FRAMES AND SUBSPACES

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FRAMES AND SUBSPACES

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Frames and Subspaces

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ABSTRACT

This thesis will consist of three parts. In the first part we find the closest probabilistic Parseval frame to a given probabilistic frame in the 2 Wasserstein Distance. It is known that in the traditional ℓ^2 distance the closest Parseval frame to a frame $\Phi = \{\varphi_i\}_{i=1}^N \subset \mathbb{R}^d$ is $\Phi^\dagger = \{\varphi_i^\dagger\}_{i=1}^N = \{S^{-1/2}\varphi_i\}_{i=1}^N$ where S is the frame operator of Φ. We use this fact to prove a similar statement about probabilistic frames in the 2 Wasserstein metric.

In the second part, we will associate a complex vector with a rank 2 real projection. Using this association we will answer many open questions in frame theory . In particular we will prove Edidin's theorem in phase retrieval in the complex case, answer a question on mutually unbiased bases, a question on equiangular lines, and a question on fusion frames.

In the last part we will give a way to calculate the exact constant for the $\ell_1 - \ell_2$ inequality and use this method to prove a couple of interesting theorems.

Chapter 1

Introduction to Chapter 2

The first part of this thesis will be devoted to probabilistic frames. In this part we will find the closest probabilistic Parseval frame to a given probabilistic frame in the 2 Wasserstein metric. This result has applications to quantum information theory.

The notion of probabilistic frames was first introduced in [33] in the setting of probability measures on the unit sphere, and was later generalized to probability measures on $\mathbb{R}^d$ in [31]. In essence, this theory is a generalization of the theory of finite frames which has seen a wealth of activities in recent years, [21, 27, 25, 26, 42].

The main contribution of this portion of the thesis is Theorem 2.2.13 where $\mu^\dagger$ is the push-forward of μ through $S_\mu^{-1/2}$, that is

$$\mu^\dagger(B) = \mu(S^{1/2}B)$$

To prove this we first start with a discrete probabilistic frame $\mu_{\Phi,w} = \sum_{i=1}^N w_i \delta_{\varphi_i}$ where $\Phi = \{\varphi_i\}_{i=1}^N \subset \mathbb{R}^d$, and $w = \{w_i\}_{i=1}^N \subset [0, \infty)$ is a set of weights such that $\sum_{i=1}^N w_i = 1$ and define a new frame $\Phi_w = \{\sqrt{w_i}\varphi_i\}_{i=1}^N$ and identify the two. We know that $\Phi^\dagger = \{\varphi_i^\dagger\}_{i=1}^N = \{S^{-1/2}\varphi_i\}_{i=1}^N$ is the unique closest Parseval frame to Φ 2.1.1.

To deal with the permutations of the 2 Wasserstein distance, given a frame $\Phi =$

$\{\varphi_k\}_{k=1}^N$ for any $i \in \{1, 2, \ldots, N\}$ we consider the new set of vectors

$$\Phi_i = \{\varphi_k\}_{k=1, k\neq i}^N \cup \{a_j\varphi_i\}_{j=1}^p = \{\varphi_k'\}_{k=1}^{N+p-1}$$

where $\sum_{j=1}^p a_j^2 = 1$. We know the distance of Φ to its canonical Parseval frame and Φ_i to its canonical Parseval frame is the same (Lemma 2.2.6). Doing this procedure for i from 1 to N and replacing Φ with the new set of vectors Φ_i we obtained after each procedure, we prove that $\Phi^\dagger = \{S^{-1/2}\varphi_i\}_{i=1}^N$ is the closest discrete probabilistic Parseval frame to Φ (Proposition 2.2.7).

To deal with continuous probabilistic frames we estimate using discrete frames. In particular we show via Theorem 2.2.8 given μ a probabilistic frame with frame bounds A and B, and $\epsilon > 0$. There exists a finite probabilistic μ_Φ with frame bounds A', B' such that $A' \geqslant A - \epsilon$, $B' \leqslant B + \epsilon$ and

$$\|\mu - \mu_\Phi\|_{W_2} < \epsilon$$

We also consider the map

$$F(\Phi) = F(\{\varphi_i\}_{i=1}^N) = S_\Phi^{-1/2}(\{\varphi_i\}_{i=1}^N) = \{S_\Phi^{-1/2}\varphi_i\}_{i=1}^N.$$

on the set of all discrete probabilistic frames. In particular we prove an uniform continuity property of F by Theorem 2.2.14 which will allow us to prove that F extends uniquely to the set of all probabilistic frames. We then prove that F also coincides with the push-forward of μ through $S_\mu^{-1/2}$ on the set of all probabilistic frames.

We also prove that $F(\mu)$ is the unique closest probabilistic Parseval frame to μ by Theorem 2.2.16.

Chapter 2

Optimal Properties of the Canonical Tight Probabilistic Frame

2.1 Introduction

The notion of probabilistic frames was first introduced in [33] in the setting of probability measures on the unit sphere, and was later generalized to probability measures on $\mathbb{R}^d$ in [31]. In essence, this theory is a generalization of the theory of finite frames which has seen a wealth of activities in recent year, [21, 27, 25, 26, 42].

2.1.1 Review of finite frame theory

Before we give the definition and some elementary properties of probabilistic frames, we recall that a set $\Phi = \{\varphi_i\}_{i=1}^N \subset \mathbb{R}^d$ is a frame for $\mathbb{R}^d$ if and only if there exist $0 < A \leqslant B < \infty$ such that

$$A\|x\|^2 \leqslant \sum_{i=1}^N \langle x, \varphi_i\rangle^2 \leqslant B\|x\|^2 \qquad \forall\, x \in \mathbb{R}^2.$$

The frame Φ is a *tight frame* if we can choose $A = B$. Furthermore, if $A = B = 1$, Φ is called a Parseval frame. In the sequel the set of frames for $\mathbb{R}^d$ with N vectors will be denoted by $\mathcal{F}(N, d)$, and when the context is clear simply $\mathcal{F}$. The subset of frames with frame bounds $0 < A \leqslant B < \infty$ will be denoted $\mathcal{F}_{A,B}(N,d)$, or simply $\mathcal{F}_{A,B}$. We equip the set $\mathcal{F}(N,d)$ with the metric

$$d(\Phi,\Psi) = \sqrt{\sum_{i=1}^{N} \|\varphi_i - \psi_i\|^2} = \sqrt{\sum_{i=1}^{d} \|R_i - P_i\|^2} \tag{2.1}$$

where $\Phi = \{\varphi_i\}_{i=1}^N, \Psi = \{\psi_i\}_{i=1}^N) \in \mathcal{F}(M,d)$, $\{R_i\}_{i=1}^d \subset \mathbb{R}^N$ and $\{P_i\}_{i=1}^d \subset \mathbb{R}^N$ denote the rows of Φ, and those of Ψ, respectively.

Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame for $\mathbb{R}^d$. Throughout the paper we shall abuse notation and denote the *synthesis matrix* of the frame by Φ, the $d \times N$ matrix whose i^{th} column is φ_i. The matrix

$$S := S_\Phi = \Phi\Phi^T = \sum_{i=1}^{N} \langle \cdot, \varphi_i \rangle \varphi_i$$

is the *frame matrix*. It is known that $\Phi = \{\varphi_i\}_{i=1}^N$ is a frame for $\mathbb{R}^d$ if and only if S is a positive definite matrix. Moreover, the smallest eigenvalue of S is the optimal lower frame bound, while its largest eigenvalue is the optimal upper frame bound. Φ is a tight frame if and only if S is a multiple of the $d \times d$ identity matrix. In particular, Φ is a Parseval frame if and only if $S = I$.

If Φ is a frame, then S is positive definite and thus invertible. Consequently,

$$\Phi^\dagger = \{\varphi_i^\dagger\}_{i=1}^N = \{S^{-1/2}\varphi_i\}_{i=1}^N$$

is a Parseval frame, leading to following reconstruction formula:

$$x = \sum_{i=1}^{N} \left\langle x, \varphi_i^\dagger \right\rangle \varphi_i^\dagger = \sum_{i=1}^{N} \left\langle x, \varphi_i^\dagger \right\rangle \varphi_i^\dagger \, \forall \, x \in \mathbb{R}^d.$$

In addition, $\Phi^\dagger$ is the unique Parseval frame which solves the following problem [22, Theorem 3.1]:

$$\min\{d(\Phi,\Psi)^2 = \sum_{i=1}^N \|\varphi_i - \psi_i\|^2 : \Psi = \{\psi_i\}_{i=1}^N \subset \mathbb{R}^d,\ Parseval\, frame\}. \tag{2.2}$$

To be specific,

Theorem 2.1.1. *[22, Theorem 3.1]*

If $\Phi = \{\varphi_i\}_{i=1}^N$ is a frame for $\mathbb{R}^d$, then $\Phi^\dagger = \{\varphi_i^\dagger\}_{i=1}^N = \{S^{-1/2}\varphi_i\}_{i=1}^N$ is the unique solution to (2.2)*.*

In Section 2.2, and for the sake of completeness, we give a new and simple proof of this result and we refer to [4, 14, 16] for related results.

2.1.2 Probabilistic frames

The main goal of this paper is to characterize the minimizers of an optimal problem analog of (2.2) for probabilistic frames. To motivate the definition of a probabilistic frame, we note that given a frame $\Phi = \{\varphi_i\}_{i=1}^N \subset \mathbb{R}^d$, the discrete probability measure

$$\mu_\Phi = \tfrac{1}{N}\sum_{k=1}^N \delta_{\varphi_k}$$

has the property that its support ($\{\varphi_k\}_{k=1}^N$) spans $\mathbb{R}^d$ and that it has finite second moment, i.e.,

$$\int_{\mathbb{R}^d} \|x\|^2 d\mu_\Phi(x) = \tfrac{1}{N}\sum_{k=1}^N \|\varphi_k\|^2 < \infty.$$

The probability measure μ_Φ is an example of a probabilistic frame that was introduced in [33, 31].

More specifically, a Borel probability measure μ is a *probabilistic frame* if there exist $0 < A \leqslant B < \infty$ such that for all $x \in \mathbb{R}^d$ we have

$$A\|x\|^2 \leqslant \int_{\mathbb{R}^d} |\langle x, y\rangle|^2 d\mu(y) \leqslant B\|x\|^2. \tag{2.3}$$

The constants A and B are called *lower and upper probabilistic frame bounds*, respectively. When $A = B$, μ is called a *tight probabilistic frame*. In particular, when $A = B = 1$, μ is called a *Parseval probabilistic frame*.

A special class of probabilistic frames that will be considered in the sequel consists of discrete measures $\mu_{\Phi,w} = \sum_{i=1}^N w_i \delta_{\varphi_i}$ where $\Phi = \{\varphi_i\}_{i=1}^N \subset \mathbb{R}^d$, and $w = \{w_i\}_{i=1}^N \subset [0, \infty)$ is a set of weights such that $\sum_{i=1}^N w_i = 1$. A probability measure such as $\mu_{\Phi,w}$ will be termed *finite probabilistic frame*, if and only if it is a probabilistic frame for $\mathbb{R}^d$. When the context is clear we will simply write μ for $\mu_{\Phi,w}$. We shall also identify a finite probabilistic frame $\mu_{\Phi,w}$ with the frame $\Phi_w = \{\sqrt{w_i}\varphi_i\}_{i=1}^N$, as both have the same frame bounds. We refer to the surveys [32, 43] for an overview of the theory of probabilistic frames.

We shall prove an analog of Theorem 2.1.1 by endowing the set of probabilistic frames with the Wasserstein metric. Let $\mathcal{P} := \mathcal{P}(\mathcal{B}, \mathbb{R}^d)$ denote the collection of probability measures on $\mathbb{R}^d$ with respect to the Borel σ-algebra $\mathcal{B}$. Let

$$\mathcal{P}_2 := \mathcal{P}_2(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P} : M_2^2(\mu) := \int_{\mathbb{R}^d} \|x\|^2 d\mu(x) < \infty \right\}$$

be the set of all probability measures with finite second moments. For $\mu, \nu \in \mathcal{P}_2$, let $\Gamma(\mu, \nu)$ be the set of all Borel probability measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ whose marginals are μ and ν, respectively, i.e., $\gamma(A \times \mathbb{R}^d) = \mu(A)$ and $\gamma(\mathbb{R}^d \times B) = \nu(B)$ for all Borel subsets A, B in $\mathbb{R}^d$. The space $\mathcal{P}_2$ is equipped with the 2-*Wasserstein metric* given by

$$W_2^2(\mu, \nu) := \min \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 d\gamma(x, y), \gamma \in \Gamma(\mu, \nu) \right\}. \tag{2.4}$$

The minimum defined by (2.4) is achieved at a measure $\gamma_0 \in \Gamma(\mu, \nu)$, that is:

$$W_2^2(\mu, \nu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 d\gamma_0(x, y).$$

We refer to [2, Chapter 7], and [49, Chapter 6] for more details on the Wasserstein spaces.

2.1.3 Our contributions

The investigation of probabilistic frames is still at its initial stage. For example, in [52] the authors introduced the notion of transport duals and used the setting of the Wasserstein metric to investigate the properties of such probabilistic frames. In particular, this setting offers the flexibility to find (non-discrete) probabilistic frames which are duals to a given probabilistic frame. Transport duals are the probabilistic analogues of alternate duals in frame theory [27, 41]. The main contribution of this paper (Theorem 2.2.13) is to investigate the properties of the canonical Parseval probabilistic frame associated to a given probabilistic frame, see Section 2.2 for definitions. To prove this result we approximate a given probabilistic frame with one that is compactly supported and whose frame bounds are controlled in a precise way (Theorem 2.2.8). In the process of proving our main result, we prove a number of results that are of interest on their own right. For example, in Section 2.2 we establish a number of new results about the canonical Parseval frame $\Phi^\dagger$ associated to a frame Φ.

2.2 Optimal Parseval probabilistic frames

Before proving our main result in Section 2.2.3, we revisit the canonical Parseval frame $\Phi^\dagger$ associated to a given frame $\Phi = \{\varphi_k\}_{k=1}^N \subset \mathbb{R}^d$. In particular, Section 2.2.1

considers the continuity properties of the map $F(\Phi) = \Phi^{\dagger}$. In Section 2.2.2 we show how a probabilistic frame can be approximated in the 2-Wasserstein metric by a sequence of finite frames whose bounds are controlled by those of the initial probabilistic frame. While such approximation for probability measures in the 2-Wasserstein metric is well known [49, Theorem 6.18], our key contribution here is the control of the frame bounds of the approximating sequence.

2.2.1 Continuity properties of the canonical Parseval frame

In this section we revisited the canonical Parseval frame $\Phi^{\dagger}$ associated to a given frame $\Phi = \{\varphi_k\}_{k=1}^N \subset \mathbb{R}^d$. First, we give a new and elementary proof of Theorem 2.1.1.

Proof. Proof of Theorem 2.1.1 We first note that a frame $\Psi \subset \mathbb{R}^d$ is Parseval if the rows of its synthesis matrix are orthonormal. Furthermore, $\Psi \subset \mathbb{R}^d$ is a Parseval frame if and only if $U\Psi$ is a Parseval frame for any $d \times d$ orthogonal matrix U.

Now let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame for $\mathbb{R}^d$. Write $S = \Phi\Phi^T = UDU^T$ for some orthogonal matrix U. Observe that $U^T\Phi$ is the matrix of Φ written with respect to the orthonormal basis given by the rows of U^T. In addition, the rows of $U^T\Phi$ are pairwise orthogonal. Let $\Psi = \{\psi_i\}_{i=1}^N \subset \mathbb{R}^d$ be any Parseval frame, then

$$d^2(\Phi, \Psi) = d^2(U^T\Phi, U^T\Psi) = \sum_{i=1}^{d} \|R_i - P_i\|^2,$$

where $\{R_i\}_{i=1}^d \subset \mathbb{R}^N$ and $\{P_i\}_{i=1}^d \subset \mathbb{R}^N$ denote respectively the rows of $U^T\Phi$ and $U^T\Psi$. Consequently, finding

$$\min\{d(\Phi, \Psi)^2 = \sum_{i=1}^{N} \|\varphi_i - \psi_i\|^2 : \Psi = \{\psi_i\}_{i=1}^N \subset \mathbb{R}^d,\ \mathrm{P}arseval\,frame\}$$

is equivalent to finding

$$\min\{\sum_{i=1}^{d} \|R_i - P_i\|^2 : \{P_i\}_{i=1}^N \subset \mathbb{R}^N,\ orthonormal\ set\}$$

where $\{R_i\}_{i=1}^d$ form an orthogonal set of vectors in $\mathbb{R}^N$.

But $\Phi^\dagger = \{S^{-1/2}\varphi_i\}_{i=1}^N$ is a Parseval frame, so its rows form an orthonormal set in $\mathbb{R}^N$. Consequently, $\Phi^\dagger$ is a solution to (2.2). The uniqueness follows by observing that the (unique) closest orthonormal set to a given orthogonal set of vectors $\{u_i\}_{i=1}^d \subset \mathbb{R}^N$ is $\{\frac{u_i}{\|u_i\|}\}_{i=1}^d$.

Consequently,

$$\min\{d(\Phi,\Psi)^2 = \sum_{i=1}^{N} \|\varphi_i - \psi_i\|^2 : \Psi = \{\psi_i\}_{i=1}^N \subset \mathbb{R}^d,\ \mathrm{P}arseval\ frame\} = \sum_{k=1}^{d}(1-\lambda_k^{-1/2})^2$$

where $\{\lambda_k\}_{k=1}^d \subset (0,\infty)$ are the eigenvalues of $S = \Phi\Phi^T$.

∎

In the remaining part of section we study the continuity properties of the functions that maps a given frame to its canonical Parseval frame. This map

$$F : \mathcal{F}(N,d) \to \mathcal{F}(N,d)$$

given by

$$F(\Phi) = F(\{\varphi_i\}_{i=1}^N) = S_\Phi^{-1/2}(\{\varphi_i\}_{i=1}^N) = \{S_\Phi^{-1/2}\varphi_i\}_{i=1}^N. \tag{2.5}$$

In fact, our results show that for $0 < A \leqslant B$, F is uniformly continuous on $\mathcal{F}_{A,B}$, the set of frames with frame bounds between A and B. More specifically,

Theorem 2.2.1. *Let $0 < A \leqslant B < \infty$, and $\delta > 0$ be given. Then there exists $\epsilon > 0$ such that given any frame $\Phi = \{\varphi_i\}_{i=1}^N$, with frame bounds between A and*

B, and $N := N_\Phi \geqslant 2$, for any frame $\Psi = \{\psi_i\}_{i=1}^N$ such that $d(\Phi, \Psi) < \epsilon$ we have $d(F(\Phi), F(\Psi)) < \delta$.

