

**STABILITY AND INSTABILITY RESULTS FOR THE 2D α -EULER
EQUATIONS**

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Stability and instability results for the 2D α -Euler equations

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ABSTRACT

We study stability and instability of time independent solutions of the two dimensional α -Euler equations and Euler equations; the α -Euler equations are obtained by replacing the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in the classical Euler equations of inviscid incompressible fluid by the term $(\mathbf{v} \cdot \nabla)\mathbf{u}$, where \mathbf{v} is the regularized velocity satisfying $(1 - \alpha^2 \Delta)\mathbf{v} = \mathbf{u}$, and $\alpha > 0$.

In the first part of the thesis, for the α -model, we develop analogues of the classical Arnol'd type stability criteria based on the energy-Casimir method for several settings including multi connected domains, periodic channels, and others. In the second part of the thesis, we study stability of a particular steady state, the unidirectional solution of the α -Euler equation on the two dimensional torus, having only one non zero mode in its Fourier decomposition. Using continued fractions, we give a proof of instability of the steady state under fairly general conditions. In the third part of the thesis we study various properties of a family of elliptic operators introduced by Zhiwu Lin in his work on instability of steady state solutions of the two dimensional Euler equations. This involves Birman-Schwinger type operators associated with the linearization of the Euler equations about the steady state and certain perturbation determinants.

Chapter 1

Introduction

This thesis is concerned with the study of some questions arising in the stability and instability of the two dimensional α -Euler and Euler equations of ideal fluids. The α -Euler equations arise as a regularization of the Euler equations. The classical Euler equations of incompressible inviscid fluid read

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0, \quad (1.0.1)$$

where \mathbf{u} is the velocity and p is the pressure, while the α -Euler model is as follows:

$$\mathbf{u}_t + (\mathbf{v} \cdot \nabla)\mathbf{u} + (\nabla\mathbf{v})^\top \mathbf{u} + \nabla\pi = 0, \quad \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0, \quad (1.0.2)$$

where

$$(1 - \alpha^2 \Delta)\mathbf{v} = \mathbf{u}, \quad \pi = p - \frac{1}{2}|\mathbf{v}|^2 - \frac{\alpha^2}{2}|\nabla\mathbf{v}|^2, \quad \alpha > 0, \quad (1.0.3)$$

and \mathbf{v} is called the regularized, or filtered, velocity. It is the presence of \mathbf{v} in the nonlinear term $(\mathbf{v} \cdot \nabla)\mathbf{u}$ that regularizes the equations. More details regarding the nature of these equations are given in the introduction to Chapter 2. For $\alpha = 0$ the α -model becomes the classical Euler equations.

The α -Euler and the closely related α -Navier-Stokes models were introduced in the foundational papers [CFHOTW98, FHT01] and since then have been intensively

studied, see [BLT08, BLT10, IT03, LT10, SR10, V12, LNTZ14, LNTZ15] and the bibliography therein. In spite of that, almost nothing is known about the stability of the steady state solutions to the α -Euler equations, cf. [PS], and the main objective of this thesis is to fill the gap.

The two dimensional inviscid Euler equations are, of course, the most basic and fundamental model to study fluid motion and its importance both to mathematics and physics and engineering cannot be overstated. It was introduced more than 250 years ago by Euler. Straddling a wide swath of areas in and around mathematics, the Euler equations have borrowed ideas and enriched areas such as geometry, partial differential equations, dynamical systems and functional analysis in pure mathematics and different fields of engineering such as mechanical, chemical, aerospace and other areas such as geophysical fluid dynamics. Even though the Euler equations have been known for a long time, there are many open problems regarding the study of these equations. We point the reader to [C07] for an overview and a list of open mathematical questions concerning the Euler equations.

A particular direction of questions concerns the *stability* of steady state solutions, i.e., time independent solutions, to the Euler equations. This class of problems has a long and storied history going back at least to Kelvin, Helmholtz and Rayleigh 150 years ago. Not only are these problems of interest in mathematics but they are of paramount importance in aerospace engineering and geophysical fluid dynamics. For an overview of mathematical questions regarding stability, we refer the reader to [FY99], [Fr], [SH12], [Y03] and we refer to [DZ04] for an applied mathematics and engineering perspective. Excellent reviews of the more recent literature on the subject

can be found in [FS05, FS01], as well as in the important papers [FSV97, FVY00, FSV99].

Informally, the basic problem can be stated as follows: given a time independent solution of the Euler equations, is it stable to small perturbations, i.e., if one starts “close” to the steady state does one remain “close” to the steady state for all time? Since the Euler equations are nonlinear, this question can be further simplified to studying the stability of steady state solutions to the *linearized* Euler equations. One starts with a steady state to the Euler equations and linearizes about this steady state. One is then left with spectral questions regarding the linearized operator. Relations between the linear and nonlinear stability and instability for the Euler equations are rather subtle issues that continue to be the subject of intensive studies. To the best of our knowledge, thus far the most advanced results in this direction, [FSV97, ZL04a], state that under appropriate assumptions the existence of unstable eigenvalues (that is, not pure imaginary) of the operator obtained by linearizing 2D Euler equations about the steady state implies the nonlinear instability of the steady state. This makes the classical problem of studying the spectrum of the linearized operator even more important. Regarding advances on this challenging spectral problem we refer to [BFY99, GC14, F71, Fr, FH98, FS05, ZL03, ZL04, ZL05, SL05, SL03, S08, L, LLS, MS, WDM1, WDM2] and to the bibliography given therein.

Due to the relatively recent substantial breakthroughs in our understanding of the essential spectrum of the linearized operator, see [FS05, FS01, SL05, SL03], the main focus of current research is still on obtaining effective sufficient conditions for the existence of unstable discrete eigenvalues of the linearized operator for a given steady

state. Although the list of known steady states is not that long, even for the solutions on this list the question of stability or instability is not always resolved. In particular, despite of much effort, [BFY99, F71, FH98], the instability is still an issue, say, even for the simplest generalizations of the classical Kolmogorov flows, see [MS] and also [L, LLS, WDM1, WDM2]. In the current thesis, we made some progress in studying instability of these generalizations, see Chapter 3; the instability results there appear to be new even in the context of the Euler equation.

Going back to the regularized model, we note again that the nonlinearity of the Euler equations have made it difficult to study. Because of that, and also due to their tremendous technological importance in areas such as turbulence modeling, various regularized models were introduced to simplify the study of these equations. These models simplify the nonlinearity to make the equations more tractable. In this thesis we focus on one of these models, the α -Euler model, introduced by [FHT01, CFHOTW99]. We combine the two threads discussed above, i.e., the study of stability of solutions and the study of regularized fluid models, and thus discuss the stability of steady state solutions to the 2D α -Euler equations. Very broadly, we would like to study the effect of the regularization on various stability criteria that have been developed for the Euler equations. Does the regularization make it easier to study stability? If so, can one study the stability of regularized models and then have something useful to say about the stability of solutions to the Euler equations themselves (by taking limits in a suitable sense)? Are there any particular examples of the steady states solutions to the 2D α -Euler equations when one can prove the existence of unstable eigenvalues?

The dissertation is structured as follows. In Chapter 2, we obtain criteria similar to the well-known Rayleigh, Fjortoft and Arnold criteria for the Euler equations but in the context of the α -Euler equations. Arnold, in 1965, see [A65], provided a new direction in the study of stability via geometric methods. Later, this became known as the energy-Casimir method, see [AK98, CSS06, HMR98, HMRW85, MP94, S00, T10, WG98, GW95, GW96, HMH14, MPSW01] as well as the literature cited therein. The literature on various aspects of Arnold's energy-Casimir method for the Euler equations is quite vast, but, to the best of our knowledge, almost nothing is known about this method in the context of the regularized α -Euler models. In fact, even the simplest and most elementary results such as the classical Rayleigh and Fjortoft criteria for α -Euler have not been recorded, not to mention the Arnold stability theorem. In Chapter 2, we fill this gap. We explore the Arnold stability criteria for the α -Euler equations. We have studied this criteria in various domains such as the periodic channel, multi connected domain and the torus and provide a unified way to apply the Arnold stability criteria in these domains. Via the use of the Rayleigh-Ritz type principle we tidy up the Arnold criteria and provide a way of studying it in a unified way describing the Arnold stability results for both Euler and α -Euler equations.

Chapter 3 concerns the study of instability of the so called unidirectional flows. These flows are simple generalizations of the two-dimensional Kolmogorov flow [MS] given by the velocity whose x -component is $\cos x$ and y component is zero; they have been studied by many authors, see [BFY99, BW, FVY00, L, LLS, WDM1, WDM2] and the literature therein. In fact, unidirectional flows are induced by vorticities hav-

ing exactly one non zero mode in its Fourier decomposition determined by a fixed two dimensional vector \mathbf{p} with integer coordinates (so that $\mathbf{p} = (1, 0)$ for the Kolmogorov flow). Starting with the paper [L], some work was done to establish possible instability of the unidirectional flows, see [LLS, WDM1, WDM2]. However, the question if the unidirectional flows are unstable for arbitrary vectors \mathbf{p} still remained open, and is settled in the current thesis under some additional but reasonable assumptions on \mathbf{p} . Our main contribution is the proof of the instability theorem, see Theorem 3.9, that shows that the unidirectional flows are linearly unstable due to the presence of a positive isolated eigenvalue in the spectrum of the operator obtained by linearizing the α -Euler equations in vorticity variables about the steady state. The proof is based on the use of continued fractions, cf.[FH98, L, MS]. Besides this, we also study the essential spectrum and prove a spectral mapping theorem for the strongly continuous group generated by the linearized α -Euler operator.

Our concluding Chapter 4 deals with the properties of Lin's elliptic dispersion operators, A_λ , see [ZL04]. In this important paper, Zhiwu Lin noted, in particular, that λ is a positive eigenvalue of the linearized Euler equations in vorticity form if and only if certain elliptic operator, A_λ , has a non zero null space. We study properties of this operator using ideas from operator theory such as the Birman-Schwinger principle and perturbation determinants. In particular, we introduce a novel holomorphic function of the spectral parameter, a certain perturbation determinant, whose zeros are exactly the isolated eigenvalues of the linearized Euler operator. Using these tools, we also give a new version of the proof of an important Lin's instability theorem that gives a sufficient condition for the instability of the steady state in terms of the spectra

of the operator A_0 .

Chapter 2

Stability criteria for the 2D α -Euler equations

In this chapter, we derive analogues of the classical Rayleigh, Fjortoft and Arnold stability-instability theorems in the context of the 2D α -Euler equations.

2.1 Introduction: Setup and equations of motion

Our objectives in this chapter are to develop analogues of the classical stability results for the α -Euler equations, i.e., we develop analogues of the Rayleigh, Fjortoft and Arnold criteria for the incompressible, inviscid α -Euler equations. The α -Euler equations are an inviscid regularization of the Euler equations of fluid dynamics which shall be explained below.

Throughout we use boldface to denote (2×1) column vectors and vector-valued functions, and the following standard notation:

$$\nabla = \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix}, \nabla^\perp = \begin{bmatrix} -\partial_y \\ \partial_x \end{bmatrix}, \operatorname{div} = [\partial_x \quad \partial_y], \operatorname{curl} = [-\partial_y \quad \partial_x], \quad (2.1.1)$$

so that $\operatorname{curl}(\nabla^\perp) = \Delta = \partial_x^2 + \partial_y^2$ and $\nabla^\perp f \cdot \nabla g = -\nabla^\perp g \cdot \nabla f$. We consider the incompressible inviscid flow satisfying the Euler equations

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0, \quad \operatorname{div} \mathbf{u} = 0, \quad (2.1.2)$$

in a two-dimensional domain D . We either assume that $D = \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and then impose 2π -periodic boundary conditions, or that $D \subset \mathbb{R}^2$ is with boundary ∂D , and then impose on ∂D the boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0, \tag{2.1.3}$$

where \mathbf{n} is the unit outer normal vector. Of interest to us are domains $D \subset \mathbb{R}^2$ bounded and multiconnected with boundary ∂D , the periodic channel $\mathbb{T} \times [-1, 1]$ (periodic in x direction and having the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ on the “walls” $y = -1$ and $y = 1$) and the channel $\mathbb{R} \times [-1, 1]$.

The α -Euler equations are an inviscid regularization of the Euler equations given as follows:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{v} \cdot \nabla) \mathbf{u} + (\nabla \mathbf{v})^\top \mathbf{u} + \nabla \pi &= 0, \\ (1 - \alpha^2 \Delta) \mathbf{v} &= \mathbf{u}, \\ \operatorname{div} \mathbf{u} &= \operatorname{div} \mathbf{v} = 0 \end{aligned} \tag{2.1.4}$$

where \mathbf{v} is the so called filtered velocity and \mathbf{u} is the actual fluid velocity. Here $\alpha > 0$ is a positive real number related to the filter of the flow and $(\nabla \mathbf{v})^\top$ represents the transpose of the Jacobi matrix of partial derivatives $\partial \mathbf{v}_i / \partial x_j$. The pressure π is related to the actual fluid pressure p as

$$\pi = p - \frac{1}{2} |\mathbf{v}|^2 - \frac{\alpha^2}{2} |\nabla \mathbf{v}|^2.$$

On domains D with a boundary ∂D , these equations are supplemented by the boundary conditions, see, for example [HMR98] (formula 8.27, page 65),

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad (\mathbf{n} \cdot \nabla \mathbf{v}) \parallel \mathbf{n}. \tag{2.1.5}$$

A comment about the boundary conditions is in order. The kinetic energy of the α -Euler fluid is given by $\frac{1}{2} \int_D \mathbf{v} \cdot \mathbf{u} d\mathbf{x}$. Upon using Green's formula, we see that,

$$\int_D \mathbf{v} \cdot \mathbf{u} d\mathbf{x} = \int_D \mathbf{v} \cdot (1 - \alpha^2 \Delta) \mathbf{v} d\mathbf{x} = \int_D \mathbf{v} \cdot \mathbf{v} d\mathbf{x} + \alpha^2 \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x} - \int_{\partial D} \mathbf{v} \cdot (\mathbf{n} \cdot \nabla) \mathbf{v} ds. \quad (2.1.6)$$

One would like the boundary term to vanish. This can be achieved by either setting $\mathbf{v} = 0$ on the boundary ∂D or by requiring that $(\mathbf{n} \cdot \nabla \mathbf{v}) \parallel \mathbf{n}$. $(\mathbf{n} \cdot \nabla \mathbf{v}) \parallel \mathbf{n}$ implies that $\mathbf{v} \cdot (\mathbf{n} \cdot \nabla) \mathbf{v} = 0$ since $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary. The kinetic energy is thus given by the expression,

$$\frac{1}{2} \int_D \mathbf{v} \cdot \mathbf{u} d\mathbf{x} = \frac{1}{2} \int_D \mathbf{v} \cdot (1 - \alpha^2 \Delta) \mathbf{v} d\mathbf{x} = \frac{1}{2} \int_D \mathbf{v} \cdot \mathbf{v} d\mathbf{x} + \frac{\alpha^2}{2} \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x}. \quad (2.1.7)$$

In this thesis we choose to work with the boundary conditions (2.1.5). It is also possible to work with the Navier-Stokes-like no-slip boundary conditions,

$$\mathbf{v} = 0 \text{ on } \partial D. \quad (2.1.8)$$

We shall however not deal with the boundary condition (2.1.8) in this thesis. The α -Euler equations are also known as Lagrangian Averaged Euler equations and their viscous counterpart, α -Navier Stokes, has found use in turbulence modeling by being able to reproduce the salient features of turbulent flows in pipes and channels (see, for example, [CFHOTW98]).

Due to the second equation in (2.1.2), there is a stream function ψ such that equation $\mathbf{u} = -\nabla^\perp \psi$ holds. Similarly, since $\text{div } \mathbf{v} = 0$, we see that $\mathbf{v} = -\nabla^\perp \phi$ for a stream function ϕ . We introduce the vorticity $\omega = \text{curl } \mathbf{u}$ so that $\omega = -\Delta \psi$, similarly, we have the vorticity of the smoothed velocity \mathbf{v} , $q = \text{curl } \mathbf{v}$ so that $q = -\Delta \phi$. We note

that the relationship between the smoothed quantities \mathbf{v}, ϕ, q and the corresponding physical quantities \mathbf{u}, ψ, ω are given as follows:

$$\begin{aligned}(1 - \alpha^2 \Delta) \mathbf{v} &= \mathbf{u}, \\ (1 - \alpha^2 \Delta) \phi &= \psi, \\ (1 - \alpha^2 \Delta) q &= \omega\end{aligned}\tag{2.1.9}$$

Applying curl in (2.1.2), one obtains the Euler equation in vorticity form,

$$\omega_t + \mathbf{u} \cdot \nabla \omega = 0.\tag{2.1.10}$$

Analogously, the α -Euler equation in vorticity form is,

$$\omega_t + \mathbf{v} \cdot \nabla \omega = 0.\tag{2.1.11}$$

Equation (2.1.11) can be derived from (2.1.4) as follows. Note that the α -Euler equations (2.1.4) can be rewritten as, see [HMR98, Eq 8.33, page 67],

$$\partial_t \mathbf{u} - \mathbf{v} \times (\nabla \times \mathbf{u}) + \nabla(\mathbf{v} \cdot \mathbf{u} - \frac{1}{2} |\mathbf{v}|^2 - \frac{\alpha^2}{2} |\nabla \mathbf{v}|^2 + p) = 0.\tag{2.1.12}$$

From the following identity, see [CM93, page 160, Vector Identity 12]

$$\nabla \times (\mathbf{v} \times \omega) = \mathbf{v} \nabla \cdot \omega - \omega \nabla \cdot \mathbf{v} + (\omega \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \omega,\tag{2.1.13}$$

using the fact that $\text{div } \mathbf{v} = 0$ and $\text{div } \text{curl } \mathbf{u} = 0$ (see [CM93, page 160, Vector Identity 10]), we see that

$$\nabla \times (\mathbf{v} \times \omega) = (\omega \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \omega.\tag{2.1.14}$$

Note also, that $\omega = \nabla \times \mathbf{u}$ and $\text{curl } \nabla f = 0$ for any scalar valued function f . Apply curl to (2.1.12) and using (2.1.14) and the fact that in 2D $(\omega \cdot \nabla) \mathbf{v} = 0$ to obtain

$$\partial_t \omega + \mathbf{v} \cdot \nabla \omega = 0.$$

We note that the α -Euler equation has a smoother nonlinear term $\mathbf{v} \cdot \nabla \omega$ compared to the $\mathbf{u} \cdot \nabla \omega$ in the Euler equation. The Euler equation in stream function variables is,

$$\Delta \psi_t - \nabla^\perp \psi \cdot \nabla (\Delta \psi) = \Delta \psi_t - \psi_x \Delta \psi_y + \psi_y \Delta \psi_x = 0. \quad (2.1.15)$$

Similarly, the α -Euler equation in stream function variables is,

$$\Delta \psi_t - \nabla^\perp \phi \cdot \nabla (\Delta \psi) = \Delta \psi_t - \phi_x \Delta \psi_y + \phi_y \Delta \psi_x = 0, \quad (2.1.16)$$

where ϕ and ψ are related by the second equation in (2.1.9), i.e., $\psi = (1 - \alpha^2 \Delta) \phi$.

Let us consider a smooth steady state solution $\omega^0 = \text{curl } \mathbf{u}^0 = -\Delta \psi^0$ of the Euler equation. In particular,

$$\nabla^\perp \psi^0 \cdot \nabla \omega^0 = (-\psi_y^0 \partial_x + \psi_x^0 \partial_y) \omega^0 = 0, \quad (2.1.17)$$

and thus $\nabla \psi^0$ and $\nabla (\Delta \psi^0)$ are parallel.

Analogously, consider a steady state solution $\omega^0 = \text{curl}(1 - \alpha^2 \Delta) \mathbf{v}^0 = -\Delta \psi^0$ of the α -Euler equation. Here

$$\mathbf{v}^0 = -\nabla^\perp \phi^0, \quad \psi^0 = (1 - \alpha^2 \Delta) \phi^0 \text{ and } \omega^0 = -(1 - \alpha^2 \Delta) \Delta \phi^0.$$

In particular,

$$\nabla^\perp \phi^0 \cdot \nabla \omega^0 = (-\phi_y^0 \partial_x + \phi_x^0 \partial_y) \omega^0 = 0, \quad (2.1.18)$$

and thus $\nabla \phi^0$ and $\nabla (\Delta \psi^0)$ are parallel.

We linearize the Euler equations (2.1.2)–(2.1.15) about the steady state \mathbf{u}^0 :

$$\mathbf{u}_t + \mathbf{u}^0 \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}^0 + \nabla p = 0, \quad \text{div } \mathbf{u} = 0, \quad (2.1.19)$$

$$\omega_t + \mathbf{u}^0 \cdot \nabla \omega + \text{curl}^{-1} \omega \cdot \nabla \omega^0 = 0, \quad (2.1.20)$$

$$\Delta \psi_t - \psi_x^0 \Delta \psi_y + \psi_y^0 \Delta \psi_x - \psi_x \Delta \psi_y^0 + \psi_y \Delta \psi_x^0 = 0. \quad (2.1.21)$$

Here, $\mathbf{u} = \text{curl}^{-1} \omega$ denotes the unique solution of the system $\text{curl} \mathbf{u} = \omega$, $\text{div} \mathbf{u} = 0$ with appropriate boundary conditions. We let ϕ^0 be the stream function such that $(1 - \alpha^2 \Delta) \phi^0 = \psi^0$ and q^0 be such that $(1 - \alpha^2 \Delta) q^0 = \omega^0$.

The corresponding linearized equations for the α -Euler model about the steady state \mathbf{v}^0 , \mathbf{u}^0 are as follows:

$$\mathbf{u}_t + \mathbf{v}^0 \cdot \nabla \mathbf{u} + \mathbf{v} \cdot \nabla \mathbf{u}^0 + (\nabla \mathbf{v}^0)^\top \mathbf{u} + (\nabla \mathbf{v})^\top \mathbf{u}^0 + \nabla \pi = 0, \quad (2.1.22)$$

$$\text{div} \mathbf{u} = \text{div} \mathbf{v} = 0,$$

$$\omega_t + \mathbf{v}^0 \cdot \nabla \omega + \mathbf{v} \cdot \nabla \omega^0 = 0, \quad (2.1.23)$$

$$\Delta \psi_t - \phi_x^0 \Delta \psi_y + \phi_y^0 \Delta \psi_x - \phi_x \Delta \psi_y^0 + \phi_y \Delta \psi_x^0 = 0, \quad (2.1.24)$$

where $\mathbf{v} = \text{curl}^{-1}(1 - \alpha^2 \Delta)^{-1} \omega$ solves the system of equations $\text{curl}(1 - \alpha^2 \Delta) \mathbf{v} = \omega$, $\text{div} \mathbf{v} = 0$ with appropriate boundary conditions.

2.2 Rayleigh and Fjortoft criteria for the α -Euler equations

In this section, we derive the classical Rayleigh criterion and the Fjortoft criterion in the context of the α -Euler equations. We label the two dimensional spatial coordinates as (x, y) . Our basic setup is the two dimensional channel, infinitely long in the x direction and bounded in the y direction, with walls at $y = A_1$ and $y = A_2$, where $-\infty < A_1 < A_2 < +\infty$, i.e., $D = \mathbb{R} \times [A_1, A_2]$.

We shall be working with the so called $2D$ plane parallel shear flows in the channel. Note that for the two dimensional Euler equations on the domain D , any steady state

velocity of the form $\mathbf{u}^0(x, y) = (U(y), 0)$ and constant pressure $p(x, y) = p^0$ will solve the Euler equations (2.1.2) where $U : [A_1, A_2] \rightarrow \mathbb{R}$ is any real valued function and p^0 is a real constant. Note that the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ is automatically satisfied by a steady state of the type $(U(y), 0)$.

We now obtain the analogous steady state for the α -Euler equations (2.1.4). Let $V : [A_1, A_2] \rightarrow \mathbb{R}$ be a real valued function such that $V'(A_1) = V'(A_2) = 0$. Define $U(y) = V(y) - \alpha^2 V''(y)$. $(U(y), 0)$ as computed and p^0 are steady state solutions of the Euler equations (2.1.2). It can be readily verified (see [LNTZ15], Proposition 2, page 60) that, $\mathbf{u}^0(x, y) = (U(y), 0)$, $\mathbf{v}^0(x, y) = (V(y), 0)$ and

$$\pi(x, y) = p_0 - \frac{1}{2}V(y)^2 + \frac{\alpha^2}{2}(V'(y))^2 \quad (2.2.1)$$

(note that π is a function of y alone) is a steady state solution to the α -Euler equations (2.1.4). Since $\mathbf{n} = (0, \pm 1)$ on the boundaries $y = A_1$ and $y = A_2$, $(\mathbf{n} \cdot \nabla)\mathbf{v}^0 = \partial_y(V(y), 0) = (V'(y), 0)$. Since $(\mathbf{n} \cdot \nabla)\mathbf{v}^0 \cdot \mathbf{t} = (V'(y), 0) \cdot (1, 0) = 0$ on the boundary $y = A_1$ and $y = A_2$, this reduces to $V'(A_1) = V'(A_2) = 0$. Another way to obtain a steady state is to start with an arbitrary profile $U(y)$ and compute $V(y)$ by solving the ODE:

$$V(y) - \alpha^2 V''(y) = U(y), \quad V'(A_1) = V'(A_2) = 0. \quad (2.2.2)$$

In either case we just proved the following Lemma.

Lemma 2.1. *Let $V(y)$ be any function satisfying $V'(A_1) = V'(A_2) = 0$, and define $U(y) = V(y) - \alpha^2 V''(y)$, $\pi(x, y) = p_0 - \frac{1}{2}V(y)^2 + \frac{\alpha^2}{2}(V'(y))^2$. Then $\mathbf{u}^0(x, y) = (U(y), 0)$, $\mathbf{v}^0(x, y) = (V(y), 0)$ and $\pi(x, y) = p_0 - \frac{1}{2}V(y)^2 + \frac{\alpha^2}{2}(V'(y))^2$ is a steady state solution to the α -Euler equations (2.1.4) on the domain $\mathbb{R} \times [A_1, A_2]$.*

Remark 2.2. We note that an arbitrary profile $V = V(y)$ cannot be a steady state for the α -Euler equations. Only profiles that satisfy the boundary condition $V'(A_1) = V'(A_2) = 0$ can be steady states. This is in contrast with the Euler case where an arbitrary profile $U = U(y)$ with no boundary conditions is a steady state for Euler.

We let ψ^0 and ϕ^0 be the steady state stream functions. These are functions real valued, associated with the respective steady state velocities U and V respectively, i.e we have, $\partial_y \phi^0(y) = V(y)$ and $\partial_y \psi^0(y) = U(y)$ and $\psi^0 = (1 - \alpha^2 \partial_{yy}) \phi^0$. Note that ψ^0 and ϕ^0 are functions of y alone. The boundary condition $V'(A_1) = V'(A_2) = 0$ implies that $\phi_{yy}^0(A_1) = \phi_{yy}^0(A_2) = 0$. We work with the α -Euler equations in stream function formulation (2.1.21). Linearizing (2.1.21) about the steady state ψ^0, ϕ^0 we obtain the linearized equation for the perturbations $\tilde{\phi} = \tilde{\phi}(x, y, t)$ and $\tilde{\psi} = \tilde{\psi}(x, y, t)$ of the stream function (2.1.24) of the form

$$\Delta \tilde{\psi}_t - \phi_x^0 \Delta \tilde{\psi}_y + \phi_y^0 \Delta \tilde{\psi}_x - \tilde{\phi}_x \Delta \psi_y^0 + \tilde{\phi}_y \Delta \psi_x^0 = 0. \quad (2.2.3)$$

Since we have the relation $(1 - \alpha^2 \Delta) \tilde{\phi} = \tilde{\psi}$ one can consider the equation above as an equation for $\tilde{\phi}$ alone. We note that this equation is supplemented by the following boundary conditions: no normal flow across the boundaries, so $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary ∂D , which are the two walls at $y = A_1$ and $y = A_2$, and \mathbf{n} is the unit normal vector on ∂D and $(\mathbf{n} \cdot \nabla) \mathbf{v}$ is parallel to \mathbf{n} (see, for example, [HMR98] (formula 8.27, page 65)). Since, $\mathbf{v} = -\nabla^\perp \tilde{\phi}$, we see that the boundary conditions for $\tilde{\phi}$ are $\nabla \tilde{\phi} \cdot \mathbf{t} = 0$ on ∂D where \mathbf{t} is the unit tangent vector on ∂D and $(\mathbf{n} \cdot \nabla)(-\nabla^\perp \tilde{\phi}) \cdot \mathbf{t} = 0$ on ∂D .

Since $\phi_x^0 = \psi_x^0 = 0$, equation (2.2.3) reduces to

$$\Delta \tilde{\psi}_t + \phi_y^0 \Delta \tilde{\psi}_x - \tilde{\phi}_x \Delta \psi_y^0 = 0, \quad (2.2.4)$$

on D with $\nabla \tilde{\phi} \cdot \mathbf{t} = 0$ and $(\mathbf{n} \cdot \nabla)(-\nabla^\perp \tilde{\phi}) \cdot \mathbf{t} = 0$ on ∂D .

Similarly to the analysis for the Euler equations, see [DZ04], Chapter 4, Sections 20-22, pages 124-133, we look for solutions to (2.2.4) of the form

$$\tilde{\psi}(x, y, t) = \psi(y)e^{ik(x-ct)} \text{ and } \tilde{\phi}(x, y, t) = \phi(y)e^{ik(x-ct)}, \quad (2.2.5)$$

where $\psi : [A_1, A_2] \rightarrow \mathbb{C}$ and $\phi : [A_1, A_2] \rightarrow \mathbb{C}$ are complex valued functions of y , $k \in \mathbb{R}$ is the wave number, which is real in this case, and c is the wave speed which is complex valued, $c = c_r + ic_i$. If $c_i > 0$, equation (2.2.5) gives an exponentially growing (in time) solution to (2.2.4). We will now derive an α -Euler version of the Rayleigh stability equation.

Lemma 2.3. *Let $\mathbf{u}^0(x, y) = (U(y), 0)$, $\mathbf{v}^0(x, y) = (V(y), 0)$ and $\pi(x, y) = p_0 - \frac{1}{2}V(y)^2 + \frac{\alpha^2}{2}(V'(y))^2$ be a steady state solution to the α -Euler equations (2.1.4) on the domain $\mathbb{R} \times [A_1, A_2]$. Suppose the linearized stream function equation (2.2.4) about this steady state has a solution of the form (2.2.5) with $c_i > 0$, and some $k \in \mathbb{R}$. Then $\phi : [A_1, A_2] \rightarrow \mathbb{C}$ satisfies the following ODE:*

$$\begin{aligned} -\alpha^2 \phi'''' + (1 + 2\alpha^2 k^2) \phi'' - k^2(1 + \alpha^2 k^2) \phi - \frac{U'' \phi}{V - c} &= 0 \\ \phi(0) = \phi(2\pi) &= 0 \end{aligned} \quad (2.2.6)$$

$$\phi''(0) = \phi''(2\pi) = 0.$$

Proof. Since $\tilde{\psi}$ and $\tilde{\phi}$ are related via $(1 - \alpha^2 \Delta) \tilde{\phi} = \tilde{\psi}$, we have the following relation between ψ and ϕ ,

$$\psi(y) = (1 + \alpha^2 k^2) \phi(y) - \alpha^2 \phi''(y). \quad (2.2.7)$$

Using (2.2.7), $\tilde{\phi}(x, y, t) = \phi(y)e^{ik(x-ct)}$, $\tilde{\psi}(x, y, t) = \psi(y)e^{ik(x-ct)}$, $\partial_y \phi^0(y) = V(y)$ and $\partial_y \psi^0(y) = U(y)$, and letting prime denote differentiation with respect to y , we

compute,

$$\begin{aligned}
\Delta\tilde{\psi} &= (-k^2\psi + \psi'')e^{ik(x-ct)}, \\
(\Delta\tilde{\psi})_t &= -ikc(-k^2\psi + \psi'')e^{ik(x-ct)}, \\
(\Delta\tilde{\psi})_x &= ik(-k^2\psi + \psi'')e^{ik(x-ct)}, \\
(\Delta\psi^0)_y &= U''.
\end{aligned} \tag{2.2.8}$$

Using (2.2.8) we see that (2.2.4) becomes,

$$[-ikc(-k^2\psi + \psi'') + ikV(-k^2\psi + \psi'') + ikU''\phi]e^{ik(x-ct)} = 0, \tag{2.2.9}$$

i.e,

$$(\psi'' - k^2\psi)(V - c) + U''\phi = 0. \tag{2.2.10}$$

The boundary conditions for $\tilde{\phi}$ are $\nabla\tilde{\phi}\cdot\mathbf{t} = 0$ on ∂D , where the boundary corresponds to $y = A_1$ and $y = A_2$. Since $\mathbf{t} = (1, 0)$ on ∂D , this means that $\partial_x\tilde{\phi} = 0$ at $y = 0$ and $y = 2\pi$. Since $\partial_x\tilde{\phi}(x, y, t) = ik\phi(y)e^{ik(x-ct)} = 0$, when $y = 0$ and $y = 2\pi$, we have that

$$\phi(0) = 0 \text{ and } \phi(2\pi) = 0.$$

The boundary condition corresponding to $\mathbf{n}\cdot\nabla(-\nabla^\perp\tilde{\phi})\cdot\mathbf{t} = 0$ is computed as follows: we note that $\mathbf{n}\cdot\nabla = (0, 1)\cdot(\partial_x, \partial_y) = \partial_y$. Thus $\mathbf{n}\cdot\nabla(-\nabla^\perp\tilde{\phi}) = \partial_y(\partial_y\tilde{\phi}, -\partial_x\tilde{\phi}) = (\tilde{\phi}_{yy}, \tilde{\phi}_{xy})$. Thus, $\mathbf{n}\cdot\nabla(-\nabla^\perp\tilde{\phi})\cdot\mathbf{t}$ becomes $(\tilde{\phi}_{yy}, \tilde{\phi}_{xy})\cdot(1, 0) = (\tilde{\phi}_{yy}, 0)$. Thus, we get that, $\mathbf{n}\cdot\nabla(-\nabla^\perp\tilde{\phi})\cdot\mathbf{t} = 0$ becomes $\tilde{\phi}_{yy}(x, y, t) = 0$ when $y = A_1$ and $y = A_2$. Since, $\tilde{\phi}_{yy}(x, y, t) = \phi''(y)e^{ik(x-ct)}$, this corresponds to

$$\phi''(A_1) = \phi''(A_2) = 0.$$

Using (2.2.7) we note that (2.2.10) becomes,

$$((1 + \alpha^2 k^2)\phi - \alpha^2 \phi'')'' - k^2(1 + \alpha^2 k^2)\phi + \alpha^2 k^2 \phi'' - \frac{U''\phi}{V - c} = 0. \quad (2.2.11)$$

Since k^2 appears in the equation, we can assume without loss of generality, that $k \geq 0$.

We note that if $c_i > 0$, since V is real valued, dividing by $V - c$ does not produce any singularity. We thus obtain equation (2.2.6). ■

The analogous equation for the Euler equation is called the Rayleigh stability equation. The Rayleigh stability equation, is also known as the inviscid Orr-Sommerfeld equation [MP94], (page 122, Equation 4.6).

Remark 2.4. We note that $\tilde{\phi}(x, y, t) = \phi(y)e^{ik(x-ct)} = \phi(y)e^{ikx}e^{-ikct} = \hat{\phi}(x, y)e^{-ikct}$, where $\hat{\phi}(x, y) = \phi(y)e^{ikx}$. The smoothness of $\tilde{\phi}$ depends on the smoothness of ϕ . If we consider ϕ to be in the space

$$Y := \{\phi \in H^4([0, 2\pi]; \mathbb{C}), \phi(0) = \phi(2\pi) = 0, \phi'(0) = \phi'(2\pi) = 0\},$$

to solve (2.2.6), then we obtain a solution to (2.2.4) and we measure instability for (2.2.4) in the space

$$\hat{Y} := \{\hat{\phi}(\cdot, \cdot) : \sup_{x \in \mathbb{R}} \|\hat{\phi}(x, \cdot)\|_{H^4([0, 2\pi]; \mathbb{C})} < \infty\}.$$

We shall consider (2.2.4) as an (linear) evolution equation for $\tilde{\phi} = \tilde{\phi}(\cdot, \cdot, t)$ in the space \hat{Y} , and obtain conditions for existence of a growing eigenmode solution, i.e for spectral instability to the linearized equation (2.2.4) for the perturbation stream function. All references to instability in this subsection is to be regarded in the sense described above. We also note the following: heuristically, if we write (2.2.4) as an linear evolution equation of the form $\tilde{\phi}_t = L\tilde{\phi}$ in the space \hat{Y} , for a linear operator

L , and look for exponentially growing in time solutions of the form $e^{\lambda t} \widehat{\phi}$ one obtains the eigenvalue problem $L\widehat{\phi} = \lambda\widehat{\phi}$ which corresponds to (2.2.9) with $\lambda = -ikc$.

Proposition 2.5. (Rayleigh α -Euler) *Let $\mathbf{u}^0(x, y) = (U(y), 0)$, $\mathbf{v}^0(x, y) = (V(y), 0)$ and $\pi(x, y) = p_0 - \frac{1}{2}V(y)^2 + \frac{\alpha^2}{2}(V'(y))^2$ be a steady state solution to the α -Euler equations (2.1.4) on the domain $\mathbb{R} \times [A_1, A_2]$. If the linearized stream function equation (2.2.4) about this steady state has solutions of the form (2.2.5) with $c_i > 0$, then $U''(y) = 0$ for at least one point $y \in [A_1, A_2]$.*

Remark 2.6. A solution of the form $\widetilde{\phi}(x, y, t) = \phi(y)e^{ik(x-ct)}$, with $\phi \in Y$, with $c_i > 0$ to equation (2.2.4) grows exponentially in time. This corresponds to spectral instability for (2.2.4) in space \widehat{Y} .

Proof. By assumption $c_i > 0$, we have $V - c \neq 0$ because V is real valued and thus (2.2.6) is non-singular. We multiply the first equation of (2.2.6) by ϕ^* , the complex conjugate of ϕ , integrate by parts (using the boundary conditions, i.e the second and third equations of (2.2.6)) to obtain,

$$\begin{aligned} -\alpha^2 \int_{A_1}^{A_2} |\phi''(y)|^2 dy - \int_{A_1}^{A_2} (1 + 2\alpha^2 k^2) |\phi'(y)|^2 dy - \int_{A_1}^{A_2} k^2 (1 + \alpha^2 k^2) |\phi(y)|^2 dy \\ - \int_{A_1}^{A_2} \frac{U''(y)}{V(y) - c} |\phi(y)|^2 dy = 0. \end{aligned} \quad (2.2.12)$$

where we have performed integration by parts twice for the first term and once for the second term in (2.2.12).

When doing integration by parts, the boundary terms vanish via the boundary conditions. We can check this as follows:

$$\int_{A_1}^{A_2} \phi''(y) \phi^*(y) dy = \phi^*(y) \phi'(y) \Big|_{A_1}^{A_2} - \int_{A_1}^{A_2} \phi'(y) (\phi^*)'(y) dy = - \int_{A_1}^{A_2} \phi'(y) (\phi^*)'(y) dy$$

since $\phi^*(A_1) = \phi^*(A_2) = 0$ and $\phi(A_1) = \phi(A_2) = 0$. We also have that, using $\phi^*(A_1) = \phi^*(A_2) = 0$ and $\phi''(A_1) = \phi''(A_2) = 0$,

$$\begin{aligned} \int_{A_1}^{A_2} \phi''''(y)\phi^*(y)dy &= \phi^*(y)\phi''''(y)|_0^{2\pi} - \int_{A_1}^{A_2} \phi''''(y)(\phi^*)'(y)dy \\ &= - \int_{A_1}^{A_2} \phi''''(y)(\phi^*)'(y)dy = -(\phi^*)'(y)\phi''(y)|_0^{2\pi} + \int_{A_1}^{A_2} \phi''(y)(\phi^*)''(y)dy \\ &= \int_{A_1}^{A_2} \phi''(y)(\phi^*)''(y)dy. \end{aligned}$$

Taking the imaginary part of (2.2.12) gives

$$c_i \int_{A_1}^{A_2} \frac{U''(y)}{|V(y) - c|^2} |\phi(y)|^2 dy = 0 \quad (2.2.13)$$

Since $c_i > 0$, this forces $U''(y) = 0$ for at least one point $y \in [A_1, A_2]$. ■

We thus have that if U does not have an inflection point in $[A_1, A_2]$ then (2.2.4) cannot have an exponentially growing eigensolution of the form $\phi(y)e^{ik(x-ct)}$.

Example 2.7. Let $U(y)$ be a profile with no inflection point. Compute $V(y)$ by solving the ODE: $V(y) - \alpha^2 V''(y) = U(y)$ subject to the boundary conditions $V'(A_1) = V'(A_2) = 0$. The steady state thus computed, with the pressure given by equation (2.2.1), is stable by Proposition 2.5. This class of steady states include profiles $U(y)$ that are symmetric about the center point with no inflection points, for example, $U(y) = 1 - y^2$ on the interval $[-1, 1]$ and it also includes linear steady states like the Couette flow $U(y) = y$ on $[0, 1]$.

We now derive Fjortoft's theorem for the α -Euler equations.

Proposition 2.8. (Fjortoft α -Euler) Let $\mathbf{u}^0(x, y) = (U(y), 0)$, $\mathbf{v}^0(x, y) = (V(y), 0)$ and $\pi(x, y) = p_0 - \frac{1}{2}V(y)^2 + \frac{\alpha^2}{2}(V'(y))^2$ be a steady state solution to the α -Euler

equations (2.1.4) on the domain $\mathbb{R} \times [A_1, A_2]$. If the linearized stream function equation (2.2.4) about this steady state has solutions of the form (2.2.5) with $c_i > 0$, then $U''(y)(V(y) - V(y_s)) < 0$ for at least one point $y \in [A_1, A_2]$. Here $y_s \in [A_1, A_2]$ is a point such that $U''(y_s) = 0$.

Proof. By Proposition 2.5 there exists a $y_s \in [A_1, A_2]$ such that $U''(y_s) = 0$. Let $V_s = V(y_s)$. Consider the real part of (2.2.12). We have,

$$\begin{aligned} \int_{A_1}^{A_2} \frac{U''(y)(V(y) - c_r)}{|V(y) - c|^2} |\phi(y)|^2 dy &= -\alpha^2 \int_{A_1}^{A_2} |\phi''(y)|^2 dy \\ &\quad - \int_{A_1}^{A_2} (1 + 2\alpha^2 k^2) |\phi'(y)|^2 dy - \int_{A_1}^{A_2} k^2 (1 + \alpha^2 k^2) |\phi(y)|^2 dy \end{aligned} \quad (2.2.14)$$

Adding

$$(c_r - V_s) \int_{A_1}^{A_2} \frac{U''(y)}{|V(y) - c|^2} |\phi(y)|^2 dy = 0, \quad (2.2.15)$$

to (2.2.14) we then obtain,

$$\begin{aligned} \int_{A_1}^{A_2} \frac{U''(y)(V(y) - V_s)}{|V(y) - c|^2} |\phi(y)|^2 dy &= -\alpha^2 \int_{A_1}^{A_2} |\phi''(y)|^2 dy \\ &\quad - \int_{A_1}^{A_2} (1 + 2\alpha^2 k^2) |\phi'(y)|^2 dy - \int_{A_1}^{A_2} k^2 (1 + \alpha^2 k^2) |\phi(y)|^2 dy < 0, \end{aligned} \quad (2.2.16)$$

since the integrand on the right hand side is non negative. From this our desired conclusion follows, i.e $U''(y)(V(y) - V(y_s)) < 0$ for at least one point $y \in [A_1, A_2]$. ■

Remark 2.9. Note that in the proof of the above theorem since the integral in (2.2.15) is zero, the coefficient in front of the integral in (2.2.15) can be replaced by any number. The statement of Fjortoft criterion can be generalized to the following fact: for every real number z , a necessary condition for instability is the existence of at least one point $y \in [A_1, A_2]$ such that

$$(V(y) - z)U''(y) < 0. \quad (2.2.17)$$

In fact, it is readily seen that the Fjortoft criterion implies the Rayleigh criterion by choosing points z_1 and z_2 such that $V(y) - z_1 > 0$ and $V(y) - z_2 < 0$ for all $y \in [A_1, A_2]$. Then Fjortoft criterion says that there exists at least one point y_1 and y_2 , respectively, such that $U''(y_1) < 0$ and $U''(y_2) > 0$ thereby ensuring that $U''(y)$ changes sign somewhere in the flow.

Remark 2.10. We recall that Fjortoft criterion for the Euler equation says that a necessary condition for instability is that $U''(y)(U(y) - U(y_s)) < 0$ for at least one point $y \in [A_1, A_2]$. For the Euler equations, Fjortoft criterion is sharper than the Rayleigh criterion. The Fjortoft criterion was used to study steady states $U(y)$ with a monotone profile and one inflection point y_s . If U'' and $U - U(y_s)$ have the same sign everywhere in the flow, then $U(y)$ is a stable steady state even though it has an inflection point y_s . The analogous result for α -Euler is as follows. Suppose we have a profile $U(y)$ which is monotone and has one inflection point y_s . We compute $V(y)$ using equation (2.2.2). If $U''(y)$ and $V(y) - V(y_s)$ have the same sign everywhere, then the steady state is stable for α -Euler.

Example 2.11. Let us consider $V(y) = y - y^3$ on the interval $[-1/\sqrt{3}, 1/\sqrt{3}]$. Note that V is monotone with one inflection point at $y = 0$ and $V'(-1/\sqrt{3}) = V'(1/\sqrt{3}) = 0$. One can compute $U(y) = V(y) - \alpha^2 V''(y) = y - y^3 + 6\alpha^2 y$. One can see that $U''(y) = -6y$. $U(y)$ thus has one inflection point at $y = 0$. Note that $V(0) = 0$. Thus $U''(y)(V(y) - V(y_s)) = -6y(y - y^3) = -6y^2(1 - y^2) < 0$ since $(1 - y^2)$ is everywhere positive in the interval $[-1/\sqrt{3}, 1/\sqrt{3}]$. Proposition 2.8 says that this steady state, with pressure as in (2.2.1) is possibly unstable for the α -Euler equations.

2.3 Arnold stability theorems for the Euler equations

In this section we provide a brief overview of the Arnold criterion for the Euler equation, see, for example, [A65], [MP94] (Section 3.2, page 104 onwards), [AK98] (Chapter 2, Section 4, page 88 onwards), [HMRW85] for a more mathematical perspective, [T10], (Section 1), [B15] (Section 3, page 7), [S00] (Section 4.5, page 114 onwards) for a more applied perspective. We shall work on the following domains of \mathbb{R}^2 : bounded multi connected domain with n holes, simply connected domain, the two torus \mathbb{T}^2 and the periodic channel $\mathbb{T} \times [-1, 1]$.

We have synthesized the presentation in all these references and we present streamlined formulations and proofs of Arnold's results that work on all of these domains modulo changes in the corresponding function spaces of the perturbations. Our eventual goal is to prove versions of these results for the α -Euler equations.

2.3.1 Arnold's theorems in a multi connected domain

Let $D \subset \mathbb{R}^2$ be a bounded multiply connected domain, bounded by smooth curves $(\partial D)_i$ where $i = 0, \dots, n$. Let ∂D denote the boundary of D where the outer boundary curve is denoted by $(\partial D)_0$ and $(\partial D)_i, i = 1, \dots, n$ denote the n inner boundaries. We shall consider the Euler equations (2.1.2) on D ,

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = 0, \quad \operatorname{div} \mathbf{u} = 0,$$

supplemented by the boundary condition (2.1.3),

$$\mathbf{u} \cdot \mathbf{n} = 0,$$

on the boundary ∂D , where \mathbf{n} is the unit normal vector on the boundary. We also recall that the vorticity $\omega = \text{curl } \mathbf{u}$, and in the vorticity form, the Euler equations are,

$$\omega_t + \mathbf{u} \cdot \nabla \omega = 0.$$

Since $\text{div } \mathbf{u} = 0$, there exists the so called stream function ψ , such that $\mathbf{u} = -\nabla^\perp \psi$. From this, we see that $\omega = \text{curl } \mathbf{u} = \text{curl}(-\nabla^\perp \psi) = -\Delta \psi$. Since $\mathbf{u} \cdot \mathbf{n} = 0$ on the boundary ∂D , we have, on ∂D , that $\nabla^\perp \psi \cdot \mathbf{n} = 0$, i.e., $\nabla \psi \cdot \mathbf{t} = 0$, where \mathbf{t} is the unit tangent vector on the boundary. Thus $\nabla \psi$ is orthogonal to the boundary, i.e., each connected piece of the boundary curve is a level sets of ψ and thus $\psi|_{(\partial D)_i}$ is a constant on each connected boundary piece $(\partial D)_i$ where $0 \leq i \leq n$.

Let H_c be a functional defined on the space

$$X := \{\mathbf{u} \in H^1(D; \mathbb{R}^2), \text{div } \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D\}$$

as follows,

$$H_c(\mathbf{u}) = \frac{1}{2} \int_D \mathbf{u} \cdot \mathbf{u} dx + \int_D C(\text{curl } \mathbf{u}) dx + \sum_{i=1}^n a_i \int_{(\partial D)_i} \mathbf{u} \cdot d\mathbf{s}. \quad (2.3.1)$$

Here, $C : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function and $a_i \in \mathbb{R}$. In (2.3.1), the first term represents the kinetic energy, the second term is the so called Casimir and the third term is the weighted sum of circulations. We first note that if $\mathbf{u}(\cdot, t)$ solves (2.1.2) then $\frac{d}{dt} H_c(\mathbf{u}(\cdot, t)) = 0$, i.e., $H_c(\mathbf{u}(\cdot, t))$ is conserved along solutions to the Euler equation (2.1.2).

Lemma 2.12. *If $\mathbf{u}(\cdot, t)$ solves (2.1.2) then $\frac{d}{dt} H_c(\mathbf{u}(\cdot, t)) = 0$.*

Proof. We first observe the following. We note that, (see [A90], Appendix, page 348, identity A.9),

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2}\nabla(|\mathbf{u}|^2). \quad (2.3.2)$$

Taking the dot product with \mathbf{u} and noting that $\mathbf{u} \cdot \{(\nabla \times \mathbf{u}) \times \mathbf{u}\} = 0$, we see that,

$$\mathbf{u} \cdot (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{u} \cdot \frac{1}{2}\nabla(|\mathbf{u}|^2). \quad (2.3.3)$$

For any scalar function f , we have, (see, [CM93], Identity 8, page 160)

$$\nabla \cdot (f\mathbf{u}) = f(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)f. \quad (2.3.4)$$

Using the fact that $\nabla \cdot \mathbf{u} = 0$, (2.3.4) reduces to

$$\nabla \cdot (f\mathbf{u}) = (\mathbf{u} \cdot \nabla)f. \quad (2.3.5)$$

By Gauss divergence theorem, (and using the facts that $\mathbf{u} \cdot \mathbf{n} = 0$ on the boundary ∂D , $\nabla \cdot \mathbf{u} = 0$ on D and (2.3.5)), we have

$$\int_D (\mathbf{u} \cdot \nabla)f dx = \int_D \nabla \cdot (f\mathbf{u}) dx = \int_{\partial D} f\mathbf{u} \cdot \mathbf{n} ds = 0. \quad (2.3.6)$$

We thus have that

$$\frac{d}{dt} \frac{1}{2} \int_D \mathbf{u} \cdot \mathbf{u} dx = \int_D \mathbf{u} \cdot \partial_t \mathbf{u} dx = \int_D \mathbf{u} \cdot (-\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p dx.$$

Using (2.3.3), (2.3.5), we see that,

$$\int_D \mathbf{u} \cdot (-\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p dx = \int_D \left\{ -\mathbf{u} \cdot \frac{1}{2}\nabla(|\mathbf{u}|^2) - \mathbf{u} \cdot \nabla p \right\} dx.$$

We can use (2.3.6) to conclude that,

$$\int_D \left\{ -\mathbf{u} \cdot \frac{1}{2}\nabla(|\mathbf{u}|^2) - \mathbf{u} \cdot \nabla p \right\} dx = 0.$$

We have thus proved that $\frac{d}{dt} \frac{1}{2} \int_D \mathbf{u} \cdot \mathbf{u} dx = 0$.

