COMPENSATION OF SAMPLED-DATA SYSTEMS

L. M. Benningfield
Assistant Professor of Electrical Engineering

G. V. Lago
Associate Professor of Electrical Engineering

Reprinted from
Proceedings of the National Electronics Conference, Volume XIII
Hotel Sherman, Chicago, Illinois, October 7, 8, 9, 1957
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COMPENSATION OF SAMPLED-DATA SYSTEMS

L. M. BENNINGFIELD and G. V. LAGO
University of Missouri, Columbia, Missouri

Abstract.—Compensation of sampled-data systems is straightforward if the compensation network can be separated from the rest of the system by samplers. However, use of directly connected continuous networks presents more of a problem. Existing theory does not adequately cover such compensation.

This paper examines the above situation using $z$-transform theory and continuous network realizability conditions. Lack of a general correlation between the number of $z$-plane and $s$-plane zeros presents the major problem. This difficulty becomes apparent when attempting to find a principle Laplace transform for the final system impulse response following $z$-plane compensation. By imposing certain restrictions on $z$-plane pole locations and by approximating the desired system impulse response in the $s$-plane, this paper demonstrates the use of directly connected RC networks in lieu of discrete networks or digital computers for compensating sampled-data systems.

Studies are also made concerning the requirements necessary to eliminate the need for approximating the final impulse response. Graphs are presented to allow the solution of this problem for third order systems.

I. INTRODUCTION

Numerous papers published during the last few years

 have considerably advanced the field of sampled-data systems. Continuous system techniques for using the system open-loop frequency locus (Nyquist criterion), maximum moduli contours (M circles), root-locus, and Bode plots have all been extended in some form to sampled-data systems. Only limited success has been obtained in devising a direct sampled-data system synthesis procedure.

The form of the root-locus technique as now applied to sampled-data systems is still limited to providing, in all but simple cases, only discrete network compensation or compensation networks separated by samplers from the rest of the system. It is the purpose of this paper to extend the root-locus technique so that continuous RC networks can be utilized as compensation elements. In accomplishing the above purpose, the physical significance of certain $z$-transform operations is investigated, difficulties with $z$-transform zeros are studied, RC transfer function requirements in the $z$-plane are determined, and a procedure for arriving at the overall objective is given.

II. Z-TRANSFORM RELATIONS

Use of the $z$-transformation provides one of the most useful approaches to sampled-data system study. As previously mentioned, root-locus methods have been adapted to the open-loop $z$-plane pole-zero configuration for systems of the form shown in Fig. 1. In this figure, $g_1(t)$ and $g_2(t)$ are the impulse responses of the compensation network and the fixed part of the system respectively, while $G_1(s)$ and $G_2(s)$ are the corresponding Laplace transforms, and $G_1(z)$ and $G_2(z)$ are the accompanying $z$-transforms.

If the loop is closed on only the fixed portion of the system, the root-locus in the $z$-plane is determined by exactly the same set of rules as used for continuous $s$-plane plots. Of course, the behavior of the system in terms of $z$-plane poles and zeros has to be interpreted in terms of $z$-transform theory.

The loci can be reshaped by inserting new poles and zeros in the open-loop system transfer function with the view to obtaining better system response.
The usual interpretation of this procedure is to lump the added poles and zeros into $G_1(z)$, the $z$-transform of the compensation network, as previously noted. This gives the complete open-loop system function as $G(z) = G_1(z)G_2(z)$. Multiplication in this manner implies that the compensation network is separated from the rest of the system by a sampler as illustrated in Fig. 1.

As an example to illustrate the above procedure, let it be supposed that

$$G_2(s) = \frac{K}{s(s + 1)}$$

and that the sampling period, $T$, is 1 second from which the fixed system $z$-transform is

$$G_2(z) = \frac{0.632Kz}{(z - 1)(z - 0.368)}$$

The root-locus for this system is shown in Fig. 2. Let it also be supposed that, as $K$ is varied, no set of closed-loop pole locations gives a response that is satisfactory. Therefore a new pole and zero are inserted leading to

$$G_1(z) = 1.67 \frac{(z - 0.4)}{(z - e^{-10})}$$

The overall open-loop function is now

$$G(z) = G_1(z)G_2(z) = \frac{K_0z(z - 0.4)}{(z - 1)(z - 0.368)(z - e^{-10})}$$

with its corresponding root-locus shown in Fig. 3. The assumption is that the gain, $K_0$, can now be set so that the response of the system satisfies the specifications.

