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ADDITIONS TO Z-TRANSFORMATION THEORY FOR SAMPLE-DATA SYSTEMS

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Additions to z-Transformation Theory for Sampled-Data Systems

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MEMBER AIEE

THERE is one important class of feedback control systems which until recently has received but limited attention. The systems in this class are known as sampled-data systems. The distinguishing feature of this class of systems is that at one or more points in the system, signal information has the form of a train of pulses.

One of the most powerful mathematical methods for analysis of sampled-data systems is known as the z-transformation method. Ragazzini and Zadeh[1] give an excellent summary of this method. The present z-transformation theory is in its infancy and, as additions to this theory are developed, it will become an even more powerful tool for analysis and design of sampled-data systems. The purpose of this paper is to present a number of additions to the present z-transformation theory in the hope that they will add to the usefulness of the z-transformation method.

Because this paper presents a number of related topics rather than a single development, each of these topics will be discussed briefly before a more detailed analysis is presented. After a brief review of the z-transformation theory, a situation is discussed wherein the present z-transformation theory can lead to erroneous results. By use of the method described in the section entitled "Suggested Notation," this situation can be avoided.

The final and initial value theorems are obvious additions to z-transformation theory and are presented for use in later developments.

One of the major criticisms of z-transformation theory is that the behavior of the system between sampling instants is unknown. This uncertainty can be removed by a number of rather tedious devices. When these devices are used, the basic simplicity of the z-transformation method is lost. This paper outlines a

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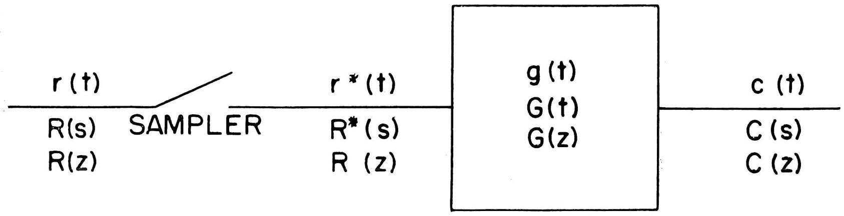

Fig. 1. Sampler followed by a network

method whereby the output at any point in the sampling period can be found by using a technique which preserves this basic simplicity.

This paper also outlines a technique for finding the pole-zero configuration of a continuous system coincides with a sampled-date system at the sampling instants. The current literature contains much material concerning the step-function response of continuous systems having various pole-zero configurations. It would be possible to duplicate this material for sampled-data systems in terms of z-transformation theory. However, the technique presented here eliminates this need and permits the designer already familiar with continuous systems to use his knowledge in determining the response of a sampled-data system.

The paper concludes by developing the equations for error coefficients in terms of the z-transforms. These equations play the same role in sampled-data theory as the conventional equations for error coefficients do in continuous systems.

Brief Review of z-Transformation Theory

The sampler is the component which distinguishes a sampled-data system from a continuous system. The input signal $r(t)$ to the sampler of Fig. 1 is continuous, whereas the output $r^*(t)$ is a train of pulses with a width T_1 seconds and a height proportional to the amplitude of $r(t)$ at the sampling instants. It is convenient mathematically to approximate this train of pulses by a corresponding train of impulses weighted so that the area of an impulse equals the pulse amplitude.[1] $R^*(s)$, the Laplace transform of $r^*(t)$, is given by two equivalent forms[2]

$$R^*(s)=\sum_{n=0}^{\infty} r(nt)\epsilon^{-nTs} \tag{1}$$

and

$$R^*(s)=\frac{1}{T}\sum_{n=-\infty}^{+\infty} R(s+jn\omega_s)+\frac{r(0)}{2} \tag{2}$$

where

$R(s)$ = the Laplace transform of $r(t)$
T = sampling period
$\omega_s=2\pi/T$

$C^*(s)$, the Laplace transform of the output $c(t)$ of the network at the sampling instants, can be expressed as

$$C^*(s)=R^*(s)G^*(s)$$
$$=\frac{1}{T}\sum_{n=-\infty}^{+\infty} C(s+jn\omega_s)+\frac{c(0)}{2} \tag{3}$$

where $G^*(s)$ is defined as

$$G^*(s)=\frac{1}{T}\sum_{n=-\infty}^{+\infty} G(s+jn\omega_s)+\frac{g(0)}{2} \tag{4}$$

