HIGH SCHOOL STUDENTS' INTERPRETATIONS AND USE OF DIAGRAMS IN GEOMETRY PROOFS

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HIGH SCHOOL STUDENTS' INTERPRETATIONS AND USE OF DIAGRAMS IN

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Dedicated to my husband, Bayram, who always has faith in me.

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ABSTRACT

In high school geometry, proving theorems and applying them to geometry problems is an expectation from high school students (CCSSI, 2010). Diagrams are considered as an essential part of the geometry proofs because diagrams are included in a typical geometric statement such as a claim or problem (Manders, 2008; Shin et al., 2001). This interview-based study investigated how high school students interpret and use diagrams during the process of proving geometric claims. Particular attention is given to the semiotic resources such as symbols, visuals, and gestures that students draw from the diagrams to develop their proving activities. Hence, the goal of the current study is to contribute to the mathematics education field by providing insights into the details of semiotic aspects of diagrammatic reasoning. Study participants were grade 10–12 high school students and data was collected through one-on-one task- based clinical interviews. In general, students focused on the figural properties of the diagrams more frequently than the conceptual properties of the diagrams in their proofs even when they produced a new diagram or multiple diagrams. Regarding the semiotic structure of students' proving process, gesture resources were prominent in the semiotic structure of students' proving process in diagram-given tasks. The findings also suggested that, in general, some visual resources such as drawing a new figure or multiple figures occurred regularly in particular tasks such as diagram-free tasks with non- diagrammatic register or truth-unknown features. Overall, the frameworks used in this study showed how important it is to consider the mathematics as multi semiotic, understanding the role of gestures in students' geometrical reasoning.

CHAPTER 1: INTRODUCTION

Rationale

Mathematics educators in the United States give value to proving and working with mathematical theorems, problems, and axioms in high schools as part of developing students' critical and analytic thinking in both mathematics and the social world. Proofs are essential in mathematics because they are the recognized ways to accept and develop concepts in mathematics. As Rav (1999) asserts, "Proofs … are the heart of mathematics, the royal road to creating analytic tools and catalysing growth" (p. 6). Proving is a significant activity in school mathematics to develop students' logical thinking and deductive or inductive reasoning for mathematical arguments.

In particular, Geometry is a subject in which high school students are expected to develop their skills on proving geometric theorems such as claims regarding the congruency and similarity of triangles (Common Core State Standards Initiative, 2010). A typical geometrical proof often consists of two parts: a diagram and discursive text, and discursive text that address the features of the diagram to construct the step in the process of proof (Manders, 2008). Consequently, a diagram in a geometrical proof either is used as a step, or its features are referred via verbal text in the demonstration of proof. In this study, diagrams or drawings are considered as "materialized models of the mental entities with which the mathematician deals" (Fischbein, 1993, p.141).

Diagrams, as part of the geometric statements, have historically played a significant role as a tool for exploring a mathematical claim or to solve a geometric problem (Dove, 2002; Shin et al., 2001). Miller (2007) mentioned that "geometric diagrams are found among some of the oldest preserved examples of written mathematics" (p. 5). Hence, in the field of geometry, visual representation in its different forms are indispensable (Arcavi, 2003). That is, visual images such as diagrams in geometry are an important object of the proof since they transmit information, and are used to make interpretations and conclusions in the proof (Giaquinto, 2008).

Throughout history, mathematicians have adopted diagrams to acknowledge and communicate geometric concepts and proofs because geometric claims imply both conceptual and spatial (diagrammatic) properties (Fischbein & Nachlieli, 1998). Similarly, Miller (2007) states that diagrammatic reasoning is a significant method of proof in geometry. In fact, Euclidean Geometry is the basis of the concepts of proof and logic in mathematics (Miller, 2007) in general, and forms the foundation of the high school Geometry curriculum in particular. In geometry, diagrams have been used not only to demonstrate geometric claims but also to move forward with the steps included in the geometric solutions. Manders (2008) argues that diagrams have a unique contribution to the geometric statements, one that verbal text and symbols may not offer alone. For example, creating two triangles inside a rectangle by drawing a diagonal is a diagrammatic conclusion which may not be inferred from a verbal text. Fischbein (1993) argues that diagrams in geometry have multiple meanings since a diagram in a geometrical reasoning can both represent a concept and exemplify the image of similar objects.

Current research on geometric diagrams focus on the meanings that geometric diagrams in textbooks express (Dimmel & Herbst, 2015), the way students employed geometric diagrams in both theoretical and spatial-graphical ways (Laborde, 2005), and descriptions of various modes of interactions with diagrams (Herbst, 2004). Therefore, it is important to understand what meanings diagrams convey in different forms of the geometric tasks and how students construe or construct meanings from the diagrams.

Statement of the Problem

Meaning making with mathematical representations such as algebraic symbols, diagrams, and gesture has become increasingly prevalent as an essential goal in mathematics education and therefore semiotics—the study of meaning making—has arisen as a means to appreciate the role of mathematics in solving real-world problems and how it evolved throughout the history (Lemke, 2003). Specifically, the meaning and use of diagrams in Euclidean geometry proofs are exposed to interpretation and subjectivity (Miller, 2007). Moreover, Alshwaikh (2010) asserted that diagrams contain various meanings in different resources such as the connections between a diagram and the person who constructed the diagram and between a diagram and the text supplementing the diagram. These studies showed that diagrams in geometry have multi-semiotic functions. In particular, Alshwaikh stated that "I consider diagrams as a semiotic mode of representation and communication which enables us to construct mathematical meaning" (p. 69, 2010).

Within geometric proof, however, there is an ambiguity for the role of diagrams because there are differing views on the legitimacy of diagrams in proof. On one hand, several philosophers (e.g., Brown, 1997; Mumma, 2009) support that pictures, figures, or diagrams are more than a heuristic tool and they can prove general theorems in geometry. On the other hand, Dove (2002) argued that pictures or diagrams are only helpful to understand a proof, but not to produce a formal proof because diagrams illustrate individual examples. In spite of the debate on the diagram validity in proof, the fact that diagrammatic reasoning appears in geometry is indisputable (Shin et al., 2001). Drawing on the two different views on the role of diagrams, Laborde (2005) discussed two properties of the diagrams in geometry: *theoretical* (conceptual) and *spatio-graphical* (perceptual) properties. Theoretical properties can be presented in various languages to address the relationship between geometrical objects. However, spatial properties refer to interpretations or perceptions of the physical representation of geometrical objects. For example, in the proof of the equality of the base angles in an isosceles triangle, one may consider the concepts of point, side, angle, and triangle in a verbal explanation which construct the theoretical properties; in contrast, spatial properties include using a pictorial representation and its figural information such as disassembling the triangle and overlapping with the original triangle (Fischbein, 1993). He claims that when solving a geometry problem, two properties are used and there is a need for further research on students' actions on the diagrams with conceptual and spatial properties.

In that vein, research has paid considerable attention to the implication of diagrams in geometric statements and the meanings that the diagrams convey (e.g., Alshwaikh, 2008, 2010; Dimmel & Herbst, 2015), however, more consideration is needed on the students' interpretation and use of diagrams. There is a gap in understanding the way in which students use and interpret diagrams in geometry proofs. Hence, it is worth considering not only what meanings were imposed on the diagrams either in textbooks or during the problem and proving activities in the classrooms, but also what meanings students draw from them.

Purpose of the Study

Geometry diagrams are multi semiotic because they are pictorial representations that are based on abstract geometric notions and contain geometric objects and symbols. As an illustration, a triangle is a set of points that constructs particular theoretical properties (i.e., a plane figure with three angles and three even line) and includes figural properties such as the length values of the sides or symbols that represent vertexes and points in the triangle. Therefore, Fischbein (1993) argued that a geometry diagram contains both conceptual and figural properties.

To prove geometric claims is challenging, especially for students, because it requires reasoning with conceptual properties and figural images simultaneously (Fischbein & Nachlieli, 1998). Moreover, the multi semiotic functions of the geometry diagrams can influence students' use of diagrams in geometrical reasoning. Hence, the research on geometry diagrams and how students employ them in geometrical contexts is a contemporary research theme (Dimmel & Herbst, 2015).

Given that, this study investigated how high school students interpret and use diagrams during the process of proving geometric claims. Particular attention is given to the semiotic resources such as symbols, visuals, and gestures that students draw from the diagrams to develop their proving activities. Hence, the goal of the current study is to contribute to the mathematics education field by providing insights into the details of semiotic aspects of diagrammatic reasoning.

The focus of the study was to analyze high school students' (whose grade levels ranged between 10–12) interpretation and use of diagrams while working on several geometric claims. Specifically, this study explored the interpretations students made about the diagrams given in the geometric theorems through the analysis of semiotic resources and proving actions in students' diagrammatic reasoning. The present study was situated in one-on-one interactions with high school students in task-based clinical interviews to understand the students' interpretations better.

Research Questions

This study investigated how high school students who had taken geometry classes at high school used diagrams in geometry proof tasks and engaged in semiotic resources—visual, symbolic, and gesture—in their proving actions. To determine students' use of diagrams while working with geometry proof tasks, some of which included given diagrams while some did not, the present study specifically addressed the following research questions:

1. *What semiotic resources (in diagrams) do high school students use to prove geometric claims? How do the semiotic resources relate to the quality of reasoning students provide?*
2. *How do high school students interpret and use geometric diagrams to prove* ***diagram-given*** *geometric claims, and what is the semiotic structure of their proving process?*
3. *How do high school students produce and use (if at all) diagrams to prove* ***diagram-free*** *geometric claims, and what is the semiotic structure of their proving process (whether or not they produced a diagram)?*

Research Question 1 examines the semiotic structure of students' diagrammatic reasoning and the relations students endow on the semiotic resources and the arguments they make during their work proving geometric claims. Likewise, with Research Questions 2 and 3, the study aimed to investigate the ways in which students' interpretation and construction of diagrams underlie their engagement with the geometric claims.

Significance of the Study

Previous research has focused on the effectiveness of diagrammatic proofs in successfully proving the geometry theorems (Dove, 2002; Giaquinto, 1994; Miller, 2007). Researchers have stressed the importance of meanings of diagrams to understand the role they play in proofs (Laborde, 2005) and students' works (Alshwaikh, 2008). In line with these ideas, Dimmel and Herbst (2015) suggest that students should learn how to read and understand geometry diagrams in the geometrical texts.

Drawing on that work, this study will contribute to the field of mathematics education by providing insights into students' meaning making with diagrams, examining how students interpret and use geometry diagrams in geometrical proofs. The study reports students' different use of diagrams depending on the type of the geometric claim. The focus on the semiotic resources that students employed on the diagram makes a valuable addition to the current research on the gesture and the semiotic characteristics of diagrams in geometry. Additionally, by understanding the semiotic properties of valid proofs in geometry, we can better understand how to support high school students to adopt successful proof practices. In other words, understanding how semiotic resources in a proof can contribute to geometrical reasoning may offer effective instructional and educational implications.

Furthermore, O'Halloran (2008) argued for the importance of understanding mathematics as multisemiotic in the following way:

> The view of mathematics as a multisemiotic discourse is significant in a pedagogical context as often teachers and students do not seem to be aware of the grammatical systems for mathematical symbolism and visual display, and the types of metaphorical construals which take place in mathematics texts and in the classroom. (pp. 16–17)

Hence, results of this study have potential implications for the role of visuals, symbols, and gestures in geometry instruction. Results may influence teacher education and professional development practices that help teachers understand and develop in students the ability to interpret diagrams in geometry.

Another potential outcome of this study is to offer insights for curriculum developers. In the view of Lemke (2003):

> The mathematics curriculum and education for mathematics teaching need to give students and teachers much greater insight into the historical contexts and intellectual development of mathematical meanings, as well as the intimate practical connections of mathematics with natural language and visual representation. I hope that I have shown how a semiotic perspective can contribute to reconceptualizing mathematics not simply

> as a system of signs, but as an integral component of a much larger system of semiotic resources for making mathematical meaning in real, historical contexts. (p. 27)

In particular, understanding the high school students' interpretation and implementation of diagrams with visual, symbolic, and gestural resources may be useful in curricular design because of potential implications related to instruction and curriculum materials.

Theoretical Framework

Mathematics learning requires a process of understanding the meanings of the mathematics knowledge that are culturally accumulated. From this viewpoint, the present study draws from semiotic (O'Halloran, 2008) perspective. Semiotics concerns meaning and, in association with the sociocultural perspective, one central assertion is that cultural meanings represented by language, symbols, and actions are the essential part of cultural knowledge. However, at the same time, personal meanings, which contribute personal perspectives and insights into the social activities, are as important as cultural meaning. That is, cultural and personal meanings are intertwined in a way that personal meanings are constructed over cultural objects, and cultural meanings are blended with personal concepts (van Oers, 1996).

In particular, the third theme of sociocultural theory (Vygotsky, 1987), mediation, refers to the semiotics, tools and signs that mediate the cognitive development of humans on both social and individual planes. Tools are externally oriented and are processed by physical and social reality. For example, pen and paper are tools that are included in the behavior and may influence the structure of the behavior. However, signs are internally oriented such as language, algebraic symbol systems, and diagrams. Some examples of semiotics may include languages, written texts, diagrams, and mathematical symbol systems created through human history. Moreover, symbols and signs overlap when they both represent written or oral language and symbol systems such as mathematical symbols since they are also a type of language that is internally oriented.

Signs and tools construct the base for behavior in a mediated activity. By internalizing the signs and symbols, an individual can perform their activity in jointly-constructed social interactions (Davydov & Kerr, 1995). Additionally, the language in its written and spoken formats are important semiotic activities embedded in social interactions (Cobb et al., 1996). Even further, Vygotsky's theory of mind claimed that thought is inner (internalized) speech or conversation (Ernest, 1994). In summary, the semiotics, tools, and signs underlie the development of jointly-constructed knowledge, and by internalizing that knowledge, individuals may apply it in a future activity such as problem-solving.

Similarly, Godino and Batanero (2003) discussed the personal (individual meaning-making) and institutional implications (jointly constructed meaning-makings) to indicate their distinct roles in the analysis of meaningful actions in mathematical learning. As they suggest, it is important to ponder semiotic aspects at the personal and cultural levels in mathematics education because there are various kinds of objects, actions, and contexts included in the semiotic processes in mathematical activities. For example, the theoretical properties of a geometric diagram may be considered in the cultural level because they are jointly constructed meanings which are valid in every condition (e.g., the definition of a triangle). Regarding the personal level, one student can interpret a triangle as a right triangle by relying on the visual appearance of the diagram without doing any theoretically or figurally represented operations that would guarantee that one of the angles is 90-degrees. As far as considering the sociocultural view of semiotics in mathematics education, signs, symbols, and diagrams used in mathematical activities, mathematical discourse, and personal meanings take an important and integrated place in the learning of mathematics (Lemke, 2003; Seeger, 2008).

In general, symbol formations (signs, words, and terms) in activities and use of language (spoken signs) in communication with others have an essential role in the process of meaning making in either cultural or personal level and can be accepted as a semiotic activity (van Oers, 1996). Mathematics is considered as “an intrinsic symbolic activity” (Radford et al., 2008) where signs and symbols (e.g., algebraic symbols, written or oral words, physical signs) signify mental and physical entities (Lemke, 2003; Godino & Batanero, 2003). Notably, it is worth stating here that signs, symbols, visuals (e.g., diagrams and parts of the diagrams such as angles and sides), and language are intermingled in the sense that sometimes mathematical signs represent the role of linguistic, and sometimes linguistic signs convey visual or mathematical meanings. In this study, visuals are considered as an image that illustrate or stand for something such as a geometric diagram or a line.

Likewise, O’Halloran (2008) stated that mathematics is multisemiotic. Connecting the ideas of Lemke (2003) and Godino and Batanero (2003) on semiotics and mathematics, O’Halloran described multisemitoic with three semiotic resources: language, visual image, and mathematical symbolism. Accepting mathematics as multisemitoic has potential to influence the research in the field of mathematics to boost mathematics learning.

Furthermore, previous studies suggest that gesturing—hand and arm movements—should also be considered with all these ideas to gain a complete meaning of the multisemiotic perspective (de Freitas & Sinclair, 2012; Williams-Pierce et al., 2017). Although gesturing is assumed to exist only in communication, in her discussion about the semiotics of gestures, Sabena (2008) asserted that gesturing also has an important place in understanding because language and gesture are intertwined in a way such that they both impact cognition. De Freitas and Sinclair (2014) argued that gesture has the potential to influence the process of mathematics learning. As an example, in the study of Williams-Pierce and colleagues (2017), undergraduate students were encouraged to produce gestures before providing a mathematical proof and concluded that students' gesturing and mathematical reasoning had a significant relationship. Consequently, Williams-Pierce and colleagues (2017) pointed that gesture was part of meaning making about mathematical ideas and could clue a student's mathematical thinking.

Drawing on the theory of semiotics discussed above (Godino & Batanero, 2003; Lemke, 2003; O'Halloran, 2008; Williams-Pierce et al. 2017), the present study adopted a semiotic perspective on diagrams to understand the language, gesture, mathematical symbols, and visual representations in an integrated way. The study of meaning making is complex in its nature, but it may contribute to the field of mathematics education with regard to a better understanding of learning processes in mathematics (Godino & Batanero, 2003). From the semiotic perspective, the current study has potential to contribute to the field in the way that meaningful actions made by individuals in geometrical activities will encourage instructors to support those actions in the similar activities.

Dissertation Outline

In the following chapters, I lay out the background literature about the historical roles and the semiotic meanings of diagrams in geometry, research methods employed in the study, key findings of the investigation, and the conclusion and the implications of the study. Chapter 2 contains a review of literature related to proof in geometry, the role of diagrams in geometry, and students' interpretation of diagrams from significant sources. Chapter 3 provides a detailed description of the setting, participants, data collection, and the process of the analysis. In Chapter 4, I present the results of the data analysis and the findings of this study. Finally, Chapter 5 includes summary discussion of the findings as well as the limitations, implications, and significance of the study.

CHAPTER 2: LITERATURE REVIEW

This chapter focuses chiefly on the review of literature under eight major threads that inform the conceptualization of this study. The eight major threads are as following: reasoning and proving in geometry, the role of diagrams in geometry, research on students' interaction with diagrams, sample proving frameworks, semiotics and geometry, visuals and symbols, gesture, and sample semiotic frameworks. Moreover, I provide studies that discuss the semiotic resources—visuals, symbols, and gestures—in learning mathematics and the need for further research on students' interpretation of diagrams. The detailed review of the literature provides insights from various studies that are related to the focus of this study. Finally, I conclude with a summary that highlights the overall points from the literature to situate my research questions in relation to the hitherto literature.

Research on Geometry Proof and Diagrams

Geometry has a long history, thousands of years, base rooted in Greek mathematics and in particular, in the study of Euclid, *Elements*. Miller (2007) argued that *Elements* is one of the first works in mathematics because the conceptions of mathematical proof and the use of geometric diagrams first appeared in it. As Miller (2007) suggested, "In Greek mathematics, geometry was viewed as the foundation for all other branches of mathematics, and so the Greek theories of arithmetic and algebra were based on their theory of geometry" (p. 7).

In today's high school curriculum, Geometry is a core subject, and proof is especially prevalent in geometry (Herbst, 2002; Schoenfeld, 1994). Current research on geometry education has focused on main themes such as visual reasoning, the role of diagrams, and teaching and learning of proving process (Sinclair et al., 2016). In this section, I discuss the research related to students' reasoning and proving in geometry, the role of the diagrams, students' interaction with diagrams, and sample proving frameworks in the following four sub-sections.

Reasoning and Proving in Geometry

Proving is an important notion in today's mathematics classes and students at almost all levels are expected to develop their reasoning and proving skills (CCSSI, 2010; NCTM, 2000). It is worth mentioning here that proof may be considered as a mathematical argument or demonstrating evidence for the truth of a statement, and proving can be defined as "a formal way of expressing particular kinds of reasoning and justification" (NCTM, p.56). In another way, Rav (1999) mentions that "Proofs are the mathematician's way to display the mathematical machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution" (p.13).

To improve their proving proficiencies, students should develop their skills in making claims about abstract and general concepts in mathematics. In particular, according to *Principles and Standards for School Mathematics*, "by the end of the secondary school, students should be able to understand and produce mathematical proofs" (NCTM, 2000, p. 56). Likewise, Schoenfeld (1994) maintains that "proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels" (p. 27).

In his study of proof and proving in school mathematics, Stylianides (2007) highlighted the meaning of proofs in K–12 mathematics, in particular with a focus on the elementary grades, and claims that teachers have a significant role in determining which arguments could be accepted as proofs. Teachers usually do not appreciate the central role of proofs in school mathematics and believe that only a small number of students can develop a conception of proof (Harel & Sowder, 2005). This suggestion is consistent with the view they expressed in an earlier study of Harel and Sowder (1998), as follows: "A major reason that students have serious difficulties understanding, appreciating, and producing proofs is that we, their teachers, take for granted what constitutes evidence in their eyes. Rather than gradually refining students' conception of what constitutes evidence and justification in mathematics, we impose on them proof methods and implication rules that in many cases are utterly extraneous to what convinces them" (p. 237). Similarly, Knuth (2002) maintains that teachers' conceptions of proof must be improved to place proofs in a central position in school mathematics. While these studies stressed the major role that teachers play in developing conceptions of proof, students' beliefs about reasoning and their struggle with proving are also significant issues.

That said, Sinclair and colleagues (2016) showed that the research on reasoning in geometry has increased. There are a variety of studies that focused on the processes of argumentation, reasoning, and proving in geometry (Arzarello & Sabena, 2011; Chazan, 1993; Herbst & Brach, 2006; Senk, 1985). The common theme that emerges from these studies is that although proof is an essential concept in teaching and learning mathematics in general, students struggle in writing proofs in geometry, especially when they attempt to produce deductive arguments. Harel and Sowder (2005) noted that students struggle with the distinction between inductive and deductive arguments. They stated that even writing or starting simple proofs is a challenge for students and they provide specific examples to prove mathematical claims. Particularly in geometry, "Students base their responses on the appearances in drawings, and mental pictures alone constitute the meaning of geometric terms" (Harel & Sowder, 2005, p. 48).

