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Engineering Experiment Station
Columbia, Missouri

TREE GENERATION

M. T. Jong, Research Assistant in Electrical Engineering
University of Missouri

H. C. Lau, Research Assistant in Electrical Engineering
University of Missouri

George W. Zobrist, Assistant Professor in Electrical Engineering
University of Missouri

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TREE GENERATION

An efficient method of generating all trees of a connected graph by using the concept of a cutset is presented. Any connected graph can be divided into simpler separate subgraphs by a cutset operation whose forests can be easily found. The trees of the whole graph can then be generated with ease and efficiency.

Hakimi and Green* have recently given an efficient method of generating all trees of a connected graph.† In this letter, a logical extension of theorem 3 by Hakimi and Green is considered, which allows the subgraphs to be unconnected. Then all trees of a connected graph G are generated by taking one of the subgraphs in the theorem to be a cutset.

In Fig. 1*a*, let T_1'' be the set of all trees of a connected

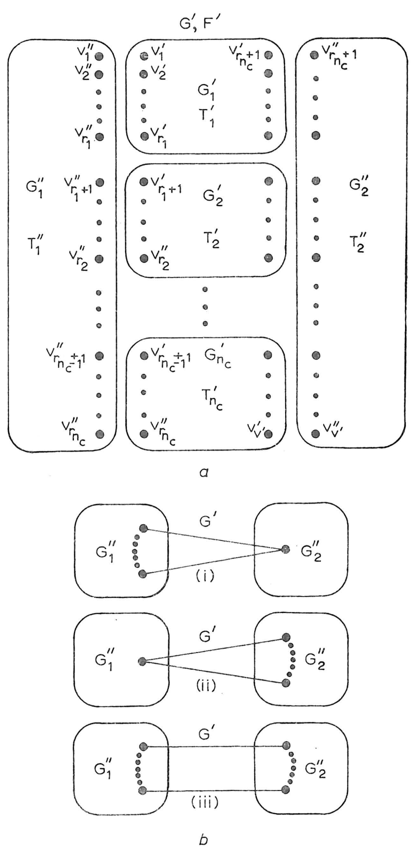

Fig. 1
a Illustration for decomposition method
b Types of paths

graph G_1'' with V_1'' vertexes. Let G' be a graph composed of n_c disjoint subgraphs, each subgraph G_i' with V_i' vertexes being connected ($i = 1, 2, \ldots n_c$). Then $G' = \bigcup_{i=1}^{n_c} G_i'$, where $\bigcap_{i=1}^{n_c} G_i' = \emptyset$. $V' = \sum_{i=1}^{n_c} V_i'$ is the number of vertexes in G'. If $n_c = 1$, G' is connected. Let T_i' be the set of all trees of G_i'; then $F' = \times_{i=1}^{n_c} T_i'$ is the set of all forests of G', where $\times$ denotes the Cartesian product. Let $G_i^* = G_{i-1}^* \cup G_i'$, $i = 1, 2, \ldots n_c$, be formed by superimposing vertexes $v''_{ri-1+1}, v''_{ri-1+2}, \ldots v''_{ri}$ in G_1'' on vertexes $v'_{ri-1+1}, v'_{ri-1+2}, \ldots v'_{ri}$ in G_i', and denote these vertexes in G_i^* by $v^*_{ri-1+1}, v^*_{ri-1+2} \ldots v^*_{ri}$, where $G_0^* \equiv G_1''$, $r_0 \equiv 0$, and $r_i - r_{i-1} \leqslant \min(V_1'', V_i')$. Then each G_i^* is connected for $i = 1, 2, \ldots n_c$.