Here, $C(s)$ and $G(s)$ are the Laplace transforms of $c(t)$ and $g(t)$ respectively

$R(z)$, the z-transform of $r(t)$, is obtained by substituting z for ϵ^{sT} in equation 1 as

$$R(z)=\sum_{n-0}^{\infty} r(nT)\frac{1}{z^n} \tag{5}$$

The z-transform of $r(t)=\epsilon^{-at}$ is taken to demonstrate the technique. The equivalent form to equation 1 is

$$R^*(s)=1+\epsilon^{-aT}\epsilon^{-1T}+\epsilon^{-2aT}\epsilon^{-2sT}+\ldots \tag{6}$$

which can be written in closed form as

$$R^*(s)=\frac{\epsilon^{sT}}{\epsilon^{sT}-\epsilon^{-aT}} \tag{7}$$

With the change in variables $z=\epsilon^{sT}$, equation 7 becomes

$$R(z)=\frac{z}{z-\epsilon^{-aT}} \tag{8}$$

A short list of z-transforms is given in Table I.

Suggested Notation

The usual interpretation placed upon equation 4 is that $G(z)$ is taken from $g(t)$ in exactly the same manner as $R(z)$ is taken from $r(t)$. This means that $g(t)$ is first represented by a train of impulses at the sampling instants and then the usual steps are performed on the train of impulses to obtain the corresponding z-transform. As can be seen by the example in Appendix, this interpretation breaks down when $g(t)$ contains an impulse. This difficulty arises because the mathematics cannot distinguish between the impulse used to represent an amplitude and the true impulse.

Another interpretation of equation 4 is possible. The signal $r(t)$ becomes a train of impulses $r^*(t)$ as a result of the action of the sampler and not as a result of the fact that the z-transformation is taken. Since $g(t)$ is continuous (except for a possible impulse at $t=0$) and is not sampled, it does not follow that it must be converted to a train of impulses before taking the z-transform. One possible solution would be to let the points at the sampling instants, Fig. 2(B), be used to represent the $g(t)$ curve; see Fig. 2(A). However, since it is impossible to take the Laplace transform of a set of points, this cannot be done. If each point is held for a short

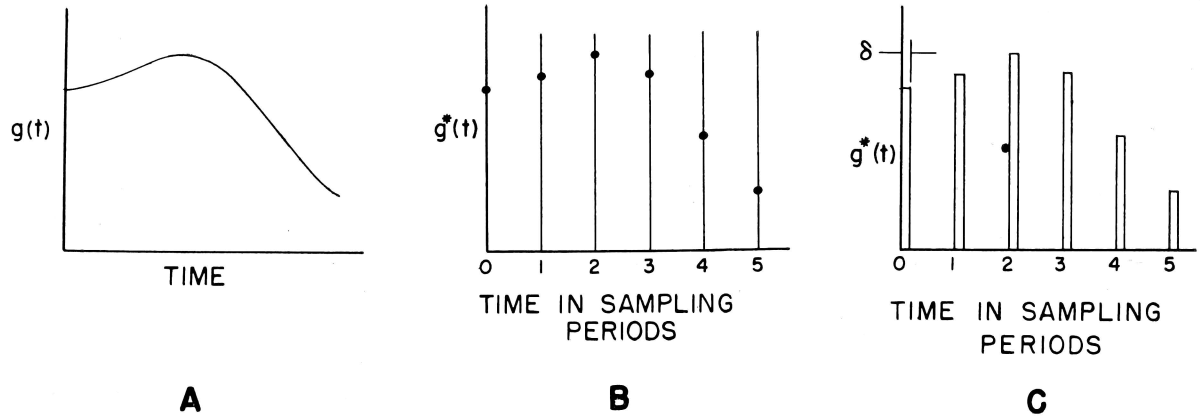

Fig. 2. Graphical depiction of g(t) and how g(t) can be represented by a train of points or pulses

