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A FEW REMARKS CONCERNING THE FLOW OF FLUID UNDER A SLUICE GATE

by

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ABSTRACT

The two-dimensional, irrotatational flow of incompressible fluid under a sluice gate in a gravitational field has been treated by de Boor (Ref. 1) using the integral equation method of Birkhoff and Zarantonello (Ref. 2). However, de Boor was not able to satisfactorily satisfy the constant pressure condition on the free streamline in the neighborhood of the point at infinity. He established (using certain hypotheses) that this difficulty occured because the singularity in the velocity potential at this point did not have a continuous derivative. The author has determined the apparent form of this singularity. By removing part of it, he has obtained solutions which appear to be substantially more accurate than those obtained by deBoor.

PROBLEM DEFINITION

The sluice gate model has an infinite reservoir without an upstream free surface. The coefficient of contraction $C_c=b/a$[1] will be a function of the Froude number $F^2=U^2/gb$ (see Fig. 1). Gravity acts in the negative y direction.

A complex velocity potential $\chi(z)$ is sought which is analytic and possesses a nonvanishing derivative $\frac{d\chi}{dz}$ in the domain of the flow field. Moreover, $\frac{d\chi}{dz}$ is to be continuous and nonvanishing of the boundary except at the boundary point (1) at infinity. Lastly, χ is to be univalent in the domain of the flow field. If these conditions are satisfied then the complex velocity

$$\zeta(z) = \frac{1}{U}(u-iv)$$

which is related to χ by the expression

$$\zeta = \frac{1}{U}\frac{d\chi}{dz} \qquad (1)$$

will be analytic in the domain of the flow field.

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1 Symbols are either defined when first encountered or may be found in appendix D.

SOLUTION

A necessary step in the solution is the mapping (in principle) of the domain $\Gamma = \left(|t|<1, \mathrm{Im}(t)>0\right)$ in the complex t plane onto the domain of the flow field. This mapping function $z = f(t)$ will be unique if three points on the boundary are specified. Hence, it is permissible to require the mapping of point (1) into $t = 0$, point (2) into $t = -1$, and point (3) into $t = 1$. Moreover, $f(t)$ exists and is continuous for $t \in (\overline{\Gamma}, t \neq o, 1)$ if χ exists. To see this we first note that $\chi(z)$ possesses an inverse $z(\chi)$ because it is univalent. Provided we take $\chi(ai) = Qi$, this inverse will map the strip $0<\mathrm{Im}(\chi)<Q$ onto the domain of the flow field. The upper half of the T plane is mapped onto the strip $0<\mathrm{Im}(\chi)<Q$ by the function

$$\chi = -\frac{Q}{\pi}\log(Te^{-\pi i}) \tag{2}$$

and in turn the domain Γ in the t plane is mapped onto the upper half of the T plane by the function

$$T = -\frac{(1-t)^2}{4t} \tag{3}$$

Hence, we conclude, since $f(t)$ is unique, that

$$f(t) = z\left\{\frac{-Q}{\pi}\log\left(\frac{(1-t)^2}{4t}\right)\right\} \tag{4}$$

and thus establish the existence of $f(t)$. The fact that $f(t)$ is continuous for $t \in (\overline{\Gamma}, t \neq 0, 1)$ is apparent when we note that $z(\chi)$ is continuous on the boundary of the strip $0<\mathrm{Im}(\chi)<Q$.

From Eq. (4) we see that

$$\chi\left(f(t)\right) \equiv \chi(t) = -\frac{Q}{\pi}\log\left(\frac{(1-t)^2}{4t}\right) \tag{5}$$

and the potential function for the t plane has been determined.

<u>The $\Omega(t)$ Function</u>

It is instructive to examine the logarithm of the complex velocity ζ. For this purpose let us define

$$\omega(t) = \log\left\{\zeta\left(f(t)\right)\right\}$$

For real values of t we see that

$$-1<t<0 \qquad \mathrm{Im}\left(\omega(t)\right) = \pi/2$$
$$0<t<1 \qquad \mathrm{Im}\left(\omega(t)\right) = 0$$

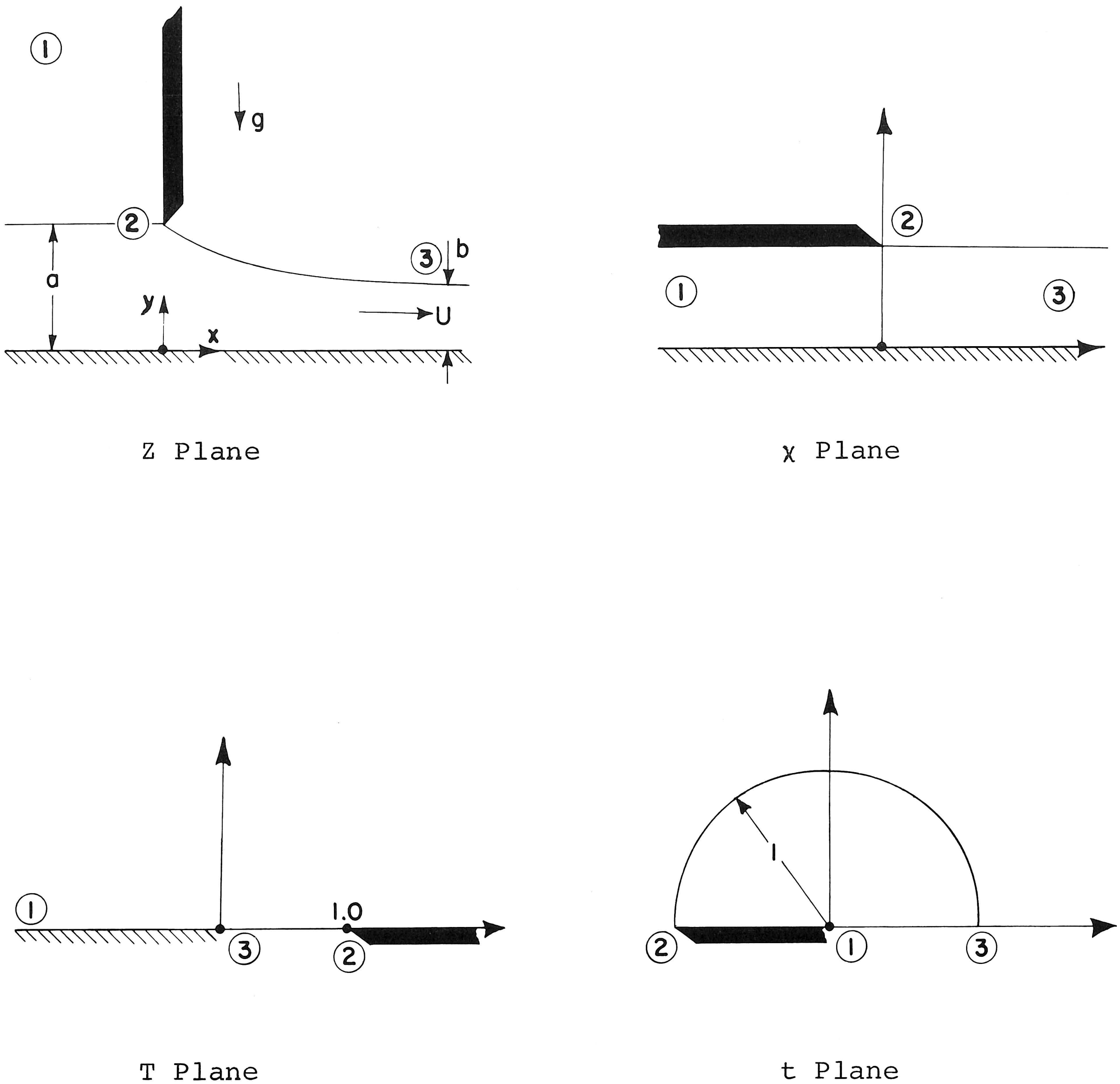

Fig. 1 PHYSICAL AND AUXILIARY PLANES

Let us now define

$$\Omega(t) = \omega(t) - \tfrac{1}{2}\log(t) \tag{6}$$

For real values of t we obtain

$$-1<t<0 \qquad \mathrm{Im}\big(\Omega(t)\big) = 0$$
$$0<t<1 \qquad \mathrm{Im}\big(\Omega(t)\big) = 0$$

Because Ω is real-valued and continuous along the real axis $t \neq 0$, it can be analytically continued into the domain $0<|t|<1$. Since Ω is single valued here it possesses a Laurent series about the point $t = 0$. Hence, we have

$$\Omega(t) = \sum_{n=o}^{\infty} a_n t^n + \sum_{n=1}^{\infty} b_n t^{-n}$$

Rearranging equation (6) produces

$$\zeta(t) \equiv \zeta\big(f(t)\big) = \sqrt{t}\, e^{\Omega(t)} \tag{7}$$

If any of the coefficients b_n are nonzero, then we observe that ζ will have an essential singularity at the point $t=0$. Because the behavior of a function near an essential singularity is irregular, we discard the possibility that ζ can possess such a singularity on the grounds that it is not physically realistic. We conclude, therefore, that $b_n=0$ for all n, i.e.,

$$\Omega(t) = \sum_{n=o}^{\infty} a_n t^n \tag{8}$$

and Ω is analytic in $|t|<1$.