In order to prove that $\frac{d}{dt} \int_D C(\omega(\mathbf{x}, t)) dx = 0$, we use (a slight variant of) the following idea from [CSS06] (see Appendix, page 162). By (2.1.10) we have,

$$\partial_t C(\omega(\mathbf{x}, t)) = C'(\omega(\mathbf{x}, t)) \partial_t \omega(\mathbf{x}, t) = -C'(\omega(\mathbf{x}, t)) \mathbf{u} \cdot \nabla \omega(\mathbf{x}, t), \quad (2.3.7)$$

and also, $-\mathbf{u} \cdot \nabla(C(\omega(\mathbf{x}, t))) = -C'(\omega(\mathbf{x}, t)) \mathbf{u} \cdot \nabla \omega(\mathbf{x}, t)$. Therefore,

$$\partial_t C(\omega(\mathbf{x}, t)) = -\mathbf{u} \cdot \nabla(C(\omega(\mathbf{x}, t))). \quad (2.3.8)$$

We thus have that, using (2.3.6) and (2.3.8),

$$\begin{aligned} \frac{d}{dt} \int_D C(\omega(\mathbf{x}, t)) dx &= \int_D \partial_t C(\omega(\mathbf{x}, t)) dx \\ &= \int_D -\mathbf{u} \cdot \nabla C(\omega(\mathbf{x}, t)) dx = 0. \end{aligned}$$

The fact that the time derivative of the circulations is equal to zero,

$$\frac{d}{dt} \int_{(\partial D)_i} \mathbf{u} \cdot \mathbf{ds} = 0,$$

$0 \leq i \leq n$, is a consequence of the Kelvin Circulation theorem for the Euler equations.

For a proof, see [CM93, pp.21-22]. ■

Let \mathbf{u}^0 denote a steady state solution of (2.1.2), ψ^0 denote its stream function, so that $\mathbf{u}^0 = -\nabla^\perp \psi^0$ and ω^0 denote its vorticity, so that $\omega^0 = \text{curl } \mathbf{u}^0 = -\Delta \psi^0$. We also note that since $\nabla^\perp \psi^0 \cdot \nabla \omega^0 = 0$, we have that $\nabla \psi^0$ is parallel to $\nabla \omega^0$ and thus ψ^0 and ω^0 have the same level curves. It follows that, locally, ψ^0 is a function of ω^0 . We shall impose the following global condition.

Assumption 2.13. Assume that there exists a differentiable real function F defined on the closed interval $\left[\min_{(x,y) \in \bar{D}} \omega^0(x, y), \max_{(x,y) \in \bar{D}} \omega^0(x, y) \right]$, such that, $\psi^0(x, y) = F(\omega^0(x, y))$ for every $(x, y) \in D$.

In particular, we have that $\nabla\psi^0(x, y) = F'(\omega^0(x, y))\nabla\omega^0(x, y)$. Notice that, the function F' , a priori, can have singularities at the critical points of ω^0 . The discussion following (2.1.10) showed that ψ^0 is a constant on each connected piece of the boundary. Assumption 2.13 then implies that $F(\omega^0)$ restricted to each connected piece $(\partial D)_i$, $0 \leq i \leq n$, is a constant. We will now specify C and a_j , such that the first variation $\delta H_c(\mathbf{u}^0)\delta\mathbf{u} := \frac{d}{d\epsilon}H_c(\mathbf{u}^0 + \epsilon\delta\mathbf{u})|_{\epsilon=0}$ of the functional H_c from (2.3.1) is zero.

Lemma 2.14. *Let \mathbf{u}^0, ω^0 be a steady state solution of (2.1.2) satisfying Assumption 2.13, where $\omega^0 = \text{curl } \mathbf{u}^0$. Let C be the antiderivative of $-F$ so that*

$$C'(\omega^0(x, y)) = -F(\omega^0(x, y)), \quad (2.3.9)$$

for every (x, y) in D . Let $a_i = F(\omega^0)|_{(\partial D)_i}$, $i = 0, \dots, n$. Then $\delta H_c(\mathbf{u}^0)\delta\mathbf{u} = 0$, i.e., \mathbf{u}^0 is a critical point of H_c from (2.3.1) with the choice of C and a_i just made.

Proof. The first variation of H_c at \mathbf{u}^0 is given by the following expression,

$$\begin{aligned} \delta H_c(\mathbf{u}^0)\delta\mathbf{u} &= \frac{d}{d\epsilon}H_c(\mathbf{u}^0 + \epsilon\delta\mathbf{u})|_{\epsilon=0} \\ &= \int_D \mathbf{u}^0 \cdot \delta\mathbf{u} d\mathbf{x} + \int_D C'(\omega^0)\delta\omega d\mathbf{x} + \sum_{i=0}^n a_i \int_{(\partial D)_i} \delta\mathbf{u} \cdot d\mathbf{s}, \end{aligned} \quad (2.3.10)$$

where $\delta\omega = \text{curl}(\delta\mathbf{u})$. We will be using the following identity (see [MP94], Eq 2.14, page 108),

$$C'(\omega^0)\delta\omega = \text{curl}(C'(\omega^0)\delta\mathbf{u}) - C''(\omega^0)\nabla^\perp\omega^0 \cdot \delta\mathbf{u}, \quad (2.3.11)$$

which follows from the identity $\text{curl}(f\mathbf{v}) = \nabla^\perp f \cdot \mathbf{v} + f \text{curl } \mathbf{v}$. Noting that, by Stokes' theorem,

$$\int_D \text{curl}(C'(\omega^0)\delta\mathbf{u}) d\mathbf{x} = \sum_{i=0}^n \int_{(\partial D)_i} C'(\omega^0)\delta\mathbf{u} \cdot d\mathbf{s}, \quad (2.3.12)$$

we see that, using (2.3.11) and (2.3.12),

$$\begin{aligned} \int_D C'(\omega^0) \delta \omega d\mathbf{x} &= - \int_D C''(\omega^0) \nabla^\perp \omega^0 \cdot \delta \mathbf{u} d\mathbf{x} \\ &+ \sum_{i=0}^n \int_{(\partial D)_i} C'(\omega^0) \delta \mathbf{u} \cdot d\mathbf{s}. \end{aligned} \quad (2.3.13)$$

We thus have that,

$$\begin{aligned} \delta H_c(\mathbf{u}^0) \delta \mathbf{u} &= \int_D \mathbf{u}^0 \cdot \delta \mathbf{u} d\mathbf{x} - \int_D C''(\omega^0) \nabla^\perp \omega^0 \cdot \delta \mathbf{u} d\mathbf{x} \\ &+ \sum_{i=0}^n \int_{(\partial D)_i} C'(\omega^0) \delta \mathbf{u} \cdot d\mathbf{s} + \sum_{i=0}^n a_i \int_{(\partial D)_i} \delta \mathbf{u} \cdot d\mathbf{s}. \end{aligned} \quad (2.3.14)$$

Since $\mathbf{u}^0 \cdot \nabla \omega^0 = 0$, and \mathbf{u}^0 is tangent to the boundary, this means that $\nabla \omega^0$ is orthogonal to the boundary and thus ω^0 is a constant on the boundary. This then implies that,

$$\begin{aligned} \delta H_c(\mathbf{u}^0) \delta \mathbf{u} &= \int_D \mathbf{u}^0 \cdot \delta \mathbf{u} d\mathbf{x} - \int_D C''(\omega^0) \nabla^\perp \omega^0 \cdot \delta \mathbf{u} d\mathbf{x} \\ &+ \sum_{i=0}^n C'(\omega^0)|_{(\partial D)_i} \int_{(\partial D)_i} \delta \mathbf{u} \cdot d\mathbf{s} + \sum_{i=0}^n a_i \int_{(\partial D)_i} \delta \mathbf{u} \cdot d\mathbf{s}. \end{aligned} \quad (2.3.15)$$

Since, by (2.3.9), C is chosen such that $C'(\omega^0(x, y)) = -F(\omega^0(x, y))$ for every $(x, y) \in D$, then, $\mathbf{u}^0 = -\nabla^\perp \psi^0 = -F'(\omega^0) \nabla^\perp \omega^0 = C''(\omega^0) \nabla^\perp \omega^0$. Since we have chosen $a_i = \psi^0|_{(\partial D)_i}$ for all $0 \leq i \leq n$, (note that ψ^0 is a constant on the boundary curves) which then implies that $a_i = -C'(\omega^0)|_{(\partial D)_i}$, then the first variation $\delta H_c(\mathbf{u}^0) \delta \mathbf{u}$ is indeed zero. ■

One is now ready to prove Arnold's first stability theorem.

Theorem 2.15. *Let \mathbf{u}^0 be a steady state solution of the Euler equations (2.1.2) on the multi connected domain D satisfying Assumption 2.13. Suppose that*

$$0 < \inf_{(x,y) \in D} (-F'(\omega^0(x, y))) \leq \sup_{(x,y) \in D} (-F'(\omega^0(x, y))) < +\infty. \quad (2.3.16)$$

Then there exists a constant $K > 0$, such that if $\mathbf{u}(\cdot, t) = \mathbf{u}^0 + \delta\mathbf{u}(\cdot, t)$, $t \in I$, solves the Euler equations (2.1.2) on D , then one has the following estimate for all times $t \in I$,

$$\|\mathbf{u}(\cdot, t) - \mathbf{u}^0\|_2^2 + \|\omega(\cdot, t) - \omega^0\|_2^2 \leq K(\|\mathbf{u}(\cdot, 0) - \mathbf{u}^0\|_2^2 + \|\omega(\cdot, 0) - \omega^0\|_2^2), \quad (2.3.17)$$

where $\omega^0 = \text{curl } \mathbf{u}^0$ and $\omega(\cdot, t) = \text{curl } \mathbf{u}(\cdot, t)$.

Remark 2.16. Note that $\delta\mathbf{u}(\cdot, t)$ does not solve the Euler equations (2.1.2) but $\mathbf{u}(\cdot, t) = \mathbf{u}^0 + \delta\mathbf{u}(\cdot, t)$ solves (2.1.2).

Proof. Put

$$c_1 = \inf_{(x,y) \in D} (-F'(\omega^0(x, y))) \quad \text{and} \quad c_2 = \sup_{(x,y) \in D} (-F'(\omega^0(x, y))).$$

Let H_c be defined as in (2.3.1) and choose C and a_i as in Lemma 2.14. Since the range of ω^0 is a connected set, (2.3.16) is equivalent to,

$$0 < c_1 \leq -F'(\xi) \leq c_2 < \infty, \quad (2.3.18)$$

for all $\xi \in [\min_{(x,y) \in \bar{D}} \omega^0(x, y), \max_{(x,y) \in \bar{D}} \omega^0(x, y)]$. Using (2.3.16), we may extend C from the range of ω^0 to \mathbb{R} such that,

$$c_1 \leq C''(\xi) \leq c_2, \quad (2.3.19)$$

for all $\xi \in \mathbb{R}$. Indeed, we first extend F linearly outside

$$[\min_{(x,y) \in \bar{D}} \omega^0(x, y), \max_{(x,y) \in \bar{D}} \omega^0(x, y)]$$

to all of \mathbb{R} . We then choose C so that $C'(\xi) = -F(\xi)$ for all $\xi \in \mathbb{R}$.

By Lemma 2.14, one has that $\delta H_c(\mathbf{u}^0)\delta\mathbf{u} = 0$. We now consider the following functional,

$$\mathcal{H}(\delta\mathbf{u}) = H_c(\mathbf{u}^0 + \delta\mathbf{u}) - H_c(\mathbf{u}^0) - \delta H_c(\mathbf{u}^0)\delta\mathbf{u}. \quad (2.3.20)$$

If $\mathbf{u}(\cdot, t) = \mathbf{u}^0 + \delta\mathbf{u}(\cdot, t)$ solves (2.1.2), then the function $t \mapsto \mathcal{H}(\delta\mathbf{u}(\cdot, t))$ is constant in time because, by Lemma 2.12, H_c is conserved along solutions and the last term in (2.3.20) is zero by Lemma 2.14. Using (2.3.10), we have that

$$\begin{aligned} \mathcal{H}(\delta\mathbf{u}) &= H_c(\mathbf{u}^0 + \delta\mathbf{u}) - H_c(\mathbf{u}^0) - \delta H_c(\mathbf{u}^0)\delta\mathbf{u} \\ &= \frac{1}{2} \int_D (\mathbf{u}^0 + \delta\mathbf{u}) \cdot (\mathbf{u}^0 + \delta\mathbf{u}) d\mathbf{x} - \frac{1}{2} \int_D \mathbf{u}^0 \cdot \mathbf{u}^0 d\mathbf{x} + \int_D C(\omega^0 + \delta\omega) d\mathbf{x} \\ &\quad - \int_D C(\omega^0) d\mathbf{x} + \sum_{i=0}^n a_i \int_{(\partial D)_i} (\mathbf{u}^0 + \delta\mathbf{u}) \cdot d\mathbf{s} - \sum_{i=0}^n a_i \int_{(\partial D)_i} \mathbf{u}^0 \cdot d\mathbf{s} \\ &\quad - \int_D \mathbf{u}^0 \cdot \delta\mathbf{u} d\mathbf{x} - \int_D C'(\omega^0) \delta\omega d\mathbf{x} - \sum_{i=0}^n a_i \int_{(\partial D)_i} \delta\mathbf{u} \cdot d\mathbf{s} \\ &= \frac{1}{2} \int_D \delta\mathbf{u} \cdot \delta\mathbf{u} d\mathbf{x} + \int_D (C(\omega^0 + \delta\omega) - C(\omega^0) - C'(\omega^0)\delta\omega) d\mathbf{x}. \end{aligned} \quad (2.3.21)$$

We now use the following elementary fact. If $\rho : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 function and if $a \leq \rho''(\xi) \leq b$ for all $\xi \in \mathbb{R}$, then, for any $\xi, \xi^0 \in \mathbb{R}$, we have the inequality,

$$\frac{a}{2}(\xi)^2 \leq \rho(\xi_0 + \xi) - \rho(\xi^0) - \rho'(\xi^0)\xi \leq \frac{b}{2}(\xi)^2. \quad (2.3.22)$$

This fact can be easily proved by considering the Taylor's formula, for any $\xi, \xi^0 \in \mathbb{R}$, there exists an η between ξ and ξ^0 such that $\rho(\xi + \xi_0) = \rho(\xi^0) + \rho'(\xi^0)\xi + \frac{1}{2}\rho''(\eta)\xi^2$.

Using (2.3.22) in the second integrand of (2.3.21), we have, by (2.3.19),

$$\frac{1}{2} \|\delta\mathbf{u}\|_2^2 + \frac{c_1}{2} \|(\delta\omega)^2\|_2^2 \leq \mathcal{H}(\delta\mathbf{u}) \leq \frac{1}{2} \|\delta\mathbf{u}\|_2^2 + \frac{c_2}{2} \|(\delta\omega)^2\|_2^2.$$

We now use the fact that $t \mapsto \mathcal{H}(\delta\mathbf{u}(\cdot, t))$ is an invariant of motion to conclude that

$$\begin{aligned} \frac{1}{2}\|\delta\mathbf{u}(\cdot, t)\|_2^2 + \frac{c_1}{2}\|\delta\omega(\cdot, t)\|_2^2 &\leq \mathcal{H}(\delta\mathbf{u}(t)) = \mathcal{H}(\delta\mathbf{u}(0)) \\ &\leq \frac{1}{2}\|\delta\mathbf{u}(\cdot, 0)\|_2^2 + \frac{c_2}{2}\|\delta\omega(\cdot, 0)\|_2^2. \end{aligned}$$

Let $K_1 = \min(1/2, c_1/2)$ and $K_2 = \max(1/2, c_2/2)$. We thus see that,

$$K_1 \left(\|\delta\mathbf{u}(\cdot, t)\|_2^2 + \|\delta\omega(\cdot, t)\|_2^2 \right) \leq K_2 \left(\|\delta\mathbf{u}(\cdot, 0)\|_2^2 + \|\delta\omega(\cdot, 0)\|_2^2 \right), \quad (2.3.23)$$

from which (2.3.17) follows by putting $K = K_2/K_1$ (we note that $K_1, K_2 > 0$ and thus $K > 0$). ■

Remark 2.17. Arnold's first theorem also works on the unbounded channel, i.e., the domain $D = \mathbb{R} \times [-1, 1]$. The proof is similar and is omitted.

Remark 2.18. Condition (2.3.16) in Arnold's first theorem is never satisfied in a domain without a boundary, see, [MP94], Section 3.2, page 112. To demonstrate this, let us assume (2.3.16). Then, since F is monotone, there exists its inverse function denoted by G , i.e., $G = F^{-1}$, and since (2.3.16) holds, one has the relationship

$$\frac{1}{c_2} \leq -G'(\xi) \leq \frac{1}{c_1}, \quad (2.3.24)$$

for all ξ in the range of $F(\omega^0(\cdot, \cdot))$, i.e., for all ξ in the range of $\psi^0(\cdot, \cdot)$. In particular, G' is negative everywhere. Assume without loss of generality that $\partial_x \psi^0 \neq 0$ (if it is, then in the argument below replace $\partial_x \psi^0$ by $\partial_y \psi^0$, we exclude the trivial case $\psi^0 =$ constant everywhere in D). We have that $-\Delta \psi^0 = \omega^0 = G(\psi^0)$. From this we see that, $\partial_x(-\Delta \psi^0) = G'(\psi^0)\partial_x \psi^0$. Multiplying this by $\partial_x \psi^0$ and integrating this over the domain D , we get,

$$-\int_D \partial_x \psi^0 \partial_x \Delta \psi^0 d\mathbf{x} = \int_D G'(\psi^0) (\partial_x \psi^0)^2 d\mathbf{x}.$$

Integrating the first term of the left hand side by parts, we get,

$$\int_D (\nabla \partial_x \psi^0)^2 d\mathbf{x} - \int_{\partial D} (\partial_x \psi^0) \mathbf{n} \cdot \nabla (\partial_x \psi^0) ds = \int_D G'(\psi^0) (\partial_x \psi^0)^2 d\mathbf{x}.$$

Note that the first term on the left hand side is positive and the term in the right hand side is negative by (2.3.24) which leads to a contradiction in the absence of the boundary term in the left hand side.

Computing the second variation of H_c yields,

$$\begin{aligned} \delta^2 H_c(\mathbf{u}^0)(\delta \mathbf{u}, \delta \mathbf{u}) &:= \frac{d^2}{d\varepsilon^2} H_c(\mathbf{u}^0 + \varepsilon \delta \mathbf{u}) \Big|_{\varepsilon=0} \\ &= \int_D \delta \mathbf{u} \cdot \delta \mathbf{u} d\mathbf{x} + \int_D C''(\omega^0) \delta \omega \delta \omega d\mathbf{x}, \end{aligned} \tag{2.3.25}$$

where $\delta \omega = \text{curl } \delta \mathbf{u}$. Indeed,

$$\begin{aligned} \frac{d}{d\varepsilon} H_c(\mathbf{u}^0 + \varepsilon \delta \mathbf{u}) &= \int_D (\mathbf{u}^0 + \varepsilon \delta \mathbf{u}) \cdot \mathbf{u}^0 dx + \int_D C'(\text{curl}(\mathbf{u}^0 + \varepsilon \delta \mathbf{u})) \text{curl } \delta \mathbf{u} dx \\ &\quad + \sum_{i=0}^n a_i \int_{(\partial D)_i} \delta \mathbf{u} \cdot d\mathbf{s}, \end{aligned}$$

and

$$\frac{d^2}{d\varepsilon^2} H_c(\mathbf{u}^0 + \varepsilon \delta \mathbf{u}) = \int_D \delta \mathbf{u} \cdot \delta \mathbf{u} d\mathbf{x} + \int_D C''(\text{curl}(\mathbf{u}^0 + \varepsilon \delta \mathbf{u})) \text{curl } \delta \mathbf{u} \text{curl } \delta \mathbf{u} d\mathbf{x},$$

from which (2.3.25) follows, where $\delta \omega = \text{curl } \delta \mathbf{u}$ and $\omega^0 = \text{curl } \mathbf{u}^0$.

Remark 2.19. We note that the quadratic form defined in (2.3.25) is bounded and positive definite on the space $\{\mathbf{u} \in H^1(D; \mathbb{R}^2), \text{div } \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D\}$ under assumption (2.3.16). We will now give a slightly different proof of Arnold's first stability theorem 2.15 based on an estimate for the second variation of H_c (see also the stability algorithm presented in [HMRW85, pp 7-11]). For any $t \geq 0$, we have, as $\varepsilon \rightarrow 0$,

$$\begin{aligned}
H_c(\mathbf{u}^0 + \varepsilon \delta \mathbf{u}) - H_c(\mathbf{u}^0) &= \left(\frac{d}{d\varepsilon} H_c(\mathbf{u}^0 + \varepsilon \delta \mathbf{u}) \Big|_{\varepsilon=0} \right) \varepsilon \\
&+ \frac{1}{2} \left(\frac{d^2}{d\varepsilon^2} H_c(\mathbf{u}^0 + \varepsilon \delta \mathbf{u}) \Big|_{\varepsilon=0} \right) \varepsilon^2 + o(\varepsilon^2) \\
&= \left(\delta H_c(\mathbf{u}^0) \delta \mathbf{u} \right) \varepsilon + \frac{1}{2} \left(\delta^2 H_c(\mathbf{u}^0) (\delta \mathbf{u}, \delta \mathbf{u}) \right) \varepsilon^2 + o(\varepsilon^2).
\end{aligned}$$

If \mathbf{u}^0 is such that $\delta H_c(\mathbf{u}^0) \delta \mathbf{u} = 0$ for every $\delta \mathbf{u}$, then,

$$\begin{aligned}
o(\varepsilon^2) + \frac{1}{2} \left(\delta^2 H_c(\mathbf{u}^0) (\delta \mathbf{u}(\cdot, t), \delta \mathbf{u}(\cdot, t)) \right) \varepsilon^2 &= H_c(\mathbf{u}^0 + \varepsilon \delta \mathbf{u}(\cdot, t)) - H_c(\mathbf{u}^0) \\
&= H_c(\mathbf{u}^0 + \varepsilon \delta \mathbf{u}(\cdot, 0)) - H_c(\mathbf{u}^0) = o(\varepsilon^2) + \frac{1}{2} \left(\delta^2 H_c(\mathbf{u}^0) (\delta \mathbf{u}(\cdot, 0), \delta \mathbf{u}(\cdot, 0)) \right) \varepsilon^2,
\end{aligned} \tag{2.3.26}$$

where we have used that H_c is an invariant of motion by Lemma 2.14. From (2.3.25) and using the fact that $c_1 \leq C'''(\xi) \leq c_2$, for every $\xi \in \mathbb{R}$, we have,

$$\begin{aligned}
\int_D \delta \mathbf{u} \cdot \delta \mathbf{u} d\mathbf{x} + c_1 \int_D \delta \omega \delta \omega d\mathbf{x} &\leq \delta^2 H_c(\mathbf{u}^0) (\delta \mathbf{u}, \delta \mathbf{u}) \\
&\leq \int_D \delta \mathbf{u} \cdot \delta \mathbf{u} d\mathbf{x} + c_2 \int_D \delta \omega \delta \omega d\mathbf{x}.
\end{aligned}$$

From this it follows that,

$$\|\delta \mathbf{u}\|_2^2 + c_1 \|\delta \omega\|_2^2 \leq \delta^2 H_c(\mathbf{u}^0) (\delta \mathbf{u}, \delta \mathbf{u}) \leq \|\delta \mathbf{u}\|_2^2 + c_2 \|\delta \omega\|_2^2.$$

Using this in (2.3.26), we see that,

$$\frac{o(\varepsilon^2)}{\varepsilon^2} + \left(\|\delta \mathbf{u}(\cdot, t)\|_2^2 + c_1 \|\delta \omega(\cdot, t)\|_2^2 \right) \leq \frac{o(\varepsilon^2)}{\varepsilon^2} + \left(\|\delta \mathbf{u}(\cdot, 0)\|_2^2 + c_2 \|\delta \omega(\cdot, 0)\|_2^2 \right).$$

Taking $\varepsilon \rightarrow 0$, one obtains inequality (2.3.23).

Remark 2.20. Note that the quadratic form computed in (2.3.25) cannot be negative definite on a subspace of $\{\mathbf{u} \in H^1(D; \mathbb{R}^2), \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D\}$ where the null space of the curl operator is non trivial. In other words, if there exists a $\delta \mathbf{u} \neq$

0 such that $\text{curl } \delta \mathbf{u} = 0$, then the value of the quadratic form given by (2.3.25) reduces to $\int_D \delta \mathbf{u} \cdot \delta \mathbf{u} d\mathbf{x}$ and this cannot be negative. In the proof of Arnold's second theorem, we will need the quadratic form defined by (2.3.25) to be negative definite on some subspace of perturbations. We would like to restrict the perturbations $\delta \mathbf{u}$ to a subspace such that the curl operator is one to one and thus the quadratic form can be negative definite under appropriate assumptions on C'' . To achieve this the perturbation stream function is restricted to the following subspace of $H^2(D; \mathbb{R})$ given by,

$$Y_0 = \left\{ \psi : H^2(D; \mathbb{R}) : \psi|_{(\partial D)_0} = 0; \int_{(\partial D)_i} -\nabla^\perp \psi \cdot d\mathbf{s} = 0, 1 \leq i \leq n; \right. \\ \left. \psi|_{(\partial D)_i} \text{ is constant}, 1 \leq i \leq n \right\}.$$

We note that we do not specify the exact values of the constant that ψ takes along the inner boundary curves. We also choose for the velocity perturbations, the subspace

$$X_0 := \left\{ \mathbf{u} \in H^1(D; \mathbb{R}^2), \text{div } \mathbf{u} = 0 \text{ in } D, \int_{(\partial D)_i} \mathbf{u} \cdot d\mathbf{s} = 0, 1 \leq i \leq n, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D \right\}$$

of the function space $X = \{ \mathbf{u} \in H^1(D; \mathbb{R}^2), \text{div } \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D \}$. We first have the following elementary Lemma.

Lemma 2.21. $-\nabla^\perp : Y_0 \rightarrow X_0$ is bijective. That is, given $\mathbf{u} \in X_0$, there exists a unique $\psi \in Y_0$ such that $\mathbf{u} = -\nabla^\perp \psi$.

Remark 2.22. Note that in the proof of this lemma, the conditions on the circulations in the definitions of both spaces X_0 and Y_0 is not used. We will need it in the proof of Lemma 2.23.

Proof. We first prove that $-\nabla^\perp$ is surjective. Let $\mathbf{u} = (u_1, u_2) \in X_0$ be given. Fix a point $(x_0, y_0) \in D$ and let (x, y) be any other point in D . Let C be any path

connecting (x_0, y_0) to (x, y) that lies entirely in D . Define $\psi(x, y) = \int_C -u_2 dx + u_1 dy$. We first claim that the line integral is independent of the path C . Indeed, let C' be another path connecting (x_0, y_0) and (x, y) that lies entirely in D so that $C \cup (-C')$ is a closed loop. Assume, without loss of generality that $C \cup (-C')$ is a simple closed loop. If not, break it up into a union of simple closed loops. Let S denote the region in D enclosed by $C \cup (-C')$. Then, by Gauss divergence theorem, and using $\operatorname{div} \mathbf{u} = 0$,

$$0 = \int_S \operatorname{div} \mathbf{u} dx dy = \int_{\partial S} \mathbf{u} \cdot \mathbf{n} ds = \int_{\partial S} -u_2 dx + u_1 dy.$$

In case $C \cup (-C')$ encloses no hole, $\partial S = C \cup (-C')$ and we thus have that $\int_{C \cup (-C')} -u_2 dx + u_1 dy = 0$. In case $C \cup (-C')$ encloses a hole $(\partial D)_i$ for some i between 1 and n , we know that

$$\int_{(\partial D)_i} -u_2 dx + u_1 dy = \int_{(\partial D)_i} \mathbf{u} \cdot \mathbf{n} ds = 0,$$

since $\mathbf{u} \cdot \mathbf{n} = 0$ on $(\partial D)_i$. We thus have that

$$\begin{aligned} 0 &= \int_{\partial S} -u_2 dx + u_1 dy = \int_{\partial S} \mathbf{u} \cdot \mathbf{n} ds = \int_{(\partial D)_i} \mathbf{u} \cdot \mathbf{n} ds + \int_{C \cup (-C')} \mathbf{u} \cdot \mathbf{n} ds \\ &= \int_{C \cup (-C')} \mathbf{u} \cdot \mathbf{n} ds = \int_{C \cup (-C')} -u_2 dx + u_1 dy. \end{aligned}$$

Thus, in both cases,

$$\int_C -u_2 dx + u_1 dy = \int_{C'} -u_2 dx + u_1 dy,$$

and the line integral defining $\psi(x, y)$ is thus independent of the path. Differentiate $\psi(x, y)$ in y to see that $\psi_y = u_1$ and differentiate in x to get $\psi_x = -u_2$. Thus, $\mathbf{u} = -\nabla^\perp \psi$ for some ψ . Note that at this stage, no condition on the circulation is imposed. However, $-\nabla^\perp$ preserves the circulation. Indeed, note that, since $\mathbf{u} \in X_0$,

we have that $\int_{(\partial D)_i} \mathbf{u} \cdot d\mathbf{s} = 0, 1 \leq i \leq n$ and thus $\int_{(\partial D)_i} -\nabla^\perp \psi \cdot d\mathbf{s} = 0, 1 \leq i \leq n$. Note that since $\mathbf{u} \cdot \mathbf{n} = 0$ on $(\partial D)_i$, we have that $-\nabla^\perp \psi \cdot \mathbf{n} = 0$ on $(\partial D)_i$. Thus $\nabla \psi \cdot \mathbf{t} = 0$ on $(\partial D)_i$ and thus ψ is constant on $(\partial D)_i, 0 \leq i \leq n$. Add a constant to ψ such that $\psi|_{(\partial D)_0} = 0$. Thus $\psi \in Y_0$. We prove that $-\nabla^\perp$ is one to one by noting that $-\nabla^\perp \psi = 0$ yields $\psi = \text{const} = 0$ due to the condition $\psi|_{(\partial D)_0} = 0$. ■

We now need another Lemma.

Lemma 2.23. $-\Delta : Y_0 \rightarrow L_2$ is one to one. That is, if $-\Delta \psi = 0$, for some $\psi \in Y_0$, then, $\psi = 0$.

Proof. Note that ψ satisfies,

$$-\Delta \psi = 0, \tag{2.3.27}$$

$$\psi(x, y)|_{(\partial D)_0} = 0, \tag{2.3.28}$$

$$\psi|_{(\partial D)_i}(x, y) = c_i, \text{ for } 1 \leq i \leq n, \tag{2.3.29}$$

$$\int_{(\partial D)_i} -\nabla^\perp \psi(s) \cdot d\mathbf{s} = 0 \text{ for } 1 \leq i \leq n. \tag{2.3.30}$$

Multiply (2.3.27) by ψ and integrate over the domain to get

$$\begin{aligned} 0 &= \int_D \psi(-\Delta \psi) d\mathbf{x} = -\sum_{i=0}^n \int_{(\partial D)_i} \psi \mathbf{n} \cdot \nabla \psi ds + \int_D |\nabla \psi|^2 d\mathbf{x} \\ &= \sum_{i=0}^n \psi|_{(\partial D)_i} \int_{(\partial D)_i} \nabla^\perp \psi \cdot d\mathbf{s} + \int_D |\nabla \psi|^2 d\mathbf{x} \\ &= \int_D |\nabla \psi|^2 d\mathbf{x}, \end{aligned} \tag{2.3.31}$$

where we have used Green's formula and (2.3.28), (2.3.29) and (2.3.30) and we have used the fact that $\mathbf{n} \cdot \nabla \psi = -\mathbf{t} \cdot \nabla^\perp \psi$. From this it follows that $\nabla \psi = 0$ on D . Then ψ is a constant, and is equal to 0 by (2.3.28). ■

We thus have the following Lemma.

Lemma 2.24. *The curl operator $\text{curl} : X_0 \rightarrow L_2$ is one to one. That is, if $\text{curl } \mathbf{u} = 0$, for some $\mathbf{u} \in X_0$, then $\mathbf{u} = 0$.*

Proof. Let $\mathbf{u} \in X_0$ be such that $\text{curl } \mathbf{u} = 0$. By Lemma 2.21, we have that there exists a $\psi \in Y_0$ such that $\mathbf{u} = -\nabla^\perp \psi$. Then $\text{curl } \mathbf{u} = \text{curl}(-\nabla^\perp \psi) = -\Delta \psi = 0$. By Lemma 2.23, we have that $\psi = 0$. Thus $\mathbf{u} = -\nabla^\perp \psi = 0$. ■

A slightly different proof of Lemma 2.24 can also be found in [MP94, Section 1.2, Theorems 2.1 and 2.2].

Remark 2.25. We comment on the Rayleigh-Ritz formula frequently used in hydrodynamics, see, e.g. ([S00], Lemma 4.16, page 111). Let A be a positive operator with compact resolvent acting in a Hilbert space \mathcal{H} . The minimal eigenvalue $\lambda_{\min}(A)$ can be computed by the following Rayleigh-Ritz formula:

$$\lambda_{\min}(A) = \min_{0 \neq \phi \in \text{dom } A} \frac{\|A\phi\|^2}{\langle A\phi, \phi \rangle}, \quad (2.3.32)$$

where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ are the norm and the scalar product in \mathcal{H} . Indeed, as the following argument shows, (2.3.32) is a consequence of the standard (Hilbert-Schmidt-Courant-Fischer) minimax principle: Let $B = A^{-1}$. By assumptions, B is a compact positive operator whose maximal eigenvalue is given by the formula (see, e.g. Reed / Simon IV, Section XIII.1, page 76 onwards)

$$\lambda_{\max}(B) = \max_{\psi \neq 0} \frac{\langle B\psi, \psi \rangle}{\|\psi\|^2}. \quad (2.3.33)$$

Letting $\psi = A\phi$ and using the spectral mapping theorem $sp(A) = (sp(B))^{-1}$, we obtain,

$$\lambda_{\min}(A) = (\lambda_{\max}(B))^{-1} = \left(\max_{0 \neq \phi \in \text{dom } A} \frac{\langle \phi, A\phi \rangle}{\|A\phi\|^2} \right)^{-1} = \min_{0 \neq \phi \in \text{dom } A} \frac{\|A\phi\|^2}{\langle A\phi, \phi \rangle},$$

yielding (2.3.32).

We will apply (2.3.32) for the following situation. First, let $A = -\Delta$ on $L^2(D)$ with the domain $\text{dom}A = Y_0$, see Remark 2.20 above. Due to the choice of the boundary condition, calculation (2.3.31) yields

$$\langle A\phi, \phi \rangle_{L^2} = \langle -\Delta\phi, \phi \rangle_{L^2} = \|\nabla\phi\|_2^2 \quad (2.3.34)$$

for all $\phi \in Y_0$. On the other hand,

$$\|A\phi\|_2^2 = \langle -\Delta\phi, -\Delta\phi \rangle_{L^2} = \|\Delta\phi\|_2^2. \quad (2.3.35)$$

Now (2.3.32), (2.3.34), (2.3.35) give the Rayleigh-Ritz formula (see, e.g., [S00, Eq. 4.49])

$$\lambda_{\min} = \min_{0 \neq \psi \in Y_0} \frac{\|\Delta\psi\|_2^2}{\|\nabla\psi\|_2^2} = \min_{0 \neq \mathbf{u} \in X_0} \frac{\|\text{curl } \mathbf{u}\|_2^2}{\|\mathbf{u}\|_2^2}. \quad (2.3.36)$$

We now prove Arnold's second stability theorem.

Theorem 2.26. *Let \mathbf{u}^0 be a steady state solution of the Euler equations (2.1.2) on the multi connected domain D , satisfying Assumption 2.13. Let $\lambda_{\min} > 0$ be the minimum eigenvalue of the operator $-\Delta : L^2(D) \rightarrow L^2(D)$ with the domain $\text{dom}(-\Delta) = Y_0$. Suppose that*

$$0 < \frac{1}{\lambda_{\min}} < \inf_{(x,y) \in D} F'(\omega^0(x,y)) \leq \sup_{(x,y) \in D} F'(\omega^0(x,y)) < +\infty. \quad (2.3.37)$$

Then, there exists a constant $K > 0$, such that if $\mathbf{u}(\cdot, t) = \mathbf{u}^0 + \delta\mathbf{u}(\cdot, t)$, $t \in I$ solves the Euler equations (2.1.2) on D , with $\delta\mathbf{u}(\cdot, 0) \in X_0$ then one has the following estimate for all times $t \in I$,

$$\|\mathbf{u}(\cdot, t) - \mathbf{u}^0\|_2^2 + \|\omega(\cdot, t) - \omega^0\|_2^2 \leq K(\|\mathbf{u}(\cdot, 0) - \mathbf{u}^0\|_2^2 + \|\omega(\cdot, 0) - \omega^0\|_2^2),$$

where $\omega^0 = \text{curl } \mathbf{u}^0$ and $\omega(\cdot, t) = \text{curl } \mathbf{u}(\cdot, t)$.

Remark 2.27. Inequality (2.3.37) is sometimes formulated (see [MP94, Theorem 2.3, p110]) as follows: there exists a $\beta \in (0, c_1)$ such that

$$\|\mathbf{u}\|_2^2 \leq \beta \|\omega\|_2^2, \quad (2.3.38)$$

for all $\mathbf{u} \in X_1$ and where $c_1 = \inf_{(x,y) \in D} F'(\omega^0(x, y))$.

Remark 2.28. Note that if $\delta\mathbf{u}(\cdot, 0) \in X_1$, then $\delta\mathbf{u}(\cdot, t) \in X_1$ for all $t \in I$ since the circulation of the solution $\mathbf{u}^0 + \delta\mathbf{u}(\cdot, t)$ is t independent by the Kelvin Circulation Theorem.

Remark 2.29. Sometimes Arnold's theorems are formulated (see, for example, [AK98], Theorem 4.3, page 91) with the assumption that $\mathbf{u}^0 + \delta\mathbf{u}$ has the same circulation on boundary curves $(\partial D)_i$, $1 \leq i \leq n$, as \mathbf{u}^0 , which is exactly the same as picking perturbations $\delta\mathbf{u}$ from subspace Y_1 with zero circulation. The reason for taking perturbations with zero circulation is that Lemma 2.24 required that condition.

Proof. We proceed similarly to the proof of Arnold Theorem I until (2.3.21), i.e., choose C and a_i as in Lemma 2.14. Here, we put

$$c_1 = \inf_{(x,y) \in D} F'(\omega^0(x, y)) \quad \text{and} \quad c_2 = \sup_{(x,y) \in D} F'(\omega^0(x, y)).$$

Note that $c_1 - 1/\lambda_{min} > 0$ by (2.3.37). Now (2.3.19) becomes,

$$c_1 \leq -C''(\xi) \leq c_2, \quad (2.3.39)$$

for all $\xi \in \mathbb{R}$. We use (2.3.21), (2.3.22) and (2.3.39) to obtain,

$$-\frac{1}{2}\|\delta\mathbf{u}\|_2^2 + \frac{c_1}{2}\|\delta\omega\|_2^2 \leq -\mathcal{H}(\delta\mathbf{u}) \leq -\frac{1}{2}\|\delta\mathbf{u}\|_2^2 + \frac{c_2}{2}\|\delta\omega\|_2^2. \quad (2.3.40)$$

Using (2.3.36) twice, with \mathbf{u} replaced by $\delta\mathbf{u}$ and ω replaced by $\delta\omega$, we infer,

$$\begin{aligned} \frac{(c_1 - 1/\lambda_{min})\lambda_{min}}{4} \|\delta \mathbf{u}\|_2^2 + \frac{c_1 - 1/\lambda_{min}}{4} \|\delta \omega\|_2^2 &\leq \frac{c_1 - 1/\lambda_{min}}{2} \|\delta \omega\|_2^2 \\ &\leq \frac{c_1}{2} \|\delta \omega\|_2^2 - \frac{1}{2} \|\delta \mathbf{u}\|_2^2. \end{aligned}$$

Thus, by the first inequality in (2.3.40), we conclude that

$$\frac{(c_1 - 1/\lambda_{min})\lambda_{min}}{4} \|\delta \mathbf{u}\|_2^2 + \frac{c_1 - 1/\lambda_{min}}{4} \|\delta \omega\|_2^2 \leq -\mathcal{H}(\delta \mathbf{u}).$$

We recall that $\mathcal{H}(\delta \mathbf{u}(\cdot, t))$ is t -independent as noted in the discussion following (2.3.20) in the proof of Arnold's first theorem, Theorem 2.15. Using the second inequality in (2.3.40) we have,

$$\begin{aligned} \frac{(c_1 - 1/\lambda_{min})\lambda_{min}}{4} \|\delta \mathbf{u}(\mathbf{x}, t)\|_2^2 + \frac{c_1 - 1/\lambda_{min}}{4} \|\delta \omega(\mathbf{x}, t)\|_2^2 \\ \leq -\mathcal{H}(\delta \mathbf{u}(t)) = -\mathcal{H}(\delta \mathbf{u}(0)) \leq \frac{1}{2} \|\delta \mathbf{u}(\mathbf{x}, 0)\|_2^2 + \frac{c_2}{2} \|\delta \omega(\mathbf{x}, 0)\|_2^2. \end{aligned}$$

Now let $K_1 = \min\left(\frac{(c_1 - 1/\lambda_{min})\lambda_{min}}{4}, \frac{c_1 - 1/\lambda_{min}}{4}\right)$ and $K_2 = \max\left(\frac{1}{2}, \frac{c_2}{2}\right)$. Combining the last two estimates, one can get the inequality

$$K_1 \left(\|\delta \mathbf{u}(\mathbf{x}, t)\|_2^2 + \|\delta \omega(\mathbf{x}, t)\|_2^2 \right) \leq K_2 \left(\|\delta \mathbf{u}(\mathbf{x}, 0)\|_2^2 + \|\delta \omega(\mathbf{x}, 0)\|_2^2 \right), \quad (2.3.41)$$

and finish the proof of the theorem similar to Arnold's first theorem. ■

Remark 2.30. We would like to expand upon the following remark in [MP94], Remark 2, page 111, "Inequality (2.3.38) is relative to the lowest eigenvalue of the Laplace operator. The value of β depends on the form and size of the domain D ". We note that β is simply $1/\lambda_{min}$. Observe that the second variation $\delta^2 H_c(\mathbf{u}^0)$ in (2.3.25) generates a quadratic form $Q(\mathbf{u}^0)$ on the space X_0 given by the expression,

$$Q(\mathbf{u}^0)(\mathbf{u}, \mathbf{u}) = \int_D \mathbf{u} \cdot \mathbf{u} dx + \int_D C''(\omega^0)(\text{curl } \mathbf{u})^2 dx.$$

We first comment on the relationship between the quadratic form $Q(\mathbf{u}^0)(\mathbf{u}, \mathbf{u})$ and the negative Laplace operator $-\Delta$. $Q(\mathbf{u}^0)(\mathbf{u}, \mathbf{u})$ is the sum of the two quadratic forms

$$D^2H(\mathbf{u}^0)(\mathbf{u}, \mathbf{u}) := \int_D \mathbf{u} \cdot \mathbf{u} d\mathbf{x}$$

and

$$D^2K(\mathbf{u}^0)(\mathbf{u}, \mathbf{u}) := \int_D C''(\omega^0)(\text{curl } \mathbf{u})^2 d\mathbf{x}.$$

At the level of the perturbation stream function $\psi \in Y_0$, the quadratic form $D^2H(\mathbf{u}^0)(\mathbf{u})$ can be expressed as $\int_D \nabla\psi \cdot \nabla\psi d\mathbf{x}$. An integration by parts yields, since the boundary terms vanish for all $\psi \in Y_0$, that

$$\int_D \nabla\psi \cdot \nabla\psi d\mathbf{x} = \int_D \psi(-\Delta\psi) d\mathbf{x},$$

c.f. (2.3.31). Therefore, the operator associated with the quadratic form

$$D^2H(\mathbf{u}^0)(\mathbf{u}, \mathbf{u}) = \int_D \nabla\psi \cdot \nabla\psi d\mathbf{x} = \int_D \psi(-\Delta\psi) d\mathbf{x} = \langle \psi, -\Delta\psi \rangle_{L^2}$$

is the negative Laplacian $-\Delta$, acting on the space $L^2(D)$ with the domain Y_0 . Under the assumptions of Theorem 2.26 we have

$$\begin{aligned} Q(\mathbf{u}^0)(\mathbf{u}, \mathbf{u}) &= \int_D \mathbf{u} \cdot \mathbf{u} d\mathbf{x} + \int_D C''(\omega^0)(\text{curl } \mathbf{u})^2 d\mathbf{x} \\ &\leq (1/\lambda_{\min} - c_1) \int_D (\text{curl } \mathbf{u})^2 d\mathbf{x} < 0. \end{aligned}$$

where we have used (2.3.36), (2.3.37) and (2.3.39). The proof of Arnold's second theorem shows that the quadratic form $Q(\mathbf{u}^0)(\mathbf{u})$ is negative definite since

$$\begin{aligned} Q(\mathbf{u}^0)(\mathbf{u}, \mathbf{u}) &= \|\mathbf{u}\|_2^2 + \int_D C''(\omega^0)(\text{curl } \mathbf{u})^2 d\mathbf{x} \leq (1/\lambda_{\min} - c_1) \|\text{curl } \mathbf{u}\|_2^2 \\ &\leq \frac{\lambda_{\min}(1/\lambda_{\min} - c_1)}{2} \|\mathbf{u}\|_2^2 + \frac{1/\lambda_{\min} - c_1}{2} \|\text{curl } \mathbf{u}\|_2^2, \end{aligned}$$

where we have used (2.3.36) again. The last expression can be estimated by $A(\|\mathbf{u}\|_2^2 + \|\operatorname{curl} \mathbf{u}\|_2^2)$, where

$$A = \min \left(\frac{\lambda_{\min}(1/\lambda_{\min} - c_1)}{2}, \frac{1/\lambda_{\min} - c_1}{2} \right). \quad (2.3.42)$$

We thus have an estimate of the form

$$Q(\mathbf{u}^0)(\mathbf{u}) \leq A(\|\mathbf{u}\|_2^2 + \|\omega\|_2^2), \quad (2.3.43)$$

where $A < 0$.

Example 2.31. See [AK98], Example 4.5, page 92. Consider a circular motion in the annulus $R_1 \leq \rho \leq R_2$. Suppose $\psi^0(\rho)$ is a steady state such that the ratio

$$\frac{(\psi^0)'}{((\psi^0)'' + \frac{1}{\rho}(\psi^0)')}$$

does not change sign, then the flow is stable. That is, on an annulus, circular flows with no inflection points are stable.

2.3.2 Arnold's theorems in a bounded, simply connected domain

In this section we consider the domain $D \subset \mathbb{R}^2$ to be a bounded, simply connected region with a smooth boundary ∂D . The analysis is quite similar to the previous section and we merely indicate the differences. The functional (2.3.1), is now given by,

$$H_c(\mathbf{u}) = \frac{1}{2} \int_D \mathbf{u} \cdot \mathbf{u} dx + \int_D C(\operatorname{curl} \mathbf{u}) dx + a \int_{\partial D} \mathbf{u} \cdot \mathbf{ds}. \quad (2.3.44)$$

Lemma 2.12 holds and we impose Assumption 2.13. Lemma 2.14 also holds, where we now set $a = F(\omega^0)|_{\partial D}$. The expression for the second variation given in (2.3.25)

remains unchanged. Theorem 2.15 also holds in this case. We now expand upon remark 2.20.

Remark 2.32. Our Lemmas 2.21, 2.23 and 2.24 in Remark 2.20 will work where the subspace for the stream function perturbations is now $Y_0 := H^2(D; \mathbb{R}) \cap H_0^1(D; \mathbb{R})$ and the subspace for the velocity perturbations is $X_0 := \{\mathbf{u} \in H^1(D; \mathbb{R}^2), \operatorname{div} \mathbf{u} = 0 \text{ in } D, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D\}$. The proofs are similar and are omitted. Arnold's second theorem then follows as stated.

Lemma 2.33. *On a simply connected domain, if $\operatorname{div} \mathbf{u} = 0$ then $\|\mathbf{u}\|_2^2 + \|\omega\|_2^2$ is equivalent to $\|\mathbf{u}\|_{H^1}^2$.*

Proof. We note that $\|\mathbf{u}\|_{H^1}^2 = \|\mathbf{u}\|_2^2 + \|\nabla \mathbf{u}\|_2^2$ where $\|\nabla \mathbf{u}\|_2^2 = \|\partial_x u_1\|_2^2 + \|\partial_y u_1\|_2^2 + \|\partial_x u_2\|_2^2 + \|\partial_y u_2\|_2^2$. Also,

$$\begin{aligned} \|\omega\|_2^2 &= \|\operatorname{curl} \mathbf{u}\|_2^2 = \int_D \left((\partial_x u_2)^2 + (\partial_y u_1)^2 - 2(\partial_x u_2)(\partial_y u_1) \right) dx dy \\ &\leq \int_D \left((\partial_x u_2)^2 + (\partial_y u_1)^2 + (\partial_x u_2)^2 + (\partial_y u_1)^2 \right) dx dy \leq 2\|\nabla \mathbf{u}\|_2^2. \end{aligned}$$

To estimate $\|\nabla \mathbf{u}\|_2^2$ from above, we first note that

$$\|\nabla \mathbf{u}\|_2^2 = \|\partial_x \partial_y \psi\|_2^2 + \|\partial_y^2 \psi\|_2^2 + \|\partial_x^2 \psi\|_2^2 + \|\partial_x \partial_y \psi\|_2^2 \leq \|\psi\|_{H^2}^2,$$

where $\mathbf{u} = -\nabla^\perp \psi$ and $\omega = -\Delta \psi$. Since $\psi \in H^2(D) \cap H_0^1(D)$, we can use the boundary H^2 -regularity of the solution of the Poisson equation with Dirichlet boundary conditions, see e.g. [[E10], Theorem 6.3.4], to conclude that

$$\|\nabla \mathbf{u}\|_2^2 \leq \|\psi\|_{H^2}^2 \leq C(\|\Delta \psi\|_2^2 + \|\psi\|_2^2) \leq C(\|\Delta \psi\|_2^2 + \|\nabla^\perp \psi\|_2^2) = C(\|\omega\|_2^2 + \|\mathbf{u}\|_2^2),$$

where we also used Poincaré's inequality $\|\nabla \psi\|_2^2 \geq c\|\psi\|_2^2$ for $\psi \in H_0^1(D)$. ■

Lemma 2.33 shows that form $Q(\mathbf{u}^0)$, see (2.3.43), is negative definite on $H^1(D)$.