Table I. Short List of z-Transforms

Time Function	Laplace Transform	z-Transform
$u_0(t)$	1	1
$u_{-1}(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
ϵ^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-\epsilon^{-aT}}$

time, δ seconds, Fig. 2(C), the Laplace transformation can be taken. A new table of z-transforms can be derived based upon this interpretation which is demonstrated with the example $g(t)=\epsilon^{-at}$. $G^*(s)$ can be written as

$$G^*(s)=1\left(\frac{1-\epsilon^{-\delta s}}{s}\right)+\epsilon^{-aT}\epsilon^{-sT}\left(\frac{1-\epsilon^{-\delta s}}{s}\right)+\epsilon^{-2aT}\epsilon^{-2sT}\left(\frac{1-\epsilon^{-\delta s}}{s}\right)+\ldots \quad (9)$$

The term in the parentheses can be factored out, the infinite series term can be summed, and with the change of variable $z=\epsilon^{sT}$ equation 9 can be written as

$$G(z)=\frac{z}{z-\epsilon^{-aT}}V \quad (10)$$

where

$$\left.\frac{1-\epsilon^{-\delta s}}{s}\right|_{s=\frac{1}{T}\ln q}=V \quad (11)$$

The symbol V can be thought of as an operator signifying that $G(z)$ represents values instead of impulses. It can be seen that the new table of z-transforms is exactly the same as Table I with the exception that each term is multiplied by V. The Appendix contains an example demonstrating the use of this notation. The only situation where this notation needs to be used is that in which the numerator and denominator of $G(s)$ are of the same degree in s. In the remainder of this paper it is assumed that the numerator of $G(s)$ is of lower degree than the denominator.

Final and Initial Value Theorems

Any z-transform of the output of a system $C(z)$ can be expanded in partial fractions as

$$C(z)=\frac{A_0z}{z-1}+\frac{A_1z}{z-\epsilon^{-a_1T}}+\frac{A_2z}{z-\epsilon^{-a_2T}}+\ldots \quad (12)$$

The inverse transform into the time domain is

$$c(t)=A_0+A_1\epsilon^{-a_1T}+A_2\epsilon^{-a_2T}+\ldots \quad (13)$$

If the Laplace transform of equation 13 has no poles in the right half of the s-plane or on the $j\omega$ axis, all terms of equation 13 go to zero as t goes to infinity with the exception of the A_0 term. Therefore, under these conditions (these same conditions apply to the final value theorem in the Laplace transformation theory) the final value of $c(t)$ can be found directly from $C(z)$ as

$$\lim_{z\to 1}\frac{(z-1)}{z}C(z)=\lim_{t\to\infty}c(t) \quad (14)$$

The initial value theorem can be derived by considering that $C(z)$ can be expanded in an infinite series in descending powers of z as

$$C(z)=B_0+\frac{B_1}{z}+\frac{B_2}{z^2}+\ldots \quad (15)$$

where

B_0 = value of $c(t)$ at $t=0$
B_1 = value of $c(t)$ at $t=T$, etc.

B_0 can be found as

$$\lim_{z\to\infty}C(z)=\lim_{t\to 0}c(t) \quad (16)$$

If the initial value of $c(t)$ is zero, an extension of the theorem is possible to give the value of $c(t)$ at $t=T$ in the form

$$\lim_{z\to\infty}(z)C(z)=c(t) \text{ at } t=T \quad (17)$$

The Output at Any Point in the Sampling Period

In the past, z-transformation theory has suffered from the uncertainty surrounding the behavior of the system between sampling instants. This uncertainty is partially removed by the procedure of finding the output at a submultiple of the sampling period.[3] The technique to be presented can be used to find the output at any time, Tx seconds, after the sampling instants. The system to be considered is that of Fig. 3(A) while Fig. 3(B) shows the hypothetical circuit after the artifices to be used in this technique are added. The signal $e^*(t)$ is a train of impulses at the sampling instants. $E(z)$ contains all the information that is contained in $e^*(t)$. However, all information except that at the sampling instants is discarded when $G(z)$ is obtained from $g(t)$. When $E(z)$ and $G(z)$ are multiplied together in the z domain, the corresponding occurrence in the time domain is the convolution of the train of impulses of $E(z)$ with the train of impulses (or values) of $G(z)$. Consequently, the same type of operation can be performed by obtaining a new function $G_{Tx}(z)$ from values of $g(t)$ at a set of points that occur T_x seconds after the sampling instants. The example $g(t)=\epsilon^{-aT}$ is used to demonstrate this method. The infinite series form for $G_{Tx}{}^*s$ is