Before proving this theorem, we establish a number of preliminary results and make the following remark that will be used in the sequel.

Remark 2.2.2. Let $\Phi = \{\varphi_i\}_{i=1}^N \in \mathcal{F}(N, d)$ be a frame. Then, $S = \Phi\Phi^T = ODO^T$ where O is a $d \times d$ orthogonal matrix and D is a positive definite diagonal matrix. Fix the orthonormal basis of $\mathbb{R}^d$ whose columns form the matrix O and write each frame vector φ_i in this basis. The synthesis matrix of the frame Φ in the basis O is

$$[\Phi]_O = O^T\Phi.$$

Let $\{R_i\}_{i=1}^d$ be the rows of $[\Phi]_O$. We shall refer to $\{R_i\}_{i=1}^d$ as simply the rows of Φ.

Lemma 2.2.3. *Let $\Phi = \{\varphi_i\}_{i=1}^N \in \mathcal{F}(N, d)$. Denote by $\{R_i\}_{i=1}^d$ the rows of Φ as described by Remark 2.2.2. Let $\epsilon > 0$ and $\Psi = \{\psi_i\}_{i=1}^N \in \mathcal{F}(N, d)$ be such that $d(\Phi, \Psi) < \epsilon$. Denote by $\{P_i\}_{i=1}^d$ the rows of Ψ when written in the orthonormal basis O. Then*

(a) $\big|\| R_i\| - \|P_i\|\big| < \epsilon$. *Furthermore,* $\sqrt{A} - \epsilon < \|P_i\| < \sqrt{B} + \epsilon$ *for each* $i = 1, 2, \ldots, d$.

(b)

$$d(\Phi, F(\Phi)) \geqslant \sqrt{\sum_{i=1}^d \|R_i - \frac{R_i}{\|R_i\|}\|^2}.$$

(c) For each $i \in \{1, 2 \ldots, d\}$ *we have*

$$\left\| \frac{P_i}{\|P_i\|} - \frac{R_i}{\|R_i\|} \right\| < \frac{2\epsilon}{\sqrt{A}}.$$

(d) For each $i \in \{1, 2 \ldots, d\}$ *we have*

$$0 \leqslant \|P_i - \tfrac{R_i}{\|R_i\|}\|^2 - \|P_i - \tfrac{P_i}{\|P_i\|}\|^2 \leqslant \tfrac{4\epsilon}{\sqrt{A}}c + \tfrac{4\epsilon^2}{A},$$

where $c = max(1 - \sqrt{A} + \epsilon, \sqrt{B} + \epsilon - 1)$.

Proof. (a) This is trivial so we omit it.

(b) This follows immediately from the fact that the rows of a Parseval frame are an orthonormal set when written with respect to any orthonormal basis and $\frac{R_i}{\|R_i\|}$ is the closest unit norm vector to R_i.

(c) Since, $d(\Phi, \Psi) < \epsilon$, we know that $\big|\|P_i\| - \|R_i\|\big| < \epsilon$. Hence

$$\left\|\frac{P_i}{\|P_i\|}\cdot\|R_i\| - R_i\right\| \leqslant \left\|\frac{P_i}{\|P_i\|}\cdot\|R_i\| - P_i\right\| + \|P_i - R_i\| = \big|\|P_i\| - \|R_i\|\big| + \|P_i - R_i\| < 2\epsilon.$$

The result follows by recalling that $\|R_i\| \geqslant \sqrt{A}$.

(d) It is clear that $\|P_i - \frac{P_i}{\|P_i\|}\| = |\|P_i\| - 1| \leqslant max(1 - \sqrt{A} + \epsilon, \sqrt{B} + \epsilon - 1) = c$. By part (c) we know that $\|\frac{P_i}{\|P_i\|} - \frac{R_i}{\|R_i\|}\| < \frac{2\epsilon}{\sqrt{A}}$. Using the fact hat $\frac{P_i}{\|P_i\|}$ is the closest unit norm vector to P_i, we see that

$$\|P_i - \frac{P_i}{\|P_i\|}\| \leqslant \|P_i - \frac{R_i}{\|R_i\|}\| \leqslant \|P_i - \frac{P_i}{\|P_i\|}\| + \frac{2\epsilon}{\sqrt{A}}.$$

The result follows by squaring the last inequality.

∎

Finally, we have the following technical lemma, that contains the key argument in the proof of Theorem 2.2.1.

Notation 2.2.4. *In the following lemma, Q will be the projection of R'_k onto the span of P_k. η will be the distance of $\frac{P_k}{\|P_k\|}$ to Q (ie $\|Q - \frac{P_k}{\|P_k\|}\|$).*

Lemma 2.2.5. *Given $0 < A \leqslant B < 0$, fix $\Phi = \{\varphi_i\}_{i=1}^N \in \mathcal{F}_{A,B}$. Let $\epsilon, \delta > 0$ be such that $\frac{\delta}{\sqrt{d}} - \frac{2\epsilon}{\sqrt{A}} > 0$ and $\sqrt{A} - \epsilon > 0$. Let $\Psi = \{\psi_i\}_{i=1}^N$ be such that $d(\Phi, \Psi) < \epsilon$, and*

$d(S_\Phi^{-1/2}\Phi, S_\Psi^{-1/2}\Psi) = d(\Phi^\dagger, \Psi^\dagger) > \delta$. Then,

$$\sum_{i=1}^{d}(\|P_i - R_i'\|^2 - \|P_i - \frac{P_i}{\|P_i\|}\|^2) \geqslant min(Cd'^2, C^2),$$

where $d' = \frac{\delta}{\sqrt{d}} - \frac{2\epsilon}{\sqrt{A}}$, $C = \min(\sqrt{A} - \epsilon, 1)$, and $\{R_i'\}_{i=1}^d \subset \mathbb{R}^d$ is the set of the rows of $S_\Psi^{-1/2}\Psi$.

Proof. We first show that there exists k with

$$\|P_k - R_k'\|^2 - \|P_k - \frac{P_k}{\|P_k\|}\|^2 \geqslant min(\|R_k' - \frac{P_k}{\|P_k\|}\|^2 \cdot min(\|P_k\|, 1), \|P_k\|^2).$$

Since $d(S_\Phi^{-1/2}\Phi, S_\Psi^{-1/2}\Psi) \geqslant \delta$, then $\|\frac{R_k}{\|R_k\|} - R_k'\| \geqslant \frac{\delta}{\sqrt{d}}$ for some k. By Lemma 2.2.3 we know that $\|\frac{P_k}{\|P_k\|} - \frac{R_k}{\|R_k\|}\| < \frac{2\epsilon}{\sqrt{A}}$. It follows from the triangle inequality that

$$\|\frac{P_k}{\|P_k\|} - R_k'\| \geqslant \frac{\delta}{\sqrt{d}} - \frac{2\epsilon}{\sqrt{A}} = d'.$$

Suppose that $C = \min(\sqrt{A} - \epsilon, 1) = 1$, or equivalently, $\sqrt{A} - \epsilon \geqslant 1$. Hence, by Lemma 2.2.3 we have $\|P_i\| \geqslant 1$ for each for all i.

Since the angle $\widehat{R_k' \frac{P_k}{\|P_k\|} P_i} > \pi/2$, it follows that

$$\|P_k - R_k'\|^2 > \|P_k - \frac{P_k}{\|P_k\|}\|^2 + \|\frac{P_k}{\|P_k\|} - R_k'\|^2.$$

But since, $\|\frac{P_k}{\|P_k\|} - R_k'\| \geqslant \frac{\delta}{\sqrt{d}} - \frac{2\epsilon}{\sqrt{A}}$, we conclude that

$$\|P_k - R_k'\|^2 - \|P_k - \frac{P_k}{\|P_k\|}\|^2 > \|\frac{P_k}{\|P_k\|} - R_k'\|^2 \geqslant d'^2 = Cd'^2$$

and we are done.

Assume now $C = \sqrt{A} - \epsilon < 1$ and $\|P_k\| + \eta \leqslant 1$.

Figure 2.1: TBA

Then,

$$\left\| P_k - R_k' \|^2 - \| P_k - \frac{P_k}{\|P_k\|} \right\|^2 = (1 - (\|P_k\| + \eta))^2 + 2\eta - \eta^2 - (1 - \|P_k\|)^2$$
$$= 2\eta \|P_k\|$$
$$= \left\| \frac{P_k}{\|P_k\|} - R_k' \right\|^2 \|P_k\|.$$

The conclusion follows from $\|\frac{P_k}{\|P_k\|} - R_k'\|^2 \geqslant d'^2$.

Now assume $\|P_k\| + \eta > 1$ and $\eta \leqslant 1$,

$$\left\| P_k - R_k' \|^2 - \| P_k - \frac{P_k}{\|P_k\|} \right\|^2 = ((\|P_k\| + \eta) - 1)^2 + 2\eta - \eta^2 - (1 - \|P_k\|)^2 = 2\eta \|P_k\|$$

and the rest of the proof is similar to the one given above.

Figure 2.2: TBA

If $\eta > 1$ then the angle $\angle P_k 0 R_k' > \frac{\pi}{2}$ hence $\|P_k - R_k'\|^2 > \|P_k\|^2 + 1$. We know $\|P_k - \frac{P_k}{\|P_k\|}\|^2 \leqslant 1$ hence

$$\|P_k - R_k'\|^2 - \|P_k - \frac{P_k}{\|P_k\|}\|^2 > \|P_k\|^2 \geqslant C^2$$

Figure 2.3: TBA

∎

We are now ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Assume by way of contradiction that there exists $\delta > 0$ such that for all $\epsilon > 0$ there exist $\Phi_\epsilon = \{\varphi_{i,\epsilon}\}_{i=1}^{N_\epsilon}, \in \mathcal{F}_{A,B}$. and $\Psi_\epsilon = \{\psi_{i,\epsilon}\}_{i=1}^{N_\epsilon}$

such that

$$d(\Phi_\epsilon, \Psi_\epsilon) < \epsilon$$

and

$$d(S_{\Phi_\epsilon}^{-1/2}\Phi_\epsilon, S_{\Psi_\epsilon}^{-1/2}\Psi_\epsilon) > \delta.$$

Furthermore, choose ϵ small enough so that $\dfrac{\delta}{\sqrt{d}} - \dfrac{2\epsilon}{\sqrt{A}} > 0$ and $\sqrt{A} - \epsilon > 0$ and

$$\sum_{i=1}^{d}(\|P_i - \frac{R_i}{\|R_i\|}\|^2 - \|P_i - \frac{P_i}{\|P_i\|}\|^2) < min(Cd'^2, C^2)$$

where C and d'^2 are as in Lemma 2.2.5(such ϵ exists by Lemma 2.2.3).

Hence

$$\sum_{i=1}^{d}(\|P_i - \frac{R_i}{\|R_i\|}\|^2 - \|P_i - \frac{P_i}{\|P_i\|}\|^2) < \sum_{i=1}^{d}(\|P_i - R_i'\|^2 - \|P_i - \frac{P_i}{\|P_i\|}\|^2)$$

Consequently, $\sum_{i=1}^{d} \|P_i - \dfrac{R_i}{\|R_i\|}\|^2 < \sum_{i=1}^{d} \|P_i - R_i'\|^2$ contradicting that R_i' are the rows of the closest Parseval frame to $\Psi_\epsilon = \{\psi_{i,\epsilon}\}_{i=1}^{N_\epsilon}$. ∎

2.2.2 Approximation of probabilistic frames in the $2-$ Wasserstein metric

In this section we prove some of the technical results needed to establish our main result. The key idea is that a probabilistic frame μ with frame bounds A, B can be approximated in the Wasserstein metric by a finite probabilistic frame whose bounds are arbitrarily closed to A, B. We prove this statement in Proposition 2.2.8 and point out that it is a refinement of a well-known result, e.g., [49, Theorem 6.18]. But first, we prove a few new results about finite probabilistic frames that are of interest in their own right. In particular, Lemma 2.2.6 will be a very useful technical tool that we shall often use. It shows that given a finite frame we may replace any frame vector by a finite number of new vectors so as to leave unchanged the frame operator. More specifically,

Lemma 2.2.6. *Given a frame $\Phi = \{\varphi_i\}_{i=1}^N$ with frame operator S_Φ. Fix $i \in \{1, 2, \dots, N\}$ and consider the new set of vectors*

$$\Phi_i = \{\varphi_k\}_{k=1, k\neq i}^N \cup \{a_j \varphi_i\}_{j=1}^p = \{\varphi'_k\}_{k=1}^{N+p-1}$$

where $\sum_{j=1}^p a_j^2 = 1$. Then, $\Phi_i \in \mathcal{F}(N+p-1, d)$, that is, Φ_i is a frame for $\mathbb{R}^d$ and its frame operator is S_Φ. Furthermore,

$$\sum_{k=1}^N \|\varphi_k - \varphi_k^\dagger\|^2 = \sum_{k=1}^{N+p-1} \|\varphi'_k - \varphi_k^{'\dagger}\|^2$$

where $\varphi_k^\dagger = S^{-1/2}\varphi_k$ and $\varphi_k^{'\dagger} = S^{-1/2}\varphi'_k$

Proof. It is easy to see that for each $x \in \mathbb{R}^d$ we have

$$\sum_{k=1}^N |\langle x, \varphi_k\rangle|^2 = \sum_{k=1}^{i-1} |\langle x, \varphi_k\rangle|^2 + \sum_{j=1}^p a_j^2 |\langle x, \varphi_i\rangle|^2 + \sum_{k=i+1}^N |\langle x, \varphi_k\rangle|^2.$$

∎

We now use Lemma 2.2.6 and Theorem 2.1.1 to find the closest Parseval frame to a finite probabilistic frame in the 2-Wasserstein metric.

Proposition 2.2.7. *Let $\mu_{\Phi,w}$ be a finite probabilistic frame with bounds A and B, where $\Phi = \{\varphi_i\}_{i=1}^N \subset \mathbb{R}^d$ and $w = \{w_i\}_{i=1}^N \subset [0, \infty)$. Then the closest finite Parseval probabilistic frame to Φ is $\Phi^\dagger = \{S^{-1/2}\varphi_i\}_{i=1}^N$ and it satisfies*

$$W_2(\mu_{\Phi,w}, \mu_{\Phi^\dagger,w}) = \sqrt{\sum_{i=1}^N w_i \|\varphi_i - \tilde{\varphi}_i\|^2} \leqslant \sqrt{d\, \max((\sqrt{A}-1)^2, (\sqrt{B}-1)^2)}$$

where $\tilde{\varphi}_i = S^{-1/2}\varphi_i$.

Proof. We first prove that $W_2^2(\mu_{\Phi,w}, \mu_{\Phi^\dagger,w}) \leqslant d \max((\sqrt{A}-1)^2, (\sqrt{B}-1)^2)$.

Let $\Phi_w = \{\sqrt{w_i}\varphi_i\}_{i=1}^N$. Let $S = \Phi_w \Phi_w^T = ODO^T$ be the frame operator of Φ_w. Consider the columns of O as an orthonormal basis for $\mathbb{R}^d$. Writing the vectors $\sqrt{w}_k\varphi_k$ with respect to this basis leads to $\Phi'_w = O^T\Phi_w$ where

$$\Phi_w = \begin{pmatrix} | & \dots & | \\ \sqrt{w_1}\varphi_1 & \dots & \sqrt{w_n}\varphi_m \\ | & \dots & | \end{pmatrix}$$

Let $\{P_{k,w}\}_{k=1}^d$ and $\{R_{k,w}\}_{k=1}^d$ respectively denote the rows of Φ'_w and Φ_w. Notice that

$$\sqrt{A} \leqslant \|P_{k,w}\| \leqslant \sqrt{B}, \quad \forall\, k = 1, 2, \dots, d.$$

It is easily seen that

$$\min_{u\in\mathbb{R}^d, \|u\|=1} \|P_{k,w} - u\|^2 = \|P_{k,w} - \tfrac{P_{k,w}}{\|P_{k,w}\|}\|^2 = |\|P_{k,w}\| - 1|^2 \leqslant \max((\sqrt{A}-1)^2, (\sqrt{B}-1)^2).$$

But by construction, $\langle P_{k,w}, P_{\ell,w}\rangle = 0$ for $k \neq \ell$, and $\frac{P_{k,w}}{\|P_{k,w}\|} = \lambda_k^{-1/2}P_{k,w}$ where λ_k is the k^{th} eigenvalue of S. Consequently, $\{\lambda_k^{-1/2}P_{k,w}\}_{k=1}^d$ represents the rows of the canonical tight frame $S^{-1/2}\Phi_w$ written in the orthonormal basis O. Therefore,

$$d(\Phi_w, S^{-1/2}\Phi_w)^2 = \sum_{k=1}^d \|P_{k,w} - \lambda_k^{-1/2}P_{k,w}\|^2 \leqslant d\max((\sqrt{A}-1)^2, (\sqrt{B}-1)^2).$$

Clearly,

$$W_2^2(\mu_{\Phi,w}, \mu_{S^{-1/2}\Phi,w}) \leqslant \sum_{i=1}^N w_i\|\varphi_i - S^{-1/2}\varphi_i\|^2 =$$
$$d(\Phi_w, S^{-1/2}\Phi_w)^2 \leqslant d\max((\sqrt{A}-1)^2, (\sqrt{B}-1)^2).$$

Suppose there exists a finite probabilistic Parseval frame $\mu_{\Psi,v}$ where

$\Psi = \{\psi_i\}_{i=1}^M \subset \mathbb{R}^d$, $v = \{v_i\}_{i=1}^M \subset [0,\infty)$ such that

$$W_2^2(\mu_{\Phi,w}, \mu_{\Psi,v}) < \sum_{i=1}^N w_i\|\varphi_i - S^{-1/2}\varphi_i\|^2.$$

Let $\gamma \in \Gamma(\mu_{\Phi,w}, \mu_{\Psi,v})$ be such that

$$W_2^2(\mu_{\Phi,w}, \mu_{\Psi,v}) = \iint_{\mathbb{R}^{2d}} \|x-y\|^2 d\gamma(x,y).$$

Note that γ is a discrete measure with $\gamma(x,y) = \sum_{i,j} w'_{i,j}\delta_{\varphi_i}(x)\delta_{\psi_i}(y)$ with $\sum_j w'_{i,j} = w_i$ and $\sum_i w'_{i,j} = v_j$.