2.3.3 Arnold's second theorem on the two torus

Note that by remark 2.18, we do not have Arnold's first theorem on the two torus.

We would like to consider Arnold's second theorem on the two torus \mathbb{T}^2 , thus in this subsection, $D := \mathbb{T}^2$. We consider the Euler equations (2.1.2) on the two torus \mathbb{T}^2 , where $\mathbf{u} : \mathbb{T}^2 \rightarrow \mathbb{R}$ is such that $\operatorname{div} \mathbf{u} = 0$. We also have the vorticity equation (2.1.10), where $\omega : \mathbb{T}^2 \rightarrow \mathbb{R}$. The Hamiltonian H_c is now given by,

$$H_c(\mathbf{u}) = \frac{1}{2} \int_D \mathbf{u} \cdot \mathbf{u} d\mathbf{x} + \int_D C(\operatorname{curl} \mathbf{u}) d\mathbf{x}. \quad (2.3.45)$$

Lemma 2.12 remains true in this setting. We also impose Assumption 2.13. Lemma 2.14 is modified as follows.

Lemma 2.34. *Let \mathbf{u}^0, ω^0 be a steady state solution of (2.1.2) satisfying Assumption 2.13, where $\omega^0 = \operatorname{curl} \mathbf{u}^0$. Let C be a smooth function such that $C'(\omega^0(x, y)) = -F(\omega^0(x, y))$ for every (x, y) in D . Then $\delta H_c(\mathbf{u}^0) \delta \mathbf{u} = 0$, i.e., \mathbf{u}^0 is a critical point of H_c .*

Proof. The first variation of H_c at \mathbf{u}^0 is given by the following expression, see (2.3.10),

$$\begin{aligned} \delta H_c(\mathbf{u}^0) \delta \mathbf{u} &= \left. \frac{d}{d\varepsilon} H_c(\mathbf{u}^0 + \varepsilon \delta \mathbf{u}) \right|_{\varepsilon=0} \\ &= \int_D \mathbf{u}^0 \cdot \delta \mathbf{u} d\mathbf{x} + \int_D C'(\omega^0) \delta \omega d\mathbf{x}, \end{aligned}$$

where $\delta \omega = \operatorname{curl}(\delta \mathbf{u})$. We will be using the following identity (see [MP94], Eq 2.14, page 108),

$$C'(\omega^0) \delta \omega = \operatorname{curl}(C'(\omega^0) \delta \mathbf{u}) - C''(\omega^0) \nabla^\perp \omega^0 \cdot \delta \mathbf{u}. \quad (2.3.46)$$

(2.3.46) follows from the identity, $\operatorname{curl}(f\mathbf{v}) = \nabla^\perp f \cdot \mathbf{v} + f \operatorname{curl} \mathbf{v}$. Noting that, by

Stokes' theorem,

$$\int_D \operatorname{curl}(C'(\omega^0)\delta\mathbf{u})d\mathbf{x} = 0, \quad (2.3.47)$$

we see that, using (2.3.46) and (2.3.47)

$$\int_D C'(\omega^0)\delta\omega d\mathbf{x} = - \int_D C''(\omega^0)\nabla^\perp\omega^0 \cdot \delta\mathbf{u}d\mathbf{x}. \quad (2.3.48)$$

We thus have that,

$$\delta H_c(\mathbf{u}^0)\delta\mathbf{u} = \int_D \mathbf{u}^0 \cdot \delta\mathbf{u}d\mathbf{x} - \int_D C''(\omega^0)\nabla^\perp\omega^0 \cdot \delta\mathbf{u}d\mathbf{x}. \quad (2.3.49)$$

If one chooses C such that $C'(\omega^0(x, y)) = -F(\omega^0(x, y))$, then, $\mathbf{u}^0 = -\nabla^\perp\psi^0 = -F'(\omega^0)\nabla^\perp\omega^0 = C''(\omega^0)\nabla^\perp\omega^0$, then the first variation $\delta H_c(\mathbf{u}^0)\delta\mathbf{u} = 0$. ■

The expression for the second variation remains the same as (2.3.25). The statement of Arnold's first theorem remains the same as in Theorem 2.15. We shall now expand upon Remark 2.20.

Remark 2.35. Our space for the perturbation stream function is now given by

$$Y_0 := \{\psi \in H^2(\mathbb{T}^2); \int_{\mathbb{T}^2} \psi d\mathbf{x} = 0\}, \quad (2.3.50)$$

and for the velocities is given by $X_0 := \{\mathbf{u} \in H^1(\mathbb{T}^2; \mathbb{R}^2); \int_{\mathbb{T}^2} \mathbf{u}d\mathbf{x} = 0; \operatorname{div} \mathbf{u} = 0\}$.

Lemmas 2.21, 2.23 and 2.24 in Remark 2.20 are true in this setting with minor modifications in the proof. We restate and prove Lemma 2.23 in this case.

Lemma 2.36. $-\Delta : Y_0 \rightarrow L_2$ is one to one. That is, if $-\Delta\psi = 0$, for some $\psi \in Y_0$, then, $\psi = 0$.

Proof. Note that ψ satisfies,

$$-\Delta\psi = 0, \quad (2.3.51)$$

$$\int_{\mathbb{T}^2} \psi dx dy = 0. \quad (2.3.52)$$

Multiply (2.3.51) by ψ and integrate over the domain to get

$$0 = \int_D \psi(-\Delta\psi) d\mathbf{x} = \int_D |\nabla\psi|^2 d\mathbf{x},$$

where we have used Green's formula. From this it follows that $\nabla\psi = 0$ on D . But ψ is thus a constant and is equal to 0 by (2.3.52). ■

Arnold's second theorem then follows as stated in Theorem 2.26.

We now prove the equivalence of the energy-entropy norm, $\|\mathbf{u}\|_2 + \|\operatorname{curl} \mathbf{u}\|_2$ to the H^1 norm, $\|\mathbf{u}\|_2 + \|\nabla\mathbf{u}\|_2$ on the two torus.

Lemma 2.37. *If $\operatorname{div} \mathbf{u} = 0$, then $\|\mathbf{u}\|_{L_2}^2 + \|\omega\|_{L_2}^2$ is equivalent to $\|\mathbf{u}\|_{H^1}^2$.*

Proof. Denote $\mathbf{u} = (u_1, u_2)$. We have that, since $\operatorname{div} \mathbf{u} = 0$,

$$0 = (\operatorname{div} \mathbf{u})^2 = (\partial_1 u_1 + \partial_2 u_2)^2 = (\partial_1 u_1)^2 + (\partial_2 u_2)^2 + 2(\partial_1 u_1)(\partial_2 u_2).$$

Thus,

$$\int_{\mathbb{T}^2} (\partial_1 u_1)^2 + (\partial_2 u_2)^2 = -2 \int_{\mathbb{T}^2} (\partial_1 u_1)(\partial_2 u_2) = -2 \int_{\mathbb{T}^2} (\partial_2 u_1)(\partial_1 u_2), \quad (2.3.53)$$

where we have integrated the second term by parts twice. Also, $\omega^2 = (\partial_1 u_2 - \partial_2 u_1)^2$.

Thus, using (2.3.53),

$$\begin{aligned} \|\omega\|_2^2 &= \int_{\mathbb{T}^2} \omega^2 = \int_{\mathbb{T}^2} (\partial_1 u_2 - \partial_2 u_1)^2 = \int_{\mathbb{T}^2} (\partial_1 u_2)^2 + (\partial_2 u_1)^2 - 2(\partial_2 u_1)(\partial_1 u_2) \\ &= \int_{\mathbb{T}^2} (\partial_1 u_2)^2 + (\partial_2 u_1)^2 + (\partial_1 u_1)^2 + (\partial_2 u_2)^2 = \|\nabla\mathbf{u}\|_2^2. \end{aligned}$$

Thus,

$$\|\mathbf{u}\|_{L_2}^2 + \|\omega\|_{L_2}^2 = \|\mathbf{u}\|_{L_2}^2 + \|\nabla\mathbf{u}\|_{L_2}^2 = \|\mathbf{u}\|_{H^1}^2.$$

■

Remark 2.38. On the two torus, by Lemma 2.37, the stability estimate we obtained in (2.3.17) can be rewritten as,

$$\|\mathbf{u}(\cdot, t) - \mathbf{u}^0\|_{H^1}^2 \leq K \|\mathbf{u}(\cdot, 0) - \mathbf{u}^0\|_{H^1}^2. \quad (2.3.54)$$

Example 2.39. See [MP94, Sec 3.2, p 111]. Let us consider the following steady state $\mathbf{u}^0 = (\sin y, 0)$, $\psi^0 = -\cos y$ and $\omega^0 = \text{curl } \mathbf{u}^0 = \sin y$. Thus, $F(\omega^0) = \omega^0$ and $F' = 1$. One can directly compute the spectrum of $-\Delta$ on the space $L^2(D)$ to be

$$\{k_1^2 + k_2^2 : (k_1, k_2) \in \mathbb{Z}^2 \setminus (0, 0)\}.$$

From this, one sees that $\lambda_{min} = 1$. Since it is not possible to check the inequality $1/\lambda_{min} < \inf_{(x,y) \in D} (F'(\omega^0(x, y)))$ one cannot use Arnold's second theorem to study the stability of this steady state. One can obtain *conditional stability* in the following sense, see [MP94, Sec 3.2, p 111]: consider subspaces generated by the modes corresponding to the \mathbf{k} values corresponding to λ_{min} , i.e. the modes generated by $(\pm 1, 0), (0, \pm 1)$. Consider perturbations which lie in subspaces that are orthogonal to the subspaces generated by the above modes. Then, with perturbations in the above subspaces, if one starts a solution, then one has stability provided the evolution preserves the motion along these subspaces.

2.3.4 Arnold's theorems in a periodic channel

We would now like to formulate of Arnold's theorems on the periodic channel $D = \mathbb{T} \times [-1, 1]$, so that the boundary conditions are periodic in the x direction with no penetration boundary condition $\mathbf{u} \cdot \mathbf{n}$ at the "walls" $y = 1$ and $y = -1$. (see [B15]). We note that since the domain is translationally invariant in the x direction, the x momentum is conserved, i.e., we have that, if $\mathbf{u}(t, \cdot) = (u_1(t, \cdot), u_2(t, \cdot))$ solves Euler

equation (2.1.2), then

$$M_x = \int_{-1}^1 \int_{\mathbb{T}} u_1(t, x, y) dx dy = \int_{-1}^1 \int_{\mathbb{T}} u_1(0, x, y) dx dy \quad (2.3.55)$$

is an invariant of the motion.

Lemma 2.40. *Suppose $\mathbf{u}(t, \cdot) = (u_1(t, \cdot), u_2(t, \cdot))$ solves Euler equation (2.1.2), on the domain $\mathbb{T} \times [-1, 1]$. Then*

$$\frac{d}{dt} M_x = \frac{d}{dt} \int_{-1}^1 \int_{\mathbb{T}} u_1(t, x, y) dx dy = 0. \quad (2.3.56)$$

Proof. We first note that the boundary conditions $\mathbf{u} \cdot \mathbf{n}|_{y=\pm 1} = 0$ imply that

$$u_2(x, -1) = u_2(x, 1) = 0. \quad (2.3.57)$$

We also note the following,

$$(\mathbf{u} \cdot \nabla) u_1 = u_1 \partial_x u_1 + u_2 \partial_y u_1 = \frac{1}{2} \partial_x (u_1^2) + u_2 \partial_y u_1. \quad (2.3.58)$$

We thus have that,

$$\begin{aligned} \frac{d}{dt} M_x &= \frac{d}{dt} \int_{-1}^1 \int_{\mathbb{T}} u_1(t, x, y) dx dy = \int_{-1}^1 \int_{\mathbb{T}} \partial_t u_1(t, x, y) dx dy \\ &= \int_{-1}^1 \int_{\mathbb{T}} (- (\mathbf{u} \cdot \nabla) u_1 - \partial_x p) dx dy = \int_{-1}^1 \int_{\mathbb{T}} -\frac{1}{2} \partial_x (u_1^2) - u_2 \partial_y u_1 - \partial_x p dx dy. \end{aligned}$$

We analyze this term by term. In the first term, the inner integral $\int_{\mathbb{T}} -\frac{1}{2} \partial_x (u_1^2) dx = 0$ because u_1 is periodic in x . The inner integral of the third term is also zero, $\int_{\mathbb{T}} -\partial_x p dx = 0$. We thus look at the second term. First interchange order of integration and then do integration by parts in y variable and use (2.3.57), we get,

$$\begin{aligned} \int_{-1}^1 \int_{\mathbb{T}} u_2 \partial_y u_1 dx dy &= \int_{\mathbb{T}} \int_{-1}^1 u_2 \partial_y u_1 dy dx = - \int_{\mathbb{T}} \int_{-1}^1 u_1 \partial_y u_2 dy dx \\ &= \int_{\mathbb{T}} \int_{-1}^1 u_1 \partial_x u_1 dy dx, \end{aligned}$$

where in the last step, we have used that $\operatorname{div} \mathbf{u} = \partial_x u_1 + \partial_y u_2 = 0$. But,

$$\int_{\mathbb{T}} \int_{-1}^1 u_1 \partial_x u_1 dy dx = \int_{-1}^1 \int_{\mathbb{T}} u_1 \partial_x u_1 dx dy = \int_{-1}^1 \int_{\mathbb{T}} \frac{1}{2} \partial_x (u_1^2) dx dy = 0.$$

■

Since $u_1 = \psi_y$, consider the following,

$$\frac{1}{2\pi} \int_{\mathbb{T}} u_1(t, x, y) dx = \frac{1}{2\pi} \int_{\mathbb{T}} \psi_y(t, x, y) dx.$$

Integrating this in y , and using (2.3.55), we see that,

$$\begin{aligned} \frac{1}{2\pi} M_x &= \frac{1}{2\pi} \int_{-1}^1 \int_{\mathbb{T}} u_1(t, x, y) dx dy = \frac{1}{2\pi} \int_{\mathbb{T}} \int_{-1}^1 u_1(t, x, y) dy dx = \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \int_{-1}^1 \psi_y(t, x, y) dy dx = \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t, x, -1) dx - \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t, x, 1) dx. \end{aligned}$$

By Lemma 2.40, $M_x/2\pi$ is a fixed number in time representing the mean flow rate across the torus. Since $\psi(x, -1)$ and $\psi(x, 1)$ are constants, one can simply take the difference to be $M_x/2\pi$. Thus in solving the Poisson equation for the stream function we can set $\psi(x, -1) = 0$ and $\psi(x, 1) = M_x/2\pi$. One thus solves the following Poisson problem to recover the stream function from the vorticity.

$$\begin{aligned} -\Delta \psi &= \omega, \text{ in } D, \\ \psi(x, -1) &= 0, \\ \psi(x, 1) &= -M_x/2\pi. \end{aligned} \tag{2.3.59}$$

Since this must hold for both the steady state ψ^0 and the perturbed flow $\psi^0 + \delta\psi$, we see that, the Poisson equation satisfied by the perturbation stream function $\delta\psi$ satisfies Dirichlet boundary conditions.

$$-\Delta \delta\psi = \omega, \text{ in } D,$$

$$\delta\psi(x, -1) = 0,$$

$$\delta\psi(x, 1) = 0. \tag{2.3.60}$$

The subspace for the perturbation stream function is now the following:

$$Y_0 = \left\{ \psi : H^2((\mathbb{T} \times [-1, 1]); \mathbb{R}) : \psi(x, 1) = 0; \psi(x, -1) = 0 \right\}.$$

One then defines the subspace for the perturbations of velocity as

$$X_0 := \left\{ \mathbf{u} : H^1((\mathbb{T} \times [-1, 1]); \mathbb{R}^2); \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } y = -1 \text{ and } y = 1; \int_{-1}^1 \int_{\mathbb{T}} u_1(x, y) dx dy = 0 \right\},$$

where $\mathbf{u} = (u_1, u_2)$. Lemma 2.21 follows as stated. One can also easily check that if

$\psi(x, 1) = \psi(x, -1) = 0$, then $\int_{\mathbb{T}} \int_{-1}^1 u_1(x, y) dy dx = 0$. Indeed,

$$\begin{aligned} \int_{\mathbb{T}} \int_{-1}^1 u_1(x, y) dy dx &= \int_{\mathbb{T}} \int_{-1}^1 \psi_y(x, y) dy dx = \int_{\mathbb{T}} (\psi(x, 1) - \psi(x, -1)) dx \\ &= \int_{\mathbb{T}} (0 - 0) dx = 0. \end{aligned}$$

The proof of Lemma 2.23 is modified as follows.

Proof. Note that ψ satisfies,

$$-\Delta\psi = 0 \text{ on } D, \tag{2.3.61}$$

$$\psi(x, -1) = \psi(x, 1) = 0. \tag{2.3.62}$$

Multiply (2.3.61) by ψ and integrate over the domain to get

$$\begin{aligned} 0 &= \int_D \psi(-\Delta\psi) dx = \int_{\partial D} \psi \mathbf{n} \cdot \nabla \psi ds + \int_D |\nabla \psi|^2 dx \\ &= \int_{-1}^1 \psi(0, y) \mathbf{n} \cdot \nabla \psi(0, y) dy + \int_0^{2\pi} \psi(x, 1) \mathbf{n} \cdot \nabla \psi(x, 1) dx \end{aligned}$$

$$\begin{aligned}
& - \int_{-1}^1 \psi(2\pi, y) \mathbf{n} \cdot \nabla \psi(2\pi, y) dy - \int_0^{2\pi} \psi(x, -1) \mathbf{n} \cdot \nabla \psi(x, -1) dx \\
& + \int_D |\nabla \psi|^2 d\mathbf{x} = \int_D |\nabla \psi|^2 d\mathbf{x},
\end{aligned}$$

where we have used Green's formula, and the facts that on the boundary pieces $y = -1$ and $y = 1$ we have $\psi(x, -1) = \psi(x, 1) = 0$, and for every $y \in [-1, 1]$ we have $\psi(0, y) = \psi(2\pi, y)$. From this it follows that $\nabla \psi = 0$ on D . ψ is thus a constant and is equal to 0 by (2.3.62). ■

Lemma 2.24 follows as stated and Arnold's second theorem also follows as stated.

Remark 2.41. Choosing perturbations in the subspace X_0 is sometimes stated as perturbations preserving the mean flow rate in the x direction and circulations, see [HMRW85, p23].

Example 2.42. *Shear flows with no point of inflection:* On the periodic channel, consider a steady state of the form $\mathbf{u}^0 = (U(y), 0)$, the so called plane parallel shear flow. Assume that $U'' \neq 0$, without loss of generality assume $U'' > 0$. From the fact that $\psi^0 = F(\omega^0)$, we have that $\mathbf{u}^0 = -\nabla^\perp \psi^0 = -F'(\omega^0) \nabla^\perp \omega^0$. Since $\omega^0 = -\partial_y U(y)$, we see that $\nabla^\perp \omega^0 = (U_{yy}(y), 0)$. We thus have that,

$$(U(y), 0) = -F'(\omega^0)(U_{yy}(y), 0).$$

Note that one can move to an inertial system of coordinates where U is always positive. In this case F' satisfies (2.3.16) and by Arnold's first theorem one has stability of the steady state. Thus one can prove nonlinear stability of a plane parallel shear flow with no inflection points.

Example 2.43. *Flows with one inflection point such that $U(y)$ and $U''(y)$ have the same sign for all y .* Let us now consider flows $\mathbf{u}^0 = (U(y), 0)$ with one inflection

point but such that $U(y)$ has the same sign as $U''(y)$. Denote by y_s the point of inflection. Also, assume that $U(y_s) = 0$. Also assume that $U(y)/U''(y)$ approaches a finite positive limit as y approaches y_s . In this case, by the same analysis as above, we have, $(U(y), 0) = -F'(\omega^0)(U_{yy}(y), 0)$, and we have that $-F'$ is bounded. One can thus use Arnold's second theorem to conclude stability. Using the same analysis one can prove the stability of a steady state with finitely many inflection points but such that $U(y)$ and $U''(y)$ have the same sign for all y and as y approaches an inflection point the ratio $U(y)/U''(y)$ approaches a finite positive limit.

Remark 2.44. Since $\mathbf{u}^0 = -\nabla^\perp \psi^0 = -F'(\omega^0)\nabla^\perp \omega^0$, sometimes, in the literature, (see, for example, [MP94], Theorem 2.2, Equation 2.4, page 106, or see [AK98], Section 4A, page 89, discussion below Equation 4.1) $-F'(\omega^0(x, y))$ is denoted by ratio of the vectors $\frac{\mathbf{u}^0(x, y)}{\nabla^\perp \omega^0(x, y)}$ or by its equivalents $\frac{-\nabla^\perp \psi^0(x, y)}{\nabla^\perp \omega^0(x, y)}$ and $\frac{-\nabla \psi^0(x, y)}{\nabla \Delta \psi^0(x, y)}$.

Example 2.45. *First eigenfunction of the Laplacian.* Suppose $\omega^0(x, y) = G(\psi^0(x, y))$ for some smooth function $G : \mathbb{R} \rightarrow \mathbb{R}$. Then the level sets of ω^0 and ψ^0 coincide and, in particular, $\nabla^\perp \psi^0$ is orthogonal to $\nabla \omega^0$. Thus, ψ^0, ω^0 is a steady state solution of the Euler equations (2.1.10). This shows, for instance, that all eigenfunctions φ^0 of the negative Laplace operator $-\Delta$ are steady state solutions of the Euler equations (2.1.10), where $\omega^0 = -\Delta \varphi^0 = \lambda \varphi^0$. We thus have that $F' = 1/\lambda$. By Arnold's second theorem, if we have that $1/\lambda_{min} < F'$, then one has stability. This strict inequality does not hold for any eigenfunction of the Laplace operator. One cannot use Arnold's theorems to conclude stability of these steady states.

Remark 2.46. The stability results reviewed above implicitly assume that a solution exists for all times t . If not, then one has the stability estimate for all times for which

the solution exists. This is sometimes referred to in the literature as *conditional stability* (see, for example, [HMRW85], page 7).

2.4 Arnold stability theorems for α -Euler equations

We now derive Arnold stability theorems for the α -Euler equations.

2.4.1 Arnold's theorems in a multi connected domain

Let $D \subset \mathbb{R}^2$ be a bounded domain, bounded by smooth curves $(\partial D)_i$ where $i = 0, \dots, n$. Denote the outer boundary of D by $(\partial D)_0$ and $(\partial D)_i, i = 1, \dots, n$ are the n inner boundaries. ∂D is used to denote the boundary of D .

We shall consider the α -Euler equations (2.1.4) on the domain D ,

$$\begin{aligned} \mathbf{u}_t + (\mathbf{v} \cdot \nabla) \mathbf{u} + (\nabla \mathbf{v})^\top \mathbf{u} + \nabla \pi &= 0, \\ (1 - \alpha^2 \Delta) \mathbf{v} &= \mathbf{u}, \\ \operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{v} &= 0 \end{aligned}$$

where \mathbf{v} satisfies the following boundary conditions: no normal flow along the boundaries, i.e $\mathbf{v} \cdot \mathbf{n} = 0$ on $(\partial D)_i, i = 0, \dots, n$ where \mathbf{n} is the unit normal vector on the boundary, and (see, for example [HMR98] (formula 8.27, page 65)) $(\mathbf{n} \cdot \nabla) \mathbf{v}$ is parallel to \mathbf{n} . That is, on the boundaries $(\partial D)_i, i = 0, \dots, n$, we have (2.1.5),

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad (\mathbf{n} \cdot \nabla \mathbf{v}) \parallel \mathbf{n}.$$

We recall that the vorticity form of the α -Euler equations are given as,

$$\omega_t + \mathbf{v} \cdot \nabla \omega = 0,$$

where the vorticity

$$\omega = \operatorname{curl}(1 - \alpha^2 \Delta) \mathbf{v}. \tag{2.4.1}$$

Since $\operatorname{div} \mathbf{v} = 0$, there exists a stream function ϕ , such that $\mathbf{v} = -\nabla^\perp \phi$. Thus,

$$\omega = \operatorname{curl}(1 - \alpha^2 \Delta) \mathbf{v} = \operatorname{curl}(1 - \alpha^2 \Delta)(-\nabla^\perp \phi) = -\Delta(1 - \alpha^2 \Delta) \phi.$$

Note that in the above computation, we used the fact that the curl operator commutes with the operator $(1 - \alpha^2 \Delta)$. We also note here that $(1 - \alpha^2 \Delta)$ commutes with the operators $-\nabla^\perp$ and div . Since $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary ∂D , we have, on ∂D , that $\nabla^\perp \phi \cdot \mathbf{n} = 0$, i.e., $\nabla \phi \cdot \mathbf{t} = 0$, where \mathbf{t} is the unit tangent vector on the boundary. Thus $\nabla \phi$ is orthogonal to the boundary, i.e., each connected piece of the boundary curve is a level sets of ϕ and thus $\phi|_{(\partial D)_i}$ is a constant on each connected boundary piece $(\partial D)_i$ where $0 \leq i \leq n$. We shall work on the following Sobolev space

$$X := \left\{ \mathbf{v} \in H^3(D; \mathbb{R}^2), \nabla \cdot \mathbf{v} = 0, \text{ on } D, \mathbf{v} \cdot \mathbf{n} = 0, (\mathbf{n} \cdot \nabla \mathbf{v}) \parallel \mathbf{n}, \text{ on } (\partial D)_i, 0 \leq i \leq n \right\}, \quad (2.4.2)$$

consisting of divergence free vector fields on D , tangent to the boundary and satisfying $(\mathbf{n} \cdot \nabla \mathbf{v}) \parallel \mathbf{n}$ on the boundary, so that $\mathbf{u} \in H^1(D; \mathbb{R}^2)$, $\omega = \operatorname{curl} \mathbf{u} \in L^2(D; \mathbb{R})$, $\phi \in H^4(D; \mathbb{R})$ and $\psi \in H^2(D; \mathbb{R})$.

A word about the notation for integrals: $\int_D f d\mathbf{x}$ will denote the volumetric integral in the region D of a scalar valued function $f : D \rightarrow \mathbb{R}$ and $\int_{(\partial D)_i} \mathbf{f} \cdot d\mathbf{s}$ will denote the line integral of a vector valued function $\mathbf{f} : (\partial D)_i \rightarrow \mathbb{R}^2$ on the curve $(\partial D)_i$ and $\int_{(\partial D)_i} f ds$ will denote the line integral of a scalar valued function $f : (\partial D)_i \rightarrow \mathbb{R}$ on the curve $(\partial D)_i$.

We shall have occasion to use the following integration by parts formula for vector valued functions

$$\int_D \mathbf{v} \cdot \Delta \mathbf{v} d\mathbf{x} = \sum_{i=0}^n \int_{(\partial D)_i} \mathbf{v} \cdot (\mathbf{n} \cdot \nabla) \mathbf{v} ds - \int_D |\nabla \mathbf{v}|^2 d\mathbf{x}. \quad (2.4.3)$$

Here $\mathbf{v} = (v_1, v_2)$, $|\nabla\mathbf{v}|^2 = \text{tr}(\nabla\mathbf{v} \cdot (\nabla\mathbf{v})^T)$, where $(\cdot)^T$ denotes the transpose, and tr denotes trace, i.e. $|\nabla\mathbf{v}|^2 = \sum_{i=1}^2 [(\partial_x v_i)^2 + (\partial_y v_i)^2]$. We shall also, at times, denote $|\nabla\mathbf{v}|^2$ by $\nabla\mathbf{v} \cdot \nabla\mathbf{v}$. Equation (2.4.3) follows from Green's identity for scalar valued functions f and g , see, e.g., [E10, Theorem 3 (ii), page 712],

$$\int_D f \Delta g d\mathbf{x} = \sum_{i=0}^n \int_{(\partial D)_i} f(\mathbf{n} \cdot \nabla) g ds - \int_D \nabla f \cdot \nabla g d\mathbf{x}. \quad (2.4.4)$$

applied to each component of \mathbf{v} and summation of the resulting identities. By assumption, for any \mathbf{v} that satisfies (2.1.5), $\mathbf{n} \cdot \nabla\mathbf{v}$ is parallel to \mathbf{n} and $\mathbf{v} \cdot \mathbf{n} = 0$, on the boundary, we have,

$$\mathbf{v} \cdot (\mathbf{n} \cdot \nabla\mathbf{v}) = 0,$$

on the boundary, i.e., we have,

$$\int_D \mathbf{v} \cdot \Delta\mathbf{v} d\mathbf{x} = - \int_D |\nabla\mathbf{v}|^2 d\mathbf{x}. \quad (2.4.5)$$

We denote by $\frac{D}{Dt} := \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$, the material derivative, i.e, the rate of change of a quantity following the smoothed fluid velocity $\mathbf{v}(\mathbf{x}, t)$. The vorticity equation (2.1.10) for the two dimensional α -Euler fluid, $\partial_t \omega(\mathbf{x}, t) + \mathbf{v} \cdot \nabla \omega = 0$ can be rewritten using the notation for the material derivative as $\frac{D}{Dt} \omega(\mathbf{x}, t) = 0$. It follows that the vorticity $\omega(\mathbf{x}, t)$ of individual fluid parcels is conserved. From this, we also observe that for any smooth function $C : \mathbb{R} \rightarrow \mathbb{R}$, we have,

$$\frac{D}{Dt} C(\omega(\mathbf{x}, t)) = (\partial_t + \mathbf{v} \cdot \nabla) C(\omega(\mathbf{x}, t)) = C'(\omega(\mathbf{x}, t)) (\partial_t + \mathbf{v} \cdot \nabla) \omega(\mathbf{x}, t) = 0. \quad (2.4.6)$$

i.e, any function $C(\omega(\mathbf{x}, t))$ of the vorticity is conserved along individual fluid trajectories. In what follows, we will need the Kelvin Circulation Theorem for the α -Euler equations. We recall that the Kelvin Circulation theorem for the Euler equations

(see, for example, [CM93], Section 1.2, pages 21-22) basically says that for any loop C_t that moves along the flow generated by the fluid \mathbf{u} , the circulation

$$\Gamma_{C_t}^e := \int_{C_t} \mathbf{u} \cdot d\mathbf{s},$$

remains constant in time. More precisely, we have the following: let C be a simple closed contour in the fluid at time $t = 0$. Let C_t be the contour carried along by the flow, i.e $C_t = \chi_t(C)$ where χ_t is the flow map associated with the fluid velocity \mathbf{u} , χ_t denotes the map $\mathbf{x} \mapsto \chi(\mathbf{x}, t)$ where the velocity field \mathbf{u} satisfies, $\frac{d}{dt}\chi(\mathbf{x}, t) = \mathbf{u}(\chi(\mathbf{x}, t), t)$. Then the circulation Γ_{C_t} defined above remains constant in time, i.e

$$\frac{d}{dt}\Gamma_{C_t}^e = 0.$$

In the case of the α -Euler equations we have the following modification of the Kelvin Circulation theorem. We define the circulation, as

$$\Gamma_{C_t} := \int_{C_t} (1 - \alpha^2 \Delta) \mathbf{v} \cdot d\mathbf{s},$$

where the loop C moves with the smoothed fluid velocity \mathbf{v} , i.e, χ_t is the flow map associated with the velocity \mathbf{v} , i.e the flow map χ_t denotes the map $\mathbf{x} \mapsto \chi(\mathbf{x}, t)$ where the velocity field \mathbf{v} satisfies, $\frac{d}{dt}\chi(\mathbf{x}, t) = \mathbf{v}(\chi(\mathbf{x}, t), t)$. We note that the flow map χ_t is a diffeomorphism, in particular, it maps simple closed curves C into simple closed curves C_t .

Lemma 2.47. (*Kelvin Circulation Theorem α -Euler*) *Let C be a simple closed contour in the fluid at time $t = 0$. Let C_t be the contour carried along by the flow following the smoothed velocity \mathbf{v} , i.e $C_t = \chi_t(C)$, where χ_t is the flow map associated with the velocity \mathbf{v} , where \mathbf{v} satisfies (2.1.4). The circulation around C_t is defined to*

be

$$\Gamma_{C_t} = \int_{C_t} (1 - \alpha^2 \Delta) \mathbf{v} \cdot d\mathbf{s}. \quad (2.4.7)$$

Then Γ_{C_t} is constant in time. That is,

$$\frac{d}{dt} \Gamma_{C_t} = 0.$$

Proof. We first note that $\mathbf{u} = (1 - \alpha^2 \Delta) \mathbf{v}$ and Γ_{C_t} is given by $\Gamma_{C_t} = \int_{C_t} \mathbf{u} \cdot d\mathbf{s}$. In order to prove this theorem, we first need to prove the following formula. We claim that,

$$\frac{d}{dt} \int_{C_t} \mathbf{u}(\mathbf{s}) \cdot d\mathbf{s} = \int_{C_t} \left(\frac{\partial}{\partial t} \mathbf{u}(\mathbf{s}) + \mathbf{v}(\mathbf{s}) \cdot \nabla \mathbf{u}(\mathbf{s}) + (\nabla \mathbf{v}(\mathbf{s}))^T \mathbf{u}(\mathbf{s}) \right) \cdot d\mathbf{s} \quad (2.4.8)$$

The proof of (2.4.8) is as follows. Let $\mathbf{x}(s)$ be a parametrization of the loop C , $0 \leq s \leq 1$. Then the corresponding parametrization of C_t is given by $\chi(\mathbf{x}(s), t)$, $0 \leq s \leq 1$. We thus have the following,

$$\begin{aligned} \frac{d}{dt} \int_{C_t} \mathbf{u}(\mathbf{s}) \cdot d\mathbf{s} &= \frac{d}{dt} \int_0^1 \mathbf{u}(\chi(\mathbf{x}(s), t), t) \cdot \frac{\partial}{\partial s} \chi(\mathbf{x}(s), t) ds \\ &= \int_0^1 \frac{\partial}{\partial t} \mathbf{u}(\chi(\mathbf{x}(s), t), t) \cdot \frac{\partial}{\partial s} \chi(\mathbf{x}(s), t) ds \\ &\quad + \int_0^1 \mathbf{u}(\chi(\mathbf{x}(s), t), t) \cdot \frac{\partial}{\partial t} \frac{\partial}{\partial s} \chi(\mathbf{x}(s), t) ds \\ &= \int_0^1 \left(\nabla \mathbf{u}(\chi(\mathbf{x}(s), t), t) \cdot \frac{\partial \chi(\mathbf{x}(s), t)}{\partial t} + \frac{\partial \mathbf{u}(\chi(\mathbf{x}(s), t), t)}{\partial t} \right) \cdot \frac{\partial}{\partial s} \chi(\mathbf{x}(s), t) ds \\ &\quad + \int_0^1 \mathbf{u}(\chi(\mathbf{x}(s), t), t) \cdot \frac{\partial}{\partial s} \frac{\partial}{\partial t} \chi(\mathbf{x}(s), t) ds \\ &= \int_0^1 \left(\nabla \mathbf{u}(\chi(\mathbf{x}(s), t), t) \cdot \mathbf{v}(\chi(\mathbf{x}(s), t), t) + \frac{\partial \mathbf{u}(\chi(\mathbf{x}(s), t), t)}{\partial t} \right) \cdot \frac{\partial}{\partial s} \chi(\mathbf{x}(s), t) ds \\ &\quad + \int_0^1 \mathbf{u}(\chi(\mathbf{x}(s), t), t) \cdot \frac{\partial}{\partial s} \mathbf{v}(\chi(\mathbf{x}(s), t), t) ds \\ &= \int_0^1 \left(\nabla \mathbf{u}(\chi(\mathbf{x}(s), t), t) \cdot \mathbf{v}(\chi(\mathbf{x}(s), t), t) + \frac{\partial \mathbf{u}(\chi(\mathbf{x}(s), t), t)}{\partial t} \right) \cdot \frac{\partial}{\partial s} \chi(\mathbf{x}(s), t) ds \\ &\quad + \int_0^1 \mathbf{u}(\chi(\mathbf{x}(s), t), t) \cdot \nabla \mathbf{v}(\chi(\mathbf{x}(s), t), t) \cdot \frac{\partial \chi(\mathbf{x}(s), t)}{\partial s} ds \end{aligned}$$

$$= \int_{C_t} \left(\frac{\partial}{\partial t} \mathbf{u}(\mathbf{s}) + \mathbf{v}(\mathbf{s}) \cdot \nabla \mathbf{u}(\mathbf{s}) \right) \cdot d\mathbf{s} + \int_{C_t} ((\nabla \mathbf{v}(\mathbf{s}))^T \mathbf{u}(\mathbf{s})) \cdot d\mathbf{s},$$

where we have used the facts that $\frac{\partial}{\partial t} \chi(\mathbf{x}(s), t) = \mathbf{v}(\mathbf{x}(s), t)$ and $\mathbf{u} \cdot \nabla \mathbf{v} = (\nabla \mathbf{v})^T \mathbf{u}$.

We thus have that,

$$\begin{aligned} \frac{d}{dt} \int_{C_t} \mathbf{u}(\mathbf{s}) \cdot d\mathbf{s} &= \int_{C_t} \left(\frac{\partial}{\partial t} \mathbf{u}(\mathbf{s}) + \mathbf{v}(\mathbf{s}) \cdot \nabla \mathbf{u}(\mathbf{s}) + (\nabla \mathbf{v}(\mathbf{s}))^T \mathbf{u}(\mathbf{s}) \right) \cdot d\mathbf{s} \\ &= - \int_{C_t} \nabla \pi(\mathbf{s}) \cdot d\mathbf{s} = 0, \end{aligned}$$

using (2.1.4) and the fundamental theorem of vector calculus since C_t is a simple closed loop. ■

Remark 2.48. One of the original motivations in adding the additional term $(\nabla \mathbf{v})^T \mathbf{u}$ to the α -Euler equations (2.1.4) was to prove the Kelvin Circulation Theorem above (see [FHT01], Section 2, page 507).

A useful corollary to the above theorem is the following. In a multi connected domain, the boundary curves get mapped to themselves by the flow map χ_t associated with velocity \mathbf{v} as a consequence of (2.1.5), because there is no normal flow across the boundaries (which means that the image of the boundary is a subset of the boundary) and by the fact that χ_t is a diffeomorphism (which implies that the image is onto: i.e, the boundary gets mapped into all of the boundary). From this it follows that the circulation along the boundary curves remains constant.

Corollary 2.49. *Suppose we have a multiply connected bounded domain $D \subset \mathbb{R}^2$, bounded by finitely many number of smooth curves $(\partial D)_i$ where $i = 0, \dots, n$. Then the circulation along each connected boundary curve remains constant in time, i.e,*

for every $0 \leq i \leq n$, we have

$$\frac{d}{dt} \int_{(\partial D)_i} (1 - \alpha^2 \Delta) \mathbf{v} \cdot d\mathbf{s} = \frac{d}{dt} \int_{(\partial D)_i} \mathbf{u} \cdot d\mathbf{s} = 0.$$

Let us now consider the following functionals $H, Q : X \rightarrow \mathbb{R}$ on X , given by

$$H(\mathbf{v}) = \frac{1}{2} \int_D \mathbf{v}(\mathbf{x}) \cdot (1 - \alpha^2 \Delta) \mathbf{v}(\mathbf{x}) d\mathbf{x}, \quad (2.4.9)$$

$$Q(\mathbf{v}) = \int_D C(\text{curl}(1 - \alpha^2 \Delta) \mathbf{v}(\mathbf{x})) d\mathbf{x}, \quad (2.4.10)$$

where $C : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function. Let us define the functional $H_c : X \rightarrow \mathbb{R}$ as follows,

$$H_c(\mathbf{v}) = H(\mathbf{v}) + Q(\mathbf{v}) + \sum_{i=0}^n a_i \int_{(\partial D)_i} (1 - \alpha^2 \Delta) \mathbf{v} \cdot d\mathbf{s}, \quad (2.4.11)$$

where $a_i \in \mathbb{R}$ for $i = 0, \dots, n$. We note that $H(\mathbf{v})$ is the kinetic energy of the α -Euler fluid, C is known as the Casimir function, the last term in (2.4.11) corresponds to the weighted sum of circulations along the boundary. The following lemma says that if $\mathbf{v}(t, \cdot)$ that solves the α -Euler equation (2.1.4), H_c is a temporal invariant of motion, i.e. $\frac{d}{dt} H_c(\mathbf{v}(t, \cdot)) = 0$.

Lemma 2.50. *Let \mathbf{v} solve the α -Euler equations (2.1.4). Then*

$$\begin{aligned} H_c(\mathbf{v}(t, \cdot)) &= \frac{1}{2} \int_D \mathbf{v}(t, \mathbf{x}) \cdot (1 - \alpha^2 \Delta) \mathbf{v}(t, \mathbf{x}) d\mathbf{x} + \int_D C(\text{curl}(1 - \alpha^2 \Delta) \mathbf{v}(t, \mathbf{x})) d\mathbf{x} \\ &\quad + \sum_{i=0}^n a_i \int_{(\partial D)_i} (1 - \alpha^2 \Delta) \mathbf{v}(\mathbf{s}, t) \cdot d\mathbf{s}, \end{aligned} \quad (2.4.12)$$

is an invariant of motion, that is, $\frac{d}{dt} H_c(\mathbf{v}(t, \cdot)) = 0$.

Proof. We note that

$$H_c(\mathbf{v}(t, \cdot)) = H(\mathbf{v}(t, \cdot)) + Q(\mathbf{v}(t, \cdot)) + \sum_{i=0}^n a_i \int_{(\partial D)_i} (1 - \alpha^2 \Delta) \mathbf{v}(\mathbf{s}, t) \cdot d\mathbf{s}.$$

Also, note that, $\mathbf{u} = (1 - \alpha^2 \Delta) \mathbf{v}$ and $\omega = \text{curl } \mathbf{u} = \text{curl}(1 - \alpha^2 \Delta) \mathbf{v}$.

We shall first show that $\frac{d}{dt} H(\mathbf{v}(t, \cdot)) = 0$. Note that

$$H(\mathbf{v}(t, \cdot)) = \frac{1}{2} \int_D \mathbf{v}(t, \mathbf{x}) \cdot (1 - \alpha^2 \Delta) \mathbf{v}(t, \mathbf{x}) d\mathbf{x} = \frac{1}{2} \int_D \mathbf{v}(t, \mathbf{x}) \cdot \mathbf{u}(t, \mathbf{x}) d\mathbf{x}.$$

We will now show that,

$$\frac{d}{dt} \frac{1}{2} \int_D \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x}, t) d\mathbf{x} = 0.$$

We start with the α -Euler equations (2.1.4),

$$\partial_t \mathbf{u} + \mathbf{v} \cdot \nabla \mathbf{u} + (\nabla \mathbf{v})^\top \mathbf{u} = -\nabla \pi. \quad (2.4.13)$$

We now use the following identity (see [HMR98], Equation 7.34)

$$(\mathbf{v} \cdot \nabla) \mathbf{u} + (\nabla \mathbf{v})^\top \mathbf{u} = -\mathbf{v} \times (\nabla \times \mathbf{u}) + \nabla(\mathbf{u} \cdot \mathbf{v}),$$

to rewrite (2.4.13) as

$$\partial_t \mathbf{u} - \mathbf{v} \times (\nabla \times \mathbf{u}) + \nabla(\mathbf{u} \cdot \mathbf{v}) = -\nabla \pi. \quad (2.4.14)$$

We take the dot product of (2.4.14) with \mathbf{v} , noting that $-\mathbf{v} \times (\nabla \times \mathbf{u}) \cdot \mathbf{v} = 0$, and integrate over the domain D to get,

$$\int_D \mathbf{u}_t(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x} = 0, \quad (2.4.15)$$

where we have used the fact that

$$\int_D \nabla(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{v} d\mathbf{x} = 0$$

and

$$\int_D \nabla \pi \cdot \mathbf{v} d\mathbf{x} = 0.$$

Indeed, for any scalar valued function f , we have that

$$(\nabla f) \cdot \mathbf{v} = \nabla \cdot (f\mathbf{v}) - f\nabla \cdot \mathbf{v} = \nabla \cdot (f\mathbf{v}), \quad (2.4.16)$$

(because \mathbf{v} is divergence free, $\nabla \cdot \mathbf{v} = 0$) and by Gauss' Divergence Theorem,

$$\int_D \nabla \cdot (f\mathbf{v})d\mathbf{x} = \sum_{i=0}^n \int_{(\partial D)_i} f\mathbf{v} \cdot \mathbf{n}ds = 0 \quad (2.4.17)$$

by (2.1.5).

We modify $H(\mathbf{v})$ as follows:

$$\begin{aligned} H(\mathbf{v}) &= \frac{1}{2} \int_D \mathbf{u} \cdot \mathbf{v}d\mathbf{x} = \frac{1}{2} \int_D (1 - \alpha^2 \Delta)\mathbf{v} \cdot \mathbf{v}d\mathbf{x} \\ &= \frac{1}{2} \int_D \mathbf{v} \cdot \mathbf{v}d\mathbf{x} - \frac{\alpha^2}{2} \int_D \Delta\mathbf{v} \cdot \mathbf{v}d\mathbf{x} \\ &= \frac{1}{2} \int_D \mathbf{v} \cdot \mathbf{v}d\mathbf{x} + \frac{\alpha^2}{2} \int_D \nabla\mathbf{v} \cdot \nabla\mathbf{v}d\mathbf{x} - \frac{\alpha^2}{2} \sum_{i=0}^n \int_{(\partial D)_i} \mathbf{v} \cdot (\mathbf{n} \cdot \nabla)\mathbf{v}ds \\ &= \frac{1}{2} \int_D \mathbf{v} \cdot \mathbf{v}d\mathbf{x} + \frac{\alpha^2}{2} \int_D \nabla\mathbf{v} \cdot \nabla\mathbf{v}d\mathbf{x} \end{aligned} \quad (2.4.18)$$

where the boundary terms vanish since $\mathbf{v} \cdot (\mathbf{n} \cdot \nabla)\mathbf{v} = 0$ via (2.1.5). We thus have

that, by (2.4.18),

$$\begin{aligned} \frac{d}{dt}H(\mathbf{v}(t, \cdot)) &= \frac{1}{2} \int_D (\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t))_t d\mathbf{x} + \frac{\alpha^2}{2} \int_D (\nabla\mathbf{v}(\mathbf{x}, t) \cdot \nabla\mathbf{v}(\mathbf{x}, t))_t d\mathbf{x} \\ &= \int_D \mathbf{v}_t(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) d\mathbf{x} + \alpha^2 \int_D \nabla\mathbf{v}_t(\mathbf{x}, t) \cdot \nabla\mathbf{v}(\mathbf{x}, t) d\mathbf{x} \\ &= \int_D \mathbf{v}_t(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) d\mathbf{x} - \alpha^2 \int_D \Delta\mathbf{v}_t(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) d\mathbf{x} \\ &\quad + \alpha^2 \sum_{i=0}^n \int_{(\partial D)_i} \mathbf{v}(s, t) \cdot (\mathbf{n}(s) \cdot \nabla)\mathbf{v}_t(s, t) ds \\ &= \int_D \mathbf{v}_t(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) d\mathbf{x} - \alpha^2 \int_D \Delta\mathbf{v}_t(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) d\mathbf{x} \\ &= \int_D (\mathbf{v}_t(\mathbf{x}, t) - \alpha^2 \Delta\mathbf{v}_t(\mathbf{x}, t)) \cdot \mathbf{v}(\mathbf{x}, t) d\mathbf{x} \\ &= \int_D \mathbf{u}_t(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) d\mathbf{x} = 0 \end{aligned}$$

by (2.4.15), where the boundary terms in the fourth equality disappear as we use the fact that $\mathbf{v} \cdot (\mathbf{n} \cdot \nabla) \mathbf{v}_t = 0$ on the boundary because of (2.1.5).

We thus have,

$$\frac{d}{dt} H(\mathbf{v}(t, \cdot)) = 0.$$

In order to prove that $\frac{d}{dt} \int_D C(\omega(\mathbf{x}, t)) dx = 0$, we use (a slight variant of) the following idea from [CSS06] (see Appendix, page 162). We first note that

$$\partial_t \omega(\mathbf{x}, t) + \mathbf{v} \cdot \nabla \omega(\mathbf{x}, t) = 0.$$

We have,

$$\partial_t C(\omega(\mathbf{x}, t)) = C'(\omega(\mathbf{x}, t)) \partial_t \omega(\mathbf{x}, t) = -C'(\omega(\mathbf{x}, t)) \mathbf{v} \cdot \nabla \omega(\mathbf{x}, t), \quad (2.4.19)$$

and also,

$$-\mathbf{v} \cdot \nabla (C(\omega(\mathbf{x}, t))) = -C'(\omega(\mathbf{x}, t)) \mathbf{v} \cdot \nabla \omega(\mathbf{x}, t).$$

Therefore,

$$\partial_t C(\omega(\mathbf{x}, t)) = -\mathbf{v} \cdot \nabla (C(\omega(\mathbf{x}, t))). \quad (2.4.20)$$

Using (2.4.16), (2.4.17), (2.4.19) and (2.4.20), we infer

$$\begin{aligned} \frac{d}{dt} \int_D C(\omega(\mathbf{x}, t)) dx &= \int_D \partial_t C(\omega(\mathbf{x}, t)) dx \\ &= \int_D -\mathbf{v} \cdot \nabla C(\omega(\mathbf{x}, t)) dx = 0. \end{aligned}$$

The fact that

$$\frac{d}{dt} \int_{(\partial D)_i} \mathbf{u} \cdot d\mathbf{s} = 0,$$

$i = 0, \dots, n$, is a consequence of Corollary 2.49. Combining these relations proves the lemma. ■

Let \mathbf{v}^0 denote a steady state solution of (2.1.11), ϕ^0 denote its stream function, so that $\mathbf{v}^0 = -\nabla^\perp \phi^0$ and ω^0 denote its vorticity, so that

$$\omega^0 = \text{curl}(1 - \alpha^2 \Delta) \mathbf{v}^0 = -\Delta(1 - \alpha^2 \Delta) \phi^0.$$

We also note that since $\nabla^\perp \phi^0 \cdot \nabla \omega^0 = 0$ by (2.1.11), we have that $\nabla \phi^0$ is parallel to $\nabla \omega^0$ and thus locally, ϕ^0 is a function of ω^0 . We shall impose the following global condition.

Assumption 2.51. Assume that there exists a differentiable function F defined on the closed interval $\left[\min_{(x,y) \in \bar{D}} \omega^0(x,y), \max_{(x,y) \in \bar{D}} \omega^0(x,y) \right]$, such that, $\phi^0(x,y) = F(\omega^0(x,y))$ for every $(x,y) \in D$.