$$G_{Tx}{}^*s=\epsilon^{-aTx}+\epsilon^{-a(T+Tx)}\epsilon^{-sT}+\epsilon^{-a(2T+Tx)}\epsilon^{-2sT}+\ldots \quad (18)$$

When the ϵ^{-aTx} term is factored out, the infinite series term can be summed and, with the change of variable $z=\epsilon^{sT}$, equation 18 can be written as

Table II. Short List of Modified z-Transforms

$G(z)$	$G_{Tx}(z)$
$\frac{z}{z-1}$	$\frac{z}{z-1}$
$\frac{Tz}{(z-1)^2}$	$\frac{Txz}{z-1}+\frac{Tz}{(z-1)^2}$
$\frac{z}{z-\epsilon^{-aT}}$	$\frac{\epsilon^{-aTx}z}{z-\epsilon^{-aT}}$

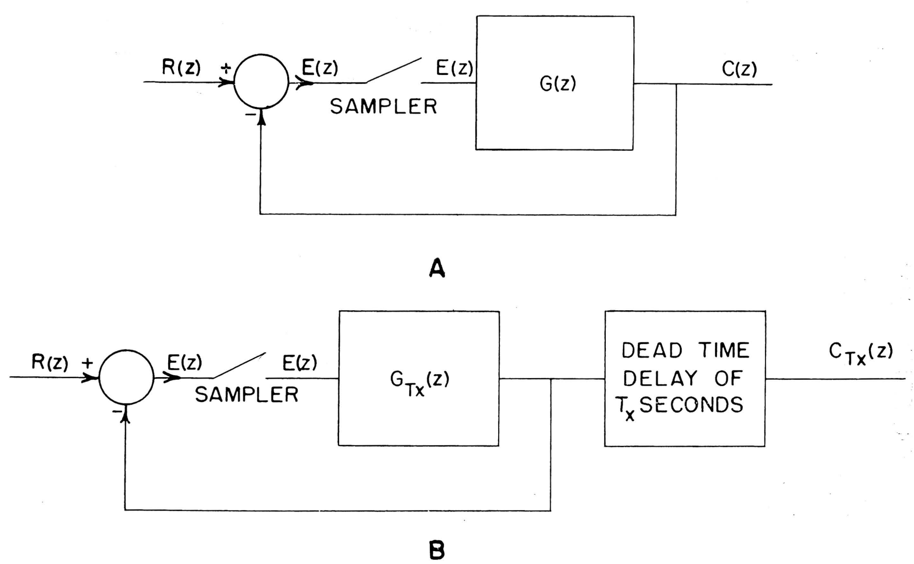

Fig. 3. Sampled-data system and hypothetical system used to find output at any time

$$G_{Tx}(z) = \frac{\epsilon^{-aTx} z}{z - \epsilon^{-aT}} \tag{19}$$

A short list of z-transforms and the corresponding modified z-transforms are given in Table II.

The example to be considered to demonstrate this technique is the circuit of Fig. 3(B) with

$$G(s) = \frac{13.4}{s(s+10)} = 1.34\left(\frac{1}{s} - \frac{1}{s+10}\right); \quad T = 0.1 \text{ second} \tag{20}$$

$G(z)$ is found as

$$G(z) = 1.34\left(\frac{z}{z-1} - \frac{z}{z-\epsilon^{-1}}\right) = \frac{0.846z}{z^2 - 1.368z + 0.368} \tag{21}$$