Rav (1999) claimed that the reason why it is hard to write a proof is because of the formal logic that must be used in the proof to draw valid conclusions. Hence, Rav stated that students may believe that they have to remember the formal rule of logic or mathematical theories to construct a proof. In a similar vein, Schoenfeld (1994) argued that based on his observations, the emphasis on the format of the proof such as the two-column format for

Euclidean proof influences students' views about proof. Schoenfeld continues his argument by stating that "Students believe that proof-writing is a ritual to be engaged in, rather than a productive endeavor" (1994, p. 27).

Given the difficulties that students face in reasoning and proving in geometry and the limited support from teachers because of the lack of understanding of proofs, it is unclear how to prepare better students to produce logical reasoning (even in an informal way) in proofs (Harel & Sowder, 2005). Reasoning and proving in geometry require a visual engagement with the tasks that are provided in different manifestations of the geometric claims. Sinclair and colleagues (2016) proposed that visual reasoning—the activity of imagining static or dynamic objects and acting on them (mentally rotating, stretching, etc.)— is an important skill for mathematics in general, and it can be emphasized in geometry as a way to also support visual reasoning in other topic areas, as well. In a similar vein, Presmeg (2006) highlighted the role of visualization, presenting mental images into spatial formats, in doing proofs.

According to Presmeg (2006), the notion of visual representations consists of the image presented and features of the concept that it represents. A triangle, for example, has both spatial and conceptual properties, which are distinct categories of mental assets. The spatial parts of the triangle are the components of the triangle image such as the angles, sides, or points on the triangle; on the other hand, the features of the concept of a triangle includes having a closed figure with three straight sides and three angles. Likewise, Fischbein and Nachlieli (1998) stated that geometrical models have both spatial and conceptual features. They referred to a *concept* and an *image* separately: "A concept is usually defined as an abstract, general representation (an idea) of a category of objects or events. On the other hand, an image (especially a visual image) is a sensorial representation of an object or event" (p. 1193).

In an earlier study, Fischbein (1993) examined students aged 14–17 years who focused on defining and identifying geometric figures, and he claimed that during the process of reasoning, concepts and images were intertwined. Fischbein used purposefully selected tasks that included visual images and were not complex or unusual problems so that students could show their typical mathematical reasoning with both conceptual and figural properties. As an illustration for intertwined conceptual and figural properties of geometry diagrams, consider a student who is asked to find the base angles of an isosceles triangle. In the case of working on a particular triangle, the figural properties of the triangle are incidental; that is, the exact values of the base angles are related to that particular triangle. However, the sides and angles are conceptual properties when the student uses the properties of the isosceles triangle—two sides of equal length and two base angles of equal degrees. Thus, the figural properties of the triangle, values of the base angles in that example, depends on the problem; whereas, the student follows the properties of the isosceles triangle (i.e., the conceptual properties) while solving the problem.

Building on the study of Fischbein (1993), Laborde (2005) proposed two properties of diagrams, *theoretical* and *spatio-graphical* properties, based on the theoretical knowledge and intuitive visual features of the diagram, and the use of these two features in solving school problems. Table 2.1 shows two properties of diagrams, conceptual/theoretical and figural/spatio-graphical properties, which Fischbein and Laborde presented. Apparently, the features of the visuals included in proving and reasoning took a notable attention. Moreover, proving in geometry requires logical reasoning with figures even if the figures are not created correctly (Poincare, 1963). The proof of a geometric claim is valid for various diagrams as long as the diagrams share the same topological features such as intersection between lines, perpendicular lines, and containing certain points (Mumma, 2008).

Table 2.1

Two properties of geometry diagrams (Fischbein, 1993; Laborde, 2005)

Terms	Definitions	Examples
Conceptual/Theoretical Properties	• Imposed by, or derived from definitions in the realm of certain axiomatic system. (p. 141, Fischbein, 1993). • The theoretical referents in a geometrical theory, to theoretical objects, relations and operations on these objects as well as to judgements about them that can be expressed in various languages (p. 161, Laborde, 2005).	Sides, angles, points, a triangle, a circle.
Figural/Spatio-graphical Properties	• Sensorial representations that reflect concrete operations (p. 140, Fischbein, 1993). • The graphical entities on which it is possible to perform physical actions, and about which it is possible to express ideas, interpretations, opinions, judgements (p. 161, Laborde, 2005).	The length value of a side, the color or width of a side, the degree value of an angle.

The Role of Diagrams in Geometry

Brown (1997) claimed that *picture* (diagrams) can be considered more than merely heuristic tools and hence, they can prove theorems. He highlighted that it is important to know how to make sense of the pictures in mathematics since they are as valuable as verbal/symbols in mathematical proofs. In other words, a geometric proof is not merely a syntactic piece since it includes interpretation of sentences and diagrams involved in the proof (Giaquinto, 2008).

Consequently, using visuals like images and representation of geometric concepts provides unique aid for how to reason and do proofs in geometry.

Affordances and Limitations of Diagrams in Geometry

Various studies have discussed the affordances and limitations of diagrams for geometrical argumentation (Brown, 1997; Dove, 2002; Mumma, 2010), but one aspect of diagrams that is widely agreed upon is their pervasiveness in geometric definitions, claims, and problems. Reasoning with diagrams has had an essential place in geometric thinking throughout the history of mathematics, and diagrams still have a crucial part in the modern proofs and geometrical reasoning for making inferences about the geometrical concepts (Avigad, 2008). Indeed Poincare (1963) stated that reasoning in geometry requires a meaningful and logical thinking with diagrams even if they are not drawn to scale. Teachers, as well as students, generally use diagrams as an action or a drawing method in the solutions (Herbst et al., 2016).

Researchers in mathematics education have examined the power of visualization in doing geometric proofs. In some cases, the visual figures may provide real properties of concepts with a stronger meaning than the words could represent (Fischbein, 1993; Presmeg, 2006). In other words, as Brown (1997) argued, there are some cases where a diagram alone has potential to prove a theorem or to validate the proof of a theorem in mathematics. Furthermore, of the visualization of geometric concepts can include infinite variations of a geometric shape. For example, even when a particular triangle is used to show a property or claim that applies to all types of triangles, such as “The medians of a triangle are concurrent,” that particular triangle can be thought of as representing an endless number of triangle objects (Fischbein, 1993).

Studies also consider the contributions made by diagrams and text-based arguments in geometrical reasoning. For instance, in his discussion of the role of diagrams in Euclidean

Geometry, Manders (2008) claimed that diagrams and text-based arguments were the core components of the geometrical reasoning. In a similar vein, Larkin and Simon (1987) discussed the sentential (i.e., sentences that are used in the problem description) and diagrammatic (i.e., parts of a diagram which are used to describe the problem) representations and maintained that the diagrammatic representations can hold the geometric information about the problem. Moreover, Brown (1997) argued, "[Pictures] are mainly a form of evidence—a different form, to be sure, than verbal/symbolic proofs; but they have the same ability to provide justification, sometimes with and sometimes without the bonus of insight and understanding" (p. 177). In their analysis of the diagrams in geometry textbooks, Dimmel and Herbst (2015) made assertions that geometry diagrams could be considered as visual texts that provided a different aid on working with mathematical problems than standard written text.

The emphasis on the role of visuals, especially diagrams, has brought to the surface the complexities and advantages of diagram use in learning mathematics. Using pictures provided in the problems, in some cases recording the work on the diagrams, may help students to produce a valid solution to the problem and aid communication of the reasoning (Heck, 2015). As Larkin and Simon (1987) suggest, a diagram in a geometric problem presents all the features of the internal imaging process that help comprehend and solve the problem. Similarly, Arcavi (2003) noted that despite the concerns about the visualization in mathematics education, visualization has a central role in learning and doing mathematics such as in reasoning, proving, and solving a problem. Additionally, Hegarty and Kozhevnikov (1999) divided visualization into two categories: *schematic* and *pictorial representations*, which are similar to the figural and spatio-graphical properties discussed above. Schematic representations refer to spatial features of the objects, and pictorial representations refer to the appearance of the objects. They claim that students who use schematic representations in mathematical problem solving are more successful than students who use the pictorial representations.

Indeed, the research about the diagram use in learning mathematics has shown that the complexities of the role of diagrams and meaning of reasoning with diagrams might be because of the different and flexible features diagrams have. Fischbein and Nachlieli (1998) claimed that geometric diagrams contained both conceptual and spatial properties. That is, geometric diagrams denoted the theoretical aspects of geometrical concepts and their visual/spatial representations. In the case of geometrical reasoning with diagrams, the conceptual and spatial features were interwoven to produce a valid and rigorous argument (Fischbein, 1993). These studies imply that understanding the proofs in geometry requires making reasonable inferences with the conceptual and spatial features of the diagrams produced and text-based arguments provided in the proofs.

While some researchers have found certain usefulness in geometric diagrams, other studies have discussed the potential limitations of using diagrams in reasoning when they were not constructed with theoretical knowledge. For instance, Laborde (2005) stated that sometimes students only consider practical and visual aspects of a diagram and avoid engaging with theoretical or conceptual relations. Likewise, Chazan (1993) reached a similar conclusion that high school geometry students hold the beliefs about evidence being proof, or sometimes they do not appreciate the generic features of diagrams in deductive arguments and accept proofs as mere evidence. Furthermore, Dove (2002) noted that despite their effectiveness as a heuristic tool in proofs, diagrams are only helpful for explaining that particular proof. Accordingly, these researchers suggest that diagrams are not very useful to draw general conclusions.

Concerning understanding students' diagrammatic thinking, Giaquinto (2008) argued that the diagrammatic thinking occurring while students prove is not reliable in the sense that diagrams used in reasoning do not always align with the entity that the diagram represented and hence, diagrams are not readily generalizable. For example, visual reasoning with an acute triangle may not be valid on obtuse triangles. That means producing a generalization based on a particular diagram includes significant risks regarding deducing general geometrical reasoning and potentially weakens the precision in mathematical arguments (Avigad, 2008; Manders, 2008). Furthermore, in her study, Stylianou (2011) pointed out students' representation use and showed that images such as drawings, diagrams, and symbols have various roles—*a means to understand the information, a recording tool, tools that facilitate exploration, and monitoring and evaluating devices*—in problem solving. Therefore, in particular with proving geometric claims, it is important to consider students' interpretations and inferences to better understand the meaning of the geometrical reasoning they produced.

Diagrammatic Register and Task Presentations in Geometry Research

Much of the past research on reasoning and proving in geometry has used task-based interviews, and so the selection of tasks and the task features are important to consider. As an illustration, Hollebrands (2007) investigated how tenth-grade Honors Geometry students interpret and reflect on their solution strategies on geometric transformations tasks and hence, develop their geometrical reasoning. The tasks students were given varied with regard to whether a diagram was accompanying the problem or not. In an earlier study, Senk (1985) explored the way secondary school geometry students in the United States write geometry proofs on the tasks that cover the congruent and similar triangles, parallel lines, and quadrilaterals. These items included a diagram, givens, and a statement to prove. Senk's study showed that students showed a low performance on a successful proof writing.

Similarly, in another study, Herbst and Brach (2006) researched geometry students' argumentation and proving in geometric contexts, and they used particular tasks that involved different forms—theorem, problem, and exercise—to engage students' interest and attention on different tasks. Additionally, Hilbert and colleagues (2008) used *heuristic examples*— worked-out examples that demonstrate heuristic steps towards a solution—to investigate student teachers' understanding of proof in geometry. In both of these studies, the researchers included a diagram accompanying the problem, theorem, or exercise statements.

One way to view all these variations is to recognize that the tasks students are engaging in geometry activities can be varied regarding the diagrammatic register. Herbst, Dimmel, and Erickson (2016) described *diagrammatic register* as the following:

> "(1) Proof problems are accompanied by a diagram; (2) which is a rather accurate representation of the givens and the proposition to be proved; (3) the statement of givens refers to the objects in the diagram via the labels for points, which requires that (4) points to be used in the proof be given labels in the diagram at the outset; and (5) properties stated in the givens do not include those of collinearity, incidence, or separation, which are relied upon in the diagram." (p. 8)

In other words, diagrammatic register is a feature that a typical geometry proof problem contains in general because in most cases of the typical geometry problems, there is a diagram accompanying to the problem in which the givens are either presented or referred to certain parts of the diagram through labels. Because this study focuses on students' use of diagrams in geometry proofs, students approach on the problems that include the non-diagrammatic register feature took a particular attention. Drawing on this conceptualization the tasks used in the present study are described with respect to their adherence (or lack thereof) of the diagrammatic register task features. Figure 2.1 presents a geometry problem implemented in this study with both diagrammatic register and non-diagrammatic register features. Considering the diagrammatic register feature in the task, in fact, provided potentials for a better understanding of how students used diagrams and if they employed diagrammatic register in their proof productions.

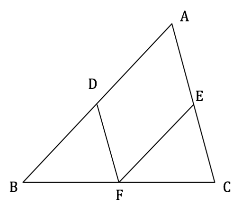
Geometry Problem in Diagrammatic Register	**Geometry Problem in Non-diagrammatic Register**
 Given: D, E, F midpoints. Prove: Triangle BDF is congruent to triangle FEC. (Δ BDF $\cong$ Δ FEC) ***Clue Theorem*:** The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.	Given a triangle, the midpoint of a side constructs two segments by joining with the other two midpoints on the two other sides of the triangle and hence, two inner triangles occur in the triangle. Prove that these inner triangles are congruent. ***Clue Theorem*:** The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

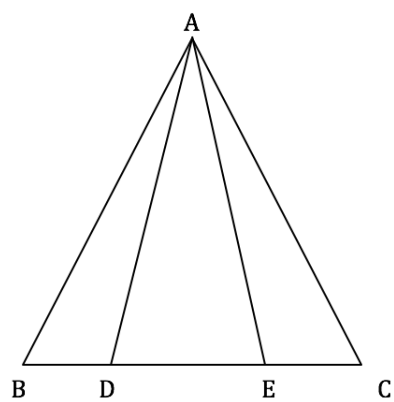
Figure 2.1. Two sample geometry problems in diagrammatic register and non-diagrammatic register.

Moreover, researchers, as mentioned above, analyzed their data mostly with a focus on the students' utterances in the interview transcripts about their written solutions to the tasks. However, particular attention has not often been given to the way students were engaged with the diagram in the analysis of students work on the diagram and how the way they were engaged in the diagram shaped their reasoning did not receive particular attention for the analysis. Additionally, as Stylianou (2011) noted that although experts and students both show evidence in using the representations (diagrams, symbols, words) in exploration, students seem less sophisticated in representation use. Similar conclusions were found in an earlier study where although students value visuals in problem solving, they showed less evidence than experts regarding creating visuals in exploration and solution of the problems (Stylianou & Silver, 2004). Thus, future research may focus on improving students' understanding of the role of the representations in mathematics and how to use them effectively in learning mathematics.

Research on Students' Interaction with Diagrams

As discussed above, geometric diagrams convey both conceptual and spatial meanings in geometrical reasoning (Fischbein & Nachlieli, 1998). Laborde (2005) addressed the same distinct features of the diagrams by naming them as theoretical and spatio-graphical properties of diagrams. Her study investigated mathematical problems, definitions, and proofs about the relationships between theoretical and spatial-graphical features. Laborde claimed that diagrams were important information sources for doing proofs. She states as follows: "[S]tudents must be able not only to distinguish representation from their theoretical referent, but also to know that in some cases they are allowed to use properties of the spatio-graphical representation without justifying them by theoretical arguments" (Laborde, 2005, p.164). When students implement theoretical and spatio-graphical properties of diagrams, they also engage in the diagrammatic register (Herbst et al., 2016) to some degree.

As an illustration, in a typical geometry proof exercise, the statement of the theorem usually includes or implies a diagrammatic object with the iconic signs in that diagram. The diagram with its icons is used to reason and prove the claim. Given a triangle, for instance, the segments are used to present its sides, and the place where two segments meet represents a vertex or a point in the triangle (Figure 2.2). The claim might provide numerical values of some icons such as segment length, vertex value, and label of a point and then ask for the equality of particular segments or value of an angle. When a student engages with such a task to prove the claim, the student might use symbols like dashes to mark the relations between the lengths of the sides, or they could label the angles to see the relationships between the angles and sides. The student would likely create a diagram figure and try to display the icons on the diagram as stated in the claim even if a diagram did not accompany the proof exercise. In both cases, the student interacts with the visual and text given in the claim, and their reasoning process involves the interpretation of the diagram and iconic signs in it.



Given: $\overline{AB}=\overline{AC}$ < BAD ≅ <CAE

Prove: Triangle ADE is isosceles.

Figure 2.2. A diagram with iconic signs in it.

Herbst and Brach (2006) pointed out the necessity of diagrams in both expressing the problems and developing concepts that might help to solve the problem or construct proofs. As far as the interactions with diagrams in mathematics, many researchers suggest further research on how to use and understand the images better in mathematics. For example, Brown (1997) pointed out that pictures like diagrams in mathematics are helpful tools in mathematical observations and we must learn how to use them. Similarly, Arcavi (2003) claimed that the field of mathematics education needs more insights into the role of visualization and how to interact with them effectively to improve reasoning and sense making in mathematics. In a similar vein, Giaquinto (1994) suggested the expertise on what Giaquinto called visualization to gain more benefit from using it.

The research on the role of diagrams in geometrical reasoning and relations between the spatial and theoretical features of the diagrams provides several implications for future studies. In particular, Laborde (2005) argues that there is a need for conducting further studies focusing on students' inferences on the conceptual/theoretical and figural/spatial features of the diagrams with consideration of students' spoken, written, and gestural works on doing proofs in geometry (Laborde, 2005). Furthermore, Larkin and Simon (1987) discuss that a diagram may be more useful than verbal explanations since diagrams can hold much information such as symbolic labels and conclusions about a mathematical problem. Hence, they propose further research on understanding how to use diagrams more successfully.

Dimmel and Herbst (2015) concluded that students should improve their skills on how to both read and construct texts and diagrams in the geometrical reasoning. That said, reading a diagram and making inferences from discursive text can help build a reasoned conjecture and deductive argument (Avigad, 2008; Manders, 2008). Therefore, diagram and discursive text should be studied together to appreciate the meanings construed during reasoning and the students' interpretations of the diagrams in doing proofs. Additionally, it is important to understand the nature of the process of students' various interactions with diagrams (Herbst, 2004) because little is known about what meanings students draw from diagrams to use them as a tool to reason spatially and theoretically. Having scholarship on this inquiry may be helpful to develop useful ideas that will meet with "the need to teach students how, why, and when they can transform a diagram in a proof" (Senk, 1985, p. 455).

Sample Proving Frameworks

In consideration of the use of diagrams in geometry proofs, some current research provides insights into how and why we, as mathematics educators, should deepen our understandings of the nature of students' geometrical reasoning with diagrams. In this respect, there are frameworks developed in recent studies to understand students reasoning in mathematics in general, and meaning making with diagrams in geometry in particular. Stylianides (2008) proposed a comprehensive analytic framework to study reasoning and proving in school mathematics with mathematical, psychological, and pedagogical components (Figure 2.3). Although the framework was developed for a broad purpose, Stylianides (2008) claimed that it might not identify students' engagement in problem solving or reasoning and the framework does not include a semiotic perspective. Thus, while this type of framework may help understand the reasoning and proving actions, it does not provide aid with regards to identifying students' engagement in a geometrical proof.

<table>
<tr><td></td><td colspan="4">Reasoning-and-proving</td></tr>
<tr><td rowspan="3">Mathematical Component</td><td colspan="2">Making Mathematical Generalizations</td><td colspan="2">Providing Support to Mathematical Claims</td></tr>
<tr><td>Identifying a Pattern</td><td>Making a Conjecture</td><td>Providing a Proof</td><td>Providing a Non-proof Argument</td></tr>
<tr><td>• Plausible Pattern
• Definite Pattern</td><td>• Conjecture</td><td>• Generic Example
• Demonstration</td><td>• Empirical Argument
• Rationale</td></tr>
<tr><td>Psychological Component</td><td colspan="4">What is the solver's perception of the mathematical nature of a pattern / conjecture / proof / non-proof argument?</td></tr>
<tr><td>Pedagogical Component</td><td colspan="4">How does the mathematical nature of a pattern / conjecture / proof / non-proof argument compare with the solver's perception of this nature?
How can the mathematical nature of a pattern / conjecture / proof / non-proof argument become transparent to the solver?</td></tr>
</table>

Figure 2.3. The Analytic Framework developed by Stylianides (2008, p. 10).

In another study, Van Meter and Garner (2005) presented a theoretical framework for *learner-generated drawing* and point out that a learner must show the components of an image with a symbolic verbal representation. They mentioned about three cognitive processes—*selection, organization, and integration*— as the core components for mental modeling in which verbal and nonverbal representations are combined. The selection component requires the learners to select basics from the text and other representations such as visuals. Organization implies a logical representation of verbal and nonverbal illustrations as well as inner thinking. Lastly, integration includes a meaningful combination of organized verbal representation with nonverbal representations.

Regarding the individual actions in a proof, specifically in geometry, Otten, Bleiler-Baxter, and Engledowl (2017) studied authority in a high school geometry classroom by investigating actions occurred during proving process. Otten and colleagues developed a framework (Figure 2.4) to code actions employed by the classroom teacher and students in a high school geometry classroom setting. Although this framework was used for the teacher-student interaction during proving, it was also claimed that it might serve for the analysis of students' actions while proving (Otten et al., 2017).

Proving Actions					
CLAIM	Stating the overall claim to be proved or disproved	INVEST	Investigating or guessing the truth value of a claim	REFINE	Making a refinement or modification of a claim
START	Calling for a proof argument	END	Stating that the proof is complete	ACCEP	Stating an accepted definition or prior result
STRUC	Identifying the structure of the argument	STEP	Providing a step in the argument	JUST	Justifying a step in the argument
SUM	Summarizing all or part of the argument	ABOUT	Making a general statement about proof		
Proving Interactions					
REQ	Requesting claims or arguments	CLAR	Clarifying claims or arguments	CRIT	Critiquing claims or arguments
ALT	Offering an alternative argument	CONF	Confirming the validity of arguments		

Figure 2.4. Proving actions framework developed by Otten and colleagues (2017).

The frameworks discussed thus far highlighted the stages that may be involved in a typical proving process, and particularly the framework developed by Otten and colleagues

(2017) is utilized for this study since it provides a practical scheme for coding actions that appear moment by moment in the proving process.