Furthermore, by assumption

$$W_2^2(\mu_{\Phi,w}, \mu_{\Psi,v}) = \sum_{i,j} w'_{i,j}\|\varphi_i - \psi_j\|^2 < \sum_{i=1}^{N} w_i\|\varphi_i - S^{-1/2}\varphi_i\|^2.$$

Notice since $\sum_i w'_{i,j} = v_j$ the frame $\Psi' = \{\sqrt{w'_{i,j}}\psi_j\}_{i,j}$ is a Parseval frame. Since $\sum_j w'_{i,j} = w_i$, it easy to see that $\sum_j \frac{w'_{i,j}}{w_i} = 1$. We now use Lemma 2.2.6. For each i, replace $\sqrt{w_i}\varphi_i$ with $\{\sqrt{w'_{i,j}}\varphi_i\}_j$. This results in a frame $\Phi' = \{\sqrt{w'_{i,j}}\varphi_i\}_{i,j}$. Consequently, $d(\Phi', \Psi') = d(\Phi_w, \Psi_v) < d(\Phi_w, \Phi_w^\dagger)$ where Ψ_v is a Parseval frame. This is a contradiction. ∎

The next result is one of our key technical results. It allows us to approximate a probabilistic frame in the 2-Wasserstein metric with a compactly supported finite probabilistic frame whose bounds are controlled by those of the original probabilistic frame.

Theorem 2.2.8. *Let μ be a probabilistic frame with frame bounds A and B, and $\epsilon > 0$. Then, there exists a finite probabilistic μ_Φ with frame bounds A', B' such that $A' \geqslant A - \epsilon$, $B' \leqslant B + \epsilon$ and*

$$\|\mu - \mu_\Phi\|_{W_2} < \epsilon.$$

To establish this result we first prove the following two Lemmas.

Lemma 2.2.9. *Let μ be a probabilistic frame with frame bound A and B. Given $\epsilon > 0$, there exists a probabilistic frame ν with compact support and frame bounds A', B' such that*

(a) $W_2^2(\mu, \nu) < \epsilon$,

(b) $A' \geqslant A - \epsilon$, *and* $B' = B$.

Proof. (a) Let μ be a probabilistic frame with frame bound A and B. Given $\epsilon > 0$, there exists $R_1 > 0$ such that

$$\int_{\mathbb{R}^d \setminus B(0,R_1)} \|x\|^2 d\mu(x) < \epsilon.$$

Let ν be the measure defined for each Borel set $A \subset \mathbb{R}^d$ by

$$\nu(A) = \mu(A \bigcap B(0, R_1) + \mu(\mathbb{R}^d \setminus B(0, R_1))\delta_0.$$

Clearly, ν is a probabilistic measure with compact support.

We consider the marginal γ of μ and ν defined for each Borel sets $A, B \subset \mathbb{R}^d$ by

$$\gamma(A \times B) = \begin{cases} \mu(A \bigcap B(0, R_1) \bigcap B) + \mu(A \bigcap B^c(0, R_1) & if \quad 0 \in B \\ \mu(A \bigcap B(0, R_1) \bigcap B) & if \quad 0 \notin B \end{cases}$$

Since ν is supported in $B(0, R_1)$

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} \|x - y\|^2 d\gamma(x, y) &= \iint_{\mathbb{R}^d \times B(0,R_1)} \|x - y\|^2 d\gamma(x, y) \\ &= \iint_{B(0,R_1) \times B(0,R_1)} \|x - y\|^2 d\gamma(x, y) \\ &+ \iint_{B^c(0,R_1) \times B(0,R_1)} \|x - y\|^2 d\gamma(x, y). \end{aligned}$$

However, we know

$$\int_{B(0,R_1)\times B(0,R_1)} \|x-y\|^2 d\gamma(x,y) = 0$$

since, when restricted to $B(0,R_1)\times B(0,R_1)$, γ is supported only on the diagonal where $\|x-y\| = 0$. Moreover,

$$\begin{aligned}
\int_{B^c(0,R_1)\times B(0,R_1)} \|x-y\|^2 d\gamma(x,y) &= \iint_{B^c(0,R_1)\times B(0,R_1)\setminus\{0\}} \|x-y\|^2 d\gamma(x,y) \\
&\quad + \iint_{B^c(0,R_1)\times\{0\}} \|x-y\|^2 d\gamma(x,y) \\
&= 0 + \iint_{B^c(0,R_1)\times\{0\}} \|x-y\|^2 d\gamma(x,y) \\
&< \epsilon.
\end{aligned}$$

Therefore, $W_2^2(\mu,\nu) < \epsilon$.

(b) The upper bound B is obtained trivially as ν is μ restricted to $B(0,R_1)$.

For $x \in \mathbb{R}^d$ we have $\int |\langle x,y\rangle|^2 d\nu(y) = \int_{B(0,R_1)} |\langle x,y\rangle|^2 d\mu(y)$. From the fact that $\int_{\mathbb{R}^d\setminus B(0,R_1)} \|x\|^2 d\mu(x) \leqslant \epsilon$ it follows that

$$\int_{\mathbb{R}^d\setminus B(0,R_1)} |\langle x,y\rangle|^2 d\mu(y) \leqslant \|x\|^2\epsilon.$$

∎

Suppose that μ is a probabilistic frame supported in a ball $B(0,R)$. Let $r>0$ and consider $Q=[0,r)^d$. Choose points $\{c_k\}_{k=1}^M \subset \mathbb{R}^d$ with $c_1 = 0$ such that $B(0,R) = \cup_{k=0}^M Q_k$ where $Q_k = c_k + Q$. Observe that $Q_k \cap Q_\ell = \emptyset$ whenever $k \neq \ell$. Let $\mu_{1,Q} = \sum_{k=1}^M \mu(Q_k)\delta_{c_k}$.

Next partition each cube Q_k uniformly into cube of size $r/2$ and construct the probability measure $\mu_{2,Q}$ as above. Iterate this process to construct a sequence of probability measures $\mu_{n,Q}$.

Lemma 2.2.10. *Let μ be a probabilistic frame with bounds A and B, which is supported in a ball $B(0,R)$. For $r>0$ let $\{\mu_{n,Q}\}_{n=1}^{\infty}$ be a sequence of probability measures as constructed above. Then,*

$$lim_{n\to\infty} W_2(\mu,\mu_{n,Q})=0.$$

Furthermore, there exists N such that for all $n \geqslant N$, $\mu_{n,Q}$ is a finite probabilistic frame whose bounds are arbitrarily close to those of μ.

Proof. Let $d=\max_{x\in Q_k}\|x-c_k\|$. Given, $x\in Q_k$, $x=c_k+a_k$, where $\|a_k\|\leqslant d$.

For any $x\in\mathbb{R}^d$,

$$\begin{aligned}
&\left|\int_{B(0,R)} \langle x,y\rangle^2\,d\mu(y)-\sum_{k=1}^{M}\langle x,c_k\rangle^2\mu(Q_k)\right| \\
&=\left|\sum_{k=1}^{M}\int_{Q_k}\langle x,y\rangle^2\,d\mu(y)-\sum_{k=1}^{M}\langle x,c_k\rangle^2\mu(Q_k)\right| \\
&=\left|\sum_{k=1}^{M}\int_{Q_k}(\langle x,y\rangle^2-\langle x,c_k\rangle^2)d\mu(y)\right| \\
&\leqslant \sum_{k=1}^{M}\int_{Q_k}\left|\,\langle x,y\rangle^2-\langle x,c_k\rangle^2\,\right|d\mu(y) \\
&=\sum_{k=1}^{M}\int_{Q_k}\left|\,\langle x,c_k+a_k\rangle^2-\langle x,c_k\rangle^2\,\right|d\mu(y) \\
&=\sum_{k=1}^{M}\int_{Q_k}\left|\,\langle x,a_k\rangle^2+2\langle x,c_k\rangle\langle x,a_k\rangle\,\right|d\mu(y) \\
&\leqslant \|x\|^2\sum_{k=1}^{M}\mu(Q_k)(\|a_k\|^2+2\|c_k\|\|a_k\|) \\
&\leqslant (d^2+2d(R+d))\|x\|^2.
\end{aligned}$$

Note that by the iterative construction of $\mu_{n,Q}$ we get that for each $x\in\mathbb{R}^d$

$$\left|\int_{\mathbb{R}^d}\langle x,y\rangle^2\,d\mu(y)-\int_{\mathbb{R}^d}\langle x,y\rangle^2\,d\mu_{n,Q}(y)\right|\leqslant (d_n^2+2d_n(R+d_n))\|x\|^2$$

where $\lim_{n\to\infty} d_n = 0$. It follows that given $\epsilon > 0$, we can find $N > 1$ such that for all $n \geqslant N$,

$$\int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu_{n,Q}(y) > \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu(y) - \epsilon\|x\|^2 > \|x\|^2(A-\epsilon)$$

which concludes that $\mu_{n,Q}$ is a a finite probabilistic frame whose lower bound is at least $A - \epsilon$. Furthermore,

$$\int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu_{n,Q}(y) < \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu(y) + \epsilon\|x\|^2 \leqslant \|x\|^2(B+\epsilon)$$

which implies that the upper frame bound $\mu_{n,Q}$ is at most $B + \epsilon$.

Next, fix $n \geqslant N$ and let $\gamma_n(x, y)$ be the measure on $\mathbb{R}^d \times \mathbb{R}^d$ defined for any Borel sets $A, B \subset \mathbb{R}^d$ by:

$$\gamma_n(A \times B) = \sum_{k: c_k \in B} \mu(A \cap Q_k) = \sum_{k=1}^{M} \mu_{|Q_k} \times \delta_{c_k}(A \times B)$$

where A, B c_k denoting the centers of the cubes Q_k. It is easy to see that $\gamma_n \in \Gamma(\mu, \mu_{n,Q})$ and so

$$\begin{aligned}
W_2^2(\mu, \mu_{n,Q}) &\leqslant \iint \|x-y\|^2 d\gamma_n(x,y) \\
&= \sum_{k=1}^{M} \iint \|x-y\|^2 d(\mu_{|Q_k} \times \delta_{c_k})(x,y) \\
&= \sum_{k=1}^{M} \int_{Q_k} \|x-c_k\|^2 d\mu(x) \\
&\leqslant \sum_{k=1}^{M} \mu(Q_k) \int_{Q_k} d_n^2 d\mu(x) \\
&\leqslant d_n^2
\end{aligned}$$

and the result follows from the fact that $\lim_{n\to\infty} d_n = 0$.

∎

We can now give a proof of Theorem 2.2.8.

Proof of Theorem 2.2.8. Let μ be a probabilistic frame with frame bounds A and B, and $\epsilon > 0$. By Lemma 2.2.9 let ν be a compactly supported probabilistic frame with frame bounds between $A - \epsilon/2$ and B and such that $W_2(\mu, \nu) < \epsilon/2$.

By Lemma 2.2.10 we know there exists a finite probabilistic frame $\mu_{\Phi,w}$ whose frame bounds are within $\epsilon/2$ of that of ν and such that $W_2(\nu, \mu_{\Phi,w}) < \epsilon/2$. Consequently, $W_2(\mu, \mu_{\Phi,w}) < \epsilon$ which concludes the proof. ∎

Corollary 2.2.11. *Let μ be a probabilistic Parseval frame and $\epsilon > 0$. Then, there exists a finite Parseval probabilistic frame $\mu_{\Phi,w}$ with*

$$W_2(\mu, \mu_{\Phi,w}) < \epsilon.$$

Proof. This follows from Proposition 2.2.7 and Theorem 2.2.8. ∎

Remark 2.2.12. Since the set of finite Parseval frames is dense in the set of all Parseval frames in the Wasserstein metric, by Proposition 2.6 since there is no finite Parseval frame closer than $\Phi^\dagger = \{S^{-1/2}\varphi_i\}_{i=1}^N$, there are no Parseval frames closer than $\Phi^\dagger = \{S^{-1/2}\varphi_i\}_{i=1}^N$.

2.2.3 The closest Parseval frame in the $2-$Wasserstein distance

In this section we state and prove our main result, Theorem 2.2.13. We recall that if μ is a probabilistic frame for $\mathbb{R}^d$, then its probabilistic frame operator (equivalently, the matrix of second moments associated to μ)

$$S_\mu : \mathbb{R}^d \to \mathbb{R}^d, \qquad S_\mu(x) = \int_{\mathbb{R}^d} \langle x, y \rangle \, y d\mu(y)$$

is positive definite, and thus $S_\mu^{-1/2}$ exists. We define the push-forward of μ through $S_\mu^{-1/2}$ by

$$\mu^\dagger(B) = \mu(S^{1/2}B)$$

for each Borel set in $\mathbb{R}^d$. Alternatively, if f is a continuous bounded function on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} f(y)d\mu^\dagger(y) = \int_{\mathbb{R}^d} f(S_\mu^{-1/2}y)d\mu(y).$$

It then follows that

$$x = S_\mu^{-1/2}S_\mu S_\mu^{-1/2}(x) = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1/2}x, y\right\rangle\, S_\mu^{-1/2}y\, d\mu(y) = \int_{\mathbb{R}^d} \langle x, y\rangle\; y\, d\mu^\dagger(y)$$

implying that $\mu^\dagger$ is a Parseval probabilistic frame [32, 43]. In particular, $S_{\mu^\dagger} = I$ where I is the identity matrix on $\mathbb{R}^d$. As was the case with the canonical Parseval frame $\Phi^\dagger$ of a given frame Φ, $\mu^\dagger$ is the (unique) closest Parseval probabilistic frame to μ.

Theorem 2.2.13. *Let μ be a probabilistic frame on $\mathbb{R}^d$ with probabilistic frame operator S_μ. Then $\mu^\dagger$ is the (unique) closest probabilistic Parseval frame to μ in the 2−Wasserstein metric, that is*

$$\mu^\dagger = arg\min W_2^2(\mu, \nu) \tag{2.6}$$

where ν ranges over all Parseval probabilistic frames.

Before proving this theorem, we need to establish a few preliminary results. We start by extending Theorem 2.2.1 to finite probabilistic frames in the Wasserstein metric. In particular, this extension allows use to deal with finite probabilistic frames of different cardinalities.

Theorem 2.2.14. *Let $0 < A \leqslant B < \infty$, and $\delta > 0$ be given. Then there exists $\epsilon > 0$ such that given any finite probabilistic frame $\mu_{\Phi,w} = \sum_{i=1}^{N} w_i \delta_{\varphi_i}$ with frame bounds between A and B, $N := N_\Phi \geqslant 2$, $\Phi = \{\varphi_i\}_{i=1}^{N} \subset \mathbb{R}^d$, and weights $w = \{w_i\}_{i=1}^{N} \subset [0,\infty)$, for any finite probabilistic frame $\mu_{\Psi,\eta} = \sum_{i=1}^{M} \eta_i \delta_{\psi_i}$, $M := M_\Psi \geqslant 2$, where $\Psi = \{\psi_i\}_{i=1}^{M} \subset \mathbb{R}^d$, and weights $\eta = \{\eta_i\}_{i=1}^{N} \subset [0,\infty)$ if $W_2(\mu_{\Phi,w}, \mu_{\Psi,\eta}) < \epsilon$, then we have*

$$W_2(F(\mu_{\Phi,w}), F(\mu_{\Psi,\eta})) < \delta.$$

Proof. Fix $\delta > 0$. By Theorem 2.2.1 we know that there exists ϵ such that given a frame $X = \{x_i\}_{i=1}^{M}$ ($M \geqslant 2$ is arbitrary) with frame bounds between A and B, and $Y = \{y_i\}_{i=1}^{M}$ is a frame such that

$$d(X,Y) = \sqrt{\sum_{i=1}^{M} \|x_i - y_i\|^2} < \epsilon$$

then

$$d(F(X), F(Y)) = d(S_X^{-1/2}X, S_Y^{-1/2}Y) < \delta.$$

Let $\mu_{\Phi,w} = \sum_{i=1}^{N} w_i \delta_{\varphi_i}$ be a finite probabilistic frame with frame bounds between A and B, $N \geqslant 2$, $\Phi = \{\varphi_i\}_{i=1}^{N} \subset \mathbb{R}^d$, and weights $w = \{w_i\}_{i=1}^{N} \subset [0,\infty)$. Then by Theorem 2.2.7, $\mu_{\Phi^\dagger,w}$ where $\Phi^\dagger = \{S_\Phi^{-1/2}\varphi_i\}_{i=1}^{N}$ is the closest Parseval frame to $\mu_{\Phi,w}$.

