THE UNIVERSITY OF MISSOURI

Bulletin

ENGINEERING REPRINT SERIES

Reprint Number 60

Engineering Experiment Station
Columbia, Missouri

Discussion of an Article by G. S. Ramaswamy and M. Ramaiah

Characteristic Equation of Cylindrical Shells --- A Simplified Method of Solution

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Reprinted from
Journal of The American Concrete Institute
Concrete Briefs, pages 1505-1509, October 1962

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THE UNIVERSITY OF MISSOURI BULLETIN

VOL. 64, NO. 12 ENGINEERING REPRINT SERIES NO. 60

Published three times monthly by the University of Missouri, Office of Publications, Columbia, Missouri. Second Class postage paid at Columbia, Missouri.

500
April 26, 1963

Discussion of a Concrete Brief by G. S. Ramaswamy and M. Ramaiah:

Characteristic Equation of Cylindrical Shells — A Simplified Method of Solution†

By A. PAUW and W. M. SANGSTER, MARIO G. SALVADORI, M. G. TAMHANKAR, and AUTHORS

By A. PAUW‡ and W. M. SANGSTER§

Messrs. Ramaswamy and Ramaiah suggest an elegant procedure for determining the roots of the eighth-degree characteristic equation. The method consists of converting the eighth-degree equation into a biquadratic and then forming its cubic resolvent equation which in turn is solved by the use of Steinman's rule.†† The object of this discussion is to show that the roots of the biquadratic can be obtained directly by using an iterative procedure to determine the quadratic factors. This procedure eliminates the need for operating with complex numbers.

ITERATIVE SOLUTION OF QUADRATIC FACTORS

The procedure is essentially an extension of Newton's method of tangents for real roots. A convenient method of applying this rule is to use synthetic division. If the equation to be solved is

$$f(x) = \sum_{i=0}^{n} A_i x^i = 0 \qquad (1)$$

and $x = a$ is a trial root, then

$$f(a) = \sum_{i=0}^{n} A_i a^i \qquad (2)$$

can be evaluated by synthetic division as shown below

$$\begin{array}{c|cccccccc} & A_n & A_{n-1} & A_{n-2} & \dots & \dots & A_2 & A_1 & A_0 \\ +a & & aB_n & aB_{n-1} & \dots & \dots & aB_3 & aB_2 & aB_1 \\ \hline & B_n & B_{n-1} & B_{n-2} & \dots & \dots & B_2 & B_1 & B_0 \end{array}$$

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†ACI JOURNAL, *Proceedings* V. 58, No. 4, Oct. 1961, p. 471.

‡Member American Concrete Institute, Professor of Civil Engineering, University of Missouri, Columbia.

§Professor of Civil Engineering, University of Missouri, Columbia.

††Steinman, D. B., "Simple Formula Solves All Higher Degree Equations," *Civil Engineering*, V. 21, No. 2, Feb. 1951.

where

$$B_n = A_n,\ B_{n-1} = A_{n-1} + aB_n$$

and in general

$$B_i = A_i + aB_i$$

If a is an exact root of Eq. (1), $B_0 = f(a)$ is equal to zero; if not, B_0 represents a residual which must be liquidated by changing the trial root. The operator for computing a correction to the trial value is determined by a second synthetic division; viz.,

$$\begin{array}{c|cccccccc} & B_n & B_{n-1} & B_{n-2} & \cdots & \cdots & B_2 & B_1 \\ +a & & aC_n & aC_{n-1} & \cdots & \cdots & aC_3 & aC_2 \\ \hline & C_n & C_{n-1} & C_{n-2} & \cdots & \cdots & C_2 & C_1 \end{array}$$

It may be shown that†

$$C_i = \frac{d\,B_i}{d\,a} \qquad (3)$$

and hence the desired correction to the trial root by Newton's method is

$$\Delta a = \frac{-B_0}{C_1} \qquad (4)$$

The new trial value is therefore

$$x = a + \Delta a = a - \frac{B_0}{C_1}$$

This procedure can be extended to the solution for quadratic factors. Consider as a quadratic factor a pair of complex conjugate roots $x = \alpha \pm i\beta$; i.e.,

$$(x - [\alpha + i\beta])(x - [\alpha - i\beta]) = x^2 - 2\alpha x + \alpha^2 + \beta^2$$