Research on Semiotics

This section describes the semiotic perspective and its role in geometry in particular. In the next sub-sections, I discuss the relations between semiotics and geometry, visuals and symbols, gesture, and sample semiotic frameworks.

Semiotics and Geometry

There is an increasing trend in mathematics education of studying how students learn through the semiotic perspective (Godino & Batanero, 2003). Over the last couple decades, research on visualization in mathematics gave extensive attention to studying the semiotic (meaning-making) aspects of mathematical visualization (Presmeg, 2006). Mathematics can be viewed as a multimodal/multi-semiotic discourse because there are various types of interactions in mathematics such as using language, mathematical symbols, visuals, gestures, and so forth. Lemke (2003) claimed that semiotics is necessary to better identify the purposes of mathematical meanings and their works. That is, through the perspective of semiotics, the values of intertwined mathematical functions such as language, mathematics, and visual representations can be revealed.

In particular, considering the role of the semiotics in geometry, in their review of the literature on geometry education, Sinclair and colleagues (2016) highlighted the increased attention in the current research in geometry education about the relationship between semiotics and geometry learning. Sinclair and colleagues mentioned the figural concepts as one of the major themes of teaching and learning of geometry. Furthermore, figural concepts like diagrams used in teaching and learning of geometry involve both visual and abstract properties. Similarly, as Fischbein (1993) suggested, “A geometrical figure may, then, be described as having *intrinsically* conceptual properties. Nevertheless, a geometrical figure is *not* a mere concept. It is an image, a visual image. It possesses a property which usual concepts do not possess, namely, it includes the mental representation of space property." (p. 141).

Arzarello and Sabena (2011) discussed the semiotic and theoretical progress made by students in diagrammatic reasoning and doing proofs. Specifically, they pointed out the relationship between semiotic and theoretic aspects in reasoning with diagrams in students' proofs. As an illustration, they investigated high school students' interpretation of signs in a graph given with a function and its derivative and primitive lines, connections of the signs with the arguments, and identification of a mathematical theory to clarify the relationship between arguments and signs. They concluded that some students identified the maximum or minimum points in the function graph and hypothesized a line as being the primitive of the given function in the graph because the max point of the given function corresponded to the point of inflection in the primitive of function. The students then identified the relationship between max/min points and the derivative or antiderivative lines of the given function.

The claim that was made by Godino and Batanero (2003) about understanding the meaning of a geometrical topic has a high connection with the sense that the spatial features convey. Namely, it is reasonable to claim that visuals like diagrams have an important semiotic role in learning geometry because geometrical reasoning involves the meanings of diagrams and discursive texts presented in the arguments (Manders, 2008). Moreover, Neto and colleagues (2009) suggest a semiotic approach to appreciate the mathematical reasoning. In this vein, recent studies considered the semiotic terms of diagrams to understand their unique roles in geometrical reasoning. Although the field of mathematics education has begun to address the meanings of the geometric diagrams, many efforts have focused on how diagrams convey the meaning rather than attending to understanding what meanings students draw from the diagrams. As a result, we know little about how students interpret and use diagrams in geometric claims.

Visuals and Symbols

Previous research has pointed to particular factors that influence the efficacy of drawings and visuals in students' mathematical learning. As one example, although Van Meter and Garner (2005) discuss drawing in a broad sense within different fields such as the social studies, the sciences, and the language arts, they also consider drawing in the mathematics education literature and conclude that drawing has a positive impact on solving mathematical problems. Similarly, Presmeg (1986) discussed the strengths and limitations of visuals in high school mathematics, and defined a visual solution of mathematical problems as "one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed" (p. 42). In their review of the literature on drawing, Van Meter and Garner stressed the benefits of learner-generated drawing and stated that "[L]earner-generated drawing is defined as a strategy in which learners construct drawing(s) to achieve a learning goal" (2005, p. 287).

In particular, Rellensmann, Schukajlow, and Leopold (2017) studied students' awareness about the characteristics of a drawing that is appropriate for a given problem and have stressed two types of drawings that students employ in mathematical modeling: *situational drawing* and *mathematical drawing*. The *situational drawing* carries the pictorial features of a problem; however, a *mathematical drawing* reflects fundamental features of the problem that will be helpful for the solution. Although both types of drawings are considered as useful methods for mathematical modeling, they provide different aids in the solutions of the problem. As an illustration, situational drawing helps shape the situation described in the problem. However, mathematical drawing helps model the best mathematical approach to solve the mathematical problem and hence, requires a complicated thinking process (Rellensmann et al., 2017). As an example, in a typical geometry problem, using the given information on the diagram such as labeling the equal side lengths may be considered as situational drawings. Whereas, mathematical drawing implies using the relevant information that is inferred from the task or followed by situational drawing such as identifying the value of an angle in the triangle by using the givens and previously known facts.

It is pointed out that drawings include a combination of nonverbal and verbal representations, although nonverbal representations pull from verbal descriptions (Van Meter and Garner, 2005). In a similar vein, Presmeg (1986) counted verbal and mathematical symbols as well as visual imagery in the definition of visuals and argued that a visual imagery may be useful in its dynamic and abstract use. Additionally, according to Nathan (2008) mathematical symbols and mental representations are also involved in gestures, especially in instructional settings, to support meaning making.

In addition to the role of symbols and visuals in meaning making, Arcavi (1994) expressed sense making with symbols in mathematics and defined symbol sense as a mindful decision for when to use or not use symbols. He argued that "[s]ymbol sense should include, beyond the relevant invocation of symbols and their proper use, the appreciation of the elegance, the conciseness, the communicability and the power of symbols to display and prove relationships in a way that arithmetic cannot" (p. 26). In fact, from his point of view, Arcavi described characteristics of symbols sense such as expressing the verbal and graphical information and their unique roles in different situations.

The studies discussed thus far support the idea that visuals or drawings like diagrams in geometry can be viewed as an integration of symbols, verbal, and gesture in the process of meaning making. As an illustration, a recent study about geometric diagrams focused on the symbols and visuals embedded on diagrams (Dimmel & Herbst, 2015). In the study, a semiotic catalog was used to identify the symbols and other visuals that appeared on the diagrams in the textbooks. Dimmel and Herbst (2015) proposed that the semiotic framework may be used to explore students' interactions with diagrams in geometry.

Although current research appreciates the need for studying the meanings of diagrams in geometrical reasoning and proving, students' interpretation of diagrams and the meanings they construct during their engagement with geometric tasks remain unclear. As far as emphasizing students' role on meaning-making with diagrams, Godino and Batanero (2003) stated a semiotic act (interpretation/ understanding) to highlight the equal significance of the meaning that mathematical symbols, texts, or visuals convey and how an individual comprehend that meaning. In particular, students have a significant responsibility in reading and understanding the implications in diagrams and deducing conclusions through their process of reasoning with discursive texts and diagrams (Manders, 2008).

Van Meter and Garner (2005) propose three hypotheses for further research after reviewing the literature on drawing in various fields. These hypotheses are as follows:

> "1. The accuracy of constructed representations is predictive of performance on outcome assessments.
>
> 2. Learners require support to use drawing effectively.
>
> 3. Higher-order, but not lower-order, assessments are sensitive to the effects of learner-generated drawing" (p. 299).

Furthermore, Dimmel and Herbst (2015) pointed out that students' ability to read diagrams in geometry and employing useful diagrams in instruction are the issues that might be studied with a grasp of the semiotic structure of diagrams.

On the other hand, it is suggested that the relationship between the problem type and the student-generated drawing type (Rellensmann et al., 2017), as well as the features of the powerful teaching to develop students drawing strategy may be studied (Van Meter & Garner, 2005). Thus, recent studies have suggested further research on the relationships between and among the student-generated drawing strategies, problem type, and characteristics of support and instruction to develop drawing strategy.

Gesture

The field of mathematics education has recently paid considerable attention to the role of gestures in learning and teaching mathematics (Nathan, 2008; Williams-Pierce et al., 2017). In particular, there is a growing number of studies that focus on the gesture in the construction of meaning making in mathematics (Marrongelle, 2007; Pier et al., 2014, Williams-Pierce et al., 2017). In Williams-Pierce and colleagues' (2017) study, gestures—*the hand and arm movements that accompany speech*—are acknowledged as one of the semiotic resources that students use when communicating their ideas in geometry proofs.

Gestures and Proofs

As far as the role of gesture in mathematical thinking, recent studies speak in favor of the view that gestures support students' reasoning abilities and production of proofs. As an illustration, Nathan and colleagues (2014) argued that gestures play a critical supportive role in mathematical reasoning and proving. For example, Nathan and colleagues gave the following problem to undergraduate students. "Mary came up with the following conjecture: For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side. Provide a justification as to why Mary's conjecture is true or false" (p. 185, Nathan et al., 2014). Although students were not allowed to use paper and pencil, they were able to produce valid mathematical reasoning when they performed task relevant gestures. Similarly, Williams-Pierce and colleagues (2017) claimed that gesture could hint at students' mathematical thinking and new ways of reasoning. Especially, they suggest that "attending to students' gestures in tandem with language can contribute significantly to understanding students' mathematical reasoning and proving practices, particularly when attending to dynamic gestures depicting relationships that may be difficult to communicate verbally" (Williams-Pierce et al., 2017, p. 257).

Williams and colleagues (2012) claimed that proof works did not merely consist of written work and speech; that is, the gesture was as important as speech and written work in the proof production. In other words, gesturing—*hand movements*— is an important part of the learning process, and they do not only represent knowledge but also help produce new insights (Goldin-Meadow et al., 2009). In a similar vein, Goldin-Meadow and Beilock (2010) said that gestures both present what people think and impact the production of thinking. In another study, Marrongelle (2007) focused on undergraduate students' algorithmatization by attending to students' use of graphs and gestures and concluded that both figures and gestures contribute to students' reasoning.

Roth (2001) focused on the relationship between gesture and speech in education by reviewing the studies in various fields such as linguistic, anthropology, mathematics, and science. In specific to mathematics and science, Roth argued that gesture has a crucial role in learning and teaching a concept as gestures were integrated with a speech in the process of meaning making. Similarly, Pier and colleagues (2014) studied students' expression and use of gestures when they worked on mathematical proofs and investigated the relationships between speech, gesture and valid proofs. The study concluded that gesture has a unique contribution to students' mathematical knowledge and production of proofs (Pier et al., 2014).

Gestures have also been regarded as a driving force in the development of mathematical knowledge in the context of teaching. Gestures are involved in mathematical communication to reflect mathematical thinking, and teachers and students use gestures to demonstrate their mathematical knowledge (Alibali & Nathan, 2012). However, Roth (2001) encouraged further research both in qualitative and quantitative setting to better understand the potential of gesture in learning and teaching. Similarly, Williams-Pierce et al. (2017) recommended studying the instructional implications drawing on the gestures since they provide insights into students' thinking.

Walkington and colleagues (2012) stated that the influence of gesture in mathematical communication is worth considering and mathematics educators should investigate students' gestures in their mathematical thinking. Göksun, Goldin-Meadow, Newcombe, and Shipley (2013) found that individuals who have different performances on certain tasks also showed differences in the performance of gesture. Thus, gesture type or common gestures that appear on certain tasks are worth considering as Marrongelle (2007) suggested further research on identifying common gestures in mathematics so that teachers can better understand students' thinking. In a similar vein, Sinclair and colleagues (2016) pointed out the importance of understanding the kinds of gestures. They indicated that "[e]xisting research suggests that the more teachers gesture, the more students will, but future work could provide insight into the types of gestures that might be helpful and the modalities in which students are invited to gesture as well" (p. 701). It follows that identifying useful student gestures may help teachers address and remove students' confusions related to proof activities (Pier et al., 2014).

However, Pier and colleagues also argued that considering merely the types of gestures may not be sufficient to understand the proof production because "there may be other components of language that are important to proof construction that our tools were not able to capture" (2014, p. 655). This idea fosters the consideration of symbols and visuals as other components of language. Indeed, Nathan (2008) stated that gestures in instructional settings involve symbols and mental representations. Hence, symbols and visuals like images, as involved in gesture, are mental entities that possess meaning making. Nathan stated that instructional settings have potential to present meaning for signs in new concepts and images.

The recent studies have provided information about the role of gesture by investigating it in the context of mathematical reasoning and proving. The findings from the recent research have fueled calls for studying gesture that boost students' mathematical thinking. That is, studies covered thus far propose that research should attend to gesture in the evaluation of students' mathematical reasoning. Although the current studies have not yet addressed the types of gestures that lead to better mathematical reasoning, they have suggested that the future research should examine this point in more detail. Thus, in the present study, symbols and visuals in addition to gesture are also considered as the components of the mathematical language that students use while proving.

Gestures and Diagrams

As discussed above, both gestures and diagrams are important facets of mathematical thinking and learning. Although they were reflected independently in the literature, there are recent studies that focus on the relationship between gestures and diagrams (e.g., Châtelet, 2000;

de Freitas & Sinclair, 2012). Châtelet (2000), the philosopher and mathematician, discussed characteristics of gestures while interacting with a diagram such as a gesture being flexible, leading to other gestures, and stimulating other gestures. Regarding the relationship between diagrams and gesture, Châtelet argued that diagrams "capture [gestures] mid-flight" and "transfix a gesture" (2000, p. 10).

Drawing on the work of Châtelet, de Freitas and Sinclair (2012) examined how diagrams and gestures are embodied acts through the samples of student diagrams. In fact, de Freitas and Sinclair give a student diagram as an example (Figure 2.5), drawing the movement of a circle, and state that

> "We will not read these diagrams as only representations of mathematical objects (in this instance, possibly two sets of concentric arcs), nor as simply an aid to solving a given problem. Rather, we use Châtelet to help us analyze student diagramming and gesturing as inventive and creative acts by which 'immovable mathematics' can come to be seen as a deeply material enterprise" (2012, p. 134).

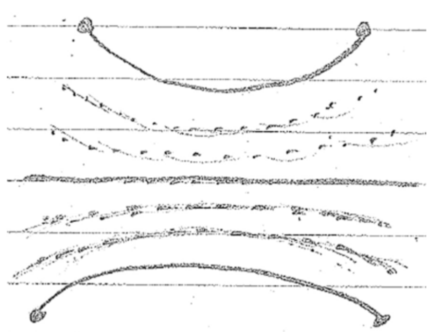


Figure 2.5. Student's diagram with dotted and solid arcs in de Freitas and Sinclair (2012, p. 135).

The same example is used in their later study (de Freitas & Sinclair, 2014) where they stressed the movement of the gesture on the paper. By taking ideas from Châtelet on diagrams and gestures, de Freitas and Sinclair (2012) pointed out that the interaction between gestures and diagrams develops new mathematical concepts. Likewise, Sinclair and colleagues (2016) verified some of Châtelet's assertions by offering evidence through student examples to indicate that gestures and diagrams help to construct ideas in mathematics. Thus, it worth considering the gesture as an important semiotic resource in the diagram construction during the proving process and in using a diagram already constructed in the previously produced works.

Sample Semiotic Frameworks

In consideration of semiotic resources used in a geometry proof, some studies discuss the semiotic phases in students' proofs or develop frameworks to investigate certain representations students produced and engaged. As an example, Arzarello and Sabena (2011) pointed out investigating three semiotic steps in students' works to understand students' reasoning in doing proofs. These steps were:

"1) Interpretation of signs;

2) Identification of relationships between the interpreted signs, and checking with arguments;

3) Elaboration of arguments that explain the 'why': towards a theory" (Arzarello & Sabena, 2011, p.193).

Moreover, Alshwaikh (2008) developed a framework to read the features in diagrams that appeared within students' works or textbooks; however, he suggested investigating the relationships between the verbal and the visual in students' works and the reasons for their choice of using or starting the work either by verbal or visual. Besides, Presmeg (2006) indicated that studying the role of gesture in learning mathematics should take place in the current research along with the semiotic perspective implementation. These studies imply the importance of considering verbal, visuals, and gestures when studying diagrams with a semiotic perspective.

Because the mathematical meanings had a complicated nature and individual discernment can lead to various reasoning types, Godino and Batanero (2003) suggested paying more attention to individual student's interpretations to expand the knowledge on identifying how specific implications from reading diagrams impact students' actions in the processes of geometrical reasoning and proving. Consequently, these studies addressed at least some aspects of the crucial role that individual's interpretation of semiotic resources, such as symbols, texts, visuals (e.g., diagrams), and gesture, has in the process of doing geometry proofs.

Summary

The present study draws from two major areas of the literature—research on geometry learning, with particular focus on reasoning and proof, and research on semiotics, with particular focus on diagrams and gesture. The review of the recent research in mathematics education has shown that there is an increased attention on these phenomena, and studies in mathematics education have documented the contributions of visuals like images and drawing in geometrical reasoning and the challenges that students face in meaningful use of representations in proof production. Therefore, the way that students interpret the diagrams in geometry proofs and the relations with their proving actions is not articulated. Although several studies addressed these issues separately, there has not been an attempt to address the visual, symbolic, and gesture semiotic resources in geometrical reasoning. Additionally, the future research in geometry education suggested by Sinclair and colleagues (2016) implies that there is a need for understanding students' visual reasoning with frameworks that incorporate both reasoning and diagrammatic components. The recent research also implies consideration of different task features such as diagrammatic register that Herbst and colleagues defined (2016).

Considering the current studies and future research calls, this study aims to examine high school students' interpretation and use of diagrams in geometry proofs through the use of semiotic resources and proving actions frameworks. This study investigates how visuals, symbols, and gesture contribute to students' justifications in geometrical proofs by conducting one-on-one clinical interviews with several high school students. Moreover, the tasks used in the clinical interviews include varying features to identify better the relationship between students' proving actions and use of semiotic resources in geometry proofs. The next chapter discusses the research methods and task features in detail. The findings from this study can be useful to both scholars and teachers in geometry education.

CHAPTER 3: METHOD

In this chapter, I describe the design of the study, which is task-based clinical interviews conducted for the purpose of understanding high school students' use of diagrams and their reasoning as they work on geometric claims. Hence, this chapter includes the research questions, research design, data sources and data collection, pilot study reflection, and a description of data analysis procedures. Finally, at the end of this chapter, I present a reliability check of the data coding.

Research Questions

This study investigated how high school students who had taken geometry classes used diagrams in geometry proof tasks and how they engaged with semiotic resources—visual, symbolic, and gestural—in their proving actions. In order to determine students' use of diagrams while working with geometry proof tasks, whether including diagrams or not, the present study is guided by the following research questions:

1. *What semiotic resources (in diagrams) do high school students use to prove geometric claims? How do the semiotic resources relate to the quality of reasoning students provide?*
2. *How do high school students interpret and use geometric diagrams to prove* ***diagram-given*** *geometric claims, and what is the semiotic structure of their proving process?*
3. *How do high school students produce and use (if at all) diagrams to prove* ***diagram-free*** *geometric claims, and what is the semiotic structure of their proving process (whether or not they produced a diagram)?*

The first question is an overarching question asking about students' use of semiotic resources and reasoning they provide when proving geometric claims. The second and the third questions focus specifically on particular types of tasks, diagram-given and diagram-free, and the semiotic structure of students' proofs on these tasks.

Research Design

In this study, I explored high school students' proving actions and use of diagrams in geometry proofs. In considering how to approach this topic, Dimmel and Herbst (2015) stated that geometry diagrams have a potential to convey multiple meanings. That means students may construct different meanings for the same diagram at the same time, and Presmeg (2006) indicated that qualitative research methods were prevalent regarding gaining more information about visual thinking of individuals. Thus, a qualitative research design is appropriate for my study because, as Merriam (2009) suggested, qualitative researchers are interested in people's interpretation of an experience or meaning making with their experiences. Hence, this study employs qualitative methods to examine the semiotic structures of students' use of diagrams and to identify students' proving actions in their geometric work.

This study is situated along semiotic perspective. The semiotic perspective allows me to investigate an activity, working with a geometric diagram, that would reveal the student's thinking during that activity (Ackerman, 1995) because individual thinking has its roots in social factors such as tools, objects, and diagrams. Hence, this affords observations of the role diagrams play within individual thinking and its contribution to socially-constructed knowledge. Furthermore, exploring students' written and spoken language with different semiotic resources is associated with a semiotic perspective (Cobb et al., 1996). A semiotic perspective also allows me to investigate how interactions with a socially-constructed tool (in this case, diagrams) might relate to the meaning constructed by the student (Ackerman, 1995; Ernest, 1996).

Clinical Interview Method

This study employed a clinical interview method for the investigation of high school students' thinking as they proved geometric claims. The clinical interview allows a flexible approach to each individual's thinking because every person is unique (Ginsburg, 1981). In this study, the aim of examining student's thinking is to discover and identify students' use of geometric diagrams and their construction of mathematical ideas as they attempt to prove geometric claims. Mathematical thinking is a complicated process, and any attempts to understand necessarily involve inference-making. Ginsburg (1981) stated that to understand the cognitive processes and to explore the approaches involved in the processes, one may employ a naturalistic observation method, but it includes limitations—for instance, thought is private, usually not appearing in public. In contrast, the clinical interview method with semi-structured and open-ended procedures may afford opportunities for students to express their natural thinking (Ginsburg, 1981; Ginsburg, 1997; Maher & Sigley, 2014).

Seidman (2013) stated that interviews help the researcher access individuals' minds and understand the meaning given by the interviewee to a particular experience. With a clinical interview method, the researcher intends to observe the actions of students as they engage in solving mathematical problems and to gain insights into students' thinking and meaning making during the interview (Ginsburg, 1997; Goldin, 1997). Consequently, with a clinical interview, the sense of students' actions, the student's level of thinking, and thinking processes in geometry could be inferred because they could reflect or explain the meaning of an experience they performed.

Developing tasks that require logical reasoning is a typical procedure for a clinical interview (Maher & Sigley, 2014). Thus, task-based interviews are a form of clinical interviews that are used to understand students’ mathematical knowledge and reasoning. In particular, Goldin (1997) discussed five major characteristics of task-based interviews as follows:

1. Accessibility. Interview tasks should embody mathematical ideas and structures appropriate for the subjects being interviewed.

2. Rich representational structure. Mathematical tasks should embody meaningful semantic structures capable of being represented imagistically, formal symbolic structures capable of notational representation, and opportunities to connect these.

3. Free problem solving. Subjects should engage in free problem solving wherever possible to allow an observation of spontaneous behaviors and reasons for spontaneous choices.