In particular, we have that $\nabla \phi^0(x,y) = F'(\omega^0(x,y)) \nabla \omega^0(x,y)$. We thus have that,

$$\mathbf{v}^0 = -\nabla^\perp \phi^0 = -F'(\omega^0) \nabla^\perp \omega^0.$$

Remark 2.52. We note that sometimes, in the literature, see, for example, [MP94], (page 106, Remark 1) the function $-F'$ is denoted by the ratio of the vectors $\frac{\mathbf{v}^0}{\nabla^\perp \omega^0}$.

Notice that, F' , a priori, can have singularities at the critical points of ω^0 . The discussion following (3.1.1) showed that ϕ^0 is a constant on each connected component of the boundary. Assumption 2.51 then implies that $F(\omega^0)$ restricted to each connected piece $(\partial D)_i$, $0 \leq i \leq n$, is a constant.

Also, since $\mathbf{v}^0 \cdot \nabla \omega^0 = 0$ and \mathbf{v}^0 is tangent to the boundary, we have that $\nabla \omega^0$ is orthogonal to the boundary, which implies that ω^0 is constant on the boundary $(\partial D)_i$, $i = 0, \dots, n$, i.e., we shall denote by $\omega^0|_{(\partial D)_i}$ the value of ω^0 on the boundary $(\partial D)_i$. This also implies that $C'(\omega^0(x,y))$ is constant for all $(x,y) \in (\partial D)_i$ and we shall denote this by $C'(\omega^0)|_{(\partial D)_i}$.

We return to (2.4.11). We will now specify C and a_j , such that the first variation $\delta H_c(\mathbf{v}^0)\delta\mathbf{v} := \frac{d}{d\epsilon}H_c(\mathbf{v}^0 + \epsilon\delta\mathbf{v})|_{\epsilon=0}$ is zero.

Lemma 2.53. *Let \mathbf{v}^0, ω^0 be a steady state solution of (2.1.4), satisfying Assumption 2.51, where $\omega^0 = \text{curl}(1 - \alpha^2\Delta)\mathbf{v}^0$. Let C be a smooth function so that*

$$C'(\omega^0(x, y)) = -F(\omega^0(x, y)), \quad (2.4.21)$$

for every $(x, y) \in D$. Let $a_i = F(\omega^0)|_{(\partial D)_i}$, $i = 0, \dots, n$. Then $\delta H_c(\mathbf{v}^0)\delta\mathbf{v} = 0$, i.e \mathbf{v}^0 is a critical point of H_c .

Proof. Note first that $H_c(\mathbf{v})$ can be expressed, using $\mathbf{u} = (1 - \alpha^2\Delta)\mathbf{v}$, as

$$H_c(\mathbf{v}) = \frac{1}{2} \int_D \mathbf{v} \cdot \mathbf{u} d\mathbf{x} + \int_D C(\omega) d\mathbf{x} + \sum_{i=0}^n a_i \int_{(\partial D)_i} \mathbf{u} \cdot d\mathbf{s}. \quad (2.4.22)$$

The first variation of H_c at \mathbf{v}^0 is given by the following expression,

$$\begin{aligned} \delta H_c(\mathbf{v}^0)\delta\mathbf{v} &= \frac{d}{d\epsilon} H_c(\mathbf{v}^0 + \epsilon\delta\mathbf{v})|_{\epsilon=0} \\ &= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta\mathbf{u} d\mathbf{x} + \frac{1}{2} \int_D \mathbf{u}^0 \cdot \delta\mathbf{v} d\mathbf{x} + \int_D C'(\omega^0)\delta\omega d\mathbf{x} + \sum_{i=0}^n a_i \int_{(\partial D)_i} \delta\mathbf{u} \cdot d\mathbf{s}, \end{aligned} \quad (2.4.23)$$

where $\delta\mathbf{u} = (1 - \alpha^2\Delta)\delta\mathbf{v}$ and $\delta\omega = \text{curl}\delta\mathbf{u}$. We will be using the following identity (see [MP94], Eq 2.14, page 108),

$$C'(\omega^0)\delta\omega = \text{curl}(C'(\omega^0)\delta\mathbf{u}) - C''(\omega^0)\nabla^\perp\omega^0 \cdot \delta\mathbf{u}, \quad (2.4.24)$$

which follows from the identity $\text{curl}(f\mathbf{v}) = \nabla^\perp f \cdot \mathbf{v} + f \text{curl}\mathbf{v}$. Noting that, by Stokes' theorem,

$$\int_D \text{curl}(C'(\omega^0)\delta\mathbf{u}) d\mathbf{x} = \sum_{i=0}^n \int_{(\partial D)_i} C'(\omega^0)\delta\mathbf{u} \cdot d\mathbf{s}, \quad (2.4.25)$$

we see that, using (2.4.24) and (2.4.25)

$$\int_D C'(\omega^0)\delta\omega d\mathbf{x} = - \int_D C''(\omega^0)\nabla^\perp\omega^0 \cdot \delta\mathbf{u}d\mathbf{x} + \sum_{i=0}^n \int_{(\partial D)_i} C'(\omega^0)\delta\mathbf{u} \cdot d\mathbf{s}. \quad (2.4.26)$$

In the following computation, we use the integration by parts formula, (2.4.3), in the second and fourth line, and the observation that, by the boundary conditions, (2.1.5), we have

$$\int_{(\partial D)_i} \delta\mathbf{v} \cdot (\mathbf{n} \cdot \nabla\mathbf{v}^0)ds = 0, \text{ and } \int_{(\partial D)_i} \mathbf{v}^0 \cdot (\mathbf{n} \cdot \nabla\delta\mathbf{v})ds = 0,$$

$i = 0, \dots, n$ because, on a boundary curve $(\partial D)_i$, $\delta\mathbf{v} \cdot (\mathbf{n} \cdot \nabla\mathbf{v}^0) = 0$ and $\mathbf{v}^0 \cdot (\mathbf{n} \cdot \nabla\delta\mathbf{v}) = 0$

for all $\delta\mathbf{v} \in X$. Thus,

$$\begin{aligned} \frac{1}{2} \int_D \mathbf{u}^0 \cdot \delta\mathbf{v}d\mathbf{x} &= \frac{1}{2} \int_D (\mathbf{v}^0 - \alpha^2\Delta\mathbf{v}^0) \cdot \delta\mathbf{v}d\mathbf{x} \\ &= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta\mathbf{v}d\mathbf{x} - \frac{\alpha^2}{2} \int_D \Delta\mathbf{v}^0 \cdot \delta\mathbf{v}d\mathbf{x} \\ &= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta\mathbf{v}d\mathbf{x} + \frac{\alpha^2}{2} \int_D \nabla\mathbf{v}^0 \cdot \nabla\delta\mathbf{v}d\mathbf{x} - \frac{\alpha^2}{2} \sum_{i=0}^n \int_{(\partial D)_i} \delta\mathbf{v} \cdot (\mathbf{n} \cdot \nabla\mathbf{v}^0)ds \\ &= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta\mathbf{v}d\mathbf{x} + \frac{\alpha^2}{2} \int_D \nabla\mathbf{v}^0 \cdot \nabla\delta\mathbf{v}d\mathbf{x} \\ &= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta\mathbf{v}d\mathbf{x} - \frac{\alpha^2}{2} \int_D \mathbf{v}^0 \cdot \Delta\delta\mathbf{v}d\mathbf{x} + \sum_{i=0}^n \frac{\alpha^2}{2} \int_{(\partial D)_i} \mathbf{v}^0 \cdot (\mathbf{n} \cdot \nabla\delta\mathbf{v})ds \\ &= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta\mathbf{v}d\mathbf{x} - \frac{\alpha^2}{2} \int_D \mathbf{v}^0 \cdot \Delta\delta\mathbf{v}d\mathbf{x} \\ &= \frac{1}{2} \int_D \mathbf{v}^0 \cdot (\delta\mathbf{v} - \alpha^2\Delta\delta\mathbf{v})d\mathbf{x} = \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta\mathbf{u}d\mathbf{x}. \end{aligned} \quad (2.4.27)$$

Using (2.4.27) and (2.4.26), we see that (2.4.23) is given by,

$$\begin{aligned} \delta H_c(\mathbf{v}^0)\delta\mathbf{v} &= \int_D \mathbf{v}^0 \cdot \delta\mathbf{u}d\mathbf{x} - \int_D C''(\omega^0)\nabla^\perp(\omega^0)\delta\mathbf{u}d\mathbf{x} \\ &\quad + \sum_{i=0}^n \int_{(\partial D)_i} C'(\omega^0)\delta\mathbf{u} \cdot d\mathbf{s} + \sum_{i=0}^n a_i \int_{(\partial D)_i} \delta\mathbf{u} \cdot d\mathbf{s}. \end{aligned} \quad (2.4.28)$$

Since $\mathbf{v}^0 \cdot \nabla\omega^0 = 0$, and \mathbf{v}^0 is tangent to the boundary, this means that $\nabla\omega^0$ is orthogonal to the boundary and thus ω^0 is a constant on the boundary. This then

implies that,

$$\begin{aligned} \delta H_c(\mathbf{v}^0)\delta\mathbf{v} &= \int_D \mathbf{v}^0 \cdot \delta\mathbf{u}d\mathbf{x} - \int_D C''(\omega^0)\nabla^\perp(\omega^0)\delta\mathbf{u}d\mathbf{x} \\ &\quad + \sum_{i=0}^n C'(\omega^0)|_{(\partial D)_i} \int_{(\partial D)_i} \delta\mathbf{u} \cdot d\mathbf{s} + \sum_{i=0}^n a_i \int_{(\partial D)_i} \delta\mathbf{u} \cdot d\mathbf{s}, \end{aligned} \quad (2.4.29)$$

from which we see that $\delta H_c(\mathbf{v}^0)\delta\mathbf{v} = 0$ provided,

$$\mathbf{v}^0(x, y) = C''(\omega^0(x, y))\nabla^\perp\omega^0(x, y), \quad (2.4.30)$$

and

$$a_i = -C'(\omega^0), \quad i = 0, \dots, n. \quad (2.4.31)$$

Since, by (2.4.21), C is chosen such that $C'(\omega^0(x, y)) = -F(\omega^0(x, y))$ for every $(x, y) \in D$, then,

$$\mathbf{v}^0 = -\nabla^\perp\phi^0 = -F'(\omega^0)\nabla^\perp\omega^0 = C''(\omega^0)\nabla^\perp\omega^0,$$

i.e., (2.4.30) holds. Since we have chosen $a_i = \psi^0|_{(\partial D)_i}$ for all $0 \leq i \leq n$, (note that ψ^0 is a constant on the boundary curves) which then implies that $a_i = -C'(\omega^0)|_{(\partial D)_i}$, then the first variation $\delta H_c(\mathbf{v}^0)\delta\mathbf{v} = 0$.

■

We are now ready to prove Arnold's first stability theorem for α -Euler.

Theorem 2.54. *Let \mathbf{v}^0 be a steady state solution of the α -Euler equations (2.1.4) on the multi connected domain D , satisfying Assumption 2.51. Suppose that*

$$0 < \inf_{(x,y) \in D} (-F'(\omega^0(x, y))) \leq \sup_{(x,y) \in D} (-F'(\omega^0(x, y))) < +\infty. \quad (2.4.32)$$

Then there exists a constant $K > 0$, such that if $\mathbf{v}(\cdot, t) = \mathbf{v}^0 + \delta\mathbf{v}(\cdot, t)$, $t \in I$ solves the α -Euler equations (2.1.4) on D then one has the following estimate for all times $t \in I$,

$$\begin{aligned} & \|\mathbf{v}(\cdot, t) - \mathbf{v}^0\|_2^2 + \alpha^2 \|\nabla(\mathbf{v}(\cdot, t) - \mathbf{v}^0)\|_2^2 + \|\omega(\cdot, t) - \omega^0\|_2^2 \\ & \leq K(\|\mathbf{v}(\cdot, 0) - \mathbf{v}^0\|_2^2 + \alpha^2 \|\nabla(\mathbf{v}(\cdot, 0) - \mathbf{v}^0)\|_2^2 + \|\omega(\cdot, 0) - \omega^0\|_2^2), \end{aligned} \quad (2.4.33)$$

where $\omega^0 = \text{curl}(1 - \alpha^2 \Delta)\mathbf{v}^0$ and $\omega(\cdot, t) = \text{curl}(1 - \alpha^2 \Delta)\mathbf{v}(\cdot, t)$.

Proof. Let $K_1 := \inf_{(x,y) \in D}(-F'(\omega^0(x, y)))$ and $K_2 := \sup_{(x,y) \in D}(-F'(\omega^0(x, y)))$. Let H_c be defined as in (2.4.11), and choose C and a_i as in Lemma 2.53. Since the range of ω^0 is a connected set, (2.4.32) is equivalent to,

$$K_1 \leq -F'(\xi) \leq K_2, \quad (2.4.34)$$

for $\xi \in [\min_{(x,y) \in \bar{D}} \omega^0(x, y), \max_{(x,y) \in \bar{D}} \omega^0(x, y)]$. Using (2.4.32), we may extend C from the range of ω^0 to \mathbb{R} such that,

$$K_1 \leq C''(z) \leq K_2, \quad (2.4.35)$$

holds for every $z \in \mathbb{R}$. Indeed, we first extend F linearly outside

$$[\min_{(x,y) \in \bar{D}} \omega^0(x, y), \max_{(x,y) \in \bar{D}} \omega^0(x, y)]$$

to all of \mathbb{R} . We then choose C such that $C'(\xi) = -F(\xi)$ for all $\xi \in \mathbb{R}$.

By Lemma 2.50, H_c is an invariant of motion and by Lemma 2.53, \mathbf{v}^0 is a critical point of H_c , i.e., $\delta H_c(\mathbf{v}^0)\delta\mathbf{v} = 0$. We first observe the following,

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} - \mathbf{u}^0 \cdot \mathbf{v}^0 &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u}^0 \cdot \mathbf{v} + \mathbf{u}^0 \cdot \mathbf{v} - \mathbf{u}^0 \cdot \mathbf{v}^0 = (\mathbf{u} - \mathbf{u}^0) \cdot \mathbf{v} + \mathbf{u}^0 \cdot (\mathbf{v} - \mathbf{v}^0) \\ &= (\mathbf{u} - \mathbf{u}^0) \cdot (\mathbf{v} - \mathbf{v}^0 + \mathbf{v}^0) + \mathbf{u}^0 \cdot (\mathbf{v} - \mathbf{v}^0) \end{aligned}$$

$$= (\mathbf{u} - \mathbf{u}^0) \cdot (\mathbf{v} - \mathbf{v}^0) + \mathbf{v}^0 \cdot (\mathbf{u} - \mathbf{u}^0) + \mathbf{u}^0 \cdot (\mathbf{v} - \mathbf{v}^0).$$

Using this formula, (2.4.11) gives

$$\begin{aligned} H_c(\mathbf{v}) - H_c(\mathbf{v}^0) &= \frac{1}{2} \int_D \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x} - \frac{1}{2} \int_D \mathbf{u}^0(\mathbf{x}) \cdot \mathbf{v}^0(\mathbf{x}) d\mathbf{x} \\ &\quad + \int_D (C(\omega(\mathbf{x})) - C(\omega^0(\mathbf{x}))) d\mathbf{x} \\ &\quad + \sum_{i=0}^n a_i \int_{(\partial D)_i} (\mathbf{u}(\mathbf{s}) - \mathbf{u}^0(\mathbf{s})) \cdot d\mathbf{s} \\ &= \frac{1}{2} \int_D (\mathbf{u}(\mathbf{x}) - \mathbf{u}^0(\mathbf{x})) \cdot (\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) d\mathbf{x} \\ &\quad + \frac{1}{2} \int_D (\mathbf{v}^0(\mathbf{x}) \cdot (\mathbf{u}(\mathbf{x}) - \mathbf{u}^0(\mathbf{x})) + \mathbf{u}^0(\mathbf{x}) \cdot (\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x}))) d\mathbf{x} \\ &\quad + \int_D C'(\omega^0(\mathbf{x}))(\omega(\mathbf{x}) - \omega^0(\mathbf{x})) d\mathbf{x} \\ &\quad + \frac{1}{2} \int_D C''(\xi)(\omega(\mathbf{x}) - \omega^0(\mathbf{x}))^2 d\mathbf{x} \\ &\quad + \sum_{i=0}^n a_i \int_{(\partial D)_i} (\mathbf{u}(\mathbf{s}) - \mathbf{u}^0(\mathbf{s})) \cdot d\mathbf{s}, \end{aligned}$$

where we have Taylor expanded $C(\omega(\mathbf{x})) - C(\omega^0(\mathbf{x}))$ and ξ depends on $\omega^0(\mathbf{x})$ and $\omega(\mathbf{x})$. By virtue of the fact that the first variation is zero at \mathbf{v}^0 , using (2.4.28), with $\delta\mathbf{v} = \mathbf{v} - \mathbf{v}^0$ and $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}^0$ we have that,

$$\begin{aligned} &\frac{1}{2} \int_D \left(\mathbf{v}^0(\mathbf{x}) \cdot (\mathbf{u}(\mathbf{x}) - \mathbf{u}^0(\mathbf{x})) + \mathbf{u}^0(\mathbf{x}) \cdot (\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) \right) d\mathbf{x} \\ &\quad + \int_D C'(\omega^0(\mathbf{x}))(\omega(\mathbf{x}) - \omega^0(\mathbf{x})) d\mathbf{x} + \sum_{i=0}^n a_i \int_{(\partial D)_i} (\mathbf{u}(\mathbf{s}) - \mathbf{u}^0(\mathbf{s})) \cdot d\mathbf{s} = 0, \end{aligned}$$

whence,

$$\begin{aligned} H_c(\mathbf{v}) - H_c(\mathbf{v}^0) &= \frac{1}{2} \int_D (\mathbf{u}(\mathbf{x}) - \mathbf{u}^0(\mathbf{x})) \cdot (\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) d\mathbf{x} \\ &\quad + \frac{1}{2} \int_D C''(\xi)(\omega(\mathbf{x}) - \omega^0(\mathbf{x}))^2 d\mathbf{x}. \end{aligned} \tag{2.4.36}$$

Using the fact that $\mathbf{u} - \mathbf{u}^0 = (1 - \alpha^2 \Delta)(\mathbf{v} - \mathbf{v}^0)$ and integrating the second term by

parts,

$$\int_D \Delta(\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) \cdot (\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) d\mathbf{x} = - \int_D \nabla(\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) \cdot \nabla(\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) d\mathbf{x},$$

where the boundary term vanishes as a result of $(\mathbf{v} - \mathbf{v}^0) \cdot (\mathbf{n} \cdot \nabla(\mathbf{v} - \mathbf{v}^0)) = 0$ on the boundary by (2.1.5) we see that,

$$\begin{aligned} H_c(\mathbf{v}) - H_c(\mathbf{v}^0) &= \frac{1}{2} \int_D (\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) \cdot (\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) d\mathbf{x} \\ &+ \frac{1}{2} \int_D \alpha^2 \nabla(\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) \cdot \nabla(\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})) d\mathbf{x} \\ &+ \frac{1}{2} \int_D C''(\xi) (\omega(\mathbf{x}) - \omega^0(\mathbf{x}))^2 d\mathbf{x}. \end{aligned} \quad (2.4.37)$$

Since (2.4.35) holds, we have that,

$$K_1 \leq C''(\xi) \leq K_2,$$

for every $\xi \in \mathbb{R}$. Thus,

$$\begin{aligned} &\frac{1}{2} \int_D |\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})|^2 d\mathbf{x} + \frac{\alpha^2}{2} \int_D |\nabla(\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x}))|^2 d\mathbf{x} + \\ &\frac{K_1}{2} \int_D (\omega(\mathbf{x}) - \omega^0(\mathbf{x}))^2 d\mathbf{x} \leq H_c(\mathbf{v}) - H_c(\mathbf{v}^0) \leq \frac{1}{2} \int_D |\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x})|^2 d\mathbf{x} \\ &+ \frac{\alpha^2}{2} \int_D |\nabla(\mathbf{v}(\mathbf{x}) - \mathbf{v}^0(\mathbf{x}))|^2 d\mathbf{x} + \frac{K_2}{2} \int_D (\omega(\mathbf{x}) - \omega^0(\mathbf{x}))^2 d\mathbf{x}. \end{aligned}$$

If we now let $\beta_1 = \min(\frac{1}{2}, \frac{K_1}{2})$ and $\beta_2 = \max(\frac{1}{2}, \frac{K_2}{2})$ we see that,

$$\begin{aligned} &\beta_1 (\|\mathbf{v} - \mathbf{v}^0\|_2^2 + \alpha^2 \|\nabla(\mathbf{v} - \mathbf{v}^0)\|_2^2 + \|\omega - \omega^0\|_2^2) \leq H_c(\mathbf{v}) - H_c(\mathbf{v}^0) \\ &\leq \beta_2 (\|\mathbf{v} - \mathbf{v}^0\|_2^2 + \alpha^2 \|\nabla(\mathbf{v} - \mathbf{v}^0)\|_2^2 + \|\omega - \omega^0\|_2^2). \end{aligned} \quad (2.4.38)$$

We now use Lemma 2.50, (the fact that the Hamiltonian is a temporal invariant of the motion) to get, for any time t ,

$$(\|\mathbf{v}(t) - \mathbf{v}^0\|_2^2 + \alpha^2 \|\nabla(\mathbf{v}(t) - \mathbf{v}^0)\|_2^2 + \|\omega(t) - \omega^0\|_2^2)$$

$$\begin{aligned}
&\leq \beta_1^{-1}(H_c(\mathbf{v}(t)) - H_c(\mathbf{v}^0)) = \beta_1^{-1}(H_c(\mathbf{v}(0)) - H_c(\mathbf{v}^0)) \\
&\leq \beta_2\beta_1^{-1}(\|\mathbf{v}(0) - \mathbf{v}^0\|_2^2 + \alpha^2\|\nabla(\mathbf{v}(0) - \mathbf{v}^0)\|_2^2 + \|\omega(0) - \omega^0\|_2^2).
\end{aligned}$$

From this, (2.4.33) follows by putting $K = \beta_2\beta_1^{-1}$. ■

We now compute the second variation of H_c , where $\delta\mathbf{u} = (1 - \alpha^2\Delta)\delta\mathbf{v}$ and $\delta\omega = \text{curl}(1 - \alpha^2\Delta)\delta\mathbf{v}$,

$$\begin{aligned}
\delta^2 H_c(\mathbf{v}^0)(\delta\mathbf{v}, \delta\mathbf{v}) &:= \frac{d^2}{d\varepsilon^2} H(\mathbf{v}^0 + \varepsilon\delta\mathbf{v}) \Big|_{\varepsilon=0} \\
&= \frac{1}{2} \int_D \delta\mathbf{v} \cdot \delta\mathbf{u} d\mathbf{x} + \frac{1}{2} \int_D \delta\mathbf{u} \cdot \delta\mathbf{v} d\mathbf{x} + \int_D C''(\omega^0) \delta\omega \delta\omega d\mathbf{x} \\
&= \int_D \delta\mathbf{v} \cdot \delta\mathbf{u} d\mathbf{x} + \int_D C''(\omega^0) \delta\omega \delta\omega d\mathbf{x} \\
&= \int_D (\delta\mathbf{v}) \cdot (\delta\mathbf{v} - \alpha^2\Delta\mathbf{v}) d\mathbf{x} + \int_D C''(\omega^0) \delta\omega \delta\omega d\mathbf{x} \\
&= \int_D (\delta\mathbf{v}) \cdot \delta\mathbf{v} d\mathbf{x} + \alpha^2 \int_D \nabla\delta\mathbf{v} \cdot \nabla\delta\mathbf{v} d\mathbf{x} \\
&\quad + \int_D C''(\omega^0) \delta\omega \delta\omega d\mathbf{x}, \tag{2.4.39}
\end{aligned}$$

where, in the second term of the fourth line, we have used the following integration by parts formula, following (2.4.3),

$$\begin{aligned}
\int_D \delta\mathbf{v} \cdot \Delta\delta\mathbf{v} d\mathbf{x} &= - \int_D \nabla\delta\mathbf{v} \cdot \nabla\delta\mathbf{v} d\mathbf{x} \\
&\quad + \sum_{i=0}^n \int_{(\partial D)_i} \delta\mathbf{v} \cdot (\mathbf{n} \cdot \nabla\delta\mathbf{v}) ds = - \int_D \nabla\delta\mathbf{v} \cdot \nabla\delta\mathbf{v} d\mathbf{x},
\end{aligned}$$

and the fact that $\int_{(\partial D)_i} \delta\mathbf{v} \cdot (\mathbf{n} \cdot \nabla\delta\mathbf{v}) ds = 0$ because of $\delta\mathbf{v} \cdot \mathbf{n} \cdot \nabla\delta\mathbf{v} = 0$ on $(\partial D)_i$ by (2.1.5).

Remark 2.55. We note that the second variation defines the following quadratic form $K(\mathbf{v}^0)$ on X .

$$K(\mathbf{v}^0)(\mathbf{v}, \mathbf{v}) := D^2 H_c(\mathbf{v}^0)(\mathbf{v}, \mathbf{v}) = \int_D \mathbf{v} \cdot \mathbf{v} d\mathbf{x} \tag{2.4.40}$$

$$+ \alpha^2 \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x} + \int_D C''(\omega^0) (\operatorname{curl}(1 - \alpha^2 \Delta) \mathbf{v})^2 d\mathbf{x},$$

for every $\mathbf{v} \in X$. Under assumption (2.4.32), the second variation defined by (2.4.40) is bounded and positive definite on the space X . Note that if there exists a $\mathbf{v} \neq 0 \in X$ such that $\operatorname{curl}(1 - \alpha^2 \Delta) \mathbf{v} = 0$, then the value of the quadratic form reduces to $\int_D \mathbf{v} \cdot \mathbf{v} d\mathbf{x} + \alpha^2 \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x}$ and this cannot be negative definite. In the proof of Arnold's second theorem given below, we will require the quadratic form defined by (2.4.40) to be negative definite. We would like to restrict the perturbations $\delta \mathbf{v}$ to a subspace of X such that the operator $\operatorname{curl}(1 - \alpha^2 \Delta)$ is one to one and thus the quadratic form (2.4.40) can be negative definite under appropriate assumptions on C'' . We thus restrict the perturbation stream function to the following subspace,

$$Y_\alpha = \left\{ \phi : H^4(D; \mathbb{R}) : \phi|_{(\partial D)_0} = 0; \int_{(\partial D)_i} -\nabla^\perp(1 - \alpha^2 \Delta) \phi \cdot d\mathbf{s} = 0, 1 \leq i \leq n; \right. \\ \left. (\mathbf{n} \cdot \nabla)(\nabla^\perp \phi) \parallel \mathbf{n} \text{ on } \partial D; \phi|_{(\partial D)_i} \text{ is constant}, 1 \leq i \leq n \right\}. \quad (2.4.41)$$

We note that we do not specify the exact values of the constant that ϕ takes along the inner boundary curves. We also choose for the velocity perturbations the subspace of X given by the formulas

$$X_\alpha := \left\{ \mathbf{v} \in H^3(D; \mathbb{R}^2), \operatorname{div} \mathbf{v} = 0 \text{ in } D, \int_{(\partial D)_i} (1 - \alpha^2 \Delta) \mathbf{v} \cdot d\mathbf{s} = 0, 1 \leq i \leq n, \right. \\ \left. \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial D, (\mathbf{n} \cdot \nabla) \mathbf{v} \parallel \mathbf{n} \text{ on } \partial D \right\}.$$

Lemma 2.56. *The operator $-\nabla^\perp : Y_\alpha \rightarrow X_\alpha$ is bijective. That is, given $\mathbf{v} \in X_\alpha$, there exists a unique $\phi \in Y_\alpha$ such that $\mathbf{v} = -\nabla^\perp \phi$.*

Proof. The proof is exactly the same as the proof of Lemma 2.21 and is thus omitted.

■

Lemma 2.57. *The operator $-\Delta(1 - \alpha^2\Delta) : Y_\alpha \rightarrow L_2$ is one to one. That is, if $-\Delta(1 - \alpha^2\Delta)\phi = 0$, for some $\phi \in Y_\alpha$, then, $\phi = 0$.*

Proof. Note that ϕ satisfies,

$$-\Delta(1 - \alpha^2\Delta)\phi = 0, \quad (2.4.42)$$

$$\phi(x, y)|_{(\partial D)_0} = 0, \quad (2.4.43)$$

$$\phi|_{(\partial D)_i}(x, y) = c_i, \text{ for } 1 \leq i \leq n, \quad (2.4.44)$$

$$\int_{(\partial D)_i} -\nabla^\perp(1 - \alpha^2\Delta)\phi(s) \cdot ds = 0 \text{ for } 1 \leq i \leq n, \quad (2.4.45)$$

$$(\mathbf{n} \cdot \nabla)\nabla^\perp\phi \cdot \mathbf{t} = 0. \quad (2.4.46)$$

Multiply (2.4.42) by ϕ and integrate over the domain to get

$$\begin{aligned} 0 &= \int_D \phi(-\Delta(1 - \alpha^2\Delta)\phi)d\mathbf{x} = -\sum_{i=0}^n \int_{(\partial D)_i} \phi \mathbf{n} \cdot (1 - \alpha^2\Delta)\nabla\phi ds \\ &\quad + \int_D \nabla\phi \cdot (1 - \alpha^2\Delta)\nabla\phi d\mathbf{x} \\ &= \sum_{i=0}^n \phi|_{(\partial D)_i} \int_{(\partial D)_i} (1 - \alpha^2\Delta)\nabla^\perp\phi \cdot ds + \int_D \nabla\phi \cdot (1 - \alpha^2\Delta)\nabla\phi d\mathbf{x} \\ &= \int_D \nabla\phi \cdot (1 - \alpha^2\Delta)\nabla\phi d\mathbf{x}, \end{aligned} \quad (2.4.47)$$

where we have used Green's formula and (2.4.43), (2.4.44) and (2.4.45) and we have used the fact that $\mathbf{n} \cdot \nabla\phi = -\mathbf{t} \cdot \nabla^\perp\phi$. But,

$$\int_D \nabla\phi \cdot (1 - \alpha^2\Delta)\nabla\phi d\mathbf{x} = \int_D \nabla\phi \cdot \nabla\phi d\mathbf{x} - \alpha^2 \int_D \nabla\phi \cdot \Delta\nabla\phi d\mathbf{x}.$$

By (2.4.5), we have that

$$\int_D \nabla\phi \cdot \Delta\nabla\phi d\mathbf{x} = \int_D -\nabla^\perp\phi \cdot \Delta(-\nabla^\perp\phi)d\mathbf{x} = \int_D \mathbf{v} \cdot \Delta\mathbf{v}d\mathbf{x} = -\int_D \nabla\mathbf{v} \cdot \nabla\mathbf{v}d\mathbf{x},$$

where $\mathbf{v} \in X_\alpha$ is the unique solution to $\mathbf{v} = -\nabla^\perp\phi$, via Lemma 2.56. Thus, we see

that,

$$\begin{aligned} \int_D \nabla \phi \cdot (1 - \alpha^2 \Delta) \nabla \phi d\mathbf{x} &= \int_D \nabla \phi \cdot \nabla \phi d\mathbf{x} + \alpha^2 \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x} \\ &= \int_D \mathbf{v} \cdot \mathbf{v} d\mathbf{x} + \alpha^2 \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x} = 0, \end{aligned}$$

from which we conclude that $\mathbf{v} = 0$.

From this it follows by Lemma 2.56 that $-\nabla^\perp \phi = 0$ on D and hence $\nabla \phi = 0$.

Then ϕ is a constant, and is equal to 0 by (2.4.43). ■

We thus have the following Lemma.

Lemma 2.58. *The operator $\text{curl}(1 - \alpha^2 \Delta) : X_\alpha \rightarrow L_2$ is one to one. That is, if $\text{curl}(1 - \alpha^2 \Delta)\mathbf{v} = 0$, for some $\mathbf{v} \in X_\alpha$, then $\mathbf{v} = 0$.*

Proof. Let $\mathbf{v} \in X_\alpha$ be such that $\text{curl}(1 - \alpha^2 \Delta)\mathbf{v} = 0$. By Lemma 2.56, we have that there exists a $\phi \in Y_\alpha$ such that $\mathbf{v} = -\nabla^\perp \phi$. Then $\text{curl}(1 - \alpha^2 \Delta)\mathbf{v} = \text{curl}(1 - \alpha^2 \Delta)(-\nabla^\perp \phi) = -\Delta(1 - \alpha^2 \Delta)\phi = 0$. By Lemma 2.57, we have that $\phi = 0$. Thus $\mathbf{u} = -\nabla^\perp \phi = 0$. ■

Remark 2.59. We will now apply formula (2.3.32) for the following situation. Let $A = -\Delta(1 - \alpha^2 \Delta)$ with the domain $\text{dom } A = Y_\alpha \subset L^2(D)$, see (2.4.41) for the definition of Y_α . We note that $Y_\alpha \subset L^2(D)$ is compactly embedded in $L^2(D)$ by the standard Sobolev embedding. Due to the choice of the boundary conditions, calculations in the proof of Lemma 2.56 yield:

$$\langle A\phi, \phi \rangle_{L^2} = \langle \phi, -\Delta(1 - \alpha^2 \Delta)\phi \rangle_{L^2} = \|\mathbf{v}\|_2^2 + \alpha^2 \|\nabla \mathbf{v}\|_2^2 \quad (2.4.48)$$

for all $\phi \in Y_\alpha$ and $\mathbf{v} = -\nabla^\perp \phi \in X_\alpha$. In particular, the operator A is positive. On the other hand, the formula,

$$\text{curl}(1 - \alpha^2 \Delta)\mathbf{v} = -\Delta(1 - \alpha^2 \Delta)\phi \quad (2.4.49)$$

yields

$$\|A\phi\|_2^2 = \|-\Delta(1 - \alpha^2\Delta)\phi\|_2^2 = \|\operatorname{curl}(1 - \alpha^2\Delta)\mathbf{v}\|_2^2. \quad (2.4.50)$$

Combining (2.3.32), (2.4.48), (2.4.50), we obtain the following analog of the Rayleigh-Ritz formula for the α -Euler equation:

$$\lambda_{min,\alpha} = \min_{0 \neq \mathbf{v} \in X_\alpha} \frac{\|\operatorname{curl}(1 - \alpha^2\Delta)\mathbf{v}\|_2^2}{\|\mathbf{v}\|_2^2 + \alpha^2\|\nabla\mathbf{v}\|_2^2}, \quad (2.4.51)$$

where $\lambda_{min,\alpha}$ is the minimum eigenvalue of the operator $-\Delta(1 - \alpha^2\Delta)$ with the domain $\operatorname{dom} A = Y_\alpha \subset L^2(D)$.

By equation (2.4.51), for every $\mathbf{v} \in X_\alpha$ we have,

$$\lambda_{min,\alpha} \left(\int_D \mathbf{v} \cdot \mathbf{v} d\mathbf{x} + \alpha^2 \int_D \nabla\mathbf{v} \cdot \nabla\mathbf{v} d\mathbf{x} \right) \leq \int_D (\operatorname{curl}(1 - \alpha^2\Delta)\mathbf{v})^2 d\mathbf{x}. \quad (2.4.52)$$

We shall now prove Arnold's second stability theorem for α -Euler.

Theorem 2.60. *Let \mathbf{v}^0 be a steady state solution of the α -Euler equations (2.1.4) on the multi connected domain D , satisfying Assumption 2.51. Let $\lambda_{min,\alpha} > 0$ be the minimum eigenvalue of the operator $-\Delta(1 - \alpha^2\Delta) : L^2(D) \rightarrow L^2(D)$ with the domain $\operatorname{dom}(-\Delta(1 - \alpha^2\Delta)) = Y_\alpha$. Suppose,*

$$0 < \frac{1}{\lambda_{min,\alpha}} < \inf_{(x,y) \in D} F'(\omega^0(x,y)) \leq \sup_{(x,y) \in D} F'(\omega^0(x,y)) < +\infty. \quad (2.4.53)$$

Then there exists a constant $K > 0$, such that if $\mathbf{v}(\cdot, t) = \mathbf{v}^0 + \delta\mathbf{v}(\cdot, t)$, $t \in I$ solves the α -Euler equations (2.1.4) on D , with $\delta\mathbf{v} \in X_\alpha$ then one has the following estimate for all times $t \in I$,

$$\begin{aligned} & \|\mathbf{v}(\cdot, t) - \mathbf{v}^0\|_2^2 + \alpha^2\|\nabla(\mathbf{v}(\cdot, t) - \mathbf{v}^0)\|_2^2 + \|\omega(\cdot, t) - \omega^0\|_2^2 \\ & \leq K(\|\mathbf{v}(\cdot, 0) - \mathbf{v}^0\|_2^2 + \alpha^2\|\nabla(\mathbf{v}(\cdot, 0) - \mathbf{v}^0)\|_2^2 + \|\omega(\cdot, 0) - \omega^0\|_2^2), \end{aligned}$$

where $\omega^0 = \operatorname{curl}(1 - \alpha^2\Delta)\mathbf{v}^0$ and $\omega(\cdot, t) = \operatorname{curl}(1 - \alpha^2\Delta)\mathbf{v}(\cdot, t)$.

Proof. Let $K_1 := \inf_{(x,y) \in D} F'(\omega^0(x,y))$ and $K_2 := \sup_{(x,y) \in D} F'(\omega^0(x,y))$. From (2.4.53), we see that,

$$K_1 \leq F'(\omega^0(x,y)) \leq K_2, \quad (2.4.54)$$

for every $(x,y) \in D$, where $K_1, K_2 > 0$ are positive constants. Using (2.4.54) and the fact that $C''(\omega^0(x,y)) = -F'(\omega^0(x,y))$ for every $(x,y) \in D$, we see that

$$K_1 \leq -C''(\omega^0(x,y)) \leq K_2,$$

for every $(x,y) \in D$. We first extend C to all of \mathbb{R} such that

$$K_1 \leq -C''(\xi) \leq K_2, \quad (2.4.55)$$

holds for every $\xi \in \mathbb{R}$. Proceeding similarly to the proof of Arnold's first theorem, we obtain that, cf. (2.4.37),

$$\begin{aligned} H_c(\mathbf{v}) - H_c(\mathbf{v}^0) &= \frac{1}{2} \int_D (\mathbf{v} - \mathbf{v}^0) \cdot (\mathbf{v} - \mathbf{v}^0) d\mathbf{x} \\ &+ \frac{1}{2} \int_D \alpha^2 \nabla(\mathbf{v} - \mathbf{v}^0) \cdot \nabla(\mathbf{v} - \mathbf{v}^0) d\mathbf{x} + \frac{1}{2} \int_D C''(\xi)(\omega - \omega^0)^2 d\mathbf{x}. \end{aligned}$$

Since (2.4.55) holds, we have that,

$$\begin{aligned} & - \frac{1}{2} \int_D |\mathbf{v} - \mathbf{v}^0|^2 d\mathbf{x} - \frac{\alpha^2}{2} \int_D |\nabla(\mathbf{v} - \mathbf{v}^0)|^2 d\mathbf{x} \\ & + \frac{K_1}{2} \int_D (\omega - \omega^0)^2 d\mathbf{x} \leq H_c(\mathbf{v}^0) - H_c(\mathbf{v}) \leq -\frac{1}{2} \int_D |\mathbf{v} - \mathbf{v}^0|^2 d\mathbf{x} \\ & - \frac{\alpha^2}{2} \int_D |\nabla(\mathbf{v} - \mathbf{v}^0)|^2 d\mathbf{x} + \frac{K_2}{2} \int_D (\omega - \omega^0)^2 d\mathbf{x}. \end{aligned} \quad (2.4.56)$$

By (2.4.52), we have that,

$$\int_D (\mathbf{v} - \mathbf{v}^0) \cdot (\mathbf{v} - \mathbf{v}^0) d\mathbf{x} + \alpha^2 \int_D \nabla(\mathbf{v} - \mathbf{v}^0) \cdot \nabla(\mathbf{v} - \mathbf{v}^0) d\mathbf{x}$$

$$\leq \frac{1}{\lambda_{min,\alpha}} \int_D (\omega - \omega^0)^2 d\mathbf{x}, \quad (2.4.57)$$

This then means that the left hand side of (2.4.56) can be estimated from below by

$$\begin{aligned} 0 &< \frac{(K_1 - 1/\lambda_{min,\alpha})}{2} \int_D ((\omega - \omega^0)^2) d\mathbf{x} \leq -\frac{1}{2} \int_D |\mathbf{v} - \mathbf{v}^0|^2 d\mathbf{x} \\ &- \frac{\alpha^2}{2} \int_D |\nabla(\mathbf{v} - \mathbf{v}^0)|^2 d\mathbf{x} + \frac{K_1}{2} \int_D (\omega - \omega^0)^2 d\mathbf{x} \leq H_c(\mathbf{v}^0) - H_c(\mathbf{v}). \end{aligned}$$

Thus, obviously, splitting the LHS, we obtain

$$\begin{aligned} &\frac{1}{4}(K_1 - 1/\lambda_{min,\alpha}) \int_D ((\omega - \omega^0)^2) d\mathbf{x} \\ &+ \frac{1}{4}(K_1 - 1/\lambda_{min,\alpha}) \int_D ((\omega - \omega^0)^2) d\mathbf{x} \leq H_c(\mathbf{v}^0) - H_c(\mathbf{v}). \end{aligned}$$

Using (2.4.52) again, we see that

$$\begin{aligned} &\frac{\lambda_{min,\alpha}(K_1 - 1/\lambda_{min,\alpha})}{4} \left(\int_D |\mathbf{v} - \mathbf{v}^0|^2 d\mathbf{x} + \alpha^2 \int_D |\nabla(\mathbf{v} - \mathbf{v}^0)|^2 d\mathbf{x} \right) \\ &\leq \frac{1}{4}(K_1 - 1/\lambda_{min,\alpha}) \int_D ((\omega - \omega^0)^2) d\mathbf{x} \end{aligned}$$

We thus have

$$\begin{aligned} &\frac{\lambda_{min,\alpha}(K_1 - 1/\lambda_{min,\alpha})}{4} \left(\int_D |\mathbf{v} - \mathbf{v}^0|^2 d\mathbf{x} + \alpha^2 \int_D |\nabla(\mathbf{v} - \mathbf{v}^0)|^2 d\mathbf{x} \right) \\ &+ \frac{1}{4}(K_1 - 1/\lambda_{min,\alpha}) \int_D ((\omega - \omega^0)^2) d\mathbf{x} \leq H_c(\mathbf{v}^0) - H_c(\mathbf{v}). \end{aligned} \quad (2.4.58)$$

On the other hand the right hand side of (2.4.56) can be estimated as follows:

$$\begin{aligned} H_c(\mathbf{v}^0) - H_c(\mathbf{v}) &\leq \frac{-1}{2} \int_D |\mathbf{v} - \mathbf{v}^0|^2 d\mathbf{x} - \frac{\alpha^2}{2} \int_D |\nabla(\mathbf{v} - \mathbf{v}^0)|^2 d\mathbf{x} \\ &+ K_2 \int_D (\omega - \omega^0) d\mathbf{x} \leq \frac{1}{2} \int_D |\mathbf{v} - \mathbf{v}^0|^2 d\mathbf{x} \\ &+ \frac{\alpha^2}{2} \int_D |\nabla(\mathbf{v} - \mathbf{v}^0)|^2 d\mathbf{x} + K_2 \int_D (\omega - \omega^0) d\mathbf{x}. \end{aligned} \quad (2.4.59)$$

Now let $\beta_1 = \min\left\{\frac{\lambda_{min,\alpha}(K_1-1/\lambda_{min,\alpha})}{4}, \frac{1}{4}(K_1 - 1/\lambda_{min,\alpha})\right\}$ and $\beta_2 = \max\{\frac{1}{2}, K_2\}$ and we obtain, cf. (2.4.38),

$$\begin{aligned} \beta_1(\|\mathbf{v} - \mathbf{v}^0\|_2^2 + \alpha^2\|\nabla(\mathbf{v} - \mathbf{v}^0)\|_2^2 + \|\omega - \omega^0\|_2^2) &\leq H_c(\mathbf{v}^0) - H_c(\mathbf{v}) \\ &\leq \beta_2(\|\mathbf{v} - \mathbf{v}^0\|_2^2 + \alpha^2\|\nabla(\mathbf{v} - \mathbf{v}^0)\|_2^2 + \|\omega - \omega^0\|_2^2). \end{aligned}$$

We can thus finish the proof as in Arnold's first theorem. ■

2.4.2 Arnold's theorems in a bounded, simply connected domain

In this section we consider the domain $D \subset \mathbb{R}^2$ to be a bounded, simply connected region with a smooth boundary ∂D . The functional (2.4.11), is now given by,

$$H_c(\mathbf{v}) = \frac{1}{2} \int_D \mathbf{v} \cdot (1 - \alpha^2 \Delta) \mathbf{v} dx + \int_D C(\text{curl}(1 - \alpha^2 \Delta) \mathbf{v}) dx + a \int_{\partial D} (1 - \alpha^2 \Delta) \mathbf{v} \cdot \mathbf{ds}. \quad (2.4.60)$$

Lemma 2.50 holds and we impose Assumption 2.51. Lemma 2.53 also holds, where we now set $a = F(\omega^0)|_{\partial D}$. The expression for the second variation given in (2.4.39) remains unchanged. Theorem 2.54 also holds in this case. We now expand upon remark 2.55.

Remark 2.61. Our Lemmas 2.56, 2.57 and 2.58 in Remark 2.20 will work where the subspace for the stream function perturbations is now $Y_\alpha := \{\phi \in H^4(D; \mathbb{R}) \cap H_0^1(D; \mathbb{R}); (\mathbf{n} \cdot \nabla)(\nabla \phi) \cdot \mathbf{n} = 0 \text{ on } \partial D\}$ and the subspace for the velocity perturbations is

$$X_\alpha := \left\{ \mathbf{v} \in H^3(D; \mathbb{R}^2), \text{div } \mathbf{v} = 0 \text{ in } D, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial D \text{ and } (\mathbf{n} \cdot \nabla) \mathbf{v} \text{ parallel to } \mathbf{n} \text{ on } \partial D \right\}$$

The proofs are similar and are omitted. Arnold's second theorem then follows as stated.

2.4.3 Arnolds second theorem on the two torus

By Remark 2.62 below, we do not expect that Arnold's first theorem holds on the two torus.

Remark 2.62. As in Remark 2.18 for the Euler case, we note that Condition (2.4.32) in Arnold's first theorem 2.54 is never satisfied in a domain without a boundary. To demonstrate this, let us assume (2.4.32). Then, since F is monotone, there exists its inverse function denoted by G , i.e., $G = F^{-1}$, and since (2.4.32) holds, one has the relationship

$$\frac{1}{c_2} \leq -G'(\xi) \leq \frac{1}{c_1}, \quad (2.4.61)$$

for all ξ in the range of $F(\omega^0(\cdot, \cdot))$, i.e., for all ξ in the range of $\phi^0(\cdot, \cdot)$. In particular, G' is negative everywhere. Assume without loss of generality that $\partial_x \phi^0 \neq 0$ (if it is, then in the argument below replace $\partial_x \phi^0$ by $\partial_y \phi^0$, we exclude the trivial case $\phi^0 =$ constant everywhere in D). We have that $-\Delta(1 - \alpha^2 \Delta)\phi^0 = \omega^0 = G(\phi^0)$. From this we see that, $\partial_x(-\Delta(1 - \alpha^2 \Delta)\phi^0) = G'(\phi^0)\partial_x \phi^0$. Multiplying this by $\partial_x \phi^0$ and integrating this over the domain D , we get,

$$-\int_D \partial_x \phi^0 \partial_x \Delta(1 - \alpha^2 \Delta)\phi^0 d\mathbf{x} = \int_D G'(\phi^0)(\partial_x \phi^0)^2 d\mathbf{x}. \quad (2.4.62)$$

Integrating the left hand side by parts, we get,

$$\begin{aligned} & -\int_D \partial_x \phi^0 \partial_x \Delta(1 - \alpha^2 \Delta)\phi^0 d\mathbf{x} = -\int_D \partial_x \phi^0 \Delta(1 - \alpha^2 \Delta)\partial_x \phi^0 d\mathbf{x} \\ & = \int_D \nabla(\partial_x \phi^0) \cdot \nabla((1 - \alpha^2 \Delta)\partial_x \phi^0) d\mathbf{x} - \int_{\partial D} (\partial_x \phi^0) \mathbf{n} \cdot \nabla(\partial_x(1 - \alpha^2 \Delta)\phi^0) ds. \end{aligned}$$

But

$$\begin{aligned}
& \int_D \nabla(\partial_x \phi^0) \cdot \nabla((1 - \alpha^2 \Delta) \partial_x \phi^0) d\mathbf{x} = \int_D (\nabla \partial_x \phi^0)^2 d\mathbf{x} - \\
& \alpha^2 \int_D \nabla(\partial_x \phi^0) \cdot \nabla(\Delta \partial_x \phi^0) d\mathbf{x} = \int_D (\nabla \partial_x \phi^0)^2 d\mathbf{x} \\
& + \alpha^2 \int_D |\nabla(\nabla \partial_x \phi^0)|^2 d\mathbf{x} - \int_{\partial D} \nabla(\partial_x \phi^0) \mathbf{n} \cdot \nabla(\nabla \partial_x (1 - \alpha^2 \Delta) \phi^0) ds.
\end{aligned}$$

Thus, rewriting the left side of (2.4.62) one obtains

$$\begin{aligned}
& \int_D (\nabla \partial_x \phi^0)^2 d\mathbf{x} + \alpha^2 \int_D |\nabla(\nabla \partial_x \phi^0)|^2 d\mathbf{x} \\
& - \int_{\partial D} \nabla(\partial_x \phi^0) \mathbf{n} \cdot \nabla(\nabla \partial_x (1 - \alpha^2 \Delta) \phi^0) ds \\
& - \int_{\partial D} (\partial_x \phi^0) \mathbf{n} \cdot \nabla(\partial_x (1 - \alpha^2 \Delta) \phi^0) ds = \int_D G'(\phi^0) (\partial_x \phi^0)^2 d\mathbf{x}.
\end{aligned}$$

Note that the first two terms on the left hand side are positive and the term in the right hand side is negative by (2.4.61) which leads to a contradiction in the absence of the boundary terms in the left hand side.

We would like to consider Arnold's second theorem on the two torus \mathbb{T}^2 , thus in this subsection, $D := \mathbb{T}^2$. We consider the α -Euler equations (2.1.4) on the two torus \mathbb{T}^2 , where $\mathbf{v} : \mathbb{T}^2 \rightarrow \mathbb{R}$ and $\mathbf{u} : \mathbb{T}^2 \rightarrow \mathbb{R}$ are such that $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{v} = 0$ and $\mathbf{u} = (1 - \alpha^2 \Delta) \mathbf{v}$. We also have the vorticity equation (2.1.11), where $\omega : \mathbb{T}^2 \rightarrow \mathbb{R}$. The Hamiltonian H_c is now given by,

$$H_c(\mathbf{v}) = \frac{1}{2} \int_D \mathbf{v} \cdot (1 - \alpha^2 \Delta) \mathbf{v} d\mathbf{x} + \int_D C(\operatorname{curl}(1 - \alpha^2 \Delta) \mathbf{v}) d\mathbf{x}. \quad (2.4.63)$$

Lemma 2.50 remains true in this setting. We also impose Assumption 2.51. Lemma 2.53 is modified as follows.