Let $\mu_{\Psi,v}$ where $\Psi = \{\psi_i\}_{i=1}^{M}$, $M \geqslant 2$ such that $W_2(\mu_{\Phi,w}, \mu_{\Psi,\eta}) < \epsilon$. Choose $\gamma \in \Gamma(\mu_{\Phi,w}, \mu_{\Psi,v})$ such that

$$W_2(\mu_{\Phi,w}, \mu_{\Psi,\eta})^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 d\gamma(x,y) < \epsilon^2.$$

Identify γ with $\{w_{i,j}\}_{i,j=1}^{N,M}$. Then,

$$W_2(\mu_{\Phi,w}, \mu_{\Psi,\eta})^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 d\gamma(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} w_{i,j} \|\varphi_i - \psi_j\|^2 < \epsilon^2.$$

Observe that $\Phi' = \{\sqrt{w_{i,j}}\varphi_i\}_{i,j=1}^{M,N}$ is a frame whose frame bounds are the same as those for $\mu_{\Phi,w}$. Similarly, $\Psi' = \{\sqrt{w_{i,j}}\psi_j\}_{i,j=1}^{M,N}$ is a frame whose frame bounds are the same as those for $\mu_{\Psi,\eta}$. Furthermore,

$$d(\Phi', \Psi') = W_2(\mu_{\Phi,w}, \mu_{\Psi,\eta}) < \epsilon$$

which implies that

$$d(F(\Phi'), F(\Psi'))^2 = \sum_{i,j=1}^{M,N} \|S_\Phi^{-1/2}(\sqrt{w_{i,j}}\varphi_i) - S_\Psi^{-1/2}(\sqrt{w_{i,j}}\psi_j)\|^2 < \delta^2.$$

However,

$$\sum_{i,j} \|S_\Phi^{-1/2}(\sqrt{w_{i,j}}\varphi_i) - S_\Psi^{-1/2}(\sqrt{w_{i,j}}\psi_j)\|^2 = \sum_{i,j} w_{i,j}\|S_\Phi^{-1/2}\varphi_i - S_\Psi^{-1/2}\psi_j\|^2$$

But since $w_{i,j} = \gamma(\{\varphi_i\}, \{\psi_j\})$ we have $\sum_j w_{i,j} = w_i$ and $\sum_i w_{i,j} = v_j$ we see that

$$W_2^2(F(\mu_{\Phi,w}), F(\mu_{\Psi,\eta})) = W_2^2(\mu_{\Phi^\dagger,w}, \mu_{\Psi^\dagger,v}) \leqslant \sum_{i,j} w_{i,j}\|S_\Phi^{-1/2}\varphi_i - S_\Psi^{-1/2}\psi_j\|^2.$$

∎

Let $DPF(A, B)$ denote the set of all discrete (finite) probabilistic frames in $\mathbb{R}^d$ whose lower frame bounds are less than or equal to A and whose upper bounds are greater or equals to B. It follows from the above result that F is uniformly continuous from $DPF(A, B)$ into itself when equipped with the Wasserstein metric. Consequently, we can prove the following result.

Proposition 2.2.15. *Let μ be a probabilistic frame with frame bounds A and B. Let $\mu_k := \mu_{\Phi_k,w_k}$, where $\Phi_k := \Phi_{k,w_k} = \{\varphi_k\}_{k=1}^{N_k}$ and $\nu_k := \mu_{\Psi_k,v_k}$, where $\Psi_k := \Psi_{k,v_k} = \{\psi_k\}_{k=1}^{M_k}$ be two sequences of finite probabilistic frames in $\mathbb{R}^d$ such that*

$\lim_{k\to\infty} W_2(\mu, \mu_{\Phi_k}) = \lim_{k\to\infty} W_2(\mu, \mu_{\Psi_k}) = 0$. *Furthermore, suppose that the frame bounds of* μ_{Φ_k} *are between* $A/2$ *and* $B + A/2$. *Then*

$$\lim_{k\to\infty} F(\mu_{\Phi_k}) = \lim_{k\to\infty} F(\mu_{\Psi_k}).$$

Proof. Theorem 2.2.8 ensures the existence of the finite probabilistic frames μ_{Φ_k}.

Let $\delta > 0$ be given. By Theorem 2.2.14 there exists $\epsilon > 0$ such that for any finite probabilistic frame ν and any $k \geqslant 1$,

$$W_2(\mu_{\Phi_k}, \nu) < \epsilon \implies W_2(F(\nu), F(\mu_{\Phi_k})) < \delta.$$

Choose $N_\epsilon > 1$ such that for all $k > N_\epsilon$, $W_2(\mu, \mu_{\Phi_k}) < \frac{\epsilon}{2}$ and $W_2(\mu, \mu_{\Psi_k}) < \frac{\epsilon}{2}$. Thus, for $k \geqslant N_\epsilon$, $W_2(\mu_{\Phi_k}, \mu_{\Psi_k}) < \epsilon$, which implies that for all $k \geqslant N_\epsilon$, $W_2(F(\mu_{\Phi_k}), F(\mu_{\Psi_k})) < \delta$. It easily follows that $\lim_{k\to\infty} F(\mu_{\Phi_k}) = \lim_{k\to\infty} F(\mu_{\Psi_k})$.

∎

We can now use this proposition to extend the definition of the map F to all probabilistic frames. Let μ be a probabilistic frame with bounds $0 < A \leqslant B < \infty$. Let $\{\mu_{\Phi_k}\}_{k=1}^{\infty}$ be a sequence of finite probabilistic frames with bounds between $A/2$ and $B + A/2$ such that $lim_{k\to\infty} W_2(\mu_{\Phi_k}, \mu) = 0$. Then,

$$F(\mu) = \lim_{k\to\infty} F(\mu_{\Phi_k})$$

is well-defined. Before proving Theorem 2.2.13 we first identify the minimizer of (2.6) with $F(\mu)$.

Theorem 2.2.16. *Let* μ *be a probabilistic frame on* $\mathbb{R}^d$ *with probabilistic frame operator* S_μ. *Then* $F(\mu)$ *is the unique closest probabilistic Parseval frame to* μ *in the* $2-$*Wasserstein metric, that is* $F(\mu)$ *is the unique solution to* (2.6).

Proof. Set $Q = \min W_2(\mu, \nu)$ where ν ranges over all Parseval probabilistic frames.

Let $\delta > 0$, and μ be a probabilistic frame wth frame bounds A and B. By Theorem 2.2.8, there exists a sequence of finite probabilistic frame μ_{Φ_k} with frame bounds between $\frac{A}{2}$ and $B+\frac{A}{2}$ where $\Phi_k := \Phi_{k,w(k)} = \{\varphi_k\}_{k=1}^{N_k} \subset \mathbb{R}^d$, $w(k) = \{w_n\}_{n=1}^{N_k} \subset (0, \infty)$, and $N_k \geqslant 2$ such that $\lim_{k\to\infty} W_2(\mu, \mu_{\Phi_k}) = 0$.

Observe that for all $k \geqslant 1$,

$$W_2(\mu, F(\mu_{\Phi_k})) \leqslant W_2(\mu, F(\mu)) + W_2(F(\mu), F(\mu_{\Phi_k})).$$

Choose $\epsilon > 0$ as in Theorem 2.2.14 and pick $K \geqslant 1$ such that $W_2(\mu, \mu_{\Phi_K}) < \epsilon$. Thus, $W_2(F(\mu), F(\mu_{\Phi_K})) < \delta$. Consequently,

$$W_2(\mu, F(\mu_{\Phi_K})) \leqslant W_2(\mu, F(\mu)) + W_2(F(\mu), F(\mu_{\Phi_K})) < W_2(\mu, F(\mu)) + \delta.$$

Since $F(\mu_{\Phi_K})$ is a Parseval frame we conclude that $F(\mu)$ minimizes (2.6).

We now prove that $F(\mu)$ is the unique minimizer of (2.6) by considering three cases.

Case 1. If μ is a finite frame $\Phi = \{\varphi_i\}_{i=1}^N \subset \mathbb{R}^d$, it is known that $S^{-1/2}\Phi$ is the (unique) closest Parseval frame to Φ, see Theorem 2.1.1, and [22, Theorem 3.1].

Case 2. If $\mu = \mu_{\Phi,w}$, where $\Phi = \{\varphi_i\}_{i=1}^N \subset \mathbb{R}^d$, and $w = \{w_i\}_{i=1}^N \subset [0, \infty)$. Then, $\mu_{\Phi^\dagger,w}$ where $\Phi^\dagger = S^{-1/2}\Phi$ is the unique closest Parseval probabilistic frame to Φ. Indeed, we already know that $\mu_{\Phi^\dagger,w}$ achieves the minimum distance by Proposition 2.2.7. We now prove that it is unique. We argue by contradiction and assume that there exists another Parseval probabilistic frame ν that achieves this distance.

First, we assume that $\nu = \mu_{\kappa,v}$ where $\kappa = \{\kappa_i'\}_{i=1}^M \subset \mathbb{R}^d$ with weights $v = \{v_i\}_{i=1}^M \subset$

$[0, \infty)$. Let $\gamma \in \Gamma(\mu, \nu)$ such that

$$W_2(\mu,\nu)^2 = \iint \|x-y\|^2 d\gamma(x,y).$$

For all i, j let $w_{i,j} = \gamma(\varphi_i, \kappa'_j)$. Let $Q = \sum_{i=1}^N w_i \|\varphi_i - \varphi_i^\dagger\|^2$, where $\varphi_i^\dagger = S^{-1/2}\varphi_i$. Since κ also achieved this distance we clearly have $Q = \sum_{i,j} w_{i,j}\|\varphi_i - \kappa'_j\|^2$.

We now use Lemma 2.2.6. For each i, we replace the vector φ_i and its weight w_i by M copies of itself (i.e., φ_i) each weighted by $w_{i,j}$. Apply the same procedure to $\Phi^\dagger$, and to κ, except that for the latter we break each vector κ'_j into N copies of itself with weights $w_{i,j}$. Denote by F_1, F_2, and F_3 the three resulting frames. We note that the vectors in each of these frames can be considered to have weight 1.

It follows from Theorem 2.1.1 that the finite frame $F_3 = \{\sqrt{w_{i,j}}\kappa'_j\}_{i,j}$ is the (unique) closest Parseval frame to $F_1 = \{\sqrt{w_{i,j}}\varphi_i\}_{i,j}$, which we also know is $F_2 = \{\sqrt{w_{i,j}}\varphi_i^\dagger\}_{i,j}$. Therefore, $\mu_{\kappa,v} = \mu_{\Phi^\dagger,w}$.

Next, we assume that ν is not discrete. Choose a sequence of finite Parseval frames $\{\nu_n\}_n^\infty$ such that $\lim_{n\to\infty} W_2(\nu_n, \nu) = 0$. Hence,

$$Q = W_2(\mu, F(\mu)) = W_2(\mu_{\Phi,w}, \nu) = \lim_{n\to\infty} W_2(\mu_{\Phi,w}, \nu_n).$$

We now prove that

$$\lim_{n\to\infty} W_2(\nu_n, \mu_{\Phi^\dagger,w}) = 0.$$

Let $\delta > 0$ and choose $N \geqslant 1$ such that for all $n > N$

$$W_2(\nu_n, \mu_{\Phi,w}) < Q + \delta.$$

Suppose by contradiction that $\lim_{n\to\infty} W_2(\nu_n, \mu_{\Phi^\dagger,w}) > 0$. Thus, there is $\epsilon > 0$ such for all $k \geqslant 1$, there exists $n > \max(k, N)$ such that

$$W_2(\nu_n, \mu_{\Phi^\dagger,w}) > \epsilon.$$

For n given above, let $\gamma_n \in \Gamma(\nu_n, \mu_{\Phi,w})$ be such that

$$W_2^2(\nu_n, \mu_{\Phi,w}) = \iint_{\mathbb{R}^d} \|x - y\|^2 d\gamma_n(x, y).$$

Since ν_n is a finite probabilistic frame we may assume further that $\nu_n = \mu_{u_n,v}$ where $u_n = \{\psi_i\}_{i=1}^M \subset \mathbb{R}^d$ and $v = \{v_i\}_{i=1}^M \subset [0, \infty)$. For the sake of simplicity in notations, we omit the dependence of both ψ_i and v_i on n. Let $w_{n,j,k} = \gamma_n(\varphi_j, \psi_k)$.

Now consider the finite frames $\{u'_j\}_j = \{\sqrt{w_{n,j,k}}\psi_k\}_{j,k}$ and $\Phi' = \{\sqrt{w_{n,j,k}}\varphi_j\}_{j,k}$.

Note that $W_2(\mu_{\Phi'}, \mu_{\Phi'^\dagger}) = Q$. Now we consider the rows of these frames written with respect to the eigenbasis of the frame operator $S := S_{\Phi'}$ of Φ'.

Because, $W_2(\nu_n, \mu_{\Phi^\dagger,w}) > \epsilon$, then $\sum_{j,k} w_{n,j,k}\|\psi_k - S^{-1/2}\varphi_j\|^2 > \epsilon$.

Using this and Lemma 2.2.5 we have the following estimates:

$$W_2^2(\mu_{\Phi,w}, \nu_n) \geqslant W_2^2(\mu_{\Phi,w}, \nu) + \min(\tfrac{\epsilon^2}{d} \cdot M, M^2)$$

where A is the lower frame bound of Φ and $M = \min(1, \sqrt{A})$.

Consequently,

$$W_2^2(\mu_{\Phi,w}, \nu_n) - Q^2 \geqslant \min(\tfrac{\epsilon^2}{d} \cdot M, M^2) > 0.$$

But, this contradicts the fact that $Q = W_2(\mu_{\Phi,w}, \nu) = \lim_{n\to\infty} W_2(\nu_{\Phi,w}, \nu_n)$. Hence, $\lim_{n\to\infty} W_2(\nu_n, \mu_{S^{-1/2}\Phi,w}) = 0$, and $\nu = \mu_{\Phi^\dagger,w}$.

Case 3: Next, we suppose that μ is a non discrete probabilistic frame with frame bounds A, and B. Let $\{\mu_n\}_{n=1}^\infty = \{\mu_{\Phi_n,w(n)}\}_{n=1}^\infty$ be a sequence of finite probabilistic frames with bounds between $A/2$ and $B+A/2$ such that $\lim_{n\to\infty} W_2(\mu_n, \mu) = 0$. Then $F(\mu) = \lim_{n\to\infty} F(\mu_n)$ is such that $Q = W_2(F(\mu), \mu)$. Suppose there exists another Parseval frame ν such that $Q = W_2(\nu, \mu)$. Choose a sequence of finite Parseval frames $\{\nu_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \nu_n = \nu$.

Observe that $Q = \lim_{n\to\infty} W_2(\mu_n, F(\mu_n)) = \lim_{n\to\infty} W_2(\nu_n, \mu_n)$. Write $\Phi_n = \{\varphi_{n,j}\}_{j=1}^{M}$ and $w(n) = \{w_j\}_{j=1}^{M}$, where for simplicity we omit the dependence of M on n. Similarly, $\{\nu_n\}_{n=1}^{\infty} = \{\psi_{n,j}\}_{j=i}^{M'}$ with weights $v(n) = \{v_j\}_{j=1}^{M'}$.

Let $\gamma_n \in \Gamma(\mu_n, \nu_n)$ be such that

$$W_2^2(\mu_n, \nu_n) = \iint \|x - y\|^2 d\gamma_n(x, y).$$

Set

$$w_{j,k} = \gamma_n(\varphi_{n,j}, \psi_{n,k})$$

We know that

$$\begin{aligned} W_2^2(\mu_n, F(\mu_n)) &= \sum_{j=1}^{M} w_j \|\varphi_{n,j} - \varphi_{n,j}^{\dagger}\|^2 = \sum_{j,k} w_{j,k} \|\varphi_{n,j} - \varphi_{n,j}^{\dagger}\|^2 \\ &= \sum_{j,k} \|\sqrt{w_{j,k}}\varphi_{n,j} - \sqrt{w_{j,k}}\varphi_{n,j}^{\dagger}\|^2 \end{aligned}$$

We also know that

$$W_2^2(\mu_n, \nu_n) = \sum_{j,k} w_{j,k} \|\varphi_{n,j} - \psi_{n,k}\|^2 = \sum_{j,k} \|\sqrt{w_{j,k}}\varphi_{n,j} - \sqrt{w_{j,k}}\psi_{n,k}\|^2$$

Suppose that $\lim_{n\to\infty} W_2(F(\mu_n), \nu_n) > 0$. Thus, there exists $\epsilon > 0$ and and integer $n > 1$ such that $W_2(F(\mu_n), \nu_n) > \epsilon$. Consequently,

$$\epsilon < \sum_{j,k} w_{j,k} \|\varphi_{n,j}^{\dagger} - \psi_{n,k}\|^2 = \sum_{j,k} \|\sqrt{w_{j,k}}\varphi_{n,j}^{\dagger} - \sqrt{w_{j,k}}\psi_{n,k}\|^2$$

Hence

$$d(\Phi_n'^{\dagger}, \Psi_n') > \epsilon$$

where $\Psi_n' = \{\sqrt{w_{j,k}}\psi_{n,k}\}$.

By the same argument as in Lemma 2.2.5 we conclude that $W_2^2(\mu_n, \nu_n) - W_2^2(\mu_n, F(\mu_n)) \geqslant min(M\frac{\epsilon^2}{d}, M^2)$. where $M = \min(1, \sqrt{\frac{A}{2}})$

This contradicts the fact that Since $\lim_{n\to\infty} W_2(\mu_n, \nu_n) = Q = \lim_{n\to\infty} W_2(\mu_n, F(\mu_n))$. Thus $\lim_{n\to\infty} W_2(F(\mu_n), \nu_n) = 0$ and so $F(\mu) = \nu$.

∎

By Proposition 2.2.15 it follows that given a probabilistic frame μ and any sequence $\Phi_k := \Phi_{k,w_k} = \{\varphi_k\}_{k=1}^{N_k}$ of finite probabilistic frames in $\mathbb{R}^d$ such that $\lim_{k\to\infty} W_2(\mu, \mu_{\Phi_k}) = 0$, then $F(\mu) = \lim_{k\to\infty} F(\mu_{\Phi_k})$. Furthermore, it is proved in [53] that if $\{\mu_n\}_{n\geqslant 1} \subset \mathcal{P}_2$ converges in the Wassertein metric to $\mu \in \mathcal{P}_2$, then

$$\|S_\mu - S_{\mu_n}\| \leqslant C W_2(\mu_n, \mu).$$

All that is needed to prove Theorem 2.2.13 is to show that $F(\mu) = \mu^\dagger$.

Proof of Theorem 2.2.13. Let μ be a probabilistic frame with bounds A, B. Let $0 < \epsilon < A/2$ and choose a compactly supported probabilistic frame ν_ϵ as in Lemma 2.2.9. In particular ν_ϵ is supported on $B(0, R_\epsilon)$ with frame bounds between $A/2$ and $B + A/2$, where $R_\epsilon > 0$ is such that

$$\int_{\mathbb{R}^d \setminus B(0,R_\epsilon)} \|x\|^2 dx < \epsilon/3.$$

We know there exists a finite probabilistic frame μ_ϵ with frame bounds between $\frac{A}{2}$ and $B + \frac{A}{2}$ which is within $\frac{\epsilon}{3}$ of ν_ϵ in the Wasserstein metric.