The Free Streamline

The condition of constant pressure intensity on the free streamline expressed by Bernoulli's law requires that

$$\frac{q^2}{2g} + y = \frac{U^2}{2g} + b \tag{9}$$

where q is the speed and y is the elevation at a point on the free streamline.

For $t = e^{i\sigma}$ we define

$$\Omega(e^{i\sigma}) = \phi(\sigma) + i\varepsilon(\sigma) \tag{10}$$

and obtain from Eq. (7)

$$\frac{q(\sigma)}{U} = |\zeta(e^{i\sigma})| = e^{\phi(\sigma)}$$

or

$$\frac{q^2(\sigma)}{U^2} = e^{2\phi}(\sigma) \qquad (11)$$

Eq. (5) becomes

$$\chi(e^{i\sigma}) = -\frac{Q}{\pi}\log\left(\frac{1-\cos\sigma}{2}\right)$$

and from it we obtain

$$\frac{d\chi}{d\sigma} = -\frac{Q}{\pi}\cot(\sigma/2) \qquad (12)$$

From Eq. (1) we find that

$$\frac{dz}{d\chi} = \frac{1}{U\zeta}$$

Since

$$\frac{dz}{d\chi} = \frac{dz}{d\sigma}\frac{d\sigma}{d\chi}$$

we obtain

$$\frac{dz}{d\sigma} = \frac{d\chi}{d\sigma}\frac{1}{U\zeta}$$

Combining the above expression with Eqs. (7), (10), and (12) produces

$$\frac{dz}{d\sigma} = -\frac{Q}{\pi U}\cot(\sigma/2)e^{-\phi-i\varepsilon-i\sigma/2}$$

where z is a point on the free streamline. Integrating the above expression yields

$$z(\sigma) = \frac{Q}{\pi U}\int_{\sigma}^{\pi}\cot(\xi/2)e^{-\phi-i\varepsilon-i\xi/2}d\xi+ia$$

By separating the above expression into real and imaginary parts, we obtain

$$x(\sigma) = \frac{Q}{\pi U}\int_{\sigma}^{\pi}\cot(\xi/2)e^{-\phi}\cos(\varepsilon+\xi/2)d\xi \qquad (13)$$

$$y(\sigma) = -\frac{Q}{\pi U}\int_{\sigma}^{\pi}\cot(\xi/2)e^{-\phi}\sin(\varepsilon+\xi/2)d\xi \qquad (14)$$

Combining Eqs. (9), (11) and (14) produces the integral equation

$$e^{2\phi}-\frac{2}{\pi F^2}\int_{\sigma}^{\pi}\cot(\xi/2)e^{-\phi}\sin(\varepsilon+\xi/2)d\xi = 1+\frac{2}{F^2}\left(1-\frac{1}{Cc}\right) \qquad (15)$$

By differentiating and rearranging Eq. (15), we obtain

$$\phi' = -\frac{1}{\pi F^2}\cot(\sigma/2)e^{-3\phi}\sin(\varepsilon+\sigma/2) \qquad (16)$$

De Boor's Solution of Equation (16)

De Boor, using the power series expansion of Ω (Eq. (8)), expressed ϕ and ε as

$$\phi(\sigma) = \sum_{\kappa=0}^{\infty} A_{\kappa} \operatorname{Cos}(\kappa\sigma)$$

$$\varepsilon(\sigma) = \sum_{\kappa=1}^{\infty} A_{\kappa} \operatorname{Sin}(\kappa\sigma)$$

He substituted these expansions into Eq. (16) and then satisfied the resulting expression at a finite number of points. This procedure produced a finite system of simultaneous nonlinear equations. De Boor solved these equations by successive approximation to obtain approximate values for a finite number of lead coefficients a_n.

When the accuracy of these approximate solutions were checked, it was found to be unsatisfactory for small values of σ. De Boor showed (using several hypotheses) that $\lim_{\sigma \to +0} \phi'(\sigma) \neq 0$. This condition requires that $\phi'(\sigma)$ be discontinuous at $\sigma=0$. To see this we have from the reflection principle

$$\Omega(\bar{t}) = \bar{\Omega}(t) \qquad |t| \leq 1^{2}$$

It follows that

$$\phi(\sigma) = \phi(-\sigma)$$

Hence, in order for ϕ to possess a continuous derivative at $\sigma=0$ it is necessary that its derivative vanish. Otherwise, $\phi'(\sigma)$ will be discontinuous. Because the cosine expansion of ϕ will surely converge poorly in the neighborhood of the point $\sigma= 0$ it appears that the deBoor difficulties stem from truncation error and not from the procedure he used to obtain the coefficients a_n.

Behavior of $\Omega(t)$ Near t=1

It is conjectured in appendix A of this work that in the T plane ζ possesses an asymptotic expansion about the point T=0 of the form

$$P(T) \sim A_{1,0}(Te^{-\pi i})^{\lambda} + A_{2,0}(Te^{-\pi i})^{2\lambda} + \sum_{n=3}^{\infty} (Te^{-\pi i})^{n\lambda} \sum_{\kappa=0}^{\infty} A_{n,\kappa} \log^{\kappa}(Te^{-\pi i}) \tag{17}$$

where $\lambda>0$ is a root of the equation

$$\pi\lambda F^2 = \tan(\pi\lambda) \tag{18}$$

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2 If we assume Ω is continuous at t=1 . This assumption excludes a system of standing waves which extend to infinity.

and $$P(T) = \log(\zeta)$$

From Eqs. (6) and (3) we obtain

$$P\left(-\frac{(1-t)^2}{4t}\right)=\Omega(t)+\tfrac{1}{2}\log(t)$$

Combining the above expression with equation (17) gives

$$\Omega(t)+\tfrac{1}{2}\log(t) \sim \frac{A_{1,0}(1-t)^{2\lambda}}{(4t)^{\lambda}} + \frac{A_{2,0}(1-t)^{4\lambda}}{(4t)^{2\lambda}}$$

$$+ \sum_{n=3}^{\infty} \sum_{\kappa=0}^{n} \frac{(1-t)^{2n\lambda}A_{n,\kappa}}{(4t)^{n\lambda}}\log^{\kappa}\left(\frac{(1-t)^2}{4t}\right)$$

or

$$\Omega(t)+\tfrac{1}{2}\log(t) \sim \frac{A_{1,0}(1-t)^{2\lambda}}{(4t)^{\lambda}} + \frac{A_{2,0}(1-t)^{4\lambda}}{(4t)^{2\lambda}} + O\left((1-t)^{4\lambda}\right)$$

where $O\left((1-t)^{4\lambda}\right)$ indicates terms which approach zero as $t\to 1$ when divided by $(1-t)^{4\lambda}$. By differentiating the above expression, we obtain

$$\Omega'(t)+\frac{1}{2t} \sim -\frac{A_{1,0}(1-t)^{2\lambda-1}}{(4t)^{\lambda}t^{\lambda}}\left(2\lambda t^{\lambda}+(1-t)\lambda t^{\lambda-1}\right)$$

$$-\frac{A_{2,0}(1-t)^{4\lambda-1}}{4^{2\lambda}t^{4\lambda}}\left(4\lambda t^{2\lambda}+(1-t)2\lambda t^{2\lambda-1}\right)+ O\left((1-t)^{4\lambda-1}\right)$$

If we now assume $\lambda>\tfrac{1}{4}$ (it is sufficient to take $F^2>\frac{4}{\pi}$, $\lambda>0$) then

$$\lim_{t\to 1}\left(\Omega'(t)+ \frac{2\lambda A_{1,0}}{4^{\lambda}}(1-t)^{2\lambda-1}\right) = -\tfrac{1}{2}$$

The Ω^* Function

Let us define

$$\Omega^*(t) = \Omega(t)- \frac{A_{1,0}(1-t)^{2\lambda}}{4^{\lambda}} \qquad (19)$$

Then

$$\Omega^{*\prime}(t) = \Omega'(t) + \frac{2\lambda A_{1,0}}{4^{\lambda}}(1-t)^{2\lambda-1}$$

and

$$\lim_{t\to 1} \Omega^{*\prime}(t) = -\tfrac{1}{2} \qquad (20)$$

For $t=e^{i\sigma}$ let

$$\phi^*(\sigma) = \mathrm{Re}(\Omega^*)$$

$$\varepsilon^*(\sigma) = \mathrm{Im}(\Omega^*)$$

Then

$$\phi^{*\prime}(\sigma) = \mathrm{Re}\left(\frac{d\Omega^*}{d\sigma}\right) = \mathrm{Re}\left(\frac{d\Omega^*}{dt} ie^{i\sigma}\right)$$

By using Eq. (20), we obtain

$$\lim_{\sigma\to 0} \phi^{*\prime}(\sigma) = 0$$

$$\lim_{\sigma\to 0} \varepsilon^{*\prime}(\sigma) = -\tfrac{1}{2} \qquad (21)$$

Just as with Ω, the function Ω^* is analytic in $|t|<1$ and is real-valued and continuous for $-1<t<1$. From the reflection principle we have that

$$\Omega^*(\bar{t}) = \overline{\Omega^*(t)}$$

Hence, we conclude that $\phi^*(\sigma)$ has a continuous derivative for $\sigma=0$.

