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Sequence Summation Factors

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Professor of Civil Engineering

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SEQUENCE SUMMATION FACTORS

by Adrian Pauw, A.M. ASCE

STRUCTURAL DIVISION

{Discussion open until December 1, 1955}

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SEQUENCE SUMMATION FACTORS

Adrian Pauw,[1] A. M. ASCE

SYNOPSIS

This paper defines a new structural parameter called "Sequence-summation Factor" which may be used in the systematic calculation of "over-relaxation" factors for moment distribution processes. These sequence-summation factors are computed directly from the distribution and carry-over factors used in the moment distribution method. The new parameter furnishes a device whereby the rate of convergence of the moment distribution process is greatly accelerated and whereby, in certain special cases, mathematically exact answers can be obtained. A procedure for computing the total joint relaxation moments is also introduced. With these joint moments the final support moments may be calculated in a single stage of distribution and carry-over. As a result, computational errors are easily detected and corrected, and the degree of accuracy of the results can readily be evaluated.

INTRODUCTION

Since the initial introduction of the "Moment Distribution" method by Professor Hardy Cross, Hon. M. ASCE,(1) a large number of refinements to the basic procedure have been published. The primary purpose of most of these modifications has been to increase the rate of convergence of the numerical sequences which arise in order to make the task of computation less arduous. These refinements tend to fall into three principal categories:

1) Relaxation procedures, i.e. relaxation of joint moments on the basis of "worst first";(2)
2) Modification of distribution and carry-over factors, including relative fixity methods and the application of symmetry and anti-symmetry;(3)(4)(5)
3) Over-relaxation and block-relaxation procedures.(6)

Each of these refinements has its advantages and disadvantages compared to the basic procedure developed by Professor Cross. Although "Relaxation" generally leads to more rapid convergence, the simplicity of the methodical basic scheme is lost; the efficiency of the method depends a great deal on the skill and experience of the computor. The use of relative fixity factors involves the computation of modified constants thereby increasing the possibility of accidental computational errors which are difficult to detect. Most over-relaxation and block-relaxation methods require considerable foresight and experience for effective use. It is the purpose of this paper to present a systematic procedure for over-relaxation based on the use of sequence summation factors. It is not the intention that these factors replace previously

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developed modifications of the Cross method—they can be applied with equal effectiveness to any relaxation process for which systematically converging sequences must be summed.

Nomenclature

As far as practical symbols are in accordance with American Standard Letter Symbols for Structural Analysis (Z 10.8 - 1949). New parameters are defined where they first appear. For convenience of the reader symbols are summarized in Appendix I.

The Summed Sequence Concept

Consider the support moment induced in the system of members shown in Fig. 1 by a moment M applied at joint A. Let D_{AB}, D_{BA}, C_{AB}, C_{BA} be the distribution factors and carry-over factors for the member AB at joints A and B respectively. For one relaxation cycle we have:

$$\begin{array}{lcr} \underline{A} & & \underline{B} \\ +D_{AB}M & \longrightarrow & +C_{AB}D_{AB}M \\ -C_{BA}D_{BA}C_{AB}D_{AB}M & \longleftarrow & -D_{BA}C_{AB}D_{AB}M \end{array}$$

At the end of this relaxation cycle the residual unbalanced moment at A is equal to $-\alpha_{AB}M$, where α_{AB} is defined as:

$$\alpha_{AB} = C_{BA}D_{BA}C_{AB}D_{AB} \tag{1}$$

If this residual unbalanced moment is in turn relaxed, we obtain for the second relaxation cycle:

$$\begin{array}{lcr} \underline{A} & & \underline{B} \\ +D_{AB}\alpha_{AB}M & \longrightarrow & +C_{AB}D_{AB}\alpha_{AB}M \\ -C_{BA}D_{BA}C_{AB}D_{AB}\alpha_{AB}M & \longleftarrow & -D_{BA}C_{AB}D_{AB}\alpha_{AB}M \end{array}$$

The residual unbalanced moment at the end of the second cycle is therefore $-\alpha_{AB}{}^2M$. It follows that if the relaxation procedure were to be continued for n cycles, the residual unbalanced moment at the end of the n-th cycle would be $-\alpha_{AB}{}^nM$. The sum of all the moments relaxed at A would then be given by the sequence,

$$(1 + \beta_A^B)M = M + \alpha_{AB}M + \alpha_{AB}^2M + \alpha_{AB}^3M + \ldots + \alpha_{AB}^nM \tag{2}$$

where,

$$\beta_A^B = \sum_{m=1}^{m=n} \alpha_{AB}^m .$$

It can readily be shown that when n becomes infinitely large, the value of β_A^B is given by:

$$\beta_A^B = \frac{\alpha_{AB}}{1 - \alpha_{AB}} . \tag{3}$$

β_A^B is defined as a "sequence-summation" factor. It is the factor which when multiplied by the initial moment relaxed at A, gives the sum of the additional moments which must be relaxed at A to bring the system of Fig. 1 into exact moment balance.

The factor $(1 + \beta_A^B)$ is defined as the "over-relaxation" factor. By multiplying the initial applied moment at A by this factor before relaxation, the effective moment relaxed in one cycle, (i.e., the initial moment $(1 + \beta_A^B)$ M less the residual unbalanced moment at the end of the cycle) is precisely equal to the applied moment M. This can readily be proved, for, at the end of one cycle the effective relaxed moment is

$$(1 + \beta_A^B)M - \alpha_{AB}(1 + \beta_A^B)M .$$

But,

$$1 + \beta_A^B = 1 + \frac{\alpha_{AB}}{1 - \alpha_{AB}} = \frac{1}{1 - \alpha_{AB}},$$

therefore, by substituting in the above expression, we have

$$\left(\frac{1}{1 - \alpha_{AB}}\right)M - \left(\frac{\alpha_{AB}}{1 - \alpha_{AB}}\right)M \equiv M$$

It should be noted that the rotation of the joints A and B can now be calculated directly. By definition, the stiffness factor K is the moment which must be applied at the end of a member to rotate that end a unit angle. To produce unit rotation of a joint, a moment equal to the sum of the stiffness factors of the members framing into the joint must be applied. Since the total moment relaxed at joint A is $(1 + \beta_A^B)$ M, it follows that

$$\theta_A = \frac{(1 + \beta_A^B)M}{\Sigma K_A}, \qquad (4)$$

where:

θ_A = the rotation of joint A, and

ΣK_A = the sum of the stiffness factors of the members framing into joint A.

Similarly the total moment relaxed at B is $-C_{AB}D_{AB}(1 + \beta_A^B)$ M, hence

$$\theta_A = -\frac{C_{AB}D_{AB}(1 + \beta_A^B)M}{\Sigma K_A} \qquad (5)$$

The use of the sequence-summation factors can best be illustrated by a simple example. Consider the frame shown in Fig. 2a. The fixed-end moments, distribution factors and carry-over factors for this frame are shown in Fig. 2b. The simplified treatment is used for the pin end at A.[2] Positive moment indicates clockwise support moment.

The unbalanced moment at joint B is:

$$M_B^U = +64 - 150 = -86 \text{ k ft},$$

2. "Theory of Modern Steel Structures" Vol. II by L. E. Grinter Macmillan, 1949, p. 114.

and the unbalanced moment at joint C is $+150^{k\ ft}$. Relaxation of the unbalanced moment at B induces a moment at C equal to:

$$M^{C}_{BC} = +86\, D_{BC}\, C_{BC} = (86)(0.5)(0.5) = +21.5^{k\ ft}$$

With the moments at B in balance, the total unbalanced moment at joint C is then $+171.5^{k\ ft}$.

The sequence-summation factor for joint C is computed from equations (1) and (2).

$$\alpha_{CB} = (0.6)(0.5)(0.5)(0.5) = 0.075$$

$$\beta^{B}_{C} = \frac{0.075}{0.925} = 0.081$$

The moment to be relaxed at joint C must therefore be increased by:

$$+171.5\,\beta = +13.91^{k\ ft}$$

and hence

$$\Sigma K_C \theta_C = 171.5 + 13.91 = 185.41^{k\ ft}$$

The total moment relaxed at joint B is the sum of the initial unbalanced moment plus the moment induced by the relaxation of $\Sigma K_C \theta_C$. Thus,

$$\Sigma K_B \theta_B = -86 - (0.6)(0.5)(185.41) = -141.62^{k\ ft}.$$

These computations and the final cycle of moment distribution required to compute the support moments are shown in tabular form in Fig. 2c.