Taking a sequence $\{\epsilon_i\}_{i=1}^{\infty}$ that goes to 0 and letting $\mu_{\epsilon_i} = \mu_i$ for convenience we have $\lim_{n\to\infty} W_2(\mu_n, \mu) = 0$ which implies that $\lim_{n\to\infty} S_{\mu_n} = S_\mu$ in the operator norm. Thus, $\lim_{n\to\infty} S_{\mu_n}^{-1/2} = S_\mu^{-1/2}$

We recall that $\lim_{n\to\infty} W_2(\mu_n, \mu) = 0$ is equivalent to

$$\lim_{n\to\infty} \int f \, d\mu_n(x) = \int f \, d\mu(x)$$

for all continuous function f such that $|f(x)| \leqslant C(1 + \|x - x_0\|^2)$ for some $x_0 \in \mathbb{R}^d$ [49, Theorem 6.9]

We know that $\lim_{n\to\infty} F(\mu_n) = \lim_{n\to\infty} \mu_n^\dagger = F(\mu)$ in the Wasserstein metric. We would like to show that $\lim_{n\to\infty} F(\mu_n) = \lim_{n\to\infty} \mu_n^\dagger = \mu^\dagger$.

We first show that for all continuous function f such that $|f(x)| \leqslant C(1+\|x-x_0\|^2)$ for some $x_0 \in \mathbb{R}^d$

$$\lim_{n\to\infty} \int f\, d\mu_n^\dagger(x) = \int f\, d\mu^\dagger(x).$$

$$\begin{aligned}
|\int f\, d\mu_n^\dagger(x) - \int f\, d\mu^\dagger(x)| &= |\int f(S_{\mu_n}^{-1/2} x)\, d\mu_n(x) - \int f(S_{\mu}^{-1/2} x)\, d\mu(x)| \\
&\leqslant \int |f(S_{\mu_n}^{-1/2} x) - f(S_{\mu}^{-1/2} x)|\, d\mu_n(x) + \\
&\quad |\int f(S_{\mu}^{-1/2} x)\, d\mu_n(x) - \int f(S_{\mu}^{-1/2} x)\, d\mu(x)|
\end{aligned}$$

Let f be continuous with $|f(x)| \leqslant C(1 + \|x - x_0\|^2)$ for some $x_0 \in \mathbb{R}^d$. Then, $f(S_\mu^{-1/2})$ is continuous and satisfies

$|f(S_\mu^{-1/2}x)| \leqslant C(1+\|x_0-S_\mu^{-1/2}x\|^2) \leqslant C(1+\|S_\mu^{-1/2}\|^2\|S_\mu^{1/2}x_0-x\|^2) \leqslant C'(1+\|(S_\mu^{1/2}x_0-x\|^2))$.

Consequently, we can find N_1 such that for all $n \geqslant N_1$,

$$|\int f(S_{\mu}^{-1/2} x)\, d\mu_n(x) - \int f(S_{\mu}^{-1/2} x)\, d\mu(x)| < \epsilon/3.$$

Since f is continuous, there exists $\delta > 0$ such that for all $x, y \in B(0, R')$, $\|x-y\| < \delta$ implies that $|f(x) - f(y)| < \epsilon/3$, where $R' > 0$ is chosen so as to guarantee that for large n, and $x \in B(0, R)$, $S_{\mu_n}^{-1/2}x, S_\mu^{-1/2}x \in B(0, R)$. Since, $\lim_{n\to\infty} S_{\mu_n}^{-1/2} = S_\mu^{-1/2}$, there exists N_2 such that for all $n \geqslant N_2$,

$$\|S_{\mu_n}^{-1/2}x - S_\mu^{-1/2}x\| \leqslant \|S_{\mu_n}^{-1/2} - S_\mu^{-1/2}\|\|x\| \leqslant R\|S_{\mu_n}^{-1/2} - S_\mu^{-1/2}\| < \delta.$$

Therefore, for $n \geqslant N_2$, $|f(S_{\mu_n}^{-1/2}x) - f(S_{\mu}^{-1/2}x)| < \epsilon/3$ for all $x \in B(0,R)$. Consequently,

$$
\begin{aligned}
\int |f(S_{\mu_n}^{-1/2}x) - f(S_{\mu}^{-1/2}x)|\, d\mu_n(x) &= \int_{B(0,R)} |f(S_{\mu_n}^{-1/2}x) - f(S_{\mu}^{-1/2}x)|\, d\mu_n(x) \\
&\quad + \int_{\mathbb{R}^d \setminus B(0,R)} |f(S_{\mu_n}^{-1/2}x) - f(S_{\mu}^{-1/2}x)|\, d\mu_n(x) \\
&< \epsilon/3 + \int_{\mathbb{R}^d \setminus B(0,R)} |f(S_{\mu_n}^{-1/2}x) - f(S_{\mu}^{-1/2}x)|\, d\mu_n(x) \\
&< \epsilon/3 + M \int_{\mathbb{R}^d \setminus B(0,R)} \|x\|^2\, d\mu_n(x) \\
&< 2\epsilon/3
\end{aligned}
$$

where > 0 is a constant that depends only on f, and μ.

It follows that for all $n \geqslant \max(N_1, N_2)$, we have

$$
|\int f\, d\mu_n^{\dagger}(x) - \int f\, d\mu^{\dagger}(x)| < \epsilon
$$

which implies that $\lim_{n\to\infty} \int f\, d\mu_n^{\dagger}(x) = \int f\, d\mu^{\dagger}(x)$.

∎

Chapter 3

Introduction to Chapter 4

In the third part of this thesis we show that every vector $v \in \mathbb{C}^n$ corresponds with a rank 2 projection in $\mathbb{R}^{2n}$. In particular given $v = (a_1 + ib_1, ...a_n + ib_n)$ in $\mathbb{C}^n$ let $v' = (a_1, b_1, ..., a_n, b_n)$ and $v'' = (-b_1, a_1, ..., -b_n, a_n)$ be vectors in $\mathbb{R}^{2n}$. We will associate v with the rank two projection onto the subspace spanned by v' and v'' (Notation 4.4.1).

This association will answer many problems. First we will prove the complex analog of Edidin's theorem in phase retrieval which states that a family of subspaces $S = \{S_i\}_{i=1}^m \subset \mathbb{R}^n$ does phase retrieval if and only if for any nonzero vector $w \in \mathbb{R}^n$ the projection of w onto the subspaces in S spans $\mathbb{R}^n$ [30]. This is Theorem 4.5.11.

We will also apply this association to mutually unbiased bases and show that mutually unbiased bases in $\mathbb{C}^n$ correspond to mutually unbiased rank 2 projections in $\mathbb{R}^{2n}$ (Corollary 4.6.3). This is done by showing that the absolute value of the inner product of two complex vectors determines the trace of the product of the two corresponding rank 2 projections (Proposition 4.6.2).

Using the above result on traces, we will also show that equiangular, and more generally, k-angular complex vectors associate with equiangular (respectively k-angular) rank 2 real projections (Theorem 4.9.2, Theorem 4.9.3).

It is also known that in $\mathbb{R}^3$ there does not exist a tight fusion frame of 2 2-dimensional subspaces (even with unequal weights). We will show using this association that in even dimensions (ie $\mathbb{R}^n$ where n is even) for all $m \geqslant \frac{n}{2}$ there exists a tight fusion frame of m 2-dimensional subspaces with equal weights (Corollary 4.7.3).

Chapter 4

4.1 Introduction

We will show that vectors in $\mathbb{C}^n$ have natural analogs as rank 2 projections in $\mathbb{R}^{2n}$. The strength of this association is that it carries norm properties with it. In particular, if $v, w \in \mathbb{C}^n$ with analogs in $\mathbb{R}^{2n}$ of v', w', let P_v, P_w be the associated rank 2 projections on $\mathbb{R}^{2n}$. Then $|\langle v, w\rangle| = \|v\|\|P_v w'\|$ and if v, w are unit vectors then $tr\ P_v P_w = 2|\langle v, w\rangle|^2$. This allows us to move many properties from $\mathbb{C}^n$ to corresponding properties of rank 2 projections in $\mathbb{R}^{2n}$. In particular, families of mutually unbiased bases (MUBs) in $\mathbb{C}^n$ will associate with families of mutually unbiased rank 2 projections in $\mathbb{R}^{2n}$. The importance of this association is that there are very few mutually unbiased bases in $\mathbb{R}^n$. It is known that in $\mathbb{C}^n$ there are at most $n+1$ MUBs and in $\mathbb{R}^n$ there are at most $\frac{n}{2}+1$ MUBs [28]. But these limits are rarely reached. It is known that the limit is reached in $\mathbb{R}^n$ if $n = 4^p$ and the limit is reached in $\mathbb{C}^n$ if $n = p^t$ where p is a prime (see [18] for n=p, and [54] for p^t). We now see that there are substantially more cases where $\mathbb{R}^n$ reaches its maximum for mutually unbiased rank two projections. For researchers who can live with mutually unbiased rank 2 projections instead of mutually unbiased bases, they now have a large number of possible families to work with.

This association carries fusion frames in $\mathbb{C}^n$ to fusion frames in $\mathbb{R}^{2n}$ with the same fusion frame bounds. Using this we will answer a longstanding problem in fusion frame theory by showing that for every $m \geqslant n$, there is a tight fusion frame of m 2-dimensional subspaces in $\mathbb{R}^{2n}$.

We will also see that this association transfers equiangular and biangular tight frames into equiangular and biangular tight fusion frames.

This association will also carry with it the notion of phase retrieval in $\mathbb{C}^n$. In certain engineering applications, the phase of a signal is lost during collection and processing. This gave rise to a need for methods to recover the phase of a signal. Phase retrieval in engineering is over 100 years old and has application to a large number of areas including speech recognition [9, 44, 45], and applications such as X-ray crystallography [8, 35, 36]. The concept of *phase retrieval* for Hilbert space frames was introduced in 2006 by Balan, Casazza, and Edidin [6] and since then it has become an active area of research.

Phase retrieval has been defined for projections as well as for vectors. *Phase retrieval by projections* occur in real life problems, such as crystal twinning [29], where the signal is projected onto some lower dimensional subspaces and has to be recovered from the norms of the projections of the vectors onto the subspaces. We refer the reader to [24] for a detailed study of phase retrieval by projections. Phase retrieval has been significantly generalized in [51].

A fundamental result concerning phase retrieval by projections due to Edidin [30] is that a family of projections $\{P_i\}_{i=1}^m$ does phase retrieval in $\mathbb{R}^n$ if and only if for all $0 \neq x \in \mathbb{R}^n$, the vectors $\{P_i x\}_{i=1}^m$ span $\mathbb{R}^n$. It has been an open question whether there is a complex analog to this theorem. We will answer this question in this paper.

First, we will show that complex phase retrieval by vectors in $\mathbb{C}^n$ is equivalent to a problem of real phase retrieval by rank two projections in $\mathbb{R}^{2n}$. Next, we will give a new *geometric* proof of this theorem. Finally, we combine these two techniques to give the complex analog of Edidin's Theorem.

Finally, we will give a classification of norm retrieval which is an analog of Edidin's theorem.

The idea of **realization** of complex vectors was first done by Balan [5] in a much more general setting than this.

4.2 Preliminaries

In this section we will introduce the concepts which will be used throughout the paper. For notation, we write $\mathbb{H}^n$ for a real or complex n-dimensional Euclidean space. If we need to restrict ourselves to one of these choices, we will write $\mathbb{R}^n$ or $\mathbb{C}^n$.

Note: We will rely heavily on the fact that given $x, y \in \mathbb{R}^n$, $x - y \perp x + y$ if and only if $\|x\| = \|y\|$.

Definition 4.2.1. *A family of vectors $\{v_i\}_{i=1}^m$ is a* **frame** *for $\mathbb{H}^n$ if there are constants $0 < A \leqslant B < \infty$ so that*

$$A\|x\|^2 \leqslant \sum_{i=1}^{m} |\langle x, v_i\rangle|^2 \leqslant B\|x\|^2, \text{ for all } x \in \mathbb{H}^n.$$

If $A = B$ this is a **tight frame** *and if $A = B = 1$ it is a* **Parseval frame**.

We will be working with phase retrieval and phase retrieval by projections.

Definition 4.2.2. *A family of vectors $\{v_i\}_{i=1}^m$ does* **phase retrieval** *on $\mathbb{H}^n$ if when-*

ever $x, y \in \mathbb{H}^n$ satisfy

$$|\langle x, v_i\rangle| = |\langle y, v_i\rangle|, \text{ for all } i = 1, 2, \ldots, m,$$

we have that $x = cy$ for some $|c| = 1$.

We say a family of subspaces $\{S_i\}_{i=1}^m$ (or their orthogonal projections $\{P_i\}_{i=1}^m$) on $\mathbb{H}^n$ do **phase retrieval** *if whenever $x, y \in \mathbb{H}^n$ satisfy*

$$\|P_i x\| = \|P_i y\|, \text{ for all } i = 1, 2, \ldots, m,$$

then $x = cy$ for some $|c| = 1$.

Remark 4.2.3. We will find it convenient to work with the contrapositive. I.e. $\{P_i\}_{i=1}^m$ does phase retrieval if and only if whenever $x \neq cy$ for any $|c| = 1$, there is a $1 \leqslant j \leqslant m$ so that $\|P_j x\| \neq \|P_j y\|$. Also, whenever $\|P_j x\| \neq \|P_j y\|$, for some j, we say that $\{P_j\}_{j=1}^m$ **distinguishes between** x, y.

We now have:

Lemma 4.2.4. *Given a set of projections $\{P_j\}_{j=1}^m$ on $\mathbb{H}^n$, the set of vectors they cannot distinguish are those v, w for which $v - w \perp P_j(v + w)$.*

Proof. We compute:

$$\langle v - w, P_j(v + w)\rangle = \langle P_j v - P_j w, P_j v + P_j w\rangle = \|P_j v\|^2 - \|P_j w\|^2.$$

So $v - w \perp P_j(v + w)$ if and only if $\|P_j v\| = \|P_j w\|$. ∎

We will need the complement property for families of vectors.

Definition 4.2.5. *A family of vectors $\{v_i\}_{i=1}^m$ in $\mathbb{H}^n$ has the* **complement property** *if whenever $I \subset \{1, 2, \ldots, m\}$, either span $\{v_i\}_{i \in I} = \mathbb{H}^n$ or span $\{v_i\}_{i \in I^c} = \mathbb{H}^n$.*

A fundamental result in this area [6] is:

Theorem 4.2.6. *If vectors $\{v_i\}_{i=1}^m$ do phase retrieval in $\mathbb{H}^n$ then they have the complement property.*

It was also shown in [6] that a family of vectors with complement property in $\mathbb{R}^n$ does phase retrieval but this implication does not hold in $\mathbb{C}^n$. A family of vectors $\{v_i\}_{i-=1}^m$ is **full spark** if for every $I \subset \{1, 2, \ldots, m\}$ with $|I| = n$, span $\{v_i\}_{i\in I} = \mathbb{H}^n$. So a full spark family with $m \geqslant 2n-1$ has the complement property.

Edidin [30] gave a fundamental classification of phase retrieval by projections for $\mathbb{R}^n$ in terms of the spans of $\{P_i x\}_{i=1}^m$, for $x \in \mathbb{R}^n$.

Theorem 4.2.7 (Edidin)**.** *Let $\{S_i\}_{i=1}^m$ (with respective projections $\{P_i\}_{i=1}^m$) be subspaces of $\mathbb{R}^n$. The following are equivalent:*

1. *$\{P_i\}_{i=1}^m$ does phase retrieval.*
2. *For every $0 \neq x \in \mathbb{R}^n$, span $\{P_i x\}_{i=1}^m = \mathbb{R}^n$.*

The corresponding result for frames (I.e. rank one projections) has been done in [5]. The necessity of the condition for frames also appeared in [7]. For $\mathbb{C}^n$, (2) does not imply (1) in the theorem in general. For example, if $\{v_i\}_{i=1}^5$ is a full spark family of vectors in $\mathbb{C}^3$, then it has complement property and so if P_i is the projection onto span v_i then for every $0 \neq x \in \mathbb{C}^3$, span $\{P_i x\}_{i=1}^5 = \mathbb{C}^3$. But any family doing phase retrieval in $\mathbb{C}^3$ must contain at least 8 vectors [30]. However, (1) does imply (2) in $\mathbb{C}^n$ [7].

Theorem 4.2.8. *If projections $\{P_i\}_{i=1}^m$ do phase retrieval on $\mathbb{H}^n$ then for every $0 \neq x \in \mathbb{H}^n$, span $\{P_i x\}_{i=1}^m = \mathbb{H}^n$.*

Proof. We will prove the contrapositive. So assume there is an $0 \neq x \in \mathbb{H}^n$ so that span $\{P_i x\}_{i=1}^m \neq \mathbb{H}^n$. Choose $0 \neq y \in \mathbb{H}^n$ so that $y \perp P_i x$ for all $i = 1, 2, \ldots, m$. Then

$$\langle P_i x, y \rangle = \langle P_i x, P_i y \rangle = 0, \text{ for all } i = 1, 2, \ldots, m.$$

Let $w = x + y$ and $v = x - y$. Then for all i,

$$\|P_i w\|^2 = \|P_i x + P_i y\|^2 = \|P_i x\|^2 + \|P_i y\|^2 = \|P_i x - P_i y\|^2 = \|P_i v\|^2.$$

But, $w \neq cv$ for any $|c| = 1$. Since if $w = cv$ then either $c = \pm 1$ and so either x=0 or y=0, or $x = \frac{1+c}{1-c} y$ which combined with $P_i x \perp P_i y$ implies again x=0. I.e. $\{P_i\}_{i=1}^m$ fails to do phase retrieval. ∎

4.3 A New Proof of Edidin's Theorem

In this section we establish a new proof of Theorem 4.2.7. This proof is **geometric** and it will allow us to generalize it to the complex case. It will also direct us to introduce a natural association between vectors in $\mathbb{C}^n$ and two dimensional subspaces of $\mathbb{R}^{2n}$.

To give our geometric proof of Theorem 4.2.7, we will need a sequence of results.