Lemma 2.63. *Let \mathbf{v}^0, ω^0 be a steady state solution of (2.1.4), satisfying Assumption*

2.51, where $\omega^0 = \text{curl}(1 - \alpha^2 \Delta) \mathbf{v}^0$. Let C be a smooth function so that

$$C'(\omega^0(x, y)) = -F(\omega^0(x, y)),$$

for every $(x, y) \in D$. Then $\delta H_c(\mathbf{v}^0) \delta \mathbf{v} = 0$, i.e \mathbf{v}^0 is a critical point of H_c .

Proof. Note first that $H_c(\mathbf{v})$ can be expressed, using $\mathbf{u} = (1 - \alpha^2 \Delta) \mathbf{v}$, as

$$H_c(\mathbf{v}) = \frac{1}{2} \int_D \mathbf{v} \cdot \mathbf{u} d\mathbf{x} + \int_D C(\omega) d\mathbf{x}. \quad (2.4.64)$$

The first variation of H_c at \mathbf{v}^0 is given by the following expression,

$$\begin{aligned} \delta H_c(\mathbf{v}^0) \delta \mathbf{v} &= \left. \frac{d}{d\varepsilon} H_c(\mathbf{v}^0 + \varepsilon \delta \mathbf{v}) \right|_{\varepsilon=0} \\ &= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta \mathbf{u} d\mathbf{x} + \frac{1}{2} \int_D \mathbf{u}^0 \cdot \delta \mathbf{v} d\mathbf{x} + \int_D C'(\omega^0) \delta \omega d\mathbf{x}, \end{aligned} \quad (2.4.65)$$

where $\delta \mathbf{u} = (1 - \alpha^2 \Delta) \delta \mathbf{v}$ and $\delta \omega = \text{curl} \delta \mathbf{u}$.

We will be using the following identity (see [MP94], Eq 2.14, page 108),

$$C'(\omega^0) \delta \omega = \text{curl}(C'(\omega^0) \delta \mathbf{u}) - C''(\omega^0) \nabla^\perp \omega^0 \cdot \delta \mathbf{u}, \quad (2.4.66)$$

which follows from the identity $\text{curl}(f \mathbf{v}) = \nabla^\perp f \cdot \mathbf{v} + f \text{curl} \mathbf{v}$. Noting that, by Stokes' theorem,

$$\int_D \text{curl}(C'(\omega^0) \delta \mathbf{u}) d\mathbf{x} = 0, \quad (2.4.67)$$

we see that, using (2.4.66) and (2.4.67)

$$\int_D C'(\omega^0) \delta \omega d\mathbf{x} = - \int_D C''(\omega^0) \nabla^\perp \omega^0 \cdot \delta \mathbf{u} d\mathbf{x}. \quad (2.4.68)$$

In the following computation, we integrate by parts in the second and fourth line to get,

$$\begin{aligned}
\frac{1}{2} \int_D \mathbf{u}^0 \cdot \delta \mathbf{v} d\mathbf{x} &= \frac{1}{2} \int_D (\mathbf{v}^0 - \alpha^2 \Delta \mathbf{v}^0) \cdot \delta \mathbf{v} d\mathbf{x} \\
&= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta \mathbf{v} d\mathbf{x} - \frac{\alpha^2}{2} \int_D \Delta \mathbf{v}^0 \cdot \delta \mathbf{v} d\mathbf{x} \\
&= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta \mathbf{v} d\mathbf{x} + \frac{\alpha^2}{2} \int_D \nabla \mathbf{v}^0 \cdot \nabla \delta \mathbf{v} d\mathbf{x} \\
&= \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta \mathbf{v} d\mathbf{x} - \frac{\alpha^2}{2} \int_D \mathbf{v}^0 \cdot \Delta \delta \mathbf{v} d\mathbf{x} \\
&= \frac{1}{2} \int_D \mathbf{v}^0 \cdot (\delta \mathbf{v} - \alpha^2 \Delta \delta \mathbf{v}) d\mathbf{x} = \frac{1}{2} \int_D \mathbf{v}^0 \cdot \delta \mathbf{u} d\mathbf{x}.
\end{aligned} \tag{2.4.69}$$

Using (2.4.69) and (2.4.68), we see that (2.4.65) is given by,

$$\delta H_c(\mathbf{v}^0) \delta \mathbf{v} = \int_D \mathbf{v}^0 \cdot \delta \mathbf{u} d\mathbf{x} - \int_D C''(\omega^0) \nabla^\perp(\omega^0) \delta \mathbf{u} d\mathbf{x}.$$

from which we see that $\delta H_c(\mathbf{v}^0) \delta \mathbf{v} = 0$ provided,

$$\mathbf{v}^0(x, y) = C''(\omega^0(x, y)) \nabla^\perp \omega^0(x, y), \tag{2.4.70}$$

Since, by (2.4.21), C is chosen such that $C'(\omega^0(x, y)) = -F(\omega^0(x, y))$ for every $(x, y) \in D$, then,

$$\mathbf{v}^0 = -\nabla^\perp \phi^0 = -F'(\omega^0) \nabla^\perp \omega^0 = C''(\omega^0) \nabla^\perp \omega^0,$$

i.e., (2.4.70) holds. Thus the first variation $\delta H_c(\mathbf{v}^0) \delta \mathbf{v} = 0$. ■

The expression for the second variation remains the same as (2.4.39). The statement of Arnold's first theorem remains the same as in Theorem 2.54. We shall now expand upon Remark 2.55.

Remark 2.64. Our space for the perturbation stream function is now given by

$$Y_\alpha := \left\{ \phi \in H^4(\mathbb{T}^2); \int_{\mathbb{T}^2} \phi d\mathbf{x} = 0 \right\}, \tag{2.4.71}$$

and for the velocities is given by $X_\alpha := \{\mathbf{v} \in H^3(\mathbb{T}^2; \mathbb{R}^2); \int_{\mathbb{T}^2} \mathbf{v} d\mathbf{x} = 0; \operatorname{div} \mathbf{v} = 0\}$. Lemmas 2.56, 2.57 and 2.58 in Remark 2.55 are true in this setting with minor modifications in the proof. We prove Lemma 2.57 in this case.

Proof. Note that ϕ satisfies,

$$-\Delta(1 - \alpha^2 \Delta)\phi = 0, \quad (2.4.72)$$

$$\int_{\mathbb{T}^2} \phi dx dy = 0. \quad (2.4.73)$$

Multiply (2.4.72) by ϕ and integrate over the domain to get

$$0 = \int_D \phi(-\Delta(1 - \alpha^2 \Delta)\phi) d\mathbf{x} = + \int_D \nabla \phi \cdot (1 - \alpha^2 \Delta) \nabla \phi d\mathbf{x}, \quad (2.4.74)$$

But,

$$\int_D \nabla \phi \cdot (1 - \alpha^2 \Delta) \nabla \phi d\mathbf{x} = \int_D \nabla \phi \cdot \nabla \phi d\mathbf{x} - \alpha^2 \int_D \nabla \phi \cdot \Delta \nabla \phi d\mathbf{x}.$$

Since (2.4.5) also holds true on the torus, we have that

$$\int_D \nabla \phi \cdot \Delta \nabla \phi d\mathbf{x} = \int_D -\nabla^\perp \phi \cdot \Delta(-\nabla^\perp \phi) d\mathbf{x} = \int_D \mathbf{v} \cdot \Delta \mathbf{v} d\mathbf{x} = - \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x},$$

where $\mathbf{v} \in X_\alpha$ is the unique solution to $\mathbf{v} = -\nabla^\perp \phi$, via Lemma 2.56. Thus, we see that,

$$\begin{aligned} \int_D \nabla \phi \cdot (1 - \alpha^2 \Delta) \nabla \phi d\mathbf{x} &= \int_D \nabla \phi \cdot \nabla \phi d\mathbf{x} + \alpha^2 \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x} \\ &= \int_D \mathbf{v} \cdot \mathbf{v} d\mathbf{x} + \alpha^2 \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x} = 0, \end{aligned}$$

from which we conclude that $\mathbf{v} = 0$.

From this it follows by Lemma 2.56 that $-\nabla^\perp \phi = 0$ on D and hence $\nabla \phi = 0$.

Then ϕ is a constant, and is equal to 0 by (2.3.52). ■

Arnold's second theorem then follows as stated in Theorem 2.60.

2.4.4 Arnold's theorems on the periodic channel

We would now like to formulate Arnold's theorems on the periodic channel $D = \mathbb{T} \times [-1, 1]$, so that the boundary conditions are periodic in the x direction with boundary conditions $\mathbf{v} \cdot \mathbf{n}$ and $(\mathbf{n} \cdot \nabla)\mathbf{v}$ parallel to \mathbf{n} at the "walls" $y = 1$ and $y = -1$. We will prove that since the domain is translationally invariant in the x direction, the x momentum is conserved, i.e., we will prove that, if $\mathbf{v}(t, \cdot) = (v_1(t, \cdot), v_2(t, \cdot))$, $\mathbf{u}(t, \cdot) = (u_1(t, \cdot), u_2(t, \cdot))$ solve the α -Euler equation (2.1.4), then

$$M_x = \int_{-1}^1 \int_{\mathbb{T}} u_1(t, x, y) dx dy = \int_{-1}^1 \int_{\mathbb{T}} u_1(0, x, y) dx dy \quad (2.4.75)$$

is an invariant of the motion. Here $\mathbf{u} = (1 - \alpha^2 \Delta)\mathbf{v}$.

Lemma 2.65. *Suppose $\mathbf{v}(t, \cdot) = (v_1(t, \cdot), v_2(t, \cdot))$, $\mathbf{u}(t, \cdot) = (u_1(t, \cdot), u_2(t, \cdot))$ solve the α -Euler equation (2.1.4), on the domain $\mathbb{T} \times [-1, 1]$. Then*

$$\frac{d}{dt} M_x = \frac{d}{dt} \int_{-1}^1 \int_{\mathbb{T}} u_1(t, x, y) dx dy = 0. \quad (2.4.76)$$

Proof. We first note that the boundary conditions $\mathbf{v} \cdot \mathbf{n}|_{y=\pm 1} = 0$ imply that

$$v_2(x, -1) = v_2(x, 1) = 0. \quad (2.4.77)$$

Also, the boundary condition $\mathbf{n} \cdot \nabla \mathbf{v}$ parallel to \mathbf{n} implies that $\mathbf{n} \cdot \nabla \mathbf{v} \cdot \mathbf{t} = 0$. On the boundaries $y = -1$ and $y = 1$, $\mathbf{n} = (0, \pm 1)$ and thus $\mathbf{n} \cdot \nabla = \partial_y$.

Thus $\mathbf{n} \cdot \nabla \mathbf{v} = \partial_y \mathbf{v} = (\phi_{yy}, -\phi_{yx})$. Thus $(\phi_{yy}, -\phi_{yx}) \cdot (1, 0) = 0$ implies that $\phi_{yy} = 0$ on the boundary, i.e.,

$$\phi_{yy}(x, -1) = \phi_{yy}(x, 1) = 0. \quad (2.4.78)$$

We note that the α -Euler equations (2.1.4) can be rewritten as, see [HMR98, Eq 8.33,

page 67],

$$\partial_t \mathbf{u} - \mathbf{v} \times (\nabla \times \mathbf{u}) + \nabla(\mathbf{v} \cdot \mathbf{u} - \frac{1}{2}|\mathbf{v}|^2 - \frac{\alpha^2}{2}|\nabla \mathbf{v}|^2 + p) = 0. \quad (2.4.79)$$

Denote $f = \mathbf{v} \cdot \mathbf{u} - \frac{1}{2}|\mathbf{v}|^2 - \frac{\alpha^2}{2}|\nabla \mathbf{v}|^2 + p$. Also note that $\nabla \times \mathbf{u} = (\partial_x u_2 - \partial_y u_1)\mathbf{k}$, where \mathbf{k} is the unit vector pointing out of the plane of flow. Thus $\mathbf{v} \times (\nabla \times \mathbf{u}) = (v_2 \partial_x u_2 - v_2 \partial_y u_1)\mathbf{i} - (v_1 \partial_x u_2 - v_1 \partial_y u_1)\mathbf{j}$. We thus have that, using (2.4.79) and the computations above,

$$\begin{aligned} \frac{d}{dt} M_x &= \frac{d}{dt} \int_{-1}^1 \int_{\mathbb{T}} u_1(t, x, y) dx dy = \int_{-1}^1 \int_{\mathbb{T}} \partial_t u_1(t, x, y) dx dy \\ &= \int_{-1}^1 \int_{\mathbb{T}} (v_2 \partial_x u_2 - v_2 \partial_y u_1 - \partial_x f) dx dy. \end{aligned}$$

We analyze this term by term. Notice first that the inner integral of the third term is zero, $\int_{\mathbb{T}} -\partial_x f dx = 0$. We look at the first term

$$v_2 \partial_x u_2 = v_2 \partial_x v_2 - \alpha^2 v_2 \partial_x \Delta v_2.$$

Rewriting $v_2 \partial_x v_2 = \partial_x (\frac{1}{2} v_2^2)$ we get $\int_{\mathbb{T}} v_2 \partial_x v_2 dx = \int_{\mathbb{T}} \partial_x (\frac{1}{2} v_2^2) dx = 0$. Also,

$$\begin{aligned} \int_{-1}^1 \int_{\mathbb{T}} v_2 \partial_x \Delta v_2 dx dy &= \int_{\mathbb{T}} \int_{-1}^1 v_2 \Delta (\partial_x v_2) dy dx \\ &= - \int_{\mathbb{T}} \int_{-1}^1 \nabla v_2 \cdot \nabla (\partial_x v_2) dy dx = - \int_{-1}^1 \int_{\mathbb{T}} \partial_x (\frac{1}{2} (\nabla v_2)^2) dx dy = 0. \end{aligned}$$

where we integrate by parts in y and boundary terms disappear by using boundary condition (2.4.78) and then switch order of integration and use the fact that

$$\nabla v_2 \cdot \nabla (\partial_x v_2) = (\frac{1}{2} (\nabla v_2)^2).$$

Thus, $\int_{-1}^1 \int_{\mathbb{T}} v_2 \partial_x u_2 dx dy = 0$. We now look at the second term,

$$\int_{-1}^1 \int_{\mathbb{T}} v_2 \partial_y u_1 dx dy = \int_{\mathbb{T}} \int_{-1}^1 v_2 \partial_y u_1 dy dx$$

$$= - \int_{\mathbb{T}} \int_{-1}^1 \partial_y v_2 u_1 dy dx = \int_{\mathbb{T}} \int_{-1}^1 \partial_x v_1 u_1 dy dx$$

where we integrate by parts and the boundary terms vanish using boundary condition

(2.4.78) and since $\operatorname{div} \mathbf{v} = 0$, we have that $\partial_x v_1 = -\partial_y v_2$. Notice that

$$\partial_x v_1 u_1 = \partial_x v_1 v_1 - \alpha^2 \partial_x v_1 \Delta v_1.$$

Since $\partial_x v_1 v_1 = \partial_x (\frac{1}{2} v_1^2)$, we have that

$$\int_{-1}^1 \int_{\mathbb{T}} \partial_x v_1 v_1 dx dy = \int_{-1}^1 \int_{\mathbb{T}} \partial_x (\frac{1}{2} v_1^2) dx dy = 0.$$

Finally,

$$\begin{aligned} \int_{-1}^1 \int_{\mathbb{T}} \partial_x v_1 \Delta v_1 dx dy &= - \int_{-1}^1 \int_{\mathbb{T}} \nabla \partial_x v_1 \cdot \nabla v_1 dx dy \\ &= - \int_{-1}^1 \int_{\mathbb{T}} \partial_x (\frac{1}{2} (\nabla v_1)^2) dx dy = 0. \end{aligned}$$

■

Since $u_1 = \psi_y$, consider the following,

$$\frac{1}{2\pi} \int_{\mathbb{T}} u_1(t, x, y) dx = \frac{1}{2\pi} \int_{\mathbb{T}} \psi_y(t, x, y) dx.$$

Integrating this in y , and using (2.3.55), we see that,

$$\begin{aligned} \frac{1}{2\pi} M_x &= \frac{1}{2\pi} \int_{-1}^1 \int_{\mathbb{T}} u_1(t, x, y) dx dy = \frac{1}{2\pi} \int_{\mathbb{T}} \int_{-1}^1 u_1(t, x, y) dy dx \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \int_{-1}^1 \psi_y(t, x, y) dy dx = \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t, x, -1) dx - \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t, x, 1) dx. \end{aligned}$$

Since $\psi = \phi - \alpha^2 \Delta \phi$, and by the boundary condition (2.4.78), $\phi_{yy}(x, \pm 1) = 0$, we

have that

$$\psi(x, \pm 1) = \phi(x, \pm 1) - \partial_{xx} \phi_{xx}(x, \pm 1).$$

Also note that,

$$\int_T \phi_{xx}(x, \pm 1) dx = \int_T \partial_x \phi_x(x, \pm 1) dx = 0$$

to conclude that

$$\begin{aligned} \frac{1}{2\pi} M_x &= \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t, x, -1) dx - \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t, x, 1) dx \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \phi(t, x, -1) dx - \frac{1}{2\pi} \int_{\mathbb{T}} \phi(t, x, 1) dx. \end{aligned} \quad (2.4.80)$$

By Lemma 2.65, $M_x/2\pi$ is a fixed number in time. Since $\phi(x, -1)$ and $\phi(x, 1)$ are constants, one can simply take the difference to be $M_x/2\pi$. Thus in solving the Poisson equation for the stream function we can set $\phi(x, -1) = 0$ and $\phi(x, 1) = M_x/2\pi$. One thus solves the following Poisson problem to recover the stream function from the vorticity.

$$\begin{aligned} -\Delta(1 - \alpha^2 \Delta)\phi &= \omega, \text{ in } D, \\ \phi(x, -1) &= 0, \\ \phi(x, 1) &= -M_x/2\pi. \end{aligned} \quad (2.4.81)$$

Since this must hold for both the steady state ϕ^0 and the perturbed flow $\phi^0 + \delta\phi$, we see that, the Poisson equation satisfied by the perturbation stream function $\delta\phi$ satisfies Dirichlet boundary conditions.

$$\begin{aligned} -\Delta(1 - \alpha^2 \Delta)\delta\phi &= \omega, \text{ in } D, \\ \delta\phi(x, -1) &= 0, \\ \delta\phi(x, 1) &= 0. \end{aligned} \quad (2.4.82)$$

The subspace for the perturbation stream function is now the following:

$$Y_\alpha = \left\{ \phi : H^4(\mathbb{T} \times [-1, 1]); \mathbb{R} : \phi(x, 1) = 0; \phi(x, -1) = 0 \right\}.$$

One then defines the subspace for the perturbations of velocity as

$$X_\alpha := \left\{ \mathbf{u} : H^3(\mathbb{T} \times [-1, 1]); \mathbb{R}^2); \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } y = -1 \text{ and } y = 1; \right. \\ \left. \mathbf{n} \cdot \nabla \mathbf{v} \text{ is parallel to } \mathbf{n} \text{ on } y = -1 \text{ and } y = 1; \right. \\ \left. \int_{-1}^1 \int_{\mathbb{T}} (1 - \alpha^2 \Delta) v_1(x, y) dx dy = 0 \right\},$$

where $\mathbf{v} = (v_1, v_2)$. Lemma 2.56 follows as stated. One can also easily check that if $\phi(x, 1) = \phi(x, -1) = 0$, then $\int_{\mathbb{T}} \int_{-1}^1 u_1(x, y) dy dx = 0$. Indeed, using (2.4.80),

$$\int_{\mathbb{T}} \int_{-1}^1 u_1(x, y) dy dx = \int_{\mathbb{T}} \int_{-1}^1 \psi_y(x, y) dy dx = \int_{\mathbb{T}} (\psi(x, 1) - \psi(x, -1)) dx \\ = \int_{\mathbb{T}} (\phi(x, 1) - \phi(x, -1)) dx = \int_{\mathbb{T}} (0 - 0) dx = 0.$$

The proof of Lemma 2.57 is modified as follows.

Proof. Note that ϕ satisfies,

$$-\Delta(1 - \alpha^2 \Delta)\phi = 0, \tag{2.4.83}$$

$$\phi(x, -1) = \phi(x, 1) = 0, \tag{2.4.84}$$

$$\phi_{yy}(x, -1) = \phi_{yy}(x, 1) = 0. \tag{2.4.85}$$

Multiply (2.4.83) by ϕ and integrate over the domain to get

$$0 = \int_{\mathbb{T}} \int_{-1}^1 \phi(-\Delta(1 - \alpha^2 \Delta)\phi) dy dx = \int_{\mathbb{T}} \int_{-1}^1 \nabla \phi \cdot (\nabla(1 - \alpha^2 \Delta)\phi) dy dx,$$

where boundary terms vanish by (2.4.84).

Since (2.4.5) also holds true in this case, we have that

$$\int_D \nabla \phi \cdot \Delta \nabla \phi d\mathbf{x} = \int_D -\nabla^\perp \phi \cdot \Delta(-\nabla^\perp \phi) d\mathbf{x} = \int_D \mathbf{v} \cdot \Delta \mathbf{v} d\mathbf{x} = - \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x},$$

where $\mathbf{v} \in X_\alpha$ is the unique solution to $\mathbf{v} = -\nabla^\perp \phi$, via Lemma 2.56. Thus, we see that,

$$\begin{aligned} \int_D \nabla \phi \cdot (1 - \alpha^2 \Delta) \nabla \phi d\mathbf{x} &= \int_D \nabla \phi \cdot \nabla \phi d\mathbf{x} + \alpha^2 \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x} \\ &= \int_D \mathbf{v} \cdot \mathbf{v} d\mathbf{x} + \alpha^2 \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x} = 0, \end{aligned}$$

from which we conclude that $\mathbf{v} = 0$.

From this it follows by Lemma 2.56 that $-\nabla^\perp \phi = 0$ on D and hence $\nabla \phi = 0$. Then ϕ is a constant, and is equal to 0 by (2.4.84). ■

Lemma 2.58 follows as stated and Arnold's second theorem also follows as stated.

2.4.5 Examples

Example 2.66. *Plane parallel shear flows and inflection points:*

1. Plane parallel shear flows V such that $U = V - \alpha^2 V''$ does not have an inflection point: Suppose we have a plane parallel shear flow on $\mathbb{T} \times [-L_1, L_2]$ consisting of a profile $\mathbf{v}^0 = (V(y), 0)$, with $V'(-L_1) = V'(L_2) = 0$, $\mathbf{u}^0 = (U(y), 0)$ where $U = V - \alpha^2 V''$ (primes denote differentiation with respect to y). We assume that $U(y)$ has no inflection point on $[-L_1, L_2]$, i.e., $U''(y) \neq 0$ for every $y \in [-L_1, L_2]$. We compute $-F'(\omega^0(x, y)) = V(y)/U''(y)$. Therefore, as long as $U''(y) \neq 0$, we can always move to a reference frame where V has the same sign as U'' , i.e., find a constant c such that $V(y) + c$ has the same sign as U'' . Thus, one can see that $-F'$ satisfies (2.4.34) and one has stability of this steady state by Arnold's first stability theorem 2.54. We thereby have a sufficient condition for stability for a shear flow V , where U does not have any inflection point. Rayleigh criterion for α -Euler, see Proposition 2.5 and example 2.7, guarantees

linear stability for flows such that U has no inflection point. Arnold's stability theorem 2.54 guarantees nonlinear Lyapunov stability in the norm in (2.4.33) thus generalizing appropriately the Rayleigh criterion.

2. Plane parallel shear flows V such that $U = V - \alpha^2 V''$ has inflection points but V and U'' have the same sign everywhere: We can also prove stability of a steady state $\mathbf{v}^0 = (V(y), 0)$, with $V'(-L_1) = V'(L_2) = 0$, $\mathbf{u}^0 = (U(y), 0)$ where $U = V - \alpha^2 V''$ such that U'' changes sign, but $V(y)/U''(y)$ has the same sign. Then, the ratio $-F'(\omega^0(x, y)) = V(y)/U''(y)$ is positive everywhere and one obtains stability of this steady state by Arnold's first stability theorem 2.54. Note that this generalizes the Fjortoft criterion, see Proposition 2.8 and example 2.10, which guaranteed linear stability of these steady states.

Example 2.67. *Effect of regularization:* This example illustrates the effect of regularization on the Arnold criterion. We present an example such that the Arnold stability Theorem 2.54 can be applied to conclude stability of the steady states for α -Euler for every $\alpha > 0$ but the corresponding Arnold Theorem 2.26 for Euler cannot be applied to conclude stability of the steady state for Euler equation obtained as the limit as $\alpha \rightarrow 0$ of the steady states for the α -Euler. Suppose we have a plane parallel shear flow on $\mathbb{T} \times [-L_1, L_2]$ consisting of a profile $\mathbf{v}^0 = (V(y), 0)$, with $V'(-L_1) = V'(L_2) = 0$, $\mathbf{u}^0 = (U(y), 0)$ where $U = V - \alpha^2 V''$ (primes denote differentiation with respect to y). Let ϕ^0 be the stream function associated with \mathbf{v}^0 , i.e., $V(y) = (\phi^0)'(y)$. The boundary condition $V'(-L_1) = V'(L_2) = 0$ implies that $(\phi^0)''(L_1) = (\phi^0)''(L_2) = 0$. Also, assume that $\phi^0 = (1 + \alpha^2)\omega^0$. Thus, $F' = (1 + \alpha^2)$.

Notice that $\omega^0 = \text{curl}(1 - \alpha^2 \Delta) \mathbf{v}^0 = -U'(y)$ Thus

$$\omega^0 = -\partial_y((1 - \alpha^2 \partial_{yy})V(y)) = -\partial_y((1 - \alpha^2 \partial_{yy})\partial_y \phi^0),$$

and,

$$\omega^0 = \alpha^2(\phi^0)'''' - (\phi^0)''.$$

Since $\omega^0 = \frac{1}{1+\alpha^2}\phi^0$, we see that ϕ^0 must satisfy the following differential equation,

$$\alpha^2(\phi^0)'''' - (\phi^0)'' - \frac{1}{1+\alpha^2}\phi^0 = 0, \quad (\phi^0)''(L_1) = (\phi^0)''(L_2) = 0. \quad (2.4.86)$$

Choose the difference $L_2 - L_1$ in such a way that $-\Delta$ has minimum eigenvalue 1 on the appropriate space Y_α . Thus $-\Delta(1 - \alpha^2 \Delta)$ will have minimum eigenvalue $1 + \alpha^2$. Since we have the inequality $\frac{1}{\lambda_{min,\alpha}} = \frac{1}{1+\alpha^2} < F' = 1 + \alpha^2$ for all $\alpha > 0$, by Arnold's second stability theorem 2.60, one has stability of this steady state for all values of $\alpha > 0$. Notice that if we formally let $\alpha \rightarrow 0$ and consider this as a steady state for the Euler equations, $\phi^0 = \psi^0$ and $\phi^0 = (1 + \alpha^2)\omega^0$ goes to $\psi^0 = \omega^0$. Thus $F' = 1$ and since the minimum eigenvalue of $-\Delta$ in the appropriate subspace is 1, the inequality $1/\lambda_{min} \leq F'$ cannot be checked and stability of this steady state cannot be concluded by Arnold's second stability theorem for the Euler equations.

Example 2.68. *Sinusoidal flows:* One class of steady states for which the regularization seems to have no effect in terms of the Arnold criterion are the oscillating sinusoidal flows, i.e., steady states of the form $\phi^0(y) = \sin y$ and $\phi^0(y) = \sin my$ where $m > 1$ is an integer. The Arnold stability theorems cannot be used to conclude stability of these steady states for both Euler and α -Euler. Consider the domain to be the two torus \mathbb{T}^2 . For example, if $\phi^0(y) = \sin y$, then $\mathbf{v}^0(y) = (\cos y, 0)$, $\mathbf{u}^0(y) =$

$(1 - \alpha^2 \partial_{yy}) \mathbf{v}^0(y) = ((1 + \alpha^2) \cos y, 0)$. $\omega^0(y) = -\partial_y(1 + \alpha^2) \cos y = (1 + \alpha^2) \sin y$. From this we can see that $F' = 1/(1 + \alpha^2)$ and in order to check for stability we need $1/\lambda_{min,\alpha} < 1/(1 + \alpha^2)$ which does not hold because $\lambda_{min,\alpha}$ of the operator $-\Delta(1 + \alpha^2 \Delta)$ with domain Y_α as in Equation (2.4.71) is equal to $1 + \alpha^2$. One thus cannot conclude stability of this steady state via Arnold's second stability Theorem 2.60. We note here the regularization does not have any effect whatsoever because for the Euler equation if $\psi^0(y) = \sin y$, $\mathbf{u}^0 = (\cos y, 0)$. $\omega^0(y) = \sin y$. Thus $F' = 1$ and λ_{min} of the negative Laplacian $-\Delta$ acting on the appropriate subspace Y_0 in Equation (2.3.50) is also 1 and thus one cannot check that $1/F' < \lambda_{min}$ which is required for stability. Thus the regularization doesn't seem to affect the ability of Arnold criterion to predict the stability of the steady state $\sin y$. Similarly, if $\phi^0(y) = \sin my$, then $\mathbf{v}^0(y) = (m \cos my, 0)$. Then, $\mathbf{u}^0(y) = (1 - \alpha^2 \partial_{yy}) \mathbf{v}^0(y) = (m(1 + \alpha^2 m^2) \cos my, 0)$. $\omega^0(y) = -\partial_y m(1 + \alpha^2 m^2) \cos my = m^2(1 + m^2 \alpha^2) \sin my$. From this we can see that $F' = 1/m^2(1 + m^2 \alpha^2)$. One can check that the minimum eigenvalue of $-\Delta(1 - \alpha^2 \Delta)$ on the subspace Y_α described in Section 2.4.3 is given by $1 + \alpha^2$. Thus, in order to check for stability we need $1/\lambda_{min,\alpha} = 1/(1 + \alpha^2) < F' = 1/(m^2(1 + m^2 \alpha^2))$. This inequality cannot be checked for $m > 1$ and thus one cannot conclude stability by Arnold's second stability theorem 2.60. We note here the regularization does not have any effect whatsoever because in the case of the Euler equation if $\psi^0(y) = \sin my$, $\mathbf{u}^0 = (m \cos my, 0)$. $\omega^0(y) = m^2 \sin my$. Thus $F' = 1/m^2$ and λ_{min} of the negative Laplacian $-\Delta$ acting on the appropriate subspace Y_0 as in (2.3.50) is also 1. Thus we need to check if $1 < 1/m^2$ which cannot be true if $m > 1$, and thus, similar to the α -Euler criterion, even for the Euler case, stability cannot be concluded via the Arnold's

second stability theorem. Thus the regularization doesn't seem to affect the ability of Arnold criterion to predict the stability of the steady state $\sin my$. This leads us to conjecture that these steady states are unstable even for the regularized α -Euler equations. For more regarding stability of sinusoidal flows for the Euler equations, see [BFY99].

Chapter 3

Instability of unidirectional flows for the 2D α -Euler equations

In this chapter we study the stability of a special steady state, the unidirectional flow, of the 2D α -Euler equations on the torus written for the Fourier coefficients of vorticity. The unidirectional steady state has exactly one nonzero Fourier mode corresponding to a two dimensional vector $\mathbf{p} \in \mathbb{Z}^2$ with integer components. We linearize the α -Euler equation and write the linearized operator L_B in $\ell^2(\mathbb{Z}^2)$ as a direct sum of one dimensional difference operators $L_{B,\mathbf{q}}$ in $\ell^2(\mathbb{Z})$ parametrized by some vectors \mathbf{q} such that the set $\{\mathbf{q} + n\mathbf{p} : n \in \mathbb{Z}\}$ covers the entire grid \mathbb{Z}^2 , see [L, LLS, WDM1]. The set $\{\mathbf{q} + n\mathbf{p} : n \in \mathbb{Z}\}$ might have zero, one or two points inside the disk of radius $\|\mathbf{p}\|$. We consider primarily the second case, and apply continued fractions to the study of spectral properties of the respective difference operator $L_{B,\mathbf{q}}$, cf. [L, FH98, MS]. We show the existence of a positive eigenvalue for $L_{B,\mathbf{q}}$ in this case, which implies that L_B has unstable spectrum. Our main result says that, under the assumptions specified below, the unidirectional steady state (3.2.2) is spectrally unstable, i.e., L_B has a positive eigenvalue. More details and a precise formulation are given in Theorem 3.9.

3.1 Basic setup and governing equations

We consider two dimensional α -Euler equations for incompressible ideal fluid on the torus written in vorticity form,

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = 0, \quad \nabla \cdot \mathbf{v} = 0, \in \mathbb{T}^2, \quad (3.1.1)$$

where ω is the vorticity of the fluid and \mathbf{v} the smoothed velocity, $\mathbf{v} = (v_1, v_2)$, $\mathbf{x} = (x, y) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. Here

$$\omega = \text{curl}(1 - \alpha^2 \Delta) \mathbf{v}, \quad (3.1.2)$$

where $\alpha > 0$ is a positive real number. Since $\nabla \cdot \mathbf{v} = 0$, there exists a stream function ϕ , such that $\mathbf{v} = -\nabla^\perp \phi$, where $\nabla^\perp = (-\partial_y, \partial_x)$. This means that

$$\omega = -\Delta(1 - \alpha^2 \Delta) \phi. \quad (3.1.3)$$

Assume that

$$\int_{\mathbb{T}^2} \omega dx dy = 0. \quad (3.1.4)$$

The fact that (3.1.4) holds implies that one can uniquely solve (3.1.3) for the stream function ϕ , and one also obtains that the zero Fourier coefficient of ϕ satisfies $\phi_{\mathbf{0}} = 0$, i.e.,

$$\int_{\mathbb{T}^2} \phi dx dy = 0.$$

Using the Fourier series

$$\omega(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} \omega_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \phi(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} \phi_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}},$$

and equation (3.1.3), one obtains the following relationship among the Fourier modes of ω and ϕ ,

$$\phi_{\mathbf{k}} = \|\mathbf{k}\|^{-2}(1 + \alpha^2\|\mathbf{k}\|^2)^{-1}\omega_{\mathbf{k}} \quad (3.1.5)$$

for every $\mathbf{k} \neq 0$. Here and in what follows, $\|\cdot\|$ denotes the standard Euclidean norm in \mathbb{R}^2 .

Our first objective is to derive the following equation for each Fourier mode $\omega_{\mathbf{k}}$ of ω ,

$$\frac{d\omega_{\mathbf{k}}}{dt} = \sum_{\mathbf{q} \in \mathbb{Z}^2 \setminus \{0\}} \beta(\mathbf{k} - \mathbf{q}, \mathbf{q})\omega_{\mathbf{k}-\mathbf{q}}\omega_{\mathbf{q}}, \quad \mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}, \quad (3.1.6)$$

where the coefficients $\beta(\mathbf{p}, \mathbf{q})$ for $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^2$ are defined as

$$\beta(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \left(\|\mathbf{q}\|^{-2}(1 + \alpha^2\|\mathbf{q}\|^2)^{-1} - \|\mathbf{p}\|^{-2}(1 + \alpha^2\|\mathbf{p}\|^2)^{-1} \right) (\mathbf{p} \wedge \mathbf{q}) \quad (3.1.7)$$

for $\mathbf{p} \neq \pm\mathbf{q}, \mathbf{p} \neq 0, \mathbf{q} \neq 0$ and $\beta(\mathbf{p}, \mathbf{q}) = 0$ otherwise. Here

$$\mathbf{p} \wedge \mathbf{q} = \det \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} \text{ for } \mathbf{p} = (p_1, p_2) \text{ and } \mathbf{q} = (q_1, q_2). \quad (3.1.8)$$

We refer to [L] for the Euler case.

Lemma 3.1. *Equation (3.1.1) holds if and only if $\omega_{\mathbf{k}}$ satisfies equation (3.1.6) for every $\mathbf{k} \neq 0$.*

Proof. Using the facts that $v_1 = \frac{\partial\phi}{\partial y}$ and $v_2 = -\frac{\partial\phi}{\partial x}$, one can rewrite equation (3.1.1)

as

$$\frac{\partial\omega}{\partial t} = -\frac{\partial\phi}{\partial y} \frac{\partial\omega}{\partial x} + \frac{\partial\phi}{\partial x} \frac{\partial\omega}{\partial y}. \quad (3.1.9)$$

Using (3.1.5), we see that,

$$\frac{\partial\phi}{\partial x} = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} \frac{ik_1\omega_{\mathbf{k}}}{\|\mathbf{k}\|^2(1 + \alpha^2\|\mathbf{k}\|^2)}, \quad \frac{\partial\phi}{\partial y} = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} \frac{ik_2\omega_{\mathbf{k}}}{\|\mathbf{k}\|^2(1 + \alpha^2\|\mathbf{k}\|^2)}.$$

Equation (3.1.9) then reads, in terms of the Fourier series,

$$\begin{aligned} \frac{\partial \omega}{\partial t} = & - \left(\sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} \frac{ik_2 \omega_{\mathbf{k}}}{\|\mathbf{k}\|^2 (1 + \alpha^2 \|\mathbf{k}\|^2)} \omega_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \right) \left(\sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} ik_1 \omega_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \right) \\ & + \left(\sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} \frac{ik_1 \omega_{\mathbf{k}}}{\|\mathbf{k}\|^2 (1 + \alpha^2 \|\mathbf{k}\|^2)} \omega_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \right) \left(\sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} ik_2 \omega_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \right). \end{aligned} \quad (3.1.10)$$

Using the identity

$$\left(\sum_{\mathbf{n}} a_{\mathbf{n}} e^{i\mathbf{n} \cdot \mathbf{x}} \right) \left(\sum_{\mathbf{l}} b_{\mathbf{l}} e^{i\mathbf{l} \cdot \mathbf{x}} \right) = \sum_{\mathbf{k}} \left(\sum_{\mathbf{q}} a_{\mathbf{q}} b_{\mathbf{k}-\mathbf{q}} e^{i\mathbf{k} \cdot \mathbf{x}} \right)$$

first for $a_{\mathbf{n}} = n_2 \|\mathbf{n}\|^{-2} (1 + \alpha^2 \|\mathbf{n}\|^2)^{-1} \omega_{\mathbf{n}}$, $b_{\mathbf{l}} = l_1 \omega_{\mathbf{l}}$ and then for $a_{\mathbf{n}} = n_1 \|\mathbf{n}\|^{-2} (1 + \alpha^2 \|\mathbf{n}\|^2)^{-1} \omega_{\mathbf{n}}$, $b_{\mathbf{l}} = l_2 \omega_{\mathbf{l}}$, equation (3.1.10) is seen to be

$$\frac{\partial \omega}{\partial t} = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} \sum_{\mathbf{q} \in \mathbb{Z}^2 \setminus \{0\}} \frac{q_2(k_1 - q_1) - q_1(k_2 - q_2)}{\|\mathbf{q}\|^2 (1 + \alpha^2 \|\mathbf{q}\|^2)} \omega_{\mathbf{k}-\mathbf{q}} \omega_{\mathbf{q}} e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (3.1.11)$$

Alternatively, using the identity

$$\left(\sum_{\mathbf{n}} a_{\mathbf{n}} e^{i\mathbf{n} \cdot \mathbf{x}} \right) \left(\sum_{\mathbf{l}} b_{\mathbf{l}} e^{i\mathbf{l} \cdot \mathbf{x}} \right) = \sum_{\mathbf{k}} \left(\sum_{\mathbf{q}} a_{\mathbf{k}-\mathbf{q}} b_{\mathbf{q}} e^{i\mathbf{k} \cdot \mathbf{x}} \right)$$

first for $a_{\mathbf{n}} = n_2 \|\mathbf{n}\|^{-2} (1 + \alpha^2 \|\mathbf{n}\|^2)^{-1} \omega_{\mathbf{n}}$, $b_{\mathbf{l}} = l_1 \omega_{\mathbf{l}}$ and then for $a_{\mathbf{n}} = n_1 \|\mathbf{n}\|^{-2} (1 + \alpha^2 \|\mathbf{n}\|^2)^{-1} \omega_{\mathbf{n}}$, $b_{\mathbf{l}} = l_2 \omega_{\mathbf{l}}$, equation (3.1.10) is seen to be

$$\frac{\partial \omega}{\partial t} = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} \sum_{\mathbf{q} \in \mathbb{Z}^2 \setminus \{0\}} \frac{q_1(k_2 - q_2) - q_2(k_1 - q_1)}{\|\mathbf{k} - \mathbf{q}\|^2 (1 + \alpha^2 \|\mathbf{k} - \mathbf{q}\|^2)} \omega_{\mathbf{k}-\mathbf{q}} \omega_{\mathbf{q}} e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (3.1.12)$$

Noticing that $\frac{\partial \omega}{\partial t} = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} \frac{d\omega_{\mathbf{k}}}{dt} e^{i\mathbf{k} \cdot \mathbf{x}}$ and taking the average of (3.1.11) and (3.1.12) we obtain that (3.1.6) for each mode $\omega_{\mathbf{k}}$ of ω holds if and only if (3.1.1) holds. ■

The choice of spaces for the sequences $(\omega_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^2}$ depends on the choice of vorticity in (3.1.1); if $\omega \in H^s(\mathbb{T}^2)$, the Sobolev space, then $(\omega_{\mathbf{k}}) \in \ell_s^2(\mathbb{Z}^2)$, the space of sequences square summable with the weight $(1 + \|\mathbf{k}\|^{2s})^{1/2}$. In what follows we will mainly consider the case $s = 0$, that is, $\omega \in L^2(\mathbb{T}^2)$ and $(\omega_{\mathbf{k}}) \in \ell^2(\mathbb{Z}^2)$.

3.2 Unidirectional flows

A *unidirectional flow* is the flow induced by a time independent solution ω^0 of (3.1.1) that has only one nonzero Fourier mode, that is,

$$\omega^0(\mathbf{x}) = \operatorname{Re}(\Gamma e^{i\mathbf{p}\cdot\mathbf{x}}) \text{ for given } \mathbf{p} \in \mathbb{Z}^2 \setminus \{0\} \text{ and } \Gamma \in \mathbb{C}. \quad (3.2.1)$$

A well-known example of the unidirectional flow is given by the Kolmogorov flow with vorticity $\omega^0(\mathbf{x}) = \cos(mx_1)$, $m = 1, 2, \dots$, (see, e.g., [MS]); this corresponds to the choice $\mathbf{p} = (m, 0)$ and $\Gamma = 1$. In the case when $m = 1$ the steady state solution of the Euler equation is called in [BW] a bar-state. The unidirectional flows have been studied by many authors, see e.g. [L, LLS, WDM1, WDM2] and the literature therein. We use notation L_B for the linearization about the steady state (3.2.1); here B stands for bar state.

Lemma 3.2. *A unidirectional flow given by the vorticity $\omega^0(\mathbf{x}) = \operatorname{Re}(\Gamma e^{i\mathbf{p}\cdot\mathbf{x}})$, where $\mathbf{p} \in \mathbb{Z}^2 \setminus \{0\}$ is fixed, i.e.,*

$$\omega_{\mathbf{k}}^0 = \begin{cases} \Gamma & \text{if } \mathbf{k} = \mathbf{p}, \\ \bar{\Gamma} & \text{if } \mathbf{k} = -\mathbf{p}, \\ 0 & \text{if } \mathbf{k} \neq \pm\mathbf{p}, \end{cases} \quad (3.2.2)$$

is a steady state solution of the α -Euler equation (3.1.1) on the torus \mathbb{T}^2 .

Proof. For every $\mathbf{k} \neq 0$ one needs to check that the right hand side of (3.1.6) is zero, where the Fourier coefficients of $\omega_{\mathbf{k}}^0$ are given by (3.2.2). Since $\omega_{\mathbf{q}}^0$ is nonzero only when $\mathbf{q} = \pm\mathbf{p}$, the right hand side of (3.2.2) reduces to

$$\beta(\mathbf{k} - \mathbf{p}, \mathbf{p})\omega_{\mathbf{k}-\mathbf{p}}^0\omega_{\mathbf{p}}^0 + \beta(\mathbf{k} + \mathbf{p}, \mathbf{p})\omega_{\mathbf{k}+\mathbf{p}}^0\omega_{\mathbf{p}}^0.$$

Now using the fact that $\omega_{\mathbf{k}-\mathbf{p}}^0$ is nonzero only when $\mathbf{k} - \mathbf{p} = \pm\mathbf{p}$ and $\omega_{\mathbf{k}+\mathbf{p}}^0$ is nonzero

only when $\mathbf{k} + \mathbf{p} = \pm \mathbf{p}$ and using (3.2.2), the above equation reduces to

$$\beta(\mathbf{p}, \mathbf{p})\Gamma^2 + \beta(-\mathbf{p}, \mathbf{p})\bar{\Gamma}\Gamma + \beta(\mathbf{p}, -\mathbf{p})\Gamma\bar{\Gamma} + \beta(-\mathbf{p}, -\mathbf{p})\bar{\Gamma}^2,$$

which is zero because $\beta(\mathbf{p}, \pm \mathbf{p}) = 0$. ■

Linearizing (3.1.6) about the unidirectional flow, we will consider in $\ell^2(\mathbb{Z}^2)$ the following operator,

$$L_B : (\omega_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^2} \mapsto (\beta(\mathbf{p}, \mathbf{k} - \mathbf{p})\Gamma\omega_{\mathbf{k}-\mathbf{p}} - \beta(\mathbf{p}, \mathbf{k} + \mathbf{p})\bar{\Gamma}\omega_{\mathbf{k}+\mathbf{p}})_{\mathbf{k} \in \mathbb{Z}^2}, \quad (3.2.3)$$

where $\bar{\Gamma}$ is the complex conjugate of Γ .

We briefly indicate how to obtain equation (3.2.3). Linearizing the right hand side of (3.1.6) about the steady state (3.2.1) reduces the right hand side of (3.1.6) to

$$\sum_{\mathbf{q} \in \mathbb{Z}^2 \setminus \{0\}} \beta(\mathbf{k} - \mathbf{q}, \mathbf{q})\omega_{\mathbf{k}-\mathbf{q}}^0\omega_{\mathbf{q}} + \sum_{\mathbf{q} \in \mathbb{Z}^2 \setminus \{0\}} \beta(\mathbf{k} - \mathbf{q}, \mathbf{q})\omega_{\mathbf{k}-\mathbf{q}}\omega_{\mathbf{q}}^0, \quad (3.2.4)$$

where in the first sum, $\omega_{\mathbf{k}-\mathbf{q}}^0 = \Gamma/2$ if $\mathbf{k} - \mathbf{q} = \mathbf{p}$, i.e., if $\mathbf{q} = \mathbf{k} - \mathbf{p}$ and $\omega_{\mathbf{k}-\mathbf{q}}^0 = \bar{\Gamma}/2$ if $\mathbf{k} - \mathbf{q} = -\mathbf{p}$, i.e., if $\mathbf{q} = \mathbf{k} + \mathbf{p}$ and zero otherwise and in the second sum, $\omega_{\mathbf{q}}^0 = \Gamma/2$ if $\mathbf{q} = \mathbf{p}$ and $\omega_{\mathbf{q}}^0 = \bar{\Gamma}/2$ if $\mathbf{q} = -\mathbf{p}$ and zero otherwise. Using these in (3.2.4), we see that it reduces to,

$$\beta(\mathbf{p}, \mathbf{k} - \mathbf{p})\frac{\Gamma}{2}\omega_{\mathbf{k}-\mathbf{p}} + \beta(-\mathbf{p}, \mathbf{k} + \mathbf{p})\frac{\bar{\Gamma}}{2}\omega_{\mathbf{k}+\mathbf{p}} + \beta(\mathbf{k} - \mathbf{p}, \mathbf{p})\frac{\Gamma}{2}\omega_{\mathbf{k}-\mathbf{p}} + \beta(\mathbf{k} + \mathbf{p}, -\mathbf{p})\frac{\bar{\Gamma}}{2}\omega_{\mathbf{k}+\mathbf{p}}.$$

Now use the facts that if $\mathbf{p} \neq \mathbf{q}$, then $\beta(\mathbf{p}, \mathbf{q}) = \beta(\mathbf{q}, \mathbf{p})$ and $\beta(-\mathbf{p}, \mathbf{q}) = -\beta(\mathbf{p}, \mathbf{q})$ in the above equation to get (3.2.3).

Our objective is to show that the spectrum of the operator L_B contains an unstable eigenvalue (having a positive real part) provided $\|\mathbf{p}\|$ is large enough.

3.3 Instability of the unidirectional flows

In this section we first review some results regarding the operator L_B defined in (3.2.3). We use the approach taken in [L, LLS, WDM1, WDM2]. Next, we show the existence of a positive eigenvalue of L_B . Our main result is Theorem 3.9 proved below.

3.3.1 Slicing and decomposition of subspaces and operators

In this subsection we follow [L, LLS, WDM1] and explain how to decompose the operator L_B acting in $\ell^2(\mathbb{Z}^2)$ into the direct sum of operators $L_{B,\mathbf{q}}$, $\mathbf{q} \in \mathcal{Q} \subset \mathbb{Z}^2$, acting in the space $\ell^2(\mathbb{Z})$, for some set $\mathcal{Q} \subset \mathbb{Z}^2$.

Let $\mathbf{p} \in \mathbb{Z}^2$ be the fixed vector from (3.2.1). Our first objective is to construct the set \mathcal{Q} such that the translated vectors of the form $\mathbf{q} + n\mathbf{p}$, with $n \in \mathbb{Z}$ and $\mathbf{q} \in \mathcal{Q}$, cover the entire grid \mathbb{Z}^2 in a way that for different \mathbf{q} and \mathbf{q}' from \mathcal{Q} the sets of the translated vectors, formed by all $n \in \mathbb{Z}$, are disjoint. To begin the construction, for any $\mathbf{q} \in \mathbb{Z}^2$ we denote $\Sigma_{B,\mathbf{q}} = \{\mathbf{q} + n\mathbf{p} : n \in \mathbb{Z}\}$ and note that the line $\{\mathbf{q} + t\mathbf{p} : t \in \mathbb{R}\}$ may contain several different sets $\Sigma_{B,\mathbf{q}'}$. For a given \mathbf{q} , we let $\tau = \tau(\mathbf{q})$ temporarily denote the radius of the smallest circle centered at zero that has a nonempty intersection with the set $\Sigma_{B,\mathbf{q}}$. The intersection consists of either one point (which we will denote by $\hat{\mathbf{q}}$) or two points (in this case we denote by $\hat{\mathbf{q}}$ one of them). In other words, for each $\mathbf{q} \in \mathbb{Z}^2$ we identify the unique vector $\hat{\mathbf{q}} = \hat{\mathbf{q}}(\mathbf{q})$ in $\Sigma_{B,\mathbf{q}}$ such that the following holds:

$$\|\hat{\mathbf{q}}\| = \min\{\|\mathbf{q} + n\mathbf{p}\| : n \in \mathbb{Z}\} \text{ and}$$

$$\hat{\mathbf{q}} = \mathbf{q} + n_{\max}\mathbf{p}, \text{ where } n_{\max} = \max\{n : \|\mathbf{q} + n\mathbf{p}\| = \|\hat{\mathbf{q}}\|\}.$$

Here the last condition just fixes one of the possibly two points in $\Sigma_{B,\mathbf{q}}$ that belong to the circle of radius $\tau = \|\widehat{\mathbf{q}}\|$. We let $\mathcal{Q} = \{\widehat{\mathbf{q}}(\mathbf{q}) : \mathbf{q} \in \mathbb{Z}^2\}$.