We have

$$G^*_{n_c} = G_1'' \cup \left(\bigcup_{j=1}^{n_c} G_j' \right) = G_1'' \cup G' \equiv G^*$$

Let $p'_{ri\,1+1, r_{i-1}+2}, p'_{r_{i-1}+2, r_{i-1}+3}, \ldots p'_{ri-1, r_i}$ be paths in G_i' between vertexes $v'_{r_{i-1}+1} - v'_{r_{i-1}+2}, v'_{r_{i-1}+2} - v'_{r_{i-1}+3}, \ldots v'_{ri-1} - v'_{ri}$; and, similarly, let $p_{r_{i-1}+1, r_{i-1}+2}, p''_{r_{i-1}+2, r_{i-1}+3}, \ldots p''_{ri-1, r_i}$ be paths in G_1'' between vertexes $v''_{r_{i-1}+1} - v''_{r_{i-1}+2}, v''_{r_{i-1}+2} - v''_{r_{i-1}+3}, \ldots v''_{ri-1} - v''_{ri}$. Then $p'_{k-1,k} \cup p''_{k-1,k}$, where $r_{i-1} + 2 \leqslant k \leqslant r_i$, is a closed path containing the vertexes v^*_{k-1} and v^*_k in G^*. There are $r \equiv r_{n_c} - n_c$ such closed paths in G^*. Without loss of generality, let us denote these closed paths by $p_1, p_2, \ldots p_r$.

Lemma 1: If T^* is the set of trees of G^*,

$$T^* = \frac{\partial^r(T_1'' \times F')}{\partial p_1 \partial p_2 \ldots \partial p_r}$$

Proof: Successive applications of Hakimi and Green's theorem 3 yield

$$T_1^* = \frac{\partial^{(r_1-1)}(T_1'' \times T_1')}{\partial(p_{12}'' \cup p_{12}') \ldots \partial(p''_{r_1-1, r_1} \cup p'_{r_1-1, r_1})} = \frac{\partial^{(r_1-1)}(T_1'' \times T_1')}{\partial p_1 \ldots \partial p_{r_1-1}}$$

$$T_2^* = \frac{\partial^{(r_2-r_1)-1}(T_1^* \times T_2')}{\partial(p''_{r_1+1, r_1+2} \cup p'_{r_1+1, r_1+2}) \ldots \partial(p''_{r_2-1, r_2} \cup p'_{r_2-1, r_2})}$$

$$= \frac{\partial^{(r_2-r_1)-1}}{\partial p_{r_1} \partial p_{r_1+1} \ldots \partial p_{r_2-2}} \left[\frac{\partial^{r_1-1}(T_1'' \times T_1')}{\partial p_1 \ldots \partial p_{r_1-1}} \times T_2' \right]$$

$$= \frac{\partial^{(r_2-2)}(T_1'' \times T_1' \times T_2')}{\partial p_1 \partial p_2 \ldots \partial p_{r_2-2}}$$

$$T^* \equiv T^*_{n_c} = \frac{\partial^{(r_{n_c}-n_c)}(T_1'' \times T_1' \times T_2' \times \ldots \times T'_{n_c})}{\partial p_1 \partial p_2 \ldots \partial p_{r_{n_c}-n_c}} = \frac{\partial^r(T_1'' \times F')^2}{\partial p_1 \partial p_2 \ldots \partial p_r}$$

where T_i^* is the set of trees of G_i^* ($i = 1, 2, \ldots n_c$).

Let T_2'' be the set of trees of a connected graph G_2'' with V_2'' vertexes, where $G_2'' \cap G_1' = \emptyset$, $V_2'' \geqslant V' - r_{n_c}$. Then $F'' = T_1'' \times T_2''$ is the set of all forests of the graph $G'' \equiv G_1'' \cup G_2''$. Let $G \equiv G^* \cup G_2''$ be formed by superimposing the $(s + 1)$ $(\equiv V' - r_{n_c})$ 'remaining' vertexes in G' on some $(s + 1)$ vertexes in G_2''; then G is connected. Let $p_{r+1}, p_{r+2}, \ldots p_{r+s}$ be closed paths formed by paths in G_2'' and G^* between these $(s + 1)$ vertexes.

Lemma 2: The set of trees of G is given by

$$T = \frac{\partial^k(F' \times F'')}{\partial p_1 \partial p_2 \ldots \partial p_k}$$

where $k = r + s$.

