STRESSES IN A UNIFORMLY LOADED CIRCULAR-ARC I-BEAM

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SYNOPSIS

Expressions are derived by which the stresses may be computed for the case of a uniformly loaded I-beam bent about its minor axis to the form of a circular-arc. This type of structural member is unusual inasmuch as bending moments are induced in the flanges. These bending moments are not apparent from the use of the equations of equilibrium but must be determined from a consideration of the distortions of the beam. An example is given to illustrate the application of the expressions derived.
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INTRODUCTION

The case in which a horizontally curved beam of I section is subjected to a concentrated load has been treated by Moorman\(^1\) and Unold.\(^2\) The case in which such a beam supports a distributed load has received some treatment by Unold\(^2\), but not sufficient to permit results to be determined readily. The following derivations follow rather closely those by Moorman\(^1\) with the exception that the terms for a uniform load are inserted where necessary.

Because of the shape of its cross section the curved beam of I form requires a special analysis. When an I-beam, fixed at its ends, is twisted, direct stresses are produced in the edges of the beam\(^3\), each flange acting as an individual beam. The bending moments induced in the top and bottom flanges by twisting are equal but of opposite sign.

In a curved I-beam fixed at its ends the stress resulting from the twisting moment may be considerably greater than the stress that a designer would ordinarily compute from the bending moment about the axis of the cross section of the I-beam perpendicular to the axis of the web of the beam. However, the stresses caused by the induced bending moments are of a localized nature and the signs of these stresses are the same at diagonally opposite corners of the I-beam. In the determination of the stress at any point it is expedient to compute the stress resulting from twisting and the stress resulting from ordinary bending and then add them algebraically.

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NOTATION

The symbols used in this bulletin are defined where they first appear, either by text or diagram and are assembled in the Appendix for convenience of reference. They conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials and American Standard Letter Symbols for Mechanics, prepared by committees of the American Standards Association and approved by the Association in 1939 and 1942, respectively.

ASSUMPTIONS

In developing the analysis it is assumed that:

1. Hooke's law (a deformation is proportional to the force producing it) applies.
2. The deformations are sufficiently small compared with the dimensions of the beam that \( \alpha = \sin \alpha = \tan \alpha \) (approximately), where \( \alpha \) is the angular deformation.
3. The angle of twist per unit length of beam varies as \( T/GK \), where \( T \) = twisting moment, \( G \) = modulus of elasticity in shear and \( K \) = torsion constant.
4. The angle of bending per unit length of beam varies as \( M/EI \), where \( M \) = bending moment, \( E \) = modulus of elasticity in tension or compression and \( I \) = moment of inertia of the section of an I-beam about the principal axis perpendicular to the web.
5. The moment of inertia of the cross section of one flange about the minor principal axis of an I-beam, \( H \), is one half of the moment of inertia of the cross section of the beam about that axis, \( I_{2-2} \).
6. The transverse shear in each flange of an I-beam acts along the outside edge of the flange.
7. The ends of the beam are fixed.

The origin of the coordinates is taken at the point of intersection of the axis of the beam with the vertical plane of symmetry.

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4 ASA—Z10a—1932.

5 ASA—Z10.3—1942.
DERIVATION

Figure 1 shows a portion of a curved I-beam with the forces acting upon it. The moments $M$ and $T$ are represented vectorially. That is, the vector representing the moment is drawn perpendicular to the plane of the moment and the moment tends to cause clockwise rotation when looking along the vector in the direction indicated by the arrowhead. The beam has a uniform load $w$ per unit length acting upon it. The distance along the axis of the beam is designated as $x$. The transverse shear in the flanges of the beam is shown as $\theta$.

Note
- $\circ$ indicates $V$ acts down
- $\bullet$ indicates $V$ acts up

Fig. 1. Forces Acting on Elemental Length of I-Beam.
If the summation of the vertical forces is zero, \( dV = wdx = wrd\phi \), and the vertical shear, \( V \), at any point is dependent on the angle \( \phi \).