We will now decompose the operator L_B in $\ell^2(\mathbb{Z}^2)$ into a direct sum of operators acting on the spaces isomorphic to $\ell^2(\mathbb{Z})$. Indeed, for each $\mathbf{q} \in \mathcal{Q}$ we denote by $X_{B,\mathbf{q}}$ the subspace of $\ell^2(\mathbb{Z}^2)$ of sequences supported in $\Sigma_{B,\mathbf{q}}$, that is, we let $X_{B,\mathbf{q}} = \{(\omega_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^2} : \omega_{\mathbf{k}} = 0 \text{ for all } \mathbf{k} \notin \Sigma_{B,\mathbf{q}}\}$. Clearly, $\ell^2(\mathbb{Z}^2) = \bigoplus_{\mathbf{q} \in \mathcal{Q}} X_{B,\mathbf{q}}$, the operator L_B leaves $X_{B,\mathbf{q}}$ invariant, and therefore $L_B = \bigoplus_{\mathbf{q} \in \mathcal{Q}} L_{B,\mathbf{q}}$ where $L_{B,\mathbf{q}}$ is the restriction of L_B onto $X_{B,\mathbf{q}}$. To stress that L_B depends on \mathbf{p} from (3.2.1), we sometimes write $L_B(\mathbf{p})$ and $L_{B,\mathbf{q}}(\mathbf{p})$. For $\mathbf{k} = \mathbf{q} + n\mathbf{p} \in \Sigma_{B,\mathbf{q}}$ we denote $w_n = \omega_{\mathbf{q}+n\mathbf{p}}$, $n \in \mathbb{Z}$, and remark that the map $(\omega_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^2} \mapsto (w_n)_{n \in \mathbb{Z}}$ is an isomorphism of $X_{B,\mathbf{q}}$ onto $\ell^2(\mathbb{Z})$. Under this isomorphism the operator $L_{B,\mathbf{q}}$ in $X_{B,\mathbf{q}}$ induces an operator in $\ell^2(\mathbb{Z})$ (that we will still denote by $L_{B,\mathbf{q}}$) given by the formula

$$L_{B,\mathbf{q}} : (w_n)_{n \in \mathbb{Z}} \mapsto (\beta(\mathbf{p}, \mathbf{q} + (n-1)\mathbf{p})\Gamma w_{n-1} - \beta(\mathbf{p}, \mathbf{q} + (n+1)\mathbf{p})\bar{\Gamma} w_{n+1})_{n \in \mathbb{Z}}. \quad (3.3.1)$$

By (3.1.7), if \mathbf{q} is parallel to \mathbf{p} then $L_{B,\mathbf{q}}(\mathbf{p}) = 0$; therefore, in what follows we will always assume that \mathbf{q} and \mathbf{p} are not parallel.

We recall that $H^s(\mathbb{T}^2)$ is the Sobolev space of 2π -periodic L^2 functions with s derivatives in L^2 . Via Fourier transform, $H^s(\mathbb{T}^2)$ is isometrically isomorphic to $\ell_s^2(\mathbb{Z}^2)$, the set of sequences $(\omega_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^2}$ which are ℓ^2 summable with the weight $(1 + \|\mathbf{k}\|^{2s})^{1/2}$. As above, we may decompose $\ell_s^2(\mathbb{Z}^2) = \bigoplus_{\mathbf{q} \in \mathcal{Q}} X_{B,\mathbf{q},s}$, where $X_{B,\mathbf{q},s}$ is the space $\ell_s^2(\mathbb{Z})$ with the weight $(1 + \|\mathbf{q} + n\mathbf{p}\|^{2s})^{1/2}$.

Our objective is to study the spectrum of $L_{B,\mathbf{q}}$ in $\ell^2(\mathbb{Z})$. From now on we assume that $\Gamma \in \mathbb{R}$. Then $L_{B,\mathbf{q}}$ can be written as $L_{B,\mathbf{q}} = (S - S^*) \text{diag}_{n \in \mathbb{Z}}\{\rho_n\}$, where

$S : (w_n)_{n \in \mathbb{Z}} \mapsto (w_{n-1})_{n \in \mathbb{Z}}$ is the shift operator in $\ell^2(\mathbb{Z})$ and we introduce the notation

$$\begin{aligned} \rho_n &= \Gamma\beta(\mathbf{p}, \mathbf{q} + n\mathbf{p}) = \frac{1}{2}\Gamma(\mathbf{q} \wedge \mathbf{p}) \\ &\times \left(\frac{1}{\|\mathbf{p}\|^2(1 + \alpha^2\|\mathbf{p}\|^2)} - \frac{1}{\|\mathbf{q} + n\mathbf{p}\|^2(1 + \alpha^2\|\mathbf{q} + n\mathbf{p}\|^2)} \right), \quad n \in \mathbb{Z}, \end{aligned} \quad (3.3.2)$$

see the definition of $\mathbf{q} \wedge \mathbf{p}$ in (3.1.8).

The spectrum of the operator $L_{B,\mathbf{q}}$ is symmetric about coordinate axes as shown in the following lemma taken from [LLS, Proposition 4, page 269].

Lemma 3.3. *The eigenvalues λ of $L_{B,\mathbf{q}}$ with non-zero real and imaginary parts are symmetric about the coordinate axes, i.e., if λ is an eigenvalue (with $\operatorname{Re}(\lambda) \neq 0$ and $\operatorname{Im}(\lambda) \neq 0$), then $-\lambda, \bar{\lambda}, -\bar{\lambda}$ are also eigenvalues.*

Proof. Note that $L_{B,\mathbf{q}} = (S - S^*) \operatorname{diag}_{n \in \mathbb{Z}} \{\rho_n\}$, where

$$(S - S^*)^* = S^* - S = -(S - S^*).$$

We thus have that,

$$\begin{aligned} \sigma(L_{B,\mathbf{q}}^*) \setminus \{0\} &= \sigma(\rho_n(S - S^*)^*) \setminus \{0\} = -\sigma(\rho_n(S - S^*)) \setminus \{0\} \\ &= -\sigma((S - S^*)\rho_n) \setminus \{0\} = -\sigma(L_{B,\mathbf{q}}) \setminus \{0\}. \end{aligned}$$

Thus $\sigma(L_{B,\mathbf{q}}) \setminus \{0\} = \overline{\sigma(L_{B,\mathbf{q}}^*)} \setminus \{0\} = -\overline{\sigma(L_{B,\mathbf{q}})} \setminus \{0\}$. Thus the eigenvalues are symmetric about the imaginary axes.

The fact that the eigenvalues are symmetric about the real axes can be proved as follows. The fact that if λ is an eigenvalue then $\bar{\lambda}$ is also an eigenvalue is a consequence of the fact that $\overline{L_{B,\mathbf{q}}\mathbf{v}} = L_{B,\mathbf{q}}\bar{\mathbf{v}}$ for any $\mathbf{v} \in \ell^2(\mathbb{Z})$. From this it follows that if λ is an eigenvalue with eigenvector \mathbf{v} , then $\bar{\lambda}$ is an eigenvalue with eigenvector $\bar{\mathbf{v}}$. This proves the Lemma.

Additionally, one can also prove the fact that if λ is an eigenvalue then $-\lambda$ is also an eigenvalue. Let \hat{J} be an operator on $\ell^2(\mathbb{Z})$ defined by $(\omega_n) \mapsto ((-1)^n \omega_n)$ and notice that $\hat{J}S = -S\hat{J}$ and $\hat{J}S^* = -S^*\hat{J}$ and $\hat{J}^2 = I$. Thus,

$$\hat{J}L_{B,\mathbf{q}}\hat{J} = \hat{J}((S - S^*) \text{diag}_{n \in \mathbb{Z}}\{\rho_n\})\hat{J} = -L_{B,\mathbf{q}}.$$

Thus,

$$\sigma(L_{B,\mathbf{q}}) = \sigma(L_{B,\mathbf{q}}\hat{J}\hat{J}) = \sigma(\hat{J}L_{B,\mathbf{q}}\hat{J}) = -\sigma(L_{B,\mathbf{q}}),$$

which concludes the proof. ■

Thus to prove spectral instability of the unidirectional flow we need to show the existence for at least one $\mathbf{q} \in \mathcal{Q}$ of an eigenvalue of $L_{B,\mathbf{q}}$ with nonzero imaginary part. In turn, this is equivalent to showing that the spectrum $\sigma(\frac{1}{c}L_{B,\mathbf{q}}) = \frac{1}{c}\sigma(L_{B,\mathbf{q}})$ of a multiple of $L_{B,\mathbf{q}}$ has an eigenvalue with nonzero imaginary part. Dividing $L_{B,\mathbf{q}}$ by an n -independent real multiple $c = \frac{\frac{1}{2}\Gamma(\mathbf{q} \wedge \mathbf{p})}{\|\mathbf{p}\|^2(1+\alpha^2\|\mathbf{p}\|^2)}$, we pass to the operator $\frac{1}{c}L_{B,\mathbf{q}}$ of the same structure as $L_{B,\mathbf{q}}$ but with the term $\frac{\frac{1}{2}\Gamma(\mathbf{q} \wedge \mathbf{p})}{\|\mathbf{p}\|^2(1+\alpha^2\|\mathbf{p}\|^2)}$ in (3.3.2) replaced by 1. In fact, this procedure is equivalent to rescaling Γ . In order to simplify notations we will assume in what follows that Γ already satisfies the condition $\frac{\frac{1}{2}\Gamma(\mathbf{q} \wedge \mathbf{p})}{\|\mathbf{p}\|^2(1+\alpha^2\|\mathbf{p}\|^2)} = 1$ in (3.3.2). Let

$$\gamma_n = -\frac{\|\mathbf{p}\|^2(1 + \alpha^2\|\mathbf{p}\|^2)}{\|\mathbf{q} + n\mathbf{p}\|^2(1 + \alpha^2\|\mathbf{q} + n\mathbf{p}\|^2)}. \quad (3.3.3)$$

With the above assumptions, we see that $\rho_n = 1 + \gamma_n$. Therefore, we want to study the spectrum of the operator

$$L_{B,\mathbf{q}} = (S - S^*) \text{diag}_{n \in \mathbb{Z}}\{1 + \gamma_n\}. \quad (3.3.4)$$

Remark 3.4. We will now classify points \mathbf{q} as follows. Recall notations \mathbf{q} and \mathcal{Q} introduced in the beginning of Subsection 3.3.1. For any $\mathbf{q} \in \mathbb{Z}^2$ the intersection of the set $\Sigma_{B,\mathbf{q}} = \{\mathbf{q} + n\mathbf{p} : n \in \mathbb{Z}\}$ with the open disc of radius $\|\mathbf{p}\|$ may have either none, or one, or two points (with integer coordinates). If this is the case then we call \mathbf{q} a point of type zero, one, or two, respectively. These are sometimes termed, respectively as \mathbf{q} being of type 0, *I* and *II*. Thus, in what follows we may assume that \mathbf{q} is either a point of type zero or one or two.

In particular, if $\mathbf{q} \in \mathbb{Z}^2$ is a point of type one, the set $\Sigma_{B,\mathbf{q}} = \{\mathbf{q} + n\mathbf{p} : n \in \mathbb{Z}\}$ contains exactly one vector $\hat{\mathbf{q}} = \hat{\mathbf{q}}(\mathbf{q})$ whose norm is strictly smaller than $\|\mathbf{p}\|$. We further classify points of type *I* as follows. We say that \mathbf{q} is of type I_0 if all other vectors in $\Sigma_{B,\mathbf{q}}$ have norms strictly larger than $\|\mathbf{p}\|$. This means that the only vector in $\Sigma_{B,\mathbf{q}}$ whose norm does not exceed $\|\mathbf{p}\|$ is located strictly inside the disk of radius $\|\mathbf{p}\|$. There are two more possibilities for $\hat{\mathbf{q}}(\mathbf{q}) \in \Sigma_{B,\mathbf{q}}$ to be strictly inside the disc of radius $\|\mathbf{p}\|$. The first possibility is when the preceding point, $\hat{\mathbf{q}}(\mathbf{q}) - \mathbf{p}$, belongs to the boundary of the disc and the second possibility is when the following point $\hat{\mathbf{q}}(\mathbf{q}) + \mathbf{p}$ belongs to the boundary of the disc. These two cases are classified as type I_- and I_+ respectively: we say that \mathbf{q} is of type I_- if $\|\hat{\mathbf{q}}(\mathbf{q})\| < \|\mathbf{p}\|$, $\|\hat{\mathbf{q}}(\mathbf{q}) - \mathbf{p}\| = \|\mathbf{p}\|$, and all other vectors in $\Sigma_{B,\mathbf{q}}$ have norms strictly larger than $\|\mathbf{p}\|$. We say that \mathbf{q} is of type I_+ if $\|\hat{\mathbf{q}}(\mathbf{q})\| < \|\mathbf{p}\|$, $\|\hat{\mathbf{q}}(\mathbf{q}) + \mathbf{p}\| = \|\mathbf{p}\|$, and all other vectors in $\Sigma_{B,\mathbf{q}}$ have norms strictly larger than $\|\mathbf{p}\|$.

Example 3.5. Let $\mathbf{p} = (1, 2)$. Then $\hat{\mathbf{q}} = (1, -1)$ is of type I_+ , while $\hat{\mathbf{q}} = (-1, 1)$ is of type I_- whereas $\hat{\mathbf{q}} = (-1, 0)$ is of type two. Let $\mathbf{p} = (2, 0)$. Then $\hat{\mathbf{q}} = (0, 1)$ is of type I_0 . See [WDM1]: If $\mathbf{p} = (3, 1)$ then $\mathbf{q} = (-1, 2)$ is of type I_0 . If $\mathbf{p} = (3, 1)$ then

$\mathbf{q} = (0, 1)$ is of type *II*.

In what follows, dealing with the operator $L_{B,\mathbf{q}}$ from (3.3.1), for a point \mathbf{q} of type *I*, we will drop hat in the notation $\hat{\mathbf{q}}$, that is, we assume that $\mathbf{q} \in \mathbb{Z}^2$ satisfies $\|\mathbf{q}\| < \|\mathbf{p}\|$. By formula (3.3.3), we have that $\rho_0 = 1 + \gamma_0$. If \mathbf{q} is of type I_0 then $\rho_n = 1 + \gamma_n > 0$ for all $n \neq 0$. If \mathbf{q} is of type I_+ , then $\rho_1 = 1 + \gamma_1 = 0 < 0$ and $\rho_n = 1 + \gamma_n > 0$ for all $n \neq 0, 1$. If \mathbf{q} is of type I_- , then $\rho_{-1} = 1 + \gamma_{-1} = 0$ and $\rho_n = 1 + \gamma_n > 0$ for all $n \neq 0, -1$.

Remark 3.6. (1) Assume $\|\mathbf{q}\| \geq \|\mathbf{p}\|$, that is, \mathbf{q} is a point of type zero. Since

$\mathbf{q} \in \mathcal{Q}$ is chosen to minimize $\|\mathbf{q} + n\mathbf{p}\|$, we know that $\|\mathbf{q} + n\mathbf{p}\| \geq \|\mathbf{p}\|$ and therefore $|\gamma_n| \leq 1$ or $1 + \gamma_n \geq 0$ for all $n \in \mathbb{Z}$.

(2) Assume $\|\mathbf{q}\| < \|\mathbf{p}\|$ and the line $\Sigma_{B,\mathbf{q}}$ has exactly one point in the open disc of radius $\|\mathbf{p}\|$ (that is, $\|\mathbf{q}\|$ is a point of type one). Then $\frac{\|\mathbf{p}\|}{\|\mathbf{q}\|} > 1$ and $\frac{(1+\alpha^2\|\mathbf{p}\|^2)}{(1+\alpha^2\|\mathbf{q}\|^2)} > 1$. So $1 + \gamma_0 < 0$ and $1 + \gamma_n \geq 0$ for all $n \neq 0$.

(3) Assume \mathbf{q} is a point of type *II*. Without loss of generality suppose that $\frac{\|\mathbf{p}\|}{\|\mathbf{q}\|} > 1$ and $\frac{\|\mathbf{p}\|}{\|\mathbf{q}+\mathbf{p}\|} > 1$ and $\frac{\|\mathbf{p}\|}{\|\mathbf{q}+n\mathbf{p}\|} \leq 1$ for all $n \in \mathbb{Z} \setminus \{0, 1\}$. Then $1 + \gamma_0 < 0$, $1 + \gamma_1 < 0$ but $1 + \gamma_n \geq 0$ for all $n \in \mathbb{Z} \setminus \{0, 1\}$.

Denote, using (3.3.3),

$$\delta_n = \begin{cases} \sqrt{1 + \gamma_n} & \text{for } 1 + \gamma_n \geq 0, \text{ when } \delta_n \in \mathbb{R}, \\ i\sqrt{|1 + \gamma_n|} & \text{for } 1 + \gamma_n < 0, \text{ when } \delta_n \in i\mathbb{R}, \end{cases} \quad (3.3.5)$$

so that $\delta_n^2 = 1 + \gamma_n$. Since $L_{B,\mathbf{q}} = (S - S^*) \text{diag}_{n \in \mathbb{Z}}\{\delta_n\} \text{diag}_{n \in \mathbb{Z}}\{\delta_n\}$ we claim that the nonzero elements of the spectrum of $L_{B,\mathbf{q}}$ coincide with the nonzero elements of the spectrum of the operator $M_{\mathbf{q}}$ defined as follows:

$$M_{\mathbf{q}} = \text{diag}_{n \in \mathbb{Z}}\{\delta_n\}(S - S^*) \text{diag}_{n \in \mathbb{Z}}\{\delta_n\}. \quad (3.3.6)$$

This claim is a consequence of the fact that the nonzero elements of the spectrum of AB coincide with the nonzero elements of the spectrum of BA , where A and B are bounded linear operators. The proof of this fact first requires the following elementary lemma.

Lemma 3.7. *$I - AB$ is invertible if and only if $I - BA$ is invertible and*

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

Proof. A simple calculation shows

$$\begin{aligned} (I - BA)[I + B(I - AB)^{-1}A] &= I - BA + (I - BA)B(I - AB)^{-1}A \\ &= I - BA + B(I - AB)(I - AB)^{-1}A = I - BA + BA = I. \end{aligned}$$

Another simple calculation shows

$$\begin{aligned} [I + B(I - AB)^{-1}A](I - BA) &= I - BA + B(I - AB)^{-1}A(I - BA) \\ &= I - BA + B(I - AB)^{-1}(I - AB)A = I - BA + BA = I. \end{aligned}$$

This concludes the proof. ■

Lemma 3.8. *Suppose $A, B : X \rightarrow X$ are bounded linear operators on a Banach space X . Then*

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}.$$

Proof. Suppose $\lambda \in \sigma(AB) \setminus \{0\}$ and $\lambda \notin \sigma(BA) \setminus \{0\}$. Then $(\lambda - BA)^{-1}$ exists. That is, $\lambda^{-1}(I - \frac{1}{\lambda}BA)^{-1}$ exists. By Lemma 3.7, we have that $\lambda^{-1}(I - \frac{1}{\lambda}AB)^{-1}$ exists, i.e., $(\lambda - AB)^{-1}$ exists which is a contradiction. ■

Because of the claim just proved we can study the spectrum of the operator $M_{\mathbf{q}}$ instead of $L_{B,\mathbf{q}}$.

If \mathbf{q} is a point of type zero then $L_{B,\mathbf{q}}$ has no unstable point spectrum by [LLS, Remark 4]. Indeed, if $\delta_n \in \mathbb{R}$ for all n , i.e., \mathbf{q} is a point of type zero and $\|\mathbf{q}\| \geq \|\mathbf{p}\|$, then $M_{\mathbf{q}}^* = -M_{\mathbf{q}}$, i.e., $M_{\mathbf{q}}$ is skew-adjoint and its spectrum is thus purely imaginary.

We now consider $M_{\mathbf{q}}$ for \mathbf{q} being of type one or two. Then two cases are possible:

- (1) $\delta_0 \in i\mathbb{R}$ and $\delta_n \in \mathbb{R}$ for all $n \neq 0$ (case 2 in the list given in Remark 3.6),
- (2) $\delta_0, \delta_1 \in i\mathbb{R}$ and $\delta_n \in \mathbb{R}$ for all $n \neq 0, 1$ (case 3 in the list given in Remark 3.6).

The operator $M_{\mathbf{q}}$ has the following structure:

$$M_{\mathbf{q}} = \begin{bmatrix} \ddots & & & & & & & & & & & \\ & 0 & -\delta_{-2}\delta_{-1} & 0 & 0 & 0 & & & & & & \\ & \delta_{-2}\delta_{-1} & 0 & -\delta_{-1}\delta_0 & 0 & 0 & & & & & & \\ & 0 & \delta_{-1}\delta_0 & \boxed{0} & -\delta_0\delta_1 & 0 & & & & & & \\ & 0 & 0 & \delta_0\delta_1 & 0 & -\delta_1\delta_2 & & & & & & \\ & 0 & 0 & 0 & \delta_1\delta_2 & 0 & & & & & & \\ & & & & & & \ddots & & & & & \end{bmatrix}.$$

We remark that $\delta_n \rightarrow 1$ and $n \rightarrow \infty$ since $\gamma_n \rightarrow 0$ and that $M_{\mathbf{q}}$ is a compact perturbation of $S - S^*$, therefore $sp_{ess}(M_{\mathbf{q}}) = sp(S - S^*) = i[-2, 2]$.

In case (1) the block

$$\begin{bmatrix} \ddots & & & & & & & & & & & \\ & 0 & -\delta_{-1}\delta_0 & 0 & & & & & & & & \\ & \delta_{-1}\delta_0 & \boxed{0} & -\delta_0\delta_1 & & & & & & & & \\ & 0 & \delta_0\delta_1 & 0 & & & & & & & & \\ & & & & \ddots & & & & & & & \end{bmatrix}$$

is self adjoint while the remaining part of $M_{\mathbf{q}}$ is skew-adjoint because $\delta_{l-1}\delta_l \in i\mathbb{R}$ only for $i = 0, 1$ and $\delta_{l-1}\delta_l \in \mathbb{R}$ for $l \neq 0, 1$.

In case (2) $\delta_0, \delta_1 \in i\mathbb{R}$ and $\delta_n \in \mathbb{R}$ for $n \neq 0, 1$ then $\delta_{l-1}\delta_l \in i\mathbb{R}$ provided that $l = 0, 2$ and $\delta_{l-1}\delta_l \in \mathbb{R}$ for $l \neq 0, 2$. This means that in case (1) or (2) we do not

know that the spectrum of $M_{\mathbf{q}}$ is purely imaginary and thus, there is a possibility that unstable eigenvalues exist.

If \mathbf{q} is a point of type one then the argument given in Subsection 3.3.2, cf. [WDM1] and based on the use of continued fractions yields the existence of an unstable eigenvalue for $L_{B,\mathbf{q}}$. In fact, we adopt to the current setting the proof from [FH98] used for the Orr-Sommerfeld operator, see also [MS].

3.3.2 Unstable eigenvalues for unidirectional flows in case of the point of type one

We will first outline an informal argument implying that if \mathbf{q} is the only point in $\Sigma_{B,\mathbf{q}} = \{\mathbf{q} + n\mathbf{p} : n \in \mathbb{Z}\}$ satisfying $\|\mathbf{q}\| < \|\mathbf{p}\|$ then $L_{B,\mathbf{q}}$ has a positive eigenvalue. Recall that by (3.3.3) the coefficients in $L_{B,\mathbf{q}}$ from (3.3.4) are given as follows:

$$\rho_n = 1 + \gamma_n = 1 - \frac{\|\mathbf{p}\|^2(1 + \alpha^2\|\mathbf{p}\|^2)}{\|\mathbf{q} + n\mathbf{p}\|^2(1 + \alpha^2\|\mathbf{q} + n\mathbf{p}\|^2)}.$$

We recall the classification given in Remark 3.4. For simplicity, we first consider a point \mathbf{q} of type I_0 . Then $\|\mathbf{q}\| < \|\mathbf{p}\|$ and $\|\mathbf{q} + n\mathbf{p}\| > \|\mathbf{p}\|$ for all $n \neq 0$. That is, $-1 < \gamma_n < 0$ for all $n \neq 0$ and $\gamma_0 < -1$. This yields that in this case (of point \mathbf{q} of type I_0)

$$\rho_0 < 0 \text{ and } \rho_n > 0 \text{ for all } n \neq 0. \quad (3.3.7)$$

We note that the eigenvalue problem

$$L_{B,\mathbf{q}}(w_n)_{n \in \mathbb{Z}} = \lambda(w_n)_{n \in \mathbb{Z}} \quad (3.3.8)$$

is equivalent to the difference equation

$$z_{n-1} - z_{n+1} = \frac{\lambda}{\rho_n} z_n, \quad \rho_n = 1 - \frac{\|\mathbf{p}\|^2(1 + \alpha^2\|\mathbf{p}\|^2)}{\|\mathbf{q} + n\mathbf{p}\|^2(1 + \alpha^2\|\mathbf{q} + n\mathbf{p}\|^2)}, \quad n \in \mathbb{Z}, \quad (3.3.9)$$

where we have denoted $z_n = \rho_n w_n$. Note that $\rho_n \rightarrow 1$ as $|n| \rightarrow \infty$. We introduce the notation $u_n = z_{n-1}/z_n$ and re-write (3.3.9) as

$$u_n = \frac{\lambda}{\rho_n} + \frac{1}{u_{n+1}} \quad \text{or} \quad u_{n+1} = -\frac{1}{\frac{\lambda}{\rho_n} - u_n}, \quad n \in \mathbb{Z}. \quad (3.3.10)$$

Forwards iterating the first equation above for $n \geq 0$ and backwards iterating the second equation for $n \leq 0$, we obtain the following two sequences depending on λ :

$$u_n^{(1)}(\lambda) = \frac{\lambda}{\rho_n} + \frac{1}{\frac{\lambda}{\rho_{n+1}} + \frac{1}{\frac{\lambda}{\rho_{n+2}} + \dots}}, \quad n = 0, 1, 2, \dots, \quad (3.3.11)$$

$$u_{n+1}^{(2)}(\lambda) = -\frac{1}{\frac{\lambda}{\rho_n} + \frac{1}{\frac{\lambda}{\rho_{n-1}} + \frac{1}{\frac{\lambda}{\rho_{n-2}} + \dots}}}, \quad n = -1, -2, \dots \quad (3.3.12)$$

Note that the two continued fractions converge, by Van Vleck Theorem, see [JT, Theorem 4.29]. As we prove in Lemma 3.11 (1) below, λ is an eigenvalue of $L_{B,\mathbf{q}}$ if and only if $u_0^{(1)}(\lambda) = u_0^{(2)}(\lambda)$, or, equivalently, if and only if

$$\frac{\lambda}{\rho_0} + f(\lambda) + g(\lambda) = 0, \quad (3.3.13)$$

where we denote

$$f(\lambda) = \frac{1}{\frac{\lambda}{\rho_1} + \frac{1}{\frac{\lambda}{\rho_2} + \dots}}, \quad g(\lambda) = \frac{1}{\frac{\lambda}{\rho_{-1}} + \frac{1}{\frac{\lambda}{\rho_{-2}} + \dots}}, \quad (3.3.14)$$

provided \mathbf{q} is of type I_0 when $\rho_1 \neq 0$ and $\rho_{-1} \neq 0$. If \mathbf{q} is of type I_- , that is, $\rho_1 \neq 0$ and $\rho_{-1} = 0$, we will use $f(\lambda)$ as in (3.3.14) and set $g(\lambda) = 0$. If \mathbf{q} is of type I_+ , that is, $\rho_1 = 0$ and $\rho_{-1} \neq 0$, we will use $g(\lambda)$ as in (3.3.14) and set $f(\lambda) = 0$. Using (3.3.7) we observe that if \mathbf{q} is of type I_0 , then both functions f and g take positive values for positive λ . We will also see that (see Lemma 3.10 (4)),

$$\lim_{\lambda \rightarrow 0^+} f(\lambda) = \lim_{\lambda \rightarrow 0^+} g(\lambda) = 1, \quad \lim_{\lambda \rightarrow +\infty} f(\lambda) = \lim_{\lambda \rightarrow +\infty} g(\lambda) = 0. \quad (3.3.15)$$

Since $\rho_0 < 0$ by (3.3.7), equation (3.3.13) must have a positive root, as claimed.

We will now proceed with a more formal proof. We recall the notation for the weighted spaces $\ell_s^2(\mathbb{Z}^2)$ and $\ell_s^2(\mathbb{Z})$ given in the discussion following (3.3.1). Our main theorem in this section is the following.

Theorem 3.9. *Assume that $\mathbf{p} \in \mathbb{Z}^2$ is such that at least one point $\mathbf{q} \in \mathcal{Q}(\mathbf{p})$ is of type I. Then the steady state $(\omega_{\mathbf{k}}^0)_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}}$ defined in (3.2.2) is linearly unstable. In particular, the operator $L_{B, \mathbf{q}}$ in the space $\ell_s^2(\mathbb{Z})$ has a positive eigenvalue and therefore L_B in $\ell_s^2(\mathbb{Z}^2)$ has a positive eigenvalue. Moreover, $\lambda > 0$ is an eigenvalue of $L_{B, \mathbf{q}}$ if and only if λ is a solution to the equation*

$$\frac{\lambda}{\rho_0} + f(\lambda) + g(\lambda) = 0 \text{ provided } \mathbf{q} \text{ is of type } I_0, \quad (3.3.16)$$

$$\frac{\lambda}{\rho_0} + f(\lambda) = 0 \text{ provided } \mathbf{q} \text{ is of type } I_-, \quad (3.3.17)$$

$$\frac{\lambda}{\rho_0} + g(\lambda) = 0 \text{ provided } \mathbf{q} \text{ is of type } I_+. \quad (3.3.18)$$

First, we consider the case when \mathbf{q} is of type I_0 . Before presenting the proof of Theorem 3.9, we will need two lemmas. The proofs of the lemmas rely on the auxiliary material on continued fractions contained in section 3.4.

Lemma 3.10. *Fix any positive λ and consider the following continued fractions*

$$u_n^{(1)}(\lambda) := \frac{\lambda}{\rho_n} + \left[\frac{\lambda}{\rho_{n+1}}, \dots \right] = \frac{\lambda}{\rho_n} + \frac{1}{\frac{\lambda}{\rho_{n+1}} + \frac{1}{\frac{\lambda}{\rho_{n+2}} + \dots}}, n = 0, 1, 2, \dots, \quad (3.3.19)$$

$$u_n^{(2)}(\lambda) := - \left[\frac{\lambda}{\rho_n}, \frac{\lambda}{\rho_{n+1}}, \dots \right] = - \frac{1}{\frac{\lambda}{\rho_n} + \frac{1}{\frac{\lambda}{\rho_{n-1}} + \dots}}, n = -1, -2, \dots \quad (3.3.20)$$

Then the following assertions hold:

(1) $u_n^{(1)}(\lambda)$ and $u_n^{(2)}(\lambda)$ are convergent continued fractions and the functions $u_n^{(1)}(\cdot)$ and $u_n^{(2)}(\cdot)$ are continuous in λ .

(2) There exist limits

$$u_\infty^{(1)}(\lambda) = \lim_{n \rightarrow \infty} u_n^{(1)}(\lambda), \quad u_{-\infty}^{(2)}(\lambda) = \lim_{n \rightarrow -\infty} u_n^{(2)}(\lambda), \quad \lambda > 0,$$

satisfying $|u_\infty^{(1)}(\lambda)| > 1$, $|u_{-\infty}^{(2)}(\lambda)| < 1$.

(3) For some $0 < q < 1$ and $C > 0$, the following hold

$$(|u_1^{(1)}(\lambda)u_2^{(1)}(\lambda) \dots u_n^{(1)}(\lambda)|)^{-1} \leq Cq^n, \text{ for all } n \geq 1, \quad (3.3.21)$$

$$(|u_n^{(2)}(\lambda) \dots u_{-2}^{(2)}(\lambda)u_{-1}^{(2)}(\lambda)|) \leq Cq^{-n}, \text{ for all } n \leq -1. \quad (3.3.22)$$

(4) $\lim_{\lambda \rightarrow 0^+} u_0^{(k)}(\lambda) = 1$, $\lim_{\lambda \rightarrow +\infty} u_0^{(k)}(\lambda) = 0$ for $k = 1, 2$.

Proof. (1) This follows from Van Vleck theorem and the Stjeltjes-Vitali Theorem, see [JT, Theorem 4.29 and Theorem 4.30], since $\lambda > 0$, and thus $\arg \lambda$ satisfies $|\arg \lambda| < \frac{\pi}{2} - \varepsilon$ and hence the continued fractions converge. In addition, Van Vleck Theorem also guarantees that the maps $\lambda \mapsto f(\lambda), g(\lambda)$ are holomorphic in λ since $|\arg \lambda| \leq \frac{\pi}{2} - \varepsilon$ implying the continuity clause.

(2) The fact that the limits $u_\infty^{(1)}(\lambda)$ and $u_{-\infty}^{(2)}(\lambda)$ exist follows from item (3) in Lemma 3.12 proved in Section 3.4. Passing to the limit as $n \rightarrow \infty$ in (3.3.19) and (3.3.20) we see that

$$u_\infty^{(1)}(\lambda) = \lambda + 1/u_\infty^{(1)}(\lambda) \text{ and } u_{-\infty}^{(2)}(\lambda) = -1/(\lambda + u_{-\infty}^{(2)}(\lambda))$$

since $\rho_n \rightarrow 1$ as $n \rightarrow \infty$. Thus, we notice that both $u_\infty^{(1)}$ and $u_{-\infty}^{(2)}$ satisfy the following quadratic equation

$$u_{\pm\infty}^2 - \lambda u_{\pm\infty} - 1 = 0.$$

The solutions of the above equations are given by $u_{\pm\infty} = \frac{\lambda}{2} \pm \sqrt{(\frac{\lambda}{2})^2 + 1}$. Notice also that $u_{\infty}^{(1)}(\lambda)$ must be positive and $u_{-\infty}^{(2)}(\lambda)$ must be negative. From these it is seen that $u_{\infty}^{(1)} = \frac{\lambda}{2} + \sqrt{(\frac{\lambda}{2})^2 + 1}$ and $u_{-\infty}^{(2)} = \frac{\lambda}{2} - \sqrt{(\frac{\lambda}{2})^2 + 1}$ and thus $|u_{\infty}^{(1)}(\lambda)| > 1$, $|u_{-\infty}^{(2)}(\lambda)| < 1$.

(3) Let $q' \in (1, u_{\infty}^{(1)}(\lambda))$. Note that from (2), since $u_{\infty}^{(1)}(\lambda) > 1$, there exists an integer $N_{q'}$ such that if $n > N_{q'}$, then $u_n^{(1)}(\lambda) > q'$. We thus have that,

$$\begin{aligned} u_1^{(1)}(\lambda)u_2^{(1)}(\lambda) \cdots u_n^{(1)}(\lambda) &= u_1^{(1)}(\lambda) \cdots u_{N_{q'}}^{(1)}(\lambda)u_{N_{q'}+1}^{(1)}(\lambda) \cdots u_n^{(1)}(\lambda) \\ &\geq u_1^{(1)}(\lambda) \cdots u_{N_{q'}}^{(1)}(\lambda)q'^{n-N_{q'}} = \frac{u_1^{(1)}(\lambda) \cdots u_{N_{q'}}^{(1)}(\lambda)}{q'^{N_{q'}}}q'^n = \frac{1}{C}q'^n, \end{aligned}$$

where we have denoted $C = C(q') = \left(\frac{u_1^{(1)}(\lambda) \cdots u_{N_{q'}}^{(1)}(\lambda)}{q'^{N_{q'}}} \right)^{-1}$. Let $q = 1/q'$ and we thus obtain (3.3.21).

Since $|u_{-\infty}^{(2)}(\lambda)| < 1$, we have that for a fixed q such that $|u_{-\infty}^{(2)}(\lambda)| < q < 1$, there exists an integer $N_q > 0$ such that if $n < -N_q$, then $|u_n^{(2)}(\lambda)| < q$. We thus have that,

$$\begin{aligned} |u_{-1}^{(2)}(\lambda)u_{-2}^{(2)}(\lambda) \cdots u_n^{(2)}(\lambda)| &= |u_{-1}^{(2)}(\lambda) \cdots u_{-N_q}^{(2)}(\lambda)u_{-N_q-1}^{(2)}(\lambda) \cdots u_n^{(2)}(\lambda)| \\ &\leq |u_{-1}^{(2)}(\lambda) \cdots u_{-N_q}^{(2)}(\lambda)|q^{n-N_q} = \frac{|u_{-1}^{(2)}(\lambda) \cdots u_{-N_q}^{(2)}(\lambda)|}{q^{N_q}}q^n = Cq^n, \end{aligned}$$

where we have denoted $C = C(q) = \frac{|u_{-1}^{(2)}(\lambda) \cdots u_{-N_q}^{(2)}(\lambda)|}{q^{N_q}}$. This proves (3.3.22).

(4) This follows from items (4) and (5) in Lemma 3.12 proved in Section 3.4. ■

Lemma 3.11. *Fix any positive $\lambda > 0$ and consider the continued fractions $u_n^{(1)}(\lambda)$ and $u_n^{(2)}(\lambda)$ given in (3.3.19) and (3.3.20). Then*

(1) $\lambda \in \sigma_{disc}(L_{B,\mathbf{q}})$ if and only if $u_0^{(1)}(\lambda) = u_0^{(2)}(\lambda)$.

(2) The respective eigenvectors $(w_n)_{n \in \mathbf{Z}}$ for $L_{B,\mathbf{q}}$ are exponentially decaying sequences and therefore belong to $\ell_s^2(\mathbf{Z})$ for any $s \geq 0$.

(3) Equation $u_0^{(1)}(\lambda) = u_0^{(2)}(\lambda)$ indeed has at least one positive root provided \mathbf{q} is of type one.

Proof. (1) If $\lambda \in \sigma_{disc}(L_{B,\mathbf{q}})$ then equations (3.3.9)-(3.3.12) imply that $u_0^{(1)}(\lambda) = u_0^{(2)}(\lambda)$. If $u_0^{(1)}(\lambda) = u_0^{(2)}(\lambda)$, then we first define $u_n^{(1)}(\lambda)$ and $u_n^{(2)}(\lambda)$ as in (3.3.19) and (3.3.20) respectively for every n , with ρ_n given by (3.3.9). We now define u_n as follows:

$$u_n = \begin{cases} u_n^{(1)}(\lambda) & \text{if } n \geq 0, \\ u_n^{(2)}(\lambda) & \text{if } n \leq 0. \end{cases} \quad (3.3.23)$$

Note that u_n is well defined for all $n \in \mathbb{Z}$ because of our assumption that $u_0^{(1)}(\lambda) = u_0^{(2)}(\lambda)$. Notice that u_n thus defined in (3.3.23) satisfies (3.3.10). Indeed, one obtains, from (3.3.11) and (3.3.23) that for every $n \geq 0$,

$$u_n = u_n^{(1)}(\lambda) = \frac{\lambda}{\rho_n} + \frac{1}{u_{n+1}^{(1)}(\lambda)} = \frac{\lambda}{\rho_n} + \frac{1}{u_{n+1}},$$

where in the second equality above, in the denominator we again used the expression from (3.3.11) for $u_{n+1}^{(1)}(\lambda)$. Similarly, from (3.3.12) and (3.3.23) that for every $n \leq 0$,

$$u_{n+1} = u_{n+1}^{(2)}(\lambda) = -\frac{1}{\lambda/\rho_n - u_n^{(2)}(\lambda)} = -\frac{1}{\lambda/\rho_n - u_n}$$

where, again, in the second equality in the denominator, we used the expression from (3.3.12) for $u_n^{(2)}(\lambda)$. This shows that u_n thus defined satisfies (3.3.10). Fix $z_0 = 1$ and for $n \geq 0$ let

$$z_n = \frac{z_0}{u_1 u_2 \dots u_n}, \quad \text{if } n \geq 0, \quad (3.3.24)$$

and for $n < 0$, we define,

$$z_n = z_0 u_0 u_{-1} u_{-2} \dots u_{n+1}, \quad \text{if } n < 0. \quad (3.3.25)$$

Notice that z_n thus defined satisfies $u_n = z_{n-1}/z_n$ for every n . Using this one can see that the sequence $(z_n)_{n \in \mathbb{Z}}$ satisfies the first equation in (3.3.9) because the sequence $(u_n)_{n \in \mathbb{Z}}$ satisfies (3.3.10). We now let

$$w_n = z_n/\rho_n$$

for every n to obtain that the sequence $(w_n)_{n \in \mathbb{Z}}$ satisfies (3.3.7) from the fact that $(z_n)_{n \in \mathbb{Z}}$ satisfies first equation in (3.3.9). It follows that $L_{B,\mathbf{q}}(w_n)_{n \in \mathbb{Z}} = \lambda(w_n)_{n \in \mathbb{Z}}$ if $u_0^{(1)}(\lambda) = u_0^{(2)}(\lambda)$. The fact that $(w_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ follows from assertion (2) in the lemma.

(2) Note that from (3.3.24), we have that,

$$z_n = \frac{z_0}{u_{-1}u_{-2} \dots u_n}, \text{ if } n \geq 0.$$

We now use (3.3.21) to conclude that

$$|z_n| \leq Cq^n, \tag{3.3.26}$$

where C is a constant and $0 < q < 1$. Note that $q^n = e^{n \ln q} = e^{-n\delta}$ for some $\delta > 0$, i.e., we have that if $n \geq 0$,

$$|z_n| \leq Ce^{-n\delta}. \tag{3.3.27}$$

Notice also, from (3.3.25), we have,

$$z_n = z_0 u_0 u_1 u_2 \dots u_{n+1}, \text{ if } n < 0.$$

We now use (3.3.22) to conclude that (3.3.26) also holds if $n < 0$. Using arguments similar to that between (3.3.26) and (3.3.27) we see that (3.3.27) holds if $n < 0$. We thus have that $(z_n)_{n \in \mathbb{Z}} \in \ell_s^2(\mathbb{Z})$ for $s \geq 0$ and since $w_n = z_n/\rho_n$ and since $(\rho_n)_{n \in \mathbb{Z}}$ is a bounded sequence, we have that $(w_n)_{n \in \mathbb{Z}} \in \ell_s^2(\mathbb{Z})$ for $s \geq 0$.

(3) The fact that $u_0^{(1)}(\lambda) = u_0^{(2)}(\lambda)$ has a positive root is equivalent to the fact that equation (3.3.13) has a positive root $\lambda > 0$. The latter fact follows from (3.3.15). Indeed, the assertion regarding the two limits in (3.3.15) follow from Lemma 3.12 (4) and (5) by replacing x and (c_n) in equation (3.4.1) by λ and (ρ_n) and (ρ_{-n}) respectively for $f(\lambda)$ and $g(\lambda)$. The fact that $\rho_0 < 0$ provided \mathbf{q} is of type one and the fact that by Van Vleck Theorem, f, g are holomorphic in λ provided that $|\arg \lambda| \leq \frac{\pi}{2} - \varepsilon$ then guarantee that (3.3.13) has a positive root $\lambda > 0$. ■

We are ready to present the proof of Theorem 3.9.

Proof. We begin with the case when \mathbf{q} is of type I_0 . We first note that (3.3.16) has a solution $\lambda > 0$ as already mentioned in the proof of Lemma 3.11 (3). This follows from the facts that the functions $\lambda \mapsto f(\lambda), g(\lambda)$ are continuous in λ (because Van Vleck Theorem [JT, Theorem 4.29] guarantees that they are in fact holomorphic for λ such that $|\arg \lambda| \leq \frac{\pi}{2} - \varepsilon$). The two limits in (3.3.15) follow from Lemma 3.12 (4) and (5) by replacing x and (c_n) in equation (3.4.1) by λ and (ρ_n) and (ρ_{-n}) respectively for $f(\lambda)$ and $g(\lambda)$. The limits in equation (3.3.15) then guarantee that there is a root for equation (3.3.13) since $\rho_0 < 0$. This then implies that $u_0^{(1)}(\lambda) = u_0^{(2)}(\lambda)$. Lemma 3.11 (1) then shows that $\lambda > 0$ is an eigenvalue of $L_{B, \mathbf{q}}$ with corresponding eigenvector $(w_n)_{n \in \mathbb{Z}}$. The fact that $(w_n)_{n \in \mathbb{Z}} \in \ell_s^2(\mathbb{Z})$ for any $s \geq 0$ is a consequence of Lemma 3.11 (2).

Next, we consider the case when \mathbf{q} is of type I_+ , see remark 3.4. Since $g(\lambda) \rightarrow 1$ as $\lambda \rightarrow 0$ and $g(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$ by Lemma 3.12, we conclude that equation (3.3.18) has at least one positive root. So, to finish the proof of the theorem in this case, it suffices to show that $\lambda > 0$ is a root of (3.3.18) if and only if $\lambda > 0$ is an

eigenvalue of the operator M_q defined in (3.3.6) (equivalently of $L_{B,q}$, cf. Lemma 3.8). Equivalently, the eigenvalue equation $M_q(w_n)_{n \in \mathbb{Z}} = \lambda(w_n)_{n \in \mathbb{Z}}$ can be re-written as the following difference equation:

$$\delta_n \delta_{n-1} w_{n-1} - \delta_n \delta_{n+1} w_{n+1} = \lambda w_n, n \in \mathbb{Z}, \quad (3.3.28)$$

where $\delta_n = (\rho_n)^{1/2} = (1 + \gamma_n)^{1/2}$, see (3.3.5). Suppose that (3.3.28) holds. Letting $z_n = \delta_n w_n$, $n \in \mathbb{Z}$, we arrive at the following equations:

$$\begin{aligned} z_{n-1} - z_{n+1} &= \frac{\lambda}{\rho_n} z_n, n \leq -1, \\ z_{-1} &= \frac{\lambda}{\rho_0} z_0, \\ z_1 &= 0, \\ -z_3 &= \frac{\lambda}{\rho_2} z_2, \\ z_{n-1} - z_{n+1} &= \frac{\lambda}{\rho_n} z_n, n \geq 3. \end{aligned} \quad (3.3.29)$$

Letting $u_n = z_{n-1}/z_n$ for $n \neq 1$ yields

$$\begin{aligned} u_n - \frac{1}{u_{n+1}} &= \frac{\lambda}{\rho_n}, n \leq -1, \\ u_0 &= \frac{\lambda}{\rho_0}, \\ -\frac{1}{u_3} &= \frac{\lambda}{\rho_2}, \\ u_n - \frac{1}{u_{n+1}} &= \frac{\lambda}{\rho_n}, n \geq 3. \end{aligned} \quad (3.3.30)$$

Solving the first equation in (3.3.30) for $u_{n+1} = -\frac{1}{\frac{\lambda}{\rho_n} - u_n}$, $n \leq -1$, and iterating, we observe that, cf. (3.3.14),

$$u_0 = -\frac{1}{\frac{\lambda}{\rho_{-1}} + \frac{1}{\frac{\lambda}{\rho_{-2}} + \dots}} = -g(\lambda).$$

Since u_0 must also satisfy the second equation in (3.3.30), we conclude that every eigenvalue $\lambda > 0$ of $L_{B,\mathbf{q}}$ satisfies equation (3.3.18).

Now we prove that every solution $\lambda > 0$ of (3.3.18) satisfies the eigenvalue problem (3.3.28). Given such λ , the third and fourth equation in (3.3.30) can be re-written as follows:

$$\begin{aligned} u_{n+1} &= -\frac{1}{\frac{\lambda}{\rho_n} - u_n}, n \geq 3, \\ u_3 &= -\frac{\rho_2}{\lambda}. \end{aligned} \tag{3.3.31}$$

Assume that $\lambda > 0$ is a solution of the equation (3.3.18). Our objective is to construct a sequence $(w_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ that satisfies (3.3.28). Then $\lambda > 0$ is proved to be an eigenvalue of $M_{\mathbf{q}}$ and therefore of $L_{B,\mathbf{q}}$, as claimed. In fact, it suffices to construct a sequence $(z_n)_{n \in \mathbb{Z}}$ satisfying (3.3.29) since then $w_n = z_n/\delta_n, n \neq 1$ and $w_1 = 0$ satisfy (3.3.28).

To construct $(z_n)_{n \in \mathbb{Z}}$, we begin by constructing a sequence $(u_n)_{n \in \mathbb{Z}}$ satisfying (3.3.30). Indeed, for $n \geq 3$ we define u_n inductively from (3.3.31). Since u_n satisfy the last equation in (3.3.30) for $n \geq 3$, we see that $u_n = u_n^{(1)}(\lambda)$ for $n \geq 3$ where $u_n^{(1)}(\lambda)$ are defined for $n \geq 3$ in (3.3.19). As a result, the sequence $(u_n)_{n \geq 3}$ enjoys all properties of $u_n^{(1)}(\lambda)$ listed in Lemma 3.10. Next, for $n \leq -1$ we define $u_{n+1} = u_n^{(2)}(\lambda)$, where $u_n^{(2)}(\lambda)$ for $n \leq -1$ is defined in (3.3.20) by means of continued fractions. Note that for $n = -1$ we obtain $u_0 = u_{-1}^{(2)}(\lambda) = -g(\lambda)$, cf. (3.3.14). Since λ satisfies (3.3.18) we just have $u_0 = \lambda/\rho_0$. As a result, we define u_n for $n \in \mathbb{Z} \setminus \{1, 2\}$. Notice, that the sequence $(u_n)_{n \leq 0}$ enjoys all properties of $u_n^{(2)}(\lambda)$ listed in Lemma 3.10.

To finish the construction of the sequence $(z_n)_{n \in \mathbb{Z}}$, we fix arbitrary z_0 and z_2 , and

define

$$z_n = \begin{cases} (u_n \cdot \dots \cdot u_3)^{-1} z_2, & n = 3, 4, 5, \dots, \\ 0, & n = 1, \\ u_{n+1} \cdot \dots \cdot u_0 z_0, & n = -1, -2, -3, \dots \end{cases} \quad (3.3.32)$$

We claim that z_n defined in (3.3.32) satisfies (3.3.29). Indeed, if $n = 3, 4, 5, \dots$ then the last equation in (3.3.30) yields

$$\begin{aligned} z_{n-1} - z_{n+1} &= (u_{n-1} \cdot \dots \cdot u_3)^{-1} z_2 - (u_{n+1} \cdot \dots \cdot u_3)^{-1} z_2 \\ &= \left(u_n - \frac{1}{u_{n+1}}\right) z_n = \frac{\lambda}{\rho_n} z_n, \end{aligned}$$

and so the last equation in (3.3.29) is proved. The last but one equation in (3.3.29) follows from (3.3.32) for $n = 3$ and the third equation in (3.3.30). A similar argument using (3.3.32) and the first two equations in (3.3.30) yields the first two equations in (3.3.29), and the sequence $(z_n)_{n \in \mathbb{Z}}$ is constructed. Note that $(z_n)_{n \in \mathbb{Z}}$ decays exponentially as $|n| \rightarrow \infty$ by Lemma 3.10 and therefore $(w_n)_{n \in \mathbb{Z}}$ also decays exponentially as $|n| \rightarrow \infty$.