From Eq. (19) we have

$$\Omega(t) = \Omega^*(t)+A(1-t)^{2\lambda} \qquad (22)$$

where A is a real constant to be determined. Expanding Ω^* in a Taylor series about $t=0$ gives

$$\Omega^*(t) = A_o^*+A_1^*t+A_2^*t^2+ \cdots$$

and

$$\phi^*(\sigma) = A_o^*+A_1^*\mathrm{Cos}\sigma+A_2^*\mathrm{Cos}2\sigma+ \cdots \qquad (23)$$

$$\varepsilon^*(\sigma) = A_1^*\mathrm{Sin}\sigma+A_2^*\mathrm{Sin}2\sigma+ \cdots \qquad (24)$$

Moreover,

$$(1-t)^{2\lambda}=1+b_1t+b_2t^2+ \cdots \qquad (25)$$

where $b_\kappa = -\dfrac{2\lambda(1-2\lambda)(2-2\lambda)\cdots(\kappa-1-2\lambda)}{\kappa!}$ $k=1,2,3 \cdots$

For $t=e^{i\sigma}$ we obtain

$$(1-e^{i\sigma})^{2\lambda} = (2-2\text{Cos}\sigma)^{\lambda} e^{-i(\pi-\sigma)\lambda} \qquad o<\sigma<\pi$$

Let

$$\delta(\sigma) = -(2-2\text{Cos}\sigma)^{\lambda}\text{Sin}(\pi-\sigma)\lambda$$

$$\Gamma(\sigma) = (2-2\text{Cos}\sigma)^{\lambda}\text{Cos}(\pi-\sigma)\lambda$$

so that

$$(1-e^{i\sigma})^{2\lambda} = \Gamma(\sigma)+i\delta(\sigma) \qquad o<\sigma<\pi$$

Combining Eqs. (22), (23), and (24) with the above expressions produces

$$\phi(\sigma) = \phi^*(\sigma)+A\Gamma(\sigma) = \sum_{n=o}^{\infty} A_n^*\text{Cos}(n\sigma)+A\Gamma(\sigma)$$

$$\varepsilon(\sigma) = \varepsilon^*(\sigma)+A\delta(\sigma) = \sum_{n=1}^{\infty} A_n^*\text{Sin}(n\sigma)+A\delta(\sigma)$$

We obtain by combining these results with Eq. (16)

$$\sum_{n=1}^{\infty} nA_n^*\text{Sin}(n\sigma)+A\Gamma'(\sigma) = \frac{1}{\pi F^2}\text{Cot}(\sigma/2)e^{-3\left(A\Gamma+\sum_{n=o}^{\infty} A_n^*\text{Cos}(n\sigma)\right)}$$

$$\text{Sin}\left(\sum_{n=1}^{\infty} A_n^*\text{Sin}(n\sigma)+\sigma/2+A\delta\right)$$

From Eq. (25) we find that for $t=e^{i\sigma}$

$$\Gamma(\sigma) = 1+b_1\text{Cos}\sigma+b_2\text{Cos}2\sigma+\cdots$$

By combining the two above expressions and solving for the Fourier coefficient on the left hand side of the resulting equation, we obtain

$$A_n^*+Ab_n = \frac{2}{\pi^2F^2n}\int_o^{\pi}\text{Cot}(\sigma/2)e^{-3\left(A\Gamma+\sum_{n=o}^{\infty} A_n^*\text{Cos}(n\sigma)\right)}$$

$$\text{Sin}\left(\sum_{n=1}^{\infty} A_n^*\text{Sin}(n\sigma)+\sigma/2+A\delta\right)\text{Sin}(n\sigma)d\sigma$$

for $n=1,2,3,\cdots$ (26)

Solution of Equations (26)

In this work Eqs. (26) were used to obtain approximate values of the coefficients a_n^*. First the Taylor series expansion of Ω^* was truncated to N terms. Next the first N of Eqs. (26) were retained to give N nonlinear equations and N+2 unknowns $A_0^*, A_1^*, \cdots A_N^*, A$. By setting $\sigma=0$ in Eq. (11), we obtain

$$\phi(0) = 0$$

or in terms of the coefficients

$$A_o^* + \sum_{\kappa=1}^{N} A_\kappa^* = 0 \tag{27}$$

The additional required equation comes from Eq. (21)

$$\varepsilon^{*\prime}(0) = -\tfrac{1}{2}$$

which becomes

$$\sum_{\kappa=1}^{N} \kappa A_\kappa^* = -\tfrac{1}{2}$$

In order to solve these equations simultaneously, Eq. (27) was used to eliminate A_o^* from the remaining N+1 equations. Residuals F_κ $\kappa=1,2,3, \ldots N+1$ were defined such that

$$F_\kappa = \frac{2}{\pi^2 F^2 \kappa} \int_0^\pi \mathrm{Cot}(\sigma/2)\, e^{-3\left\{A\Gamma - \sum_{n=1}^{N} A_N^*\left(1-\mathrm{Cos}(n\sigma)\right)\right\}}$$

$$\mathrm{Sin}\left(\sum_{n=1}^{N} A_N^* \,\mathrm{Sin}(n\sigma)+\sigma/2+A\delta\right)\mathrm{Sin}(\kappa\sigma)\,d\sigma - Ab_\kappa - A_\kappa^*$$

for $\kappa = 1,2,3 \cdots N$ and

$$F_{N+1} = \tfrac{1}{2} + \sum_{\kappa=1}^{N} \kappa A_\kappa^*$$

The actual computations were begun by assigning a value to $\tfrac{1}{4} < \lambda < \tfrac{1}{2}$ and then computing the Froude number from Eq. (18). Next the initial approximations

$$A_1^* = -\tfrac{1}{2}$$

$$A = 0$$

$$A_\kappa^* = 0 \quad \kappa = 2,3, \cdots N$$

where assigned to the unknowns. To obtain better approximations the matrix equation

$$\begin{bmatrix} \frac{\partial F_1}{\partial A_1^*} & \frac{\partial F_1}{\partial A_2^*} & & \frac{\partial F_1}{\partial A} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{\partial F_N}{\partial A_1^*} & & & \\ \frac{\partial F_{N+1}}{\partial A_1^*} & \cdots & & \frac{\partial F_{N+1}}{\partial A} \end{bmatrix} \begin{bmatrix} \Delta A_1^* \\ \Delta A_2^* \\ \cdot \\ \cdot \\ \cdot \\ \Delta A \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_{N+1} \end{bmatrix} = 0$$

was solved repeatedly for corrections ΔA_κ^*, ΔA, each time using the previously computed values of A_κ^*, A to compute $\frac{\partial F_\kappa}{\partial A_m^*}$ and F_κ.

Since the solution found in this manner is approximate, the dimensionless error function

$$\tau(\sigma) = 1-e^{2\phi} + \frac{2}{F^2}\left(1-\frac{1}{Cc} + \frac{1}{\pi}\int_\sigma^\pi \mathrm{Cot}(\xi/2)e^{-\phi}\mathrm{Sin}(\varepsilon+\xi/2)d\xi\right)$$

was used to check its accuracy. Eq. (15) shows that $\tau(\sigma)$ will vanish for an exact solution and should be small for a sufficiently accurate approximation to the solution.

Coefficient of Contraction

From Eq. (14) we obtain upon setting $\sigma=0$ the relationship

$$b = -\frac{Q}{\pi U}\int_0^\pi \mathrm{Cot}(\sigma/2)e^{-\phi}\mathrm{Sin}(\varepsilon+\sigma/2)d\sigma + a$$

By combining this expression with the coefficient of contraction $Cc = b/a$, we obtain, after some rearrangement,

$$Cc = \frac{1}{1+\frac{1}{\pi}\int_0^\pi \mathrm{Cot}(\sigma/2)e^{-\phi}\mathrm{Sin}(\varepsilon+\sigma/2)d\sigma} \qquad (28)$$

RESULTS AND DISCUSSION

It is apparent from Fig. 2 that Ω^* is represented by a polynomial (truncated power series) much better than Ω. Values of the function ϕ^* corresponding to $N=5$ and $N=7$ for $0 \leq \sigma \leq \pi$ differ by less than one digit in the third decimal place so that their plots seemingly coincide. Moreover, the error function $\tau(\sigma)$ was found to be small over the complete range $0 \leq \sigma \leq \pi$, and in no case when using $N=5$ did it exceed 0.5%. These results strongly support the validity of Eq. (20) and also support to some extent the validity of Eq. (17).

The accuracy of these solutions is seemingly impossible to determine. However, it appears from Table 1 that the coefficients of contraction may be accurate to three decimal places for $N=5$. The term $\bar{\tau}$ is the maximum value of $|\tau(\sigma)|$ in the interval $0 < \sigma < \pi$. It was found that $\bar{\tau}$ increased slowly as $\frac{1}{F^2}$ increased until it reached a maximum value of about 0.5% for the limiting case $\lambda=3.2515$. It appears, therefore, that the coefficients of contraction presented in Fig. 3 may be slightly less accurate for low Froude numbers than high ones.