Since the beam is in equilibrium the summation of the moments about the line \( a \) is zero, or

\[
T + \theta 2a - (T + dT) - (\theta + d\theta) 2a - (M + dM) d\phi = 0.
\]

If this equation is simplified and all differentials higher than the first degree (which would have negligible values) are dropped,

\[
M d\phi + dT + d\theta 2a = 0, \quad \text{and}
\]

\[
M + T' + \theta' 2a = 0. \quad (1)
\]

In this bulletin, \( T' = \frac{dT}{d\phi} \), \( T'' = \frac{d^2T}{d\phi^2} \), etc.

Similarly, if moments are taken about the line \( \beta \) and equated to zero,

\[
M - (M + dM) + (T + dT) \, d\phi + (\theta + d\theta) 2ad\phi
\]

\[
- (V + dV) \, dx + \frac{1}{2}w(dx)^2 = 0,
\]

and, since \( dx = rd\phi \),

\[
-M' + T + \theta 2a - Vr = 0. \quad (2)
\]

The coordinates at a point are expressed as \( y, v, \) and \( z \) and at a distance \( dx \) from this point are expressed by \( y + dy, v + dv, \) and \( z + dz \).

From Fig. 2 it may be seen that

\[
v = a \frac{dy}{dx} = \frac{ad}{r \, d\phi} = a \frac{y'}{r}.
\]

Under load the middle fiber of the upper flange shortens from the original length by an amount

\[
\Delta dx = v + dv - v - (r + z) d\phi + rd\phi = dv - zd\phi.
\]

Therefore, the unit strain is

\[
\epsilon = \frac{\Delta dx}{dx} = \frac{dv}{dx} - z \frac{d\phi}{dx} = \frac{dv}{rd\phi} - \frac{z}{r} = \frac{v' - z}{r},
\]

and the corresponding bending moment is

\[
M = \frac{(\epsilon E) I}{a} = (v' - z) \frac{EI}{ar}. \quad (4)
\]
Fig. 2. Displacements of \( v, y, \) and \( z. \)

The relative radial displacement between points 1 and 2 of Figs. 2 and 3, which are a distance \( dx \) apart, is \( ad\delta. \) Then the angle of twist, \( d\phi, \) between the cross section at I and the cross section at II is

\[
d\delta = \frac{v d\phi + dz}{a}, \text{ or } \delta' = \frac{v + z'}{a},
\]
and the rotation per unit length is

\[ \theta = \frac{d \delta}{dx} = \frac{d \delta}{r d \phi} = \frac{\delta'}{r}. \]

But \( \theta \) also equals \( T/GK \) and if the substitution \( q = GK/EI \) is made, then

\[ \frac{\delta'}{r} = \frac{T}{EI q}, \quad \text{or} \quad T = (v + z') \frac{EI}{ar} q. \quad (5) \]

Fig. 3. Relative Displacement of Points 1 and 2.

Fig. 4. Relation Between \( m \) and \( \theta \).

It is now necessary to determine the bending moment \( (m) \) in the flange resulting from the change in the curvature of the axis of the top flange. Figure 5 shows the points 1 and 2 in their original positions and in their displaced positions \( (1' \) and \( 2' \)). By definition

\[ dx = r d \phi, \quad \text{or} \quad \frac{d \phi}{dx} = \frac{1}{r}. \]

The new curvature may be expressed as

\[ \frac{1}{r_1} = \frac{d \phi + \Delta d \phi}{dx + \Delta dx}, \quad \text{where} \quad r_1 \text{ is the new radius.} \]

The angle between the tangent to the center line at \( 1' \) and the normal to the radius at \( 1' \) is \( dz/dx \). The corresponding angle at \( 2' \) is

\[ \frac{dz}{dx} + \frac{d^2 z}{dx^2} dx. \]
Then,

$$\Delta d\phi = -\frac{d^2 z}{dx^2} dx,$$

and

$$\Delta dx = (r + z) d\phi - r d\phi = z d\phi = \frac{z dx}{r}.$$ 

Substituting in Eq. 6 gives

$$\frac{1}{r_1} = \frac{d\phi - \frac{d^2 z}{dx^2} dx}{dx + \frac{zd\phi}{r}} = \frac{1}{r - \frac{d^2 z}{dx^2}}, \text{ or}$$

$$\frac{1}{r_1} \left(1 + \frac{z}{r}\right) = \frac{1}{r} - \frac{d^2 z}{dx^2}, \text{ or}$$

$$\frac{1}{r_1} + \frac{z}{r_1 r} = \frac{1}{r} - \frac{d^2 z}{dx^2}, \text{ or}$$

$$\frac{1}{r_1} - \frac{1}{r} = -\frac{z}{r_1 r} - \frac{d^2 z}{dx^2} = -\left(\frac{z}{r^2} + \frac{d^2 z}{dx^2}\right), \quad (7)$$
since $r^2$ is approximately equal to $rr_1$ for very small changes in the length of the radius.

