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THREE AND FOUR COIL SYSTEMS FOR HOMOGENEOUS MAGNETIC FIELDS

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THREE AND FOUR COIL SYSTEMS FOR HOMOGENEOUS MAGNETIC FIELDS

by

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Summary:

In the magnetic testing of satellites and spacecraft, very homogeneous fields are needed. One method of obtaining such fields is to use a number of circular coils on a common axis. The parameters of the best three-and four-coil systems were obtained by making zero as many terms as possible of the series for the field along the axis of the system. The parameters are presented in the form of tables and curves.

Introduction:

In thetesting of the magnetic properties and instruments of satellites and spacecraft, there is a need to cancel the earth's magnetic field and then produce a very homogeneous controlled magnetic field. There should be ample access to the working volume and the electrical design and construction should be as simple as possible. Since the working volume required is large, it appears as if air-core coils are the only practical solution. Most of the previous work done has been on circular coils and this will be followed here, although square coils might have some constructional advantages. The volume of homogeneity for two coils (Helmholtz pair) is so small that prohibitively large coils would be needed for the required uniformity. For four coils, a much larger volume of homogeneity may be obtained for a given size of the coils. A few special solutions have been given for the four-coil system but no general solution over the whole range of parameters seems to have been made. It is the purpose of this paper to present such a solution and to indicate the various optimum values. The availability of a general solution will allow the design of a system when various factors such as the size or shape of the object being tested do not allow an optimum value to be used. In preparing for the solution of the four-coil system, it was found that the solution of the three-coil system was useful. The general solution of the three-coil system is also presented here although access to the volume of uniform field is quite limited because of the position of the center coil.

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Theory:

The magnetic field of a single circular coil may be obtained by various methods such as the use of a scalar or vector magnetic potential[1,2]. The magnetic field H along the axis of the coil is

$$H = \sum_{n=0}^{\infty} a_n z^n \qquad (1)$$

where z is the distance along the axis measured from the origin 0 as shown in Figure 1, $z < b$ and

$$a_n = \frac{NI(1-x^2)}{2b^{n+1}} P'_{n+1}(x) \qquad (2)$$

N = number of turns on the coil

I = current through the coil

$$P'_{n+1}(x) = \frac{dP_{n+1}(x)}{dx}$$

where $P_n(x)$ is the nth order Legendre polynomial and

$$x = \cos\alpha .$$

Two or more of the coils on a common axis may be used to produce a more homogeneous magnetic field than is possible with one coil. This is done by making as many terms zero beyond a_0 in (1) as is possible[3,4]. The same results may be obtained by use of a potential[5,6].

For a symmetrical four-coil system as shown in Figure 2, the magnetic field along the axis is

$$H = \sum_{n=0,2,4,\ldots}^{\infty} a_n z^n \tag{3}$$

where

$$a_n = \frac{N_1 I_1 (1 - x_1^2)}{b_1^{n+1}} P'_{n+1}(x_1) + \frac{N_2 I_2 (1 - x_2^2)}{b_2^{n+1}} P'_{n+1}(x_2) \tag{4}$$

The terms for odd n's in (3) are zero because of symmetry.

<u>Three coil results</u>:

The three coil solution is obtained by setting $x_1 = 0$ or $a_1 = 90°$. This has the effect of making the two inner coils of Figure 2 become the one center coil of Figure 3. With three coils it is possible to make a_2 and a_4 of (3) zero and then a_6 may be made a minimum. The solution in more detail is carried out in the appendix and the results are shown in Figures 4 and 5. Additional results are presented in Table I. The range of x_2 in Figures 4 and 5 is from 0.4472 to 0.7651 and the ratio of the radii $b = b_2/b_1$ ranges from infinity to zero as shown in Figure 4. This indicates that when x_2 is close to 0.4472, the diameter of the center coil should be smaller than that of the two outer coils, while when x_2 is close to 0.7651, the diameter of the center coil should be larger than that of the two outer coils. From Table I all three coils should have equal diameters at $x_2 = 0.6051$ while at $x_2 = 0.6547$ the three coils should lie on the surface of a sphere. The range of the ratio of ampere turns, $I = N_2I_2/N_1I_1$ is infinity to zero as shown in Figure 4. Where $x_1 = 0$ the two inner coils of a four-coil system become the center coil of a three-coil system. The ampere turns on the center coil then would be $2N_1I_1$, and the actual ratio of ampere turns of an outer coil to the center coil is $N_2I_2/2N_1I_1 = I/2$. When x_2 is close to 0.4472, the ampere-turns of the center coil should be smaller than that of the outer coils, while when x_2 is close to 0.7651, the ampere-turns of the center coil should be larger than that of the outer coils. From Table I all three coils would have equal ampere turns at $x_2 = 0.6402$.