4. Explicit criteria. Major contingencies should be addressed in the interview design as explicitly and clearly as possible.

5. Interaction with the learning environment. Various external capabilities should be provided, which permits interaction with rich, observable learning or problem-solving environment and allows inferences about problem solvers' internal representations.” (p. 61–62)

In this study, there were two sets of task-based interviews. In the first interview, participants took part in a 30–90 minutes interview where a one-on-one relationship occurred between interviewer and each interviewee. Because the interviewees engaged with four tasks during the interviews, participants were given 5 to 10 minutes for each task, based on the time they needed to devote for each proof. However, some students used more than 10 minutes on particular tasks. The first interview occurred once with each participant. Then, a selected subset of participants was invited for a follow-up interview based on initial evaluation of first interview data and students' willingness to participate in the second interview. The second interview had the same general format as the first interview but it involved four new tasks (described below).

To select students for the second interview, I considered students' performances in the first interview. I focused on students who showed a rich and complex use of diagrams, relied on the figural features of a given or produced diagram, and produced less written work but more gestures in their first interview. As an illustration, two female students were invited to the second interview because they provided more text and explanations on the tasks and articulated flexibility in their work on all tasks in the first interview. On the other hand, one of the male students who was invited to the second interview had the shortest first interview in duration, and he did not produce texts or written explanations on the tasks. Instead, he merely employed symbols and gestures on the diagram. That made him unusual among other participating students regarding reasoning techniques and he was invited to the second interview because he did use the diagrams substantially but I wanted to gather more data about his thinking on additional tasks. Moreover, the other male student who participated in the second interview substantially relied on the figural concepts of the diagrams either given or produced in the tasks. Having a second interview with him helped gather more information about the meanings he constructed with diagrams in geometry proofs.

Hence, the purpose of the first interview was to identify all participants' reasoning processes when they worked on the tasks and to give useful information about informing the protocol for the second interview. The goal of the second interview, however, was to present more opportunities to understand students' interpretation and use of diagrams, thinking processes, and reflections on how they perceive the work they produce. Thus, the second interview increased the robustness of the data available to answer the research questions in this study.

The interviews were conducted in the Summer 2016. The interview sessions took place in a quiet meeting room at the University of Missouri campus based upon participant's agreement on being in that location. The first interviews with each participant were finished in 45 days. The second interviews were conducted 10 days after finishing the first interviews and were completed in 14 days.

Participants

Participants for this study were 9 high school students (4 female, 5 male) from two different high schools in the same urban public school district in Missouri. Because the goal of the study was to explore individual student's thinking and to understand their particular interpretation and use of diagrams, the study did not involve a large number of students but instead focused on having long, in-depth interview sessions with each participant. The primary criterion for participant selection was that the student participant had already taken a Geometry class prior to the study so that the interview tasks were accessible. They should have had previous opportunities to learn at least the basic geometric concepts such as congruency, similarity, and right triangles. Given that the Missouri Department of Elementary and Secondary Education adopted the Common Core State Standards, it is expected that students will have learned at the basic level how to construct congruency, similarity, and triangle proofs in geometry. Thus, it was expected that students who have taken the geometry class and attended different grades were able to work with the geometric claims and provided a range of proving approaches on the tasks selected for this study. It was not important, however, that the Geometry class was completed in the academic year immediately prior or more than one year earlier, but rather that it was completed sometime in the past.

A second criterion was to include some variety in the sample of students concerning gender and the teacher or school they had for Geometry class. Although the research questions are not specifically focused on gender or teacher issues, this diversity was likely to provide opportunities to see variations in the ways diagrams were used or the types of reasoning observed.

Student Interview Background

Participants took part individually in the interview session(s) with me, the interviewer. Students provided their background information on a Student Interview Background (Appendix A) form that was given at the beginning of the first interview. The form was created to collect information about student's gender, ethnicity, school, current grade level, and grade level that she/he took the geometry class. The form was also used to record the date and duration of the interviews. Later, this form helped report the above-mentioned issues accurately.

Table 3.1 shows the profiles of the participating students based on the information they provided in the Student Interview Background Form. Students' grade levels varied between 10–12 (ages 16–18), but most of the students were from 12th grade. Students indicated that they had taken geometry one or two years ago, either in Grade 9 or Grade 10. During the first interview sessions, 8 participant students stated that they were successful in Geometry class and passed with a high score; however, only one student, Lee, mentioned that he struggled in Geometry class and did not consider himself as successful as his classmates. All participants identified their ethnicities as White. It is worth noting here that the student names used in this study and appeared in the students' profiles are pseudonyms.

Table 3.1

Profile of the participant students

Participants	School	Grade Level	Geometry Taken in Grade	Gender	Ethnicity
Deborah Roberts	School A	12	9	Female	White
Lydia Upton	School B	12	9	Female	White
Martin Stewart	School B	12	10	Male	White
Megan Utley	School B	12	9	Female	White
Samantha Taylor	School B	11	9	Female	White
Flynn Thompson	School A	12	10	Male	White
Lee Irvine	School A	10	9	Male	White
Ben Jones	School A	12	10	Male	White
Nathan Dixon	School A	12	10	Male	White

The participant students' names in the table are pseudonyms.

Consent to Participate

Because this study did not involve the instructional setting or school and classroom environment of the participants, participants were recruited through personal connections to their parents or friends. In participant recruitment process, I, first, contacted to one of the parents or guardians of the student via email or phone since participants were likely to be under the age of 18. The University of Missouri Campus Institutional Review Board required me to receive the student guardian's permission to provide the details of my research involving their children.

After having the permission, I sent a detailed explanation about my research focus and invited the students to participate my study. Students who were interested in participating my study received a copy of the consent form via e-mail (Appendix B) and were asked to bring the printed copy of the consent form signed by both their parents and themselves. The IRB Office of Research at MU required getting approval from both student and their parents before conducting the interview.

In the consent forms, participants and their parents were informed about the study in detail to help them understand the expectation from participants during the study. The consent form indicated that the study involved task-based interviews with participants and hence, participants were expected to produce work on geometrical proof tasks. Moreover, in the consent form, I informed the students that the purpose of the study was not to evaluate the correctness of students' works which implied that all valid or invalid reasoning were welcomed during the interviews. Thus, the students who volunteered to be part of the study were aware of the nature of tasks and the expectation from them.

During the first interview, participating students were informed that all productions were accepted and hence, I tried to treat all incorrect and correct responses in the same way. Moreover, participants were reminded that they could ask clarification questions about any task or might request skipping one task and work on it later if they desired. I hoped that providing a detailed explanation about the expectation from the participants encouraged the students to feel comfortable with working on geometric proofs, present their thinking without hesitation, and volunteer to be a participant for the second interview. Each participant received a $30 gift card at the end of the attendance of the interview as compensation for their time. Among the participants, 4 participating students (2 female, 2 male) were selected for a second interview based on their works and communication during the first interview.

Data Sources and Data Collection

Tasks

In the clinical interview method, it is possible for a researcher to use tasks that require basic calculations or one step solution to understand student's thinking and meaning-making; however, these tasks may not reveal the complex mathematical thinking of students (Ginsburg, 1981). That said, the focus of the current study, which is how students interpret and use the diagrams given in geometric claims, required a complex task in that matter. Therefore, the tasks that were employed in this study were geometric statements that provided opportunities for justifications and required multiple steps with a complex thinking to prove.

The selection of tasks was based on their relevancy with the CCSSI so that participants were familiar with the concepts included in the claims. As mentioned above, the participants were from the state of Missouri during the time in which they had adopted the Common Core State Standards (CCSSI, 2016). Hence, the tasks were selected based on their alignment with the CCSS listed here.

"Prove geometric theorems

CCSS.Math.Content.HSG.CO.C.9

Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints*.

CCSS.Math.Content.HSG.CO.C.10

Prove theorems about triangles. *Theorems include: measures of interior angles of a* *triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

CCSS.Math.Content.HSG.CO.C.11

Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals*." (CCSSI, 2016, Retrieved from http://www.corestandards.org/Math/Content/HSG/CO/)

The tasks that were used in the first and second interviews were selected and designed according to CCSSI relevancy. In this regard, a pilot study was conducted to test the first interview tasks for this study. Details of the pilot study are discussed below.

Pilot Study Reflection

The main purpose of the pilot study was to test the tasks that would be used in the recent study. The task-based interview protocol was practiced for developing the interview questions and three phases in the data analysis before its use in the actual clinical interviews. The Isosceles, Midpoint, and Right Triangle tasks were pilot tested with two volunteered undergraduate students; one is a freshman and the other is a junior at the University of Missouri-Columbia, two months before the recent study conducted. Both students were white male and graduated from a public high school in Columbia, Missouri. The interviews were conducted at the University of Missouri Main Library and lasted almost 30 minutes for each participant. Pilot study provided opportunities to practice the tasks that would be used in the actual study, producing codes for the analysis of data, and students' reasoning on particular tasks.

First of all, the pilot study allowed a preliminary testing of the Isosceles, Midpoint, and Right Triangle tasks to investigate the approaches and ideas that might be used in the proof processes. The study was helpful to produce the codes catalog (Table 3.5) for semiotic resources in Visual, Symbolic, and Gesture categories such as VF–Draw figure, VAC–Draw angle congruent, VSC–Draw side congruency, SA–Label angle, SE–Writing an equation, SF–Label figure, and GA–Pointing at angle. Indeed, the codes generated in the pilot study appeared in the actual clinical interviews as well. Therefore, the pilot study helped create and develop codes that were used in the actual clinical interviews.

Second, participant's interpretations of diagrams included giving faulty reasoning because of using the claim to be proved as an assumption during the proving process in the Isosceles Task and having a conception of the trick because of unmatched diagram and claim in the Right Triangle Task. As an illustration, during his work on the Isosceles Task, one participant marked and labeled the base angles of the triangle ADE as congruent angles to remind himself what he needs to prove, but later he started using these symbols in his further steps. Additionally, the other student seemed to rely on the scales of the diagrams given in the Right Triangle Task and framed his argument based on the assumptions he made by the given diagram. That is, the student's reasoning in the Right Triangle Task was based on the image provided and accepting the angle was not a right angle. He also asked questions in the Isosceles Task about if the triangle given was equilateral because of its scaled image. It is worth noting that, the pilot study allowed me to realize how students used the figural features of the diagrams in their proofs. Hence, in the actual interviews, I paid particular attention to students reasoning with figural properties of the diagrams and tried to pose follow-up questions to understand better if the student relied only on the figural properties of the diagrams.

Finally, the pilot study showed that students sometimes followed complex steps and usually talked about the steps as they worked on the proofs, but they mostly showed their works on the diagram instead of writing the steps using mathematical text outside the diagram. Therefore, in the actual clinical interview, I encouraged students who asked if they should write down their proofs. That helped me address the participants' reasoning that appeared when they worked on their own at the beginning of each task because some students when they explained their proving verbally, did not provide an appropriate flow of arguments as they did in their written explanation or vice versa.

The First Interview Tasks

As noted above, the first-round interviews involved four proving tasks. These tasks, described in detail below, had different features such as the type of geometric object included in the task and whether a diagram is given (diagram-given) and accurate or not given (diagram-free). The guiding principle for the selection of tasks was the accompanying diagram in the task because the focus of the current study was to understand students' use of diagrams in diagram-given and diagram-free tasks. However, other features of the tasks were also considered to understand better the relationships between students' use of diagrams and types of the tasks. Table 3.2 shows the features of the tasks used in the first interview. The reason for the variation in features was to provide different opportunities for the participants to use (or not use) diagrams in various ways as they tried to prove geometric claims.

Table 3.2

The features of the tasks in the first interview

Task Number	Geometric Object	Truth Known	Diagrammatic Register	Accompanying Diagram
1	Isosceles Triangle	Yes	Yes	Not Given
2	Triangle Midpoints	Yes	Yes	Given/Accurate

3	Right Triangle	No	Yes	Given/Inaccurate
4	Right Triangle	Yes	Yes	Given/Accurate
1	Triangle Midpoints	Yes	No	Not Given
2	Isosceles Triangle	Yes	Yes	Given/Accurate
3	Right Triangle	No	Yes	Given/Inaccurate
4	Right Triangle	Yes	Yes	Given/Accurate

Figure 3.1 illustrated the two different sequences of tasks used in the first interview. The goal of the first interview was to have half of the participants start with a diagram-free Isosceles Task and then a diagram-given Midpoint Task while the other half have diagram-free Midpoint Task first and diagram-given Isosceles Task later. The reason for this ordering was that, by starting with a diagram-free task, I could see whether and how students used diagrams spontaneously. With regard to varying the first diagram-free task, on the one hand, the presentation of diagram-free Isosceles Task allowed me to observe how students spontaneously generated a diagram without seeing a model for a problem with a diagram. On the other hand, the diagram-free Midpoint Task was helpful for investigating whether students produced and used diagrams on tasks that were outside the diagrammatic register (Herbst et al., 2016) and examining the nature of that process. Subsequently, the third task was designed with an inaccurate diagram and truth-unknown case to investigate how students interpreted and used the diagram when the diagram did not match with the hypothesis made in the task. Finally, the fourth task was the same question with the third task except with truth-known feature and an accurate accompanying diagram. Two different sequences of the tasks that were selected for the first task-based interview (Appendix C) had flexible construction and presentation regarding the level of difficulty and format of the task to motivate students.

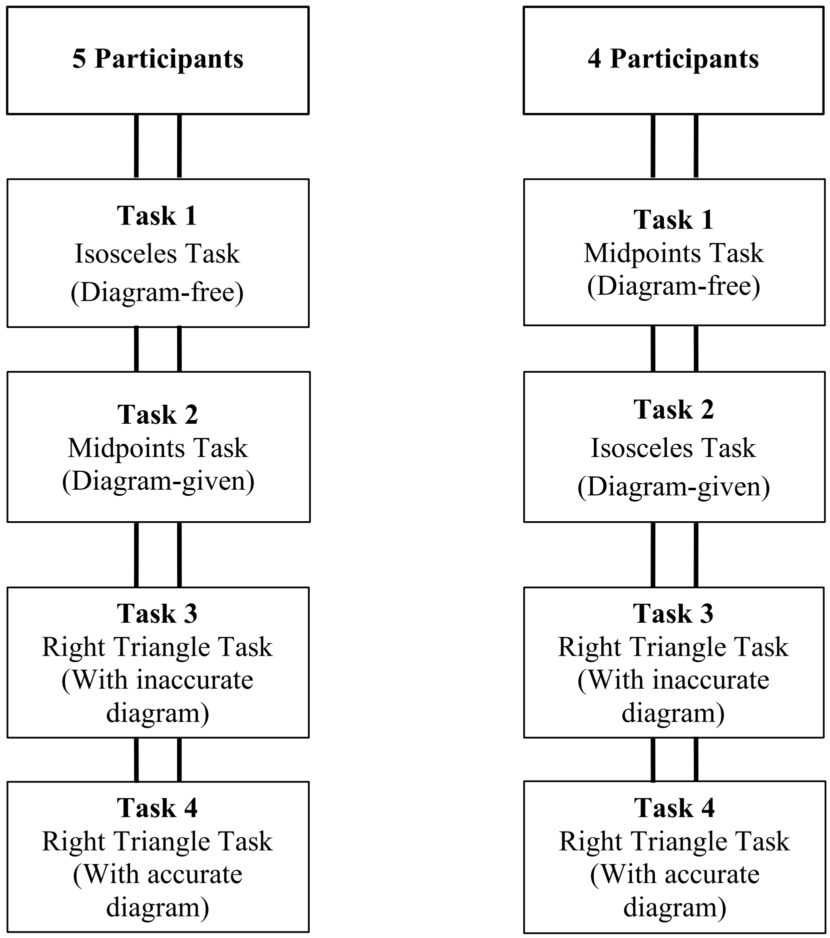
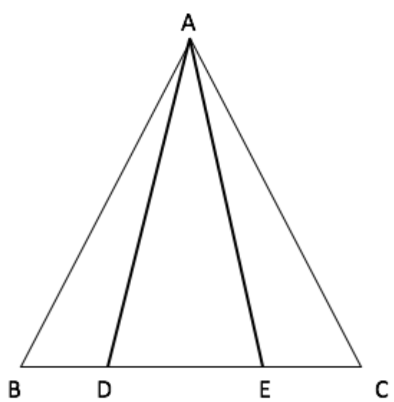


Figure 3.1. Two different sequences of tasks used in the first interview.

One of the tasks that appeared in the first interview was the Isosceles Task, which was used in an earlier study (Senk, 1985). The Isosceles Task aligned with the HSG.CO.C.10 standard because it involved proving that a triangle embedded in an isosceles triangle was itself an isosceles triangle (Figure 3.2). This task was written in both diagram-given and diagram-free formats and was given in one format to each participant. Both formats involved aspects of the diagrammatic register, which implies that the givens and the claim to be proved are accurately presented, labeled, referred to the objects in the diagram, and the task was accompanied by an appropriate diagram in the diagram-given format. However, the diagram-free version was not exactly in the full diagrammatic register because it did not have the diagram. The purpose of giving the same task in diagram-given and diagram-free formats was to gather data that would address both Research Questions 2 and 3.

Isosceles Task

A. Diagram-given



Given: $\overline{AB}=\overline{AC}$ $\quad$ < BAD ≅ <CAE

Prove: Triangle ADE is isosceles.

B. Diagram-free

Let ABC be an isosceles triangle and $\overline{AB}=\overline{AC}$. Let D and E be two points on $\overline{BC}$ such that angle <BAD is equal to angle <CAE. Prove that triangle ADE is isosceles.

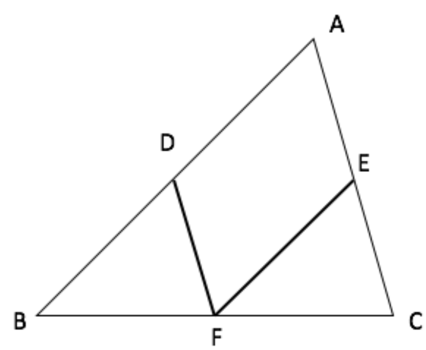
Figure 3.2. Isosceles Task in both diagram-given and diagram-free formats.

The second task was called the Midpoints Task and, similar to the first task, it was written in the diagram-given and diagram-free formats. The Midpoints Task asked students to prove that two triangles constructed with the midpoints of a larger triangle are congruent (Figure

3.3). When the diagram was given in the Midpoint Task, the task was presented in the diagrammatic register (Herbst et al., 2016). However, in the diagram-free format, the Midpoint Task was outside the diagrammatic register because no diagram, no givens such as points or segments, and no proposition to be proved were labeled or referred to the objects in the diagram. The Midpoints Task addressed the HSG.CO.C.10 and HSG.CO.C.11 standards.

Midpoints Task

A. Diagram-given



Given: D, E, F midpoints.

Prove: Triangle BDF is congruent to triangle FEC. (Δ BDF $\cong$ Δ FEC)

***Clue Theorem*:** The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

B. Diagram-free

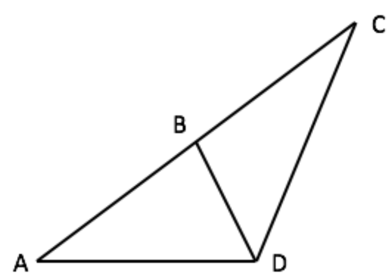
Given a triangle, the midpoint of a side constructs two segments by joining with the other two midpoints on the two other sides of the triangle and hence, two inner triangles occur in the triangle. Prove that these inner triangles are congruent.

***Clue Theorem*:** The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

Figure 3.3. Midpoints Task in both diagram-given and diagram-free formats.

The Right Triangle Task was the third task and asked students to determine whether a triangle was a right triangle based on the given features (Figure 3.4). This task was also used in Senk's (1985) study, but I made slight adaptations to have opportunities to understand better students' reasoning with a mismatched diagram. Specifically, the task included a diagram that was drawn as an obtuse triangle and, hence, challenged students regarding reasoning with an inappropriate figure. That is, the diagram was given with attributes that did not precisely match the given information (because AB appeared to be greater than BD). Moreover, the task was presented in truth-unknown format, forcing the student to decide whether it was always true, sometimes true, or never true, and providing an opportunity for the students to use the diagram in making that determination. Right Triangle Task aligned with the HSG.CO.C.10 and HSG.CO.C.11 standards.

Right Triangle Task



If B is the midpoint of $\overline{AC}$ and $\overline{AB}$=$\overline{BD}$.

It is claimed that <CDA is a right angle.

The claim is;

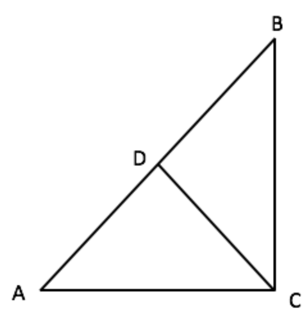
Circle one: Always True Sometimes True Never True

Prove:

Figure 3.4. Right Triangle Task with an inaccurate diagram and truth-unknown format.

Finally, the fourth task was the Right Triangle Task and, as is evident from its name, this task had a similar concept with the third task (Figure 3.5). The only changes that were included in the fourth task were the appropriate diagram and the truth-known format of the task. The goal of presenting this task was to understand students' reasoning on the task that had a similar claim as the previous task, but slight changes in the features.

Right Triangle Task



Given $\overline{AD} = \overline{DB} = \overline{DC}$, prove that angle <BCA is right angle.

Figure 3.5. Right Triangle Task with an accurate diagram and truth-known format.

The Second Interview Tasks

The second interview tasks were designed similar to the first interview tasks. One purposeful difference, though, was that the second interview tasks involved more variation with respect to the diagrammatic register. In particular, three non-diagrammatic register tasks were included in the second interview to better observe students' reasoning and use of diagrams on these tasks. Additionally, the second interview included a task that had a geometric object other than a triangle. Table 3.3 illustrates the features of the second interview tasks.

Table 3.3

The features of the tasks in the second interview

Task Number	Geometric Object	Truth Known	Diagrammatic Register	Accompanying Diagram
1	Triangle Midpoints	Yes	No	Not Given
2	Right Triangle	Yes	Yes	Given/Inaccurate
3	Isosceles Triangle	Yes	No	Given/Accurate
4	Pentagon	No	No	Not Given

The four participants who were invited to the second interview took the same sequences of tasks (Appendix D). Figure 3.6 shows the sequence of the tasks in the second interview.