Also, if it is assumed that the moment of inertia of the cross section of one flange about the principal axis through the center line of the web of an I-beam, $H$, is one half of the moment of inertia of the cross section of the beam about that axis, $I_{2-2}$,

$$\frac{1}{r_1} - \frac{1}{r} = \frac{m}{EH}.$$  \hspace{1cm} (8)

The bending moment is taken to be positive when it produces a decrease in the initial curvature; thus the right side of Eq. 8 is positive.

If Eq. 7 is substituted in Eq. 8,

$$- \frac{m}{EH} = \frac{z}{r^2} + \frac{d^2z}{dx^2} = \frac{1}{r^2} (z + z''),$$ and

$$m = - \frac{EH}{r^2} (z'' + z).$$  \hspace{1cm} (9)

The relation between $m_1$ and $\theta$ may be found by taking moments about the vertical axis through A in Fig. 4. Since the segment shown is in equilibrium,

$$m + dm - m - (\theta + d \theta) dx = 0.$$  

As the product of the two differentials, $d\theta dx$, is negligible,

$$dm = \theta dx = \theta r \, d\phi$$

and

$$\theta = \frac{m'}{r}. \hspace{1cm} (10)$$

Equations 1 to 10 form a system of simultaneous differential equations between the variable $\phi$ and the terms $y$, $v$, $z$, $V$, $M$, $T$, $m_1$, and $\theta$. It is desirable to obtain a differential equation containing only $\phi$ and $y$.

From Eqs. 3 and 4

$$M = \left( v' - z \right) \frac{EI}{ar} = \left( \frac{a}{r} y'' - z \right) \frac{EI}{ar},$$

which gives

$$z = \frac{a}{r} y'' - \frac{ar}{EI} M,$$ \hspace{1cm} (11)

and

$$z' = \frac{a}{r} y''' - \frac{ar}{EI} M'. \hspace{1cm} (12)$$
From Eqs. 3 and 5
\[ T = (v + z') \frac{EI q}{ar} = \left( \frac{a}{r} y' + z' \right) \frac{EI q}{ar}, \]
and
\[ z' = \frac{ar}{EI q} T - \frac{a}{r} y'. \]  
(13)

By equating Eqs. 12 and 13 and solving for T,
\[ \frac{a}{r} y''' - \frac{ar}{EI} M' = \frac{ar}{EI q} T - \frac{a}{r} y', \]
and
\[ T = \frac{EI q}{r^2} (y' + y''') - M'q. \]  
(14)

From Eqs. 9 and 10
\[ \vartheta = \frac{m'}{r} = -\frac{EH}{r^3} (z' + z'''). \]

Substituting the expressions for \( z' \) and \( z''' \) from Eqs. 12 and 13 gives
\[ \vartheta = -\frac{EH}{r^3} \left( \frac{a}{r} y''' - \frac{ar}{EI} M' + \frac{a}{r} y^{(5)} - \frac{ar}{EI} M'''' \right) \]
\[ = -\frac{EH a}{r^4} \left( y'''' + y^{(5)} \right) + \frac{H a}{I r^2} \left( M' + M'''' \right). \]  
(15)

To determine an expression for \( M' + M'''' \), Eq. 2, after it has been differentiated, may be subtracted from Eq. 1 as follows:
\[ M + T' + \vartheta'2a = 0 \]
\[ -M'' + T' + \vartheta'2a - V'r = 0 \]
\[ M + M'' + V'r = 0. \]

Since \( V = wr\phi, \ V' = wr \) and
\[ M + M'' = -wr^2 \]
and
\[ M' + M'' = 0. \]  
(16)