The most homogeneous field would be the one which made the remainder of the series of (3) a minimum, i.e., in

$$H = a_0\left[1 + \frac{a_6}{a_0} z^6 + \frac{a_8}{a_0} z^8 + \cdots\right] \tag{5}$$

the sum of the terms in a_6, a_8, and so on should be a minimum. Since the term containing a_6 usually is much larger than that containing a_8 and any of the succeeding terms, making the a_6 term a minimum should give a close approximation to the optimum field. This may be done in a number of ways depending upon which of the parameters or combinations of the parameters are assumed to remain constant. As an example let

$$A_6 = b_1^2 b_2^4 \frac{a_6}{a_0} = \bar{b}^6 \frac{a_6}{a_0}. \tag{6}$$

In (6) if the mean radius from the center of the system $\bar{b} = (b_1 b_2^2)^{1/3}$, for the three coils were constant, the a_6 term would have its minimum at $x_2 = 0.6163$ as shown in Figure 5 and Table I. As a second example let

$$B_6 = b_m^6 \frac{a_6}{a_0} = \begin{cases} b_2^6 \frac{a_6}{a_0} = \bar{b}^2 A_6 \text{ when } b_2 \geq b_1 \\ b_1^6 \frac{a_6}{a_0} = \bar{b}^{-4} A_6 \text{ when } b_1 \geq b_2 \end{cases} \quad (7)$$

In (7) if the larger radius b_m from the center of the system to the coils were constant, the a_6 term would have its minimum at $x_2 = 0.6547$ as shown in Figure 5 and Table I. If the system must be limited to a certain largest volume, probably the second optimum, that of B_6, would be the better of the two to use. It is possible to define other optima as well but it is believed that the two mentioned above are the most useful and practical.

Four coil results:

With a four coil system such as that shown in Figure 2, it is possible to make zero a_2, a_4 and a_6 of (3) and to make a_8 a minimum.[4] This solution is carried out in more detail in the appendix and the results are shown in Figures 6 and 7 and in Table II. The range of x_2 in Figures 6 and 7 is from 0.44721 to 0.87174 and the corresponding x_1 as shown in Figure 6 decreases from 0.20929 to a minimum of 0.20360 and then increases to a maximum of 0.44721. The ratio of the radii $b = b_2/b_1$ ranges from infinity to zero as shown in Figure 6. Thus when x_2 is close to 0.44721, the diameter of the two inner or No. 1 coils should be smaller than that of the two outer or No. 2 coils, while when x_2 is close to 0.87174, the diameter of the No. 1 coils should be larger than that of the No. 2 coils. As indicated in Table II at $x_2 = 0.68519$ all four coils have the same diameter, while at $x_2 = 0.76505$ the four coils lie on the surface of a sphere and at 0.85363 both the No. 1 and No. 2 coils lie in the same plane perpendicular to the axis of the system with the No. 1 coils having the larger diameter of 3.76797 times that of the No. 2 coils. From $x_2 = 0.85363$ to 0.87174, the No. 2 coils are closer to the center of the system than the No. 1 coils. It is interesting to notice that if the No. 1 and No. 2 coils are interchanged the two end points of Table II become identical. The range of the ratio of ampere-turns, $I = N_2 I_2 / N_1 I_1$ goes from infinity to zero as shown in Figure 6. When x_2 is close to 0.44721 the ampere-turns of the No. 1 (inner) coils would be smaller than that of the No. 2 (outer) coils while for x_2 close to 0.87174, the ampere-turns of the No. 1 coils would be larger. All coils will have equal ampere-turns at $x_2 = 0.74207$.