Proof: Using Hakimi and Green's theorem 3, and lemma 1 in this letter, we have‡

$$T = \frac{\partial^s(T^* \times T_2'')}{\partial p_{r+1} \partial p_{r+2} \ldots \partial p_{r+s}}$$

$$= \frac{\partial^s}{\partial p_{r+1} \partial p_{r+2} \ldots \partial p_{r+s}} \left[\frac{\partial^r(T_1'' \times F')}{\partial p_1 \partial p_2 \ldots \partial p_r} \times T_2'' \right]$$

$$= \frac{\partial^{(r+s)}(T_1'' \times F' \times T_2'')}{\partial p_1 \partial p_2 \ldots \partial p_{r+s}} = \frac{\partial^k(F' \times F'')}{\partial p_1 \partial p_2 \ldots \partial p_k}$$

Let G be a connected graph. Let G' be a cutset subgraph and G'' be the corresponding cutset remainder (the cutset

* HAKIMI, S. L., and GREEN, D. G.: 'Generation and realization of trees and k-trees', *IEEE Trans.*, 1964, CT-11, p. 247

† Throughout this letter, by a graph we mean a labelled linear graph.

‡ If H_1 and H_2 are two sets of subgraphs such that no subgraph in H_1 contains an element of a subgraph in H_2, and if g_1 is a subgraph such that no subgraph in H_2 contains an element of g_1, it can be shown that

$$\frac{\partial(H_1 \times H_2)}{\partial g_1} = \frac{\partial H_1}{\partial g_1} \times H_2$$

remainder is the complement of the cutset subgraph in G). Let G_1'' and G_2'' be the two separate parts of G''. Then $G'' = G_1'' \cup G_2''$, $G_1'' \cap G_2'' = \emptyset$, and each of G_1'' and G_2'' is connected. G' may not be connected, but $G' \cup G_1''$ is always connected. Let $p_1, p_2, \ldots p_k$ be the closed paths formed by the following rules:

(i) Closed paths are formed with edges from the cutset and the cutset remainder.
(ii) The total number of closed paths is equal to the number of forest branches of the cutset minus one, i.e. $k = b_c - 1$, where b_c is the number of forest branches of the cutset.
(iii) All vertexes of the cutset subgraph should be covered by the closed paths.

The closed paths fall into three categories, as shown in Fig. 1*b*.

Then, from lemma 2, we have the following:

Theorem: The set of trees of G is

$$T = \frac{\partial^k (F' \times F'')}{\partial p_1 \partial p_2 \ldots \partial p_k}$$

Example: Consider the graph of Fig. 2*a*. The cutset G' and

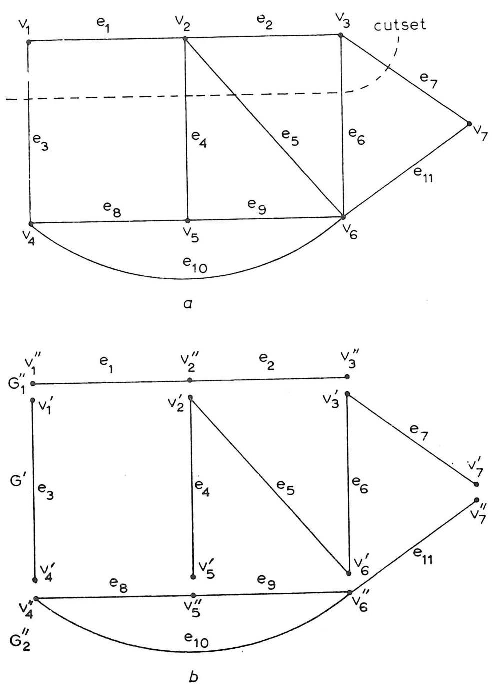

Fig. 2
a Labelled graph
b Labelled graph decomposed by cutset

cutset remainder $G'' = G_1'' \cup G_2''$ are shown in Fig. 2*b*. G' has one forest

$$F' = \{e_3 e_4 e_5 e_6 e_7\}$$

and the set of forests of G'' is

$$F'' = \{e_1 e_2 e_8 e_9 e_{11},\ e_1 e_2 e_8 e_{10} e_{11},\ e_1 e_2 e_9 e_{10} e_{11}\}$$

