(\mathbb{R}; [0, 1])$ with

$$\operatorname{supp} \Phi \subset [-2, 2], \qquad \Phi = 1 \text{ on } (-1, 1),$$

and define $\Psi \in C_{\mathrm{b}}^\infty(\mathbb{R}; [0, 1])$ by $\Psi := 1 - \Phi$. For each scale factor $s > 0$, denote $\Phi^s := \Phi(\cdot/s)$ and likewise for Ψ^s. Then there exists a universal constant $C > 0$ such that

$$\|\partial_x^j \Phi^s\|_{L^\infty}, \ \|\partial_x^j \Psi^s\|_{L^\infty} \le C s^{-k}, \qquad \text{for all } k \ge 0.$$

Let $\{s_n\}$ be a sequence with $s_n \to \infty$ and decompose

$$v_n = v_n^1 + v_n^2, \qquad v_n^1 := \Phi^{s_n}(\cdot - x_n) v_n, \ v_n^2 := \Psi^{s_n}(\cdot - x_n) v_n.$$

Thus

$$\|v_n^1\|_{L^p}^p+\|v_n^2\|_{L^p}^p = 1, \qquad \|v_n^1\|_{W^{k,p}}^p+\|v_n^1\|_{W^{k,p}}^p \leq \|v_n\|_{W^{k,p}}^p+Cs_n^{-k}, \text{ for } k=1,2,$$

for some constant $C > 0$ and where all norms are being evaluated on Ω. By the choice of R and $\{x_n\}$, we note that we also have

$$\inf_n \|v_n^1\|_{L^p(\Omega)}^p \geq \inf_n \int_{x_n-R}^{x_n+R} \int_B |v_n^1|^p \, dy \, dx > \frac{\delta}{2}.$$

Define the translated function $w_n := v_n^1(\cdot + x_n, \cdot)$, which in particular gives that $\operatorname{supp} w_n \subset (-2s_n, 2s_n) \times B$ and $\|w_n\|_{W^{k,p}(\Omega)} = \|v_n^1\|_{W^{k,p}(\Omega)}$ for $k = 0, 1, 2$. Thus there exists $w \in W^{2,p}(\Omega)$ such that $w_n \rightharpoonup w$ in $W^{2,p}(\Omega)$ and $w_n \to w$ in $C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$. Note that by construction, we have also that

$$\|w\|_{L^p(\Omega)}^p > \frac{\delta}{2} > 0,$$

and hence $w \not\equiv 0$. On the other hand, if redefine the translated coefficients

$$\tilde{a}_n^{ij} := a_n^{ij}(\cdot{+}x_n, \cdot), \quad \tilde{b}_n^i := b_n^i(\cdot{+}x_n, \cdot), \quad \tilde{c}_n := c_n(\cdot{+}x_n, \cdot), \quad \tilde{\gamma}_n := \gamma_n(\cdot{+}x_n, \cdot).$$

then

$$\begin{aligned} \partial_i\left(\tilde{a}_n^{ij}\partial_j w_n\right) + \tilde{b}_n^i \partial_i w_n + \tilde{c}_n w_n = {} & \left(\tilde{a}_n^{ij}\partial_i \Phi^{s_n} + \tilde{b}_n^j \Phi^{s_n}\right)\partial_j v_n(\cdot + x_n, \cdot) \\ & + \left(\partial_i\left(\tilde{a}_n^{ij}\partial_j \Phi^{s_n}\right) + \tilde{b}_n^i \partial_i \Phi^{s_n}\right) v_n(\cdot + x_n, \cdot) \end{aligned} \qquad \text{on } \Omega,$$

along with the conormal boundary condition

$$-\nu^i \tilde{a}_n^{ij}\partial_j w_n + \tilde{\gamma}_n w_n = -\nu^i \tilde{a}_n^{ij}\partial_i \Phi^{s_n}\partial_j v_n(\cdot + x_n, \cdot) \qquad \text{on } \mathbb{R} \times \Gamma_1,$$

and the homogeneous Dirichlet condition

$$w_n = 0 \qquad \text{on } \mathbb{R} \times \Gamma_0.$$

As before, we denote the left-hand side of the above problem as $\mathcal{L}_n w_n$. Arguing as in the compactness case, we readily confirm that $\mathcal{L}_n w_n \rightharpoonup \mathcal{L} w$. But the right-hand side terms in the above system decay in $L^p(\Omega) \cap L^\infty(\Omega)$, and so we infer that (w, λ) is in the kernel of the limiting operator (A.15). As $\lambda \in \Lambda$, this produces a contradiction.

Vanishing. This can be ruled out exactly as was done for the linear problem since the argument we gave there only depends on the fact that we have convergence, not the specific equation.

As a result of the above analysis, we have confirmed that $u_n \to u$ in $L^p(\Omega)$. To complete the argument, we note that $u_n - u$ also solves the system (A.18), and the coefficients a_n^{ij}, b_n^i, c_n, and γ_n are uniformly bounded in $L^p(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$. We therefore have the a priori estimate

$$\|u_n - u\|_{W^{2,p}(\Omega)} \leq C \|u_n - u\|_{L^p(\Omega)},$$

where $C > 0$ is independent of n. This completes the proof of the lemma. ∎

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