Eq. 15 may be rewritten
\[ \vartheta = -\frac{EH a}{r^4} \left( y'''' + y^{(5)} \right). \]  
(17)

If \( T' \) (derivative of Eq. 14) and \( \vartheta' \) (derivative of Eq. 17) are substituted in Eq. 1,
\[ M + \frac{EI q}{r^2} (y'' + y''') - M''q - \frac{2EH a^2}{r^4} (y''' + y^{(5)}) = 0. \]  
(18)
Differentiating Eq. 18 twice gives
\[ M'' + \frac{EI q}{r^2} (y''' + y^{(6)}) - M''' q - \frac{2EH a^2}{r^4} (y^{(5)} + y^{(8)}) = 0. \quad (19) \]

By adding Eqs. 18 and 19 and noting that \( M + M'' = -wr^2 \) and \( M'' + M''' = 0 \) (Eq. 16),
\[ -w r^2 + \frac{EI q}{r^2} (y'' + 2y'''' + y^{(6)}) - \frac{2EH a^2}{r^4} (y''' + 2y^{(5)} + y^{(8)}) = 0. \quad (20) \]

If terms are simplified by letting
\[ \frac{EI q}{r^2} = A, \quad \frac{2EH a^2}{r^4} = B, \text{ and } \sqrt{\frac{A}{B}} = \rho, \]
Eq. 20 becomes
\[ -w r^2 + (y'' + 2y'''' + y^{(6)}) A - (y''' + 2y^{(5)} + y^{(8)}) B = 0. \quad (21) \]

The solution of Eq. 21 is
\[ y = C_1 + C_2 \phi + C_3 \sinh \rho \phi + C_4 \cosh \rho \phi + C_5 \sin \phi + C_6 \cos \phi + C_7 \phi \sin \phi + C_8 \phi \cos \phi + \frac{w r^2}{2A} \phi^2. \quad (22) \]

Eqs. 2, 9, 11, 12, 14, 16, 17, 18, 22 and the derivatives of Eqs. 12 and 22 may be used to obtain the following equations:
\[ y = C_1 + C_2 \phi + C_3 \sinh \rho \phi + C_4 \cosh \rho \phi + C_5 \sin \phi + C_6 \cos \phi + C_7 \sin \phi + C_8 \cos \phi + \frac{w r^2}{2A} \phi^2. \quad (23) \]

\[ M = -\frac{A}{1+q} (y'' + y''') + \frac{B}{1+q} (y''' + y^{(6)}) - \frac{q}{1+q} w r^2 \]
\[ = \frac{2B (1 + \rho^2)}{1 + q} (C_7 \cos \phi - C_8 \sin \phi) - w r^2. \quad (24) \]

\[ T = A(y'' + y''') - M' q = A(y'' + y''') + A q (y''' + y^{(6)}) - B q (y^{(5)} + y^{(7)}) \]
\[ = A[C_2 + C_3 \rho (1 + \rho^2) \cos \rho \phi + C_4 \rho (1 + \rho^2) \sin \rho \phi - C_7 2 \sin \phi - C_8 2 \cos \phi] + \frac{2B (1 + \rho^2)}{1 + q} (C_7 \sin \phi + C_8 \cos \phi) + w r^2 \phi. \quad (25) \]
\[ \vartheta = -\frac{B}{2a} (y''' + y^{(v)}) = -\frac{B}{2a} \left[ C_3 \rho^3 (1 + \rho^2) \cosh \rho \phi + C_4 \rho^3 (1 + \rho^2) \sinh \rho \phi + C_7 2 \sin \phi + C_8 2 \cos \phi \right]. \quad (26) \]

\[ m_1 = -\frac{EH}{r^2} (z'' + z) = -\frac{EH}{r^2} \left[ \frac{a}{r} (y'' + y^{''''}) + \frac{w a r^3}{EI} \right] \]

\[ = -\frac{Br}{2a} (y'' + y^{''''}) - \frac{w a r H}{I} \]

\[ = -\frac{Br}{2a} \left[ C_3 \rho^3 (1 + \rho^2) \sinh \rho \phi + C_4 \rho^3 (1 + \rho^2) \cosh \rho \phi - C_7 2 \cos \phi + C_8 2 \sin \phi \right] - \frac{w a r^3}{2a \rho^2 (1 + q)}. \quad (27) \]