From the numerical procedure used in this work, it appears that the limiting case (for low Froude numbers) is reached for $0.4282 < \lambda \leq 0.4285$. This procedure produced results for $\lambda=0.4285$ but failed for $\lambda=0.4282$. De Boor felt that the speed at the separation point became arbitrarily small in the limiting case. However, for $\lambda=0.4285$ it was found that $\phi(\sigma)=-0.843$ and $\frac{q_o}{U} = 0.430$ where q_o is the speed at the separation point. Hence, if the solution does fail to exist for $0.4282 < \lambda \leq 0.4285$ it is unlikely that this conjecture is correct.

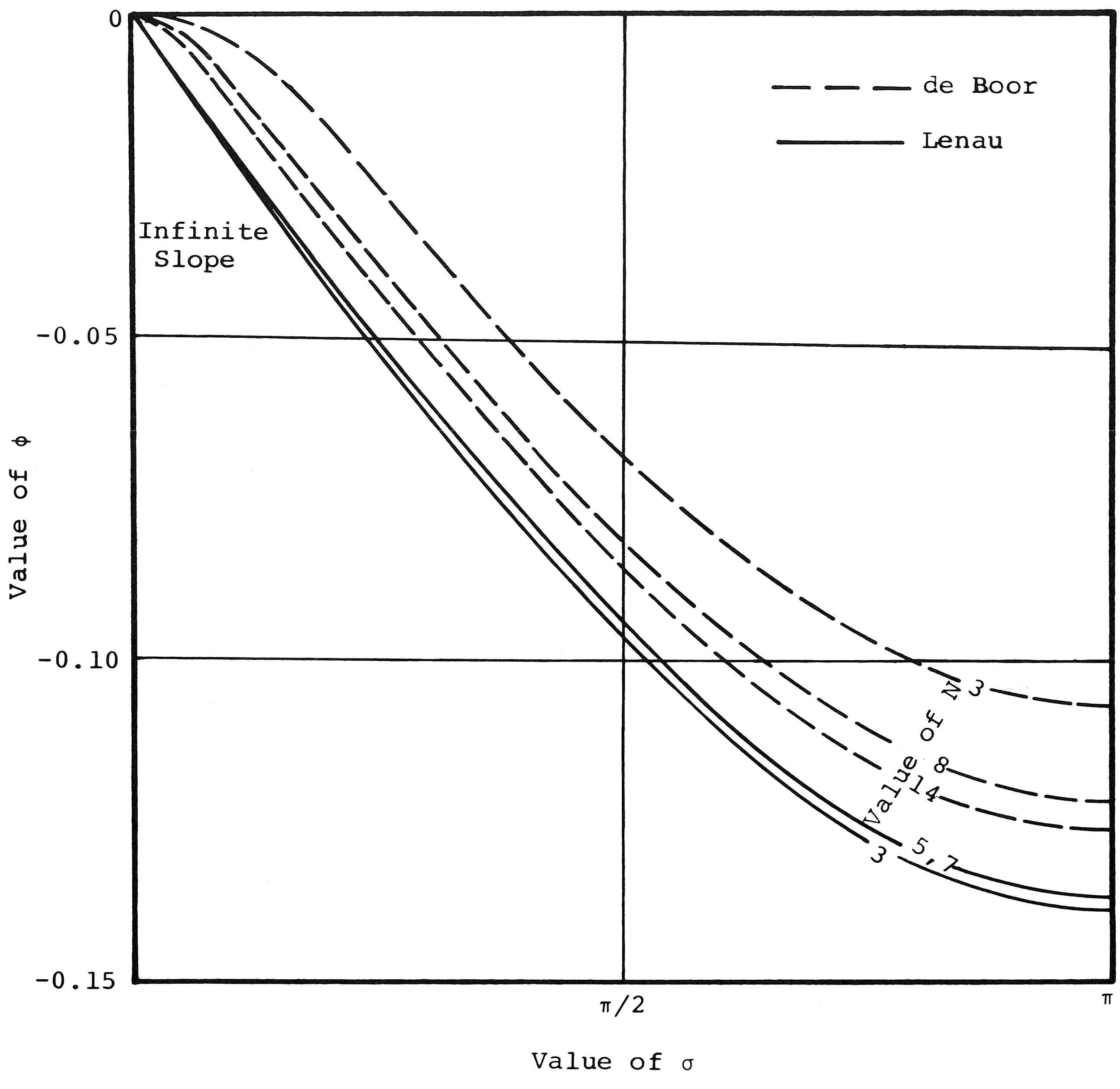


Fig. 2 A comparison of the solutions obtained by De Boor and Lenau: $F^2 = 6.37$

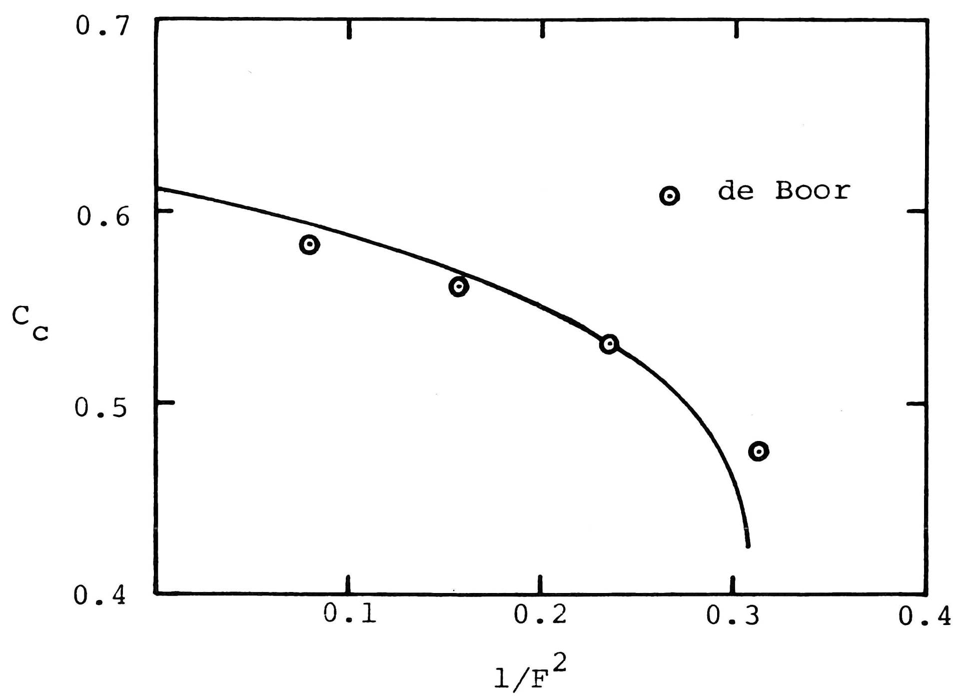


Fig. 3 A PLOT OF COEFFICIENT OF CONTRACTION VS $1/F^2$: N=5

TABLE 1. THE EFFECT OF N UPON C_c AND $\bar{\tau}$

N	F^2	C_c	$\bar{\tau}$%
3	6.371	0.5667	0.52
5	6.371	0.5676	0.21
7	6.371	0.5678	0.11

It is not difficult to show by "physical reasoning" that there exists a F_o^2 such that for all $F^2 < F_o^2$ the solution either does not exist or the jet changes its characteristic shape. Referring now to Fig. 4 we see a typical case.

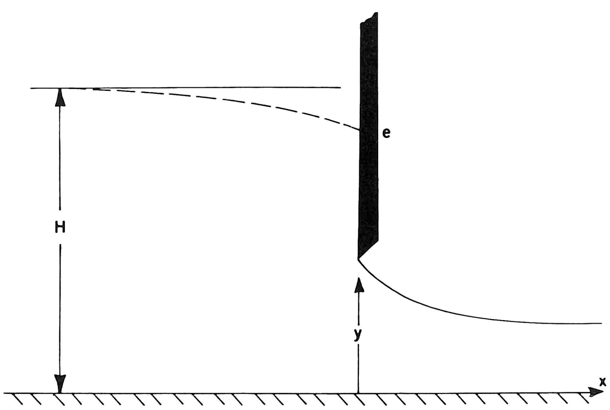


Fig. 4

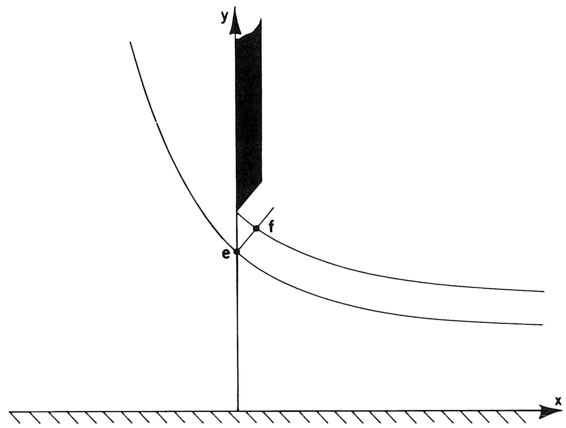


Fig. 5

It will be assumed that there exists a constant $0<K<1$ such that $C_c \leq K$. If the jet has the characteristic shape shown in Fig. 4, the intensity of pressure along the bottom of the channel would be greater than zero (if we take atmospheric pressure to be zero). Because the reservoir is infinite, there would exist within it a line of zero pressure intensity. This line would intersect the y axis at point e and would approach a horizontal line of elevation

