A SYNTHESIS PROCEDURE FOR SAMPLED-DATA SYSTEMS

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Abstract.—This paper proposes a direct synthesis procedure for the design of sampled-data systems using the z-transformation method. This procedure can be summarized as follows:

1. By methods set forth in the literature the pole-zero configuration of a continuous system is found such that the specifications are satisfied.
2. The z-transformation of the output of the sampled-data system is found such that the response of the sampled-data system coincides with the response of the continuous system at the sampling instants.
3. The closed-loop pole-zero configuration in the z-plane is determined from the transform of the output.
4. The open-loop pole-zero configuration is found from the closed-loop configuration.
5. From the open-loop pole-zero configuration the open-loop transfer function can be found in the s-plane from which the compensation network is synthesized.

This paper demonstrates that in certain cases restrictions exist concerning the location of the poles and zeros of step 1 of the above procedure. The paper concludes by summarizing the scope of usefulness and limitations of the procedure.

I. INTRODUCTION

During the last few years a major change in methods for design of continuous feedback control systems has taken place. Until this time the conventional method of design was to work in terms of the frequency-domain behavior of the open-loop transfer function. This procedure reshapes the open-loop transfer function in such a manner that when the loop is closed, it is hoped that the closed-loop transfer function will be satisfactory. In 1947 Guillemin proposed an entirely different procedure which is the basis of work done by Truxal and Aaron. This procedure can be summarized as follows:

1. The closed-loop transfer function is determined so that the specifications are satisfied.
2. From the closed-loop transfer function the corresponding open-loop transfer function is found.
3. From the open-loop transfer function and from the fixed part of the system the appropriate compensation network is synthesized.

This paper investigates the possibility of using a synthesis procedure for sampled-data systems that is analogous to the Guillemin procedure for continuous systems. The procedure for sampled-data systems is more complicated and can be summarized as follows:

1. By methods set forth in the literature the pole-zero configuration of a continuous system is found such that the specifications are satisfied.
2. The z-transformation of the output of the sampled-data system is found such that the response of the sampled-data system coincides with the response of the continuous system at the sampling instants.
3. The closed-loop pole-zero configuration in the z-plane is determined from the transform of the output.
4. The open-loop pole-zero configuration is found from the closed-loop configuration.
5. From the open-loop pole-zero configuration the open-loop transfer function can be found in the s-plane from which the compensation network is synthesized.

II. SYNTHESIS OF SECOND-ORDER SYSTEM

The system of Fig. 1 is considered first. Because of the nature of typical transfer function $G(s)$ the pole-zero configuration for the continuous system is chosen so that the response of the continuous system starts at zero at $t = 0$ and has a steady-state value equal to unity when a unit-step function is applied to the input. Therefore the pole-zero configuration for the continuous system is chosen as

$$\frac{C(s)}{R(s)} = \frac{(s + a_1)}{(s + p_1)(s + p_2)} \frac{p_1}{a_1}$$

When $r(t)$ is a unit-step function, $C(s)$ becomes

$$C(s) = \frac{(s + a_1)}{s(s + p_1)(s + p_2)a_1} \frac{p_1}{p_2}$$

Equation (2) can be expanded as

$$C(s) = \frac{1}{s} + \frac{K_1}{(s + p_1)} + \frac{K_2}{(s + p_2)}$$

where

$$K_1 = \frac{(p_1 - a_1)}{(p_2 - p_1)} \frac{p_2}{a_1} \text{ and } K_2 = \frac{(p_2 - a_1)}{(p_1 - p_2)} \frac{p_1}{a_1}$$

The $z$-transformation of the sampled-data system that coincides with the output of the continuous system at the sampling instants can be found by taking the $z$-transformation of (3) by using the transform pairs of Table I as

$$C(z) = \frac{z}{(z - 1)} + \frac{K_1z}{(z - \alpha_1)} + \frac{K_2z}{(z - \alpha_2)}$$

where $\alpha_1 = e^{\pi T}$ and $\alpha_2 = e^{\pi T}$.