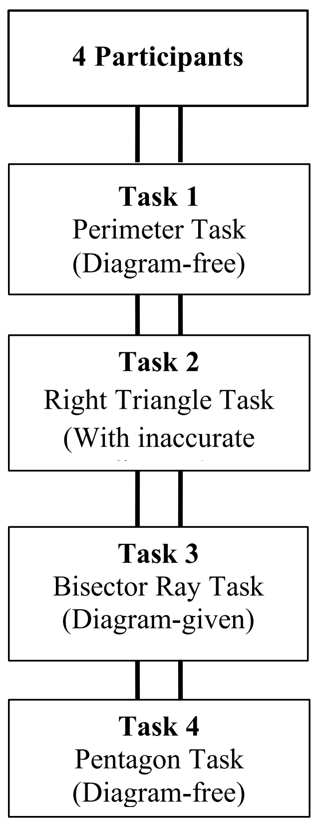


Figure 3.6. The sequence of tasks used in the second interview.

The first task that was given in the second interview was the Perimeter Task (Figure 3.7). The task was a modified version of the diagram-free Midpoints Task from the first interview in the sense that the geometric situation was similar. But instead of asking about the congruency of two inner triangles, the Perimeter Task asked students to compare the perimeter of two triangles. In this case, the problem was not presented in the diagrammatic register.

Perimeter Task

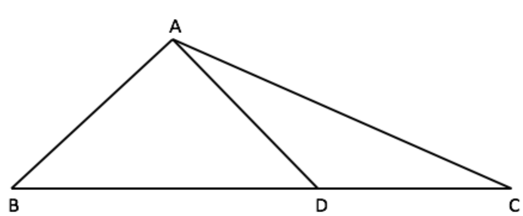
Given a triangle, the midpoint of any side constructs two segments by joining with the other two midpoints on the other two sides of the triangle and hence, one inner triangle occurs in the center of the triangle. Prove that the perimeter of the center triangle is half the perimeter of the given triangle.

***Clue Theorem*:** The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

Figure 3.7. Perimeter Task in non-diagrammatic register format.

The second task in the second interview was the Right Triangle Task, which was also similar to the Right Triangle Task in the first interview. Because most of the students failed to prove the task in the first interview, there was additional information given in the task for the second interview. Figure 3.8 represents the Right Triangle Task used in the second interview.

Right Triangle Task



Δ ABD is equilateral and D is the midpoint of BC. Prove that Δ ABC is a right triangle.

Figure 3.8. Right Triangle Task used in the second interview.

The Bisector Ray Task was the third task used in the second interview (Figure 3.9). Compared to the prior non-diagrammatic register tasks, this task had a different version of the non-diagrammatic register because, although there was a diagram given in the task, there were no labels or signs given in the written statement of the task that was referring to particular segments or parts of the given diagram. Hence, the goal was to explore students' approach to that kind of task.

Bisector Ray Task

Given a triangle, if an angle bisector ray emanating from one vertex meets the opposite side at a right angle, then prove that the overall triangle is isosceles.

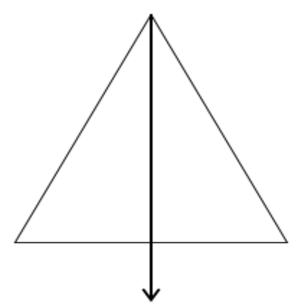


Figure 3.9. Bisector Ray Task used in the second interview.

Finally, the fourth task given in the second interview was the Pentagon Task (Figure 3.10). The goal of this task was to give participants a task that is both in truth-unknown and non-diagrammatic register formats. Moreover, the geometric object—a pentagon—was different than the triangles involved in the other tasks.

Pentagon Task

It is claimed that the sum of interior angles of a pentagon is 540.

The claim is;

Circle One: Always True Sometimes True Never True

Prove:

Figure 3.10. Pentagon Task used in the second interview.

Participants received the first and second interview tasks in the order indicated in Appendix C and D. Participant students had 5–10 minutes for each task to work silently, if they wished. When they became ready to share out, participants explained the proof (or attempted proof) they produced and the reasoning for the arguments they had in each task. After students finished working and explaining their work on a task, they were given the next task. Students were allowed to skip a task if they did not want to work at that time and were given time to work on the skipped task at the end if they wanted. However, during the first and second interviews, no participant refused to work on any of the tasks.

Interview Protocol

Ginsburg (1981) argued that giving an open-ended task and asking follow-up questions based on the student's attitude and explanations on these tasks are helpful techniques that can be used in the clinical interview method. That is, the purpose of interviews is to facilitate productive communication with the participant and reveal the verbalization of thinking processes. With a clinical interview approach, although the researcher attempts to use some common questions, the follow-up questions may differ for each student because of their responses to the questions (Ginsburg, 1997). In the current study, the interview protocol was semi-structured in the sense that it included four major parts, with each having one of the four tasks described above. The interview aimed to understand or reveal a student's reasoning and use of symbols and diagrams in different tasks (diagram-given and diagram-free). Hence, in this study, although all students were asked to explain their proof after they worked on their own for a while, depending on the personality of the student and based on the student's work, the follow-up questions varied for each student.

As an illustration, some general I used for each student were as follows: "Can you explain me what you did?", "Why do you think that?", "How do you say that?", "Would you like to talk about what you have so far?". Moreover, follow-up questions sometimes varied among students based on their different approaches to the task. That is, students' written explanations, the type of diagram they drew, or the way they used or labeled the diagram differed. Thus, some sample follow-up questions used in this study were: "Can you explain me what you mean here [pointing to student's written work]?", "When you say the points' distances from the edges are the same, which points you are referring here [Pointing to student's work]?", "How do you think they would be congruent in that case [referring to his first drawing], but not in that case [referring to his second drawing]?", "Why did you prefer labeling the diagram?".

As aforementioned, each interview contained four tasks presented above. The tasks required students to write or explain the proofs. The only difference between the task-based interviews was the sequence of the tasks in the first interview. The type of the diagram-given and diagram-free tasks varied among the individuals, but every participant first had a diagram-free and then a diagram-given task in the first interviews (Appendix C). In the second interview, participants received the same sequence of tasks (Appendix D).

Student Work

One of the primary data sources gathered from a task-based interview is the representations produced by the participants to show their thinking (Goldin, 1997). Because this study intended to capture students' use of diagrams in geometry proofs, the written work they produced by using the pencil and paper was as significant as the reasoning they provided orally. Hence, I collected the students' written work on the task paper given to them during the interviews. By collecting students' written works, I tended to investigate better (a) the symbolic resources (e.g., labeling, geometrical symbols, mathematical calculations), (b) the visual resources (e.g., drawing a new diagram or parts of the diagram), and (c) the written explanations students created on each task. Moreover, even the things students crossed out were captured in students' written work. In this respect, the copies of students' written works on each task provided a unique contribution to the process of data analysis.

Video Recordings

Video recordings are another major data source collected in this study. All the interview sessions were video recorded with two cameras—one camera focused on the student's paper to record the actions that the student produced on the tasks, the other camera capturing a broader view including the student and his/her work on the tasks together. The video files provided an accurate record of (a) how the student worked on the tasks by using symbolic and visual resources, (b) what was stated, and (c) what kind of gestures students performed during the proving process. Thus, the video recordings were the primary source to produce transcriptions and screenshots that were chiefly used in the data analysis.

Gee (2014) mentioned that a transcript is an essential part of the analysis that supports the validity of the study by working with other components of the analysis. *MAXQDA*, the software for data analysis, was used to create transcripts of the video recordings. As the researcher, I transcribed the video recordings of the student's verbalizations when they worked on tasks. In the transcription, I also reported student's drawings, symbol use, and gestures in brackets ([…]) and occasionally incorporated images of the diagram where student's performed gestures. The images were assigned to the lines in which relevant utterances occurred during that time (Norris, 2002). Figure 3.11 illustrates a sample transcription in which gestures and images were included.

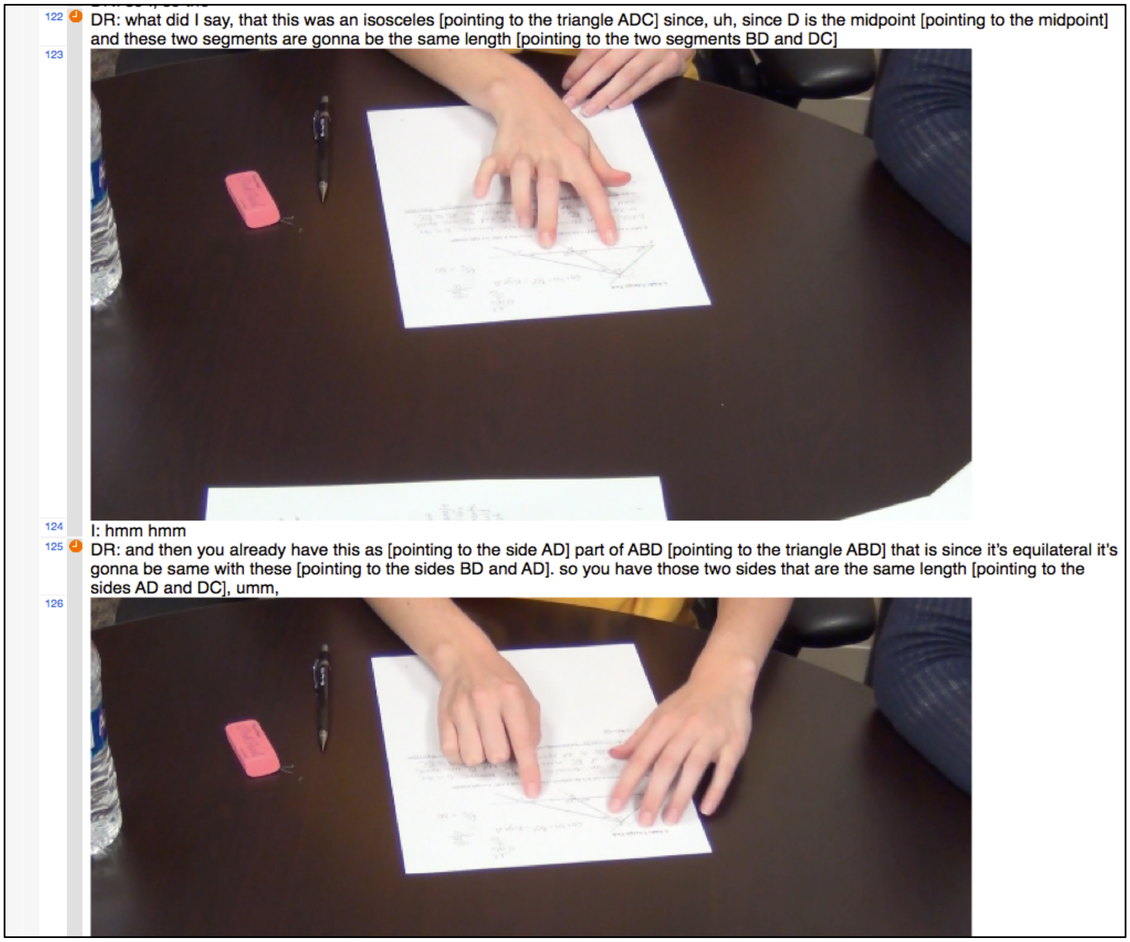


Figure 3.11. A sample transcription with images and reports of gestures in brackets from videotapes.

Data Analysis

As described above, in a clinical interview, there usually are various forms of data such as a rich verbalization, participant's activity on particular tasks, and many verbal reflections (Ginsburg, 1981). Thus, in this study, I used multi-semiotic mathematical discourse analysis to address the research questions because the multi-semiotic discourse analysis includes student's actions, representation of the mathematical entities either with symbols, signs, or written text, and the mathematical language that student speaks as a resource for explaining their thinking

(O'Halloran, 2008). It is important to consider all these aspects of the discourse together so that one can understand what meanings students draw from the diagrams by articulating various modes of the discourse.

Data Analysis Phases

Because this study focused on the links between students' interpretation and use of diagrams and students' geometrical reasoning, the method of coding considered the activity of students when they did proofs. Data analysis of this study included three phases to answer the research questions. These three phases addressed the meanings students construed with the diagrams by using the semiotic resources in the tasks and the analysis of students' geometrical reasoning by attending to their proving actions. In particular, Phase 1 focused on coding the symbolic, visual, and gesture semiotic resources in student's proving activity. Phase 2 focused on the proving actions such as steps, justifications, or refinements students applied in their proofs and if the proving actions were valid, limited, or wrong. Finally, Phase 3 involved the comparison of semiotic resources and student's proving actions by checking them in student's arguments for each task. Thus, semiotic resources coded in Phase 1 and the proving actions coded in Phase 2 were considered simultaneously in Phase 3 to understand the relationship between them. With Phase 1 and 2, I tried to answer Research Questions 2 and 3. Then, Phase 1, 2, and 3 together helped me answer the overarching question of Research Question 1. Below, each of the three phases is described in detail and Table 3.4 summarizes them.

Table 3.4

Three phases of data analysis

Phases	**Coding Activity**	**Findings**

Phase 1	Coding semiotic resources by using a framework that includes visual, semiotic, and gesture categories and types of codes under each category.	Identifying each of the symbols, visual, and gesture that student employed in the proof production.
Phase 2	Coding students' proving actions by using the framework developed by Otten et al. (2017).	Identifying valid, limited, or wrong proving actions such as steps, justifications, or refinements in students' proof work.
Phase 3	Comparing both Phase 1 and Phase 2 codes in the same proof work to find the frequencies of code types and relationship between them.	Identifying the common proving actions and semiotic resources employed by each participant under each task.

Phase 1

Phase 1 involved multiple semiotic resources that students displayed when they worked with the tasks. The proving students provided for each task coded under three main categories of the Visual, Symbolic, and Gesture semiotic resources. Regarding the semiotic resources code systems, the verbatim transcription data, as described above, was used to code what students implemented on the diagrams in their proofs. Table 3.5 shows various types of sub-codes under Visual, Symbolic, and Gesture categories.

Table 3.5.

Codes for semiotic resources in visual, symbolic, and gesture.

Codes for Three Semiotic Resources		
Visual	**Symbolic**	**Gesture**
VF–Draw a new figure/diagram	SF–Label figure/diagram	GF–Pointing at figure/diagram
VS–Draw side/segment	SS–Label side/segment	GS–Pointing at side/segment
VL–Draw line	SL–Label line	GL–Pointing at line

VP–Draw point	SP–Label point	GP–Pointing at point
VA–Draw angle marking	SA–Label angle	GA–Pointing at angle
VAC–Draw angle congruency	SV–Label vertex	GV–Pointing at vertex
VH–Draw hatch/tick marks		GH–Pointing at hatch/tick marks
VY–Draw ray		GY-Pointing at ray
	SE–Writing an equation	GC–Pointing at a calculation in the work
VR–Redraw figure	SSE–Solving an equation	GM–Referring to movement
VSC–Draw side/segment congruency	SG–Use geometric symbols	GT–Turning the paper
	SAS–Writing algebraic symbols	GPA–Acting to show perpendicular
		GG–Pointing at a given in the task
		GW–Pointing at something in the work

The first interview tasks were pilot tested with two undergraduate students because students at the university campus were more accessible than high school students to the researcher. Nearly all of the codes for visual, symbolic, and gesture resources were adequately captured and developed based on the actions that students in the pilot study carried out concerning the semiotic resources. The codes for Visual resources included what students drew either on the diagram or as a new figure. For example, when a student drew an angle marking on the base angles of the isosceles triangle, that moment in the transcription was coded as VA–Draw angle marking. Similarly, if the student connected two midpoints on the two sides of the triangle and constructed a segment inside the figure, then that part in the transcription was coded as VS–Draw side/segment. Likewise, the codes for Symbolic resources included the letters, symbols, signs that express a mathematical relation or representation. For example, when a student labeled a vertex of the triangle he/she drew, that was coded as SV–Label vertex, or when the student labeled a side of the diagram with a number or a letter, it was coded as SS–Label side. Finally, in the current study, the gesture was considered as—*the hand and arm movements that accompany speech*—(Sabena, 2008; Williams-Pierce et al., 2017). In particular, I coded the student's hand and arm movements that appeared in the diagram or the proof work. For instance, when a student pointed to a side of a diagram, a point on one side of the diagram, or an angle inside the diagram, they were coded as GS–Pointing at side/segment, GP–Pointing at point, or GA–Pointing at angle, respectively.

I used *MAXQDA*, qualitative data analysis software, to code semiotic resources in each transcription of the videotapes. In the transcriptions, I focused on the parts where students explained their proofs for each task. Therefore, the parts where students made general statements about overall tasks or talked about the things that were not related to the particular task such as the teacher or the geometry class they had in the school were excluded. For each task, by using a feature of the *MAXQDA*, I tallied up totals of semiotic resources for each student and for students who worked on the same task to get an overall sense of semiotic resources used in that particular task. Figure 3.12 shows a sample excerpt from a transcription coded with semiotic resources in *MAXQDA*.

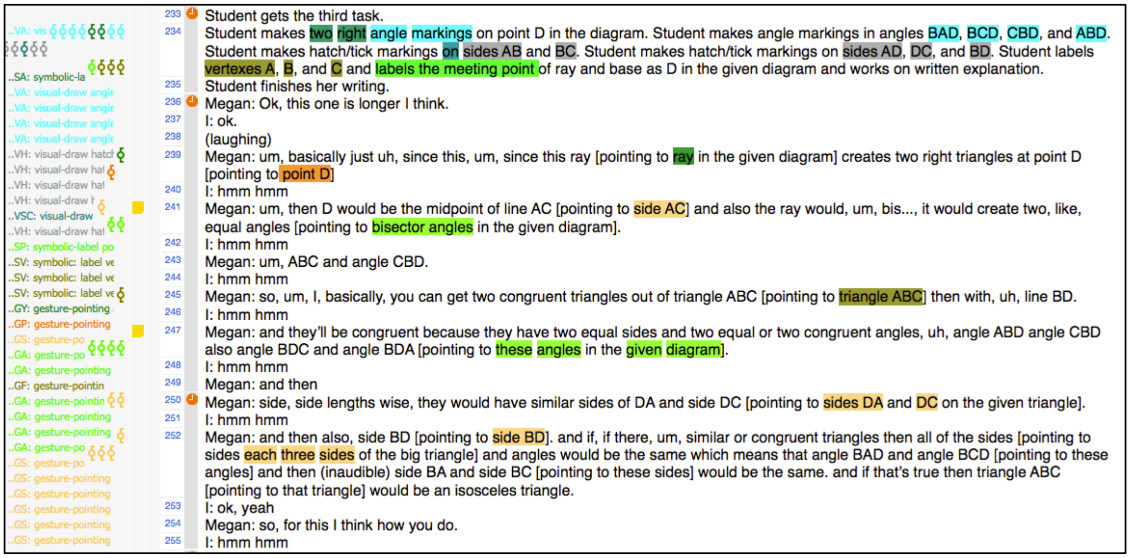


Figure 3.12. A sample excerpt coded with semiotic resources in *MAXQDA*.

Phase 2

Phase 2 involved identifying the proving actions that students employed during their work on the tasks. The purpose of this phase was to describe the steps involved in student's proof activity and to assess some degree of the validity of student's reasoning so that I might later be able to inspect whether the diagram use seemed to be helping our hindering their reasoning. To code the proving actions that were happening moment by moment during the students' works on the tasks, I adapted the coding framework developed by Otten and colleagues (2017). Table 3.6 presents the codes for identifying student's actions that appeared during the process of proving.

Table 3.6

Codes for analyzing the actions in proving process (Adapted from Otten et al., 2017)

Proving Actions					
CLAIM	Stating the overall claim to be proved or disproved	INVEST	Investigating or guessing the truth value of a problem	REFINE	Making a refinement or modification of a claim

STRUC	Identifying the structure of the argument or proof	END	Stating the end of the proof	ACCEP	Stating an accepted definition or previous result
SUM	Summarizing all or part of the argument	STEP	Providing a step in the argument	JUST	Justifying a step in the argument
ABOUT	Making a general statement about a proof				
		Proving Interactions			
CLAR	Clarifying claim or arguments	CRIT	Critiquing claims or arguments	CONF	Confirming the validity of arguments

The codes for proving actions were used to address students' proving works, and the codes for proving interactions were helpful to point out the students' interactions with the actions they already took in the proof work. For example, a student might provide a step (e.g., marking the congruent segments) in the argument and give a reasoning for that step (e.g., I marked these two segments because they are congruent). The student's previous action was coded as STEP, and her/his explanation for that action was coded as JUST. Regarding the proving interactions codes, when student provided a clarification, critique, or confirmation of an action, the action was coded with both of the codes. That is, the student's critique of a justification she/he made earlier was coded as CRIT-JUST. It is also worth stating that students' actions were not always precise and valid moves. Hence, applying Otten and colleagues' (2017) nomenclature, if some proving actions involved "wrong" or "limited" reasoning, the codes for those actions ended with W or L, respectively. For instance, a wrong step (e.g., marking two non-congruent segments as congruent) was coded as STEP-W. Limited steps or justifications refer to those that are not incorrect but not precisely correct either. Furthermore, students also employed visual, symbolic, or gesture resources when they made an argument. Because almost all proving actions were reported in the transcriptions (e.g., pointing to a side, drawing hatch/tick marks to show congruency, labeling a point on the diagram), the Visual, Symbolic, and Gesture semiotic resources were coded with the proving actions by adding V, S, or G letters to the end of the codes respectively. For example, if a student offered a step and used gestures to make a justification (e.g., these three sides are congruent [pointing to three segments in the triangle]) the report of the gesture in the transcription was coded as JUST-G. Figure 3.13 illustrates an example of the Phase 2 coding in *MAXQDA*. As similar to Phase 1, for each task, I tallied up totals of proving actions for each student and students who worked on the same task to understand salient proving actions that appeared on each task.