Thus, if we solve for a quadratic factor in the form

$$x^2 = a x - b \qquad (5)$$

then

$$\alpha = \frac{a}{2} \qquad (5a)$$

and

$$\beta = \tfrac{1}{2}\sqrt{4b - a^2} \qquad (5b)$$

The coefficients a and b are determined by an iterative procedure similar to that used for real roots. Successive synthetic division is again used to determine both the residuals and the operators required for liquidation. In general, real roots, if any, should be determined first, since the reduced equation then contains only pairs of complex conjugate roots and is therefore of even degree.

†"Basic Research in Force Relaxation Methods," *Solution of Polynomial Equations,* Part IV, Report ORD Project No. TB2-0001, University of Missouri Experiment Station, June 1958.

The two steps of synthetic division will take the form:

$$\begin{array}{c|cccccccccc}
 & A_{2n} & A_{2n-1} & A_{2n-2} & \cdots & \cdots & A_3 & A_2 & A_1 & A_0 \\
+a & & aC_{2n} & aC_{2n-1} & \cdots & \cdots & aC_4 & aC_3 & aC_2 & aC_1 \\
-b & & & -bC_{2n} & \cdots & \cdots & -bC_5 & -bC_4 & -bC_3 & -bC_1 \\
\hline
 & C_{2n} & C_{2n-1} & C_{2n-2} & \cdots & \cdots & C_3 & C_2 & C_1 & C_0 \\
+a & & aE_{2n} & aE_{2n-1} & \cdots & \cdots & aE_4 & aE_3 & aE_2 & \\
-b & & & -bE_{2n} & \cdots & \cdots & -bE_5 & -bE_4 & -bE_3 & \\
\hline
 & E_{2n} & E_{2n-1} & E_{2n-2} & \cdots & \cdots & E_3 & E_2 & E_1 &
\end{array}$$

where

$$C_i = A_i + a\,C_{i+1} - b\,C_{i+2}$$

and

$$E_i = C_i + a\,E_{i+1} - b\,E_{i+2}$$

respectively.

C_0 and C_1 vanish if, and only if

$$x^2 - a\,x + b \equiv 0$$

Hence, these terms can be considered as residuals which must be liquidated by applying correction to the trial values of a and b. It may be demonstrated that†

$$E_i = \frac{\partial\,C_{i-1}}{\partial\,a} = \frac{\partial\,C_{i-2}}{\partial\,b} \qquad (3)$$

Hence, to eliminate the residuals C_0 and C_1 we set

$$\Delta\,a\,\frac{\partial\,C_0}{\partial\,a} + \Delta\,b\,\frac{\partial\,C_0}{\partial\,b} = \Delta\,a\,E_1 + \Delta\,b\,E_2 = -\,C_0$$

and

$$\Delta\,a\,\frac{\partial\,C_1}{\partial\,a} + \Delta\,b\,\frac{\partial\,C_1}{\partial\,b} = \Delta\,a\,E_2 + \Delta\,b\,E_3 = -\,C_1$$

Solving the above equations for Δa and Δb, the desired corrections are

$$\Delta\,a = \frac{C_1\,E_2 - C_0\,E_3}{E_1\,E_3 - E_2^2} \qquad (7a)$$

and

$$\Delta\,b = \frac{C_0\,E_2 - C_1\,E_1}{E_1\,E_3 - E_2^2} \qquad (7b)$$

To illustrate the method of application consider the biquadratic equation solved by Messrs. Ramaswamy and Ramaiah, viz., Eq. (2):

$$y^4 - 0.090492\,y^3 + 0.000778\,y^2 - 0.0004028\,y + 0.9600055 = 0$$

Since the coefficients of the y^3, y^2, and y terms are small and the constant is approximately equal to unity, we take as our trial roots one of the quadratic factors of

$$y^4 + 1 = 0 \qquad (8)$$

†Ibid.