The most uniform field would be the one which made the remainder of the series of (3) a minimum, i.e., in

$$H = a_0 \left[1 + \frac{a_8}{a_0} z^8 + \frac{a_{10}}{a_0} z^{10} + \cdots \right] \quad (8)$$

the sum of the terms involving a_8, a_{10}, etc. should be made a minimum. A good approximation to this minimum should be that which makes the a_8 term a minimum. This may be done in a number of ways, one of which is to make the mean radius from the center of the system, $\bar{b} = (b_1 b_2)^{1/2}$, constant and define

$$A_8 = b_1^4 b_2^4 \frac{a_8}{a_0} = \bar{b}^8 \frac{a_8}{a_0} . \quad (9)$$

Figure 7 shows a curve of A_8 plotted against x_2 and Table II gives some values of A_8 including those at the end points. The minimum value of A_8 as given in Table II and as shown in Figure 7 is -2.21988 and occurs at $x_2 = 0.75208$ and $x_1 = 0.27505$. Another way of making the a_8 term a minimum is to put

$$B_8 = b_m^8 \frac{a_8}{a_0} = \begin{cases} b_2^8 \frac{a_8}{a_0} \text{ when } b_2 \geq b_1 \\ b_1^8 \frac{a_8}{a_0} \text{ when } b_1 \geq b_2 \end{cases} \quad (10)$$

where b_m is the larger radius from the center of the system to the coils. A curve of B_8 against x_2 is shown in Figure 7 and some values of B_8 are given in Table II. One of the values in Table II is the minimum value of $B_8 = -2.25510$ which occurs at $x_2 = 0.76505$ and $x_1 = 0.28523$. If more than one term is considered in the series of (8), the minimum depends upon the magnitude of z. For example if the a_8 and a_{10} terms are considered in (8) they may be written as

$$\left(\bar{b}^8 \frac{a_8}{a_0}\right)\left(\frac{z}{\bar{b}}\right)^8 + \left(\bar{b}^{10} \frac{a_{10}}{a_0}\right)\left(\frac{z}{\bar{b}}\right)^{10} = A_8 \left(\frac{z}{\bar{b}}\right)^8 + A_{10}\left(\frac{z}{\bar{b}}\right)^{10}$$

$$= C \left(\frac{z}{\bar{b}}\right)^8 \qquad (11)$$

where

$$A_{10} = \bar{b}^{10} \frac{a_{10}}{a_0} = b_1^5 b_2^5 \frac{a_{10}}{a_0}$$

$$C = A_8 + A_{10}\left(\frac{z}{\bar{b}}\right)^2 .$$

The minimum value of C is the same as the minimum value of A_8 when z = 0 but when $z/\bar{b} = 0.2167$, the minimum value of C occurs at $x_2 = 0.74842$ and $x_1 = 0.27235$ as given in Table II.

There are several other well known four coil solutions which do not have as great a uniformity as the solutions given in Figures 6 and 7 and in Table II because only two coefficients of (3) namely, a_2 and a_4, are made zero. As an example Neumann[6] and Fanselau[4,8], put x_1 and x_2 at the roots of $P_5'(x) = 0$ to make a_4 zero, then chose the ampere-turns of both No. 1 and No. 2 coils equal and found b by putting $a_2 = 0$. Further details are given in Table III. Fanselau[10] in another solution made a_4 zero by using the roots of $P_5'(x) = 0$. Both coils on one side of the system were put in the same plane perpendicular to the axis of the system and a_2 was made zero by choosing I = 28.2897. The ratio of the diameter of the smaller coil to that of the larger coil was 0.250495 and b = 0.372830. Fanselau also indicated that a similar solution could be made where both sets of coils would have the same radius. Several additional four coil solutions with both a_2 and a_4 zero are given by McKeehan[6]. Scott[11] used four coils with the inner pair having a smaller diameter than the outer pair. His solution had both a_2 and a_4 zero. Franzen[12] used Garrett's theory[45] to develop a theory of finite coil cross-section for a four coil system.

In Table III is presented the specifications of several coil systems along with an indication of how large a sphere about the center of the system will have a given homogeneity. Several three-and four-coil systems are given along with the two coil system of Helmholtz and the presently known six- and eight-coil systems. These are compared on a basis of an inhomogeneity of 10^{-5} and as an example choose the minimum A_8 solution for the four-coil system. Then

$$\left| A_8\left(\frac{z}{\bar{b}}\right)^8 \right| = 10^{-5} \qquad (12)$$

and

$$\left(\frac{z}{\bar{b}}\right) = 0.21677 \text{ or } 21.677\%$$

This means that if $\bar{b}$ = 10 feet and considering only the A_8 term, the inhomogeneity of the magnetic field inside a sphere of 2.1677 feet radius would be less than or equal to 10^{-5} or 0.001 %. If B_8 were used