It should be noted that in this instance, "mathematically" exact answers were obtained with a single distribution cycle. It should further be noted that the procedure is self checking in the sense that if the correct over-relaxed unbalanced moment values $\Sigma K\theta$ are used, the final support moments at each joint are in equilibrium. If an accidental error is introduced in the computation of the $\Sigma K\theta$ terms, correct final results may still be obtained by distributing the residual unbalanced moments in the usual manner.

Central Joint Sequence Summation Factor

In the above example it was possible to achieve moment balance simultaneously at joints B and C by first balancing joint B and then computing the total rotation-moment at C by means of the sequence-summation factor. More generally it is possible to achieve simultaneous moment balance in a set of joints, provided they are simply connected to a single central joint. Such a system is shown in Fig. 3.

Referring to equation (1) it may be seen that for this case a single cycle of relaxation of the applied moment M at joint A would produce a residual unbalanced moment

$$M^{C}_{A} = -(\alpha_{AB} + \alpha_{AC} + \alpha_{AD} + \alpha_{AE})\, M$$

$$= -\sum_{i} \alpha_{Ai}\, M . \qquad i = B, C, D, E \qquad (6)$$

It follows from equation (3) that the sequence-summation factor should be defined as:

$$\beta_A^{BCDE} = \frac{\sum_i \alpha_{Ai}}{1 - \sum_i \alpha_{Ai}} \tag{7}$$

With $(1 + \beta_A^{BCDE})$ as the over-relaxation factor, one cycle of relaxation will then again precisely balance the applied moment M.

The computational procedure is relatively simple. Unbalanced moments in adjacent joints are distributed and carried over leaving only a single moment at the central joint. This moment is then multiplied by the over-relaxation factor and relaxed in a single cycle. The procedure is illustrated by the solution of the previous example using the conventional treatment at the pin end at A. Computations are summarized in Fig. 4. Distribution factors and the computations for the α and β factors are shown in Fig. 4a. The selection of joint B as the central joint is an obvious choice in this case because the frame is fixed at D. Fixed-end moments at joints A and C are first balanced leaving only a single unbalanced moment at joint B, namely

$$M_B^U = +42.7 - 150.0 + 21.3 - 45.0 = -131.0^{\text{ k ft}}$$

This moment must be multiplied by the over-relaxation factor $(1 + \beta_B^{AC})$ and distributed. Thus the moment distributed at joint B is:

$$M_B^U (1 + \beta_B^{AC}) = -131.0\,(1.1712) = -153.4^{\text{k ft}}$$

The over-relaxation of joint B in turn induces unbalanced moments at joints A and C. When these moments are distributed and carried over, the moments at joints A, B and C are found to be in exact balance. Induced moments at the fixed ends D, E and F are computed in the usual manner.

As an alternate method, the total unbalanced moments to be distributed (i.e., the joint-rotation moments $\Sigma K\theta$) at each joint are first computed, and then these moments are distributed and carried over in a single cycle of moment-distribution. Computations are shown in Fig. 4c and are self-explanatory.

The advantage of the alternate procedure is that the computations for the support moments never involve more than the addition of three terms, namely: the fixed-end moment, the distributed moment, and the carry-over moment. This method also acts as a check on the accuracy of the joint-rotation moment calculations. It should be noted that in the example in Fig. 4, $\Sigma K_B \theta_B$ has a different value than found in the example in Fig. 2. This result is due to the fact that the value of K_{AB} is 2 for the conventional treatment of the hinged member AB, but only 1.5 for the special treatment of the pin at A. The angle θ_B is, of course, the same in either case.

Extension to Singly Connected Multiple Joints

For most applications the central joint β-factors will give extremely rapid convergence. Occasionally however, it may be desirable to extend the method to obtain over-relaxation factors for the simultaneous balancing of a series of singly-connected joints. Where this can be done, as is the case for multispan continuous beams and one story frames, it is then possible to obtain mathematically exact solutions with only a single cycle of moment-distribution.

The procedure for deriving the equations for the necessary coefficients is shown in Fig. 5. Relaxation of the applied moment M induces an unbalanced moment $D_{AB}C_{AB}M$ at joint B. This unbalanced moment at B is over-relaxed using the factor:

$$1 + \beta_B^C = \frac{1}{1 - \alpha_{BC}} .$$

Over-relaxation results in the simultaneous balancing of joints B and C, but induces a residual unbalanced moment at A,

$$M_A^C = - \frac{\alpha_{AB}}{1 - \alpha_{BC}} M .$$

It follows then, that if the relaxation pattern is repeated for n cycles, the total joint-rotation moment at A is given by:

$$(1 + \beta_A^{BC}) M = M \left[1 + \frac{\alpha_{AB}}{1 - \alpha_{BC}} + \left(\frac{\alpha_{AB}}{1 - \alpha_{BC}}\right)^2 + \ldots + \left(\frac{\alpha_{AB}}{1 - \alpha_{BC}}\right)^n \right]$$

From equation (3) it is easily recognized that the required sequence-summation factor is given by:

$$\beta_A^{BC} = \frac{\alpha_{AB}}{1 - \alpha_{BC}} \div \left[1 - \frac{\alpha_{AB}}{1 - \alpha_{BC}} \right] = \frac{\alpha_{AB}}{1 - \alpha_{AB} - \alpha_{BC}} \qquad (8)$$

In applying this factor the computor must not forget that a sequence-summation factor is also required for joint B to balance the moment induced by the relaxation of joint A. This factor of course is given by:

$$\beta_B^C = \frac{\alpha_{BC}}{1 - \alpha_{BC}}$$

The sequence-summation factor for a central joint of a singly-connected, multiple joint system is given by:

$$\beta_A^i = \frac{\sum_i \gamma_{Ai}}{1 - \sum_i \gamma_{Ai}} \qquad (9)$$

where the γ_{Ai} are modified α-factors for all members framing into joint A, and are given by equations of the form

$$\gamma_{AB} = \frac{\alpha_{AB}}{1 - \alpha_{BC}}$$

Similar coefficients can be determined for any number of singly-connected joints. Due to the increasing complexity of these factors their use is generally not warranted. An example where these factors can be used to advantage is given in Fig. 6. Computations for all necessary coefficients are given in the figure. Since joint A is hinged and joint F is fixed, either joint C or joint D could be selected as the central joint. Joint D has been arbitrarily selected in this case; the computations would be similar if joint C were chosen. With joint D as the central joint, the factor β_C^B must be computed. With this factor the unbalanced moment at joint C can be over-relaxed (by relaxing the moment $(1+\beta_C^B)M_{CD}^F$) resulting in simultaneous balance at joints B and C and leaving only a single unbalanced moment at joint D equal to the sum of the induced moment and the initial unbalanced moment. The total unbalanced moment at D is then over-relaxed using the sequence-summation factor β_D^{BCE} This relaxation in turn induces a moment at C which must again be over-relaxed by the use of the β_C^B factor to maintain simultaneous balance at joints B and C. Total joint-rotation moments at joints C and D are thus determined. The rotation moments at joints B and E are the moments induced at these joints by relaxation of the joint-rotation moments at C and D. The final step consists of moment-distribution, carry-over, and summation with the fixed-end moment. It may be observed that the resulting support moments are in exact equilibrium at all supports.

Indirect Determination of Sequence Summation Factors

It is evident from the foregoing discussion that the albegraic expressions for the sequence-summation factors for systems more extensive than a central "star" joint become rather complex. It should be noted however that the sequence-summation factor for any joint of a given system may be computed from the relationship

$$\beta = \frac{\Sigma \gamma}{1 - \Sigma \gamma} \tag{9}$$

where $\Sigma\gamma$ is the residual unbalanced moment at a given joint of the system, when after initial relaxation of a unit moment at the given joint, the induced moments in the other joints of the system are relaxed into balance. Sequence-summation factors can therefore be calculated by relaxation. Since we are dealing with unbalanced induced moments only, it is convenient to compute these moments directly by the use of transmission coefficients T, defined by:

$$T_{Ai} = D_{Ai}\, C_{Ai} \tag{10}$$

This procedure will be illustrated by computing the sequence-summation factors for the continuous beam of Fig. 6. The transmission coefficients are:

$$T_{BC} = (0.4)(0.5) = 0.2; \quad T_{CB} = T_{CD} = (0.5)(0.5) = 0.25;$$

$$T_{DC} = (0.4)(0.5) = 0.2; \quad T_{DE} = T_{ED} = (0.6)(0.5) = 0.30.$$

Since joint D was selected as the "central joint," the two β-values required are:

β_C^B - the summed sequence factor producing simultaneous balance at joints B and C when a moment is relaxed at joint C, and

β_D^{BCE} - the summed sequence factor producing simultaneous balance at joints B, C, and E when moment is relaxed at joint D.