TABLE A—SUMMARY OF COMPUTATIONS

			4 y^4	3 y^3	2 y^2	1 y	0 Constant
			1	−0.090 492	+0.000 778	−0.000 402 8	+0.960 005 5
a	+1.4			+1.4	+1.833 311	+1.167 724 6	−0.199 060 4
b		−1.0			−1.0	−1.309 508	−0.834 089
C			1	+1.309 508	+0.834 089	−0.142 186	−0.073 144
	+1.4			+1.4	+3.793 311	+5.078 360	
		−1.0			−1.0	−2.709 508	
E			1	+2.709 508	+3.627 400	+2.226 666	

$C_1E_2 - C_0E_3 = (-0.142\ 186)(3.627\ 400) - (-0.073\ 144)(2.709\ 508) = -0.317\ 581$

$C_0E_2 - C_1E_1 = (-0.073\ 144)(3.627\ 400) - (-0.142\ 186)(2.226\ 666) = +0.051\ 278$

$E_1E_3 - E_2^2 = (2.226\ 666)(2.701\ 508) - (3.627\ 400)^2 = -7.124\ 861$

$$\Delta a = \frac{-0.317\ 581}{-7.124\ 861} = +0.044\ 574 \qquad a = +1.4 + 0.044\ 574 = 1.444\ 574$$

$$\Delta b = \frac{+0.051\ 278}{-7.124\ 861} = -0.007\ 197 \qquad b = -1.0 - 0.007\ 197 = -1.007\ 197$$

		1	−0.090 492	+0.000 778	−0.000 402 8	+0.960 005 5
+1.444 574			+1.444 574	+1.956 072	+1.371 844 0	+0.010 998 8
	−1.007 197			−1.007 197	−1.363 827 3	−0.956 487 7
		1	+1.354 082	+0.949 653	+0.007 613 9	+0.014 516 6
+1.444 574			+1.444 574	+4.042 866	+5.757 092 5	
	−1.007 197			−1.007 197	−2.818 797 9	
		1	+2.798 656	+3.985 322	+2.945 908	

$\Delta a = +0.001\ 346 \quad a = 1.445\ 920 \quad \Delta b = -0.004\ 638 \quad b = -1.011\ 835$

		1	−0.090 492	+0.000 778	−0.000 402 8	+0.960 005 5
+1.445 920			+1.445 920	+1.959 840	+1.371 864 3	−0.000 011 6
	−1.011 835			−1.011 835	−1.371 469 5	−0.960 011 8
		1	+1.355 428	+0.948 783	−0.000 008 0	−0.000 017 9
+1.445 920			+1.445 920	+4.050 525	+5.765 567 0	
	−1.011 835			−1.011 835	−2.834 502 0	
		1	+2.801 348	+3.987 473	+2.931 057 0	

$\Delta a = -0.000\ 002\ 3 \quad a = 1.445\ 917\ 7 \quad \Delta b = +0.000\ 006\ 2 \quad b = -1.011\ 828\ 8$

		1	−0.090 492	+0.000 778	−0.000 402 8	+0.960 005 5
+1.445 917 7			+1.445 917 7	+1.959 834 0	+1.371 862 4	+0.000 001 2
	−1.011 828 8			−1.011 828 8	−1.371 458 8	−0.960 006 2
		1	+1.355 425 7	+0.948 783 2	+0.000 000 8	+0.000 000 5

$y^2 - 1.445\ 917\ 7y + 1.011\ 828\ 8 = 0$

$\alpha = -0.722\ 958\ 8 \quad \alpha^2 = +0.522\ 669\ 4 \quad \beta^2 = +0.489\ 159\ 4 \quad \beta = +0.699\ 399\ 3$