$$\left| B_8\left(\frac{z}{b_m}\right)^8 \right| = 10^{-5} \qquad (13)$$

and

$$\left(\frac{z}{b_m}\right) = 0.20944 \text{ or } 20.944\%.$$

The ampere-turns in the middle coil of the three coil system should be $2N_1I_1$. For the six-coil systems, the x's for the Neumann-Fanselau and the Williams-Cain solutions are the roots of $P_7'(x) = 0$. Similarly for the eight-coil systems, the x's for the Neumann-McKeehan and the Williams-Cain solutions are the roots of $P_7'(x) = 0$. Similar solutions using the roots of $P_{n+1}'(x) = 0$ where n is even could be made for systems containing ten, twelve or higher (n) number of coils[9]. In fact Garrett[5] mentions the fact that the solution for a sixteen-coil system could be made readily. The Williams-Cain solution always gives the optimum solution for minimum B_{2n}.

Conclusions:

Complete solutions along with tables and curves that should be useful in design work have been presented for the three and four circular-coil systems with zero cross-sectional area of the coils. Comparisons of these results have been made with the two-coil (Helmholtz) and the known six and eight coil systems. At the present time there is needed:

1. A complete study of the six-coil problem over the ranges of all of the parameters.
2. An investigation of the square or rectangular coil systems.
3. An analysis of the effect of finite cross-sectional area of the coils.

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APPENDIX

Solution of the coil systems

1. Three coil system

Put $x_1 = 0$ in (4) and let $b = b_2/b_1$ and $I = N_2 I_2/N_1 I_1$. Then for $n = 2$ and $n = 4$ in (4):

$$\left(-\frac{3}{2}\right)b^3 + I\left(1 - x_2^2\right) P_3'\left(x_2\right) = 0 \qquad \text{(A-1)}$$

$$\left(\frac{15}{8}\right)b^5 + I\left(1 - x_2^2\right) P_5'\left(x_2\right) = 0 \qquad \text{(A-2)}$$

where

$$P_3'\left(x_2\right) = \left(\frac{3}{2}\right)\left(5x_2^2 - 1\right)$$

and

$$P_5'\left(x_2\right) = \left(\frac{15}{8}\right)\left(21\, x_2^4 - 14\, x_2^2 + 1\right)$$

From (A-1) and (A-2):

$$\frac{I\left(1 - x_2^2\right)}{b^3} = \frac{\left(+\frac{3}{2}\right)}{P_3'\left(x_2\right)} = \frac{\left(-\frac{15}{8}\right)b^2}{P_5'\left(x_2\right)} \qquad \text{(A-3)}$$

From (A-3):

$$b^2 = \frac{1 - 14\, x_2^2 + 21\, x_2^4}{1 - 5x_2^2} \qquad \text{(A-4)}$$

$$I = \frac{b^3}{\left(1 - x_2^2\right)\left(5x_2^2 - 1\right)} \qquad \text{(A-5)}$$

In (A-4) b^2 must be positive so the only possible solutions must have $0.0 \le x_2 \le 0.2852$ or $0.4472 \le x_2 \le 0.7651$ where 0.4472 is the root of $P_3'\left(x_2\right) = 0$ and 0.2852 and 0.7651 are the roots of $P_5'\left(x_2\right) = 0$. The range of values of x_2 from 0.4472 to 0.7651 produces positive values of I while the range from 0.0 to 0.2852 produces negative values of I. A negative value of I means that the current direction in the center coil is reversed from those in the two outer coils.

For the a_6 term of (3) let

$$A_6 = b_1^2 b_2^4 \frac{a_6}{a_0} = b^4 b_1^6 \frac{a_6}{a_0} \qquad \text{(A-6)}$$

where a_6 and a_0 are given by (2). Then by the use of (4), (A-4) and (A-5)

$$A_6 = -\frac{7}{16}\left(\frac{15\, x_2^6 - 9\, x_2^4 + 5\, x_2^2 + 5}{x_2^2 + 1}\right) \qquad \text{(A-7)}$$

The value of x_2 at which (A-7) has its minimum value is 0.6163. This value of x_2 is in the range for which I is positive. Hence for the three-coil solution at least, making all currents flow in the same direction will produce a more homogeneous field. Another minimum value may be obtained by defining

$$B_6 = \begin{cases} b_2^6 \dfrac{a_6}{a_0} = \bar{b}^2 A_6 & \text{for } 0.4475 \le x_2 \le 0.6547 \\[2ex] b_1^6 \dfrac{a_6}{a_0} = \dfrac{A_6}{\bar{b}^4} & \text{for } 0.6547 \le x_2 \le 0.7651 \end{cases} \qquad \text{(A-8)}$$

where $\bar{b} = 1.0$ at $x_2 = 0.6547$. Then the minimum value of B_6 occurs at $x_2 = 0.6547$.