When the input $r(t)$ is a unit step function, $E(z)$ is found as

$$E(z) = \frac{R(z)}{1+G(z)} = \frac{z^2 - 0.368z}{z^2 - 0.522z + 0.368} \tag{22}$$

The usual z-transform for the output is given by

$$C(z) = E(z)G(z) = \frac{z}{(z-1)} \frac{0.846z}{(z^2 - 0.522z + 0.368)} \tag{23}$$

The first few terms of the solution are placed in evidence by dividing the denominator into the numerator as

$$C(z) = \frac{0.846}{z} + \frac{1.29}{z^2} + \frac{1.20}{z^3} + \frac{1.01}{z^4} + \ldots \tag{24}$$

The modified z-transform $G_{Tx}(Z)$ is taken term by term from $G(z)$ by use of the transform pairs of Table II. When this is done with $T_x = 0.02$ second, the result is

$$G_{0.02}(z) = 1.34\left(\frac{z}{z-1} - \frac{\epsilon^{-0.2} z}{z - \epsilon^{-1}}\right) = \frac{0.242z^2 + 0.605z}{(z-1)(z-0.368)} \tag{25}$$

The z-transform for the output 0.02 second after the sampling instants is

$$C_{0.02}(z) = E(z)G_{0.02}(z) = \frac{0.242z^3 + 0.605z^2}{z^3 - 1.522z^2 + 0.890z - 0.368} \tag{26}$$

The first few terms of the solution are placed in evidence by dividing the denominator into the numerator as

$$C_{0.02}(z) = 0.242 + \frac{0.973}{z} + \frac{1.265}{z^2} + \frac{1.149}{z^3} + \ldots \tag{27}$$

These steps are repeated with $Tx = 0.06$ second. $G_{0.06}(z)$ is

$$G_{0.06}(z) = 1.34\left(\frac{z}{z-1} - \frac{\epsilon^{-0.6} z}{z - \epsilon^{-1}}\right) = \frac{0.605z^2 + 0.243z}{(z-1)(z-0.368)} \tag{28}$$

$C_{0.06}(z)$ becomes

$$C_{0.06}(z) = \frac{0.605z^3 + 0.243z^2}{z^3 - 1.522z^2 + 0.890z - 0.368} \tag{29}$$

which when expanded is

$$C_{0.06}(z) = 0.605 + \frac{1.165}{z} + \frac{1.237}{z^2} + \frac{1.063}{z^3} + \ldots \tag{30}$$

The $C_{0.02}(z)$ points are shown in Fig. 4. by x-marks and the $C_{0.06}(z)$ points by dots. The curve of Fig. 4 is obtained by using the impulse-response approach.[3]

By the technique outlined in the preceding paragraphs, as many points as desired can be found between sampling periods. The denominator of $C_{Tx}(z)$ remains the same; however, for each value of Tx a new numerator must be found.

Continuous Systems which Coincide with Sampled-Data Systems at the Sampling Instants

The inverse z-transform of a function can be found by an infinite series expansion in descending powers of z or by a partial fraction expansion with the corresponding time functions identified by use of Table I. This latter type of expansion, when combined with methods explained in this section, adds a great deal to the designer's circumspection of the response of sampled-data systems. This method is demonstrated with an example using the circuit of Fig. 3(A) in which $G(s)$ and T are given by equation 20. When a unit step function is applied to the input, the $C(z)$ for this system is given by equation 23 and the denominator of this equation can be factored as

$$C(z) = \frac{0.846z^2}{(z-1)(z-0.261+j0.547)(z-0.261-j0.547)} \tag{31}$$

$[C(z)]/z$ can be expanded in partial fractions and the result multiplied by z to obtain $C(z)$ as

$$C(z) = \frac{z}{z-1} - \frac{0.51\underline{/-11^\circ} z}{z-0.261+j0.547} - \frac{0.51\underline{/+11^\circ} z}{z-0.261+j0.547} \tag{32}$$

The time function $c(t)$ corresponding to $C(z)$ can be found by use of Table I as

$$c(t) = 1 - 0.51\underline{/-11^\circ}\epsilon^{-(5+j11.3)t} - 0.51\underline{/+11^\circ}\epsilon^{-(5-j11.3)t} \tag{33}$$

Equation 33 can be written as

$$c(t) = 1 - \epsilon^{-5t}(\cos 11.3t - 0.194 \sin 11.3t) \tag{34}$$

The $c(t)$ of equation 34 can be plotted but it gives the correct output for the sampled-data system at the sampling instants only.