Lemma 4.3.1. *Given a subspace S of dimension d in $\mathbb{R}^n$, for any point x, either S is in the orthogonal complement of x or there exists a subspace S' of dimension d-1 contained in S that is contained in the orthogonal complement of x.*

Proof. Let C be the orthogonal complement of x in $\mathbb{R}^n$. Then C is a hyperplane so its intersection with S is either all of S or a subspace of dimension $d - 1$. ∎

From the above it is clear that for any point x either S is orthogonal to x or there exists an orthonormal basis $b_1, ...b_d$ of S such that b_1 is not orthogonal to x but the rest of the elements in the orthonormal basis are orthogonal to x.

Remark 4.3.2. Given two real numbers a and b, $|a| = |b|$ if and only if $a = b$ or $a = -b$. Hence given two vectors v and w, $| < x, v > | = | < x, w > |$ if and only if x is orthogonal to $v - w$ or $v + w$.

Another simple observation we will heavily use is:

Lemma 4.3.3. *Given two vectors $x, y \in \mathbb{H}^n$, letting $v = \frac{x-y}{2}$ and $w = \frac{x+y}{2}$, we have that $w + v = x$ and $w - v = y$.*

The next theorem is a variation of the argument of Theorem 4.8.3.

Theorem 4.3.4. *Given a family of subspaces $\{S_i\}_{i=1}^m$ of $\mathbb{R}^n$ with respective projections $\{P_i\}_{i=1}^m$, and a point $0 \neq x \in \mathbb{R}^n$, let $M = span\ \{P_i x\}_{i=1}^m$. For any $y \in M^\perp$, we have for all $1 \leqslant i \leqslant m$, $\|P_i(v)\| = \|P_i(w)\|$ for all w and v such that $w + v = x$ and $w - v = y$.*

Proof. Given any S_i, if $x = w + v \perp S_i$ then $P_i w = -P_i v$ and so $\|P_i(w)\| = \|P_i(v)\|$. If S_i is not orthogonal to x then as mentioned above there exists an orthonormal basis $b_1, ...b_d$ of S_i such that b_1 is not orthogonal to x but the rest of the vectors are orthogonal to x. Hence the projection of x onto S_i, $P_i x$, is a nonzero scalar multiple of b_1, $< w, b_1 > \neq - < v, b_1 >$ and for all $2 \leqslant i \leqslant d$, $< w, b_i >= - < v, b_i >$. Hence $\|P_i(v)\| = \|P_i(w)\|$ if and only if $< w, b_1 >=< v, b_1 >$. But, this happens only if $w - v$ is orthogonal to b_1.

Considering the above argument over all i we see if we pick any point $y \in M^\perp$ such that $w + v = x$ and $w - v = y$, then for all $1 \leqslant i \leqslant m$, $\|P_i(v)\| = \|P_i(w)\|$. ∎

Theorem 4.3.5. *Given the assumptions of Theorem 4.3.4, given $x, y \in \mathbb{R}^n$, if we can pick w and v such that $w + v = x$ and $w - v = y$ are not orthogonal to M then for some i, $\|P_i(v)\| \neq \|P_i(w)\|$*

Proof. Since M was the span of all the $\{P_i x\}_{i=1}^m$ then for some i, $w-v$ is not orthogonal to $P_i(x)$. Hence for some orthonormal basis $b_1, ...b_d$ of S_i we have $< w, b_1 > \neq < v, b_1 >$ and for all other i, $< w, b_i >= - < v, b_i >$. Clearly $\|P_i(v)\| \neq \|P_i(w)\|$ by the Pythagorean theorem. ∎

We now prove Edidin's Theorem [30].

Theorem 4.3.6. *If a set of subspaces $\{S_i\}_{i=1}^m$ of $\mathbb{R}^n$ with respective projections $\{P_i\}_{i=1}^m$ does not do real phase retrieval then for some $0 \neq x$, span $\{P_i(x)\}_{i=1}^m \neq \mathbb{R}^n$.*

Proof. Suppose there exists v and w, $v \neq \pm w$ such that for all $1 \leqslant i \leqslant m$, $\|P_i v\| = \|P_i w\|$. Clearly $w + v$ and $w - v$ are nonzero. By the above theorem choose $w + v$ to be x. Clearly $w - v$ must be orthogonal to *span* $\{P_i x\}_{i=1}^m$ by Theorem 4.3.5. ∎

The other direction of Theorem 4.2.7 is Theorem 4.2.8.

4.4 Turning vectors in $\mathbb{C}^n$ into rank 2 projections on $\mathbb{R}^{2n}$

We need a piece of notation.

Notation 4.4.1. *For the rest of the paper, for a complex vector $v = (a_1 + ib_1, ...a_n + ib_n)$ in $\mathbb{C}^n$ let $v' = (a_1, b_1, ..., a_n, b_n)$ and $v'' = (-b_1, a_1, ..., -b_n, a_n)$ be vectors in $\mathbb{R}^{2n}$. We will write $S_v =$ span $\{v', v''\}$ and P_v as the rank 2 projection of $\mathbb{R}^{2n}$ onto S_v.*

The following are immediate from the definition.

Proposition 4.4.2. *Given vectors* $v = (a_1 + ib_1, \ldots, a_n + ib_n)$ *and* $w = (a_1' + ib_1', \ldots, a_n' + b_n')$. *The following hold:*

1. *we have*

$$v' \perp v'' \text{ and } \|v\|^2 = \|v'\|^2.$$

2. *We have*

$$\langle w', v' \rangle = \sum_{j=1}^{n} (a_j a_j' + b_j b_j') \text{ and } \langle w', v'' \rangle = \sum_{j=1}^{n} (a_j b_j' - a_j' b_j).$$

3. *We have*

$$\langle w'', v'' \rangle = \langle w', v' \rangle.$$

4. *We have*

$$\langle w, v \rangle = \langle w', v' \rangle + i \langle w', v'' \rangle.$$

5. *It follows that if* $\|v\| = 1$ *then*

$$|\langle v, w \rangle| = |\langle w', v' \rangle + i \langle w', v'' \rangle| = \sqrt{|\langle w', v' \rangle|^2 + |\langle w', v'' \rangle|^2} = \|P_v w'\|.$$

6. *If* $\|v\| \neq 1$ *then*

$$|\langle v, w \rangle| = \|v\| \|P_v w'\|.$$

7. $(iw)' = w''$.

8. *Given* $z = (a_1, b_1 a_2, b_2, \ldots, a_n, b_n) \in \mathbb{R}^{2n}$, $z = v'$ *where* $v = (a_1 + ib_1, \ldots, a_n + ib_n) \in \mathbb{C}^n$. *Hence,* $\mathbb{R}^{2n} = \{v' : v \in \mathbb{C}^n\}$.

Corollary 4.4.3. *If* $\{v_j\}_{j=1}^m$ *is an orthonormal set of vectors in* $\mathbb{C}^n$ *then* $\{v_j', v_j''\}_{j=1}^m$ *is an orthonormal set of vectors in* $\mathbb{R}^{2n}$*.*

Proof. This is immediate by (1) and (4) in Proposition 4.4.2. ∎

As a consequence:

Corollary 4.4.4. *Given complex vectors* v, w_1, w_2 *let* P *be the projection onto* S_v*. The following are equivalent:*

1. $| < w_1, v > | = | < w_2, v > |$.

2. $\|Pw_1'\| = \|Pw_2'\|$.

In particular if $w_1 = cw_2$ *where* $c \in \mathbb{C}^n$ *and* $|c| = 1$ *then* $\|Pw_1'\| = \|Pw_2'\|$*.*

Lemma 4.4.5. *The rotation (multiplication by a unit complex scalar cos* $\theta +$ *isin* θ*) of a complex vector* $w = (a_1' + ib_1', ...a_n' + ib_n')$ *is the same as taking* $\cos(\theta)w + \sin(\theta)iw$*.*

Hence

$$(\cos(\theta)w + \sin(\theta)iw)' = \cos(\theta)w' + \sin(\theta)w''.$$

Remark 4.4.6. It follows that the vectors obtained by multiplying w by any unit norm complex scalar would associate with the points on the circle of radius $\|w\|$ in S_w

Theorem 4.4.7. *Given any nonzero vector* v*, and complex scalar* $c = a + bi \neq 0$*,* $[(a + ib)v]' = av' + bv''$ *and we have that* $S_v = S_{cv}$*.*

Proof. Let $V = span\ \{(cv)', (cv)''\}$. We will show that $S_v = V$. We first show V is contained in S_v We know v identifies with v' and iv identifies with v'' hence both v' and $(iv)'$ are in S_v. Next, given any vector of the form $av' + bv''$ we will show that $av + ibv$

identifies with this vector. Clearly $av + ibv = a(a_1 + ib_1, ...a_n + ib_n) + b(-b_1 + ia_1, ... - b_n + ia_n)$ which identifies with $av' + bv'' = a(a_1, b_1, ...a_n, b_n) + b(-b_1, a_1, ... - b_n, a_n)$ Hence V is contained in S_v

Now given a complex scalar $(a + bi)$. As shown above $(a + bi)v$ identifies with $av' + bv''$ which is a vector in V. Hence S_v is contained in V. It follows that $S_v = V$. ∎

We may define an equivalence relation on $\mathbb{C}^n \backslash 0$ by saying two vectors are *equivalent* if and only if one is a complex scalar multiple of the other. It is clear that two vectors v and w are in the same equivalence class if and only if $S_v = S_w$.

Theorem 4.4.8. *For any two subspaces S_v and S_w, if $S_v \bigcap S_w \neq \{0\}$ then $S_v = S_w$. Moreover, if $v \perp w$ then $S_v \perp S_w$.*

Proof. Assume there is a $0 \neq x$ with $x' \in S_v \bigcap S_w \neq 0$. Then we have that $x = av$ and $x = bw$ for some nonzero complex scalars a and b and hence $v = a^{-1}bw$. From the above v and w are in the same equivalence class and so $S_v = S_w$.

The moreover part follows from Corollary 4.4.3. ∎

Putting this altogether,

Theorem 4.4.9. *Given a set of complex vectors $\{v_i\}_{i=1}^m \in \mathbb{C}^n$ which do phase retrieval in $\mathbb{C}^n$, if we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, for any vector w, the only points that cannot be distinguished from w' by projections onto S_{v_i} are any points on the circle of radius $\|w\|$ in the subspace S_w spanned by w' and w''.*

4.5 Complex Phase Retrieval and Rank Two Projections

If we let $\{P_i\}_{i=1}^m$ be the projections onto S_{v_i}, we see for any nonzero point x and any $c \in \mathbb{R} \setminus 0$, *span* $\{P_i x\} =$ *span* $\{P_i(cx)\}_{i=1}^m$. Hence when looking at the span of the projections we will identify a point with any scalar multiple of it. We see then that y is in the orthogonal complement of the span if and only if cy is in the orthogonal complement for any $c \in \mathbb{R} \setminus 0$. Hence when we look at vectors in the orthogonal complement we will identify a vector y with any scalar multiple of it.

Theorem 4.5.1. *Given a vector $v \in \mathbb{C}^n$, let $w = iv$ and $m = (\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))v$. Then $v + w = \sqrt{2}m$, and $v' + w' = \sqrt{2}m'$.*

Proof. $v + w = (i+1)v = \sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))v = \sqrt{2}m$ ∎

Corollary 4.5.2. *For every nonzero $m' \in \mathbb{R}^{2n}$ there exists nonzero v' and w' such that $w = iv$ (hence $w' \neq v'$) and $v' + w' = m'$*

Proof. This follows from the theorem above by taking $v = \dfrac{\cos(\frac{\pi}{4}) - i\sin(\frac{\pi}{4})}{\sqrt{2}}m$. ∎

Proposition 4.5.3. *Given a vector $v \in \mathbb{C}^n$ with projection P onto S_v, for every $w' \in \mathbb{R}^{2n}$, $Pw' \perp w''$.*

Proof. By Corollary 4.5.2, there exist $0 \neq x, y \in \mathbb{C}^n$ so that $x = iy$ and $x' + y' = w'$. Hence, $\|x'\| = \|y'\|$ and $|\langle x, v\rangle| = |\langle y, v\rangle|$. So $y' \in S_x$. Hence, $w' \in S_x$ and so by Theorem 4.4.8, $S_w = S_x$. Now,

$$\langle x' - y', x' + y'\rangle = \|x'\| - \|y'\| = 0 \text{ and } x' - y' \in S_w.$$

But, w'' (and its multiples) are the only vector in S_w orthogonal to w'. So $w'' = x' - y' \perp P(x' + y') = Pw'$. ∎

Corollary 4.5.4. *Given vectors $\{v_j\}_{j=1}^m$ in $\mathbb{C}^n$ with projections $\{P_j\}_{j=1}^m$ onto $\{S_{v_j}\}_{j=1}^m$ and given any nonzero $w' \in \mathbb{R}^{2n}$ let $M = span\ \{P_j w'\}_{j=1}^m$. Then $w'' \perp M$.*

Theorem 4.5.5. *Given vectors $\{v_j\}_{j=1}^m$ in $\mathbb{C}^n$ which do phase retrieval with projections $\{P_j\}_{j=1}^m$ onto $\{S_{v_j}\}_{j=1}^m$ and given any nonzero $w' \in \mathbb{R}^{2n}$ let $M = span\ \{P_j w'\}_{j=1}^m$. Then $M^\perp = span\{w''\}$.*

Proof. Given $v' \in M^\perp$, there exists two vectors x' and y' with $x' + y' = w'$ and $x' - y' = v'$. It follows that

$$0 = \langle P_j(x'+y'), x'-y' \rangle = \langle P_j(x'+y'), P_j(x'-y') \rangle = \|P_j x'\|^2 - \|P_j y'\|^2.$$

So $\|P_j x'\| = \|P_j y'\|$ and by Corollary 4.4.4, we have that $|\langle x', v_j \rangle| = |\langle y', v_j \rangle|$ for all $j = 1, 2, \ldots, m$. Therefore, since we have phase retrieval, $x = cy$ for some $|c| = 1$ and hence $y' \in S_x$ and $\|y'\| = \|x'\| \neq 0$. Since S_x is a subspace $x' + y' = w', x' - y' \in S_x$. By Theorem 4.4.8 $S_w = S_x$. But, $\{w', w''\}$ is an orthonormal basis for S_w and $x' + y' = w' \perp x' - y'$. It follows that $x' - y'$ is a multiple of w''. Hence, $w'' \perp M$ and its scalar multiples are the only vectors orthogonal to M. ∎

Now we will see how to reformulate complex phase retrieval by vectors in $\mathbb{C}^n$ into a variant of real phase retrieval for our class of rank two projections on $\mathbb{R}^{2n}$.

Theorem 4.5.6. *Given a set of complex vectors $\{v_j\}_{j=1}^m \in \mathbb{C}^n$ let $\{P_j\}_{j=1}^m$ be the corresponding projections onto S_{v_j}. The following are equivalent:*

1. *$\{v_j\}_{j=1}^n$ does phase retrieval.*

2. *For each point $x' \in \mathbb{R}^{2n}$, x'' is the only vector in $\mathbb{R}^{2n}$ orthogonal to $M = span\ \{P_i x'\}_{i=1}^m$.*

3. *For each point $x' \in \mathbb{R}^{2n}$, $M = span\ \{P_i x'\}_{i=1}^m$ is a hyperplane.*

Proof. (1) $\Rightarrow$ (2): This is Theorem 4.5.5.

(2) $\Rightarrow$ (1): Choose vectors v, w with

$$|\langle v, v_i\rangle| = |\langle w, v_i\rangle| \tag{4.1}$$

for all $i = 1, 2, \ldots, m$. We first need to observe that $\langle v, v_i\rangle \neq 0$ for some i. For otherwise we would have that $v' \in M^\perp$ and since $v' \perp v'' \in M^\perp$, this contradicts our assumption that $dim\ M^\perp = 1$. By Corollary 4.4.4, $\|P_i v'\| = \|P_i w'\|$. It follows that

$$\langle P_i(v' + w'), v' - w'\rangle = \langle P_i(v' + w'), P_i(v' - w')\rangle = \|P_i v'\|^2 - \|P_i w'\|^2 = 0.$$

Hence, $v' - w' \perp P_i x' = P_i(v' + w')$ for all $i = 1, 2, \ldots, m$. By assumption (2), $v' - w' = cw'' \in S_w$ and so $v' \in S_w$. By Theorem 4.4.8, $S_v = S_w$ and so $v = cw$ for some c. Now by Equation 4.1, $|c| = 1$.

(2) $\Leftrightarrow$ (3): This is clear. ∎

Corollary 4.5.7. *Let $\{v_i\}_{i=1}^m$ do phase retrieval on $\mathbb{C}^n$ and for $x \in \mathbb{C}^n$ let $I = \{i : \langle x, v_i\rangle \neq 0\}$. Then*

1. *$|I| \geqslant 2n - 1$.*

2. *$span\ \{v_i\}_{i \in I} = \mathbb{C}^n$.*

Proof. (1) We proceed by way of contradiction. Let P_i be the rank two projection in $\mathbb{R}^{2n}$ onto S_{v_i}. If $|I| \leqslant 2n - 2$, then for $i \in I^c$, $\|P_i x'\| = |\langle x, v_i\rangle| = 0$. It follows that $|\{i : P_i x' \neq 0\}| \leqslant 2n - 2$ and hence $\{P_i x'\}_{i=1}^m$ does not span a hyperplane in $\mathbb{R}^{2n}$ contradicting Theorem 4.5.6.

(2) Let P_i be the rank two projection in $\mathbb{R}^{2n}$ onto S_{v_i}. We proceed by way of contradiction. So assume there exists $y \in \mathbb{C}^n$ with $y \perp v_i$ for all $i \in I$. Then, $P_i y' = 0$ for all $i \in I$. Since $y'' = (iy)'$, it follows that $P_i y'' = 0$ for all $i \in I$. Also, $P_i x = 0$ for all $i \in I^c$. So

$$M = span\ \{P_i x'\}_{i=1}^m = span\ \{P_i x'\}_{i \in I}.$$

It follows that $y', y'' \perp M$ and so M is not a hyperplane in $\mathbb{R}^{2n}$ contradicting Theorem 4.5.6.