Finally, we consider the case when \mathbf{q} is of type I_- . The arguments are similar to I_+ ; we will just highlight the differences. Equation (3.3.28) remains the same. Equation (3.3.29) takes the form

$$\begin{aligned} z_{n-1} - z_{n+1} &= \frac{\lambda}{\rho_n} z_n, n \leq -3, \\ z_{-3} &= \frac{\lambda}{\rho_{-2}} z_{-2}, \\ z_{-1} &= 0, \\ -z_1 &= \frac{\lambda}{\rho_0} z_0, \\ z_{n-1} - z_{n+1} &= \frac{\lambda}{\rho_n} z_n, n \geq 1, \end{aligned}$$

where w_n and z_n related via $z_n = \delta_n w_n$, $n \neq -1$. Equations (3.3.30) now become

$$\begin{aligned} u_n - \frac{1}{u_{n+1}} &= \frac{\lambda}{\rho_n}, n \leq -3, \\ u_{-2} &= \frac{\lambda}{\rho_{-2}}, \\ -\frac{1}{u_1} &= \frac{\lambda}{\rho_0}, \\ u_n - \frac{1}{u_{n+1}} &= \frac{\lambda}{\rho_n}, n \geq 1. \end{aligned}$$

Writing $u_n = \frac{\lambda}{\rho_n} + \frac{1}{u_{n+1}}$ for $n \geq 1$ we now obtain

$$u_1 = \frac{\lambda}{\rho_1} + \frac{1}{\frac{\lambda}{\rho_2} + \frac{1}{\frac{\lambda}{\rho_3} + \dots}} = \frac{1}{f(\lambda)} = -\frac{\rho_0}{\lambda}$$

which is (3.3.17). Similarly to (3.3.31) we obtain

$$\begin{aligned} u_n &= \frac{\lambda}{\rho_n} + \frac{1}{u_{n+1}}, n \leq -3, \\ u_{-2} &= -\frac{\lambda}{\rho_{-2}}. \end{aligned}$$

The remaining part of the proof is similar to the case of type I_+ . ■

3.4 Some auxiliary results on continued fractions

In this subsection we collect several simple facts about continued fractions needed in Subsection 3.3.2. We follow the Appendix in [FH98] and mention [JT] as a general reference. Although the results are not new we added some arguments not articulated in [FH98] in great details.

Assume that $(c_n)_{n \geq 1}$ is a sequence of positive numbers that has a positive limit.

For $x > 0$ we introduce the function

$$G(x) := [xc_1, xc_2, \dots] = \frac{1}{xc_1 + \frac{1}{xc_2 + \frac{1}{xc_3 + \dots}}} \tag{3.4.1}$$

defined by means of a continued fraction. By changing x , when necessary, we can and will assume in what follows that $\lim_{k \rightarrow \infty} c_k = 1$. We note that the continued fraction (3.4.1) converges, that is, the limit of the truncated continued fractions

$$G^{(k)}(x) = \frac{1}{xc_1 + \frac{1}{xc_2 + \frac{1}{xc_3 + \frac{1}{\ddots + \frac{1}{xc_k}}}}}$$

exists and is positive, that is, $G(x) = \lim_{k \rightarrow \infty} G^{(k)}(x)$. This follows from Van Vleck Theorem, see [JT, Theorem 4.29] since $\sum_{k=1}^{\infty} |xc_k| = \infty$ by the divergence test. Moreover, the proof of [JT, Theorem 4.29] based on the Stjeltjes-Vitali Theorem [JT, Theorem 4.30] yields that the function $G(\cdot)$ is holomorphic for $x \in \mathbb{C}$ satisfying $-\frac{\pi}{2} + \varepsilon < \arg(x) < \frac{\pi}{2} + \varepsilon$, with any $\varepsilon > 0$.

In addition we will use the notations

$$G_n(x) = [xc_n, xc_{n+1}, \dots] = \frac{1}{xc_n + \frac{1}{xc_{n+1} + \frac{1}{xc_{n+2} + \ddots}}}, \quad n = 1, 2, \dots \quad (3.4.2)$$

$$G_{\infty}(x) = [x, x, \dots] = \frac{1}{x + \frac{1}{x + \frac{1}{x + \ddots}}}, \quad (3.4.3)$$

and, given positive numbers $a, b > 0$, we denote

$$F := F(a, b) = [a, b, a, b] = \frac{1}{a + \frac{1}{b + \frac{1}{a + \frac{1}{b + \ddots}}}}, \quad (3.4.4)$$

the latter continued fractions also converges by Van Vleck Theorem.

Lemma 3.12. *Assume that $a, b > 0, c_k > 0, \lim_{k \rightarrow \infty} c_k = 1$ and $x > 0$. Then the following assertions hold:*

(1)

$$F(a, b) = \frac{\frac{b}{a}}{\sqrt{\left(\frac{b}{2}\right)^2 + \frac{b}{a} + \frac{b}{2}}} \quad (3.4.5)$$

(2) *If $0 < A \leq c_k \leq B$ for $k = 1, 2, \dots$, then*

$$\frac{\frac{A}{B}}{\sqrt{\left(\frac{xA}{2}\right)^2 + \frac{A}{B} + \frac{xA}{2}}} \leq G(x) \leq \frac{\frac{B}{A}}{\sqrt{\left(\frac{xB}{2}\right)^2 + \frac{B}{A} + \frac{xB}{2}}} \quad (3.4.6)$$

(3) *The limit $\lim_{n \rightarrow \infty} G_n(x)$ exists and is equal to*

$$G_\infty(x) = \lim_{n \rightarrow \infty} G_n(x) = \sqrt{\left(\frac{x}{2}\right)^2 + 1} - \frac{x}{2} \quad (3.4.7)$$

(4)

$$\lim_{x \rightarrow 0^+} G(x) = 1, \quad (3.4.8)$$

(5)

$$\lim_{x \rightarrow +\infty} G(x) = 0. \quad (3.4.9)$$

Proof. (1) The k -th truncated continued fraction for $F(a, b)$ are given by $F^{(2k)}(a, b) = [a, b, \dots, a, b]$, $F^{2k+1}(a, b) = [a, b, \dots, a]$ and satisfy

$$F^{(k+2)}(a, b) = \frac{1}{a + \frac{1}{b + F^{(k)}(a, b)}}, \quad k = 1, 2, \dots$$

Since the continued fraction $[a, b, \dots]$ converges, that is, $F^{(k)} \rightarrow F$ as $k \rightarrow \infty$, we conclude that

$$F(a, b) = \frac{1}{a + \frac{1}{b + F(a, b)}}$$

or $F^2(a, b) + bF(a, b) - \frac{b}{a} = 0$, yielding (3.4.5).

(2) For each k -th truncated continued fraction $G^{(k)}(x) = [xc_1, \dots, xc_k]$ we replace the odd-numbered c_j by the smaller value A and even-numbered c_j by the larger value B . Thus, $G^{(k)}(x)$ is majorated by the k -th truncation $F^{(k)}(A, B)$ of $[A, B, A, B, \dots]$. Passing to the limit as $k \rightarrow \infty$ and using (1) yields the second inequality in (3.4.6). The first inequality follows from $F^{(k)}(B, A) \leq G^{(k)}(x)$.

(3) Formula $G_\infty(x) = \sqrt{(\frac{x}{2})^2 + 1} - \frac{x}{2}$ follows from (3.4.5) with $a = b = x$. It remains to show that the limit $\lim_{n \rightarrow \infty} G_n(x)$ exists and is equal to $G_\infty(x)$. For any $\delta \in (0, 1)$ choose $N = N(\delta)$ such that for all $n \geq N$ we have $1 - \delta < c_n < 1 + \delta$. For any $n \geq N$ we apply assertion (2) with c_k replaced with c_{n+k} , $k = 1, 2, \dots$ and $A = 1 - \delta, B = 1 + \delta$. This yields

$$A(x, \delta) \leq G_n(x) \leq B(x, \delta), \text{ for all } n \geq N, \quad (3.4.10)$$

where we introduce the notations

$$\begin{aligned} A(x, \delta) &:= \frac{(1 - \delta)/(1 + \delta)}{\sqrt{(\frac{x(1-\delta)}{2})^2 + \frac{1-\delta}{1+\delta} + \frac{x(1-\delta)}{2}}}, \\ B(x, \delta) &:= \frac{(1 + \delta)/(1 - \delta)}{\sqrt{(\frac{x(1+\delta)}{2})^2 + \frac{1+\delta}{1-\delta} + \frac{x(1-\delta)}{2}}}. \end{aligned} \quad (3.4.11)$$

We note that $G_\infty(x) = \lim_{\delta \rightarrow 0} A(x, \delta) = \lim_{\delta \rightarrow 0} B(x, \delta)$, $x > 0$. For any $\varepsilon > 0$, we fix $\delta = \delta(\varepsilon) \in (0, 1)$ such that

$$G_\infty(x) - \varepsilon < A(x, \delta), G_\infty(x) + \varepsilon > B(x, \delta).$$

Then (3.4.10) yields $|G_\infty(x) - G_n(x)| < \varepsilon$ for all $n \geq N(\delta(\varepsilon))$ as claimed.

(4) Pick a small $\delta > 0$ to be determined later and choose $N = N(\delta)$ such that

(3.4.10) holds. Fix an even number $2n > N$ and notice that

$$G(x) = G_1(x) = [xc_1, xc_2, \dots, xc_{2n-1}, G_{2n}(x)] \leq [xc_1, xc_2, \dots, xc_{2n-1}, B(x, \delta)], \quad (3.4.12)$$

where we used that $G_{2n} \leq B(x, \delta)$ by (3.4.10). Clearly, $\lim_{x \rightarrow 0} B(x, \delta) = \sqrt{\frac{1+\delta}{1-\delta}}$ yielding

$$\limsup_{x \rightarrow 0} G(x) \leq \left[0, \dots, 0, \sqrt{\frac{1+\delta}{1-\delta}} \right] = \sqrt{\frac{1+\delta}{1-\delta}}.$$

A similar argument shows that $\liminf_{x \rightarrow 0} G(x) \geq \sqrt{\frac{1-\delta}{1+\delta}}$. Passing to the limit as $\delta \rightarrow 0$ proves (4).

(5) As before, we arrive at (3.4.12) and notice that $\lim_{x \rightarrow +\infty} B(x, \delta) = 0$ by (3.4.11). Then

$$\lim_{x \rightarrow +\infty} [xc_1, xc_2, \dots, xc_{2n-1}, B(x, \delta)] = 0$$

yields (5). ■

3.5 The essential spectrum and the spectral mapping theorem

In this section, we follow [LLS] and prove for the linearized α -Euler operator that the essential spectrum of the operator L_B is the imaginary axis. We also prove the spectral mapping theorem for the group $\{e^{tL_B}\}_{t \in \mathbb{R}}$ generated by the operator L_B .

First note that L_B is the direct sum of operators $L_{B,\mathbf{q}}$, i.e., $L_B = \bigoplus_{\mathbf{q} \in \mathcal{Q}} L_{B,\mathbf{q}}$, where $L_{B,\mathbf{q}}$ is given by

$$L_{B,\mathbf{q}} = (cS - \bar{c}S^*) \operatorname{diag}_{n \in \mathbb{Z}} \{1 + \gamma_n\}, \quad (3.5.1)$$

with

$$c = \frac{\frac{1}{2}\Gamma(\mathbf{q} \wedge \mathbf{p})}{\|\mathbf{p}\|^2(1 + \alpha^2\|\mathbf{p}\|^2)}, \quad (3.5.2)$$

and γ_n given by (3.3.3). We note that in general, if $\Gamma \in \mathbb{C}$, then c is a complex number.

We thus write $c = |c|e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Equation (3.5.1) then becomes,

$$L_{B,\mathbf{q}} = |c|(e^{i\theta}S - e^{-i\theta}S^*) \text{diag}_{n \in \mathbb{Z}}\{1 + \gamma_n\}.$$

Lemma 3.13. *The essential spectrum of the operator $L_{B,\mathbf{q}}$ is given by*

$$\sigma_{ess}(L_{B,\mathbf{q}}) = [-2i|c|, 2i|c|]. \quad (3.5.3)$$

Proof. We observe that the Fourier transform $\mathbb{F} : L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{Z}) : f \mapsto (w_n)_{n \in \mathbb{Z}}$ is an isometric isomorphism, where $\mathbb{F}^{-1} : \ell^2(\mathbb{Z}) \rightarrow L^2(\mathbb{T})$ is given by $(w_n) \mapsto \sum_{n \in \mathbb{Z}} w_n e^{inz}$ for $z \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. The operator $e^{i\theta}S - e^{-i\theta}S^*$ acting on $\ell^2(\mathbb{Z})$ is similar via \mathbb{F} to the operator of multiplication by $e^{i\theta}z - e^{-i\theta}\bar{z}$ acting on $L^2(\mathbb{T})$, where $z \in \mathbb{T}$. That is,

$$\mathbb{F}^{-1}(e^{i\theta}S - e^{-i\theta}S^*)\mathbb{F} = e^{i\theta}z - e^{-i\theta}\bar{z}.$$

The above equality follows from the observation that

$$\mathbb{F}^{-1}S = z\mathbb{F}^{-1} \text{ and } \mathbb{F}^{-1}S^* = \bar{z}\mathbb{F}^{-1}.$$

We will now use the fact that the spectrum of a multiplication operator on $L^2(\mathbb{T})$ is equal to its essential spectrum and is given by the closure of the range of the multiplier.

In other words, the spectrum of the operator of multiplication by $e^{i\theta}z - e^{-i\theta}\bar{z}$ on $L^2(\mathbb{T})$ is the closure of the range of $e^{i\theta}z - e^{-i\theta}\bar{z}$ as $z \in \mathbb{T}$. But this is equal to $[-2i, 2i]$.

We thus conclude that the essential spectrum of the operator $|c|(e^{i\theta}S - e^{-i\theta}S^*)$ is

$[-2i|c|, 2i|c|]$. Now, notice that the operator $L_{B,q}$ is a compact perturbation of the operator $|c|(e^{i\theta}S - e^{-i\theta}S^*)$ by the operator $|c|(e^{i\theta}S - e^{-i\theta}S^*) \text{diag}_{n \in \mathbb{Z}}\{\gamma_n\}$. Here, the operator $|c|(e^{i\theta}S - e^{-i\theta}S^*) \text{diag}_{n \in \mathbb{Z}}\{\gamma_n\}$ is compact because $|\gamma_n| \rightarrow 0$ as $|n| \rightarrow \infty$. Weyl's theorem [RS78, Lemma XIII.4.3] allows us to conclude that the essential spectrum of $L_{B,q}$ is the same as the essential spectrum of $|c|(e^{i\theta}S - e^{-i\theta}S^*)$. Thus (3.5.3) holds. ■

We now prove that the spectrum of L_B is exactly the union of the spectra of $L_{B,q}$ cf. [LLS].

Proposition 3.14. $\sigma(L_B) = \bigcup_{\mathbf{q} \in Q} \sigma(L_{B,\mathbf{q}})$.

Proof. Since $\bigcup_{\mathbf{q} \in Q} \sigma(L_{B,\mathbf{q}}) \subset \sigma(L_B)$ trivially holds, it is enough to show that

$$\sigma(L_B) \subset \bigcup_{\mathbf{q} \in Q} \sigma(L_{B,\mathbf{q}}).$$

We first split the operator $L_B = L^s + L^b$, where $L^s = \bigoplus_{\|\mathbf{q}\| \leq \|\mathbf{p}\|} L_{B,\mathbf{q}}$ and $L^b = \bigoplus_{\|\mathbf{q}\| > \|\mathbf{p}\|} L_{B,\mathbf{q}}$ correspond to \mathbf{q} with small and big norms. We have that $\sigma(L_B) = \sigma(L^s) \cup \sigma(L^b)$, and since L^s is the sum of finitely many operators we have that

$$\sigma(L_B) = \left(\bigcup_{\|\mathbf{q}\| \leq \|\mathbf{p}\|} \sigma(L_{B,\mathbf{q}}) \right) \cup \sigma(L^b).$$

It is thus enough to show that $\sigma(L^b) \subset \bigcup_{\|\mathbf{q}\| > \|\mathbf{p}\|} \sigma(L_{B,\mathbf{q}})$. Since $|c| \rightarrow \infty$ as $\|\mathbf{q}\| \rightarrow \infty$ (see (3.5.2)), and using the fact that $\sigma_{ess}(L_{B,\mathbf{q}}) = [-2i|c|, 2i|c|]$, we see that $i\mathbb{R} \subset \bigcup_{\|\mathbf{q}\| > \|\mathbf{p}\|} \sigma(L_{B,\mathbf{q}})$. It therefore suffices to show that $\sigma(L^b) \subset i\mathbb{R}$. Let us denote

$$N_{\mathbf{q}}^0 = (e^{i\theta}S - e^{-i\theta}S^*)$$

and

$$N_{\mathbf{q}} = (e^{i\theta}S - e^{-i\theta}S^*) \text{diag}_{n \in \mathbb{Z}}\{1 + \gamma_n\}.$$

Thus $N_{\mathbf{q}} = N_{\mathbf{q}}^0 \text{diag}_{n \in \mathbb{Z}} \{1 + \gamma_n\}$ and $L_{B, \mathbf{q}} = |c|N_{\mathbf{q}}$, i.e.,

$$L^b = \oplus_{\|\mathbf{q}\| > \|\mathbf{p}\|} |c|N_{\mathbf{q}}.$$

In order to show that $\sigma(L^b) \subset i\mathbb{R}$ we show that if $\lambda \notin i\mathbb{R}$, then λ is in the resolvent set of L^b . Thus, to prove the proposition, we need to show that

$$\text{if } \lambda \notin i\mathbb{R}, \quad \text{then} \quad \sup_{\|\mathbf{q}\| > \|\mathbf{p}\|} \|\lambda - |c|N_{\mathbf{q}}\|^{-1} < +\infty. \quad (3.5.4)$$

Notice that

$$(\lambda - |c|N_{\mathbf{q}})^{-1} = \frac{1}{|c|} \left(\frac{\lambda}{|c|} - N_{\mathbf{q}} \right)^{-1}.$$

Notice that $(N_{\mathbf{q}}^0)^* = -N_{\mathbf{q}}^0$, i.e., $N_{\mathbf{q}}^0$ is a bounded skew self-adjoint operator with $\|N_{\mathbf{q}}^0\| = 2$. It's spectrum lies along the imaginary axis and since $\lambda \notin i\mathbb{R}$ we have that,

$$\left\| \left(\frac{\lambda}{|c|} - N_{\mathbf{q}}^0 \right)^{-1} \right\| = \frac{|c|}{|\text{Re}(\lambda)|}. \quad (3.5.5)$$

Also

$$\begin{aligned} \frac{\lambda}{|c|} - N_{\mathbf{q}} &= \frac{\lambda}{|c|} - N_{\mathbf{q}}^0 - N_{\mathbf{q}}^0 \text{diag}_{n \in \mathbb{Z}} \{\gamma_n\} \\ &= \left(\frac{\lambda}{|c|} - N_{\mathbf{q}}^0 \right) \left[I - \left(\frac{\lambda}{|c|} - N_{\mathbf{q}}^0 \right)^{-1} N_{\mathbf{q}}^0 \text{diag}_{n \in \mathbb{Z}} \{\gamma_n\} \right]. \end{aligned} \quad (3.5.6)$$

Claim: $|c| \|\text{diag}_{n \in \mathbb{Z}} \{\gamma_n\}\| \leq \frac{K(\mathbf{p})}{\|\mathbf{q}\|(1 + \alpha^2 \|\mathbf{q}\|^2)}$, where $K(\mathbf{p}) > 0$ is a constant.

Proof of Claim: Using the definition of γ_n (see (3.3.3)) and c (see (3.5.2)) we have,

$$|c\gamma_n| = \frac{|\Gamma| |\mathbf{q} \wedge \mathbf{p}|}{2 \|\mathbf{q} + n\mathbf{p}\|^2 (1 + \alpha^2 \|\mathbf{q} + n\mathbf{p}\|^2)}.$$

Now use the fact that $\mathbf{q} \wedge \mathbf{p} = (\mathbf{q} + n\mathbf{p}) \wedge \mathbf{p}$ and the fact that $|\mathbf{q} \wedge \mathbf{p}| = |\mathbf{q} \cdot \mathbf{p}^\perp|$ and the Cauchy-Schwarz inequality to see that $|\mathbf{q} \wedge \mathbf{p}| = |(\mathbf{q} + n\mathbf{p}) \wedge \mathbf{p}| \leq \|\mathbf{q} + n\mathbf{p}\| \|\mathbf{p}\|$.

This then implies that,

$$|c\gamma_n| \leq \frac{K(\mathbf{p})}{\|\mathbf{q} + n\mathbf{p}\| (1 + \alpha^2 \|\mathbf{q} + n\mathbf{p}\|^2)}.$$

We thus have that,

$$|c| \|\text{diag}_{n \in \mathbb{Z}} \{\gamma_n\}\| \leq |c| \sup_n |\gamma_n| \leq \frac{K(\mathbf{p})}{\|\mathbf{q}\|(1 + \alpha^2 \|\mathbf{q}\|^2)},$$

which finishes the proof of the Claim.

Now choose $\|\mathbf{q}_0\| > \|\mathbf{p}\|$ so that for all $\|\mathbf{q}\| \geq \|\mathbf{q}_0\|$, the inequality

$$\frac{2K(\mathbf{p})}{|Re(\lambda) \|\mathbf{q}\|(1 + \alpha^2 \|\mathbf{q}\|^2)|} \leq \frac{1}{2} \quad (3.5.7)$$

holds. We stress that \mathbf{q}_0 depends on $Re(\lambda)$ but does not depend on $Im(\lambda)$. Denote

$Q_s := \{\mathbf{q} \in Q : \|\mathbf{q}\| \in [\|\mathbf{p}\|, \|\mathbf{q}_0\|]\}$ and $Q_b := \{\mathbf{q} \in Q : \|\mathbf{q}\| \geq \|\mathbf{q}_0\|\}$. If $\mathbf{q} \in Q_b$, using

(3.5.7), and the fact that $\|N_{\mathbf{q}}^0\| = 2$, we have,

$$\begin{aligned} \left\| \left(\frac{\lambda}{|c|} - N_{\mathbf{q}}^0 \right)^{-1} N_{\mathbf{q}}^0 \text{diag}_{n \in \mathbb{Z}} \{\gamma_n\} \right\| &\leq \frac{2|c| \|\text{diag}_{n \in \mathbb{Z}} \{\gamma_n\}\|}{|Re(\lambda)|} \\ &\leq \frac{2K(\mathbf{p})}{|Re(\lambda) \|\mathbf{q}\|(1 + \alpha^2 \|\mathbf{q}\|^2)|} \leq \frac{1}{2}. \end{aligned}$$

This proves that as long as $\mathbf{q} \in Q_b$, the operator $\left[I - \left(\frac{\lambda}{|c|} - N_{\mathbf{q}}^0 \right)^{-1} N_{\mathbf{q}}^0 \text{diag}_{n \in \mathbb{Z}} \{\gamma_n\} \right]$

is invertible and

$$\left\| \left[I - \left(\frac{\lambda}{|c|} - N_{\mathbf{q}}^0 \right)^{-1} N_{\mathbf{q}}^0 \text{diag}_{n \in \mathbb{Z}} \{\gamma_n\} \right] \right\| \leq 2.$$

Therefore, as long as $\mathbf{q} \in Q_b$, we have that

$$\|(\lambda - |c|N_{\mathbf{q}})^{-1}\| = \frac{1}{|c|} \left\| \left(\frac{\lambda}{|c|} - N_{\mathbf{q}}^0 \right)^{-1} \right\| \leq \frac{1}{|c|} \frac{|c|}{|Re(\lambda)|} 2 = \frac{2}{|Re(\lambda)|}.$$

Thus,

$$\sup_{\mathbf{q} \in Q_b} \|(\lambda - |c|N_{\mathbf{q}})^{-1}\| \leq \frac{2}{|Re(\lambda)|}. \quad (3.5.8)$$

To finish the proof, we note that the set Q_s is finite and since $(\lambda - |c|N_{\mathbf{q}})^{-1}$ is a bounded linear operator for every $\mathbf{q} \in Q_s$, it follows that $\bigoplus_{\mathbf{q} \in Q_s} \|(\lambda - |c|N_{\mathbf{q}})^{-1}\|$ is also

a bounded linear operator, where, if $\lambda = Re(\lambda) + iIm(\lambda)$, with $Re(\lambda) \neq 0$, then the resolvent operator grows as $O(1/(|Im(\lambda)|))$ as $|Im(\lambda)| \rightarrow \infty$. We have that,

$$\sup_{\mathbf{q} \in Q_s} \|(\lambda - |c|N_{\mathbf{q}})^{-1}\| < +\infty. \quad (3.5.9)$$

Since $\{\mathbf{q} : \|\mathbf{q}\| > \|\mathbf{p}\|\} = Q_s \cup Q_b$, equations (3.5.8), (3.5.9) show that (3.5.4) holds.

This proves the proposition. ■

Proposition 3.15. $\sigma_{ess}(L_B) = i\mathbb{R}$ and $\sigma_p(L_B) \setminus i\mathbb{R} = \bigcup_{\|\mathbf{q}\| \leq \|\mathbf{p}\|} (\sigma_p(L_{B,\mathbf{q}}) \setminus i\mathbb{R})$ is a bounded set with accumulation points only on $i\mathbb{R}$.

Proof. The facts that $|c| \rightarrow \infty$ as $\|\mathbf{q}\| \rightarrow \infty$ and (3.5.3), together with the fact that $\bigcup_{\mathbf{q} \in Q} \sigma_{ess}(L_{B,\mathbf{q}}) \subset \sigma_{ess}(L_B)$ imply that $i\mathbb{R} \subset \sigma_{ess}(L_B)$. It is thus enough to prove that $\sigma_{ess}(L_B) \subset i\mathbb{R}$. We have,

$$\sigma_{ess}(L_B) = \bigcup_{\|\mathbf{q}\| \leq \|\mathbf{p}\|} \sigma_{ess}(L_{B,\mathbf{q}}) \bigcup \sigma_{ess}(L^b).$$

Notice that, since $\bigoplus_{\|\mathbf{q}\| \leq \|\mathbf{p}\|} (L_{B,\mathbf{q}})$ is a sum of finitely many bounded linear operators and using (3.5.3), we have that

$$\bigcup_{\|\mathbf{q}\| \leq \|\mathbf{p}\|} \sigma_{ess}(L_{B,\mathbf{q}}) \subset i\mathbb{R}.$$

From the proof of Proposition 3.14, see Equation (3.5.4), we know that $\sigma(L^b) \subset i\mathbb{R}$, i.e., L^b does not have points in the spectrum with non zero imaginary values. Thus,

$$\sigma_{ess}(L^b) \subset \sigma(L^b) \subset i\mathbb{R}.$$

This proves that $\sigma_{ess}(L_B) \subset i\mathbb{R}$. The second statement of the Proposition follows from the above and from the fact that $\bigcup_{\|\mathbf{q}\| \leq \|\mathbf{p}\|} L_{B,\mathbf{q}}$ is a finite sum of bounded linear operators. ■

We now prove the spectral mapping theorem for the operator L_B .

Proposition 3.16. *The spectral mapping property,*

$$\sigma(e^{tL_B}) = e^{t\sigma(L_B)}, \quad t \neq 0,$$

holds for the operator L_B .

Proof. We know from Proposition 3.15, that the essential spectrum of L_B satisfies $\sigma_{ess}(L_B) = i\mathbb{R}$. This tells us that $e^{t\sigma_{ess}(L_B)} = e^{i\mathbb{R}} = \{z \in \mathbb{C} : |z| = 1\}$. Since $e^{t\sigma_{ess}(L_B)} \subseteq \sigma(e^{tL_B})$ for any semigroup, we see that $\{z \in \mathbb{C} : |z| = 1\} \subseteq \sigma(e^{tL_B})$. We want to show that $\sigma_{ess}(e^{tL_B}) \subset \{z \in \mathbb{C} : |z| = 1\}$. We use a general Gearhart-Pruss spectral mapping theorem for Hilbert spaces, see [LLS, Th.2, pg 268]. On a Hilbert space, $\sigma(e^{tL_B})$, $t \neq 0$, is the set of points $e^{\lambda t}$ such that either $\mu_n = \lambda + 2\pi n/t$ belongs to $\sigma(L_B)$ for all $n \in \mathbb{Z}$ or the sequence $\{\|R(\mu_n, L_B)\|\}_{n \in \mathbb{Z}}$ is unbounded. Suppose $\sigma_{ess}(e^{tL_B}) \not\subset \{z \in \mathbb{C} : |z| = 1\}$. Then, there exists $e^{t\lambda}$ such that $\lambda \notin i\mathbb{R}$ and either $\mu_n = \lambda + 2\pi n/t \in \sigma_{ess}(L_B)$ for all $n \in \mathbb{Z}$ or the sequence $\{\|R(\mu_n, L_B)\|\}_{n \in \mathbb{Z}}$ is unbounded. The first outcome is precluded by the fact that $\sigma_{ess}(L_B) = i\mathbb{R}$. So if $e^{t\lambda} \notin \{z \in \mathbb{C} : |z| = 1\}$ and $e^{t\lambda} \in \sigma_{ess}(e^{tL_B})$ then we must have that $\sup_{y \in \mathbb{R}} \|R(Re(\lambda) + iy, L_B)\| = +\infty$. But this is impossible because, as we prove below that for each $\lambda \notin i\mathbb{R}$, $\sup_{y \in \mathbb{R}} \|R(Re(\lambda) + iy, L_B)\| < +\infty$. So it remains to establish the following fact.

Claim: Assume $\{Re(\lambda) + iy : y \in \mathbb{R}\} \cap \sigma(L_B) = \emptyset, Re(\lambda) > 0$, then $\sup_{y \in \mathbb{R}} \|R(Re(\lambda) + iy, L_B)\| < \infty$.

Let $\lambda \notin i\mathbb{R}$ as in the proof of Proposition 3.14 and fix $Re(\lambda)$. Since Q_s is a finite set, the operator $\|R(\lambda, \oplus_{\mathbf{q} \in Q_s} L_{B, \mathbf{q}})\|$ is a bounded linear operator such that the norm of its resolvent decays as $O(1/|Im(\lambda)|)$ as $|Im(\lambda)| \rightarrow \infty$ and (3.5.9) holds, i.e., one

has $\|R(\lambda, \oplus_{\mathbf{q} \in Q_s} L_{B, \mathbf{q}})\| \leq C$. One also has that if $\mathbf{q} \in Q_b$, then (3.5.8) holds, i.e., the norm of the resolvent operator $\|R(\lambda, \oplus_{\mathbf{q} \in Q_b} L_{B, \mathbf{q}})\| \leq C/|Im(\lambda)|$. These two facts above can be combined to give

$$\|(\lambda - L_B)^{-1}\| = O(1) \quad \text{as} \quad |Im(\lambda)| \rightarrow \infty. \quad (3.5.10)$$

By estimate (3.5.10), we know that if $Re(\lambda) \neq 0$, then $e^{t\lambda}$ is not in the spectrum of e^{tL_B} . This shows that the essential spectrum of e^{tL_B} , $\sigma_{ess}(e^{tL_B})$, is contained in the unit circle. One also knows that the spectral mapping property always holds for the point spectrum. One can combine these facts to obtain the result. ■

Chapter 4

Instability of steady states for the 2D Euler equations via Birman-Schwinger and Lin's operators

In this chapter we study the discrete spectrum of the operator L_{vor} obtained by linearizing the 2D Euler equations on the torus \mathbb{T}^2 written in vorticity form about a general steady state. Our main tools are Birman-Schwinger type operators, $K_\lambda(\mu)$, and Lin's operators, A_λ . The Lin's dispersion operators were introduced and studied by Zhiwu Lin in [ZL01, ZL03, ZL04] and defined by the formulae

$$A_\lambda = -\Delta - g'(\psi^0(x, y)) + g'(\psi^0(x, y))\lambda(\lambda - L^0)^{-1}, \quad \lambda > 0,$$

$$A_0 = -\Delta - g'(\psi^0(x, y)) + g'(\psi^0(x, y))P_0, \quad \lambda = 0,$$

where ψ^0 is the stream function for the steady state, g is the real function relating the vorticity and the stream function via $\omega^0 = g(\psi^0)$, L^0 is the operator of differentiation along streamlines, and P_0 is the orthogonal projection onto the kernel of L^0 .

A remarkable property of the dispersion operators discovered by Z. Lin is that $\lambda > 0$ is an eigenvalue of the operator L_{vor} if and only if 0 is an eigenvalue of A_λ ; cf. Proposition 4.7. With this fact in mind, in this chapter we introduce a family of

Birman-Schwinger operators, $K_\lambda(\mu)$, which belong to the ideal \mathcal{B}_2 of Hilbert-Schmidt operators and satisfy the identity $A_\lambda - \mu = (I - K_\lambda(\mu))(-\Delta - \mu)$, and define the 2-modified Fredholm determinants $\mathcal{D}(\lambda, \mu) = \det_2(I - K_\lambda(\mu))$, see formulas (4.4.4) and (4.4.6) below. As a result, in this chapter, we describe the unstable eigenvalues of L_{vor} as zeros of the analytic function $\mathcal{D}(\cdot, 0)$ for $\mu = 0$, see Theorem 4.14, which is the main new result of this chapter. In addition, we give a new version of the proof of an important theorem by Z. Lin saying that if A_0 has no kernel and an odd number of negative eigenvalues, then the operator L_{vor} has at least one positive eigenvalue, see Theorem 4.10 below. The proof is based on the fact that $A_\lambda - A_0$ converges to zero strongly in $L^2(\mathbb{T}^2)$ and, as a result, that $K_\lambda(\mu) - K_0(\mu)$ converges to zero in \mathcal{B}_2 as $\lambda \rightarrow 0^+$. Due to the convergence of the respective perturbation determinants, it follows that the number of negative eigenvalues of A_0 and A_λ for small $\lambda > 0$ (which is equal to the number of zeros of $\mathcal{D}(0, \cdot)$ and $\mathcal{D}(\lambda, \cdot)$, respectively) coincide, see Claim 4.15 below. Since A_λ has no negative eigenvalues for large $\lambda > 0$, and has an even number of nonreal eigenvalues, this shows that when λ changes from small to large positive values, an eigenvalue of A_λ must cross through zero, thus proving the existence of a positive eigenvalue of L_{vor} .

We use boldface to denote (2×1) column vectors and vector-valued functions, and the following standard notation:

$$\nabla = \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix}, \nabla^\perp = \begin{bmatrix} -\partial_y \\ \partial_x \end{bmatrix}, \text{div} = [\partial_x \quad \partial_y], \text{curl} = [-\partial_y \quad \partial_x], \quad (4.0.1)$$

so that $\text{curl}(\nabla^\perp) = \Delta = \partial_x^2 + \partial_y^2$ and $\nabla^\perp f \cdot \nabla g = -\nabla^\perp g \cdot \nabla f$.

4.1 Steady states and linearized Euler operators on the torus

We consider the incompressible inviscid flow satisfying the Euler equations

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0, \quad \operatorname{div} \mathbf{u} = 0, \quad (4.1.1)$$

in the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. Due to the second equation in (4.1.1), there is a stream function ψ such that equation $\mathbf{u} = -\nabla^\perp\psi$ holds. We introduce the vorticity $\omega = \operatorname{curl} \mathbf{u}$ so that $\omega = -\Delta\psi$. Applying curl in (4.1.1), one obtains the Euler equation in vorticity form,

$$\omega_t + \mathbf{u} \cdot \nabla\omega = 0. \quad (4.1.2)$$

The Euler equation for the stream function reads as follows,

$$\Delta\psi_t - \nabla^\perp\psi \cdot \nabla(\Delta\psi) = \Delta\psi_t - \psi_x\Delta\psi_y + \psi_y\Delta\psi_x = 0. \quad (4.1.3)$$

Let us consider a smooth steady state solution $\omega^0 = \operatorname{curl} \mathbf{u}^0 = -\Delta\psi^0$ of the Euler equation. In particular,

$$\nabla^\perp\psi^0 \cdot \nabla\omega^0 = (-\psi_y^0\partial_x + \psi_x^0\partial_y)\omega^0 = 0,$$

and then $\nabla\psi^0$ and $\nabla(\Delta\psi^0)$ are parallel. We further make the following assumption.

Hypothesis 4.1. *For the given smooth steady state solution of (4.1.1) on \mathbb{T}^2 and for some differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$, the equation*

$$\omega^0(x, y) = -\Delta\psi^0(x, y) = g(\psi^0(x, y)) \quad (4.1.4)$$

holds for all $(x, y) \in \mathbb{T}^2$.

Hypothesis 4.1 in turn, implies that

$$\nabla^\perp \omega^0 = g'(\psi^0) \nabla^\perp \psi^0. \quad (4.1.5)$$

Remark 4.2. We need a few further assumptions about the steady state. We follow the notation in [GC14]. Denote by \widehat{D} , the union of the images of periodic orbits of the flow generated by \mathbf{u}^0 and by $D_0 = \{(x, y) : \nabla^\perp \psi^0(x, y) = 0\}$ the set of fixed points of the flow. We further assume that the periodic orbits together with the fixed points “fill up” the torus, i.e., more precisely, we have the following hypothesis.

Hypothesis 4.3. *In addition to Hypothesis 4.1 we assume that $\mathbb{T}^2 \setminus (\widehat{D} \cup D_0)$ has measure zero in \mathbb{T}^2 .*

For any ρ in the image of ψ^0 which is not a critical value, the level sets $\{(x, y) : \psi^0(x, y) = \rho\}$ consist of a finite number of disjoint closed curves which we denote by $\Gamma_1(\rho), \Gamma_2(\rho), \dots, \Gamma_{n(\rho)}(\rho)$. Define the set \mathcal{J} to be the disjoint union of the values assumed by the steady state ψ^0 on each periodic orbit, i.e., as in [GC14], we set

$$\mathcal{J} := \amalg_i \psi^0(x, y)|_{\Gamma_i(\rho)}. \quad (4.1.6)$$

This set assumes the values of the stream function on periodic orbits, counted with multiplicity. It is a disjoint union of open intervals. For example, if we consider the steady state $\mathbf{u}^0 = (\cos y, 0)$ on \mathbb{T}^2 , then $\mathcal{J} = (-1, 1) \amalg (-1, 1)$. One can thus define the period function $T : \mathcal{J} \rightarrow (0, \infty)$ by letting $T(\rho)$ be the time period of the orbit corresponding to $\rho \in \mathcal{J}$. Henceforth, we now denote by $\Gamma(\rho)$ the periodic orbit corresponding to $\rho \in \mathcal{J}$.

We linearize the Euler equations (4.1.1)–(4.1.3) about the steady state:

$$\mathbf{u}_t + \mathbf{u}^0 \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}^0 + \nabla p = 0, \quad \operatorname{div} \mathbf{u} = 0, \quad (4.1.7)$$

$$\omega_t + \mathbf{u}^0 \cdot \nabla \omega + \operatorname{curl}^{-1} \omega \cdot \nabla \omega^0 = 0, \quad (4.1.8)$$

$$\Delta \psi_t - \psi_x^0 \Delta \psi_y + \psi_y^0 \Delta \psi_x - \psi_x \Delta \psi_y^0 + \psi_y \Delta \psi_x^0 = 0. \quad (4.1.9)$$

Here, $\mathbf{u} = \operatorname{curl}^{-1} \omega$ denotes the unique solution of the system $\operatorname{curl} \mathbf{u} = \omega$, $\operatorname{div} \mathbf{u} = 0$ with ω having zero space average $\int_{\Omega} \omega \, dx \, dy = 0$.

We introduce the respective linear operators L_{vel} , L_{vor} , L_{str} corresponding to (4.1.7)–(4.1.9), on the following Sobolev spaces. Assume $\Omega = \mathbb{T}^2$ and periodic boundary conditions. We fix an $m \in \mathbb{Z}$, and denote by H_a^m the Sobolev space of (scalar $W_2^m(\mathbb{T}^2)$ -) functions or distributions with zero space average:

$$H_a^m = \left\{ w(x, y) = \sum_{\mathbf{k} \in \mathbb{Z}^2} w_{\mathbf{k}} e^{i\mathbf{k} \cdot (x, y)} \mid w_{\mathbf{0}} = 0, \sum_{\mathbf{k} \in \mathbb{Z}^2} (1 + |\mathbf{k}|^2)^m |w_{\mathbf{k}}|^2 < \infty \right\}, \quad (4.1.10)$$

and by \mathbf{H}_s^m the Sobolev space of (solenoidal vector valued) functions with zero divergence and zero space average:

$$\mathbf{H}_s^m = \left\{ \mathbf{v}(x, y) = \sum_{\mathbf{k} \in \mathbb{Z}^2} \mathbf{v}_{\mathbf{k}} e^{i\mathbf{k} \cdot (x, y)} \mid \mathbf{v}_{\mathbf{0}} = 0, \operatorname{div} \mathbf{v}_{\mathbf{k}} = 0, \sum_{\mathbf{k} \in \mathbb{Z}^2} (1 + |\mathbf{k}|^2)^m \|\mathbf{v}_{\mathbf{k}}\|^2 < \infty \right\}. \quad (4.1.11)$$

Setting

$$L_{\text{vel}} \mathbf{u} = -\mathbf{u}^0 \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u}^0 - \nabla p, \quad (4.1.12)$$

$$L_{\text{vor}} \omega = -\mathbf{u}^0 \cdot \nabla \omega - \operatorname{curl}^{-1} \omega \cdot \nabla \omega^0, \quad (4.1.13)$$

$$\begin{aligned} L_{\text{str}} \psi &= -\Delta^{-1} \left(-\psi_x^0 \Delta \psi_y + \psi_y^0 \Delta \psi_x - \psi_x \Delta \psi_y^0 + \psi_y \Delta \psi_x^0 \right) \\ &= \Delta^{-1} \left(-\mathbf{u}^0 \cdot \nabla (\Delta \psi) + \nabla^\perp \psi \cdot \nabla (\Delta \psi^0) \right), \\ &= \Delta^{-1} \left(-\mathbf{u}^0 \cdot (\Delta \psi) - \operatorname{curl}^{-1} (\Delta \psi) \cdot \nabla (-\Delta \psi^0) \right), \end{aligned} \quad (4.1.14)$$

we observe that the following diagrams commute:

$$\begin{array}{ccc}
H_a^{m+2} & \xrightarrow{L_{\text{str}}} & H_a^{m+1} \\
\downarrow \nabla^\perp & & \uparrow (\nabla^\perp)^{-1} \\
\mathbf{H}_s^{m+1} & \xrightarrow{L_{\text{vel}}} & \mathbf{H}_s^m, \\
\downarrow \text{curl} & & \uparrow (\text{curl})^{-1} \\
H_a^m & \xrightarrow{L_{\text{vor}}} & H_a^{m-1}
\end{array}
\quad , \quad
\begin{array}{ccc}
H_a^{m+2} & \xrightarrow{L_{\text{str}}} & H_a^{m+1} \\
\Delta \downarrow & & \uparrow (\Delta)^{-1} \\
H_a^m & \xrightarrow{L_{\text{vor}}} & H_a^{m-1}
\end{array}
\tag{4.1.15}$$

Alternatively, we can view the respective operator L as an unbounded operator in the space H^n with the domain H^{n+1} . This point of view is more consistent as then we measure the growth of the time dependent solutions of the respective linearized Euler equation in the norm of H^n although the initial value is taken in H^{n+1} .

Using diagrams (4.1.15), one finds quite natural the example in [ZL04, Theorem 4.1] of the flow which is linearly unstable in the energy norm but linearly (Lyapunov) stable in the enstrophy norm. In fact, every flow with stretching and no unstable (that is, nonimaginary) isolated eigenvalues will have this property. Indeed, we remind a result from [SL05] saying that the spectral bound of the essential spectrum of the operator L_{vor} acting on the Sobolev space H_a^n for any $n \in \mathbb{Z}$ is equal to $|n|\Lambda$, where Λ is the top Lyapunov exponent of the flow induced by the steady state \mathbf{u}^0 . Thus, the flow is linearly unstable because of the presence of the essential spectrum if and only if $|n|\Lambda > 0$.

Let us suppose that $m = 1$ in the diagram above, that is, let us consider the stream function $\psi \in H_a^2$, velocity $\mathbf{u} \in \mathbf{H}_s^1$, and vorticity $\omega \in H_a^0 = L_a^2$. The H^2 norm of the stream function is the enstrophy norm. By the result in [SL05], the essential spectrum of L_{vor} in $H_a^0 = L_a^2$ is located on the imaginary axis as $n = 0$. Therefore, if this operator does *not* have nonimaginary isolated eigenvalues then the linear (Lyapunov) stability holds in the enstrophy norm. Let us now consider $m = 0$ in the diagram

above, that is, let us consider the stream function $\psi \in H_a^1$, velocity $\mathbf{u} \in \mathbf{H}_s^0 = \mathbf{L}_s^2$, and vorticity $\omega \in H_a^{-1}$. The \mathbf{L}_s^2 -norm of velocity is the energy norm. Assume stretching, that is, assume $\Lambda > 0$. By the result in [SL05], the essential spectrum of L_{vor} in H_a^{-1} is *not* located on the imaginary axis as $n = -1$. Therefore, even though this operator does *not* have nonimaginary isolated eigenvalues, still there is no linear (Lyapunov) stability in the energy norm due to the presence of the essential spectrum. In the remaining part of this chapter we will always assume that $m = 1$, that is, the operator L_{vor} is considered in L_a^2 .

4.2 Lin's dispersion operators and their properties

In this section we define and study elliptic dispersion operators, A_λ , parametrized by the spectral parameter $\lambda \geq 0$ for the linearized Euler operator L_{vor} defined in (4.1.13). The operators A_λ have the remarkable property that $\lambda > 0$ is an isolated eigenvalue of the linearized Euler operator L_{vor} if and only if 0 is in the spectrum of A_λ . The operators A_λ are often called dispersion operators; they were introduced by Zhiwu Lin in [ZL01, ZL03, ZL04] and studied in subsequent work, see, e.g. [S08, GGVR07]. We call them Lin's operators.

Setting $m = 1$ in the diagrams (4.1.15), we will discuss the spectrum of the operator L_{vor} in L_a^2 . Thus, we will view the operator L_{vor} in (4.1.13) as an unbounded first order differential operator on L_a^2 with the domain H_a^1 , the Sobolev space of functions with zero space average. If ω is an eigenfunction of L_{vor} corresponding to the eigenvalue λ with $\text{Re}(\lambda) > 0$, then $\omega \in H_a^1$ and $L_{\text{vor}}\omega = \lambda\omega$ show that the linearized Euler equation (4.1.8) has the solution $e^{\lambda t}\omega$ whose $L^2(\mathbb{T}^2)$ -norm growth exponentially. Respectively, the enstrophy, that is, $H^2(\mathbb{T}^2)$ -norm of the corresponding stream function $e^{\lambda t}\psi$, such

that $\omega = -\Delta\psi$, grows exponentially.

Let us denote by φ^t the flow on \mathbb{T}^2 generated by the vector field $\nabla^\perp\psi^0 = -\mathbf{u}^0$. In other words, $\varphi^t(x, y) = (X(t; x, y), Y(t; x, y))$ where X and Y solve the Cauchy problem

$$X_t = -\psi_y^0(X, Y), Y_t = \psi_x^0(X, Y), X(0) = x, Y(0) = y. \quad (4.2.1)$$

The strongly continuous evolution group $\{T^t\}_{t \in \mathbb{R}}$ on $L^2(\mathbb{T}^2)$ of unitary operators defined by $T^t\phi = \phi \circ \varphi^t$ is generated by the first order differential operator L^0 defined by

$$L^0\phi = \nabla^\perp\psi^0 \cdot \nabla\phi = (-\psi_y^0\partial_x + \psi_x^0\partial_y)\phi, \quad (4.2.2)$$

with the domain $\text{dom } L^0 = H_a^1$. We have the following properties of the operator L^0 .

Proposition 4.4.

$$(i) \quad (L^0)^* = -L^0; \quad (4.2.3)$$

$$(ii) \quad \sigma(L^0) \subseteq i\mathbb{R}; \quad (4.2.4)$$

$$(iii) \quad \sigma(L^0) = i\mathbb{R}$$

$$\textit{provided } \varphi^t \textit{ has arbitrary long orbits}; \quad (4.2.5)$$

$$(iv) \quad \|(\lambda - L^0)^{-1}\|_{\mathcal{B}(L_2)} \leq |\text{Re}(\lambda)|^{-1}, \text{Re}(\lambda) \neq 0; \quad (4.2.6)$$

$$(v) \quad \|(\lambda - L^0)^{-1}\|_{\mathcal{B}(L_2)} = |\text{Re}(\lambda)|^{-1}, \text{Re}(\lambda) \neq 0;$$

$$\textit{provided } \varphi^t \textit{ has arbitrary long orbits}; \quad (4.2.7)$$

$$(vi) \quad L^0 \textit{ and } (\lambda - L^0)^{-1} \textit{ are normal operators}; \quad (4.2.8)$$

$$(vii) \quad L^0 \textit{ and the operator of multiplication by } g'(\psi^0(x, y)) \textit{ commute.} \quad (4.2.9)$$

Proof. Property (4.2.3) follows by integrating by parts and implies (4.2.4). Property (4.2.5) is proved, for example, in [SL05, Theorem 5] or [CL99]. Properties (4.2.6) and (4.2.7) are proved by passing to the self adjoint operator iL^0 . Property (4.2.8) is obvious. Property (4.2.9), see (4.1.4), follows from the fact that $\psi^0(\varphi^t(x, y))$ is t -independent due to $\nabla\psi^0 \cdot \nabla^\perp\psi^0 = 0$. ■

We remark the following identity (cf. [ZL04, Remark 3.1] for the first line in the following equation)

$$\begin{aligned} \int_{-\infty}^0 e^{\lambda t} \phi(X(-t), Y(-t)) dt &= \int_0^{+\infty} e^{-\lambda t} T^t \phi dt \\ &= (\lambda - L^0)^{-1} \phi, \quad \operatorname{Re}(\lambda) > 0, \phi \in L^2(\mathbb{T}^2). \end{aligned} \tag{4.2.10}$$

We will now show that $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ is an eigenvalue of the operator L_{vor} acting in $L_2(\mathbb{T}^2)$ if and only if 0 is an eigenvalue of a certain elliptic operator, A_λ in $L_2(\mathbb{T}^2)$, having a sufficiently smooth eigenfunction.

Definition 4.5. We define A_λ with the domain $\operatorname{dom}(A_\lambda) = \operatorname{dom} \Delta = H_a^2$ for non-imaginary λ as follows:

$$A_\lambda = -\Delta + g'(\psi^0(x, y))L^0(\lambda - L^0)^{-1} \tag{4.2.11}$$

$$= -\Delta - g'(\psi^0(x, y)) + g'(\psi^0(x, y))\lambda(\lambda - L^0)^{-1}, \operatorname{Re}(\lambda) \neq 0. \tag{4.2.12}$$

We remark that the operator A_λ is not self-adjoint; it is a perturbation of a self-adjoint operator (either $-\Delta$, see (4.2.11), or $-\Delta - g'(\psi^0)$, see (4.2.12)) by a normal, see (4.2.8), (4.2.9), relatively compact bounded operator (either $g'(\psi^0)L^0(\lambda - L^0)^{-1}$ or $\lambda g'(\psi^0)(\lambda - L^0)^{-1}$).