2. Four coil system

For $n = 2, 4, 6$ from (4) and putting $b = b_2/b_1$ and $I = N_2 I_2/N_1 I_1$:

$$b^3\left(1 - x_1^2\right) P_3'\left(x_1\right) + I\left(1 - x_2^2\right) P_3'\left(x_2\right) = 0 \qquad \text{(A-9)}$$

$$b^5\left(1 - x_1^2\right) P_5'\left(x_1\right) + I\left(1 - x_2^2\right) P_5'\left(x_2\right) = 0 \qquad \text{(A-10)}$$

$$b^7\left(1 - x_1^2\right) P_7'\left(x_1\right) + I\left(1 - x_2^2\right) P_7'\left(x_2\right) = 0 \qquad \text{(A-11)}$$

where

$$P_7'\left(x_1\right) = \frac{7}{16}\left(429\, x_1^6 - 495\, x_1^4 + 135\, x_1^2 - 5\right)$$

and P_3' and P_5' are given under (A-2). From (A-9), (A-10) and (A-11)

$$\frac{I\left(1 - x_2^2\right)}{b^3\left(1 - x_1^2\right)} = -\frac{P_3'\left(x_1\right)}{P_3'\left(x_2\right)} = -\frac{b^2 P_5'\left(x_1\right)}{P_5'\left(x_2\right)} = -\frac{b^4 P_7'\left(x_1\right)}{P_7'\left(x_2\right)} \qquad \text{(A-12)}$$

From (A-12):

$$b^2 = \frac{P_3'\left(x_1\right) P_5'\left(x_2\right)}{P_3'\left(x_2\right) P_5'\left(x_1\right)} = \frac{P_5'\left(x_1\right) P_7'\left(x_2\right)}{P_5'\left(x_2\right) P_7'\left(x_1\right)} \qquad \text{(A-13)}$$

From (A-13):

$$\frac{P_3'\left(x_1\right) P_7'\left(x_1\right)}{\left[P_5'\left(x_1\right)\right]^2} = \frac{P_3'\left(x_2\right) P_7'\left(x_2\right)}{\left[P_5'\left(x_2\right)\right]^2} = f\left(x_2\right) \qquad \text{(A-14)}$$

or

$$P_3'\left(x_1\right) P_7'\left(x_1\right) - f\left(x_2\right)\left[P_5'\left(x_1\right)\right]^2 = 0 \qquad \text{(A-15)}$$

For a given x_2, (A-15) was solved for x_1 and the ratio of the radii b was obtained from (A-13). The ampere-turn ratio I may be obtained by the use of (A-12):

$$I = -\frac{b^3\left(1 - x_1^2\right) P_3'\left(x_1\right)}{\left(1 - x_2^2\right) P_3'\left(x_2\right)} \qquad \text{(A-16)}$$

The use of a computer allowed numerous solutions to be obtained over the range for x_2 from 0.4472 which is a root of $P_3'(x) = 0$ to 0.8717 which is a root of $P_7'(x) = 0$. The corresponding range for x_1 is from 0.2036 to 0.4472. For these values I is positive. I may be negative when both x_1 and x_2 lie in the range between 0.4472 and 0.5917 where 0.5917 is a root of $P_7'(x) = 0$. The values for I negative were not calculated because the more useful values were in the range where I is positive.