β_C^B is most readily calculated by equation (3). Thus:

$$\alpha_{BC} = T_{BC}\, T_{CB} = 0.05,$$

hence,

$$\beta_C^B = \frac{0.05}{1 - 0.05} = 0.05263$$

To compute β_D^{BCE} we merely relax a unit moment at D and then relax the induced moments at E, C, and B. To eliminate decimals it is convenient to relax a moment of 10,000 at D. The induced moments at C and at E will be:

$$10{,}000\ T_{DC} = 2000 \qquad \text{at C}$$

and

$$10{,}000\ T_{DE} = 3000 \qquad \text{at E.}$$

To produce simultaneous balance at joint B and at joint C it is necessary to over-relax the induced moment at B. The increase is therefore:

$$2000\ \beta_C^B = 105.26,$$

and hence the joint-relaxation moment at B is

$$2000 + 105.26 = 2105.26.$$

Relaxation of this moment induces a residual unbalanced moment at D of:

$$-\ 2105.26\ T_{CD} = -\ 526.32.$$

Relaxation of the unbalanced moment at E induces a residual moment at D of

$$-\ 3000\ T_{ED} = -\ 900.$$

The total residual unbalanced moment at D is therefore:

$$-\ (526.32 + 900) = -\ 1426.32$$

$\Sigma \gamma_D$ is therefore

$$\frac{1426.32}{10,000} = 0.142632, \text{ and}$$

$$\beta_D^{BCE} = \frac{0.142632}{1 - 0.142632} = \frac{0.142632}{0.857368} = 0.16636$$

These calculations are summarized in Fig. 7. Slide rule accuracy will normally be adequate for computing these factors.

Application to Closed Rings and Multistory Frames

Although the techniques discussed in the previous sections can be extended to give exact results in closed rings and multistory frames, the labor involved in computing special factors is seldom justified. For such cases two cycles of relaxation using the central-joint sequence-summation factors will normally result in a residual error of less than one per cent. This high degree of accuracy is the result of the increased rate of convergence obtained with the use of these factors. To apply the process, alternate joints are selected as central joints. Sequence-summation factors and joint-rotation moments need be calculated for these joints only. The first step is to distribute the unbalanced moments at all joints so that only the central joints are unbalanced. These unbalanced moments are then increased using the β-factors, and the induced moment at adjacent central joints computed. This calculation may be expedited by computing transmission coefficients for adjacent central joints. The algebraic sum of the induced moments at each central joint is in turn the new unbalanced joint moment. This procedure can therefore be repeated as often as necessary until the residuals are negligible. Generally two or at most three cycles will be sufficient. The total joint-rotation moment at each central joint is then obtained by summing the several moments relaxed. The joint-rotation moments at the other joints are computed by adding the initial unbalanced moments at these joints to the sum of the induced moments obtained when the joint-rotation moments in adjacent central joints are relaxed.

The final step consists of distributing and carrying-over the joint-rotation moment at each joint and of adding the resulting values to the initial fixed-end moments. The complete procedure is illustrated for the two-story, two-bay frame shown in Fig. 8a. Sideway is treated by the method of artificial joint forces and superposition.

Distribution and carry-over factors for this example are shown in Fig. 8b, and α-factors and transmission coefficients in Fig. 8c. Central-joint β-factors and transmission coefficients are shown in Fig. 8d. It should be noted that the central-joint transmission coefficients are negative due to sign reversal when the induced moments at intermediate joints are relaxed. These coefficients are readily computed as products or sum of products of the transmission coefficients shown in Fig. 8c. The coefficients shown in Fig's 8b, c, and d are independent of any loading on the structure, and are therefore applicable to any loading.

The fixed-end moments for the vertical loads are shown in Fig. 8f. Support moments are first computed with sidesway prevented. Calculations for the joint-rotation moments are tabulated in Fig. 8e, and for the support moments in Fig. 8f. Story shears and the magnitude of the artificial restraint forces can be computed from the sum of the column moments.

Since the columns in each story are of equal height, the fixed-end moments due to a sidesway of either story will be proportional to the stiffness factors of the columns. Fixed-end moments equal to 100 times the relative column stiffness factor were selected. Relaxation computations for sidesway of the top and of the bottom story are shown in Fig's 8g, 8h and in Fig's 8i, 8j respectively.

The shear in each story is proportional to the sum of the column support moments in that story. The required values of the sum of the column support moments are therefore:

$\Sigma M = -(5)(12) = -60^{k\,ft}$ for the top story, and
$\Sigma M = -(5+6)(15) = -165^{k\,ft}$ for the bottom story.

The sum of the column moments in each story for the vertical loads and for the assumed sidesways are:

	Top Story	Bottom Story
Vertical Loads	+ 16.44	- 8.54
Sidesway - Top Story	- 515.56	+ 181.20
Sidesway - Bottom Story	+ 181.27	- 1295.41

We can therefore write the equations:

$$+16.44 - 515.56A + 181.27B = -60$$

$$-8.54 + 181.20A - 1295.41B = -165$$

where A and B are the proportionality factors required to compute the final support moments. Solution of these equations yields:

$$A = +0.2006$$

$$B = +0.1488$$

The required final support moments are the sum of the moments in Fig. 8f, A times the moments of Fig. 8h, and B times the moments of Fig. 8j. These computations are tabulated in Fig. 8k.

CONCLUSIONS

The greatest advantage of this method is not the reduction of labor involved in computing moments, but in making the moment-distribution process self-checking. The support moments obtained by distribution of joint-rotation moments will be in exact equilibrium at each joint, provided the rotation moments have been correctly calculated. As a byproduct, the method reduces the clutter normally associated with the solution of multistory frames by moment-distribution.

The determination of the required parameters is extremely simple once the basic concept of equation (3) is understood. The discussion in this paper has been limited to a presentation of the basic concept and use of the sequence-summation factor. It is quite evident that the method will apply equally well to continuous beams and frames with varying sections, since all the new parameters can be defined in terms of distribution and carry-over factors.

The use of the sequence-summation factor is not limited to structural problems requiring the determination of support moments. The principles involved are quite general and can be applied effectively to most problems which lend themselves to solution by relaxation methods.

ACKNOWLEDGMENTS

The concept of sequence-summation factors and their application to moment-distribution were developed as a byproduct of a basic research program in "Force Relaxation Methods" sponsored by the Office of Ordnance Research, U. S. Army.

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APPENDIX I

NOMENCLATURE

C = Carry-over factor
D = Distribution factor
K = Stiffness factor
M = Moment
T = DC = Transmission coefficient
T' = Central-joint transmission coefficient to adjacent central joints

α = Flexural parameter—defined by equation (1)
β = Sequence-summation factor—defined by equation (3)
γ = Modified flexural parameter—see equation (9)
θ = Angle of joint rotation

Σ = Symbol designating summation of a series of like factors
$\Sigma K \theta$ = Joint-rotation moment

i = Joint designator in summation formulae

Subscripts

Single subscript designates joint.
Double subscript designates member; first subscript refers to joint.

Superscripts

For moments, superscript designates source, thus:

M^F = Fixed end moment,
M^D = Distributed moment,
M^C = Carry-over moment.

For β-factors, superscripts designate joints covered, thus:

β_B^{ACD} = β-factor for joint B covering joints A, C and D.