$C(z)$ can be written as

$$C(z) = \frac{z}{(z - 1)} \frac{(Az + B)}{(z - \alpha_1)(z - \alpha_2)}$$
where

\( A = K_1\alpha_1 + K_2\alpha_2 + 1 \)

and

\( B = \alpha_1\alpha_2 + K_1\alpha_2 + K_2\alpha_1 \)

The \( z \)-transformation of the output of the system shown in Fig. 1 is

\[ C(z) + \frac{R(z)G(z)}{1 + G(z)} \]

When \( r(t) \) is a unit step function, (9) becomes

\[ C(z) = \frac{z}{(z - 1)} \frac{G(z)}{[1 + G(z)]} \]

From (6) and (10) the following identification can be made

\[ \frac{G(z)}{1 + G(z)} = \frac{Az + B}{(z - \alpha_1)(z - \alpha_2)} \]

\( G(z) \) and \( \frac{G(z)}{[1 + G(z)]} \) are the ratio of polynomials and have the form

\[ G(z) = \frac{p(z)}{q(z)} \]

and

\[ \frac{G(z)}{1 + G(z)} = \frac{p(z)}{n(z)} = \frac{p(z)}{p(z) + q(z)} \]

From (13) \( G(z) \) is given by

\[ G(z) = \frac{p(z)}{n(z) - p(z)} \]

When (14) is applied to (11), \( G(z) \) for this example is

\[ G(z) = \frac{Az + B}{z^2 + (-\alpha_1 - \alpha_2 - A)z + \alpha_1\alpha_2 - B} \]

It is possible to obtain an infinite number of different \( G(s) \) transfer functions from (15). If the system of Fig. 1 is the desired system, the term labeled \( B \) in (15) and the other preceding equations must equal zero. In such a system each term of \( G(z) \) in the partial fraction expansion has a zero at the origin; therefore, \( G(z) \) must also have a zero at the origin. Therefore the following restrictions are placed on the original continuous system

\[ B = e^{(p_1+p_2)t} + K_1e^{p_2t} + K_2e^{p_3t} = 0 \]

Physically this means that sampled-data systems of this type respond in a
more limited manner than do continuous systems. That is, for each sampled-data system there exists a continuous system such that the outputs of the two systems coincide at the sampling instants; however for each continuous system no corresponding sampled-data system of this type exists unless the restrictions of (16) are satisfied. This is not a limitation of this procedure but is simply a characteristic of sampled-data systems. Equation (16) is a function of $p_1$, $p_2$, and $a_1$. If any two of these are chosen, the third is fixed. If $p_1$ and $p_2$ are chosen, $a_1$ is fixed as

$$a_1 = \frac{p_1p_2 \left[ e^{-p_1T} - e^{-p_2T} \right]}{(p_1 - p_2) e^{(p_1+p_2)T} + p_2 e^{-p_2T} - p_1 e^{-p_1T}}$$

When the restrictions of (16) are applied to (15), it becomes

$$G(z) = \frac{A\pi}{z^2 + (-\alpha_1 - \alpha_2 - A) z + \alpha_1 \alpha_2}$$

Since in steady state $r(t)$ and $c(t)$ are chosen equal, $e(t)$ in steady state is zero. Therefore $(z - 1)$ must be a factor of the denominator of $G(z)$. Upon substitution for $A$ and factoring the denominator (18) becomes

$$G(z) = \frac{Az}{(z - 1) [z - \alpha_1 \alpha_2]}$$

When $G(z)/(z)$ is expanded in partial fractions and the result is multiplied by $z$, the following equation is obtained

$$G(z) = \frac{A}{1 - \alpha_1 \alpha_2} \left[ \frac{z}{z - 1} - \frac{z}{z - \alpha_1 \alpha_2} \right]$$

The $G(s)$ function corresponding to this $G(z)$ is

$$G(s) = \frac{A}{1 - \alpha_1 \alpha_2} \left[ \frac{1}{s} - \frac{1}{s + p_1 + p_2} \right]$$

This can be written as

$$G(s) = \frac{1 + K_1 e^{-p_1T} + K_2 e^{-p_2T}}{1 - e^{-(p_1+p_2)T}} \left[ \frac{p_1 + p_2}{s(s + p_1 + p_2)} \right]$$