\[ V = \frac{1}{r} (-M' + T + \vartheta 2a) = \frac{A}{r} C_2 + w r \phi. \quad (28) \]

\[ z = \frac{a}{r} \left[ C_3 \rho^3 \sinh \rho \phi + C_4 \rho^3 \cosh \rho \phi - C_5 \sin \phi - C_6 \cos \phi + C_7 (2 \cos \phi - \phi \sin \phi) + C_8 (-2 \sin \phi - \phi \cos \phi) \right] \]

\[- \frac{2B a r (1 + \rho^2)}{(1 + q) EI} (C_7 \cos \phi - C_8 \sin \phi) + \frac{w a r}{A} (1 + q). \quad (29) \]

\[ z' = \frac{a}{r} \left[ C_3 \rho^3 \cosh \rho \phi + C_4 \rho^3 \sinh \rho \phi - C_5 \cos \phi + C_6 \sin \phi + C_7 (-3 \sin \phi - \phi \cos \phi) + C_8 (-3 \cos \phi + \phi \sin \phi) \right] \]

\[ + \frac{2B a r (1 + \rho^2)}{(1 + q) EI} (C_7 \sin \phi + C_8 \cos \phi). \quad (30) \]

Equations 22 to 30, inclusive, may be used to determine the constants of integration in terms of the load \( w \). The condition \( V = 0 \) when \( \phi = 0 \) and Eq. 28 gives as a result \( C_3 = 0 \). Then using the conditions when \( \phi = 0, \vartheta = 0 \) (Eq. 26), \( y' = 0 \) (Eq. 23) and \( z' = 0 \) (Eq. 30) three equations are obtained containing the three constants \( C_3, C_5 \) and \( C_8 \). Upon simultaneous solution of these three equations it is found that \( C_3 = C_5 = C_8 = 0 \).

Expressions for the four remaining constants, \( C_1, C_4, C_6 \) and \( C_7 \) may be obtained by using the conditions that when \( \phi = \alpha, y = 0 \) (Eq. 22), \( y' = 0 \) (Eq. 23), \( z = 0 \) (Eq. 29) and \( z' = 0 \) (Eq. 30), and solving these four equations simultaneously.
Equations 22 to 30 may now be rewritten as

\[
y = \frac{N_{1}}{D} + \frac{N_{5}}{D} \cos \phi + \frac{N_{7}}{D} \sin \phi + \frac{w r^{2}}{2 A} \phi^{2},
\]  
\[
y' = \frac{N_{4}}{D} \sinh \rho \phi - \frac{N_{5}}{D} \sin \phi + \frac{N_{7}}{D} (\sin \phi + \phi \cos \phi) + \frac{w r^{2}}{A} \rho,
\]  
\[
M = \frac{2 B (1 + \rho^{2}) N_{7}}{(1 + q)} \cos \phi - w r^{2} = \frac{Q A}{q} \frac{N_{7}}{D} \cos \phi - w r^{2},
\]  
\[
\frac{T}{D} = \frac{N_{4}}{D} \rho (1 + \rho^{2}) \sinh \rho \phi + (Q - 2) N_{7} \sin \phi \right] + \frac{w r^{2}}{A} \varphi,
\]  
\[
\vartheta = \frac{- B}{2 a D} \left[ N_{4} \rho^{3} (1 + \rho^{2}) \sinh \rho \phi + 2 N_{7} \sin \phi \right],
\]  
\[
\frac{m}{2 a D} = \frac{N_{4}}{2 a D} \rho^{2} (1 + \rho^{2}) \cosh \rho \phi - 2 N_{7} \cos \phi \right] - \frac{w r^{3}}{2 a \rho^{3}} (1 + q),
\]  
\[
V = w r \rho \phi,
\]  
\[
z = \frac{a}{r D} \left[ N_{4} \rho^{2} \cosh \rho \phi - N_{5} \cos \phi - N_{7} (Q - 2) \cos \phi + \phi \sin \phi \right]
\]  
\[
+ \frac{w r}{A} (1 + q),
\]  
\[
z' = \frac{a}{r D} \left[ N_{4} \rho^{2} \sinh \rho \phi + N_{5} \sin \phi + N_{7} (Q - 3) \sin \phi - \phi \cos \phi \right],
\]  
where