$y = +0.722\ 958\ 8 \pm 0.699\ 399\ 3\ i$

$y^2 + 1.355\ 425\ 7y + 0.948\ 783\ 2 = 0$

$\alpha = +0.677\ 712\ 8 \quad \alpha^2 = +0.459\ 294\ 7 \quad \beta^2 = +0.489\ 488\ 5 \quad \beta = +0.699\ 634\ 5$

$y = -0.677\ 712\ 8 \pm 0.699\ 634\ 5\ i$

The roots of Eq. (8) are

$$y = \tfrac{1}{2}(\pm\sqrt{2} \pm \sqrt{2}\,i)$$

hence

$$y^2 = \pm\sqrt{2}\,y - 1 \qquad (9)$$

The computations required are shown in Table A. The first trial quadratic factor used is

$$y^2 = 1.4\,y - 1$$

After the third correction, the residuals have been reduced to an order of 10^{-6}. The real part of the roots obtained by this procedure agree exactly with the results obtained by Messrs. Ramaswamy and Ramaiah, although the absolute value of the imaginary part and the average of the absolute values of the imaginary part differ in the fourth significant figure.

While up to eight significant figures have been used in this example for each trial, it is usually more economical of computation time to round off to a lower number of significant figures in the early trials. It may also be noted that a computational error in one of the intermediate trials does not affect the final results, but merely tends to reduce the rate of convergence. Thus, it may be concluded that the procedure not only eliminates the need for resolving the biquadratic into a cubic equation, but also provides an automatic check on the accuracy of the results.

By MARIO G. SALVADORI†

The authors of this paper contribute the valuable information that the three roots of the resultant cubic equation of the shell equation are approximately +2, −2, and zero for all cases of practical importance.

It should be pointed out that today one seldom solves the characteristic equation of cylindrical shells, because of the availability of English, German, Italian, and Spanish manuals on shells.

It should, moreover, be noted that if one must solve cubic equations starting with fairly good approximations of the roots, no method of solution is as simple and as fast as Newton's method when using synthetic substitutions.

By M. G. TAMHANKAR‡

The authors have presented a simplified method which will be extremely useful for solving eighth degree equations. The authors have stated without proof that the roots of the cubic equation are approximately ±2. This can be proved as follows:

The cubic equations arrived at, in each of the three examples, are for

Shell 1

$$Z^3 = 0.004586\,Z^2 + 3.83998\,Z + 0.0000002118$$

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‡Senior Research Fellow, Structures Division, Central Building Research Institute, Roorkee, India.

Shell 2

$$Z^3 = 0.00052608\,Z^2 + 3.83984\,Z + 0.00000041806$$

Shell 3

$$Z^3 = 0.00036122\,Z^2 + 3.839644\,Z + 0.00000070079$$

Here, if the value of Z equals 1.96 is substituted to judge the relative magnitudes of the various terms in the equation then, we obtain for

Shell 1

$$Z^3 = +\,0.017618 + 7.5264 + 0.0000002118$$

Shell 2

$$Z^3 = +\,0.0024051 + 7.5261 + 0.00000041806$$

Shell 3

$$Z^8 = +0.00138767 + 7.5257 + 0.00000070079$$

It is evident that the first and the last terms on the right are insignificant compared to the second term. If these are omitted, we obtain

$$Z^3 + (a^2 - 4\,c)Z = 0$$

Therefore

$$Z = 0 \quad \text{and} \quad Z = \pm\sqrt{4c - a^2}$$

Since the second term $2\,az^2$ of the cubic equation is neglected, the term a^2 under the square root sign can also be omitted.
Hence

$$Z = \pm\,2\sqrt{c}$$

The value of c, working backwards, will be equal to

$$\left(-\frac{\nu\,k^8}{\rho^2} + k^4 \right) + (1 - \nu^2)$$

The magnitude of the first bracket above, in Example 1, is 0.0000055 whereas the magnitude of the second bracket is 0.96. Naturally the first bracket can be neglected.

Thus

$$Z = \pm\,2\sqrt{1 - \nu^2}$$

In all the examples solved, the value of ν is taken as 0.2.