A study of Table I reveals that each z-transform not only has a function of time associated with it but also a Laplace transform. Actually there are an infinite number of Laplace transforms corresponding to each z-transform because the transformation $z = \epsilon^{sT}$ is multivalued in going from the z-plane to the s-plane.

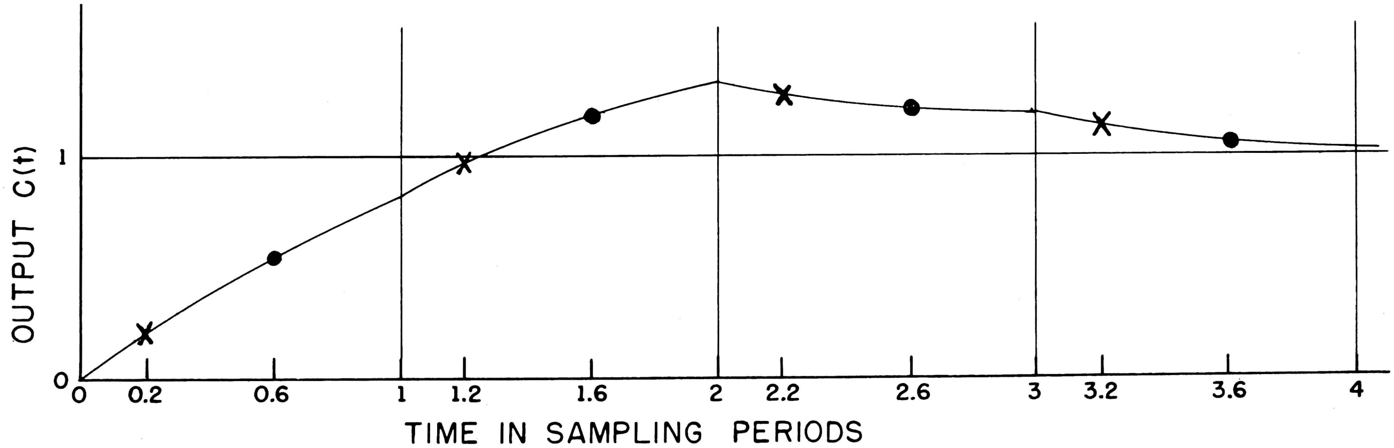

Fig. 4. Response of the system of Fig. 3 showing points for Tx = 0.2T and Tx = 0.6T

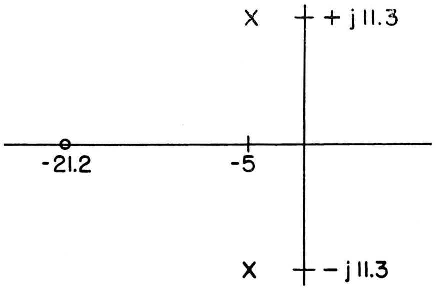

Fig. 5 (left). Pole-zero configuration of equation 37

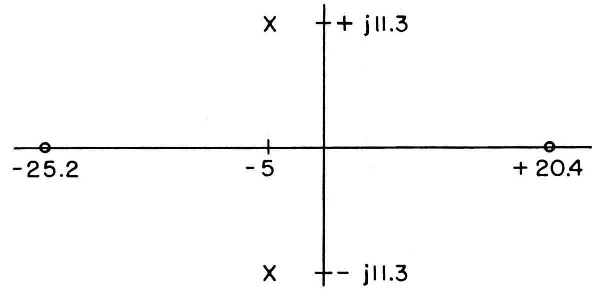

Fig. 6 (right). Pole-zero configuration of equation 38

The convenient Laplace transform to use is the one containing its poles in the primary strip in the s-plane. The Laplace transform associated with equation 32 is