The closed path in $G' \cup G_1''$ is

$$p_1 = e_2 e_5 e_6$$

and closed paths in $(G' \cup G_1'') \cup G_2''$ are

$$p_2 = e_1 e_3 e_4 e_8,\ p_3 = e_4 e_5 e_9,\ p_4 = e_6 e_7 e_{11}$$

According to the theorem, the set of all trees of G is given by

$$T = \frac{\partial^4 (F' \times F'')}{\partial(e_2 e_5 e_6)\partial(e_1 e_3 e_4 e_8)\partial(e_4 e_5 e_9)\partial(e_6 e_7 e_{11})}$$

$$= \frac{\partial^4 F'}{\partial(e_3 e_4)\partial(e_4 e_5)\partial(e_5 e_6)\partial(e_6 e_7)} \times F''$$

$$\cup \left[\frac{\partial^3 F'}{\partial(e_3 e_4)\partial(e_4 e_5)\partial(e_6 e_7)} \times \frac{\partial F''}{\partial e_2} \oplus \frac{\partial^3 F'}{\partial(e_4 e_5)\partial(e_5 e_6)\partial(e_6 e_7)} \times \frac{\partial F''}{\partial(e_1 e_8)} \oplus \frac{\partial^3 F'}{\partial(e_3 e_4)\partial(e_4 e_5)\partial(e_5 e_6)} \times \frac{\partial F''}{\partial e_{11}} \oplus \frac{\partial^3 F'}{\partial(e_3 e_4)\partial(e_5 e_6)\partial(e_6 e_7)} \times \frac{\partial F''}{\partial e_9}\right]$$

$$\cup \left[\frac{\partial^2 F'}{\partial(e_4 e_5)\partial(e_6 e_7)} \times \frac{\partial^2 F''}{\partial(e_1 e_8)\partial e_2} \oplus \frac{\partial^2 F'}{\partial(e_3 e_4)\partial(e_6 e_7)} \times \frac{\partial^2 F''}{\partial e_2 \partial e_9} \oplus \frac{\partial^2 F'}{\partial(e_3 e_4)\partial(e_4 e_5)} \times \frac{\partial^2 F''}{\partial e_2 \partial e_{11}} \oplus \frac{\partial^2 F'}{\partial(e_5 e_6)\partial(e_6 e_7)} \times \frac{\partial^2 F''}{\partial(e_1 e_8)\partial e_9} \oplus \frac{\partial^2 F'}{\partial(e_4 e_5)\partial(e_5 e_6)} \times \frac{\partial^2 F''}{\partial(e_1 e_8)\partial e_{11}} \oplus \frac{\partial^2 F'}{\partial(e_3 e_4)\partial(e_5 e_6)} \times \frac{\partial^2 F''}{\partial e_9 \partial e_{11}}\right]$$

$$\cup \left[\frac{\partial F'}{\partial(e_6 e_7)} \times \frac{\partial^3 F''}{\partial e_2 \partial(e_1 e_8)\partial e_9} \oplus \frac{\partial F'}{\partial(e_5 e_6)} \times \frac{\partial^3 F''}{\partial(e_1 e_8)\partial e_9 \partial e_{11}} \oplus \frac{\partial F'}{\partial(e_4 e_5)} \times \frac{\partial^3 F''}{\partial(e_1 e_8)\partial e_2 \partial e_{11}}\right]$$

$$\cup\ F' \times \frac{\partial^4 F''}{\partial e_2 \partial(e_1 e_8)\partial e_9 \partial e_{11}}$$

If the operation is carried out, the 159 trees of the graph will be obtained.

18th July 1966

M. T. JONG
H. C. LAU
G. W. ZOBRIST

Department of Electrical Engineering
University of Missouri
Columbia, Miss., USA

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