Hence

$$Z = \pm\,1.9596$$

Thus the roots of the cubic equation are established as

$$\pm\,2\sqrt{1 - \nu^2}$$

irrespective of the parameters of the shell. If the value of Poisson's ratio is taken as zero, then the roots would be very close to $\pm\,2$.

TABLE B—COMPARISON OF ROOTS USING MODIFIED AND AUTHORS' METHOD

Shell No.	Modified method	Authors' method
3	$\pm 1.05992 \pm 0.32928\, i$	$\pm 1.05961375 \pm 0.3303257\, i$
2	$\pm 1.00089 \pm 0.348702\, i$	$\pm 1.00034633 \pm 0.349815\, i$
1	$\pm 0.9302 \pm 0.37517\, i$	$\pm 0.929769 \pm 0.376178\, i$

Now, that the roots have been found to be

$$\pm 2\sqrt{1-\nu^2}$$

it can be shown, by working backwards, that the roots of the eighth-degree equation are

$$M_{1,2,3,4} = \pm \frac{1}{2}\left[\sqrt{\frac{\sqrt{a^2 - 2^{5/2}\, a\, b^{1/4} + 16\, b^{1/2}} + 2^{3/2}\, b^{1/4} - a}{2}} \pm i\sqrt{\frac{\sqrt{a^2 - 2^{5/2}\, a\, b^{1/4} + 16\, b^{1/2}} + a - 2^{3/2}\, b^{1/4}}{2}}\right] \quad (10)$$

and

$$M_{5,6,7,8} = \pm \frac{1}{2}\left[\sqrt{\frac{\sqrt{a^2 + 2^{5/2}\, a\, b^{1/4} + 16\, b^{1/2}} - 2^{3/2}\, b^{1/4} - a}{2}} \pm i\sqrt{\frac{\sqrt{a^2 + 2^{5/2}\, a\, b^{1/4} + 16\, b^{1/2}} + 9 + 2^{3/2}\, b^{1/4}}{2}}\right] \quad (11)$$

where

$$a = -4k + \frac{2}{\rho^2}$$

and

$$b = 1 - \nu^2$$

where k, ρ, and ν are the basic constants of the problem. The roots calculated from the formulas mentioned previously are shown compared in Table B with those obtained by the authors.

Eq. (10) and (11) give the roots of the eighth-degree equation directly. Considering the time and labor saved in adopting the modified method, the approximation involved in the preceding results is of negligible order.

AUTHOR'S CLOSURE

The method suggested by Messrs. Pauw and Sangster is based on Newton's method and it avoids operations with complex numbers.

In this procedure, the first trial quadratic factor, denoted by

$$y^2 = 1.4\, y - 1$$

is obtained by ignoring all terms except the first and the last in the bi-quadratic equation. This is analogous to Schorer's procedure. The intermediate terms are small for long shells such as the one given as an example in the authors' paper.

The dimensions and the characteristic equations for typical intermediate and short shells are given below:

Intermediate shell

Using $L = 36$ ft, $R = 60$ ft, and $t = 0.25$ ft (where $\rho = 12.829$ and $k = 0.18166$) the biquadratic equation associated with the characteristic equation is

$$y^4 - 0.71342\, y^3 + 0.58846\, y^2 - 0.12364\, y + 0.96978 = 0$$

Short shell

Using $L = 24$ ft, $R = 80$ ft, and $t = 0.25$ ft (where $\rho = 18.676$ and $k = 0.3146$) the biquadratic equation is

$$y^4 - 1.25288\, y^3 + 0.1906\, y^2 - 0.023247\, y + 0.96108 = 0$$

In the preceding cases intermediate terms are by no means small compared to unity. This would mean a longer number of cycles of computation if the method suggested in the discussion is used.

The method suggested by the authors is likely to prove less time consuming for the following reasons:

1. The method is an extension of Steinman's procedure which itself is based on Newton's method but is more rapidly convergent.