$$C(s)=\frac{1}{s}-\frac{0.51\underline{/-11^\circ}}{s+5+j11.3}-\frac{0.51\underline{/+11^\circ}}{s+5-j11.3} \tag{35}$$

which is, of course, the Laplace transform of equation 33. In other words, equation 35 is the Laplace transform of a continuous system having an output which takes on the same set of values at the sampling instants as does the output of the corresponding sampled-data system. After terms are combined, equation 35 becomes

$$C(s)=\frac{7.2(s+21.2)}{s(s+5+j11.3)(s+5-j11.3)} \tag{36}$$

The pole-zero configuration of the continuous system is placed in evidence as

$$\frac{C(s)}{R(s)}=\frac{7.2(s+21.2)}{(s+5+j11.3)(s+5-j11.3)} \tag{37}$$

and is shown in Fig. 5.

This technique enables the designer of sampled-data systems to use all the experience he has gained and all the material now available in the literature concerning the step-function response of continuous systems having various pole-zero configurations. This technique also eliminates the need for duplicating this work for sampled-data systems.

If a unit ramp function is applied to the input of this system with $t=0$ coinciding with a sampling instant, the same procedure can be followed in finding the pole-zero configuration for the continuous system which coincides with the sampled-data system at the sampling instants. The pole-zero configuration is placed in evidence as

$$\frac{C(s)}{R(s)}=\frac{-0.296(s+25.2)(s-20.4)}{(s+5+j11.3)(s+5-j11.3)} \tag{38}$$

and is shown in Fig. 6.

These two examples point out several facts. No matter what input is applied to a sampled-data system, there exists a continuous system which coincides with the sampled-data system at the sampling instants. However, there is a different continuous system for each input. Each of these continuous systems has the same poles but different zeros.

Error Coefficients in Terms of z-Transforms

For the single-loop unity feedback sampled-data system of Fig. 7(A), the error coefficients can be defined as

$$K_v=\frac{1}{\text{steady-state error with } r(t)=t} \tag{39}$$

and

$$K_a=\frac{1}{\text{steady-state error with } r(t)=\frac{1}{2}t^2} \tag{40}$$

where

K_v = velocity error constant and
K_a = acceleration error constant

When $r(t)=t$, $R(z)=(Tz)/(z-1)^2$ and $E(z)$ is given by

$$E(z)=\frac{R(z)}{1+G(z)}=\frac{Tz}{(z-1)^2[1+G(z)]} \tag{41}$$

The final value theorem can be used to find the steady-state error as

$$e_{ss}=\lim_{z\to\infty}\frac{T}{(z-1)[1+G(z)]}=\frac{T}{\lim z\to1\,[(z-1)G(z)]} \tag{42}$$

K_v is therefore

$$K_q=\frac{1}{e_{ss}}=\frac{1}{T}\lim_{z\to1}[(z-1)G(z)] \tag{43}$$

In a similar manner, K_a can be shown to be

$$K_a=\frac{1}{T^2}\lim_{z\to1}[(z-1)^2G(z)] \tag{44}$$

The error coefficients for the sampled-data system of Fig. 7(A) can be compared with the error coefficients for the continuous system of Fig. 7(B) where the two $G(s)$ functions are the same. It should be noted that $G(z)$ and $G(s)$ are the z-transform and Laplace transform of the same $g(t)$. It should also be noted that