$$H = aCc(1+\tfrac{1}{2}F^2)$$

relative to the channel bottom far upstream from the gate opening. Because $\frac{H}{a} > \frac{m}{a}$ and $Cc \leq K < 1$, we see that either the solution would fail to exist or $F^2 > 0$ can be decreased until $\frac{m}{a} < 1$ where the term m is the elevation of point e relative to the channel bottom. Let point f be a point defined by the intersection of the free streamline with a straight line which passes through point e at right angles to the streamline upon which point e is located. Then it is apparent from Fig. 5 that the pressure intensity at point e would be greater than it is at point f because of the curvature of the streamlines and the difference in elevation of the two points. Clearly then a solution would exist only if $\frac{m}{a} \geq 1$ and we would find

$$F^2 > 2\left(\frac{1}{Cc} - 1\right) \geq \left(\frac{1}{k} - 1\right) = F_o^2$$

It appears from the numerical procedure used in this work that the limiting value of $\frac{m}{a}$ is close to unity. For $\lambda=0.4285$ it was found that $\frac{H}{a}=1.130$ in which case $1.0 \leq \frac{m}{a} < 1.130$.

APPENDIX A

In this section an attempt has been made to describe the behavior of Ω in the neighborhood of the point $t=1$ for $0<F^2<\infty$. The arguments presented are not complete. Indeed, the writer has merely assumed that Ω possesses an infinite asymptotic expansion of a certain form, has demonstrated that this expansion describes the asymptotic behavior of a jet of the form we expect and has demonstrated that the coefficients of this expansion are probably defined.

It is convenient to work in the T plane for this purpose. Let Δ be the domain $\{|T|<\rho,\ \mathrm{Im}(T)>0\}$ where $0<\rho<1$. For $T\varepsilon\Delta$ $\zeta(T)$ does vanish; hence, we can define

$$\zeta(T)=e^{P(T)} \tag{A.1}$$

where P(T) is analytic for $T\varepsilon\Delta$, and continuous for $T\varepsilon\overline{\Delta}$ except perhaps for $T=o$. For $0<T<1$ let

$$P(T)=R(T)+iI(T)$$

where P and I are real-valued. From Eq. (2) we have

$$\chi = \frac{-Q}{\pi}\log\,(Te^{-\pi i})$$

or

$$\frac{d\chi}{dT} = -\frac{Q}{\pi T}$$

By combining Eqs. (1) and (A.1) with the above expression, we obtain

$$\frac{dz}{dT} = -\frac{Q}{\pi T}\,\frac{e^{-P}}{U}$$

and

$$z = -\frac{Q}{\pi U}\int\frac{e^{-P}dT}{T} + C \tag{A.2}$$

For $T>0$ we have

$$Z = -\frac{Q}{\pi U}\int \frac{e^{-R-iI}dT}{T} + C$$

or

$$y = \frac{Q}{\pi U}\int \frac{e^{-R}\mathrm{Sin}I}{T}\,dT + \mathrm{Im}(C) \qquad (A.3)$$

where y locates a point on the free streamline. The speed of a point on the free streamline is obtained from Eq. (A.1)

$$\frac{q}{U} = e^{R}$$

Upon combining this expression with Eqs. (A.3) and (9), we differentiate and rearrange the resulting expression to produce

$$R'(T) = -\frac{1}{F^2\pi}\,\frac{e^{-3R}\mathrm{Sin}I}{T} \quad 0\le T\le 1 \qquad (A.4)$$

As a first attempt to solve Eq. (A.4), let us assume

$$P(T) = A(e^{-\pi i}T)^{\lambda} \text{ for } T\varepsilon\Delta$$

where $\lambda>0$ and A are real constants to be determined. We note that P is real valued for $T<0$. For $T>0$ we obtain

$$R(T) = AT^{\lambda}\mathrm{Cos}\lambda\pi$$

$$I(T) = -AT^{\lambda}\mathrm{Sin}\lambda\pi$$

Upon substituting these expressions into the right hand side of Eq. (A.4), we obtain

$$\frac{1}{F^2\pi}\,\frac{e^{-3A\,\mathrm{Cos}\lambda\pi T^{\lambda}}\mathrm{Sin}(AT^{\lambda}\mathrm{Sin}\lambda\pi)}{T} = \frac{1}{F^2\pi}\Big(AT^{\lambda-1}\mathrm{Sin}\lambda\pi$$

$$-3A^2T^{2\lambda-1}\mathrm{Sin}\lambda\pi\ \mathrm{Cos}\lambda\pi+0(T^{2\lambda-1})\Big)$$

where the symbol $0(T^{2\lambda-1})$ indicates terms which approach zero as T approaches zero when divided by $T^{2\lambda-1}$. Upon evaluating the left hand side of Eq. (A.4), we obtain

$$R'(T) = A\lambda T^{\lambda-1}\mathrm{Cos}\pi\lambda$$

Thus we see that $P(T) = A(e^{-\pi i}T)^{\lambda}$ is a solution to (A.4) for small values of $|T|$ provided

$$\tan\lambda\pi = \lambda\pi F^2 \tag{A.5}$$

To see what this solution represents we obtain from Eq. (A.2)

$$z = -\frac{Q}{U\pi}\left(\log(T) + \frac{A(Te^{-\pi i})^{\lambda}}{\lambda} + 0(T^{\lambda})\right) + C$$

For $T>0$, we obtain

$$x = -\frac{Q}{U\pi}\log(T) - \frac{AQ}{\lambda U\pi}T^{\lambda}\cos\lambda\pi + 0(T^{\lambda}) + \mathrm{Re}(C)$$

$$y = +\frac{AQ}{\lambda U\pi}T^{\lambda}\sin\lambda\pi + 0(T^{\lambda}) + \mathrm{Im}(C)$$

for the equation of the free streamline. As $T\to 0$ $x\to+\infty$ and $y\to\mathrm{Im}(C)$. Clearly $P(T)=A(e^{-\pi i}T)^{\lambda}$ represents the asymptotic behavior of a jet since y is monotonically increasing or decreasing depending upon the sign of A . This observation suggests that perhaps $P(T)$ possesses an asymptotic expansion about the point $T=0$ of the form

$$P(T) \sim \sum_{k=1}^{\infty} (Te^{-\pi i})^{k\lambda} A_k \qquad \lambda>0$$

where the coefficients A_k are real numbers. By assuming the above equation to be correct, we obtain for $T>0$

$$R(T) \sim \sum_{k=1}^{\infty} A_k \cos(k\lambda\pi)\, T^{k\lambda}$$

$$I(T) \sim -\sum_{k=1}^{\infty} A_k \sin(k\lambda\pi)\, T^{k\lambda}$$

and

$$-\sin(I) \sim \sum_{k=1}^{\infty} r_k T^{\lambda k} \tag{A.6}$$

$$e^{3R} \sim 1 + \sum_{k=1}^{\infty} C_k T^{\lambda k} \tag{A.7}$$

where the coefficients r_k and C_k satisfy the following recursion relationships

$C_0 = 1$

$$C_1 = C_0(3A_1)\mathrm{Cos}(\pi\lambda)$$

$$2C_2 = 2C_0(3A_2)\mathrm{Cos}(2\pi\lambda) + C_1(3A_1)\mathrm{Cos}(\pi\lambda)$$

$$3C_3 = 3C_0(3A_3)\mathrm{Cos}(3\pi\lambda)+2C_1(3A_2)\mathrm{Cos}(2\pi\lambda)+C_2(3A_1)\mathrm{Cos}(1\pi\lambda)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad (A.8)$$

$$\beta_0 = 1$$

$$\beta_1 = 0$$

$$r_1 = A_1\mathrm{Sin}(\pi\lambda)$$

$$2\beta_2 = -r_1A_1\mathrm{Sin}(\pi\lambda)$$

$$2r_2 = 2\beta_0A_2\mathrm{Sin}(2\pi\lambda)+\beta_1A_1\mathrm{Sin}(\pi\lambda)$$

$$3\beta_3 = -2r_1A_2\mathrm{Sin}(2\pi\lambda)-r_2A_1\mathrm{Sin}(\pi\lambda)$$

$$3r_3 = 3\beta_0A_3\mathrm{Sin}(3\pi\lambda)+2\beta_1A_2\mathrm{Sin}(2\pi\lambda)+\beta_2A_1\mathrm{Sin}(\pi\lambda)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad (A.9)$$