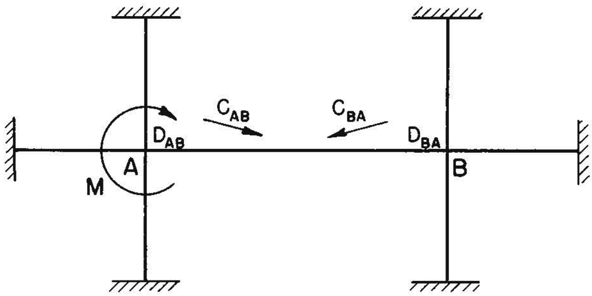


FIG. 1

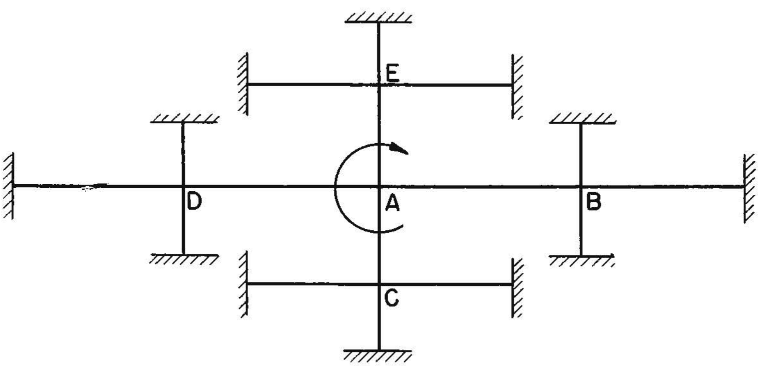


FIG. 3

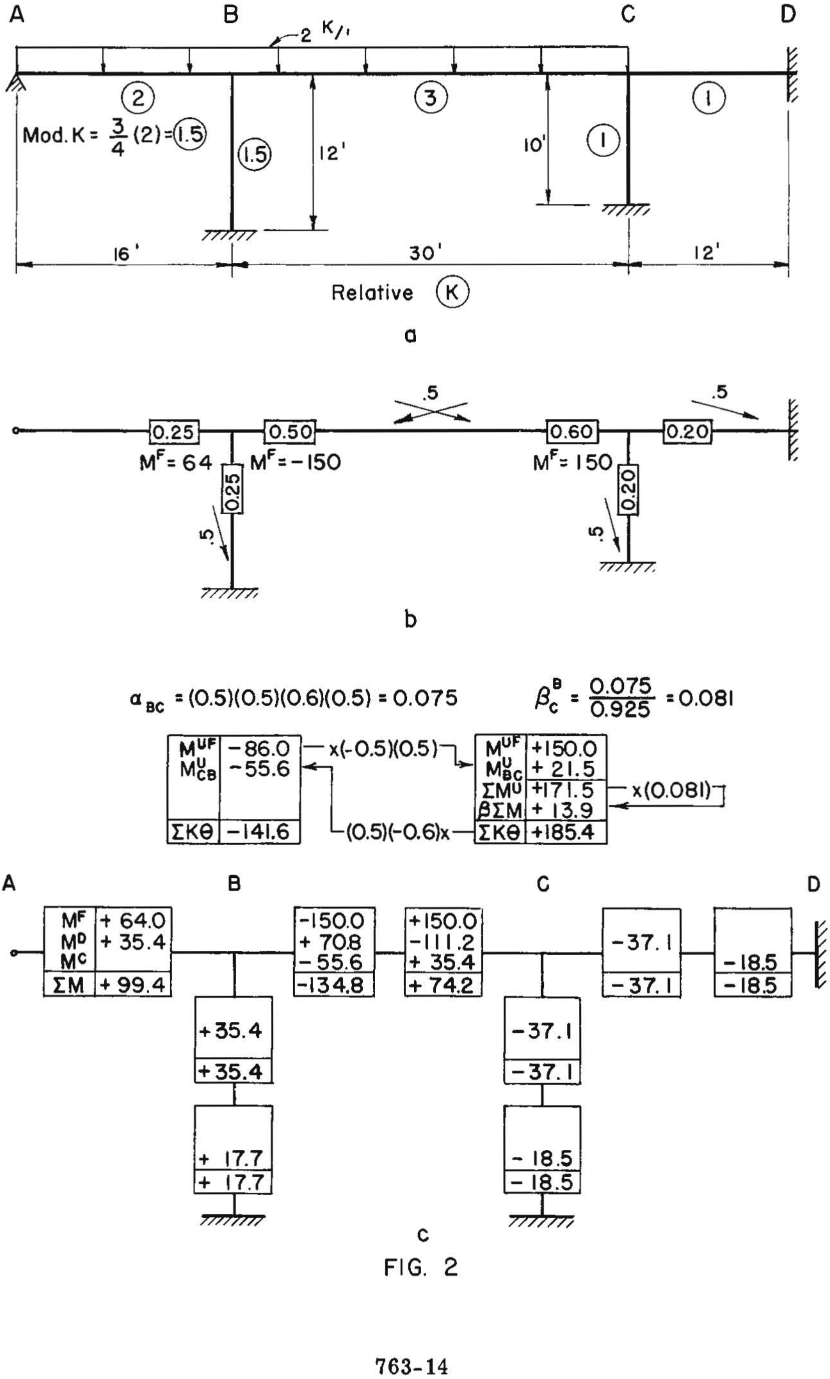
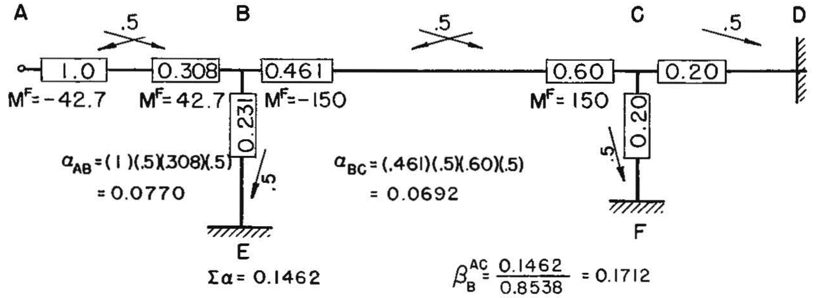

A
B
C
D
2 K/'
2
3
1
Mod. K = 3/4 (2) = 1.5
1.5
12'
10'
1
16'
30'
12'
Relative K
a
.5
.5
0.25
0.50
0.60
0.20
M^F = 64
M^F = −150
M^F = 150
0.25
0.20
.5
.5
b
α_BC = (0.5)(0.5)(0.6)(0.5) = 0.075
β_c^B = 0.075 / 0.925 = 0.081
M^UF −86.0
M^U_CB −55.6
ΣKθ −141.6
x(−0.5)(0.5)
(0.5)(−0.6)x
M^UF +150.0
M^U_BC + 21.5
ΣM^U +171.5
βΣM + 13.9
ΣKθ +185.4
x(0.081)
M^F + 64.0
M^D + 35.4
M^C
ΣM + 99.4
−150.0
+ 70.8
− 55.6
−134.8
+150.0
−111.2
+ 35.4
+ 74.2
−37.1
−37.1
−18.5
−18.5
+35.4
+35.4
+ 17.7
+ 17.7
−37.1
−37.1
− 18.5
− 18.5
c

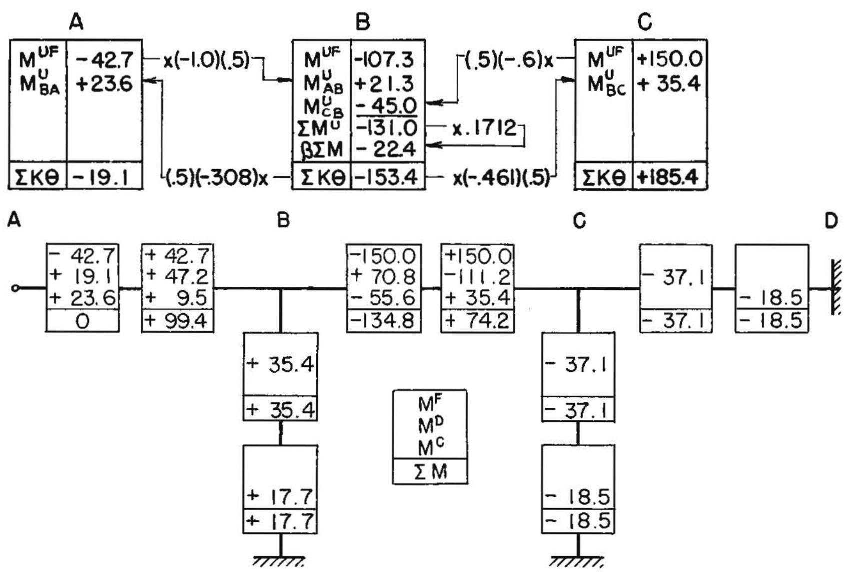
FIG. 2



a

	AB	BA	BE	BC	CB	CF	CD	DC
M^F	−42.7	+42.7		−150.0	+150.0			
M^D	+42.7				−90.0	−30.0	−30.0	
M^C		+21.3		−45.0				
$(1+\beta)M^D$		+47.2	+35.4	+70.8				
M^C	+23.6				+35.4			
M^D	−23.6				−21.2	−7.1	−7.1	
M^C		−11.8		−10.6				
ΣM	0	+99.4	+35.4	−134.8	+74.2	−37.1	−37.1	−18.5
			+17.7			−18.5		
			EB			FC		