2. The trial value involves only one term as against two in the method proposed by Messrs. Pauw and Sangster.

However, the authors are of the opinion that Messrs. Pauw and Sangster, in their discussion, have suggested a valuable independent method which will be useful for solving higher degree algebraic equations.

The authors agree with Professor Salvadori that the most important part of the paper is the finding that the roots of the cubic equation are nearly +2, —2, and zero, for almost all proportions of cylindrical shells used in practice. The authors, therefore, regard their brief paper primarily as a contribution to the knowledge on cylindrical shells.

Although several manuals on cylindrical shells in different languages have become available, situations do arise where the characteristic equation has to be solved. Thus, for instance, in the ASCE *Manual*,† values are given only for shells with R/t ratios of 100 and 200. Shells with other R/t values cannot be analyzed accurately using the tables given.

The Steinman procedure given in the paper has been compared with Newton's method described by Salvadori.‡ While it is conceded that Newton's method is simpler, it is found that three cycles are required as against two necessary with Steinman's method to get the same accuracy. As pointed out by Steinman himself, his method is based on and is identical with Newton's method.

The method suggested by Mr. Tamhanker is less time consuming; but it is also less accurate as the results correspond nearly to those given by the first cycle of iteration in the authors' method.

Since the final expressions are given in concise analytical form, the designer need not go through the procedure of solving the equations.

The method is easily amenable to programming on a digital computer.

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†*Design of Cylindrical Concrete Shell Roofs*, ASCE Manual No. 31, American Society of Civil Engineers, New York, 1956.

‡Salvadori and Miller, *The Mathematical Solution of Engineering Problems*, McGraw-Hill Book Co., New York, 1948. See also: Salvadori and Baron, *Numerical Methods in Engineering*, Prentice-Hall Inc., New York, 1955.

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$$f(x) = \sum_{i=0}^{n} A_i x^i = 0 \qquad (1)$$

and $x = a$ is a trial root, then

$$f(a) = \sum_{i=0}^{n} A_i a^i \qquad (2)$$

can be evaluated by synthetic division as shown below

$$\begin{array}{c|ccccccc} & A_n & A_{n-1} & A_{n-2} & \ldots\ \ldots & A_2 & A_1 & A_0 \\ +a & & aB_n & aB_{n-1} & \ldots\ \ldots & aB_3 & aB_2 & aB_1 \\ \hline & B_n & B_{n-1} & B_{n-2} & \ldots\ \ldots & B_2 & B_1 & B_0 \end{array}$$

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and in general

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$$\begin{array}{r|cccccc} & B_n & B_{n-1} & B_{n-2} & \dots & \dots & B_2 & B_1 \\ +a & & aC_n & aC_{n-1} & \dots & \dots & aC_3 & aC_2 \\ \hline & C_n & C_{n-1} & C_{n-2} & \dots & \dots & C_2 & C_1 \end{array}$$

It may be shown that†

$$C_i = \frac{d\,B_i}{d\,a} \qquad (3)$$

and hence the desired correction to the trial root by Newton's method is

$$\Delta a = \frac{-B_0}{C_1} \qquad (4)$$

The new trial value is therefore

$$x = a + \Delta a = a - \frac{B_0}{C_1}$$

This procedure can be extended to the solution for quadratic factors. Consider as a quadratic factor a pair of complex conjugate roots $x = \alpha \pm i\beta$; i.e.,

$$(x - [\alpha + i\beta])(x - [\alpha - i\beta]) = x^2 - 2\alpha x + \alpha^2 + \beta^2$$

Thus, if we solve for a quadratic factor in the form

$$x^2 = a\,x - b \qquad (5)$$

then

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The two steps of synthetic division will take the form:

	A_{2n}	A_{2n-1}	A_{2n-2}	. . .	. . .	A_3	A_2	A_1	A_0
$+a$		aC_{2n}	aC_{2n-1}	. . .	. . .	aC_4	aC_3	aC_2	aC_1
$-b$			$-bC_{2n}$	. . .	. . .	$-bC_5$	$-bC_4$	$-bC_3$	$-bC_1$
	C_{2n}	C_{2n-1}	C_{2n-2}	. . .	. . .	C_3	C_2	C_1	C_0
$+a$		aE_{2n}	aE_{2n-1}	. . .	. . .	aE_4	aE_3	aE_2	
$-b$			$-bE_{2n}$	. . .	. . .	$-bE_5$	$-bE_4$	$-bE_3$	
	E_{2n}	E_{2n-1}	E_{2n-2}	. . .	. . .	E_3	E_2	E_1	

where

$$C_i = A_i + a\,C_{i+1} - b\,C_{i+2}$$

and

$$E_i = C_i + a\,E_{i+1} - b\,E_{i+2}$$

respectively.

C_0 and C_1 vanish if, and only if

$$x^2 - a\,x + b \equiv 0$$

Hence, these terms can be considered as residuals which must be liquidated by applying correction to the trial values of a and b. It may be demonstrated that†

$$E_i = \frac{\partial\,C_{i-1}}{\partial\,a} = \frac{\partial\,C_{i-2}}{\partial\,b} \qquad (6)$$

Hence, to eliminate the residuals C_0 and C_1 we set

$$\Delta\,a\,\frac{\partial\,C_0}{\partial\,a} + \Delta\,b\,\frac{\partial\,C_0}{\partial\,b} = \Delta\,a\,E_1 + \Delta\,b\,E_2 = -\,C_0$$

and

$$\Delta\,a\,\frac{\partial\,C_1}{\partial\,a} + \Delta\,b\,\frac{\partial\,C_1}{\partial\,b} = \Delta\,a\,E_2 + \Delta\,b\,E_3 = -\,C_1$$

Solving the above equations for Δa and Δb, the desired corrections are

$$\Delta\,a = \frac{C_1\,E_2 - C_0\,E_3}{E_1\,E_3 - E_2^2} \qquad (7a)$$

and

$$\Delta\,b = \frac{C_0\,E_2 - C_1\,E_1}{E_1\,E_3 - E_2^2} \qquad (7b)$$

To illustrate the method of application consider the biquadratic equation solved by Messrs. Ramaswamy and Ramaiah, viz., Eq. (2):

$$y^4 - 0.090492\,y^3 + 0.000778\,y^2 - 0.0004028\,y + 0.9600055 = 0$$

Since the coefficients of the y^3, y^2, and y terms are small and the constant is approximately equal to unity, we take as our trial roots one of the quadratic factors of

$$y^4 + 1 = 0 \qquad (8)$$

†Ibid.

TABLE A—SUMMARY OF COMPUTATIONS

			4 y^4	3 y^3	2 y^2	1 y	0 Constant
			1	−0.090 492	+0.000 778	−0.000 402 8	+0.960 005 5
a	+1.4			+1.4	+1.833 311	+1.167 724 6	−0.199 060 4
b		−1.0			−1.0	−1.309 508	−0.834 089
C			1	+1.309 508	+0.834 089	−0.142 186	−0.073 144
	+1.4			+1.4	+3.793 311	+5.078 360	
		−1.0			−1.0	−2.709 508	
E			1	+2.709 508	+3.627 400	+2.226 666	

$C_1E_2 - C_0E_3 = (-0.142\ 186)(3.627\ 400) - (-0.073\ 144)(2.709\ 508) = -0.317\ 581$

$C_0E_2 - C_1E_1 = (-0.073\ 144)(3.627\ 400) - (-0.142\ 186)(2.226\ 666) = +0.051\ 278$

$E_1E_3 - E_2^2 = (2.226\ 666)(2.701\ 508) - (3.627\ 400)^2 = -7.124\ 861$

$\Delta a = \dfrac{-0.317\ 581}{-7.124\ 861} = +0.044\ 574 \qquad a = +1.4 + 0.044\ 574 = 1.444\ 574$