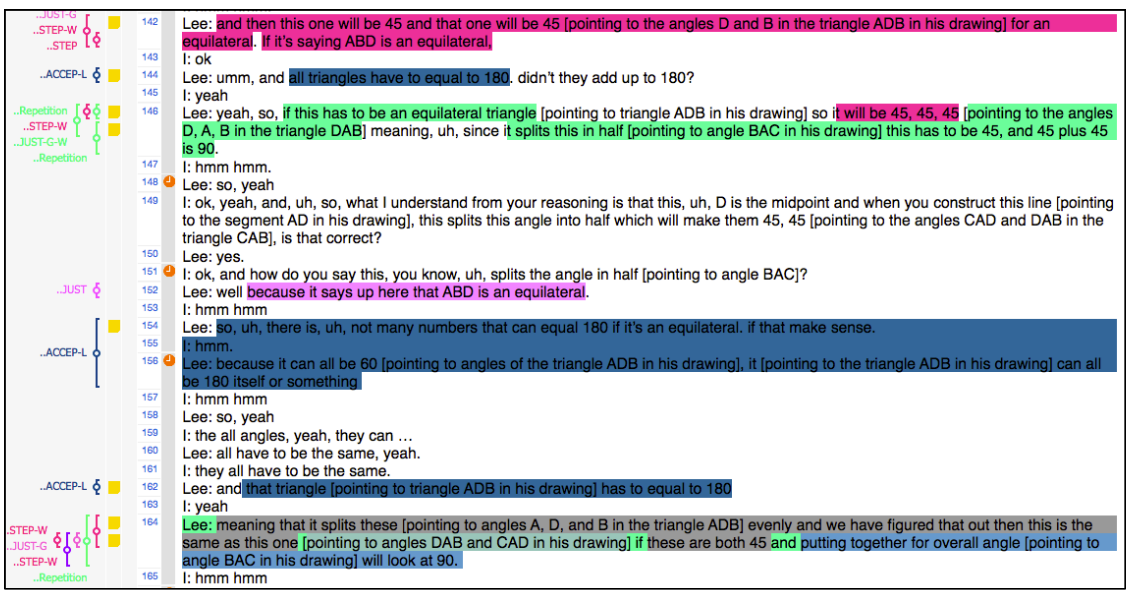


Figure 3.13. A sample excerpt coded with proving actions in *MAXQDA*.

Phase 3

Phase 3 involved considering both Phase 1 and 2 to investigate the relationship between students' use of semiotic resources in geometrical proofs by using diagrams and the argument and reasoning they developed during the proof work. Namely, this phase focused on each task individually and compared both the semiotic resources and the proving actions that students produced in each task. Thus, in Phase 3, by looking across students, I considered the range of semiotic resources and proving actions that occurred under each task, and I looked for patterns between those two dimensions.

Table 3.7 shows Phase 3 analysis of one task in a non-diagrammatic register, the Perimeter Task, by giving the frequencies of particular codes that four students employed under three semiotic resources and the proving actions.

Table 3.7

A sample Phase 3 analysis of four students on Perimeter Task

Semiotic Resources and Proving Actions	**Code Frequencies in Perimeter Task**
Visual\VS: visual-draw side/segment	18
Visual\VH: visual-draw hatch/tick marks	9
Visual\VR: visual-redraw figure	1
Visual\VF: visual-draw a new figure/diagram	6
Visual\VP: visual-draw point	12
Visual\VA: visual-draw angle marking	1
Visual\VSC: visual-draw side congruency	1
Gesture\GS: gesture-pointing at side/segment	64
Gesture\GG: gesture-pointing at a given in the task	7
Gesture\GP: gesture-pointing at a point	10
Gesture\GF: gesture-pointing at figure/diagram	10
Symbolic\SV: symbolic: label vertex	6
Symbolic\SP: symbolic-label point	6
Symbolic\SS: symbolic-label side/segment	4
Symbolic\SA: symbolic-label angle	1
Phase 2 Codes\CLAR-CLAIM	2
Phase 2 Codes\CLAR-STEP-W	2
Phase 2 Codes\CLAR-STEP	2
Phase 2 Codes\CLAR-JUST-S	1
Phase 2 Codes\CLAR-INVEST	1
Phase 2 Codes\ABOUT	5
Phase 2 Codes\INVEST	6
Phase 2 Codes\JUST	9
Phase 2 Codes\JUST-L	2
Phase 2 Codes\JUST-G	14

Phase 2 Codes\JUST-G-L	2
Phase 2 Codes\JUST-G-W	4
Phase 2 Codes\STEP	13
Phase 2 Codes\STEP-L	2
Phase 2 Codes\STEP-W	2
Phase 2 Codes\ACCEP-L	1
Phase 2 Codes\REFINE	1
Phase 2 Codes\CLAIM	1
Phase 2 Codes\Repetition	3

Although the Perimeter Task was given in non-diagrammatic register, the table 3.7 shows that students used visual resources such as VF–draw a new figure/diagram and VP–draw point. Moreover, it is apparent that gesture resources are the most common semiotic resources used in the Perimeter Task. As far as the proving actions employed in the Perimeter Task, as shown in above table, most of the justifications made by students were through gestures. Although Table 3.7 lists the patterns of semiotic resources and proving actions in one particular task, the patterns of semiotic resources and proving actions were considered across various tasks, as well.

Reliability

In a qualitative study, reliability of the data analysis process is significant to ensure that the results of the study are trustworthy (Merriam, 2009). In this study, the initial semiotic resources framework (Table 3.5) was developed during and after the pilot study. The framework, then, was used to analyze the semiotic categories in students' proof production. After coding one-third of the data, my dissertation supervisor coded two different transcriptions of students' proof work by using the semiotic resources framework. We coded individually and then met to compare our codes. Although my dissertation supervisor and I reached almost 95% agreement on the coding, we still negotiated on the discordant codes and through our discussion refined some of the code definitions. For example, VL–Draw line code was used as VLS–Draw line segment initially. However, in order to make a clear distinction for how to code a line and a segment that a student drew in the proof work, we decided to use the code VL–Draw line for the lines that extend in opposite directions, and the code VS–Draw side/segment for the sides or segments that occur between two points.

The proving actions framework (Table 3.6) I used in Phase 2 of the data analysis was adapted from the research study of Otten, Bleiler-Baxter, and Engledowl (2017). The third author of the study, Christopher Engledowl, who was also a fellow doctoral student, became the second coder to check the reliability of the codes by using the proving actions framework. He coded transcriptions of three different students' proof work. Similar to the process of the reliability check for Phase 1, he and I coded the data independently and then compared our codes. Although we had some disagreements on the codes, we had 89% alignment on our coding, and we resolved our disagreement in the assigned codes. Consequently, by checking the coding for both Phase 1 and Phase 2 with second coders, I ensured a high percentage of coding reliability in the data analysis.

Summary

Throughout this chapter, I discussed the research design, data sources and data collection, and data analysis procedures. The ultimate purpose of this study is to understand the high school students' use of diagrams with semiotic resources and its relation to students' reasoning quality. Hence, the selection of participants and conducting one-on-one clinical interviews with them by using purposefully selected tasks allow a reasonable rationale for the investigation. Moreover, applying three phases in the data analysis with unique contributions of each phase to better answer the research questions, and the reliability check of coding with second coders support the rigor and trustworthiness of this study. Next, in the following chapter, I discuss the research findings.

CHAPTER 4: FINDINGS

This chapter presents the results of the study and is organized by two main types of tasks, diagram-given and diagram-free, that students worked during the first and second interviews. Therefore, I lay out the findings in two major sections. In the first section, I discuss how students engaged in diagram-given tasks in the first and second interviews. In the second section, I describe the student work on the diagram-free tasks in the first and second interviews. The diagram-given or diagram-free tasks varied with regard to the diagrammatic register, whether the truth of the claim was known, and whether the diagram provided was accurate. In particular, in the first interview, there was only one non-diagrammatic register task which showed noteworthy data such as no diagrammatic register appeared in students' proof work. That intrigued me to consider more non-diagrammatic register tasks in the second interview and gather more data about them. Therefore, in reporting the results, themes related to diagrammatic register and truth known task features are demonstrated based on sample works of the students who took the task. In the representation of the results, frequency counts for the semiotic resources and proving actions, as well as representative excerpts from the interviews, are used.

Recall the research questions that guided this study:

1. *What semiotic resources (in diagrams) do high school students use to prove geometric claims? How do the semiotic resources relate to the quality of reasoning students provide?*
2. *How do high school students interpret and use geometric diagrams to prove* ***diagram-given*** *geometric claims, and what is the semiotic structure of their proving process?*
3. *How do high school students produce and use (if at all) diagrams to prove* ***diagram-free*** *geometric claims, and what is the semiotic structure of their proving process (whether or not they produced a diagram)?*

The results presented in the first section relates to the Research Question 2 and the findings presented in the second section relates to the Research Question 3. Finally, the summary of both sections addresses the first research question, which is the overarching research question.

Diagram-Given Tasks

In this section, I present the findings related to five diagram-given tasks from the first and second interviews. Although each participant received at least two tasks with an accompanying diagram in the first and second interviews, there were differences in the other features of the tasks such as the geometric object accompanying the task and truth-known features. Table 4.1 shows different features of the diagram-given tasks from the first and second interviews and the number of students who worked on each of them. As described in Chapter 3, there were two sequences of tasks in the first interview. That is, the first two tasks differed in two sets of four tasks. Hence, the number of students who worked on particular tasks differed in the first interview.

Table 4.1

Diagram-given tasks from the first and second interviews and number of students worked on them

Interview	Task Name	Task Features	Number of Students
1st Interview	Isosceles Task	with an accurate diagram truth-known in diagrammatic register	4
1st Interview	Midpoints Task	with an accurate diagram truth-known in diagrammatic register	5
1st Interview	Right Triangle Task	with an inaccurate diagram truth-unknown in diagrammatic register	9

1st Interview	Right Triangle Task	with an accurate diagram truth-known in diagrammatic register	9
2nd Interview	Right Triangle Task	with an inaccurate diagram truth-known in diagrammatic register	4
2nd Interview	Bisector Ray Task	with an accurate diagram truth-known in non-diagrammatic register	4

Truth-Unknown Task

Right Triangle Task (Inaccurate Diagram, Truth-Unknown)

The Right Triangle Task was given with an inaccurate diagram, the truth unknown, and diagrammatic register features (see Table 4.1). It was offered as the third task in the first interview and all the students attempted the task. Below, I present the semiotic resources students employed, the way students used the diagram, and proving actions based on the semiotic resources and diagram use in students' reasoning processes.

Semiotic Resources Used in the Right Triangle Task

Gesture

Gesture resources were the most commonly used semiotic resources in the Right Triangle Task. As shown in Table 4.2, students frequently pointed at the diagram or sides and angles of the diagram.

Table 4.2

Semiotic resources that nine students employed while proving truth-unknown Right Triangle Task

	Semiotic Resources	Frequencies of the Codes									
		Martin	**Lydia**	**Nathan**	**Samantha**	**Megan**	**Lee**	**Deborah**	**Flynn**	**Ben**	**Total**
Visual	VS: visual-draw side/segment	3	6	1	2	0	0	2	0	0	14
	VH: visual-draw hatch/tick marks	0	0	6	0	6	0	4	0	3	19
	VL: visual-draw line	0	0	0	0	0	2	0	0	0	2
	VF: visual-draw a new figure/diagram	3	12	2	1	4	0	3	1	0	**26**
	VP: visual-draw point	2	2	0	0	0	0	1	0	0	5
	VA: visual-draw angle marking	2	0	1	0	2	3	0	0	4	12
	VSC: visual-draw side congruency	0	0	3	0	2	0	0	0	1	6
	Total	10	20	13	3	14	5	10	1	8	84
Gesture	GS: gesture-pointing at side/segment	2	67	2	10	11	12	15	2	16	137
	GPA: gesture-acting to show perpendicular	0	0	0	0	0	1	0	0	0	1
	GM: gesture-referring to movement	0	0	0	0	0	0	1	0	0	1
	GV: gesture-pointing at vertex	1	0	0	0	4	0	0	0	0	5
	GH: gesture-pointing at hatch/tick marks	0	0	0	0	0	0	2	0	0	2
	GT: gesture-turning the paper	0	0	2	0	0	0	2	0	0	4
	GG: gesture-pointing at a given in the task	1	2	1	0	4	0	1	0	0	9
	GP: gesture-pointing at a point	0	2	1	0	0	0	0	0	1	4
	GF: gesture-pointing at	7	1	11	0	12	0	6	0	3	40

	figure/diagram										
	GA: gesture-pointing at angle	8	18	5	0	23	22	4	0	25	105
	Total	19	90	22	10	54	35	31	2	45	**308**
Symbolic	SV: symbolic: label vertex	10	0	9	0	0	0	0	3	0	22
	SF: symbolic-label figure	0	0	3	0	0	0	0	0	0	3
	SP: symbolic-label point	4	0	1	0	0	0	0	0	0	5
	SS: symbolic-label side/segment	2	6	0	0	0	3	0	0	0	11
	SG: symbolic-use geometric symbols	0	0	0	0	1	0	0	0	0	1
	SA: symbolic-label angle	2	1	1	0	2	1	0	0	0	7
	SE: symbolic-writing an equation	0	0	0	0	1	0	0	0	0	1
	Total	18	7	14	0	4	4	0	3	0	50

Ben's gestures on the given figure exemplify a common use of gesture resources on the diagrams in the Right Triangle Task.

> *Ben*: OK so, I guess it would be true if *this angle* [*pointing to angle BDA*] equals 45 degrees, and that's the only way it makes it true.
>
> *Interviewer*: OK.
>
> *Ben*: Because since *these are all the* [*pointing to inner angles of the triangle ACD*], since *this is the midpoint* [*pointing to point B*], that means *this is the same as that* [*pointing to sides AB and BC*]. It says that this is same as the, *BD is the same as AB* [*pointing to sides BD and AB*]. So that means *all three of these lines* [*pointing to sides BD, AB, and BC*] are the exact same length, *and this is an isosceles triangle* [*pointing to triangle BAD*] which means *these angles have to be the same* [*pointing to angles BAD and BDA*]. And *this will be the exact same isosceles triangle* [*pointing to triangle BDC*], I guess.

Ben's explanation of his proof on the Right Triangle task involved gestures together with his justifications as he specified the particular sides or angles that the pronouns referred to in his verbal explanation. Note that sometimes he gestured in conjunction with the labels from the diagrammatic register (e.g., "BD is the same as AB") but more often he gestured and did not use the explicit labels (e.g., this angle [pointing to angle BDA] equals 45-degrees"). That type of gesturing was common among almost all the students when they explained their proofs in the task.

Symbols and Visuals

In the Right Triangle Task, in addition to gesture resources, students actively applied to the visual and symbolic semiotic resources (See Table 4.2). One semiotic resource that regularly occurred in this task was drawing a new figure/diagram in the proof work. In fact, seven out of nine students drew a new diagram to supply their reasoning in the task. In the drawings, some students labeled the points or vertexes on the new diagrams, but others did not actively use symbols to name particular parts of the diagram they sketched. As an example, Figure 4.1 shows Martin and Lila's work with several drawings of diagrams and symbol use in the Right Triangle Task.

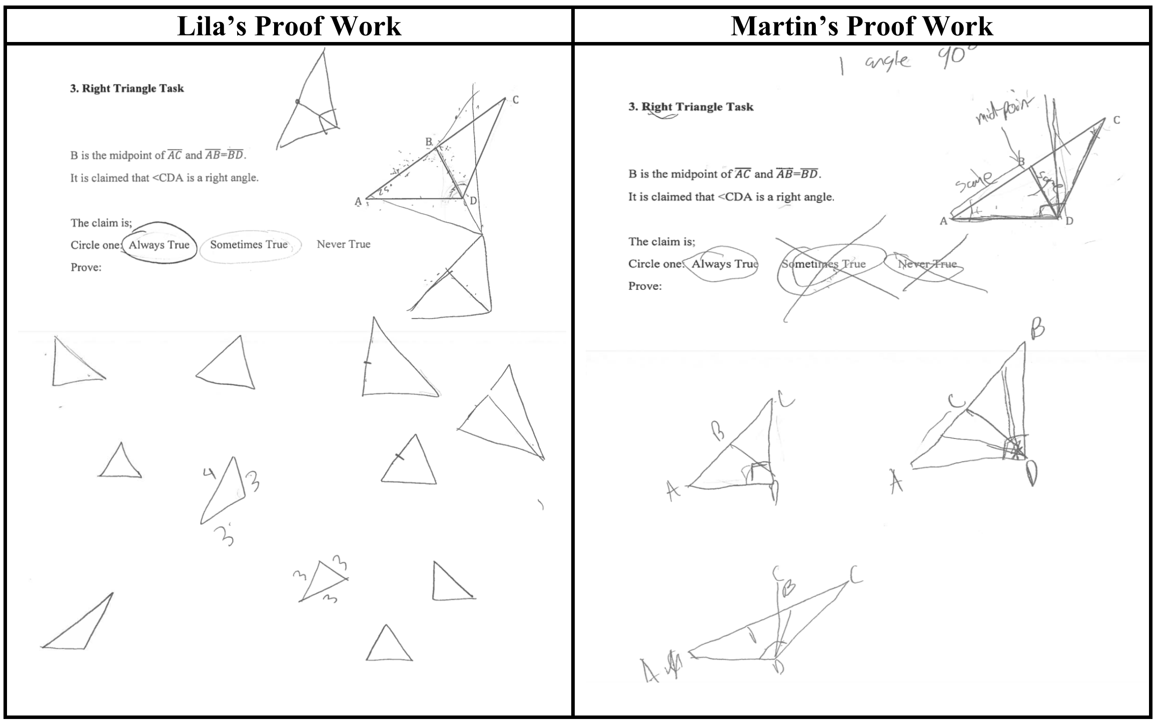


Figure 4.1. Lila's and Martin's proof works on the Right Triangle Task.

Note that both Lila and Martin drew several diagrams and even redrew some parts of the given diagram in their proof works. As far as symbolic resources, Martin labeled the vertices of all the new diagrams he drew; whereas, Lila only labeled side lengths of some diagrams she constructed.

Students' Use of Diagrams

Four noticeable types of diagram use arose when considering the students' proof works in the Right Triangle Task: Drawing a range of diagrams, holistic reasoning with a diagram, reliance on the figural properties of the diagram, and the mobility of one side of the diagram. In the first type, seven out of nine students drew a range of diagrams to test the claim of the task. This type of diagram use can be seen in above figure (Figure 4.1) with two students' various drawings to check when the claim would be valid. In the multiple drawings, a right triangle was always included, and students tested the validity of the givens on it. By considering right triangles and other types of triangles (e.g., obtuse, acute, or equilateral triangles), students apparently checked the possibilities to ensure they circled the correct choice. Drawing and working with different versions of the given diagram was a new approach that students did not perform on the previous two diagram-given tasks.

Drawing several diagrams and testing the givens on them shed light on the second type of diagram use in the Right Triangle Task which was a *holistic reasoning* with a diagram. In fact, four students used a right triangle or a different kind of triangle as the starting point in their proofs and then examined the givens in the claim to conclude when the claim would be valid. For example, Samantha argued that some changes she made on the given diagram by sketching new sides and segments "take away the given information". Hence, these four students circled *sometimes true* because *sometimes true* refers to the case in which the triangle has a 90-degree angle and the side length values are appropriate to the givens in the task.

Reliance on the figural properties of the diagram was the third prominent type of diagram use in the Right Triangle Task. In particular, three students stated that the claim is sometimes or never true based on the visual aspects of a figure. For example, Lee argues that the claim is never true because it should have implied a 90-degree angle in the given-diagram. The following excerpt shows his argument.

> *Lee*: So, I know that, umm, opposite sides are congruent [pointing to the sides AB, BC, and BD], but I don't think that, umm, angle CDA [pointing to the angle CDA] is 90 degrees.
>
> *Interviewer:* Why?
>
> *Lee*: *because, well, if it said that line CD and line AD were perpendicular [pointing to the lines CD and AD], then that would be because they would intersect at a 90-degree angle [making gesture with his two hands to show 90-degree angle], but it doesn't say that.*

In this excerpt, Lee assumes that the given diagram in the task should have represented with a 90-degree angle to conclude that angle CDA might be a right angle. As similar to Lee's argument, Nathan also made his decision based on the figural characteristics of the diagrams, as evident in the following excerpt:

> *Nathan*: So, I said it is sometimes true. I gave two examples of when it gives you true [pointing to his drawings on the paper] and the given example is [pointing to the given diagram in the task] one of the false.
>
> *Interviewer:* OK, why do you say that?
>
> *Nathan*: *So, if you look at, it says that CDA [pointing to the given information in the task], the angle is a right angle.*
>
> *Interviewer:* hmm hmm
>
> *Nathan*: *That does not look like a right angle [pointing to the angle CDA in the given triangle] because it goes up too much [pointing to side CD in given triangle].*

Interviewer: yeah

Nathan: It should go straight up [student draws a side inside the given triangle that is the straight up version of side CD] and then a right triangle [pointing to the right triangle he drew], uh, *CDA [pointing to the angle CDA in the right triangle he drew] creates a 90-degree angle, and then CDA here [pointing to the angle CDA in the obtuse triangle he drew] actually does not create a right angle*.

Interviewer: OK

Nathan: So, I think it's only true on right triangles [pointing to the right triangle he drew].

Apparently, Nathan considered two cases of diagrams in which there is a right triangle and an obtuse triangle (Figure 4.2). He reasoned that a 90-degree angle would occur in a right triangle that would make the claim true, but an obtuse triangle would be false concerning the validity of the claim and he labeled the given diagram under the false category.

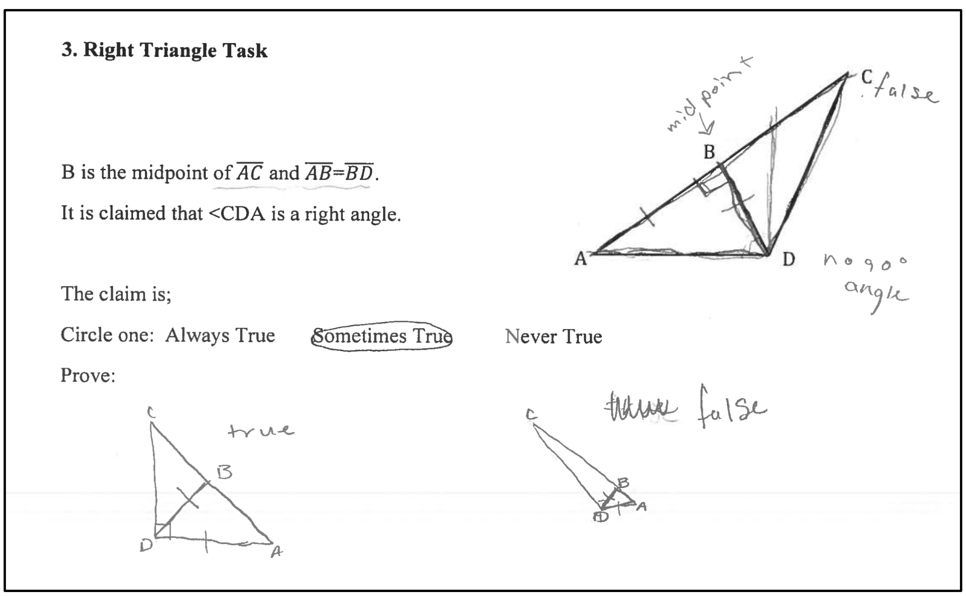


Figure 4.2. Nathan's proof work in the Right Triangle Task.