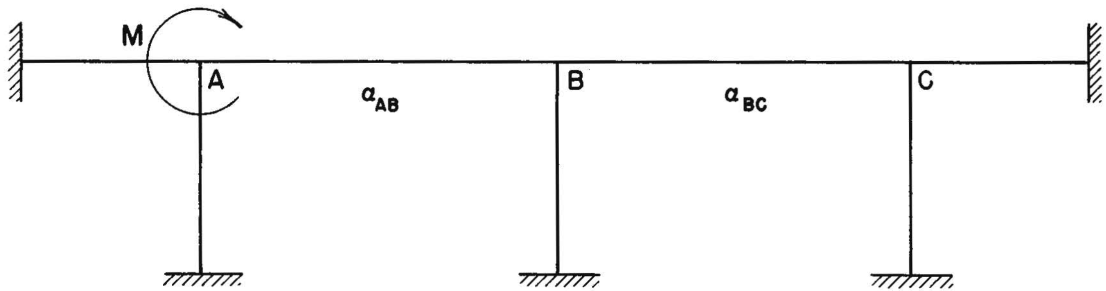
b



c

FIG. 4

763-16



A	B	C
$D_{AB}M \longrightarrow C_{AB}D_{AB}M$		
$-\frac{\alpha_{AB}}{1-\alpha_{BC}}M \longleftarrow -\frac{1}{1-\alpha_{BC}}D_{BA}C_{AB}D_{AB}M$	$-\frac{1}{1-\alpha_{BC}}D_{BC}C_{AB}D_{AB}M \longrightarrow -\frac{1}{1-\alpha_{BC}}C_{BC}D_{BC}C_{AB}D_{AB}M$	
	$+\frac{\alpha_{BC}}{1-\alpha_{BC}}C_{AB}D_{AB}M \longleftarrow +\frac{1}{1-\alpha_{BC}}D_{CB}C_{BC}D_{BC}C_{AB}D_{AB}M$	

FIG. 5

763-17

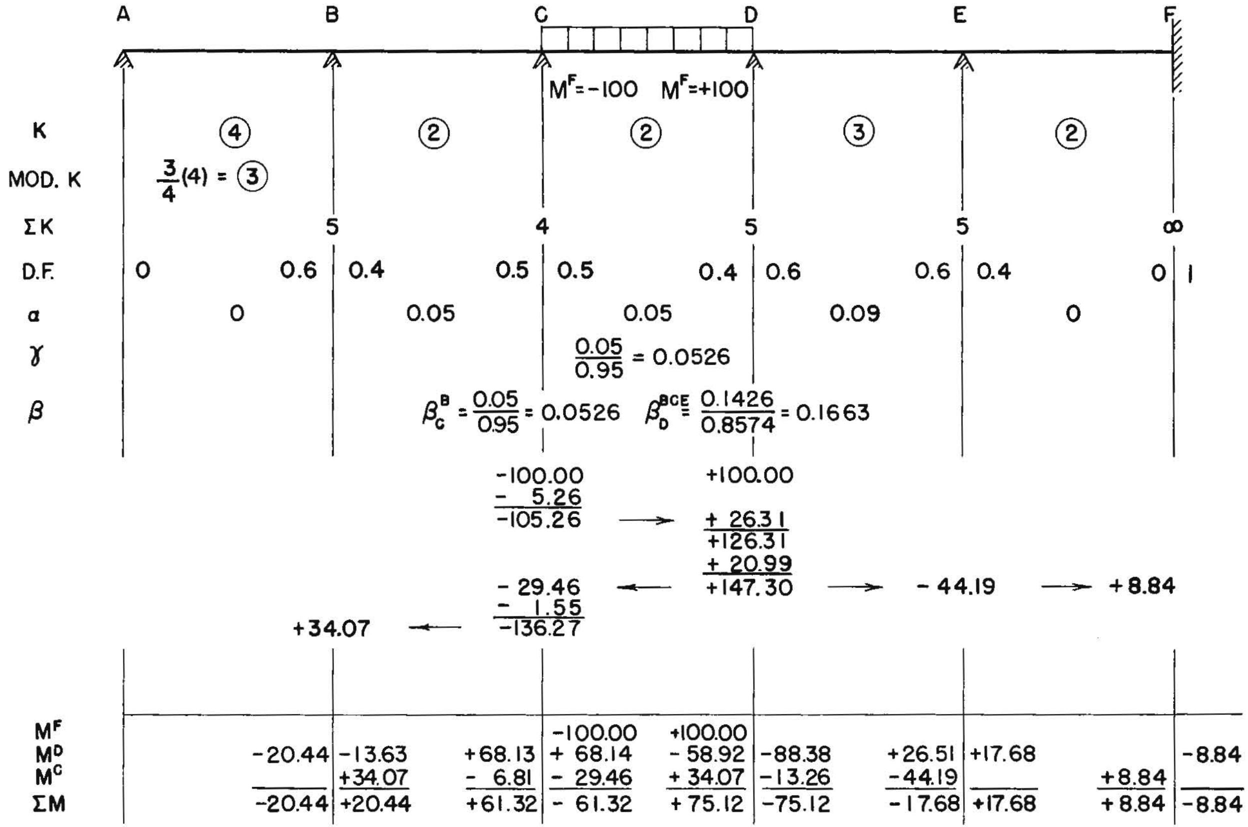
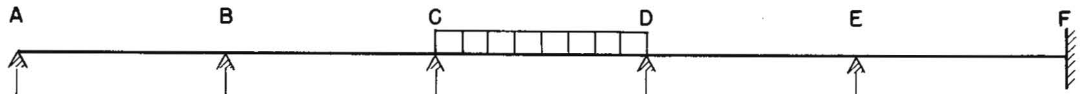

A B C D E F
M^F = -100 M^F = +100
K
4 2 2 3 2
MOD. K
3/4 (4) = 3
ΣK
5 4 5 5 ∞
D.F.
0 0.6 0.4 0.5 0.5 0.4 0.6 0.6 0.4 0 1
α
0 0.05 0.05 0.09 0
γ
0.05/0.95 = 0.0526
β
β_C^B = 0.05/0.95 = 0.0526
β_D^BCE = 0.1426/0.8574 = 0.1663
-100.00
- 5.26
-105.26
+100.00
+ 26.31
+126.31
+ 20.99
+147.30
- 29.46
- 1.55
-136.27
+34.07
- 44.19
+8.84
M^F
M^D
M^C
ΣM
-100.00 +100.00
-20.44 -13.63 +68.13 + 68.14 - 58.92 -88.38 +26.51 +17.68 -8.84
+34.07 - 6.81 - 29.46 + 34.07 -13.26 -44.19 +8.84
-20.44 +20.44 +61.32 - 61.32 + 75.12 -75.12 - 17.68 +17.68 +8.84 -8.84

FIG. 6

763-18



	A	AB	B	BC	C	CD	D	DE	E	EF	F
K		4		2		2		3		2	
MOD. K		$\frac{3}{4}(4) = 3$									
ΣK			5		4		5		5		∞
D.F.	0		0.6 | 0.4		0.5 | 0.5		0.4 | 0.6		0.6 | 0.4		0
T			0 | 0.2		0.25 | 0.25		0.2 | 0.3		0.3 | 0.2		0
α				0.05							