By substituting Eqs. (A.6) and (A.7) into equation (A.4), combining the resulting expression with the above relationships and equating coefficients of like powers of T^λ, we obtain

$$\tan(\lambda\pi) = \lambda\pi F^2$$

$$A_k = \left\{\frac{3g(\beta_{k-1},A_{k-1})}{\pi F^2 k} - \lambda f(C_{k-1},A_{k-1})\right\} \Big/ \; 3\lambda\mathrm{Cos}(k\pi\lambda)\left(k-\frac{\tan(k\pi\lambda)}{\tan(\pi\lambda)}\right) \qquad (A.10)$$

where

$$g(\beta_{k-1},A_{k-1}) = \sum_{j=1}^{k-1}(k-j)A_{kj}\mathrm{Sin}\big((k-j)\pi\lambda\big)\beta_j$$

and

$$f(C_{k-1}, A_{k-1}) = \sum_{j=1}^{k-1}(k-j)A_{k-j}\mathrm{Cos}\big((k-j)\pi\lambda\big)C_j$$

The coefficients can be computed by the following procedure. First a value is assigned to the lead coefficient A_1. From Eqs. (A.8) it is now possible to compute C_1 and from equations (A.9) r_1 and β_1. By using these coefficients, we compute $g(\beta_1,A_1)$, $f(C_1,A_1)$

and from equation (A.10) A_2 . This procedure can be repeated to determine higher coefficients A_k provided the denominator of Eq. (A.10) does not vanish, i.e., provided

$$k\tan(\pi\lambda) \neq \tan(\pi\lambda k)$$

for all $k>1$. It is possible, however, that it will vanish for some k since the roots of the equation

$$k\tan(\pi\lambda) = \tan(\pi\lambda k)$$

are numerous for large values of k. Hence, in general P(T) does not possess such an expansion since only a finite number of coefficients A_k exist. The author has found that the expansion

$$\sum_{k=1}^{\infty}\sum_{m=0}^{k} A_{k,m}(e^{-\pi i}T)^{\lambda k}\log^m(Te^{-\pi i}) \quad \lambda>0$$

contains the expansion just discussed as a special case and seems in general to have well defined real-valued coefficients $A_{k,m}$. To see this we now assume that

$$P(T)\sim A_{1,0}(Te^{-\pi i})^{\lambda}+\left(A_{2,0}+A_{2,1}\log(e^{-\pi i}T)+A_{2,2}\log^2(e^{-\pi i}T)\right)(Te^{-\pi i})^{2\lambda}+\ \ldots$$

It is clear that such an expansion is real-valued for $T<0$. For $0<t<1$ we obtain

$$R(T)\sim\sum_{n=1}^{\infty}T^{\lambda n}\sum_{j=0}^{n}b_{n,j}\log^j(T) \qquad (A.11)$$

$$I(T)\sim\sum_{n=1}^{\infty}T^{\lambda n}\sum_{j=0}^{n}C_{n,j}\log^j(T) \qquad (A.12)$$

The coefficients $b_{n,j}$ and $C_{n,j}$ are related through the coefficients $A_{n,j}$. However, it is simpler to let these expansions define R(T) and I(T) and then to require that $P(T)=R(T)+iI(T)$ be real-valued along the negative real axis. It is shown in appendix B that if R and I are real-valued (i.e., if the coefficients $b_{n,j}$ and $C_{n,j}$ are real numbers) it is necessary and sufficient that

$$R(Te^{-\pi i2})-iI(Te^{-2\pi i}) = R(T)+iI(T)$$

in order for P to be real-valued for $-\rho<T<0$. Substituting Eqs. (A.11) and (A.12) into the above expression and equating coefficients of like powers of T^{λ} produces

$$\sum_{j=0}^{n}(b_{n,j}+iC_{n,j})\log^{j}(T)=e^{-2\pi in\lambda}\sum_{j=0}^{n}(b_{n,j}-iC_{n,j})\left(\log(T)-2\pi i\right)^{j}$$

Upon equating like powers of log (T), we obtain

$$b_{n,j}+iC_{n,j} = e^{-2\pi in\lambda}\sum_{k=j}^{n}(b_{n,k}-iC_{n,k})\binom{k}{k-j}(-2\pi i)^{k-j}$$

where $\binom{k}{k-j}$ is a Binomial coefficient. Rearranging the above expression produces

$$b_{n,j}(1-e^{-2\pi in\lambda})+iC_{n,j}(1+e^{-2\pi i\lambda n}) = \Gamma_{n,j} \qquad \text{(A.13)}$$

where

$$\Gamma_{n,j} = \begin{cases} e^{-2\pi i\lambda n}\displaystyle\sum_{k=j+1}^{n}(b_{n,k}-iC_{n,k})\binom{k}{k-j}(-2\pi i)^{k-j} & j=0,1,\ldots n-1 \\ 0 & j=n \end{cases}$$

By substituting Eqs. (A.11) and (A.12) into equation (A.4) and equating like terms, we obtain

$$\lambda nb_{n,n} + \frac{1}{F^{2}\pi}C_{n,n} = \alpha_{n,n}$$

$$\lambda nb_{n,j} + \frac{1}{F^{2}\pi}C_{n,j} = -(j+1)b_{n,j+1}+\alpha_{n,j} \quad j=0,1,2\ldots n-1 \qquad \text{(A.14)}$$

where $\alpha_{n,j}$ is a polynomial in $b_{k,L}$ and $C_{k,L}$ $k=0,1,\ldots n-1;$ $L=0,1,\ldots k$ which vanish if $b_{k,L}$ and $C_{k,L}$ vanish.

Let us now examine when it is possible to solve Eqs. (A.13) and (A.14) simultaneously. Setting the determinate of the coefficient matrix equal to zero we obtain

$$\tan(\pi\lambda n) = \lambda nF^{2}\pi \qquad \text{(A.15)}$$

Combining this expression with Eq. (A.5) produces

$$\tan(\pi\lambda n) = n\tan(\pi\lambda) \tag{A.16}$$

We see that the determinate of the coefficient matrix vanishes under the same condition as for the denominator of Eq. (A.10). It also vanishes when $n=1$ in which case Eqs. (A.13) and (A.14) become

$$\lambda b_{1,1} + \frac{1}{F^2{}_\pi} C_{1,1} = 0$$

$$\lambda b_{1,0} + \frac{1}{F^2{}_\pi} C_{1,0} = -b_{1,1}$$

and

$$b_{1,1}(1-e^{-2\pi i\lambda})+iC_{1,1}(1+e^{-2\pi i\lambda}) = 0$$

$$b_{1,0}(1-e^{-2\pi i\lambda})+iC_{1,0}(1+e^{-2\pi i\lambda}) = e^{-2\pi i\lambda}(b_{1,1}-iC_{1,1})$$

By setting $C_{1,1}=b_{1,1}=0$ all four equations will be satisfied provided

$$b_{1,0} = -\frac{C_{1,0}}{\pi\lambda F^2}$$

Hence, we may select one constant arbitrarly provided it is a real number. For $N=2$ Eq. (A.16) has no root $\lambda>0$. Hence, the determinate of the coefficient matrix will not vanish. Eqs. (A.13) and (A.14) become

$$2\lambda b_{2,2} + \frac{1}{F^2{}_\pi}C_{2,2} = \alpha_{2,2}$$

$$2\lambda b_{2,1} + \frac{1}{F^2{}_\pi}C_{2,1} = -2b_{2,2} + \alpha_{2,1}$$

$$2\lambda b_{2,0} + \frac{1}{F^2{}_\pi}C_{2,0} = -b_{2,1} + \alpha_{2,0}$$

and

$$b_{2,2}(1-e^{-4\pi i\lambda})+iC_{2,2}(1+e^{-4\pi i\lambda}) = 0$$

$$b_{2,1}(1-e^{-4\pi i\lambda})+iC_{2,1}(1+e^{-4\pi i\lambda}) = \Gamma_{2,1}$$

$$b_{2,0}(1-e^{-4\pi i\lambda})+C_{2,0}(1+e^{-4\pi i\lambda}) = \Gamma_{2,0}$$

Because $C_{1,1}=b_{1,1}=0$ the terms $\alpha_{2,2}$ and $\alpha_{2,1}$ are zero. We have then $b_{2,2}=C_{2,2}=b_{2,1}=C_{2,1}=\Gamma_{2,1}=\Gamma_{2,1}=\Gamma_{2,0}=0$. Indeed all of the coefficients will vanish except for $b_{j,0}$ and $C_{j,0}$ $j=0,1,\ldots$ until the determinate of the coefficient matrix again vanishes. Suppose this occurs for $j=k$. Then we would have $\alpha_{k,j}=0$ $j=1,2,3,\ldots k$. Eqs. (A.13) and (A.14) would become

$$\lambda b_{k,k}(1-e^{-2\pi ik\lambda})+iC_{k,k}(1+e^{-2\pi ik\lambda}) = 0$$

$$kb_{k,k} + \frac{1}{F^2{}_\pi}C_{k,k} = 0$$

$$b_{k,k-1}(1-e^{-2\pi ik\lambda}) + iC_{k,k-1}(1+e^{-2\pi ik\lambda}) = \Gamma_{k,k-1}$$