The last common type of diagram use involved the mobility of one side (i.e., side DC) of the given figure. Chiefly, six students redrew the side DC in the given diagram to examine the variations of the figural aspects of the given diagram. In other words, sketches of different types of side DC led the diagram capture the movements of the particular sides of the figure that students drew. As an example, Deborah examined the mobility of side DC by redrawing the given diagram (Figure 4.3) and selected the choice sometimes true, yet she expressed that having a physical instrument to test the mobility of the given figure would ensure her choice.

Deborah: I did not get all written down. I am not sure how to explain this without like actually seeing it.

Interview: hmm hmm

Deborah: hmm, because I don't know if that's true or not. *I have to have like several different; I have to have like a physical thing to play with to see more*.

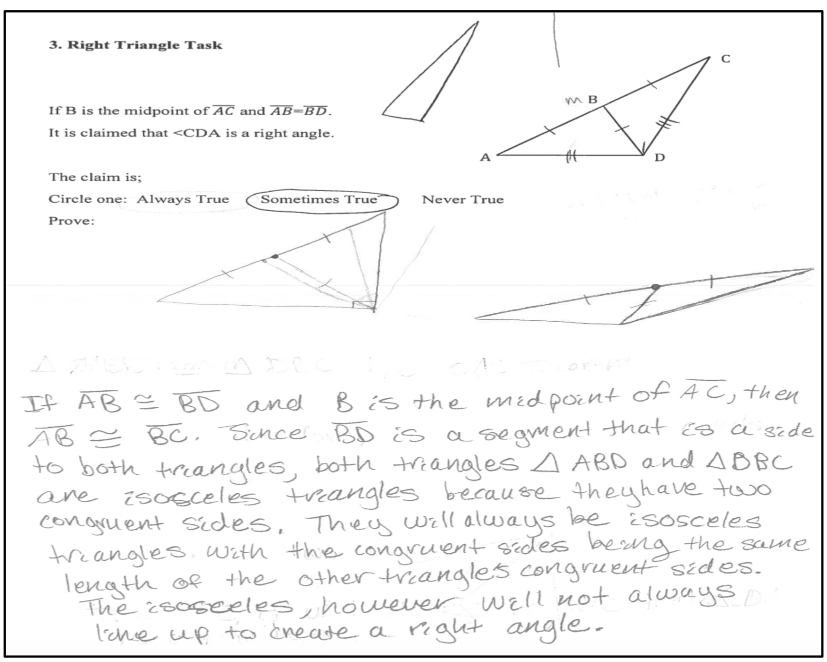


Figure 4.3. Deborah's proof work in the Right Triangle Task.

Consequently, in the Right Triangle task, unlike previous diagram-given tasks, students showed different types of diagram use such as producing multiple figures with different visual aspects and considering the mobility of certain parts of the given diagram. Saying that the Right Triangle Task had a unique feature, truth-unknown, which was not present in other diagram-given tasks in the first interview. Moreover, the given diagram in this task was not an accurate figure while the diagrams in other diagram-given tasks were correct. Thus, the production of different scenarios of the given diagram might be the outcome of two distinct features that the Right Triangle Task had.

Proving Actions of the Students

Table 4.3 shows the proving actions that students used in the Right Triangle Task. As similar to previous diagram-given tasks, gesturing on the diagrams to justify the arguments was a usual proving action employed in the Right Triangle Task. Other frequently used proving actions were types of STEP and JUST actions which were not surprising when compared to the proving actions used in the previous tasks. However, students also engaged in activities that were specific to the Right Triangle Task. As an illustration, INVEST proving action was common and exceptional to that particular task which was not surprising given the fact that students created numerous diagrams to investigate the truth value of the claim.

Table 4.3

Proving actions that nine students employed while proving the truth-unknown Right Triangle Task

Proving Actions	**Frequencies of the Codes**									
	Ben	**Flynn**	**Deborah**	**Lee**	**Megan**	**Samantha**	**Nathan**	**Lydia**	**Martin**	**Total**
CRIT-STEP	0	0	1	0	0	0	0	0	0	1
CRIT-ABOUT	0	0	0	0	0	0	0	1	0	1
CRIT-INVEST	0	0	0	0	1	0	0	0	0	1
CRIT-CLAR	0	0	0	0	2	0	0	0	0	1
CLAR-ABOUT	0	0	1	0	0	0	0	2	1	4
CLAR-STEP-L	0	0	0	0	3	0	0	0	1	4
CLAR-STEP-W	0	0	0	0	1	0	0	0	0	1
CLAR-STEP	0	0	0	0	1	0	0	0	0	1
CLAR-INVEST	0	0	0	0	1	0	1	0	1	3
STRUC	0	0	0	0	0	0	1	0	0	1
SUM	0	0	1	0	0	0	0	0	0	1
ABOUT	0	0	1	2	0	1	1	7	3	15
END	0	1	0	0	0	0	0	0	0	1
END-W	0	0	0	0	0	0	0	1	0	1
INVEST	1	0	1	1	4	3	1	6	3	**20**
JUST	1	1	2	0	1	0	0	2	0	7
JUST-L	0	0	1	1	1	0	1	3	1	8
JUST-W	2	0	1	4	2	0	2	2	2	15
JUST-G	9	2	2	2	2	1	1	10	0	**29**
JUST-G-L	0	0	4	5	5	0	2	2	2	20
JUST-G-W	7	2	1	4	3	1	2	5	1	26
JUST-S	0	0	0	0	0	0	0	1	0	1
JUST-S-L	0	0	0	0	0	0	0	0	3	3
JUST-V	0	1	0	0	0	0	1	2	0	4
JUST-V-L	0	0	3	0	0	0	1	1	1	6

JUST-V-W	0	2	0	0	0	0	0	2	1	5
STEP	8	3	2	3	0	2	2	9	0	29
STEP-L	1	1	4	1	6	0	2	3	3	21
STEP-W	5	5	3	4	5	2	1	4	3	32
ACCEP	0	0	0	0	2	0	0	0	0	2
ACCEP-L	0	0	0	1	0	0	0	0	0	1
REFINE	0	0	1	1	0	0	0	0	0	2
Repetition	1	0	2	1	3	1	0	3	0	11

The above mentioned Figures 4.1, 4.2, and 4.3 demonstrate the students' production of various diagrams. In particular, Martin drew right triangles and an obtuse triangle in his proof work (see Figure 4.1).

> *Martin*: If *I put B up here [drawing and labeling the midpoint B on side AC in his obtuse triangle drawing] it's gonna equal to same overall as what it would be up here [pointing to the given triangle]*. It's the same amount of angle. I mean because you are adding the two sides [pointing to sides AD and BD in his obtuse triangle drawing] together, so doesn't matter where B is, it will often be over 90 degrees.

In this excerpt, Martin considers an obtuse triangle that he drew and applies the givens on it to investigate whether the claim would be true or not.

Another unique thing regarding the proving actions involved in the Right Triangle Task was the proving interactions students employed. As can be seen from the students' proof works in Figures 4.1, 4.2, and 4.3, most of the students changed their answers to the problem. In other words, clarification or critique of some steps or claims made in students reasoning were noticeable proving interactions in the Right Triangle Task.

In general, students did not provide a valid proof in the task because they selected one of the *always true*, *sometimes true*, or *never true* options based on the figural properties of a diagram by considering the visual properties of the given diagram. They also tested the length values of the equal sides, delivered in the claim, on a right triangle and a non-right triangle to prove the claim. However, both approaches were limited in the sense that they did not include the theoretical features of the diagram. That is, a triangle with givens in the claim would include two inner isosceles triangles whose base angles would be equal to the sum of the inner angles of the diagram. Then the sum of each base angles of the inner triangles would be half of 180-degrees which would guarantee that the value of the asked angle would be 90-degrees.

Diagrammatic Register Tasks

In this section, I discuss four diagram-given tasks from the first and second interviews: Isosceles Task, Midpoints Task, Right Triangle Task (Accurate diagram, Truth known), and Right Triangle Task (Inaccurate diagram, Truth-unknown). Although there were several common student responses on these tasks, I present each task individually so that students' use of diagrams discussed in detail. Within each task, I describe the semiotic resources and the way students used the diagram in the proving process. Then, I present the proving actions students employed in the proofs by drawing insights from the semiotic resources students used and the way students worked with the diagram during the proving process.

Isosceles Task

As shown in Table 4.1, the Isosceles Task was given with an accurate diagram, with the truth known (in this case, a true claim), and in the diagrammatic register format. Four students worked on the task in the first interview.

Semiotic Resources Used in the Isosceles Task

Gesture

Gestures were the most commonly used semiotic resources in the Isosceles Task. Table 4.4 shows that pointing to sides or angles in the given diagram frequently happened among four students' proofs. In fact, each student who worked on the diagram-given Isosceles Task employed gestures such as pointing to the side, angle, or diagram when explaining their proving process.

Table 4.4

Semiotic resources that four students employed while proving the diagram-given Isosceles Task

Semiotic Resources		Frequencies of the Codes Nathan	Martin	Lydia	Lee	Total
Visual	VS: visual-draw side/segment	1	0	0	0	1
	VP: visual-draw point	3	0	0	0	3
	VA: visual-draw angle marking	5	0	3	0	8
	Total	9	0	3	0	12
Gesture	GW: gesture-pointing something in the work	0	1	0	0	1
	GS: gesture-pointing at side/segment	7	33	6	11	57
	GM: gesture-referring to movement	0	0	0	1	1
	GT: gesture-turning the paper	0	1	0	1	2
	GG: gesture-pointing at a given in the task	0	0	3	1	4
	GP: gesture-pointing at a point	0	0	0	3	3
	GF: gesture-pointing at figure/diagram	3	0	0	1	4
	GA: gesture-pointing at angle	6	7	19	4	36
	Total	16	42	28	22	**108**
Symbolic	SS: symbolic-label side/segment	4	5	0	0	9
	SG: symbolic-use geometric symbols	0	1	0	0	1
	SA: symbolic-label angle	2	0	0	0	2
	SAS: symbolic-writing algebraic symbols	0	1	0	0	1
	SE: symbolic-writing an equation	2	1	0	0	3
	Total	8	8	0	0	16

As an example, Martin gestured on the given diagram to refer to the claim to be proved and the provided information in the task. Figure 4.4 shows his work on the Isosceles Task.

Martin: OK, so this is the same length, this one [pointing to sides AB and AC], and so I also said BAD and CAE equal each other [pointing to angles BAD and CAE], right? *So that means that these are gonna be the same length sides [pointing to sides AD and AE]* because they are in the same amount [pointing to sides BD and EC]. So, the length is gonna be the same for these two [pointing to sides AD and AE].

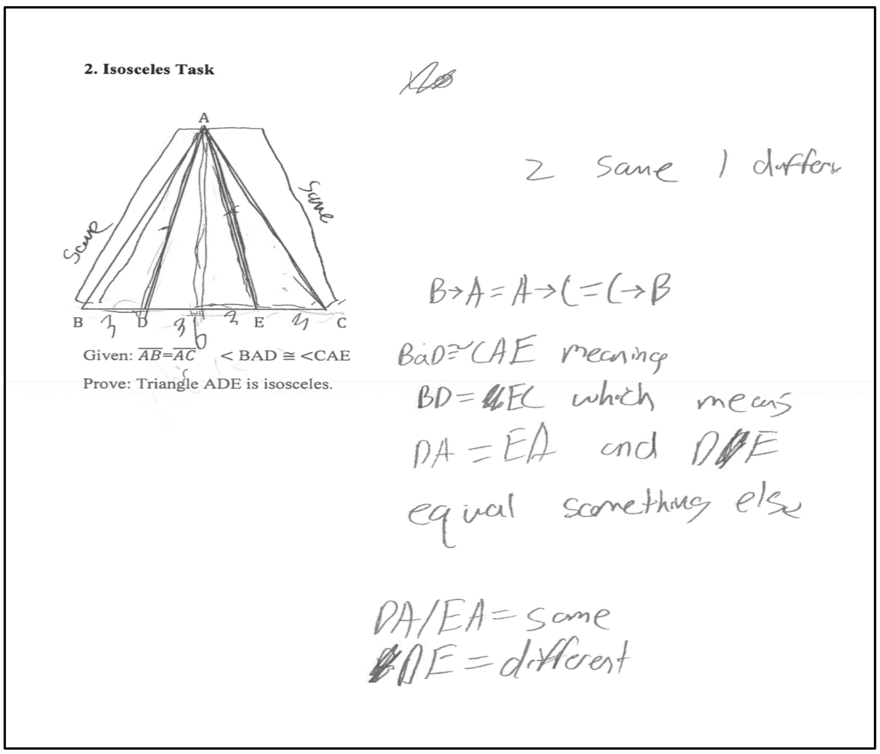


Figure 4.4. Martin's work on the diagram given Isosceles Task.

As similar to Martin' gesturing, when students stated the claim and provided the steps in their proving process, they used gesture resources to clarify the pronouns in their reasoning.

Symbols and Visuals

Although visual and symbolic resources were not as common as gestural resources, several students used them in the proofs as well. Drawings and labeling mostly occurred in students' private work time during their thinking or production of a written explanation. For example, Lee made side drawings and angle markings on his work (Figure 4.5) during his private time but did not employ any of the symbolic or visual resources in the verbal explanation.

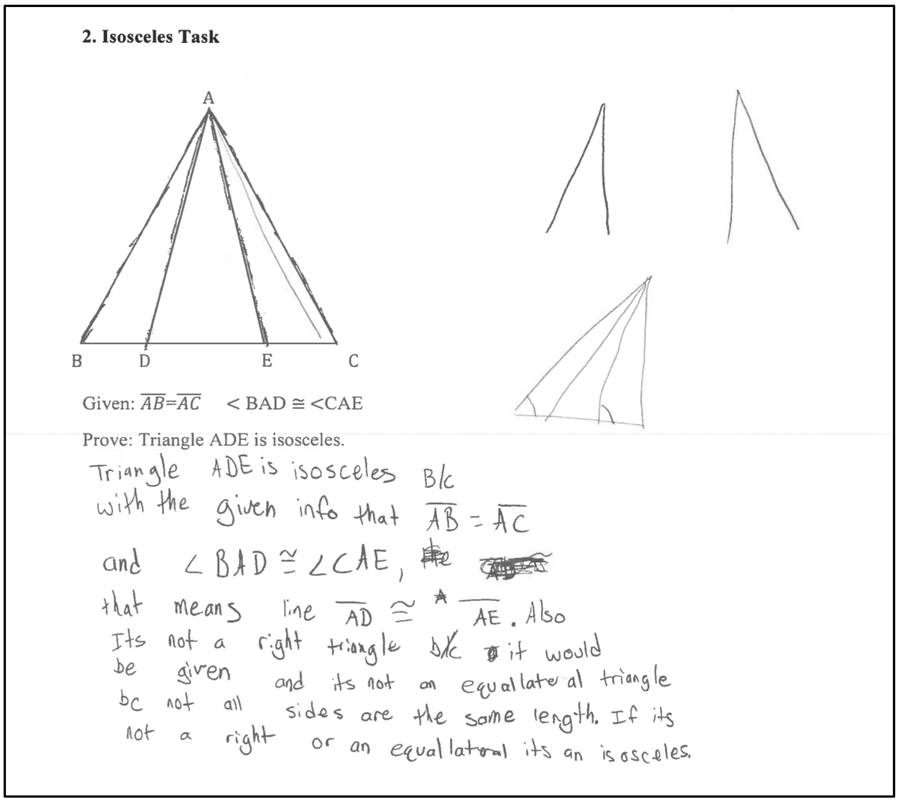


Figure 4.5. Lee's drawings and symbols in his work on the diagram given Isosceles Task.

As apparent in Lee's work, he made sketches on the similar sides of the triangles in the given diagram when he produced the written work. Moreover, in his written work, he showed the equal angles and sides by using geometric symbols.

Students' Use of Diagrams

All the students used the diagram in some way, either as a tool to show the given information or to draw a conclusion by using the figural properties of the diagram. Three students used the figural characteristics of the given diagram to prove the claim in the task. There are two students' statements as an example in the following.

> *Lee*: So, basically, I tried to like eliminate options, …, *basically, it would have been given if it was a right triangle, like it would have 90-degree symbol,* so it can't be a right triangle, and it's not an equilateral triangle ADE [pointing to triangle ADE] because *it would, it would like, umm, the, the bottom line [pointing to the bottom side of the triangle ADE] would have to be the same exact as line AD and AE [pointing to sides AD and AE], but sure it's not.* So, it pretty much has to be an isosceles triangle (See Figure 4.2).

In his statement, Lee expresses that the figural concepts included on the diagram help prove that the triangle ADE is an isosceles triangle. He concludes by only considering the appearance of the given triangle. That is, he claims that the triangle ADE cannot be a right triangle because there is no 90-degree symbol in the triangle, or it cannot be an equilateral because the length of the side DE seems less than sides AD and AE.

Whereas two students such as Lee relied on figural features, sometimes inaccurately, Lila used the diagram as a tool to provide arguments in her proving process.

> *Lila*: So, if it's already saying that AB and AC are similar [pointing to sides AB and AC] and then BAD and CAE [pointing to angles BAD and CAE] are similar. *That means that they are literally the exact same and the angle would have to be the same [pointing to angles ABC and BCA]. So that makes that one different [pointing to angle A in the triangle ABC] if it is an isosceles triangle.*

In this statement, Lila addressed the known information about the diagram by gesturing and labeling the angles ABC and BCA on the diagram. She used the term “similar” when she referred to the congruent sides and then said they are “the exact same.” She also claimed that the triangle ABC was an isosceles triangle by pointing and stating that if one angle was different in the triangle ABC. In her use of the diagram, she mostly gestured on the sides and angles and drew angle markings on the base angles of the given triangle ABC when she argued with the given information in the task.

The sample excerpts reinforce the idea that students use the diagrams as visual tools to prove geometric claims. From students’ interpretations, the figural features of the given-diagram are relevant and sometimes used as sufficient sources to demonstrate the statement that triangle ADE is an isosceles triangle in the Isosceles Task.

Two students provided a written explanation in their proofs, and they aligned with the diagrammatic register in their work (see Figures 4.4 and 4.5). That is, they used geometric notations or symbols referring to particular parts of the diagram. On the other hand, the other two students did not adhere to the diagrammatic register. They drew angle markings or sides segments on the diagram but, for example, Lila claimed that she could provide an oral explanation for the proof but not a written one. Hence, she used some aspects of the diagrammatic register in the verbal explanation, but she did not provide a written explanation with the full diagrammatic register.

Proving Actions of the Students

As shown in Table 4.5, several proving actions were evident in students’ proving process in the Isosceles Task. However, the most common proving actions are the Limited or Wrong steps and justifications students produced in their reasoning. With regard to the relations between semiotic resources and proving actions, responses related specifically to the gesturing resources because students used their hand and arm movements to illustrate the arguments they made in their reasoning. Recall that JUST-G codes applied to instances where the student was gesturing as they provided a justification.

Table 4.5

Proving actions that four students employed while proving the diagram-given Isosceles Task

Proving Actions	**Frequencies of the Codes**				
	Nathan	**Martin**	**Lydia**	**Lee**	**Total**
CONF-STEP	0	0	1	0	1
CRIT-JUST-W	1	3	0	0	4
CLAR-ABOUT	1	0	0	0	1
CLAR-JUST-W	1	0	0	1	2
STRUC	0	0	0	1	1
ABOUT	1	0	1	0	2
END-L	0	0	1	0	1
INVEST	0	1	0	0	1
JUST	1	2	0	2	5
JUST-L	0	2	1	1	4
JUST-W	2	1	0	4	7
JUST-G	1	7	4	3	15
JUST-G-L	0	8	1	5	14
JUST-G-W	2	2	0	3	7
JUST-S	0	2	0	0	2
JUST-V-L	0	0	1	0	1
JUST-V-W	1	0	0	0	1
STEP	1	5	3	2	11
STEP-L	1	7	1	6	15
STEP-W	1	1	1	2	5
ACCEP	0	1	0	0	1
ACCEP-L	0	0	2	0	2
Repetition	0	2	2	0	4

As depicted in Table 4.5, five proving actions (i.e., STEP, STEP-L, JUST-G, JUST-G-L, JUST-W and JUST-G-W) were particularly common among the four students' proving processes. These actions are reasonable after discussing the semiotic resources students engaged and the way students used the diagrams in the isosceles task. That is, gesturing frequently occurred in students' oral explanations and the limited and wrong steps and justifications were coded for the students' statements which relied only on the physical properties of the diagram given in the task and may not be deductively valid. In particular, Figure 4.6 shows an excerpt from one of the students' proving coded with STEP-L, STEP-W, and JUST-W proving actions.

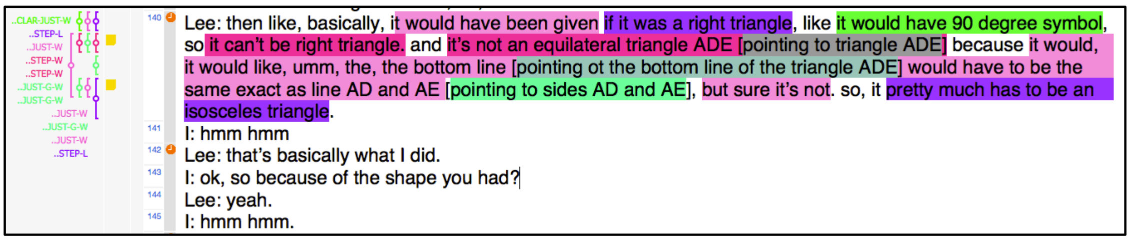


Figure 4.6. Lee's proving actions in the Isosceles Task.