$$\beta_C^B = \frac{0.05}{0.95} = 0.0526$$

2000	10,000	3000
105	− 900	
2105	− 526	
	− 1426	

$$\beta_D^{BCE} = \frac{0.1426}{0.8574} = 0.1663$$

FIG. 7

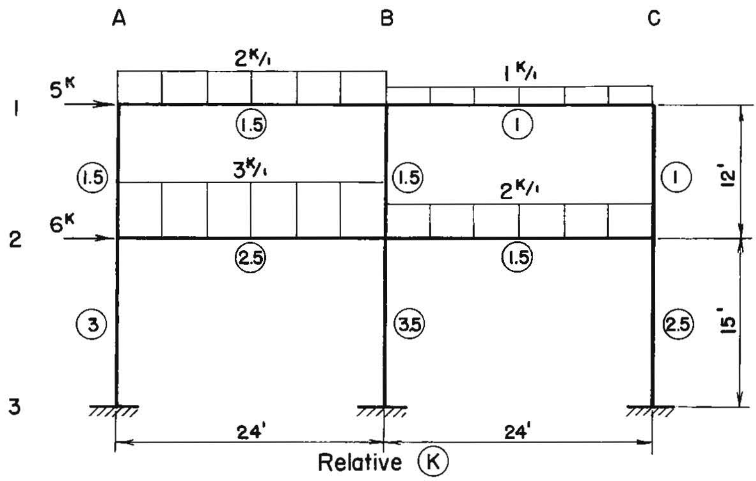


FIG. 8a

FRAME DIMENSIONS

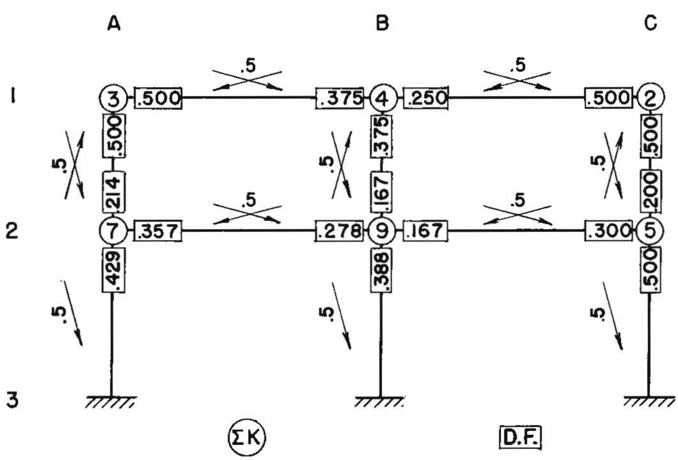


FIG. 8b

DISTRIBUTION AND CARRY-OVER FACTORS

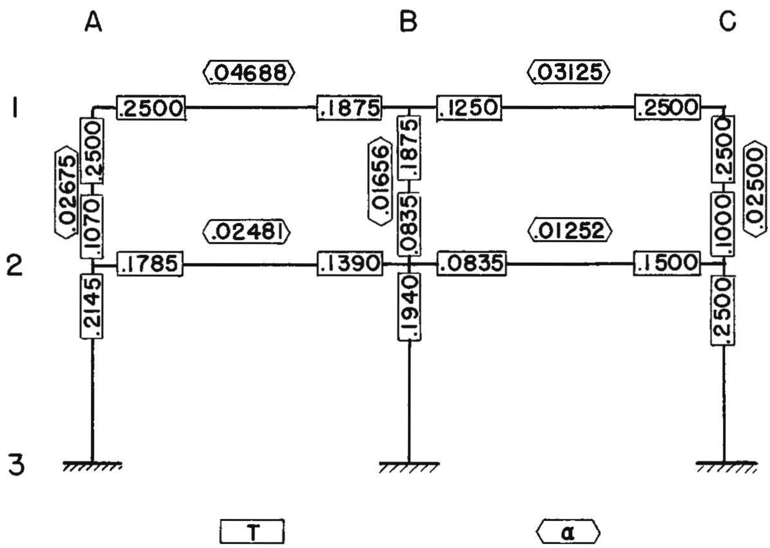


FIG. 8c

TRANSMISSION COEFFICIENTS

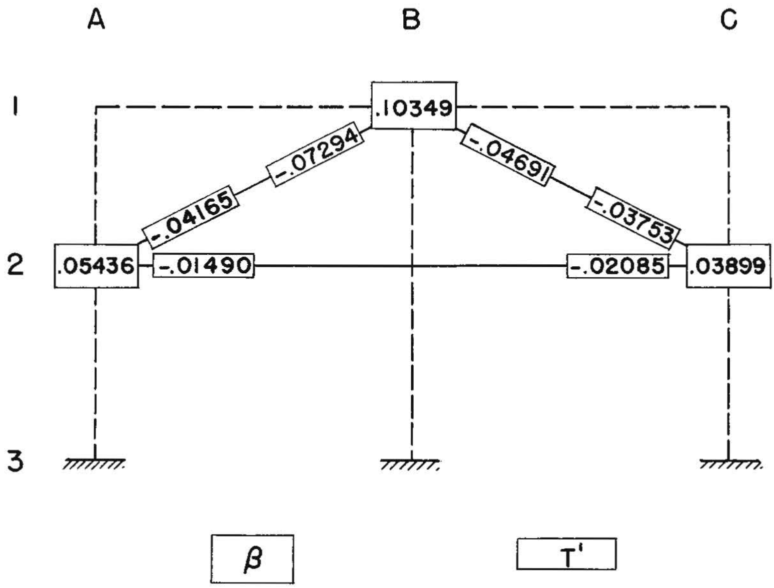


FIG. 8d

CENTRAL-JOINT COEFFICIENTS

A-1	B-1	C-1	A-2	B-2	C-2
-96.00	+48.00 +24.00 -12.00 - 4.00 - 5.56 + 3.04 + 5.53 +59.01 + .26 + .11 + .04 + .41	+48.00	-144.00 + 24.00 - 6.67 - 6.89 -133.56 + 1.69 + 4.30 + .33 + 6.32 + .06 + .03	+48.00	+96.00 -12.00 - 4.00 - 1.99 + 3.04 +81.05 + 2.77 + .09 + .11 + 2.97 + .02
-11.14 +13.61	+59.42	- 7.43 - 8.40	-127.15	- 11.14 +22.70 -12.61	+84.04
-93.53	+59.42	+32.17	-127.15	+46.95	+84.04

FIG. 8e

JOINT ROTATION MOMENTS

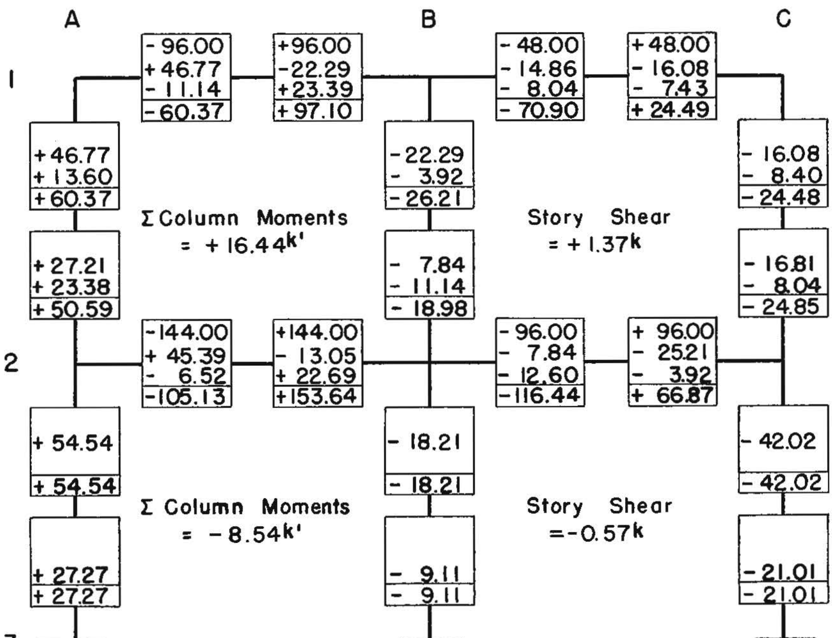


FIG. 8f

VERTICAL LOAD - MOMENT DISTRIBUTION

A-1	B-1	C-1	A-2	B-2	C-2
	-150.00		-150.00		-100.00
-150.00	+ 37.50		+ 37.50		
	+ 25.00	-100.00			+ 25.00
	+ 12.50		+ 20.83	-150.00	+ 12.50
			- 4.98		
	- 4.03		- 96.65		- 1.44
	- 8.18				
	- 87.21		- 6.36		- 4.09
					- 2.65
	- 2.65		- 1.47		- 70.68
			- .43		
	- .34		- 8.26		- .12
	- .31				
	- 3.30		- .24		- .15
					- .01
	- .01		- .01		- .28
+ 16.97	- 90.52	+ 11.32		+ 16.97	
+ 11.25			-105.16	+ 18.77	
		+ 7.10		+ 10.64	- 70.96
-121.78	- 90.52	- 81.58	-105.16	-103.62	- 70.96