\[
N_{1} = \frac{w r^{2}}{A} \left[ \alpha^{2} (1 + \rho^{2}) \cosh \rho \alpha - \alpha \rho (1 + \rho^{2}) \left( 1 + q + \frac{\alpha^{2} \rho^{2}}{2} \right) \sinh \rho \alpha
\]  
\[
+ (Q - 2) \left( 1 + q + \frac{\alpha^{2} \rho^{2}}{2} \right) \sin \alpha \cosh \rho \alpha
\]  
\[
+ \alpha (1 - \rho^{2} (Q - 3) \sin \alpha \cos \alpha \cosh \rho \alpha - \rho (Q - 1) \left[ \frac{\alpha^{2} \rho^{2}}{2} - (1 + q) \right]
\]  
\[
+ (1 + q) (2 + \rho^{2}) + \frac{\alpha^{2}}{2} \right] \sin \alpha \cos \alpha \sinh \rho \alpha
\]  
\[
- \alpha \rho^{3} (Q - 2) \cos \rho \alpha \sinh \rho \alpha \right]
\]  
\[
N_{4} = \frac{w r^{2}}{A} \left[ (1 + q) (2 - Q) \sin \alpha \cos \alpha \right],
\]
\[ N_6 = \frac{w r^2}{A} \left[ \rho \left\{ (1 + q) (3 + \rho^2 - Q) + \alpha^2 \rho^2 \right\} \sin \alpha \sinh \rho \alpha \\
+ \alpha \rho \left\{ (1 + q) (1 + \rho^2) + \rho^2 (Q - 2) \right\} \cos \alpha \sinh \rho \alpha \\
+ \alpha \rho^2 (Q - 3) \sin \alpha \cosh \rho \alpha - \alpha^2 \rho^2 \cos \alpha \cosh \rho \alpha \right], \quad (42) \]

\[ N_7 = \frac{w r^2}{A} \left[ \rho (1 + \rho^2) (1 + q) \sin \alpha \sinh \rho \alpha - \alpha \rho \cos \alpha \sinh \rho \alpha \right], \quad (43) \]

\[ D = \alpha \rho (1 + \rho^2) \sinh \rho \alpha + \rho [(Q - 1) \rho^2 + 1] \sin \alpha \cos \alpha \sinh \rho \alpha \\
+ \rho^2 (Q - 2) \sin^2 \alpha \cosh \rho \alpha, \quad (44) \]

and

\[ Q = \frac{2 (1 + \rho^2) q}{\rho^2 (1 + q)}. \]

To determine the outer fiber stress the bending moment \( M \) is substituted in the flexure formula,

\[ s_1 = \frac{M a}{I}, \]

where \( a \) is one half of the depth of the I-beam, and the bending moment \( m_b \) is substituted in the flexure formula,

\[ s_2 = \frac{m b}{H}, \]

where \( b \) is one-half the width of the flange. The unit stress in the outer fiber is the algebraic sum of \( s_1 \) and \( s_2 \).
ILLUSTRATIVE EXAMPLE

Assume that the beam shown in Fig. 6 sustains a load \( w \) per lin. in. Other data pertaining to this beam, which is a 21'' WF 112 lb., are:

\( I_{1-1} = 2620.6 \text{ in.}^4, \ I_{2-2} = 289.7 \text{ in.}^4, \ H = 144.85 \text{ in.}^4, \ K = 7.57 \text{ in.}^4, \ a=10.5 \text{ in.}, \ b=6.5 \text{ in.}, \ \alpha=60^\circ, \ r=100 \text{ in.}, \ E=30\,000\,000 \text{ lb. per sq. in.} \) and \( G=12\,000\,000 \text{ lb. per sq. in.} \). The following constants can be determined from these data: \( q=0.001\,155\,46, \ A=9084.0000, \ B=9581.8275, \ \rho=0.973\,675\,82 \) and \( Q=0.004\,743\,006\,6 \).