$$\lim_{z\to1}(z-1)G(z)=g(0) \tag{45}$$

because this is simply the final value theorem being applied to $G(z)$. The

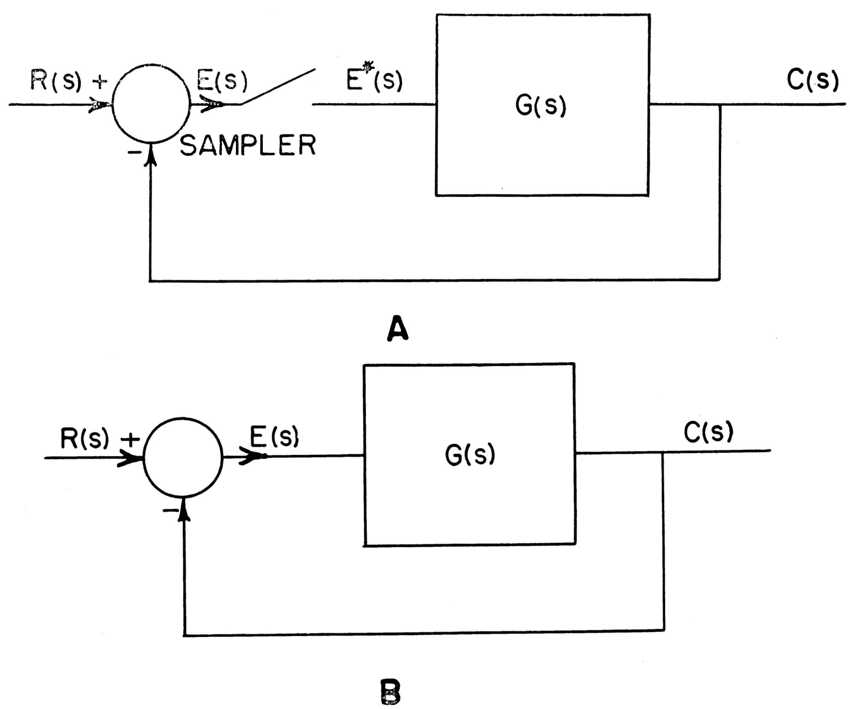

Fig. 7. Sampled-data system and continuous system containing same G(s) function

velocity constant for the continuous system is denoted by K_v^C and is found as

$$K_v^C = \lim_{s \to 0} sG(s) = g(0) \tag{46}$$

because this is the final value theorem being applied to $G(s)$. Therefore, the following relationship exists between K_v^C and K_v

$$K_v = \frac{1}{T} K_v^C \tag{47}$$

By a similar development it can be shown that

$$K_a = \frac{1}{T} K_a^C \tag{48}$$

Without critical examination, equations 47 and 48 seem to indicate that, if a sampler with a sampling period T less than one is added to a continuous system, the error constants are increased resulting in an improved system response. The fallacy in this reasoning results from failure to recognize the fact that K_v^C and K_a^C of equations 47 and 48 respectively are not the K_v^C and K_a^C that would be used in a continuous system to meet the same specifications such as maximum overshoot for which the sampled-data system is designed. It can be shown that for most purposes the response of a sampled-data system is less satisfactory than the response of a continuous system designed to meet the same maximum overshoot requirement.

Conclusions

The z-transformation theory as applied to sampled-data systems is still in its infancy. The techniques presented in this paper add to the designer's circumspection of sampled-data systems.

Appendix. Example Using Notation of the Paper

Given the circuit of Fig. 1 with

$$G(s) = \frac{s}{s+1} = 1 - \frac{1}{(s+1)} \tag{49}$$

The z-transformation will first be taken improperly to dramatize this situation. The z-transform of equation 49 is

$$G(z) = 1 - \frac{z}{z - \epsilon^{-T}} = -\frac{\epsilon^{-T}}{z - \epsilon^{-T}} \tag{50}$$

If $r(t)$ is a unit step function, $C(z)$ is

$$C(z) = R(z)G(z) = -\frac{z\epsilon^{-T}}{(z-1)(z-\epsilon^{-T})} \tag{51}$$

When equation 51 is expanded, the first three terms of the solution are placed in evidence as

$$C(z) = 0 - \frac{\epsilon^{-T}}{z} - \frac{\epsilon^{-T}(1-\epsilon^{-T})}{z^2} - \ldots \tag{52}$$

From physical reasoning it can be seen that this result is incorrect.

The example is again repeated using the notation suggested in this paper in which case the z-transform of equation 49 is

$$G(z) = 1 - \frac{z}{z - \epsilon^{-T}} V \tag{53}$$

and the z-transform for the output is

$$C(z) = R(z)G(z) = \frac{z}{z-1} - \frac{z^2}{(z-1)(z-\epsilon^{-T})} V \tag{54}$$

The first term is not multiplied by V and therefore signifies impulses while the second term is multiplied by V and therefore signifies values.

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