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**Fig. 1—Prototype sampled-data control system with discrete compensation.**

**Fig. 2—Z-plane root-locus for system described by (2).**

**Fig. 3—Z-plane root-locus for compensated function given by (4).**
Figure 4 depicts a system where the compensation network is not separated from the fixed part of the system by a sampler. If a system of this form could be realized, the added expense of the sampler could be saved, reliability improved, and ordinary continuous networks used as compensation. The following paragraphs set forth design methods for arriving at such a system.

As a start in the direction of continuous network compensation the system described by (4) can be thought of as representing the system of Fig. 4. In other words, this equation can also represent the z-transform of the product \( G(s) = G_1(s)G_2(s) \). The \( G(s) \) corresponding to (4) is

\[
G(s) = \frac{K_1(s + 0.885)}{s(s + 1)(s + 10)}
\]

By comparison of this function with the original continuous part of the system, (1), a continuous compensation function, (6), results.

\[
G_1(s) = \frac{G(s)}{G_2(s)} = \frac{K_1(s + 0.885)}{K(s + 10)}
\]

The above steps indicate a method for arriving at a continuous compensation function. Unfortunately this compensation is not generally physically realizable as an RC network except for systems that meet a certain set of restrictions. These restrictions are explained and methods for arriving at RC compensation networks in the majority of commonly encountered control systems are presented.

### III. RESTRICTIONS ON THE PROCEDURE AND RC NETWORK REQUIREMENTS

The preceding example was chosen to illustrate the proposed method to advantage but, as mentioned, in the general case this method does not lead to a compensation network that is RC realizable. The reason for this is now examined. Let it be supposed for the moment that the Laplace transform of the fixed part of the system has three poles and no finite zeros as given by

\[
G_2(s) = \frac{K}{s(s + p_1)(s + p_2)}
\]

The impulse response of this system has an initial value equal to zero, an initial slope equal to zero, and an initial second derivative equal to \( K \) as can be determined from the initial value theorem. The z-transform corresponding to functions of the form (7) and having sampling period, \( T \) seconds, in general has the form

\[
G_2(z) = \frac{K_0(z - u_1)}{(z - 1)(z - \alpha_1)(z - \alpha_2)}
\]

where \( \alpha_1 = e^{-p_1T} \), \( \alpha_2 = e^{-p_2T} \), and \( u_1 \) is a function of \( p_1 \), \( p_2 \), and \( T \). It should
be noticed that the additional zero at \( z = u_1 \) has been introduced in taking the \( z \)-transform. For all functions which have a nonzero value at the first sampling instant after \( t = 0 \) the highest power of \( z \) in the numerator of \( G_2(z) \) will be one less than the highest power of \( z \) in the denominator. This means the introduction of \( z \)-plane zeros for which no \( s \)-plane images exist in many cases.

When transformations are made from the \( s \)-plane to the \( z \)-plane, the introduction of these zeros presents no problem. However let it be supposed that compensation is to be performed in the \( z \)-plane using (8) as the fixed system \( z \)-transform. By adding an additional real pole and zero, the overall open-loop function is

\[
G(z) = \frac{K_1(z)(z - u_1)(z - u_2)}{(z - 1)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)}
\]

When this function is returned to the \( s \)-plane the only assurance is that the function will have a zero initial value. The form of \( G(s) \) corresponding to (9) is

\[
G(s) = \frac{K_2(s + a_1)(s + a_2)}{s(s + p_1)(s + p_2)(s + p_3)}
\]

The net result has been the addition of two \( s \)-plane zeros instead of one. The compensation function for this case would be

\[
G_1(s) = \frac{G(s)}{G_2(s)} = \frac{K_2(s + a_1)(s + a_2)}{K(s + p_3)}
\]

From network synthesis theory the following physical realizability conditions for RC transfer functions are obtained.