For the a_8 term of (3) let

$$A_8 = b_1^4 b_2^4 \frac{a_8}{a_0} \qquad \text{(A-17)}$$

where a_8 and a_0 are given by (2). Then by the use of (4), (A-13) and (A-16)

$$A_8 = \frac{P_3'(x_1)P_5'(x_2)P_7'(x_2)P_9'(x_1) - P_3'(x_2)P_5'(x_1)P_7'(x_1)P_9'(x_2)}{P_3'(x_2)P_5'(x_2)P_7'(x_1) - P_3'(x_1)P_5'(x_1)P_7'(x_2)} \qquad \text{(A-18)}$$

where

$$P_9'(x) = \frac{45}{128}\left(2431x^8 - 4004x^6 + 2002x^4 - 308x^2 + 7\right)$$

The minimum value of $|A_8|$ is 2.21989 and occurs at $x_1 = 0.27505$ and $x_2 = 0.75209$. If the sum of both the a_8 and a_{10} terms is minimized the minimum occurs at $x_1 = 0.27235$ and $x_2 = 0.74841$. Again another minimum may be defined by taking

$$B_8 = \begin{cases} b_2^8 \dfrac{a_8}{a_0} = \bar{b}^4 A_8 & \text{when } x_2 \leq 0.76505 \\[2ex] b_1^8 \dfrac{a_8}{a_0} = \dfrac{A_8}{\bar{b}^4} & \text{when } x_2 \geq 0.76505 \end{cases} \qquad \text{(A-19)}$$

The minimum value of B_8 occurs at $x_1 = 0.28523$ and $x_2 = 0.76505$. The next term in the series has

$$A_{10} = b_1^5 b_2^5 \frac{a_{10}}{a_0} \qquad \text{(A-20)}$$

where a_{10} and a_0 are given by (2).

If solutions of greater accuracy are needed than those available in the curves and tables here presented, they may be obtained by numerical solution of the equations given in the appendix using the present solutions as a starting point.