$$\lambda kb_{k,k-1} + \frac{1}{F^2{}_\pi}C_{k,k-1} = -kb_{k,k}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_{k,1}(1-e^{-2\pi ik\lambda}) + iC_{k,1}(1+e^{-2\pi ik\lambda}) = \Gamma_{k,1}$$

$$\lambda kb_{k,1} + \frac{1}{F^2{}_\pi}C_{k,1} = -2b_{k,2}$$

$$b_{k,0}(1-e^{-2\pi ik\lambda}) + iC_{k,0}(1+e^{-2\pi ik\lambda}) = \Gamma_{k,0}$$

$$\lambda kb_{k,0} + \frac{1}{F^2{}_\pi}C_{k,0} = -b_{k,1} + \alpha_{k,0}$$

The first $k-1$ pair of the above equations would be satisfied if we take $b_{k,j}=C_{k,j}=0$ for $j=2,3,4,\ldots k$. The last two pair of equations would then become after some rearrangement

$$b_{k,1} + C_{k,1}\mathrm{Cot}(\pi k\lambda) = 0$$

$$b_{k,1} + \frac{1}{k\lambda F^2{}_\pi}C_{k,1} = 0$$

$$b_{k,o} + C_{k,0}\mathrm{Cot}(\pi k) = \frac{e^{-2\pi i\lambda k}}{1-e^{-2\pi i\lambda k}}(b_{k,1}-iC_{k,1})(-2\pi i)$$

$$b_{k,o} + \frac{1}{k\lambda F^2\pi}C_{k,o} = (-b_{k,1} + \alpha_{k,o})\frac{1}{\lambda k}$$

The first pair of the above equations would be identical because

$$\tan(\kappa\pi\lambda) = \kappa\pi\lambda F^2$$

hence, they would both be satisfied if we take

$$C_{\kappa,1} = -\kappa\lambda F^2\pi b_{\kappa,1} \tag{A.17}$$

where $b_{\kappa,1}$ is any real number. The second pair of equations can be satisfied if we make their right hand sides equal by the proper adjustment of $b_{\kappa,1}$ and then take

$$b_{\kappa,o} = \frac{1}{\lambda\kappa}(-b_{\kappa,1}+\alpha_{\kappa,o}) - \frac{1}{\kappa\lambda F^2\pi}C_{\kappa,o}$$

where $C_{\kappa,o}$ is an arbitrary real constant. To determine the proper value of $b_{\kappa,1}$ we set

$$\frac{e^{-2\pi i\lambda\kappa}}{1-e^{-2\pi i\lambda\kappa}}(b_{\kappa,1}-iC_{\kappa,1})(-2\pi i) = \frac{1}{\lambda\kappa}(-b_{\kappa,1}+\alpha_{\kappa,o})$$

We combine it with Eqs. (A.17) and (A.15) and rearrange the resulting expression to obtain

$$b_{\kappa,1} = -\frac{\operatorname{Sin}(2\pi\lambda\kappa)\alpha_{\kappa,o}}{2\pi\lambda\kappa-\operatorname{Sin}(2\pi\lambda\kappa)}$$

because $\lambda>o$ the denominator of the above equation would not vanish and $b_{\kappa,1}$ would be well defined.

For $j>k$ the logarithmic terms would appear. The process of computing the coefficients would fail only if the determinate of the coefficients matrix vanishes a third time. In order for this to occur λ would have to be simultaneously a root of the equation

$$n\tan(\pi\lambda) = \tan(n\pi\lambda)$$

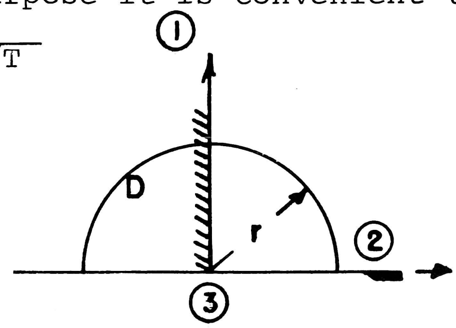
or two distinct values of $n\neq 1$. If the roots of the above equation interlace then it would be possible to continue computing the unknown coefficients indefinitely. It appears to the author that the latter situation is probably the case for $0<F^2<\infty$ and $\lambda>0$.

APPENDIX B

In this section it is shown that it is necessary and sufficient that the function $P=R+iI$ satisfied the relationship

$$R(Te^{-2\pi i})-iI(Te^{-2\pi i}) = R(T)+iI(T)$$

in order that P be real valued along the negative real axis. For this purpose it is convenient to work in the W plane where $W = \sqrt{T}$



Let D be the domain $\{0<|W|<r,\ 0<\arg(W)<\pi\}$ where $0<r<1$. Moreover, let us define

$$P^*(W) \equiv P(W^2)$$

$$R^*(W) \equiv R(W^2)$$

$$I^*(W) \equiv I(W^2)$$

It is shown in Appendix C that $P(T)$, $R(T)$ and $I(T)$ are analytic in the domain $\{0<|T|<\rho,0<\arg(T)<2\pi\}$. Hence, P^*, R^*, and I^* are analytic for $W\varepsilon D$. Because R^* and I^* are real-valued and continuous for $\arg(W)=0$, $0<|W|<r$, we have from the reflection principle

$$R^*(\overline{W}) = \overline{R^*(W)} \qquad \text{(B.1)}$$

$$I^*(\overline{W}) = \overline{I^*(W)} \qquad \text{for } W\varepsilon D \text{ (B.2)}$$

Let us define

$$P^{+}(W) = R^{*}(W) - iI^{*}(W) \qquad (B.3)$$

then we can write

$$P^{+}(\overline{W}) = R^{*}(\overline{W}) - iI^{*}(\overline{W}) \qquad W \varepsilon D$$

By combining this relationship with Eqs. (B.1) and (B.2), we obtain

$$P^{+}(\overline{W}) = \overline{R^{*}(W)} - i\overline{I^{*}(W)}$$

or

$$P^{+}(\overline{W}) = \overline{P^{*}(W)} \qquad (B.4)$$

Because $P^{*}(W)$ is real-valued for $0<|W|<r, \arg(W) = \frac{\pi}{2}$ we see from the above equation that $P^{+}(W)$ is real-valued for $0<|W|<r, \arg(W) = -\frac{\pi}{2}$. Hence we have from the reflection principle

$$P^{+}(We^{-\pi i}) = \overline{P^{+}(\overline{W})} \qquad W \varepsilon \{0<|W|<r, 0<\arg(W)<\tfrac{\pi}{2}\}$$

Combining the above relationship with Eq. (B.3) produces

$$P^{+}(We^{-\pi i}) = P^{*}(W)$$

or

$$R^{*}(We^{-\pi i}) - iI^{*}(We^{-\pi i}) = R^{*}(W) + iI^{*}(W) \qquad (B.5)$$

In the T plane equation (B.5) becomes

$$R(Te^{-2\pi i}) - iI(Te^{-2\pi i}) = R(T) + iI(T) \qquad (B.6)$$

We have just established that equation (B.6) is a necessary condition for P(T) to be real-valued along the positive imaginary axis. To see that it is sufficient we combine Eqs. (B.5), (B.1) and (B.2) to obtain

$$R^{*}(We^{-\pi i}) - iI^{*}(We^{-\pi i}) = \overline{R^{*}(\overline{W})} + i\overline{I^{*}(\overline{W})}$$

or

$$R^{*}(We^{-\pi i}) - iI^{*}(We^{-\pi i}) = \overline{\left(R^{*}(\overline{W}) - iI^{*}(\overline{W})\right)}$$

Combining with Eq. (B.3) produces

$$P^{+}(We^{-\pi i}) = \overline{P^{+}(\overline{W})}$$

Hence, we see that $P^{+}(W)$ is real-valued for $W\varepsilon\{0<|W|<r, \arg(W) = -\frac{\pi}{2}\}$; moreover, that $P^{*}(W)$ is real-valued for $W\varepsilon\{0<(W)<r, \arg(W) = \frac{\pi}{2}\}$.

APPENDIX C

In this section it is shown that $R(T)$ and $I(T)$ are analytic functions for $T\varepsilon\Delta^{*} = \{0<|T|<\rho, 0<\arg(T)<2\pi\}$ where $0<\rho<1$.