∎

Corollary 4.5.8. *If $\{v_i\}_{i=1}^{4n-4}$ does phase retrieval in $\mathbb{C}^n$ and $I \subset [4n-4]$ with $|I| = 2n-2$, then $\{v_i\}_{i \in I}$ and $\{v_i\}_{i \in I^c}$ both span $\mathbb{C}^n$.*

To prove the complex analog of Theorem 4.2.7, we need a result.

Proposition 4.5.9. *Let W be a d-dimensional subspace of $\mathbb{C}^n$. Then there is a 2d-dimensional subspace V of $\mathbb{R}^{2n}$ so that for every orthonormal basis $\{v_i\}_{i=1}^d$ for W we have span $\{S_{v_i}\}_{i=1}^d = V$. Recall that the $\{S_{v_i}\}_{i=1}^m$ is an orthogonal set of 2-dimensional subspaces of $\mathbb{R}^{2n}$.*

Proof. Choose any orthonormal basis $\{v_i\}_{i=1}^d$ for W and let $V = span\ \{S_{v_i}\}_{i=1}^m$. We want to show that for any other orthonormal basis $\{w_i\}_{i=1}^m$ for W, we have $span\ \{S_{w_i}\}_{i=1}^m = V$. It suffices to show that for any $w \in W$, $w', w'' \in V$. If we write $w = \sum_{j=1}^d (a + bi) v_i$, by Theorem 4.5.6 we have

$$w' = \sum_{j=1}^d [(a_j + ib_j)v_j]' = \sum_{j=1}^m [a_j v_j' + b_j v_j''] \in V.$$

Since $w'' = (iw)'$, the same argument shows that $w'' \in V$. ∎

We will need one more preliminary result.

Proposition 4.5.10. *Let W be a subspace of $\mathbb{C}^n$ of dimension d, let $\{v_i\}_{i=1}^d$ be an orthonormal basis of W and let $V = span\ \{S_{v_i}\}_{i=1}^d$ be the induced subspace in $\mathbb{R}^{2n}$. Let Q (respectively P) be the projection onto W (respectively V). Then for all $x \in \mathbb{C}^n$, $\|Qx\| = \|Px'\|$.*

Proof. Let P_i be the projections onto S_{v_i}. For any i, $|\langle x, v_i\rangle| = \|P_i x'\|$ and so

$$\|Qx\|^2 = \sum_{i=1}^{d} |\langle x, v_i\rangle|^2 = \sum_{i=1}^{d} \|P_i x'\|^2 = \|Px'\|^2.$$

∎

Now we give the complex analog of Edidin's Theorem [30].

Theorem 4.5.11. *Let $\{W_i\}_{i=1}^m$ be subspaces of $\mathbb{C}^n$ with projections $\{Q_i\}_{i=1}^m$ and let $\{V_i\}_{i=1}^m$ be the corresponding subspaces of $\mathbb{R}^{2n}$ given in Proposition 4.5.9 with projections $\{P_i\}_{i=1}^m$. The following are equivalent:*

1. *$\{Q_i\}_{i=1}^m$ does phase retrieval.*

2. *For every $w' \in \mathbb{R}^{2n}$, if $M = span\ \{P_i w'\}_{i=1}^m$ then $M^\perp = span\ \{w''\}$.*

Proof. (1) $\Rightarrow$ (2): Given $v' \in M^\perp$, there exist two vectors x', y' so that $x' + y' = w'$ and $x' - y' = v'$. Now,

$$0 = \langle P_i(x' + y'), x' - y'\rangle = \langle P_i x' + P_i y', P_i x' - P_i y'\rangle = \|P_i x'\|^2 - \|P_i y'\|^2.$$

By Proposition 4.5.10, $\|Q_i x\| = \|Q_i y\|$ for all $i = 1, 2, \ldots, m$. Since $\{Q_i\}_{i=1}^m$ does phase retrieval, we have that $x = cy$ for some $|c| = 1$ and $\|x'\| = \|y'\|$. Since S_x is a subspace $x' + y' = w', x' - y' \in S_x$. By Theorem 4.4.8 $S_w = S_x$. But, $\{w', w''\}$ is an orthonormal basis for S_w and $x'+y' = w' \perp x'-y'$. It follows that $x'-y'$ is a multiple of w''. Hence, $w'' \perp M$ and its scalar multiples are the only vectors orthogonal to M.

(2) $\Rightarrow$ (1): Assume $v, w \in \mathbb{C}^n$ and $\|Q_i v\| = \|Q_i w\|$ for all $i = 1, 2, \ldots, m$. By Proposition 4.5.10, $\|P_i v'\| = \|P_i w'\|$ for all i. Now,

$$\langle P_i(v' + w'), v' - w'\rangle = \langle P_i v' + P_i w', P_i v' - P_i w'\rangle = \|P_i v'\|^2 - \|P_i w'\|^2 = 0.$$

By our assumption in (2), $v' - w' = c(v'' + w'')$. It follows that $S_v = S_w$ and $v' - cv'' = w' + cw''$. Since $\|v'\| = \|v''\|$, $\|w'\| = \|w''\|$, $v' \perp v''$, $w' \perp w''$, and $v' - cv'' = w' + cw'$, a little geometry shows that $|c| = 1$ and so $v = dw$ for some $|d| = 1$. ∎

4.6 Mutually Unbiased Bases

In this section, we will see that for a family of mutually unbiased bases in $\mathbb{C}^n$ their corresponding rank 2 projections in $\mathbb{R}^{2n}$ are mutually unbiased.

Definition 4.6.1. *Two orthonormal bases $\{e_i\}_{i=1}^n$, $\{e_i'\}_{i=1}^n$ for $\mathbb{H}^n$ are said to be* **mutually unbiased** *if*

$$|\langle e_i, e_j'\rangle|^2 = \frac{1}{n}, \text{ for all } i, j = 1, 2, \ldots, n.$$

We first need a lemma.

Proposition 4.6.2. *Given v, w be unit vectors in $\mathbb{C}^n$ let P, Q be the induced rank 2 projections in $\mathbb{R}^{2n}$ onto S_v, S_w. Then, $tr\ (PQ) = 2|\langle v, w\rangle|^2$.*

Proof. We note that $P = v'v'^* + v''v''^*$ and $Q = w'w'^* + w''w''^*$. Now we compute using Proposition 4.4.2 (5):

$$\langle P, Q\rangle = tr\ (PQ)$$

$$= tr\ (v'v'^* + v''v''^*)(w'w'^* + w''w''^*)$$

$$= tr\ (v'v'^*w'w'^*) + tr\ (v'v'^*w''w''^*) + tr\ (v''v''^*w'w'^*) + tr\ v''v''^*w''w''^*)$$

$$= |\langle v', w'\rangle|^2 + |\langle v', w''\rangle|^2 + |\langle v'', w'\rangle|^2 + |\langle v'', w''\rangle|^2$$

$$= \|Pwv'\|^2 + \|P_w v''\|^2$$

$$= |\langle v, w\rangle|^2 + |\langle iv, w\rangle|^2$$

$$= 2|\langle v, w\rangle|^2.$$

∎

Corollary 4.6.3. *If* $\{v_{ij}\}_{j=1}^n$ *are mutually unbiased orthonormal bases for* $\mathbb{C}^n$*, for* $i = 1, 2, \ldots, k$*, then the rank 2 projections* $\{P_{ij}\}_{i=1,j=1}^{n,\ k}$ *onto* $\{S_{v_{ij}}\}_{i=1,j=1}^{n,\ k}$ *in* $\mathbb{R}^{2n}$ *are mutually unbiased.*

Proof. By Corollary 4.4.3, for each i, $\{S_{v_{ij}}\}_{j=1}^n$ is an orthogonal family of two dimensional subspaces in $\mathbb{R}^{2n}$. Hence, for $j \neq k$,

$$\langle P_{ij}, P_{ik}\rangle = tr\ P_{ij}P_{ik} = 0.$$

Also, by Proposition 4.6.2, if $i \neq k$,

$$\langle P_{ij}, P_{k,l}\rangle = tr\ P_{ij}P_{kl} == 2|\langle v_{ij}, v_{kl}\rangle|^2 = \frac{2}{n}.$$

∎

4.7 Fusion Frames

In this section, we will apply our association of vectors in $\mathbb{C}^n$ to rank two projections in $\mathbb{R}^n$ to answer a longstanding problem in fusion frame theory. This topic was introduced in [23]. Fusion frames have application to dimension reduction and Grassmannian packings [40].

Definition 4.7.1. *Given a family of subspaces* $\{W_i\}_{i=1}^m$ *with respective projections* $\{P_i\}_{i=1}^m$ *in* $\mathbb{R}^n$ *or* $\mathbb{C}^n$*, and given* $a_i > 0$ *for* $i = 1, 2, \ldots, m$*, we say* $(W_i, a_i)_{i=1}^m$ *(respectively,* $(P_i, a_i)_{i=1}^m$*) is a* **fusion frame** *with fusion frame bounds* $0 < A \leqslant B < \infty$ *if for all vectors* v *we have:*

$$A\|v\|^2 \leqslant \sum_{i=1}^m a_i^2 \|P_i v\|^2 \leqslant B\|v\|^2.$$

We start by showing that frames in $\mathbb{C}^n$ will associate with fusion frames of 2-dimensional subspaces of $\mathbb{R}^{2n}$.

Theorem 4.7.2. *Let* $\{v_i\}_{i=1}^m$ *be a frame in* $\mathbb{C}^n$ *and let* $W_i = S_{v_i}$ *with orthogonal projection* P_i *for* $i \in [m]$*. The following are equivalent:*

1. $\{v_i\}_{i=1}^m$ *has frame bounds* A, B*.*
2. $(W_i, \|v_i\|)_{i=1}^m$ *is a fusion frame of two dimensional subspaces for* $\mathbb{R}^{2n}$ *with fusion frame bounds* A, B*.*

Proof. This is immediate since given $w \in \mathbb{C}^n$,

$$\sum_{i=1}^m |\langle w, v_i \rangle|^2 = \sum_{i=1}^m \|v_i'\|^2 \|P_i w'\|^2.$$

∎

It is exceptionally difficult to construct tight fusion frames. Especially since they often do not exist. For example, there does not exist a tight fusion frame for $\mathbb{R}^3$ consisting of two 2-dimensional subspaces. To see this, let $(W_i, a_i)_{i=1}^2$ be a fusion frame for $\mathbb{R}^3$ with associated projections $\{P_i\}_{i=1}^2$. Let $x \in W_1 \cap W_2$. Then

$$a_1^2\|P_1 x\|^2 + a_2^2\|P_2 x\|^2 = (a_1^2 + b_1^2)\|x\|^2.$$

But, if $x \in W_1$ but not in W_2 then $\|P_2 x\|^2 < \|x\|^2$ and so

$$a_1^2\|P_1 x\|^2 + a_2^2\|P_2 x\|^2 < (a_1^2 + b_1^2)\|x\|^2.$$

I.e. This fusion frame is not tight. It is believed that this problem occurs over and over with odd numbers of two dimensional subspaces of $\mathbb{R}^3$ as well as occuring in $\mathbb{R}^{2n+1}$ for all n. It has been a longstanding open problem whether this is a problem of *odd dimensions* or does this problem show up in even dimensions also. I.e. Is there a tight fusion frame of m 2-dimensional subspaces in $\mathbb{R}^{2n}$ for all n and all $m \geqslant n$. We see that the above answers this in the affirmative.

Corollary 4.7.3. *The following fusion frames exist:*

1. *For every $m \geqslant n$,if there is a tight frame $\{v_i\}_{i=1}^m$ for $\mathbb{C}^n$ and then for $W_i = S_{v_i}$, $\{W_i, \|v_i\|\}_{i=1}^m$ is a tight fusion frame of two dimensional subspaces of $\mathbb{R}^{2n}$.*

2. *For every $m \geqslant n$, there is an equal norm Parseval frame $\{v_i\}_{i=1}^m$ for $\mathbb{C}^n$ (and so $\|v_i\|^2 = \frac{n}{m}$) and then for $W_i = S_{v_i}$, $\{W_i, \sqrt{\frac{n}{m}}\}_{i=1}^m$ is a tight fusion frame of two dimensional subspaces of $\mathbb{R}^{2n}$.*

And for the general case we have,

Theorem 4.7.4. *If $\{W_i, a_i\}_{i=1}^m$ be a fusion frame for $\mathbb{C}^n$ with dim $W_i = k_i$ for $i \in [m]$, with fusion frame bounds A, B then $\{V_i, a_i\}_{i=1}^m$, the induced $2k_i$-dimensional subspaces of $\mathbb{R}^{2n}$, form a fusion frame in $\mathbb{R}^{2n}$ with fusion frame bounds A, B.*

Proof. Let Q_i (respectively, P_i) be the projection onto W_i (respectively, V_i). By the proof of Proposition 4.5.10 we have for all $x \in \mathbb{C}^n$: $\|Q_i x\| = \|P_i x'\|$. It follows that

$$\sum_{i=1}^m a_i^2\|Q_i x\|^2 = \sum_{i=1}^m a_i^2\|P_i x'\|^2.$$

This proves the result. ∎

4.8 Classifying Norm Retrieval

We will give a theorem similar to the Edidin Theorem but which classifies norm retrieval.

Definition 4.8.1. *A family of projections $\{P_i\}_{i=1}^m$ on $\mathbb{H}^n$ does* **norm retrieval** *if whenever $x, y \in \mathbb{H}^n$ and $\|P_i x\| = \|P_i y\|$, for all $i = 1, 2, \ldots, m$, we have that $\|x\| = \|y\|$.*

It is immediate that if $\{P_i\}_{i=1}^m$ does phase retrieval then it does norm retrieval. The converse fails since orthonormal bases do norm retrieval and must fail phase retrieval. The importance of norm retrieval is that [24] if a family of projections $\{P_i\}_{i=1}^m$ does phase retrieval on $\mathbb{H}^n$ then $\{(I - P_i)\}_{i=1}^m$ does phase retrieval if and only if it does norm retrieval.

Theorem 4.8.2. *Given projections $\{P_i\}_{i=1}^m$ on $\mathbb{R}^n$ the following are equivalent:*

1. *$\{P_i\}_{i=1}^m$ does norm retrieval.*

2. *For every $0 \neq x \in \mathbb{R}^n$, we have*

$$[span\ \{P_i x\}]^{\perp} \subset x^{\perp}.$$

3. *For every $0 \neq x \in \mathbb{R}^n$, we have that $x \in span\ \{P_i x\}_{i=1}^m$.*

Proof. (2) $\Rightarrow$ (1): We will prove the contrapositive. If norm retrieval fails, then there are vectors $x, y \in \mathbb{R}^n$ with $\|P_i x\| = \|P_i y\|$ for all $i = 12, \ldots, m$ but $\|x\| \neq \|y\|$. This implies if $v = x + y$ and $w = x - y$ then v, w are not orthogonal. But $w \in [span\ \{P_i v\}_{i=1}^m]^{\perp}$, so property (2) fails.

(2) $\Leftrightarrow$ (3): This is immediate.

(2) $\Rightarrow$ (1): Again by the contrapositive, there exists v, w which are not orthogonal (so $w \neq 0$) but $w \perp \textit{span}\ \{P_i v\}_{i=1}^m$ and so $P_i w \perp P_i v$.. Write $v = x + y$ and $w = x - y$. Since v, w are not orthogonal, $\|x\| \neq \|y\|$. But, since $w \perp P_i v$,

$$\begin{aligned}\|P_i(v+w)\|^2 &= \langle P_i v + P_i w, P_i v + P_i w\rangle \\ &= \|P_i v\|^2 + \|P_i w\|^2 \\ &= \langle P_i v - P_i w, P_i v - P_i w\rangle = \|P_i(v-w)\|^2.\end{aligned}$$

So $\|P_i x\| = \|P_i y\|$ for every i, while $\|x\| \neq \|y\|$. I.e. $\{P_i\}_{i=1}^m$ fails norm retrieval. ∎

In general, it is difficult to show that projections $\{P_i\}_{i=1}^m$ do norm retrieval (especially if they fail phase retrieval) and even more difficult to show that this passes to $\{(I - P_i)\}_{i=1}^m$. It is known that, in general, $\{P_i\}_{i=1}^m$ may do norm retrieval (even phase retrieval) while $\{(I - P_i)\}_{i=1}^m$ fails norm retrieval [55]. We will now give a slightly weaker sufficient condition for norm retrieval (respectively, phase retrieval) to pass from $\{P_i\}_{i=1}^m$ to $\{(I - P_i)\}_{i=1}^m$.

Theorem 4.8.3. *Let $\{P_i\}_{i=1}^m$ be projections on $\mathbb{R}^n$. The following are equivalent:*

1. *$\{P_i\}_{i=1}^m$ does norm retrieval and for every $y \in \mathbb{R}^n$ there are scalars $\sum_{i=1}^m a_i \neq 1$ so that $y = \sum_{i=1}^m a_i P_i y$.*

2. *$\{(I - P_i)\}_{i=1}^m$ does norm retrieval and for every $y \in \mathbb{R}^n$ there are scalars $\sum_{i=1}^m a_i \neq 1$ so that $y = \sum_{i=1}^m a_i (I - P_i) y$.*

Proof. (1) $\Rightarrow$ (2): Given $0 \neq y \in \mathbb{R}^n$, choose $\{a_i\}_{i=1}^m$ to that $\sum_{i=1}^m a_i P_i y = y$ and

$\sum_{i=1}^{m} a_i \neq 1$. Then,

$$\sum_{i=1}^{m} a_i(I - P_i)y = \left(\sum_{i=1}^{m} a_i\right) y - \sum_{i=1}^{m} a_i P_i y = \left(\sum_{i=1}^{m} a_i - 1\right) y.$$

So

$$y = \sum_{i=1}^{m} \frac{a_i}{\sum_{i=1}^{m} a_i - 1}(I - P_i)y.$$

Also,

$$\sum_{i=1}^{m} \frac{a_i}{\sum_{i=1}^{m} a_i - 1} = \frac{\sum_{i=1}^{m} a_i}{\sum_{i=1}^{m} a_i - 1} \neq 1,$$

So $y \in span \ \{(I - P_i)y\}_{i=1}^{m}$. I.e. $\{(I - P_i)\}_{i=1}^{m}$ does norm retrieval.