If $G(s)$ is made up of the fixed part of the system $G_2(s)$ and the compensation network $G_1(s)$, that is $G(s) = G_1(s) G_2(s)$, then $G_1(s)$ can be found as

$$G_1(s) = \frac{G(s)}{G_2(s)}$$

If $G_2(s)$ has the form

$$G_2(s) = \frac{a}{s(s + a)}$$

the compensation network can be synthesized from
\[ G_1(s) = \frac{(1 + K_1 e^{-p_1 T} + K_2 e^{-p_2 T}) (p_1 + p_2) (s + a)}{(1 - e^{-(p_1 + p_2) T}) a (s + p_1 + p_2)} \]

As an example, the following values are chosen

\[ p_1 = 1, \quad p_2 = 2 \quad \text{and} \quad T = 1 \text{ second} \]

and the fixed part of the system is assumed to be

\[ G_2(s) = \frac{1}{s(s + 1)} \]

From (17) \( a_1 \) is found to be

\[ a_1 = 3.16 \]

From (6) \( C(z) \) can be determined as

\[ C(z) = \frac{z}{(z - 1) \left( z - 0.368 \right) \left( z - 0.135 \right)} \]

From (15) or (18) \( G(z) \) becomes

\[ G(z) = \frac{0.545z}{(z - 1) (z - 0.0498)} \]

From (22) \( G(s) \) is found to be

\[ G(s) = \frac{1.719}{s(s + 3)} \]

From (25) \( G_1(s) \) is found to be

\[ G_1(s) = \frac{1.719 (s + 1)}{(s + 3)} \]

**Fig. 2—Responses of continuous system and sampled-data system containing a hold-circuit.**

Curve A of Fig. 2 shows the response of the output of the continuous system and curve B shows the response of the sampled-data system. Curve B is obtained by use of the impulse response approach. The manner in which the outputs of these two systems coincide at the sampling instants can be seen from these curves.
III. SYNTHESIS OF SECOND-ORDER SYSTEM
WITH A ZERO-ORDER HOLD-CIRCUIT

A system containing a zero-order hold-circuit as shown in Fig. 3 can also be synthesized from the $G(z)$ of (15). The notation to be used is that $G(s)$ is given by

$$G(s) = G_3(s) G_4(s)$$

Also $G_3(s)$ contains the fixed part of the system $G_2(s)$ and the compensation network $G_1(s)$ so that

$$G_3(s) = G_1(s) G_2(s)$$

To obtain the $z$-transform of $G_3(s)$, $G(z)$ must be multiplied by the term $\frac{z}{z-1}$. The restriction of (16) that $B = 0$ does not apply to this type of system because the zero needed at the origin for synthesis is supplied by the $z$ factor in the numerator of the multiplying $\frac{z}{z-1}$ term. Physically this means that systems containing hold-circuits as shown in Fig. 3 do not respond in as limited a manner as systems without hold-circuits as shown in Fig. 1. For every continuous system, as given by (2), there exists a sampled-data system containing a hold-circuit such that the outputs of the two systems coincide at the sampling instants. However when this procedure is carried to its conclusion other restrictions may occur as will be demonstrated at the appropriate time.

![Diagram of sampled-data system](image)

The remainder of this development is demonstrated with the aid of the example previously used, as given by (26). In this example $a_1$ can have any desired value; however the value of $a_1 = 3.16$ is used again to show how the output of this system takes on the same set of values at the sampling instants as did the previous system. This system has the same $G(z)$ as given by (30).

$$G(z) = \frac{0.545z}{(z-1)(z-0.0498)}$$

To find the $z$-transform of $\frac{G_3(s)}{s}$, (35) must be multiplied by $\frac{z}{z-1}$ which gives
When (36) is divided by $z$, expanded in partial fractions, and the result multiplied by $z$, the following equation is obtained

\[
\frac{z}{s} \left[ G_3(s) \right] = \frac{0.545z^2}{(z - 1)^2 (z - 0.0498)}
\]