FIG. 8g

JOINT ROTATION MOMENTS

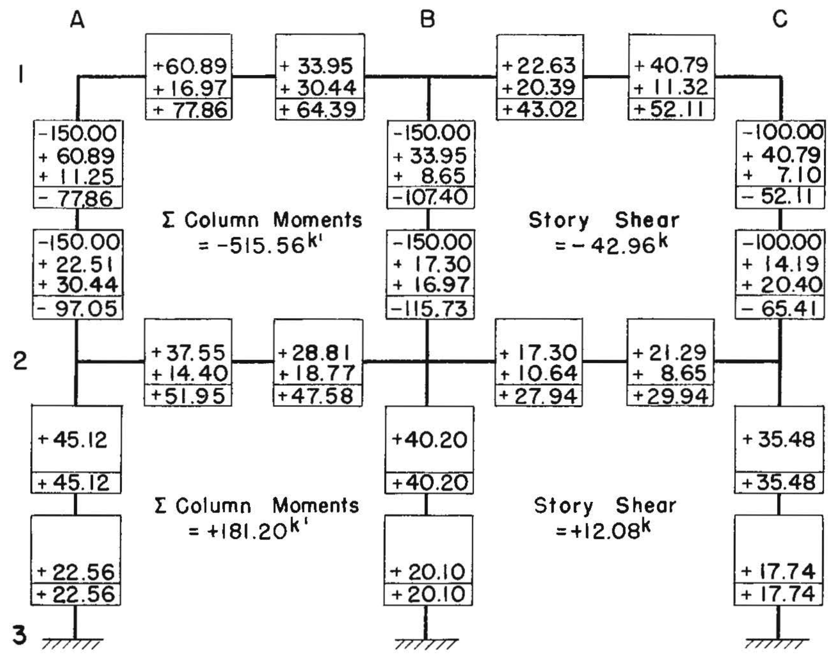


FIG. 8h

TOP STORY SIDESWAY — MOMENT DISTRIBUTION

A-1	B-1	C-1	A-2	B-2	C-2
			-300.00		-250.00
	+ 29.22		+ 48.65	-350.00	+ 29.22
			- 13.66		
	- 11.04		-265.01		- 3.95
					- 8.76
	- 8.72		- 4.87		-233.49
	+ .98				
	+ 10.44		+ .76		+ .49
			- .22		
	- .18		- 4.33		- .06
					+ .02
	+ .02		+ .01		+ .45
	- .02				
	- .18		- .01		
- 1.92	+ 10.26	- 1.28		- 1.92	
+ 28.82			-269.34	+ 48.08	
		+ 23.30		+ 34.96	-233.04
+ 26.90	+ 10.26	+ 22.02	-269.34	-268.88	

FIG. 8i

JOINT ROTATION MOMENTS

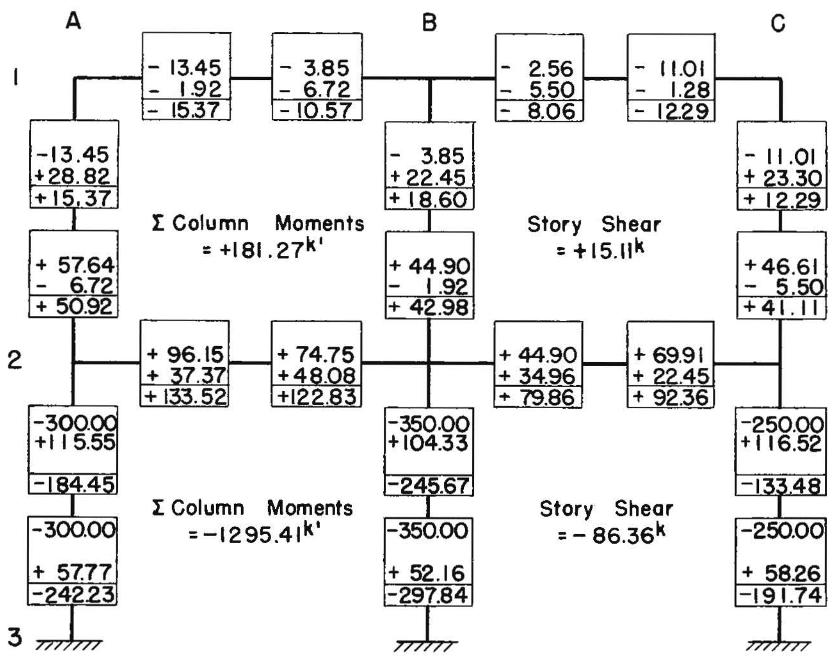


FIG. 8j

BOTTOM STORY SIDESWAY - MOMENT DISTRIBUTION

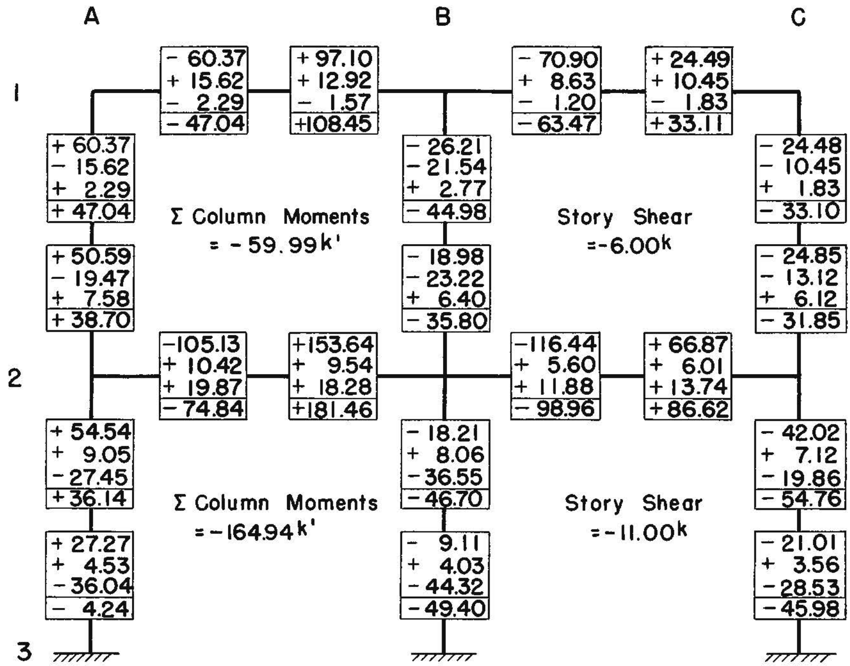


FIG. 8k

FINAL MOMENTS

PROCEEDINGS PAPERS

The technical papers published in the past year are presented below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways (WW) divisions. For titles and order coupons, refer to the appropriate issue of "Civil Engineering" or write for a cumulative price list.

VOLUME 80 (1954)

AUGUST: 466(HY), 467(HY), 468(ST), 469(ST), 470(ST), 471(SA), 472(SA), 473(SA), 474(SA), 475(SM), 476(SM), 477(SM), 478(SM)[c], 479(HY)[c], 480(ST)[c], 481(SA)[c], 482(HY), 483(HY).

SEPTEMBER: 484(ST), 485(ST), 486(ST), 487(CP)[c], 488(ST)[c], 489(HY), 490(HY), 491(HY)[c], 492(SA), 493(SA), 494(SA), 495(SA), 496(SA), 497(SA), 498(SA), 499(HW), 500(HW), 501(HW)[c], 502(WW), 503(WW), 504(WW)[c], 505(CO), 506(CO)[c], 507(CP), 508(CP), 509(CP), 510(CP), 511(CP).

OCTOBER: 512(SM), 513(SM), 514(SM), 515(SM), 516(SM), 517(PO), 518(SM)[c], 519(IR), 520(IR), 521(IR), 522(IR)[c], 523(AT)[c], 524(SU), 525(SU)[c], 526(EM), 527(EM), 528(EM), 529(EM), 530(EM)[c], 531(EM), 532(EM)[c], 533(PO).

NOVEMBER: 534(HY), 535(HY), 536(HY), 537(HY), 538(HY)[c], 539(ST), 540(ST), 541(ST), 542(ST), 543(ST), 544(ST), 545(SA), 546(SA), 547(SA), 548(SM), 549(SM), 550(SM), 551(SM), 552(SA), 553(SM)[c], 554(SA), 555(SA), 556(SA), 557(SA).

DECEMBER: 558(ST), 559(ST), 560(ST), 561(ST), 562(ST), 563(ST)[c], 564(HY), 565(HY), 566(HY), 567(HY), 568(HY)[c], 569(SM), 570(SM), 571(SM), 572(SM)[c], 573(SM)[c], 574(SU), 575(SU), 576(SU), 577(SU), 578(HY), 579(ST), 580(SU), 581(SU), 582(Index).