1. All poles must be simple and must lie on the negative real axis with no poles at infinity or zero.
2. Zeros may be anywhere and of any order.

The first requirement means that (11) is not physically realizable as an RC network transfer function. Since added \( z \)-plane poles that actually appear in the final \( G(z) \) have direct \( s \)-plane images, the first requirement above also means that such poles must be placed on the real \( z \)-plane axis from \((0,0)\) to \((1,0)\). It might be noted at this point that poles can be added as a direct cancellation of \( z \)-plane zeros and not appear in the final \( G(z) \). It is therefore possible to broaden the previous region for added \( z \)-plane poles to include the cancellation of zeros inside the unit circle.

The lack of a zero correspondence between Laplace and \( z \)-transforms, however, cannot be eliminated in the manner that was used to satisfy the compensation pole requirements. An entirely different approach is needed. In order to have reasonable freedom in the root-locus compensation procedure and still obtain a final RC realizable compensation function the following steps are, therefore, proposed:

1. Obtain the Laplace transform of the open-loop continuous system, \( G_2(s) \).
2. Find \( G_2(z) \) and its root-locus.
3. Alter the \( z \)-plane root-locus of the previous step by adding only poles on the real axis from \((0,0)\) to \((1,0)\) or to cancel zeros inside the unit circle. There is no location restriction on zeros.
4. Find the primary Laplace transform, \( G(s) \), of the \( z \)-plane pole-zero pattern resulting from Step 3.
5. Approximate the Step 4 $G(s)$ impulse response by a new impulse response, $G'(s)$, having the same relative degree between the highest power of $s$ in the numerator and denominator as in the Step 1 fixed transfer function.

6. Find the compensation function $G_1(s) = G'(s)/G_2(s)$ and realize this function as an RC transfer function by using any standard synthesis procedure.

The Step 5 approximation procedure is generally best accomplished by adding remote poles on the negative real $s$-plane axis as is done in Guillemin's direct synthesis procedure for continuous feedback systems. In fact any added poles must be on the negative real axis. It is this step that circumvents the occurrence of poles at infinity in the final transfer function $G_1(s)$ as happened in (11). The remaining steps are straightforward after one gains facility in altering root-locus shapes to improve system response.

While the $z$-plane pole restrictions may appear rather confining, it is believed that these restrictions allow reasonable freedom in compensation since similar restrictions are required in the RC network compensation of continuous systems.

**IV. EXAMPLES**

As an example of the above procedure, consider

\begin{equation}
G_2(s) = \frac{K}{s(s + 1)(s + 2)}; \quad T = 1 \text{ sec.}
\end{equation}

\begin{equation}
G_2(z) = \frac{K_0 z(z + 0.368)}{(z - 1)(z - 0.368)(z - 0.1354)}
\end{equation}

The root locus for this function is shown in Fig. 5. It is apparent that the limiting factors on rise time are the poles at $z = 0.368$, and $0.1354$. By cancelling these poles with zeros and introducing new poles at $z = e^{-5}$ and $z = e^{-10}$ it appears that improvement in rise time can be obtained. However it must be remembered that as the gain is increased a root-locus of the form shown in Fig. 6 results. Consequently there still has not been much improvement in the system transient response. By going one step further and cancelling
the zero at $z = -0.368$ and inserting a new zero at $z = 0.01$, the root-locus is modified to that shown in Fig. 7. While this may not be advisable due to the fact that a closed-loop pole will always remain on the positive real $z$-plane axis, it will at least illustrate the procedure. The complete open-loop response, $G(z)$, is now

$$G(z) = \frac{K_0 z(z - 0.01)}{(z - 1) (z - e^{-5})(z - e^{-10})}$$

(14)

$$G(s) = \frac{K_1(s + 4.02)}{s(s + 5)(s + 10)}$$

(15)

Without approximation this leads to a compensation function

$$G_1(s) = \frac{K_1(s + 4.02)(s + 1)(s + 2)}{K(s + 5)(s + 10)}$$

(16)

Equation 16, of course, is not RC realizable. Approximating the overall impulse response by

$$G'(s) = \frac{50K_1(s + 4.02)}{s(s + 5)(s + 10)(s + 50)}$$

(17)

results in the RC realizable compensation function

$$G_1'(s) = \frac{50K_1(s + 4.02)(s + 1)(s + 2)}{K(s + 5)(s + 10)(s + 50)}$$

(18)

A similar procedure can be applied to other system functions although in many cases a simpler compensation results.