(2) $\Rightarrow$ (1): By symmetry. ∎

Theorem 4.8.4. *Let projections $\{P_i\}_{i=1}^{m}$ do phase retrieval in $\mathbb{R}^n$. The following holds:*

If for every $0 \neq y$ there exist scalars $\{a_i\}_{i=1}^{m}$ so that $y = \sum_{i=1}^{m} a_i P_i y$ and $\sum_{i=1}^{m} a_i \neq 1$, then $\{(I - P_i)\}_{i=1}^{m}$ does phase retrieval.

Proof. By Theorem 4.8.3, $\{(I - P_i)\}_{i=1}^{m}$ does norm retrieval and hence phase retrieval.

∎

Corollary 4.8.5. *If $\{P_i\}_{i=1}^{m}$ does norm retrieval (respectively, phase retrieval) on $\mathbb{R}^n$ and for every $y \in \mathbb{R}^n$ either there exists $\sum_{i=1}^{m} a_i \neq 1$ and $y = \sum_{i=1}^{m} a_i P_i y$ or there exists $\sum_{i=1}^{m} a_i \neq 0$ and $\sum_{i=1}^{m} a_i P_i y = 0$, then $\{(I - P_i)\}_{i=1}^{m}$ does norm retrieval (respectively, phase retrieval.*

Proof. If $\sum_{i=1}^{m} a_i P_i y = y$ and $\sum_{i=1}^{m} a_i \neq 1$ then

$$\sum_{i=1}^{m} a_i(I - P_i)y = \left(\sum_{i=1}^{m} a_i\right) y - \sum_{i=1}^{m} a_i P_i y = \left(\sum_{i=1}^{m} a_i - 1\right) y.$$

So $y \in span\ \{(I-P_i)y\}_{i=1}^m$. If $\sum_{i=1}^m a_i \neq 0$ and $\sum_{i=1}^m a_i P_i y = 0$, then

$$\sum_{i=1}^m a_i(I-P_i)y = \left(\sum_{i=1}^m a_i\right) y - \sum_{i=1}^m a_i P_i y = \left(\sum_{i=1}^m a_i\right) y,$$

and so $y \in span\ \{(I-P_i)y\}_{i=1}^m$. ∎

4.9 Equiangular and Biangular Frames

In this section we will see how our association changes equiangular and biangular tight frames in $\mathbb{C}^n$ into equiangular and biangular tight fusion frames in $\mathbb{R}^{2n}$.

Definition 4.9.1. *A family of unit vectors $\{\phi_i\}_{i=1}^m$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is said to be k-angular if there are constants $\alpha_1 > \alpha_2 > \cdots > \alpha_k \geqslant 0$ so that*

$$\{|\langle \phi_i, \phi_j \rangle| : i \neq j\}| = \{\alpha_i : i = 1, 2, \ldots, k\}$$

Similarly, a family of subspaces $\{W_i\}_{i=1}^m$ with respective projections $\{P_i\}_{i=1}^m$ is $k - angular$ if

$$|\{\langle P_i, P_j \rangle| : i \neq j\}| = \{\alpha_i : i = 1, 2, \ldots, k\}.$$

If $k = 1$, we call this equiangular and if $k = 2$ we call this biangular.

Heath and Strohmer [46] (See also [15, 48]) made a detailed analysis of this class of frames and related this to several areas of research. Wooters and Fields [54] use them (not under this name) to obtain *unbiased* measurements to determine the state (or density operator) of a quantum system. Grassmannian frames also arise in communication theory. In [47] Grassmannian frames are used for low-bit rate channel feedback in MIMO systems. It also arises in **Welsh bound equality** sequences [50].

The main problem in this area of research is that very few equiangular/biangular tight frames are known. It is known that the number of equiangular lines in $\mathbb{R}^n$ is less than or equal to $n(n+1)/2$ [28] but these bounds are rarely achieved. For example, the maximal number of equiangular lines in $\mathbb{R}^d$ is just 28 for all $7 \leqslant d \leqslant 14$ [28]. In the complex case, the maximal number of equiangular lines in $\mathbb{C}^n$ is n^2 [28]. It is an open problem whether this number is always attained. We will now show how to transfer equiangular lines from $\mathbb{C}^n$ to equiangular fusion frames of two dimensional subspaces of (respectively, equiangular families of rank 2 projections) $\mathbb{R}^{2n}$. This provides many more equiangular sets to deal with in $\mathbb{R}^{2n}$ if researchers can live with rank 2 projections instead of vectors. For example, there are only 28 equiangular lines in $\mathbb{R}^{14}$, but there are 15 equiangular families of rank 2 projections in $\mathbb{R}^{14}$.

Theorem 4.9.2. *If a unit norm tight frame $\{\phi_i\}_{i=1}^m$ in $\mathbb{C}^n$ is equiangular, then the rank 2 projections P_i onto S_{v_i} in $\mathbb{R}^{2n}$ form an equiangular tight fusion frame in $\mathbb{R}^{2n}$.*

Proof. We just observe that

$$|\langle \phi_i, \phi_j \rangle|^2 = 2\|P_i \phi_j'\|^2 = tr\ P_i P_j = \langle P_i, P_j \rangle.$$

∎

Similarly we have:

Theorem 4.9.3. *If a unit norm tight frame $\{\phi_i\}_{i=1}^m$ in $\mathbb{C}^n$ is k-angular, then the rank 2 projections P_i onto S_{ϕ_i} in $\mathbb{R}^{2n}$ form a k-angular tight fusion frame in $\mathbb{R}^{2n}$.*

Remark 4.9.4. A very general form of **realization** of complex vectors first appeared in [3] (See Lemma 3.4). In particular, it is shown that for any linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$ admits an equivalent $Areal : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with a special structure. The spectrum of Areal is the spectrum of A with each eignevalue repeated twice (hence doubling the rank etc.). An interesting thing here is that there are many different realifications for $\mathbb{C}^n$. Possibly, these could answer many more open questions along the lines of this paper.

Chapter 5

Introduction to Chapter 6

5.1 Introduction

In the last part of this thesis we will show a method to compute the $\ell_1 - \ell_2$ inequality of a unit norm vector $v \in \mathbb{H}^n$. In particular we will show that the distance of v to the closest constant modulus vector determines its $\ell_1 - \ell_2$ inequality completely (Theorem 6.2.2). We will then use this to solve an open problem in compressed sensing by showing that given a s dimensional subspace S such that for all $y \in S$, $\|y\|_1 \leqslant \sqrt{s}\|y\|_2$, then S must be the span of s vectors in the standard orthonormal basis (Theorem 6.2.5). We will also apply this method to computing the $\ell_1 - \ell_2$ inequality of a unit norm function in $L_2[0,1]$ (Theorem 6.3.1).

Chapter 6

The exact constant for the $\ell_1 - \ell_2$ norm inequality

6.1 Introduction

The $\ell_1 - \ell_2$-norm inequality which gives that for any $x \in \mathbb{H}^n$, $\|x\|_1 \leqslant \sqrt{n}\|x\|_2$. But this is a strict inequality for all but vectors with constant modulus for their coefficients. We will give a trivial method to compute, for each x, the constant c for which $\|x\|_1 = c\sqrt{n}|x\|_2$. Since this is one of the most fundamental and most used inequalities in Hilbert space theory, we believe this will have broad application in the field. We will also show some variations of this result. For a background in this area see [19, 23].

6.2 The $\ell_1 - \ell_2$-norm Inequality

We need a definition.

Definition 6.2.1. *A vector of the form $x = \frac{1}{\sqrt{n}}(c_1, c_2, \ldots, c_n) \in \mathbb{H}^n$, with $|c_i| = 1$ for all $i = 1, 2, \ldots, n$ will be called a* **constant modulus vector**.

Theorem 6.2.2. *Let $x = (a_1, a_2, \ldots, a_n) \in \mathbb{H}^n$, a real or complex Hilbert space. The following are equivalent:*

1. *We have*

$$\|x\|_1 = (1 - \frac{c_x}{2})\sqrt{n}\|x\|_2.$$

2. *We have*

$$\sum_{i=1}^{n} \left| \frac{|a_i|}{\|x\|_2} - \frac{1}{\sqrt{n}} \right|^2 = c_x.$$

3. *The infimum of the distance from* $\frac{x}{\|x\|_2}$ *to the constant modulus vectors is* $\sqrt{c_x}$.

In particular,

$$\|x\|_1 \leqslant \sqrt{s}\|x\|_2,$$

if and only if

$$\left(1 - \frac{c_x}{2}\right)\sqrt{n} \leqslant \sqrt{s},$$

if and only if

$$1 - \frac{c_x}{2} \leqslant \sqrt{\frac{s}{n}}.$$

Proof. (1) $\Leftrightarrow$ (2): We compute:

$$\begin{aligned}\sum_{i=1}^{n} \left| \frac{|a_i|}{\|x\|_2} - \frac{1}{\sqrt{n}} \right|^2 &= \frac{1}{\|x\|_2^2}\sum_{i=1}^{n} |a_i|^2 + \sum_{i=1}^{n}\frac{1}{n} - \frac{2}{\sqrt{n}\|x\|_2}\sum_{i=1}^{n}|a_i| \\ &= 2\left(1 - \frac{1}{\sqrt{n}\|x\|_2}\sum_{i=1}^{n}|a_i|\right) = c_x.\end{aligned}$$

if and only if

$$\frac{1}{\sqrt{n}\|x\|_2}\sum_{i=1}^{n}|a_i| = 1 - \frac{c_x}{2},$$

if and only if

$$\sum_{i=1}^{n}|a_i| = (1 - \frac{c_x}{2})\sqrt{n}\|x\|_2.$$

(1) $\Leftrightarrow$ (3): We compute:

$$\inf\left\{\sum_{i=1}^{n}\left|\frac{a_i}{\|x\|_2}-\frac{c_i}{\sqrt{n}}\right|^2 : |c_i|=1\right\} =$$

$$\inf\left\{\frac{1}{\|x\|_2^2}\sum_{i=1}^{n}|a_i|^2+\sum_{i=1}^{n}\left|\frac{c_i}{\sqrt{n}}\right|^2-2\frac{1}{\|x\|_2\sqrt{n}}Re\sum_{i=1}^{n}a_i\bar{c_i} : |c_i|=1\right\} =$$

$$2-\frac{2}{\|x\|_2\sqrt{n}}\sum_{i=1}^{n}|a_i|.$$

The equality occurs when $\frac{1}{\sqrt{n}}(c_1, c_2, \ldots, c_n)$ is a constant modulus vector with $c_i = \dfrac{a_i}{|a_i|}$ if $a_i \neq 0$.

Thus

$$c_x = 2-\frac{2}{\|x\|_2\sqrt{n}}\sum_{i=1}^{n}|a_i| \text{ if and only if (1) holds}$$

∎

Now we want to look at an application of the above. For this we need two preliminary results.

Theorem 6.2.3. *Let S be a subspace of $\mathbb{H}^n$ and let P be the orthogonal projection on S. For any $x \in \mathbb{H}^n$, $\frac{Px}{\|Px\|}$ is the closest unit vector in S to x.*

Proof. Let y be a unit vector in S and extend it to be an orthonormal basis $\{y, u_1, u_2, \ldots, u_k\}$ for S. Then

$$Px = \langle x, y\rangle y + \sum_{i=1}^{k}\langle x, u_i\rangle u_i.$$

Hence

$$\|Px\|^2 = |\langle x, y\rangle|^2 + \sum_{i=1}^{k}|\langle x, u_i\rangle|^2 \geqslant |\langle x, y\rangle|^2.$$

Therefore

$$\|Px\| \geqslant |\langle x, y\rangle| \geqslant Re\langle x, y\rangle.$$

Now we have

$$\|x - \frac{Px}{\|Px\|}\|^2 = \|x\|^2 - 2\|Px\| + 1 \leqslant \|x\|^2 - 2Re\langle x, y\rangle + \|y\|^2 = \|x - y\|^2,$$

which is our claim. ∎

Next, we examine the $\ell_1 - \ell_2$-norm inequality for subspaces.

Theorem 6.2.4. *Let S be a subspace of $\mathbb{H}^n$ and let P be the projection onto S. The following are equivalent:*

1. *For every unit vector $x \in S$,*

$$\|x\|_1 \leqslant (1 - \frac{c}{2})\sqrt{n}.$$

2. *The ℓ_2 distance of any unit vector in S to any constant modulus vector is greater than or equal to $\sqrt{c}$.*

3. *For every constant modulus vector x, we have*

$$\|Px\|_2 \leqslant 1 - \frac{c}{2}.$$

Proof. (1) ⇔ (2): Let $x = (a_1, a_2, \dots, a_n)$.

$$\begin{aligned}\inf\{\sum_{i=1}^{n} |a_i - \frac{c_i}{\sqrt{n}}|^2 : |c_i| = 1\} &= \inf\{\sum_{i=1}^{n} |a_i|^2 + \sum_{i=1}^{n} \frac{1}{n} - \frac{2}{\sqrt{n}} Re \sum_{i=1}^{n} a_i \bar{c}_i : |c_i| = 1\} \\ &= 2 - \frac{2}{\sqrt{n}} \sum_{i=1}^{n} |a_i|.\end{aligned}$$

Now,

$$c \leqslant 2 - \frac{2}{\sqrt{n}} \sum_{i=1}^{n} |a_i| \text{ if and only if } \sum_{i=1}^{n} |a_i| \leqslant \left(1 - \frac{c}{2}\right)\sqrt{n}.$$

(2) $\Leftrightarrow$ (3): By Theorem 6.2.3, we need to check how close

$\frac{Px}{\|Px\|}$ is to the all one's vector x.

So we compute:

$$\left\|\frac{Px}{\|Px\|} - x\right\|^2 = 2 - \langle \frac{Px}{\|Px\|}, x\rangle - \langle x, \frac{Px}{\|Px\|}\rangle = 2 - 2\|Px\|$$

So,

$$c \leqslant \left\|\frac{Px}{\|Px\|} - x\right\|^2 \text{ if and only if } \|Px\| \leqslant 1 - \frac{c}{2}.$$

∎

Now we have the second main result. For this recall [19, 23] that if P is a projection on $\mathbb{H}^n$ with orthonormal basis $\{e_i\}_{i=1}^n$ then $\sum_{i=1}^n \|Pe_i\|^2 = dim\ P(\mathbb{H}^n)$.

Theorem 6.2.5. *Let S be a s-dimensional subspace of $\mathbb{R}^n$ with orthonormal basis $\{e_i\}_{i=1}^n$. If*

$$\|y\|_1 \leqslant \sqrt{s}\|y\|_2, \text{ for all } y \in S,$$

then there is an $I \subset [n]$ with $|I| = s$ and $S = span\ \{e_i\}_{i \in I}$.

Proof. For any $y \in S$, let c_y be defined in (2) of Theorem 6.2.2. Since

$$\|y\|_1 \leqslant \sqrt{s}\|y\|_2, \text{ for all } y \in S,$$

then

$$1 - \frac{c_y}{2} \leqslant \sqrt{\frac{s}{n}}.$$

Set

$$c = \inf\{c_y : y \in S\}$$

then

$$1 - \frac{c}{2} \leqslant \sqrt{\frac{s}{n}} \tag{6.1}$$

We will prove: $\{Pe_i\}_{i=1}^n$ is an orthogonal set. This will impliy that there is an $I \subset [n]$ so that $Pe_i = e_i$ for $i \in I$ and $Pe_i = 0$ for $i \in I^c$.

First note that $\{Pe_i\}_{i=1}^n$ is a Parseval frame for S and so

$$\sum_{i=1}^n \|Pe_i\|^2 = s.$$

Assume there are two of these vectors which are not orthogonal. By reindexing, we will assume Pe_1, Pe_2 are not orthogonal. Hence, by replacing Pe_2 by c_2Pe_2 with $|c_2| = 1$ if necessary with $Re\ c_2\langle Pe_1, Pe_2\rangle > 0$, we have

$$\|Pe_1 + c_2Pe_2\|^2 > \|Pe_1\|^2 + \|Pe_2\|^2.$$

Now, by replacing Pe_3 by c_3Pe_3 with $|c_3| = 1$ if necessary, we have

$$\|Pe_1 + c_2Pe_2 + c_3Pe_3\|^2 \geqslant \|Pe_1 + Pe_2\|^2 + \|Pe_3\|^2 > \|Pe_1\|^2 + \|Pe_2\|^2 + \|Pe_3\|^3.$$

Continuing, and letting $c_1 = 1$, we have

$$\|P\left(\sum_{i=1}^n c_ie_i\right)\|^2 > \sum_{i=1}^n \|Pe_i\|^2 = s.$$

It follows from Theorem 6.2.4,

$$\sqrt{\frac{s}{n}} < \left\|P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n c_ie_i\right)\right\| \leqslant 1 - \frac{c}{2},$$

which contradicts Equation (6.1) above. ∎

6.3 An Application to $L_p[0,1]$

It was pointed out to us by Bill Johnson that our work has application to Banach space theory. That, in general, when working with finite dimensional ℓ_p, it is better to use the $L_p[0,1]$ normalization. But applying our results, the *nasty* $n^{1/2}$ *goes away* and the expressions are independent of dimension. What is quite interesting here is the fact that if $p < s$ and $f \in L_1[0,1]$ then we can measure *how peaky f is* by seeing how small $\|f\|_p$ is. What apparently was not realized is that when $p = 1$ and $s = 2$ we get a nice equality instead of an inquality.

Theorem 6.3.1. *Let $f \geqslant 0$ be norm one in $L_2[0,1]$. The following are equivalent:*

1. *We have*

$$\|f\|_1 = (1 - \frac{c}{2}).$$

2. *We have*

$$\|f - 1\|_2^2 = c.$$

Proof. We use the parallelogram law:

$$\begin{aligned} 4 &= \|f-1\|_2^2 + \|f+1\|_2^2 \\ &= \|f-1\|_2^2 + \|f\|_2^2 + 1 + 2\int_0^1 f(t)dt \\ &= \|f-1\|_2^2 + 2 + 2\|f\|_1. \end{aligned}$$

I.e.

$$\|f - 1\|_2^2 = 2 - 2\|f\|_1.$$

It follows that

$$\|f-1\|_2^2 = c \text{ if and only if } \|f\|_1 = 1 - \frac{c}{2}.$$

∎

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