Substituting in Eqs. 40 to 44, inclusive, one obtains:

\[
N_1 = -0.2274367 \times w \]
\[
N_4 = -0.057\,134\,225 \times w \]
\[
N_6 = +0.28454425 \times w \]
\[
N_7 = +0.058\,646\,852 \times w \]
\[
D = +0.201\,247\,91. \]

The expressions for \( M, V, T, T_1, \) and \( \phi \) may now be written as

\[
M = (10866.4840 \cos \phi-10\,000) \times w, \]
\[
V = 100 \phi \times w, \]
\[
T = (-4891.64999 \sinh \rho \phi-5281.88929 \sin \phi+10\,000 \phi) \times w, \]
\[
\eta_1 = (23923.3334 \cosh \rho \phi+26593.3088 \cos \phi-50\,286.736) \times w, \]
\[
\eta_2 = (232.935713 \sinh \rho \phi-265.933\,088 \sin \phi) \times w. \]

These functions are plotted in Figs. 7 to 11, inclusive. The function is used as the ordinate with the developed axis of the beam as the abscissa.

For a uniform load of 6000 pounds per linear foot the maximum flange stress at the support may be computed as

\[
s = s_1 + s_2 = 9150 + 10860 = 20010 \text{ lb. per sq. in.} \]

where

\[
s_1 = \frac{4567 \times 500 \times 10.5}{2620.6} = 9150 \text{ lb. per sq. in.} \]

and

\[
s_2 = \frac{484 \times 500 \times 6.5}{144.85} = 10860 \text{ lb. per sq. in.} \]
Fig. 6. Curved Beam of Illustrative Example.

Fig. 7. Variation of Bending Moment, M.

Fig. 8. Variation of Shear, V.
Fig. 9. Variation of Torsional Moment, T.

Fig. 10. Variation of Moment, m.

Fig. 11. Variation of Shear, δ.
CONCLUDING REMARKS

This bulletin presents a method for computing the stresses in a uniformly loaded I-beam bent horizontally to the form of a circular-arc. Because of the nature of the expressions for the stresses it is necessary to carry out the computations to what appears an unreasonable number of significant figures.

It will be noted that the maximum stresses occur at the support and that the twisting is resisted solely by shear in the flanges at the support.
APPENDIX

NOTATION

The following notation used in the bulletin is assembled here for ready reference.

$M =$ bending moment.

$T =$ twisting moment (pure torsion).

$V =$ vertical shear.

$m =$ bending moment in the flange of an I-beam induced by twist. It is positive when compression is produced on the inside of the top flange.

$\phi =$ shear accompanying $m$ and resulting from twist in an I-beam.

$E =$ modulus of elasticity in tension or compression. In this analysis the modulus is assumed to be the same in either tension or compression.

$K =$ torsion constant (see Bethlehem Manual of Steel Construction).

$EI =$ flexural rigidity.

$GK =$ torsional rigidity.

$I =$ moment of inertia of an I-beam about the principal axis perpendicular to the web.

$I_{z-y} =$ moment of inertia of the section of an I-beam about the principal axis through the center line of the web.

$H =$ moment of inertia of one flange of an I-beam about the principal axis through the center line of the web. It is assumed to be one half of $I_{z-y}$. x = distance along the axis of an I-beam.

$y =$ displacement of the axis of a horizontally curved I-beam in a vertical direction (positive for upward movement).

$v =$ displacement of the center line of the top flange of an I-beam in a circumferential direction (positive when movement is toward the origin).

$z =$ displacement of the center line of the top flange of an I-beam in a radial direction. It is positive for outward movement.

$r =$ radius of curvature.

$a =$ one half the depth of an I-beam.

$b =$ one half the width of the flange of an I-beam.

$w =$ load per unit of length of beam.

$\epsilon =$ unit strain.

$\delta =$ elemental angle of twist.

$\alpha =$ one half the angle spanning the distance from the supports of a curved beam.
\( \phi = \) angle coordinate of a point on a curved beam.
\( \theta = \) angle of twist per unit length of I-beam.

\[
A = \frac{EIq}{r^2}
\]

\[
B = \frac{2EH\alpha^2}{r_4}.
\]

\[
\rho = \sqrt{\frac{A}{B}}.
\]

\[
q = \frac{GK}{EI}.
\]

\[
Q = \frac{2(1 + \rho^2)q}{\rho^2(1 + q)}.
\]
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