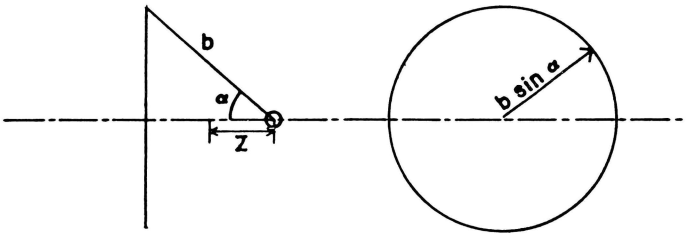

Figure 1 - The single circular coil

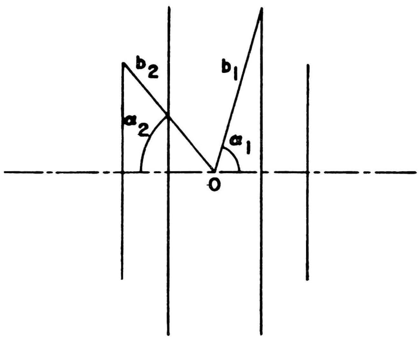

Figure 2 - The four-coil system

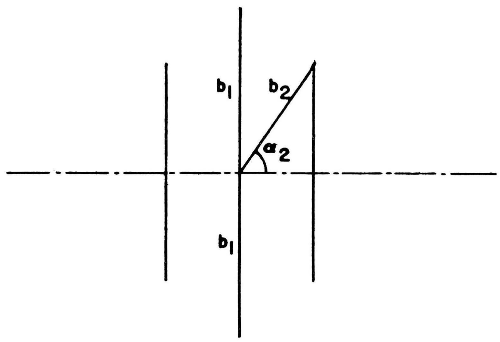

Figure 3 - The three-coil system

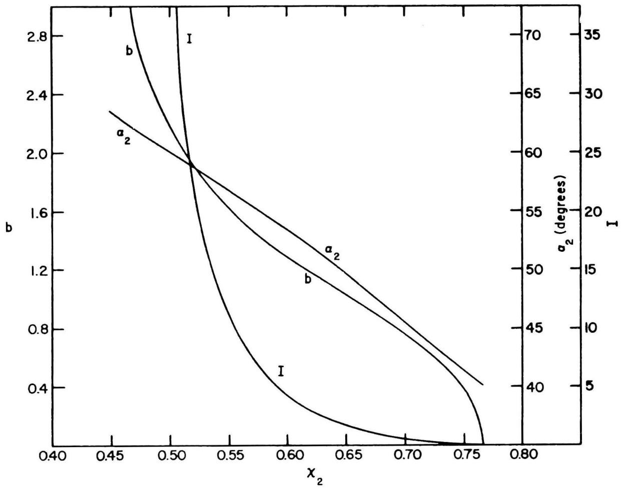

Figure 4 - Calculated values of d_2, I and b for the three-coil system

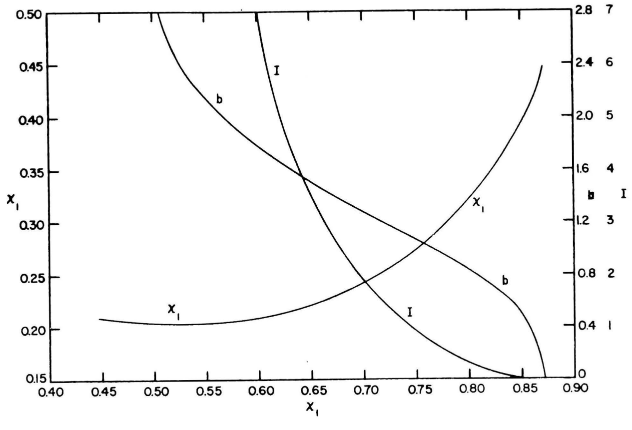

Figure 6 - Calculated values of x, I and b for the four-coil system

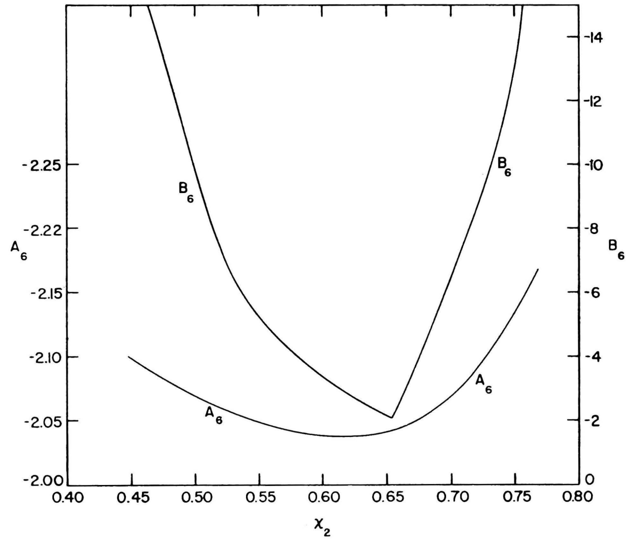

Figure 5 - Calculated values of A_6 and B_6 for the three-coil system

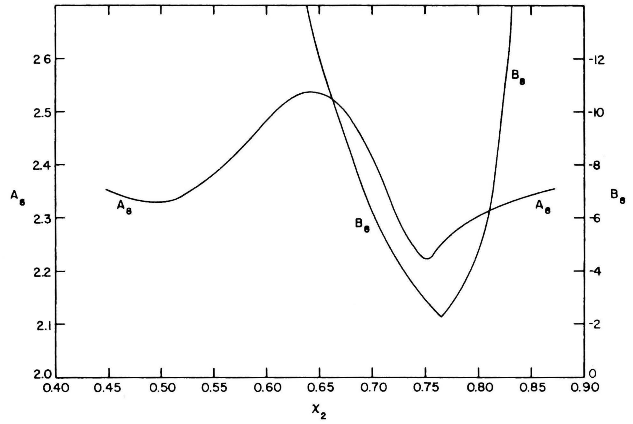

Figure 7 - Calculated values of A_8 and B_8 for the four-coil system

Table I

Particular Three-Coil Systems

x_2	α_2(degrees)	b	I	A_6	B_6	Remarks
0.4472	63.42°	∞	∞	-2.100	$-\infty$	End point
0.6051	52.76°	1.256	3.763	-2.0369	-3.214	Coils of equal diameter Barker's solution[3]
0.6163	51.96°	1.197	3.076	-2.0364	-2.925	Minimum $\lvert A_6 \rvert$
0.6402	50.19°	1.074	2.000	-2.0388	-2.351	Coils of equal ampere-turns
0.6547	49.10°	1.000	1.531	-2.043	-2.043	Minimum $\lvert B_6 \rvert$ Maxwell's solution[6] Coils on surface of sphere
0.7651	40.08°	0.000	0.000	-2.167	$-\infty$	End point