We first recall that $P(T)$ is analytic for $T\varepsilon\Delta$ and real-valued and continuous for $0<\mathrm{Im}(T) \leq \rho$. From the reflection principle we see that $P(T)$ and $\zeta=e^{P(T)}$ are analytic for $T\varepsilon\Delta^{*}$. From Eq. (A.1) we obtain

$$\zeta = e^{R}(\mathrm{Cos}I+i\mathrm{Sin}I)$$

$$\frac{1}{\zeta} = e^{-R}(\mathrm{Cos}I-i\mathrm{Sin}I)$$

By combining these two relationships, we obtain

$$\mathrm{Sin}I = \frac{1}{2i}(\zeta e^{-R}{}_{+}e^{R}/\zeta)$$

Substituting the above relationship into Eq. (A.4) produces

$$R' = \frac{1}{F^{2}\pi}\,\frac{e^{-3R}}{T}\,\frac{1}{2i}(\zeta e^{-R}+e^{R}/\zeta)$$

which may be written in the form

$$R(T) = R(r_{o})+\int_{r_{o}}^{T} K(\zeta,R)\,d\xi \qquad (C.1)$$

where $0<r_{o}<\rho$ and

$$K(\zeta,R) = \frac{1}{2\pi F^{2}i}\,\frac{e^{-3R}}{T}\,(\zeta e^{-R}+e^{R}/\zeta) \qquad (C.2)$$

Let us examine under what conditions the successive approximation scheme

$$R_{o} = R(r_{o})$$

$$R_{k+1} = R_{o} + \int_{r_{o}}^{T} K(R_{k},\zeta)\,d\xi \qquad (C.3)$$

will converge to a solution if $T\varepsilon\{0<r_o\leq|T|\leq\rho_o<\rho, 0\leq\arg(T)\leq 2\pi\}$

Because ζ is bounded and nonvanishing in the domain $T\varepsilon\Delta^*\cup\bar{\Delta}^*$ there exist a constant $M>0$ such that

$$e^{-M}<|\zeta|<e^{M} \qquad \text{for} \quad T\varepsilon\Delta^*\cup\bar{\Delta}^*$$

Hence from equation (C.3) we have

$$\|R_{k+1}\|\leq|R_o|+(\rho_o-r_o)\|K(R_k,\zeta)\| \qquad (C.4)$$

where the symbol $\|\ \|$ denotes the maximum absolute value of the function inclosed over the domain of study. From equation (C.2), we obtain

$$\|K(\zeta,R_k)\|\leq\frac{1}{2\pi F^2}\frac{e^{5\|R_k\|}}{r_o} \qquad \text{for} \quad T\varepsilon\{r_o\leq|T|<\rho_o, 0\leq\arg(T)\leq 2\pi\}$$

By combining this relationship with (C.4), we obtain

$$\|R_{k+1}\|\leq|R_o|+\frac{\rho_o-r_o}{r_o}\frac{2}{\pi F^2}e^{5\|R_k\|}$$

Suppose $\|R_k\|<M$ then we would have

$$\|R_{k+1}\|<|R_o|+\frac{\rho_o-r_o}{r_o}\frac{2}{\pi F^2}e^{5M}$$

because $|R_o|<M$ we can always choose $\frac{\rho_o-r_o}{r_o}<0$ such that $\|R_{k+1}\|<M$. This condition requires that

$$\frac{\rho_o-r_o}{r_o}<\frac{\pi F^2}{2}e^{-5M}(M-|R_o|) \qquad (C.5)$$

Hence, if the inequality (C.5) is satisfied then

$$\|R_k\|<M \qquad \text{for} \qquad k=0,1,2,\ldots$$

From Eq. (C.3) we have

$$\|R_{k+1}-R_k\|\leq(\rho_o-r_o)\|K(R_k,\zeta)-K(R_{k-1},\zeta)\|$$

It is clear from Eq. (C.2) that the kernel K satisfied the Lipschitz condition

$$|K(R_{k+1},\zeta)-k(R_k,\zeta)|\leq\frac{C}{r_o}|R_{k+1}-R_k| \qquad \text{for} \quad T\varepsilon\{r_o\leq|T|\leq\rho_o, o\leq\arg(T)<2\pi\}$$

provided $\|R_k\|<M$ for $k=o,1,2,\ldots$ Thus, we obtain

$$\|R_{k+1}-R_k\|\leq(\frac{\rho o-ro}{r_o})C\|R_k-R_{k-1}\| \qquad C>0$$

Let us define

$$\gamma = \left(\frac{\rho_o - r_o}{r_o}\right) C \tag{C.6}$$

Then we have

$$\| R_2 - R_1 \| \leq \gamma \| R_1 - R_o \|$$

$$\| R_3 - R_2 \| \leq \gamma \| R_2 - R_1 \| \leq \gamma^2 \| R_1 - R_o \|$$

$$\| R_4 - R_3 \| \leq \gamma \| R_3 - R_2 \| \leq \gamma^3 \| R_1 - R_o \|$$

and in general

$$\| R_{k+1} - R_k \| \leq \gamma^k \| R_1 - R_o \| \tag{C.7}$$

Consider now the two expressions

$$R_k = R_o + (R_1 - R_o) + (R_2 - R_1) + (R_3 - R_2) + \ldots + (R_k - R_{k-1}) \tag{C.8}$$

$$S_k = |R_o| + \| R_1 - R_o \| + \| R_2 - R_1 \| + \| R_3 - R_2 \| + \ldots + \| R_k - R_{k-1} \| \tag{C.9}$$

Let $\varepsilon > 0$ be a number such that

$$\varepsilon < \frac{1}{C}$$

$$\varepsilon < \frac{\pi F^2}{2} e^{-5M} (M - |R_o|)$$

Moreover, let r_o and ρ_o be adjusted so that

$$\frac{\rho_o - r_o}{r_o} = \varepsilon$$

Then we see that Eq. (C.5) is satisfied and from Eq. (C.5) we note that $\gamma < 1$. Hence, from equation (C.8) and (C.6) we have

$$S_k \leq |Ro| + \| R_1 - R_o \| \left(1 + \gamma + \gamma^2 + \ldots + \gamma^{k-1}\right) = |R_o| + \| R_1 - R_o \| \frac{1 - \gamma^k}{1 - \gamma}$$

and

$$k \to \infty \quad S_k < \infty$$

Thus, from Eqs. (C.8) we see that R_k approaches a limit function $\tilde{R}$ as $k \to \infty$ which is analytic for $T \varepsilon \{ r_o \leq |T| \leq \rho_o,$ $o < \arg(T) < 2\pi \}$ and continuous and real-valued for $r_o \leq T \leq \rho_o$.

This limit function $\tilde{R}$ is equal to R for $r_o \le T \le \rho_o$ because it is unique among the class of functions which satisfy Eq. (C.1) and are continuous and bounded by M for $r_o \le T \le \rho_o$. To see that this is the case suppose there are two functions R_1 and R_2 which are bounded and satisfy equation (C.1) in the intevral $r_o \le T \le \rho_o$. Then

$$R_1 = R_o + \int_{ro}^{T} k(R_1, \zeta) d\xi$$

$$R_2 = R_o + \int_{ro}^{T} k(R_2, \zeta) d\xi$$

By following the same procedure which lead to Eq. (C.7), we obtain

$$\| R_1 - R_2 \| \le \gamma \| R_1 - R_2 \|$$

This inequality is possible only if $R_1 = R_2$. It is clear that for any point in the domain Δ r_o may be adjusted so that this point is also contained in the domain $r_o \le |T| \le r_o + \varepsilon$. Hence we conclude that R is analytic for $T \varepsilon \Delta^*$. Moreover, I is also analytic here because $I = \frac{1}{i}(P-R)$.

APPENDIX D - SYMBOLS

A	real-valued coefficient of the singularity
a_o, a_1, a_2, a_3, etc.	real-valued coefficients of a power series
a_o^*, a_1^*, a_2^*, etc.	real-valued coefficients of a power series
$a_{o,o}, a_{1,o}, a_{1,1}, a_{2,o}$	real-valued coefficients of an asymptotic expansion
a	elevation of gate opening relative to channel bottom
b	elevation of free streamline at infinity relative to channel bottom
b_o, b_1, b_2, etc.	real-valued coefficients of a power series
$b_{o,o}, b_{1,o}, b_{1,1}, b_{2,o}$	real-valued coefficients of an asymptotic expansion
C_c	coefficient of contraction defined by $C_c = \frac{b}{a}$
C_o, C_1, C_2, etc.	real-valued coefficients of an asymptotic expansion
$C_{o,o}, C_{1,o}, C_{1,1}$, etc.	real-valued coefficients of an asymptotic expansion
D	simply connected domain in the W plane
$\overline{D}$	boundary of the domain D
$f(t)$	the transformation $Z=f(t)$ which maps Γ onto the domain of the flow field
F^2	Froude number defined by $F^2 = \frac{U^2 b}{g}$
g	acceleration due to gravity
H	the elevation of the energy line relative to the channel bottom
Q	discharge through sluice gate
K	positive constant

q, $q(\sigma)$	speed at a point on the free streamline
r_o, r_1, r_2, etc.	real-valued coefficients of an asymptotic expansion
u	x component of velocity in the physical plane
U	velocity of the jet far downstream from the gate opening
v	y component of velocity in the physical plane
Im( )	imaginary part of ( )
Γ	simply connected domain in the t plane
Γ	boundary of domain Γ
$\Gamma, \Gamma(\sigma)$	special function defined when first used
$\Gamma_{o,o}, \Gamma_{1,o}, \Gamma_{1,1}$, etc.	special numbers which are defined when first used
Δ	simply connected domain in the T plane
Δ^*	simply connected domain in the T plane
$\zeta, \zeta(z), \zeta(t)$	normalized complex velocity defined by $\zeta = \frac{1}{U}(u-iv)$
λ	real-valued parameter
ξ	complex valued variable
$\tau, \tau(\sigma)$	error function
ε	belongs to

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