$\Delta b = \dfrac{+0.051\ 278}{-7.124\ 861} = -0.007\ 197 \qquad b = -1.0 - 0.007\ 197 = -1.007\ 197$

		4	3	2	1	0
		1	−0.090 492	+0.000 778	−0.000 402 8	+0.960 005 5
+1.444 574			+1.444 574	+1.956 072	+1.371 844 0	+0.010 998 8
	−1.007 197			−1.007 197	−1.363 827 3	−0.956 487 7
		1	+1.354 082	+0.949 653	+0.007 613 9	+0.014 516 6
+1.444 574			+1.444 574	+4.042 866	+5.757 092 5	
	−1.007 197			−1.007 197	−2.818 797 9	
		1	+2.798 656	+3.985 322	+2.945 908	

$\Delta a = +0.001\ 346 \qquad a = 1.445\ 920 \qquad \Delta b = -0.004\ 638 \qquad b = -1.011\ 835$

		4	3	2	1	0
		1	−0.090 492	+0.000 778	−0.000 402 8	+0.960 005 5
+1.445 920			+1.445 920	+1.959 840	+1.371 864 3	−0.000 011 6
	−1.011 835			−1.011 835	−1.371 469 5	−0.960 011 8
		1	+1.355 428	+0.948 783	−0.000 008 0	−0.000 017 9
+1.445 920			+1.445 920	+4.050 525	+5.765 567 0	
	−1.011 835			−1.011 835	−2.834 502 0	
		1	+2.801 348	+3.987 473	+2.931 057 0	

$\Delta a = -0.000\ 002\ 3 \qquad a = 1.445\ 917\ 7 \qquad \Delta b = +0.000\ 006\ 2 \qquad b = -1.011\ 828\ 8$

		4	3	2	1	0
		1	−0.090 492	+0.000 778	−0.000 402 8	+0.960 005 5
+1.445 917 7			+1.445 917 7	+1.959 834 0	+1.371 862 4	+0.000 001 2
	−1.011 828 8			−1.011 828 8	−1.371 458 8	−0.960 006 2
		1	+1.355 425 7	+0.948 783 2	+0.000 000 8	+0.000 000 5

$y^2 - 1.445\ 917\ 7y + 1.011\ 828\ 8 = 0$

$\alpha = -0.722\ 958\ 8 \qquad \alpha^2 = +0.522\ 669\ 4 \qquad \beta^2 = +0.489\ 159\ 4 \qquad \beta = +0.699\ 399\ 3$

$y = +0.722\ 958\ 8 \pm 0.699\ 399\ 3\ i$

$y^2 + 1.355\ 425\ 7y + 0.948\ 783\ 2 = 0$

$\alpha = +0.677\ 712\ 8 \qquad \alpha^2 = +0.459\ 294\ 7 \qquad \beta^2 = +0.489\ 488\ 5 \qquad \beta = +0.699\ 634\ 5$

$y = -0.677\ 712\ 8 \pm 0.699\ 634\ 5\ i$

The roots of Eq. (8) are

$$y = \tfrac{1}{2}(\pm \sqrt{2} \pm \sqrt{2}\, i)$$

hence

$$y^2 = \pm \sqrt{2}\, y - 1 \qquad (9)$$

The computations required are shown in Table A. The first trial quadratic factor used is

$$y^2 = 1.4\, y - 1$$

After the third correction, the residuals have been reduced to an order of 10^{-6}. The real part of the roots obtained by this procedure agree exactly with the results obtained by Messrs. Ramaswamy and Ramaiah, although the absolute value of the imaginary part and the average of the absolute values of the imaginary part differ in the fourth significant figure.

While up to eight significant figures have been used in this example for each trial, it is usually more economical of computation time to round off to a lower number of significant figures in the early trials. It may also be noted that a computational error in one of the intermediate trials does not affect the final results, but merely tends to reduce the rate of convergence. Thus, it may be concluded that the procedure not only eliminates the need for resolving the biquadratic into a cubic equation, but also provides an automatic check on the accuracy of the results.

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