Remark 4.6. Since A_λ commutes with complex conjugation, the nonreal eigenvalues of A_λ are complex conjugate. Thus, the number of the nonreal eigenvalues of A_λ must be even for each value of λ .

The following calculation shows how the eigenfunctions of L_{vor} are related to the elements of the kernel of A_λ . Using $\mathbf{u} = \text{curl}^{-1} \omega = -\nabla^\perp \psi$ and (4.1.5), let us re-write (4.1.13) as follows:

$$L_{\text{vor}}\omega = \nabla^\perp \psi^0 \cdot \nabla \omega + \nabla^\perp \psi \cdot \nabla \omega^0 = \nabla^\perp \psi^0 \cdot \nabla \omega - \nabla^\perp \omega^0 \cdot \nabla \psi \quad (4.2.13)$$

$$= \nabla^\perp \psi^0 \cdot \nabla \omega - g'(\psi^0(x, y)) \nabla^\perp \psi^0 \cdot \nabla \psi \quad (4.2.14)$$

$$= L^0 \omega - g'(\psi^0(x, y)) L^0 \psi. \quad (4.2.15)$$

Thus,

$$L_{\text{vor}}\omega = \lambda \omega \text{ if and only if } (\lambda - L^0)\omega = -g'(\psi^0(x, y)) L^0 \psi, \lambda \in \mathbb{C}, \quad (4.2.16)$$

where ω and ψ are related via $\omega = -\Delta \psi$. Using (4.2.4) and (4.2.9), we see that

$$L_{\text{vor}}\omega = \lambda \omega \text{ if and only if } \omega = -g'(\psi^0(x, y)) L^0 (\lambda - L^0)^{-1} \psi, \text{Re}(\lambda) \neq 0. \quad (4.2.17)$$

Using $\omega = -\Delta \psi$, the second equation in (4.2.17) can be re-written as

$$A_\lambda \psi = -\Delta \psi + g'(\psi^0(x, y)) L^0 (\lambda - L^0)^{-1} \psi = 0. \quad (4.2.18)$$

We use periodic boundary conditions on $\Omega = \mathbb{T}^2$ and recall that H_a^1 is the set of functions with zero space average. We thus have the following fact first proved by a slightly different argument in [ZL04, Lemma 3.2].

Proposition 4.7. *A non imaginary λ belongs to $\sigma_p(L_{\text{vor}})$ if and only if 0 belongs to $\sigma_p(A_\lambda)$. Specifically, if $\omega \in \text{dom}(L_{\text{vor}}) = H_a^1(\Omega)$ is an eigenfunction of the operator L_{vor} and $\text{Re}(\lambda) \neq 0$ then $\psi = -\Delta^{-1} \omega \in H_a^3 \subset \text{dom}(A_\lambda)$ satisfies $A_\lambda \psi = 0$. Conversely, if $\psi \in \text{dom}(A_\lambda)$ satisfies $A_\lambda \psi = 0$ with $\text{Re}(\lambda) \neq 0$ then $\omega = -\Delta \psi \in \text{dom}(L_{\text{vor}}) = H_a^1(\Omega)$ is an eigenfunction of L_{vor} .*

Proof. By (4.2.16)-(4.2.18), if ω is an eigenfunction of L_{vor} then $\psi = -\Delta^{-1}\omega$ is an eigenfunction of A_λ . Conversely, assuming $\psi \in \ker(A_\lambda)$, we have that $\psi \in H_a^1 = \text{dom } A_\lambda$ satisfies the elliptic equation $-\Delta\psi - g'(\psi^0)\psi = \phi$, where we temporarily define $\phi = -(\lambda - L^0)^{-1}\lambda g'(\psi^0)\psi \in H_a^1$. Due to elliptic regularity the solution ψ of the equation is in fact in $H_a^{1+2} = H_a^3$, see also [ZL04a]. Then $\omega = -\Delta\psi \in H_a^1 = \text{dom}(L_{\text{vor}})$. Running (4.2.16)-(4.2.18) backwards, we have that ω is an eigenfunction of L_{vor} . ■

We will now discuss what happens to the operators A_λ when $|\text{Re}(\lambda)|$ is large, and when $\lambda \rightarrow 0^+$. We begin with large $|\text{Re}(\lambda)|$. Since

$$\|L^0\psi\|_{L^2} = \|\nabla^\perp\psi^0 \cdot \nabla\psi\|_{L^2} \leq \|\nabla\psi^0\|_{L^\infty}\|\nabla\psi\|_{L^2} \leq c\|\nabla\psi\|_{L^2}, \quad (4.2.19)$$

we infer that there is a constant c such that for all $\psi \in H_a^1$ and $\text{Re}(\lambda) \neq 0$ one has:

$$\begin{aligned} & \left| \langle g'(\psi^0)L^0(\lambda - L^0)^{-1}\psi, \psi \rangle_{L^2} \right| \\ & \leq \|g'(\psi^0)\|_{L^\infty} \|(\lambda - L^0)^{-1}\|_{\mathcal{B}(L^2)} \|\psi\|_{L^2} \|L^0\psi\|_{L^2} \\ & \leq c \|g'(\psi^0)\|_{L^\infty} |\text{Re}(\lambda)|^{-1} \|\psi\|_{L^2} \|\nabla\psi\|_{L^2} \quad (\text{by (4.2.6) and (4.2.19)}) \\ & \leq c_0 \|g'(\psi^0)\|_{L^\infty} |\text{Re}(\lambda)|^{-1} \|\nabla\psi\|_{L^2}^2 \quad (\text{by the Poincare inequality}). \end{aligned} \quad (4.2.20)$$

Thus, for all $\psi \in H_a^2$ and sufficiently large $|\text{Re}(\lambda)|$, the Poincare inequality yields

$$\begin{aligned} \text{Re} \langle A_\lambda\psi, \psi \rangle_{L^2} & \geq \|\nabla\psi\|_{L^2}^2 - c_0 \|g'(\psi^0)\|_{L^\infty} |\text{Re}(\lambda)|^{-1} \|\nabla\psi\|_{L^2}^2 \\ & \geq c_1 \|\psi\|_{L^2}^2, \end{aligned} \quad (4.2.21)$$

and thus A_λ has no eigenvalues with negative real parts provided $|\text{Re}(\lambda)|$ is large enough, cf. [ZL04, Lemma 3.4]. In particular, there exists $\lambda_\infty > 0$ such that

$$\text{if } \lambda \geq \lambda_\infty \text{ then } A_\lambda \text{ has no negative eigenvalues.} \quad (4.2.22)$$

We will now discuss what happens to the operator A_λ when $\lambda \rightarrow 0^+$. First, let us consider the operator $\lambda(\lambda - L^0)^{-1}$ from (4.2.12). To motivate the limiting behavior of this operator as $\lambda \rightarrow 0^+$, let us impose, for a second, an additional assumption that zero is an isolated eigenvalue of L^0 (this is indeed a strong assumption that holds provided all orbits of φ^t are periodic uniformly with bounded periods). Then the usual expansion of the resolvent operator, see [Ka80, Section III.6.5], around zero, $(\lambda - L^0)^{-1} = P_0\lambda^{-1} + D_0 + D_1\lambda + \dots$, yields that $P_0 = \lim_{\lambda \rightarrow 0} \lambda(\lambda - L^0)^{-1}$ is the Riesz spectral projection for L^0 onto $\ker L^0$. A remarkable fact proved in [ZL04, Lemma 3.5] (in a different form and using a different method) is that this limiting relation holds without any additional assumptions but in the sense of strong convergence. In particular, going back to the general case, we have the following fact first proved by Z. Lin in [ZL04, Lemma 3.5].

Lemma 4.8. *Assume Hypothesis 4.3 and let P_0 denote the orthogonal projection in L_a^2 onto the subspace $\ker L^0 = \{\phi \in H_a^1 : \nabla^\perp \psi^0 \cdot \nabla \phi = 0\}$. Then for any $\phi \in L_a^2$ one has $\lambda(\lambda - L^0)^{-1}\phi \rightarrow P_0\phi$ in L^2 as $\lambda \rightarrow 0^+$.*

Proof. We recall Remark 4.2 about the assumptions regarding the steady state. Hypothesis 4.3 implies that it is enough to check the lemma on $D_0 \cup \widehat{D}$. On D_0 , the operator $L^0|_{L^2(D_0)}$ is the zero operator, and the orthogonal projection $P_0|_{L^2(D_0)}$ is the identity operator and the statement of the lemma holds trivially. It is thus enough to check the lemma on \widehat{D} . Following [ZL04] and using co-area formula, see [GC14, Formula 16], the $L^2(\widehat{D})$ norm of any function $\phi \in L^2(D)$ restricted to the set \widehat{D} can be represented as

$$\|\phi\|_{L^2(\widehat{D})}^2 = \int_{-\infty}^{\infty} \left(\int_{(\psi^0)^{-1}(\rho)} \frac{|\phi|^2}{|\nabla \psi^0|} ds \right) d\rho = \int_{\mathcal{J}} \left(\int_{\Gamma(\rho)} \frac{|\phi|^2}{|\nabla \psi^0|} ds \right) d\rho. \quad (4.2.23)$$

Here, ds is the induced measure on each streamline $\Gamma(\rho)$, and $d\rho$ is the Lebesgue measure on the index set \mathcal{J} defined in (4.1.6). By assumption, each $\Gamma(\rho)$ is diffeomorphic to the unit circle \mathbb{T} with period $T(\rho)$. As time t varies from 0 to 2π , a point $\varphi^t(x, y)$ on $\Gamma(\rho)$ traces one full orbit around $\Gamma(\rho)$. One can see that $dt = ds/|\nabla\psi^0|$ because $d\varphi^t/dt = -\nabla^\perp\psi^0(\varphi^t)$. Thus, one can rewrite (4.2.23) as

$$\|\phi\|_{L^2(\widehat{D})}^2 = \int_{\mathcal{J}} \left(\int_{\Gamma(\rho)} |\phi_\rho(t)|^2 dt \right) d\rho, \quad (4.2.24)$$

where the restriction of ϕ to $\Gamma(\rho)$ is denoted by ϕ_ρ which is in the space $L^2_{per}(0, T(\rho))$, i.e., the space of L^2 functions on $(0, T(\rho))$ with periodic boundary conditions. In the language of direct integral decomposition of operators, see for example [RS78, Section XIII.16], we can represent the space $L^2(\widehat{D})$ as

$$L^2(\widehat{D}) = \int_{\mathcal{J}}^{\oplus} L^2_{per}(0, T(\rho)) d\rho. \quad (4.2.25)$$

We will use the direct integral decomposition of L^0 from [GC14]. Fix a point $(x, y) \in \Gamma(\rho)$. We first note that $L^0|_{\Gamma(\rho)} f(x, y) = -\frac{d}{dt}|_{t=0} f(\varphi^t(x, y))$, where $\varphi^t(x, y)$ is the flow generated by the velocity field \mathbf{u}^0 and f is smooth. Thus, if $\phi_\rho \in L^2_{per}(0, T(\rho))$ with the Fourier series representation $\phi_\rho(t) = \sum_{k \in \mathbb{Z}} \widehat{\phi}_\rho(k) e^{2\pi i k t / T(\rho)}$, where $t \in [0, T(\rho))$, then

$$\begin{aligned} L^0(\rho)\phi_\rho(t) &= -\frac{d}{dt}\phi_\rho(t) = -\frac{d}{dt} \sum_{k \in \mathbb{Z}} \widehat{\phi}_\rho(k) e^{2\pi i k t / T(\rho)} \\ &= -\sum_{k \in \mathbb{Z}} \frac{2\pi i k}{T(\rho)} \widehat{\phi}_\rho(k) e^{2\pi i k t / T(\rho)}, \end{aligned} \quad (4.2.26)$$

and where $L^0(\rho)$ represents the restriction of L^0 to $L^2(\Gamma(\rho))$. Using (4.2.26) one can obtain a representation for the resolvent operator as

$$(\lambda - L^0(\rho))^{-1} \left(\sum_{k \in \mathbb{Z}} \widehat{\phi}_\rho(k) e^{2\pi i k t / T(\rho)} \right) = \sum_{k \in \mathbb{Z}} \frac{1}{\lambda + \frac{2\pi i k}{T(\rho)}} \widehat{\phi}_\rho(k) e^{2\pi i k t / T(\rho)}$$

$$= \sum_{k \in \mathbb{Z}} \frac{T(\rho)}{\lambda T(\rho) + 2\pi i k} \widehat{\phi}_\rho(k) e^{2\pi i k t / T(\rho)}. \quad (4.2.27)$$

We thus have, for each Fourier coefficient $\widehat{\phi}_\rho(k)$,

$$\lim_{\lambda \rightarrow 0^+} \lambda(\lambda - L^0(\rho))^{-1} \widehat{\phi}_\rho(k) = \lim_{\lambda \rightarrow 0^+} \frac{\lambda T(\rho)}{\lambda T(\rho) + 2\pi i k} \widehat{\phi}_\rho(k) = \begin{cases} \widehat{\phi}_\rho(0) & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases} \quad (4.2.28)$$

Let us denote by $P_0(\rho)$ the orthogonal projection in $L^2(\Gamma(\rho))$ onto the kernel of $L^0(\rho)$. We note here that in the language of direct integral decomposition of spaces and operators, see [RS78, Section XIII.16], we can write, as proved in [GC14], that

$$P_0 = \int_{\mathcal{J}}^{\oplus} P_0(\rho) d\rho, \quad iL^0 = \int_{\mathcal{J}}^{\oplus} iL^0(\rho) d\rho. \quad (4.2.29)$$

From (4.2.26), we see that $L^0(\rho)\phi_\rho = 0$ if and only if $\widehat{\phi}_\rho(k) = 0$ for every $k \neq 0$. Thus we have that

$$P_0(\rho)\phi_\rho = \widehat{\phi}_\rho(0). \quad (4.2.30)$$

We now claim the following: if $\phi \in L^2(\widehat{D})$ and $\phi_\rho \in L^2(\Gamma(\rho))$ then for each ρ one has

$$\lim_{\lambda \rightarrow 0^+} \|(\lambda(\lambda - L^0(\rho))^{-1} - P_0(\rho))\phi_\rho\|_{L^2(\Gamma(\rho))}^2 = 0. \quad (4.2.31)$$

Indeed, using Parseval's theorem and formulae (4.2.30) and (4.2.27), we have,

$$\|(\lambda(\lambda - L^0(\rho))^{-1} - P_0(\rho))\phi_\rho\|_{L^2(\Gamma(\rho))}^2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\lambda^2 T^2(\rho)}{\lambda^2 T^2(\rho) + 4\pi^2 k^2} |\widehat{\phi}_\rho(k)|^2. \quad (4.2.32)$$

For every $k \neq 0$, since the following inequality holds,

$$\frac{\lambda^2 T^2(\rho)}{\lambda^2 T^2(\rho) + 4\pi^2 k^2} \leq 1, \quad (4.2.33)$$

we have that,

$$\frac{\lambda^2 T^2(\rho)}{\lambda^2 T^2(\rho) + 4\pi^2 k^2} |\widehat{\phi}_\rho(k)|^2 \leq |\widehat{\phi}_\rho(k)|^2, \quad (4.2.34)$$

and

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\lambda^2 T^2(\rho)}{\lambda^2 T^2(\rho) + 4\pi^2 k^2} |\widehat{\phi}_\rho(k)|^2 \leq \sum_{k \in \mathbb{Z}} |\widehat{\phi}_\rho(k)|^2. \quad (4.2.35)$$

Since (4.2.34) holds, one can apply Lebesgue dominated convergence theorem on the space $\ell^2(\mathbb{Z})$ to conclude that,

$$\begin{aligned} & \lim_{\lambda \rightarrow 0^+} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\lambda^2 T^2(\rho)}{\lambda^2 T^2(\rho) + 4\pi^2 k^2} |\widehat{\phi}_\rho(k)|^2 \\ &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \lim_{\lambda \rightarrow 0^+} \frac{\lambda^2 T^2(\rho)}{\lambda^2 T^2(\rho) + 4\pi^2 k^2} |\widehat{\phi}_\rho(k)|^2 = 0. \end{aligned} \quad (4.2.36)$$

Hence (4.2.31) holds, as claimed. We are now ready to prove the main assertion in the lemma that $\lambda(\lambda - L^0)^{-1}\phi \rightarrow P_0\phi$ in $L^2(\widehat{D})$ as $\lambda \rightarrow 0^+$. We need to show that

$$\lim_{\lambda \rightarrow 0^+} \|(\lambda(\lambda - L^0)^{-1} - P_0)\phi\|_{L^2(\widehat{D})}^2 = 0. \quad (4.2.37)$$

Using formula (4.2.24) this reduces to showing that

$$\lim_{\lambda \rightarrow 0^+} \int_{\mathcal{J}} \|(\lambda(\lambda - L^0(\rho))^{-1} - P_0(\rho))\phi_\rho\|_{L^2(\Gamma(\rho))}^2 d\rho = 0. \quad (4.2.38)$$

Using (4.2.32) and (4.2.35) we have that, applying Parseval's theorem twice,

$$\|(\lambda(\lambda - L^0(\rho))^{-1} - P_0(\rho))\phi_\rho\|_{L^2(\Gamma(\rho))}^2 \leq \|\phi_\rho\|_{L^2(\Gamma(\rho))}^2. \quad (4.2.39)$$

One can thus apply the Lebesgue dominated convergence theorem to the left hand side of (4.2.38) and use (4.2.31) to conclude that,

$$\begin{aligned} & \lim_{\lambda \rightarrow 0^+} \int_{\mathcal{J}} \|(\lambda(\lambda - L^0(\rho))^{-1} - P_0(\rho))\phi_\rho\|_{L^2(\Gamma(\rho))}^2 d\rho \\ &= \int_{\mathcal{J}} \lim_{\lambda \rightarrow 0^+} \|(\lambda(\lambda - L^0(\rho))^{-1} - P_0(\rho))\phi_\rho\|_{L^2(\Gamma(\rho))}^2 d\rho = 0, \end{aligned}$$

finishing the proof of the lemma. ■

We now extend the definition of A_λ in Definition 4.5 as follows.

Definition 4.9. Introduce the operator A_0 in L^2 with $\text{dom}(A_0) = \text{dom}(\Delta) = H_a^2$ by the formula

$$A_0 = -\Delta - g'(\psi^0(x, y)) + g'(\psi^0(x, y))P_0. \quad (4.2.40)$$

By Lemma 4.8 we infer that $A_\lambda\phi \rightarrow A_0\phi$ in L^2 as $\lambda \rightarrow 0^+$ for each $\phi \in \text{dom}(A_\lambda)$.

To conclude this section we formulate the following important theorem proven by Zhiwu Lin in [ZL04] and provide an outline of its proof.

Theorem 4.10. *Assume Hypothesis 4.3 and consider the dispersion Lin's operator A_0 defined in Definition 4.9. Assume that A_0 has an odd number of negative eigenvalues and no kernel. Then L_{vor} has a positive isolated eigenvalue.*

This result gives a new instability criterion for the steady state ψ^0 , cf. [ZL03]. The proof given in [ZL04] rests on an abstract theorem based on the use of the infinite determinants of the operators e^{-A_λ} . In the next section we will offer an alternative proof of Theorem 4.10 based on the use of the Birman-Schwinger type operators. In addition, these operators, associated with A_λ and $-\Delta$, provide us with an analytic function of the spectral parameter whose zeros are exactly the eigenvalues of the operator A_λ . Computing this function at the value $\mu = 0$ for the spectral parameter for A_λ , we will obtain a function of the spectral parameter for L_{vor} whose zeros are exactly the isolated eigenvalues of L_{vor} .

Remark 4.11. The strategy for the proof of Theorem 4.10 is as follows. First, assuming that A_0 has an odd number of negative eigenvalues, we will show, see Claim 4.15, that A_λ has the same number of negative eigenvalues provided $\lambda > 0$ are small

enough. This is indeed the only assertion to be proved to establish the result in Theorem 4.10 as the rest is easy. Indeed, as $\lambda > 0$ changes from small to large positive values, one of the eigenvalues of A_λ should cross through zero since A_λ has no negative eigenvalues for large $\lambda > 0$ by (4.2.22) and the eigenvalues of A_λ may leave the real line only in pairs by Remark 4.6. So, for some $\lambda > 0$ we have $0 \in \sigma_p(A_\lambda)$ and thus $\lambda \in \sigma_p(L_{\text{vor}})$ by Proposition 4.7, completing the proof of Theorem 4.10.

4.3 An application of Lin's instability theorem

In this section, we study an example of a steady state on \mathbb{T}^2 which satisfies the hypothesis of Lin's Theorem 4.10, i.e., a steady state for which the operator A_0 is invertible and has an odd number of negative eigenvalues. This steady state is thus unstable. A similar example was used by Lin, see [ZL03, Example 2.7], to study instability of steady states on the channel, and we adapted this to the case of the torus.

Example 4.12. We shall work with steady states of the form, $\mathbf{u}^0 = (\cos my, 0)$, where $m \geq 2$ is an integer. Observe that $\psi^0(x, y) = \sin my/m$ and $\omega^0(x, y) = m \sin my$, from which we obtain that $\omega^0(x, y) = g(\psi^0(x, y)) = m^2 \psi^0(x, y)$, i.e. $g' = m^2$. Let us fix a positive integer $k > 0$ to be determined later. Let j be an integer in $[1, \lfloor m/2 \rfloor]$, where $\lfloor m/2 \rfloor$ denotes the greatest integer less than or equal to $m/2$. Denote by $X_{k,j}$ the subspace of $L_a^2(\mathbb{T}^2; \mathbb{C})$ consisting of functions of the form $\phi(y)e^{ikx}$ where $\phi(y) \in Y := \text{span} \{\sin(ny)\}$, where $n = j + mp$, $p \in \mathbb{Z}$. In other words, we have that $X_{k,j} = Y \otimes E_{ikx}$, where we have denoted the one dimensional subspace by $E_{ikx} := \text{span} \{e^{ikx}\}$.

We work in the subspace $X_{k,j}$. Our first observation is as follows. The space $X_{k,j}$ is invariant under the operators $L^0, L_{\text{vor}}, A_\lambda$ defined in the previous sections. The domain of the operators L^0 and L_{vor} is the subspace of $X_{k,j}$ such that $\phi(y) \in Y \cap H_a^1(\mathbb{T})$. The domain of A_λ is the subspace of $X_{k,j}$ such that $\phi(y) \in Y \cap H_a^2(\mathbb{T})$.

Note that $\cos(my) \sin(ny) \in Y$ by the following elementary computation,

$$\cos(my) \sin(ny) = \sin(ny) \cos(my) = \frac{1}{2}(\sin(n+m)y + \sin(n-m)y).$$

Both $\sin(n+m)y$ and $\sin(n-m)y$ are in Y because $n+m = j + (p+1)m$ and $n-m = j + (p-1)m$. This also shows that $\cos(my)\phi(y) \in Y$ for every $\phi(y) \in Y$.

To see that $X_{k,j}$ is invariant under L^0 , consider a function, $f \in X_{k,j}$ such that $f(x, y) = \phi(y)e^{ikx}$. We compute,

$$L^0 f = -\mathbf{u}^0 \cdot \nabla f = -\cos(my) \partial_x (\phi(y) e^{ikx}) = -\cos(my) (ik) \phi(y) e^{ikx}.$$

The computation presented above shows that $\cos(my)\phi(y) \in Y$ and hence $L^0 f \in X_{k,j}$. From the formula, $L_{\text{vor}}\omega = L^0\omega - g'(\psi^0(x, y))L^0\psi$, we see that L_{vor} leaves $X_{k,j}$ invariant because L^0 leaves it invariant. Similarly the formula for A_λ given by $A_\lambda\psi = -\Delta\psi + g'(\psi^0(x, y))L^0(\lambda - L^0)^{-1}\psi$ shows that A_λ leaves $X_{k,j}$ invariant, since $-\Delta$ and the operator by multiplication of $g'(\psi^0(x, y))$ also leave the space $X_{k,j}$ invariant. In what follows, all of the operators will be restricted to the subspace $X_{k,j}$.

We now compute the kernel of the operator L^0 acting on the space $X_{k,j}$. Let $\psi(x, y) \in \text{dom } L^0 \cap X_{k,j}$, i.e., ψ is of the form $\psi(x, y) = \phi(y)e^{ikx}$, where $\phi(y) \in Y \cap \text{dom } L^0$. Then

$$L^0\psi(x, y) = -\mathbf{u}^0 \cdot \nabla\psi(x, y) = -\cos my \partial_x \psi(x, y) = -\cos my (ik) \phi(y) e^{ikx}.$$

We note that $-\cos my (ik) \phi(y) e^{ikx} = 0$ if and only if $\phi(y) = 0$. In particular, we see

that the kernel of the operator L^0 acting on $X_{k,j}$ is simply $\{0\}$. Thus the orthogonal projection P_0 onto the kernel of L^0 restricted to the subspace $X_{k,j}$, is the zero operator, i.e $P_0|_{X_{k,j}} = 0$ on the space $X_{k,j}$.

We now provide a decomposition of the operator A_λ as $A_{\lambda,y}$ acting on the space Y times the identity operator acting on e^{ikx} , i.e., $A_\lambda = A_{\lambda,y} \otimes I$, where I denotes the identity operator on E_{ikx} . Since

$$L^0\psi(x, y) = L^0(\phi(y)e^{ikx}) = -\cos my \partial_x \phi(y) e^{ikx} = -\cos(my)(ik)\phi(y)e^{ikx},$$

we see that

$$(\lambda - L^0)\psi(x, y) = (\lambda + ik \cos(my))\psi(x, y),$$

i.e we have, $(\lambda - L^0)^{-1}\psi(x, y) = (\lambda + ik \cos(my))^{-1}\psi(x, y)$. This tells us that $\lambda(\lambda - L^0)^{-1}\psi(x, y) = \lambda(\lambda + ik \cos(my))^{-1}\psi(x, y)$, i.e we have,

$$\lambda(\lambda - L^0)^{-1}\phi(y)e^{ikx} = \frac{\lambda}{\lambda + ik \cos(my)}\phi(y)e^{ikx}.$$

Thus,

$$\begin{aligned} A_\lambda\psi(x, y) &= A_\lambda\phi(y)e^{ikx} = [-\Delta - g'(\psi^0(x, y)) + g'(\psi^0(x, y))\lambda(\lambda - L^0)^{-1}]\phi(y)e^{ikx} \\ &= [-\phi''(y) + k^2\phi(y) - m^2\phi(y) + m^2\frac{\lambda}{\lambda + ik \cos(my)}\phi(y)]e^{ikx} = e^{ikx}[A_{\lambda,y}\phi(y)], \end{aligned}$$

where $A_{\lambda,y}\phi(y) := -\phi''(y) + k^2\phi(y) - m^2\phi(y) + m^2\frac{\lambda}{\lambda + ik \cos(my)}\phi(y)$.

We now compute the operator (4.2.40) A_0 ,

$$\begin{aligned} A_0\psi(x, y) &= [-\Delta - g'(\psi^0(x, y)) + g'(\psi^0(x, y))P_0]\psi(x, y) \\ &= [-\Delta - g'(\psi^0(x, y))]\psi(x, y) = [-\Delta - g'(\psi^0(x, y))]\phi(y)e^{ikx} \\ &= [-\phi''(y) + k^2\phi(y) - m^2\phi(y)]e^{ikx} = e^{ikx}[A_{0,y}\phi(y)] \end{aligned}$$

(since the projection $P_0 = 0$), where $A_{0,y}$ acts on the space Y , and is given by

$$A_{0,y}\phi(y) = -\phi''(y) + k^2\phi(y) - m^2\phi(y).$$

The operator A_0 splits nicely into the operator $A_{0,y}$ acting on the space Y times the identity operator acting on e^{ikx} , i.e., $A_0 = A_{0,y} \otimes I$. Our special ansatz thus reduces the above to a one dimensional problem. The spectrum of A_0 acting on the space $X_{k,j}$ thus corresponds to the spectrum of the operator $A_{0,y}$ on the space Y , that is, $\sigma(A_0; X_{k,j}) = \sigma(A_{0,y}; Y)$.

By the computations presented in (4.2.20), we know that when λ is large the operator A_λ (restricted on the space $X_{k,j}$) has no eigenvalues with negative real parts and by Lemma 4.8, we know that $A_\lambda\psi \rightarrow A_0\psi$ in L_2 as $\lambda \rightarrow 0^+$ for every $\psi \in X_{k,j}$. We also know that the non real eigenvalues of A_λ restricted to $X_{k,j}$ (which is invariant under complex conjugation) occur in complex conjugate pairs, since A_λ is an operator with real coefficients. Thus, if we can prove that A_0 acting on the space $X_{k,j}$ has an odd number of negative eigenvalues, we have proved the existence of an eigenfunction ω of the L_{vor} such that $L_{\text{vor}}\omega = \lambda\omega$, with the real part of $\lambda > 0$ and thus the spectrum of L_{vor} is unstable. Since $\sigma(A_0; X_{k,j}) = \sigma(A_{0,y}; Y)$, this amounts to showing that $A_{0,y}$ acting on the space Y has an odd number of negative eigenvalues.

We now choose k such that the operator $A_{0,y}$ acting on Y has exactly one negative eigenvalue. Since $-(\sin(ny))'' = n^2 \sin(ny)$, we have that the eigenvalues contributed by the second derivative operator acting on Y are $\{n^2 : n = j + mp, p \in \mathbb{Z}\}$ (note that $\{\sin(ny)\}$ is a basis for Y). The two smallest eigenvalues in magnitude, contributed by the second derivative operator are when $n = j$ (i.e., when $p = 0$) and $n = j - m$ (when $p = -1$). Since $A_{0,y}$ is the second derivative operator added to a multiplication

operator, the spectrum of $A_{0,y}$ is given by $\{n^2 + k^2 - m^2 : n = j + mp, p \in \mathbb{Z}\}$.

We thus need $-(k^2 - m^2)$ to lie between j^2 and $(j - m)^2$. We thus have that, $j^2 < -k^2 + m^2 < (j - m)^2$, i.e, $-j^2 > k^2 - m^2 > -(j - m)^2$ i.e, if we choose k such that $m^2 - j^2 > k^2 > m^2 - (j - m)^2$, then the operator $A_{0,y}$ acting on Y has *exactly* one negative eigenvalue (counting multiplicity). For a concrete example, choose $m = 4$, $j = 1$ and $k = 3$. Another example that works is given by the choices, $m = 7$, $j = 2$ and $k = 6$. This gives exactly one eigenvalue for $A_{0,y}$, hence for A_0 , and therefore proves the existence of a positive eigenvalue of L_{vor} .

4.4 Birman-Schwinger type operator associated with Lin's operator A_λ

In this section we introduce and study the family of Birman-Schwinger type operators $K_\lambda(\mu)$, $\mu \in \mathbb{C} \setminus \sigma(-\Delta)$, $Re(\lambda) \neq 0$, associated with Lin's dispersion operators A_λ , introduced in Section 4.2 and the negative Laplace operator $-\Delta$. As we will see, $K_\lambda(\mu) \in \mathcal{B}_2$, the set of Hilbert-Schmidt operators in L_a^2 . We will also define the respective two-modified Fredholm determinants,

$$\mathcal{D}(\lambda, \mu) = \det_2(I_{L_a^2} - K_\lambda(\mu)), \quad Re(\lambda) \neq 0, \mu \in \mathbb{C} \setminus \sigma(-\Delta), \quad (4.4.1)$$

see, e.g. [Si05, GK69]. For each λ , this will allow us to characterize the eigenvalues of A_λ as zeros of the holomorphic function $\mathcal{D}(\lambda, \cdot)$ of the spectral parameter μ . Using the relation between the spectra of the linearized Euler operator L_{vor} and A_λ described in Proposition 4.7, we will characterize the nonimaginary eigenvalues of L_{vor} as zeros of the function $\mathcal{D}(\cdot, 0)$ holomorphic in λ for $Re(\lambda) \neq 0$. These two results, given in Proposition 4.13 and Theorem 4.14 constitute the main results in this chapter.

In addition, we will introduce a family of Birman-Schwinger type operators $K_0(\mu)$, $\mu \in \mathbb{C} \setminus \sigma(-\Delta)$ associated with Lin's operator A_0 , defined in Definition 4.9, and $-\Delta$ and extend the definition of $\mathcal{D}(0, \mu)$ to $\lambda = 0$ by the same formula (4.4.1). This will allow us to characterize the eigenvalues of A_0 as zeros of the holomorphic function $\mathcal{D}(0, \cdot)$ of the spectral parameter μ . It turns out that $K_\lambda(\mu) \rightarrow K_0(\mu)$ in \mathcal{B}_2 as $\lambda \rightarrow 0^+$ uniformly in μ on compact subsets of $\mathbb{C} \setminus \sigma(-\Delta)$, see Lemma 4.18 below. Using these facts, we will show that the number of negative eigenvalues of A_0 is equal to the number of negative eigenvalues of A_λ provided $\lambda > 0$ is small enough, cf. Claim 4.15. As we have already explained in Remark 4.11, this implies the conclusion of Lin's Theorem 4.10, thus providing a new proof of this important assertion.

Our first task is to define the Birman-Schwinger operators $K_\lambda(\mu)$. We recall (4.2.12) for A_λ ,

$$A_\lambda = -\Delta - g'(\psi^0) + g'(\psi^0)\lambda(\lambda - L^0)^{-1}, \operatorname{Re}(\lambda) \neq 0. \quad (4.4.2)$$

Notice that if $\mu \in \mathbb{C} \setminus \sigma(-\Delta)$, then

$$\begin{aligned} A_\lambda - \mu &= (-\Delta - \mu) - g'(\psi^0) + g'(\psi^0)\lambda(\lambda - L^0)^{-1} \\ &= \left(I - (g'(\psi^0) - g'(\psi^0)\lambda(\lambda - L^0)^{-1})(-\Delta - \mu)^{-1} \right) (-\Delta - \mu) \\ &= (I - K_\lambda(\mu))(-\Delta - \mu), \end{aligned} \quad (4.4.3)$$

where we introduce the operators $K_\lambda(\mu)$ by the formula

$$K_\lambda(\mu) = (g'(\psi^0) - g'(\psi^0)\lambda(\lambda - L^0)^{-1})(-\Delta - \mu)^{-1}, \operatorname{Re}(\lambda) \neq 0, \mu \in \mathbb{C} \setminus \sigma(-\Delta). \quad (4.4.4)$$

We notice that

$$(g'(\psi^0) - g'(\psi^0)\lambda(\lambda - L^0)^{-1})(-\Delta - \mu)^{-1}$$

$$\begin{aligned}
&= g'(\psi^0)(I - \lambda(\lambda - L^0)^{-1})(-\Delta - \mu)^{-1} \\
&= g'(\psi^0)(I - (\lambda - L^0 + L^0)(\lambda - L^0)^{-1})(-\Delta - \mu)^{-1} \\
&= g'(\psi^0)(I - I + L^0(\lambda - L^0)^{-1})(-\Delta - \mu)^{-1} \\
&= g'(\psi^0)L^0(\lambda - L^0)^{-1}(-\Delta - \mu)^{-1}.
\end{aligned}$$

As a result, an alternate expression for (4.4.4) is thus given by the formula

$$K_\lambda(\mu) = g'(\psi^0)L^0(\lambda - L^0)^{-1}(-\Delta - \mu)^{-1}, \operatorname{Re}(\lambda) \neq 0, \mu \in \mathbb{C} \setminus \sigma(-\Delta). \quad (4.4.5)$$

We notice that $K_\lambda(\mu) \in \mathcal{B}_2$, the class of Hilbert-Schmidt operators. Indeed, the operators $g'(\psi^0)$ and $L^0(\lambda - L^0)^{-1}$ are bounded, and the operator $(-\Delta - \mu)^{-1}$ is in \mathcal{B}_2 as it is similar via the Fourier transform to the diagonal operator $\operatorname{diag}\{(|\mathbf{k}|^2 - \mu)^{-1}\}_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}}$, and the series $\sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} (|\mathbf{k}|^2 - \mu)^{-2}$ converges. Since $K_\lambda(\mu) \in \mathcal{B}_2(L^2)$, the following 2-modified determinant exists:

$$\mathcal{D}(\lambda, \mu) = \det_2(I_{L^2_a} - K_\lambda(\mu)) = \prod_{n=1}^{\infty} ((1 - \varkappa_\lambda^{(n)})e^{\varkappa_\lambda^{(n)}}), \quad (4.4.6)$$

where $\varkappa_\lambda^{(n)}$ denote the eigenvalues of the operator $K_\lambda(\mu)$, and we remark that $\varkappa_\lambda^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. We refer to [Si05, Chapter 9] or [GK69, Section IV.2] or [Y10, Sec. I.7] for properties of the two-modified determinants.

We note that for each fixed $\lambda \in \mathbb{C} \setminus \sigma(-\Delta)$ the function $K_\lambda : \mathbb{C} \setminus \sigma(-\Delta) \rightarrow \mathcal{B}_2$ of the parameter μ is holomorphic with the derivative

$$\frac{dK_\lambda}{d\mu} = K_\lambda(\mu)(-\Delta - \mu)^{-1}, \quad (4.4.7)$$

and for each fixed $\mu \in \mathbb{C} \setminus \sigma(-\Delta)$ the function $K_{(\cdot)}(\mu) : \mathbb{C} \setminus i\mathbb{R} \rightarrow \mathcal{B}_2$ is a holomorphic function of the parameter λ . We also note that $0 \notin \sigma(-\Delta)$ because the Laplace operator is being considered on the space L^2_a of functions with zero average, and is

therefore similar via Fourier transform to the diagonal operator $\text{diag}\{\|\mathbf{k}\|^2\}_{\mathbf{k}\in\mathbb{Z}^2\setminus\{0\}}$.

Thus the operators

$$K_\lambda(0) = (g'(\psi^0) - g'(\psi^0)\lambda(\lambda - L^0)^{-1})(-\Delta)^{-1}, \quad \text{Re}(\lambda) \neq 0, \quad (4.4.8)$$

are well defined.

We have the following version of the Birman-Schwinger principle.

Proposition 4.13. *Assume Hypothesis 4.1 and recall formulas (4.4.2), (4.4.4) and (4.4.6). The following assertions are equivalent for each $\lambda \in \mathbb{C}\setminus i\mathbb{R}$ and $\mu \in \mathbb{C}\setminus\sigma(-\Delta)$:*

$$(i) \quad \mu \in \sigma(A_\lambda)\setminus\sigma(-\Delta), \quad (4.4.9)$$

$$(ii) \quad 1 \in \sigma(K_\lambda(\mu)), \quad (4.4.10)$$

$$(iii) \quad \mathcal{D}(\lambda, \mu) := \det_2(I_{L^2} - K_\lambda(\mu)) = 0. \quad (4.4.11)$$

Proof. The equivalence of (i) and (ii) is a direct consequence of the formula

$$A_\lambda - \mu = (I - K_\lambda(\mu))(-\Delta - \mu) \text{ for } \mu \in \mathbb{C}\setminus\sigma(-\Delta),$$

see formula (4.4.3). The equivalence of (ii) and (iii) is just a general fact from the theory of two-modified determinants, see, e.g. [Si05, Theorem 9.2(e)]. ■

We are now ready to formulate the main result of this section, Theorem 4.14, which gives a characterization of the unstable eigenvalue of L_{vor} in terms of the zeros of the function $\mathcal{D}(\lambda, 0) := \det_{2,L^2}(I - K_\lambda(0))$ and thus gives us a new way of detecting unstable eigenvalues of the operator L_{vor} . In particular, it reduces the study of instability of the linearized vorticity operator L_{vor} to the study of the 2-modified determinant $\det_{2,L^2}(I - K_\lambda(0))$ associated with the operator $K_\lambda(0)$ from (4.4.8).

Theorem 4.14. *The following assertions are equivalent:*

$$(i) \quad \lambda \in \sigma(L_{\text{vor}}) \setminus i\mathbb{R}, \quad (4.4.12)$$

$$(ii) \quad 1 \in \sigma(K_\lambda(0)), \quad (4.4.13)$$

$$(iii) \quad \mathcal{D}(\lambda, 0) := \det_{2,L^2}(I - K_\lambda(0)) = 0. \quad (4.4.14)$$

Proof. We will offer two proofs of the main conclusion of the theorem. We note that the equivalence of (ii) and (iii) is just a general fact from the theory of two-modified determinants, see again [Si05, Theorem 9.2(e)]. Our first proof shows that (i) is equivalent to (iii) while the second proof shows that (i) is equivalent to (ii).

First proof (i) \iff (iii). By Proposition 4.7 we have that $\lambda \in \sigma(L_{\text{vor}})$ if and only if $0 \in \sigma(A_\lambda)$. We now apply the equivalence of assertions (i) and (iii) of Proposition 4.13 for $\mu = 0$ and conclude that assertions (i) and (iii) of the current theorem are equivalent.

Second proof (i) \iff (ii). We return to representation (4.2.15) of the operator L_{vor} and discuss a Birman-Schwinger operator associated with the unperturbed operator L^0 and the perturbed operator L_{vor} . Recalling that ω and ψ are related via $\omega = -\Delta\psi$, equation (4.2.15) yields

$$L_{\text{vor}}\omega = L^0(I + g'(\psi^0(x, y))\Delta^{-1})\omega, \quad (4.4.15)$$

thus implying, for any $\lambda \in \mathbb{C} \setminus \sigma(L^0)$, that

$$\begin{aligned} \lambda - L_{\text{vor}} &= \lambda - L^0 - g'(\psi^0(x, y))L^0\Delta^{-1} \\ &= (I - g'(\psi^0(x, y))L^0\Delta^{-1}(\lambda - L^0)^{-1})(\lambda - L^0) \\ &= (I - \widetilde{K}_\lambda(0))(\lambda - L^0), \end{aligned} \quad (4.4.16)$$

where we temporarily denote $\widetilde{K}_\lambda(0) = g'(\psi^0(x, y))L^0\Delta^{-1}(\lambda - L^0)^{-1}$. We write $A = g'(\psi^0(x, y))L^0\Delta^{-1}$ and $B = (\lambda - L^0)^{-1}$ and note that $\widetilde{K}_\lambda(0) = AB$ while, using (4.4.5)

for $\mu = 0$, we also have $K_\lambda(0) = BA$ since the operators $g'(\psi^0(x, y))L^0$ and $(\lambda - L^0)^{-1}$ commute. By Lemma 3.7, we have that the operator $I - \widetilde{K}_\lambda(0)$ is invertible if and only if the operator $I - K_\lambda(0)$ is invertible. Since $\lambda - L_{\text{vor}} = (I - \widetilde{K}_\lambda(0))(\lambda - L^0)$, for $\lambda \in \mathbb{C} \setminus \sigma(L^0)$ and $\sigma(L^0) \subset i\mathbb{R}$, we know that $\lambda \in \sigma(L_{\text{vor}}) \setminus i\mathbb{R}$ if and only if $1 \in \sigma(\widetilde{K}_\lambda(0))$. This shows that assertions (i) and (ii) of the current theorem are equivalent. ■

We will conclude this section by completing the new proof of Lin's Theorem 4.10 outlined in Remark 4.11. As indicated in this remark, the only missing part of the proof is the following assertion.

Claim 4.15. *Assume that the operator A_0 has an odd number of negative eigenvalues and no kernel. Then A_λ has the same number of negative eigenvalues provided $\lambda > 0$ is small enough.*

In order to prove this claim (and thus finish the proof of Lin's Theorem 4.10) we will need to involve Birman-Schwinger type operators $K_0(\mu) \in \mathcal{B}_2$ associated with the operator A_0 and $-\Delta$, and the respective determinant $\mathcal{D}(0, \mu) = \det_{2, L^2}(I - K_0(\mu))$. Recalling formula (4.2.40),

$$A_0 = -\Delta - g'(\psi^0(x, y)) + g'(\psi^0(x, y))P_0, \quad (4.4.17)$$

similarly to (4.4.3) we arrive at the formulas

$$A_0 - \mu = (I - K_0(\mu))(-\Delta - \mu), \quad (4.4.18)$$

where we introduce the operator $K_0(\mu)$ as follows:

$$K_0(\mu) = (g'(\psi^0) - g'(\psi^0)P_0)(-\Delta - \mu)^{-1}, \quad \mu \in \mathbb{C} \setminus \sigma(-\Delta). \quad (4.4.19)$$

We notice that the only difference with (4.4.4) is that the operator $\lambda(\lambda - L^0)^{-1}$ is replaced by P_0 . As before, it is easy to see that $K_0 : \mathbb{C} \setminus \sigma(-\Delta) \rightarrow \mathcal{B}_2$ is a holomorphic function and $dK_0/d\mu = K_0(\mu)(-\Delta - \mu)^{-1}$. Similarly to Proposition 4.13 one shows the following fact.

Proposition 4.16. *Assume Hypothesis 4.3 and recall formulas (4.4.17) and (4.4.19).*

The following assertions are equivalent for $\mu \in \mathbb{C} \setminus \sigma(-\Delta)$:

$$(i) \quad \mu \in \sigma(A_0) \setminus \sigma(-\Delta), \tag{4.4.20}$$

$$(ii) \quad 1 \in \sigma(K_0(\mu)), \tag{4.4.21}$$

$$(iii) \quad \mathcal{D}(0, \mu) := \det_{2, L^2}(I - K_0(\mu)) = 0. \tag{4.4.22}$$

Remark 4.17. Because $K_\lambda(\cdot) : \mathbb{C} \setminus \sigma(-\Delta) \rightarrow \mathcal{B}_2$ and $K_0(\cdot) : \mathbb{C} \setminus \sigma(-\Delta) \rightarrow \mathcal{B}_2$ are holomorphic, by the general theory of two modified Fredholm determinants we conclude that the functions $\mathcal{D}(0, \cdot)$ and $\mathcal{D}(\lambda, \cdot)$ are holomorphic functions of $\mu \in \mathbb{C} \setminus \sigma(-\Delta)$. This is seen by applying Assertion IV.1.8 in [GK69] and Lemma 9.1 in [Si05].

Recall that $\lambda(\lambda - L^0)^{-1} \rightarrow P_0$ strongly in L_a^2 as $\lambda \rightarrow 0^+$ by Lemma 4.8. Re-writing $K_\lambda(\mu)$ and $K_0(\mu)$ as $K_\lambda(\mu) = T_\lambda S(\mu)$ and $K_0(\mu) = T_0 S(\mu)$, where $T_\lambda = g'(\psi^0) - g'(\psi^0)\lambda(\lambda - L^0)^{-1}$ for $\lambda > 0$ and $T_0 = g'(\psi^0) - g'(\psi^0)P_0$, and $S(\mu) = (-\Delta - \mu)^{-1} \in \mathcal{B}_2$, we conclude that $T_\lambda \rightarrow T_0$ strongly in L_a^2 as $\lambda \rightarrow 0$. The following fact is a standard result in the theory of Hilbert-Schmidt operators, cf. [GK69, Theorem 6.3].

Lemma 4.18. *For the operators $K_\lambda(\mu)$ from (4.4.4) and $K_0(\mu)$ from (4.4.19) we have $K_\lambda(\mu) \rightarrow K_0(\mu)$ in \mathcal{B}_2 as $\lambda \rightarrow 0^+$ uniformly for μ from any compact set in $\mathbb{C} \setminus \sigma(-\Delta)$.*

Proof. Indeed, for each fixed μ assertions $T_\lambda \rightarrow T_0$ strongly as $\lambda \rightarrow 0^+$ and $S(\mu) \in \mathcal{B}_2$ yield $T_\lambda S(\mu) \rightarrow T_0 S(\mu)$ in \mathcal{B}_2 by e.g., Theorem 6.3 in [GK69]. The proof given in

[GK69] is by writing $S(\mu) = K + L$ where K is of finite rank and $L \in \mathcal{B}_2$ with a small \mathcal{B}_2 norm, and then using that $T_\lambda u_j \rightarrow T_0 u_j$ as $\lambda \rightarrow 0^+$ for a finite basis $\{u_j\}$ of $\text{range}(K)$. This proof can be easily adapted for $S = S(\mu)$ depending on μ by taking first a finite ε -dense net $\{\mu_k\}$ in the compact set so that $\|S(\mu) - S(\mu_k)\|_{\mathcal{B}_2}$ are sufficiently small, and next applying the proof of [GK69, Theorem III.6.3] for each μ_k . ■

By [GK69, Theorem IV.2.1] the map $K \rightarrow \det_2(I - K)$ is continuous on \mathcal{B}_2 . By Lemma 4.18 we then conclude that $\mathcal{D}(\lambda, \mu) \rightarrow \mathcal{D}(0, \mu)$ as $\lambda \rightarrow 0^+$ uniformly for μ on compact subsets of $\mathbb{C} \setminus \sigma(-\Delta)$; here $\mathcal{D}(0, \mu)$ is defined in (iii) of Proposition 4.16. In what follows, we will apply the argument principle to the family of holomorphic functions $\mathcal{D}(\lambda, \cdot)$, $\lambda \geq 0$, in parameter μ . The following general formula for the logarithmic derivative can be found in [Y10, sec 1.8], see there formula (18),

$$\frac{d}{d\mu} \log \mathcal{D}(\lambda, \mu) = -\text{Tr} \left((I - K_\lambda(\mu))^{-1} K_\lambda(\mu) \frac{dK_\lambda(\mu)}{d\mu} \right). \quad (4.4.23)$$

We recall that $K_\lambda(\mu)$ and $\frac{dK_\lambda(\mu)}{d\mu}$ are in \mathcal{B}_2 and their product is in \mathcal{B}_1 , see e.g. [Y10, Prop I.6.3]. This formula holds for $\lambda \geq 0$ and $\mu \in \mathbb{C} \setminus \sigma(-\Delta)$. Since Tr is a continuous functional on \mathcal{B}_1 , it follows from Lemma 4.18 that

$$\frac{d}{d\mu} \log \mathcal{D}(\lambda, \mu) \rightarrow \frac{d}{d\mu} \log \mathcal{D}(0, \mu) \text{ as } \lambda \rightarrow 0^+ \quad (4.4.24)$$

on compact subsets of $\mathbb{C} \setminus \sigma(-\Delta)$. We recall that by the argument principle the number of zeros of a holomorphic function enclosed by the contour is computed via the integral over the contour of its logarithmic derivative.

We are ready to prove Claim 4.15.

Proof. Take a long thin rectangle $\mathcal{R} = [-a, -\varepsilon] \times [-\varepsilon, \varepsilon]$ that contains all the negative eigenvalues of A_0 and does not contain any other points in $\sigma(A_0)$. By Proposition 4.16 the eigenvalues of A_0 are zeros of the holomorphic function $\mathcal{D}(0, \mu)$ for $\mu \in \mathcal{R}^0$, the interior of \mathcal{R} . Using the argument principle and (4.4.24) on the boundary of \mathcal{R} we conclude that the number of zeros of the function $\mu \mapsto \mathcal{D}(\lambda, \mu)$ inside \mathcal{R}^0 is equal to the number of zeros of $\mu \mapsto \mathcal{D}(0, \mu)$ provided that $\lambda > 0$ is small enough. By Proposition 4.13 we see that Claim 4.15 holds, and thus the proof of Lin's Theorem 4.10 is completed. ■

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