### TABLE I
Short Table of $z$-transforms

<table>
<thead>
<tr>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Function</td>
<td>Laplace Transform</td>
<td>$z$-transform</td>
</tr>
<tr>
<td><strong>a</strong></td>
<td>$u_1(t)$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td><strong>b</strong></td>
<td>$e^{-at}$</td>
<td>$\frac{1}{s + a}$</td>
</tr>
<tr>
<td><strong>c</strong></td>
<td>$t$</td>
<td>$\frac{1}{s}$</td>
</tr>
</tbody>
</table>

The Laplace transform corresponding to (37) can be found by using Table I as

\[
\frac{G_3(s)}{s} = \frac{0.573}{s^2} - \frac{0.03}{s} + \frac{0.03}{s + 3}
\]

The transfer function $G_f(s)$ can be written as

\[
G_3(s) = \frac{0.483(s + 3.55)}{s(s + 3)}
\]

Curve $A$ of Fig. 4 shows the response of the output of the continuous system and Curve $B$ shows the response of the sampled-data system containing a zero-order hold-circuit. Curve $B$ is obtained by using the square-pulse response approach, which is a procedure similar to the impulse response approach. The manner in which the outputs of these two systems coincide at the sampling instants can be seen from these curves.

It was mentioned earlier that although the restriction of (16) that $B = 0$ did not apply to sampled-data systems of this type with a hold-circuit, in
certain cases another restriction did apply. This can be seen from (39). If the fixed part of the system $G_2(s)$ has a denominator in $s$ of two degrees higher than the numerator, then the transfer function for the compensation network would have a pole at infinity. To avoid this $G_2(s)$ can be assumed to have the form

$$G_2(s) = \frac{a}{s(s + a)}$$

and the procedure may be reversed to find the restrictions on the continuous pole-zero configuration such that $G_3(s)$ has the desired form.

\[
\text{Fig. 5—Responses of continuous system and sampled-data system of Fig. 1 with poles outside the primary strip.}
\]

**IV. SYNTHESIS OF OTHER SECOND-ORDER SYSTEMS**

There are an infinite number of other sampled-data systems that can be synthesized from a given pole-zero configuration for the continuous system. As another example, the system of Fig. 1 is again used and another system is synthesized from the $G(z)$ of (30) which is repeated for convenience as

$$G(z) = \frac{0.545z}{(z - 1)(z - 0.0498)}$$

Because of the fact that the transformation

$$z = e^{sT}$$

is multivalued when going from the $z$-plane to the $s$-plane, it is possible to develop a system with poles in each pair of period strips. One such system which has poles in the period strips on each side of the primary strip leads to the following $G(s)$
\[ G(s) = \frac{27.8(s + 1)}{s(s^2 + 6s + 48.6)} \]

Curve A of Fig. 5 shows the response of the output of the continuous system and Curve B the response of the output of the sampled-data system.

In general it would seem desirable to synthesize the sample-data system by maintaining the poles in the s-plane in the primary strip.

V. SYNTHESIS OF HIGHER-ORDER SYSTEMS

General solutions to the third-order systems are possible although somewhat unwieldy. General solutions of systems of higher order than the third become extremely difficult because of the need to factor a third (or higher) degree polynomial in general terms. However for any specific pole-zero configuration involving numbers the solution can be carried out along lines similar to those already presented.

VI. CONCLUSIONS

It is not proposed that this method of synthesis in itself is sufficient to handle all sampled-data systems. However the arguments in favor of this method are the following:

1. The designer is probably already familiar with continuous system techniques, and correlating the design of sampled-data systems with continuous systems should prove helpful.

2. The more methods the designer understands, the more his knowledge about the basic response of systems is increased. For example, restrictions on the pole-zero configuration for the continuous system have been discussed. These restrictions are not in themselves the result of this method of attack but rather this method of attack simply points out that sampled-data systems respond in a more limited manner than do continuous systems.

3. It is proposed that this procedure will have its maximum usefulness when combined with other methods. For example, no matter how complicated the fixed part of the system actually is, the designer can pick as simple a pole-zero configuration as possible to satisfy the specifications and this procedure will lead to a desired impulse response (or square-pulse response) for the forward loop of the system. The problem then is to reshape the impulse response (or square-pulse response) of the fixed part of the system by adding a compensation network so that the response of the combination approximates the desired response. Thus this procedure gives the designer an insight into how he should attack the problem.

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