VOLUME 81 (1955)

JANUARY: 583(ST), 584(ST), 585(ST), 586(ST), 587(ST), 588(ST), 589(ST)[c], 590(SA), 591(SA), 592(SA), 593(SA), 594(SA), 595(SA)[c], 596(HW), 597(HW), 598(HW)[c], 599(CP), 600(CP), 601(CP), 602(CP), 603(CP), 604(EM), 605(EM), 606(EM)[c], 607(EM).

FEBRUARY: 608(WW), 609(WW), 610(WW), 611(WW), 612(WW), 613(WW), 614(WW), 615(WW), 616(WW), 617(IR), 618(IR), 619(IR), 620(IR), 621(IR)[c], 622(IR), 623(IR), 624(HY)[c], 625(HY), 626(HY), 627(HY), 628(HY), 629(HY), 630(HY), 631(HY), 632(CO), 633(CO).

MARCH: 634(PO), 635(PO), 636(PO), 637(PO), 638(PO), 639(PO), 640(PO), 641(PO)[c], 642(SA), 643(SA), 644(SA), 645(SA), 646(SA), 647(SA)[c], 648(ST), 649(ST), 650(ST), 651(ST), 652(ST), 653(ST), 654(ST)[c], 655(SA), 656(SM)[c], 657(SM)[c], 658(SM)[c].

APRIL: 659(ST), 660(ST), 661(ST)[c], 662(ST), 663(ST), 664(ST)[c], 665(HY)[c], 666(HY), 667(HY), 668(HY), 669(HY), 670(EM), 671(EM), 672(EM), 673(EM), 674(EM), 675(EM), 676(EM), 677(EM), 678(HY).

MAY: 679(ST), 680(ST), 681(ST), 682(ST)[c], 683(ST), 684(ST), 685(SA), 686(SA), 687(SA), 688(SA), 689(SA)[c], 690(EM), 691(EM), 692(EM), 693(EM), 694(EM), 695(EM), 696(PO), 697(PO), 698(SA), 699(PO)[c], 700(PO), 701(ST)[c].

JUNE: 702(HW), 703(HW), 704(HW)[c], 705(IR), 706(IR), 707(IR), 708(IR), 709(HY)[c], 710(CP), 711(CP), 712(CP), 713(CP)[c], 714(HY), 715(HY), 716(HY), 717(HY), 718(SM)[c], 719(HY)[c], 720(AT), 721(AT), 722(SU), 723(WW), 724(WW), 725(WW), 726(WW)[c], 727(WW), 728(IR), 729(IR), 730(SU)[c], 731(SU).

JULY: 732(ST), 733(ST), 734(ST), 735(ST), 736(ST), 737(PO), 738(PO), 739(PO), 740(PO), 741(PO), 742(PO), 743(HY), 744(HY), 745(HY), 746(HY), 747(HY), 748(HY)[c], 749(SA), 750(SA), 751(SA), 752(SA)[c], 753(SM), 754(SM), 755(SM), 756(SM), 757(SM), 758(CO)[c], 759(SM)[c], 760(WW)[c].

AUGUST: 761(BD), 762(ST), 763(ST), 764(ST), 765(ST)[c], 766(CP), 767(CP), 768(CP), 769(CP), 770(CP), 771(EM), 772(EM), 773(SA), 774(EM), 775(EM), 776(EM)[c], 777(AT), 778(AT), 779(SA), 780(SA), 781(SA), 782(SA)[c], 783(HW), 784(HW), 785(CP), 786(ST).

c. Discussion of several papers, grouped by Divisions.

AMERICAN SOCIETY OF CIVIL ENGINEERS

PUBLICATIONS OF THE ENGINEERING REPRINT SERIES

Copies of the complete list of publications may be secured from the Director of the Engineering Experiment Station, University of Missouri

Reprint No.

*6. Design of Low Frequency Constant Time Delay Lines, by C. M. Wallis, Professor of Electrical Engineering. Reprinted from Transactions of the A.I.E.E., Vol.71, Part 1, p. 135, April 1952.

*7. The Engineer Becomes a Professional Manager, by Harry Rubey, Professor of Civil Engineering. Reprinted from the Journal of Engineering Education, Vol. 43, p. 338, January 1953.

8. Use of the Centrifugal Governor Mechanism as a Torsional Vibration Absorber, by O. A. Pringle, Assistant Professor of Mechanical Engineering. Reprinted from the Transactions of the A.S.M.E., Vol. 75, p. 59, January 1953.

*9. How to Plan for the Safe and Adequate Highways We Need, by Harry Rubey, Professor and Chairman of Civil Engineering. Reprinted from the General Motors "Better Highways Awards", 1953.

*10. A Dynamic Analogy of Foundation - Soil Systems, by Adrian Pauw, Associate Professor of Civil Engineering. Reprinted from Symposium on Dynamic Testing of Soils, Special Technical Publication No. 156, American Society for Testing Materials, 1953.

11. Ternary System Ethyl Alcohol--n--Heptane-Water at 30^{o}C, by Joseph L. Schweppe, Research Engineer, C. F. Braun and Co. and James R. Lorah, Associate Professor Chemical Engineering. Reprinted from Industrial and Engineering Chemistry, Vol. 26, p. 2391, November 1954.
The Rectifying Property of Polarized Barium Titanate, by Donald L. Waidelich, Associate Director, Engineering Experiment Station and Professor of Electrical Engineering. Reprinted from Journal of the Acoustical Society of America, Vol. 25, p. 796, July 1953.

12. Chip Breakers Studies 1, Design and Performance of Ground Chip Breakers, Erik K. Henriksen, Associate Professor of Mechanical Engineering

    Balanced Design Will Fit the Chip Breaker to the Job, from American Machinist, April 26, 1954, pp. 117-124, Special Report No. 360

    How to Select Chip Breakers I, II, III, from American Machinist, May 10, 1954, pp. 179, 181, 183, Reference Book Sheets

    Chip Breaking-A Study of Three-Dimensional Chip Flow, from page No. 53-5-9, presented at the A.S.M.E. Spring Meeting, Columbus, Ohio, April 28-30, 1953

    Economical Chip Breakers for Machining Steel, from Technical Aids to Small Business, May 1954, pp. 1-8

13. The Design of Sampled-Data Feedback Systems by Gladwyn V. Lago, Associate Professor of Electrical Engineering and John G. Truxal, Polytechnic Institute of Brooklyn. Reprinted from Transactions of the A.I.E.E., Vol. 73, Part 2, p. 247, 1954.

14. Selection of Personnel by George W. Elliott, Assistant Professor of Mechanical Engineering. Reprinted from the 1954 Transcript of the Midwest Feed Production School.

15. Lightweight Aggregates for Structural Concrete by Adrain Pauw, Associate Professor of Civil Engineering. Reprinted from the Preceedings of the A.S.C.E., Vol. 81, Separate No. 584, January 1955.

16. Coating Thickness Measurements Using Pulsed Eddy Currents by Donald L. Waidelich, Associate Director, Engineering Experiment Station. Reprinted from the Proceedings of the National Electronics Conference, Vol. 10, February 1955.

17. Additions to Sample-Data Theory by G. V. Lago, Associate Professor of Electrical Engineering. Reprinted from the Proceedings of the National Electronics Conference, Vol. 10, February 1955.

18. Additions to Z-Transformation Theory for Sample-Data Systems by Gladwyn V. Lago, Associate Professor of Electrical Engineering. Reprinted from Transactions of the American Institute of Electrical Engineers, Vol. 74, January, 1955.

19. Tension Control for High Strength Structural Bolts by Adrian Pauw, Professor of Civil Engineering and Leonard L. Howard, Lakeland Engineering Associates, Inc., with a discussion on the Turn-of-the-Nut Method by E. J. Ruble, Association of American Railroads. Reprinted from the Proceedings of the American Institute of Steel Construction, National Engineering Conference, April 18-19, 1955.

20. Autotransformer Betters Motor Phase Conversion by Joseph C. Hogan, Associate Professor of Electrical Engineering. Reprinted from Electrical World, Vol 144, page 120, October 17, 1955.

21. Sequence Summation Factors by Adrain Pauw, Professor of Civil Engineering. Reprinted from the Proceedings of the American Society of Civil Engineers, Vol. 81, Paper No. 763, August, 1955.

*Out of Print