V. Z-PLANE ZEROS AND LIMITATIONS

It is theoretically possible to avoid the necessity of approximating the system response by properly positioning the zeros of the $z$-transform. Some study has been given to the movement of $z$-plane zeros for functions of the form

$$G(s) = \frac{K}{s(s + p_1)(s + p_2)}$$

(19)

$$G(z) = \frac{K_0 z(z + u)}{(z - 1)(z - e^{-p_1T})(z - e^{-p_2T})}$$

(20)
Figure 8 and 9 illustrate the location of \( u \) as a function of \( p_1T \) and \( p_2T \). This set of curves will give the location of the z-plane zero to be added in compensating a function of the form

\[
G_2(z) = \frac{K_0 z}{(z-1)(z - e^{-p_1T})}
\]

in order to be assured that the corresponding Laplace transform is of the form of (19).

Figure 10 does the same for functions of the form

\[
G(s) = \frac{K}{s[(s + \alpha)^2 + \beta^2]}
\]

\[
G(z) = \frac{K_0 z + u}{(z-1)(z^2 - 2z\epsilon^{-\alpha T} \cos \beta T + \epsilon^{-2\alpha T})}
\]

Further study has also been given to the movement of the z-plane zeros of higher order functions when using lead-lag compensation. The system

\[
G_2(s) = \frac{1}{s(s + 1)(s + 2)}
\]

is the basic system to which a lead-lag compensation is applied to give

\[
G(s) = \frac{(s + a)}{s(s + 1)(s + 2)(s + b)}
\]

For \( b = 10 \) and the sampling period, \( T \), equal to 0.694, Fig. 11 illustrates the movement of the z-plane zeros for the z-transform of (25) as ‘\( a \)’ is varied. Other functions of this form have been found to give similar curves. It is not feasible to determine general curves for such higher order functions.
Fig. 9—Locus of the z-plane zero, "u", as a function of "pT" for functions of the form

\[ G(s) = \frac{K}{s^2(s+p)} \quad G(z) = \frac{K_0(z+u)}{(z-1)^2(z-e^{-pT})} \]

A word of caution is now in order concerning the manipulations of z-plane zeros. While the z-plane poles of a sampled function remain the same for any instant in the sampling period, the zeros move for each different instant as can be seen from the modified z-transformation integral. Therefore a given set of z-plane zeros is valid only for the train of impulses taken at a particular time in the sampling period. If hidden oscillations exist in the system, then caution should be used in dealing with any set of z-plane zeros. For the above reason, it has been found advisable to avoid working with the z-plane zeros whenever possible.

Fig. 10—Loci of the z-plane zero, "u", as a function of "αT", "βT" for functions given by (22) and (23).
The preceding discussion has extended the existing root-locus technique to the determination of RC realizable continuous compensation for sampled-data systems. This procedure therefore eliminates the need for using discrete networks or samplers in compensating sampled-data systems. While there are certain limitations in applying the procedure, a reasonably wide class of functions are covered and reasonable freedom is available in the compensation process.

VI. CONCLUSIONS

Functions of a type not mentioned previously cannot be handled by the procedure discussed herein. These functions are those having complex zeros in $G_2(s)$. Any alteration of $G(z)$ results in the complex zeros occurring as poles in the final compensation function. A similar limitation is evident in continuous systems in that no complex zeros can be cancelled.
Additional limited study has been given the movement of $z$-plane zeros and curves are presented to eliminate the need for approximating the impulse response of certain functions in realizing RC compensation networks.

Precautions concerning the significance of $z$-plane zeros are pointed out and reference made to the modified $z$-transform for determining the complete system response in cases that might have hidden oscillations.

**ACKNOWLEDGMENT**

The authors are indebted to the National Science Foundation for making this work possible with their grant NSF-G2304.

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