Table II

Particular Four-Coil Systems

x_1	x_2	b	I	A_8	B_8	Remarks
0.20929	0.44721	∞	∞	-2.35384	$-\infty$	End point
0.20360	0.51961	2.49155	46.00025	-2.34170	-90.24299	Minimum x_1
0.23629	0.68519	1.33407	2.26058	-2.47133	- 7.82798	All coils have equal diameters Barker's solution[3]
0.26786	0.74207	1.09795	1.000	-2.23448	- 3.24721	All coils have equal ampere-turns Braunbek's solution[6,7]
0.27235	0.74842	1.07127	0.90406	-2.22196	- 2.92648	Optimum using both A_8 and A_{10} terms
0.27505	0.75208	1.05576	0.85165	-2.21988	- 2.75803	Optimum using A_8 only
0.28523	0.76505	1.000	0.68211	-2.25510	- 2.25510	All coils on surface of a sphere Optimum using B_8. McKeehan's solution[5,6]
0.39864	0.85363	0.46699	0.024569	-2.34225	-49.24825	Both No. 1 and No. 2 coils lie in the same plane perpendicular to the axis of the system
0.44721	0.87174	0.000	0.000	-2.35384	$-\infty$	End point

Table 3

Specifications of Various Coil Systems.

Source		Ampere (Reference 6)	Helmholtz (Reference 6)	Barker (Reference 3)	Maxwell (Reference 6)	Minimum A_6
Number of Coils		1	2	3	3	3
Assumptions		None	None	Equal diameter coils	Coils on surface of sphere	None
The a's that are zero		None	a_2	a_2, a_4	a_2, a_4	a_2, a_4
Coefficient of next term, A_n		−1.5	−1.8	−2.03689	−2.0428571	−2.036426
x_1		0	0.4472136	0.0	0.0	0.0
x_2		-	-	0.605108	0.6546537	0.61627
b		-	-	1.25606	1.00	1.19697
I		-	-	3.76323	1.53125	3.075846
For an inhomogeneity of 10^{-5} (percent)	$z/\bar{b}$	0.26	4.86	13.0368	13.031	13.0373
	z/b_m	0.26	4.86	12.083	13.031	12.279

Source		Neumann-Fanselau (References 6 and 8)	Braunbek (Reference 7)	Barker (Reference 3)	McKeehan (References 5 and 6)	Minimum A_8
Number of coils		4	4	4	4	4
Assumptions		Coils have equal ampere-turns	Coils have equal ampere-turns	Coils have equal diameters	Coils on surface of sphere	None
The a's that are zero		a_2, a_4	a_2, a_4, a_6	a_2, a_4, a_6	a_2, a_4, a_6	a_2, a_4, a_6
Coefficient of next term, A_n		−1.289	−2.23448	−2.47133	−2.255102	−2.2219679
x_1		0.285232	0.26786	0.23629	0.2852315	0.2723547
x_2		0.765055	0.74207	0.68519	0.7650553	0.7484183
b		1.136009	1.09795	1.33407	1.00	1.0712777
I		1.00	1.00	2.26058	0.6821109	0.9040608
For an inhomogeneity of 10^{-5} (percent)	$z/\bar{b}$	14.07	21.672	21.480	21.607	21.677
	z/b_m	13.20	20.682	18.583	21.607	20.944

Source		Neumann-Fanselau (References 6 and 8)	Braunbek-McKeehan (References 6 and 7)	Williams-Cain (Reference 9)	Neumann-McKeehan (Reference 6)	Williams-Cain (Reference 9)
Number of coils		6	6	6	8	8
Assumptions		Coils have equal ampere-turns	Coils have equal ampere-turns	Coils lie on surface of sphere	Coils have equal ampere-turns	Coils lie on surface of sphere
The a's that are zero		a_2, a_4, a_6	a_2 to a_{10}	a_2 to a_{10}	a_2 to a_8	a_2 to a_{14}
x_1		0.20929922	0.190655	0.20929922	0.1652754	0.1652754
x_2		0.59170018	0.550274	0.59170018	0.4779250	0.4779250
x_3		0.87174003	0.843307	0.87174003	0.7387739	0.7387739
x_4		-	-	-	0.9195342	0.9195342
b_2/b_1		1.071723	1.046147	1.00	1.0222398	1.00
b_3/b_1		1.242359	1.157907	1.00	1.1827288	1.00
b_4/b_1		-	-	-	1.2382935	1.00
N_2I_2/N_1I_1		1.00	1.00	0.8270469	1.00	0.891626
N_3I_3/N_1I_1		1.00	1.00	0.5108492	1.00	0.686604
N_4I_4/N_1I_1		-	-	-	1.00	0.406992
For an inhomogeneity of 10^{-5} (percent)	z/b_m	31	34	36	16	47

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