As seen in Figure 4.6, Lee's work on the Isosceles Task included limited and wrong steps as well as justifications. It was discussed earlier that Lee relied on the figural properties of the given diagram and hence, in his proof, he mostly argued with the appearance of the given triangle or what should have appeared in the given triangle. That approach led Lee to consider the given diagram as the only source for his reasoning which was led limited or wrong arguments in his proof work. In another example, Martin's proof work shows that he employed gesture and symbolic semiotic resources when providing justifications in his proof. His statement below was coded with proving actions such as STEP, STEP-L, JUST-G, JUST-S, and JUST-G-L.

> *Martin*: OK so, I figured out these are the same BA and CA [pointing to sides BA and CA], and then I figured out this is the same as this [pointing to angles BAD and CAE], correct. And this bottom is gonna be one certain length [pointing to side BC]. So, if this is the same as this [*pointing to sides BD and EC*] say it's ten for the whole bottom of this, it's ten feet long [*pointing to side BC*] and this is 3 feet and this is 3 feet [*labeling the*

sides BD and EC as 3]. That means this is 6 feet long [*labeling the side DE*] and this has to go the same exact angle [pointing to angle DAE] because these are the same exact equations [pointing to angles BAD and CAE]. So that means that this has to be the same length as that [pointing to sides AE and AD] (See Figure 4.4).

As evident in his reasoning, Martin gave numerical values on particular sides of the given triangle and used labeling to illustrate the equal sides that were given in the task. Moreover, he had gestures on the diagram (as given in brackets in above excerpt) to clarify which side or angle he was referring to in his explanation.

In summary, during their work on the Isosceles Task, the students engaged mostly in gesture semiotic resources and hence, they frequently justified their arguments by employing gestures in their reasoning (Tables 4.4 and 4.5). These students either used the diagram as an accurate figure to draw conclusions based on its figural properties (e.g., Lee) or as a tool to show the given information in the task and move forward with the arguments made in the reasoning (e.g., Lila).

Midpoints Task

Similar to the Isosceles Task, the Midpoints Task was also given with an accurate diagram and in truth-known and diagrammatic register formats (see Table 4.1). Five out of nine students from the first interview took the Midpoints Task.

Semiotic Resources Used in the Midpoints Task

Gesture

In the Midpoints Task, the most frequently used semiotic resources were gesture resources (Table 4.6). Among them, pointing to a side, angles, and a diagram were recurring gestures in students' proof works.

Table 4.6

Semiotic resources that five students employed while proving the diagram-given Midpoints Task

Semiotic Resources		Frequencies of the Codes					
		Deborah	**Ben**	**Megan**	**Flynn**	**Samantha**	**Total**
Visual	VS: visual-draw side/segment	3	1	0	0	0	4
	VH: visual-draw hatch/tick marks	0	9	4	8	0	21
	VL: visual-draw line	10	0	0	0	0	10
	VR: visual-redraw figure	0	0	2	0	0	2
	VA: visual-draw angle marking	0	6	0	0	0	6
	VSC: visual-draw side congruency	0	3	1	3	0	7
	Total	13	19	7	11	0	50
Gesture	GW: gesture-pointing something in the work	0	0	0	0	1	1
	GS: gesture-pointing at side/segment	14	19	18	20	3	74
	GV: gesture-pointing at vertex	0	0	0	0	3	3
	GH: gesture-pointing at hatch/tick marks	0	0	2	0	0	2
	GT: gesture-turning the paper	0	0	0	0	1	1
	GG: gesture-pointing at a given in the task	0	0	1	1	0	2
	GP: gesture-pointing at a point	2	0	3	2	2	9
	GF: gesture-pointing at figure/diagram	2	7	2	4	8	23
	GA: gesture-pointing at angle	21	6	0	0	0	27
	Total	39	32	26	27	18	142
Symbolic	SP: symbolic-label point	3	0	0	0	0	3
	SS: symbolic-label side/segment	0	0	11	0	0	11
	Total	3	0	11	0	0	14

Although all students employed gesture, symbolic, and visual semiotic resources in the Midpoints Task either in when they were thinking about the task or explaining to their proofs to me, one student, Flynn, did not show evidence of using symbolic and visual resources in his thinking. He applied only gesture resources when he explained his proof to the Midpoints Task. Flynn's proof explanation in the Midpoints Task is as follows:

> *Flynn*: OK so like, it says the segments joining the midpoints are parallel and half the length. So that means that, uh,
>
> *Interviewer*: hmm hmm.
>
> *Flynn*: This [*pointing to side BD*] is half of this [*pointing to side AB*], and this [pointing to side EF] is also half of this [*pointing to AB*]. So that means they [*pointing to both BD and EF*] have to be the same, and same goes for this one and this one [*pointing to DF and EC*] and then, these [*pointing to BF and FC*] have to be same because it splits at [*pointing to the point F*] in half. So that, uh, this triangle [*pointing to triangle BDF*] and this triangle [*pointing to triangle FEC*] are the same.

In this excerpt, Flynn uses gestures in his proof explanation to point to particular sides, points, and diagrams on the given diagram. Although his reasoning does not include the diagrammatic register (because he didn't state "side AB" or "point F"), Flynn performed gestures to clarify the pronouns that addressed specific sides, points, or diagrams in his proving. After working on the other tasks in the first interview, I asked Flynn to talk about his proof to Midpoints Task again. Interestingly, he employed visual resources (i.e., drawing hatch/tick marks) to show equal sides on the given diagram although he repeated the same explanation as above excerpt. Figure 4.7 shows Flynn's work on the Midpoints Task with hatch/tick marks.

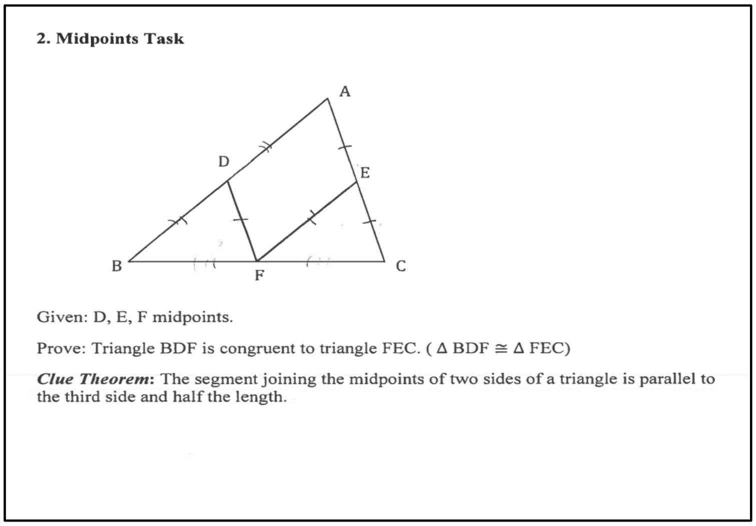
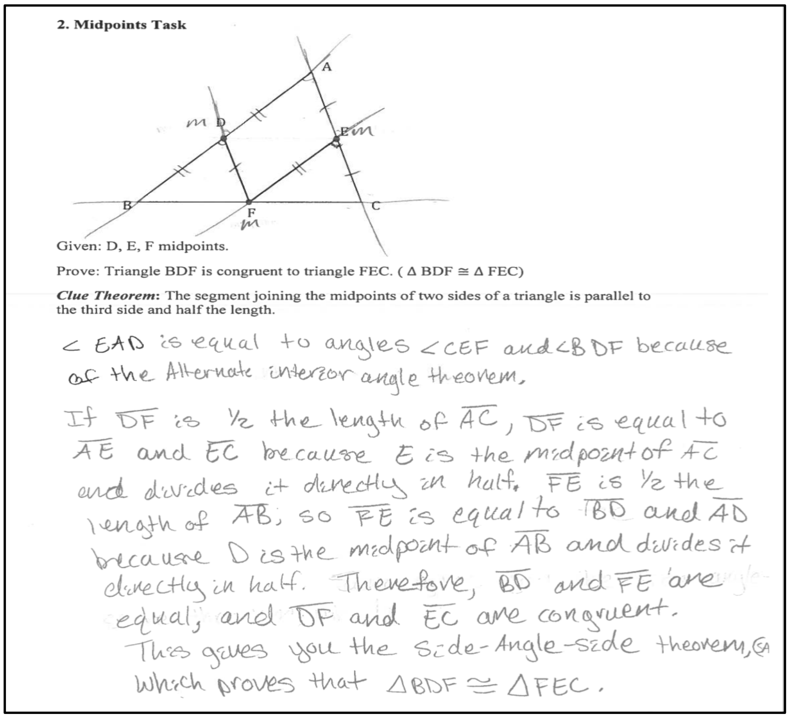


Figure 4.7. Flynn's work on the Midpoints Task with visual resources.

Although there was a precise representation of the givens in students' written proofs, most students, similar to Flynn, used pronouns accompanying with gestures in their verbal reasoning when referring to the certain parts in the diagram.

Symbols and Visuals

The visual and symbolic semiotic resources also surfaced in several students' responses to the Midpoints Task. The most common visual and symbolic resources that students used were drawing hatch/tick marks and lines as well as labeling points and side/segments (see Table 4.6). As an example, Figure 4.8 shows Deborah's work on the task. She evidently employed visual and symbolic resources in her reasoning and used the diagrammatic register, providing letters and geometric symbols to demonstrate their thinking, in the written explanation. Similar to Deborah, three other students used visual and symbolic resources along with gestures, and used the diagrammatic register in the written work of their proofs. However, Flynn did not use the diagrammatic register in his proof work since he did not produce a written explanation in the Midpoints Task (see Figure 4.7).


2. Midpoints Task

Given: D, E, F midpoints.

Prove: Triangle BDF is congruent to triangle FEC. (Δ BDF ≅ Δ FEC)

Clue Theorem: The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

< EAD is equal to angles < CEF and < BDF because of the Alternate interior angle theorem.

If $\overline{DF}$ is ½ the length of $\overline{AC}$, $\overline{DF}$ is equal to $\overline{AE}$ and $\overline{EC}$ because E is the midpoint of $\overline{AC}$ and divides it directly in half. $\overline{FE}$ is ½ the length of $\overline{AB}$, so $\overline{FE}$ is equal to $\overline{BD}$ and $\overline{AD}$ because D is the midpoint of $\overline{AB}$ and divides it directly in half. Therefore, $\overline{BD}$ and $\overline{FE}$ are equal, and $\overline{DF}$ and $\overline{EC}$ are congruent. This gives you the Side-Angle-Side theorem, (SA) which proves that $\triangle BDF \cong \triangle FEC$.

Figure 4.8. Deborah's work on the Midpoints Task with visual and symbolic resources.

Students' Use of Diagrams

All the students used the given diagram in some way in their proofs. Three students made drawings or used labeling that arose from the given information in the task and used the diagram as a tool to develop arguments that helped them prove the claim in the task (e.g., Deborah's proof work in Figure 4.8). Similar to Deborah, Megan used the diagram as a way to organize the given information in the task and hence, she concluded the congruency of two inner triangles based on the arguments constructed with the help of a diagram.

> *Megan*: So, umm, *since these are the midpoints [pointing to the points F, D, and E on the given diagram] we know that these sides are gonna be the same [drawing hatch/tick marks on the sides BF and FC] because they'll both be the half of what I named, to be the*

length of z. Yeah, so they'll both be half z, and then, umm, this also [pointing to DF in the big triangle], this, umm, these two lines [pointing to the sides DF and EC in two separate triangles] will be half y because also this clue theorem [pointing to the clue theorem], umm, like, they'll be half of that, like, there is parallel line and then, yeah same for these sides [pointing to the sides BD and EF in the big triangle], too.

Interviewer: OK.

Megan: So, then *since all of them are the same length [pointing to corresponding sides of the two inner triangles], they are congruent*.

This excerpt shows that Megan interpreted the sides of the given diagram as having various length values and used a variable for each side to indicate the variability of the side lengths. Figure 4.9 shows the proof work she produced, including the redrawing of two inner triangles with variable values on the side lengths.

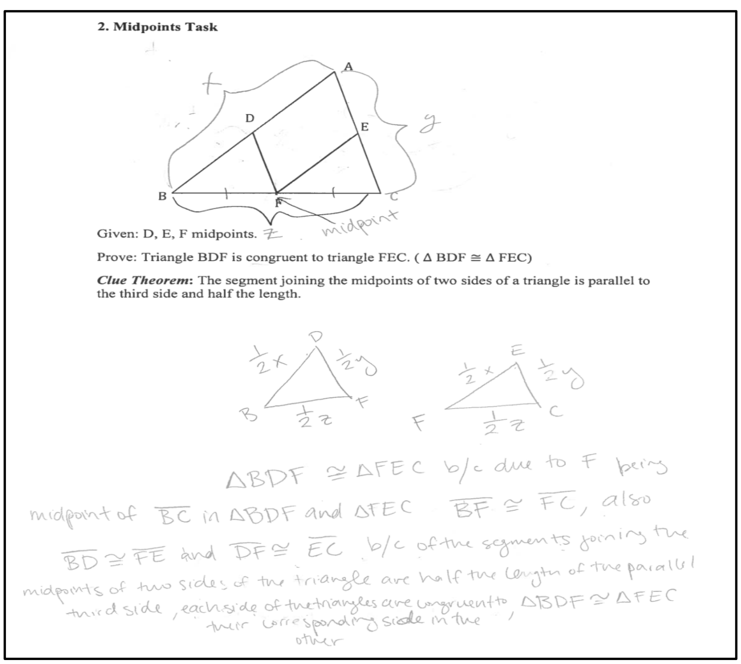


Figure 4.9. Megan's use of the diagram in the Midpoints Task.

As seen from her work, Megan used algebra in her proof by labeling the sides with variables to consider the variability of the sides. She then redrew the new versions of the two inner triangles (that had to be proven congruent) that occurred in the given triangle. Her diagram use showed that she incorporated the given information into the given diagram and collected all the information, including the clue theorem, in those new triangles by labeling their sides with variables that represented the corresponding side lengths of the given triangle.

In contrast, two students attempted to determine the congruency of two inner triangles by relying on the visual characteristics of the given diagram. As an illustration, the following excerpt expresses Samantha's use of visual features in the given diagram.

> *Samantha*: I started with those two things [pointing to the points D and E], then I connected those and saw that those were all similar [pointing to inner triangles] …because that creates lines that since they are parallel, they would end up making the same triangle [pointing to the sides and segments inside the triangle].
>
> …
>
> *Interviewer*: So, are all four triangles congruent?
>
> *Samantha*: *It looks like they are at least close to, if not*?
>
> *Interviewer*: Is it because of the figure [pointing to the given diagram] here because they [the four inner triangles] look like they are congruent?
>
> *Samantha*: *Yes*.

In this excerpt, Samantha claimed congruency of the inner triangles in the given diagram by associating them with the visual similarities. Additionally, when discussing her proof work on the Midpoints Task in general, she mentioned the difficulty of explaining a visual feature in proofs as follows: “it was hard to try and explain what I was seeing in words. I was trying to figure out how to put it.”

Proving Actions of the Students

Students exhibited various proving actions during their work on the Midpoints Task, with the most frequent being providing a step and providing a justification, with justifications often accompanied by gestures (Table 4.7).

Table 4.7

Proving actions that five students employed while proving the diagram-given Midpoints Task

Proving Actions	**Frequencies of the Codes**					
	Deborah	**Ben**	**Megan**	**Flynn**	**Samantha**	**Total**
CRIT-JUST-L	0	0	1	0	0	1
ABOUT	1	0	0	0	1	2
END	1	0	0	1	1	3
INVEST	0	0	0	0	2	2
JUST	8	1	6	2	0	17
JUST-L	1	0	1	0	2	4
JUST-W	0	2	0	0	0	2
JUST-G	14	8	5	12	0	39
JUST-G-L	0	4	0	2	3	9
JUST-G-W	0	2	0	0	0	2
JUST-V	0	1	1	4	0	6
STEP	12	7	3	18	1	41
STEP-L	0	4	1	3	2	10
STEP-W	0	1	0	0	0	1
ACCEP	4	0	0	0	0	4
CLAIM	1	0	0	0	0	1
Repetition	2	2	1	0	0	5

Figure 4.10 illustrates the proving actions that occurred when Deborah developed her reasoning by using the givens and previously known facts.

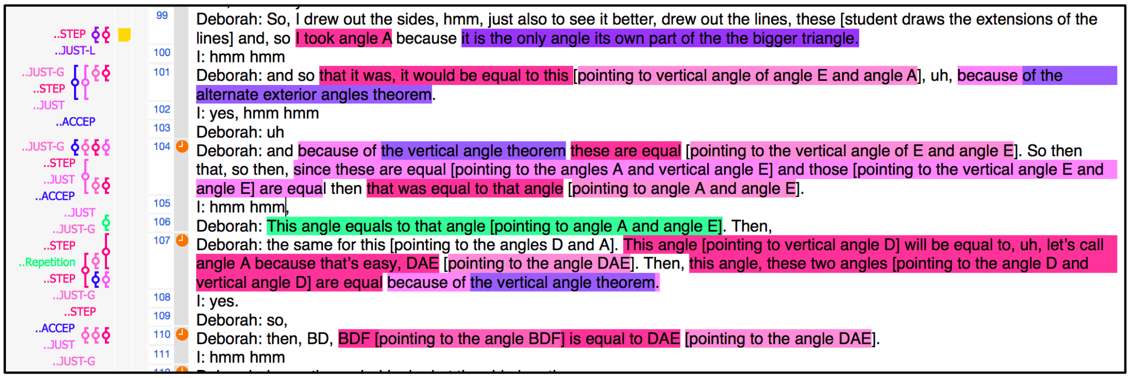

99 Deborah: So, I drew out the sides, hmm, just also to see it better, drew out the lines, these [student draws the extensions of the lines] and, so I took angle A because it is the only angle its own part of the the bigger triangle.
100 I: hmm hmm
101 Deborah: and so that it was, it would be equal to this [pointing to vertical angle of angle E and angle A], uh, because of the alternate exterior angles theorem.
102 I: yes, hmm hmm
103 Deborah: uh
104 Deborah: and because of the vertical angle theorem these are equal [pointing to the vertical angle of E and angle E]. So then that, so then, since these are equal [pointing to the angles A and vertical angle E] and those [pointing to the vertical angle E and angle E] are equal then that was equal to that angle [pointing to angle A and angle E].
105 I: hmm hmm
106 Deborah: This angle equals to that angle [pointing to angle A and angle E]. Then,
107 Deborah: the same for this [pointing to the angles D and A]. This angle [pointing to vertical angle D] will be equal to, uh, let's call angle A because that's easy, DAE [pointing to the angle DAE]. Then, this angle, these two angles [pointing to the angle D and vertical angle D] are equal because of the vertical angle theorem.
108 I: yes.
109 Deborah: so,
110 Deborah: then, BD, BDF [pointing to the angle BDF] is equal to DAE [pointing to the angle DAE].
111 I: hmm hmm

Figure 4.10. Proving actions in Deborah's proof in the Midpoints Task.

In the Figure 4.10, Deborah employed STEP and JUST-G proving actions by using the given information and gesturing on the angles in the given triangle She drew extension lines of the sides of the given triangle (see Figure 4.8) which helped her use previously known facts such as alternate exterior angles theorem and vertical angle theorem. Because of the previously known facts in her proof work, the ACCEP proving actions occurred in her reasoning.

Two students, Deborah and Megan, who provided valid steps and justifications as proving actions on the Midpoints task, were the only ones who employed symbolic resources (i.e., labeling the midpoints and sides of the given triangle) for this task. Another student, Flynn, who also produced a valid proof for the Midpoints Task, did not employ these symbolic or visual resources in his proof production.

The other students, Ben and Samantha, who explicitly relied on the visual characteristics of the given diagram in their reasoning, applied wrong or limited proving actions in their attempted proofs. The semiotic resources they included in their proofs were visual and gestural. They actively gestured when providing an oral explanation to the proofs. To exemplify, Ben's proving actions and gestures are presented in Figure 4.11.

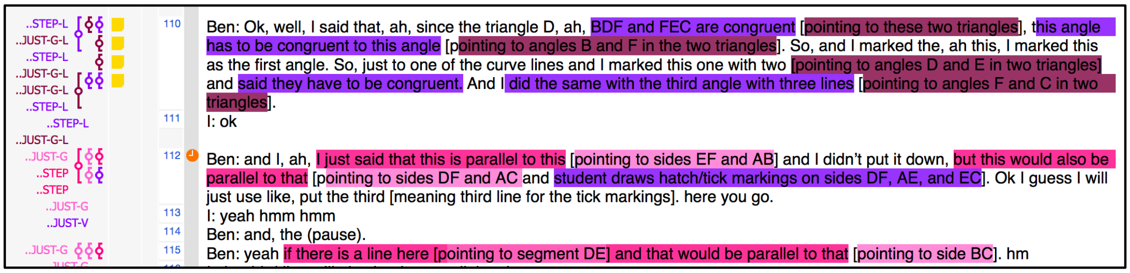


Figure 4.11. Proving actions in Ben's proof in the Midpoints Task.

Note that Ben started with the claim to be proved (i.e., Triangles BDF and FEC are congruent) which caused STEP-L and JUST-G-L proving actions at the beginning of his reasoning. Then, he based his further arguments on these previous limited step and justification and hence, his following proving actions also become limited in his reasoning.

In summary, on the Midpoints Task, the students used gesture resources more frequently than other semiotic resources. Two kinds of diagram use recurred among students—the diagram was either used as a representation of the givens and a place to show previously known facts (as Deborah and Megan did) or interpreted as an accurate figure and visually used a part of the givens (as Samantha and Ben did). It is worth noting that the given diagram in the Midpoints Task was accurate. Therefore, that probably allowed Samantha and Ben to have correct steps even though they were relying on the visual characteristics of the diagram rather than deductions from the given facts. Consequently, valid steps and gestural justifications were explicit and the most common proving actions by students on the Midpoints Task.

Right Triangle Task (Accurate Diagram, Truth-Known)

The Right Triangle Task which was given as the last task in the first interview involved an accurate diagram with truth-known and diagrammatic register features (see Table 4.1). The task had the same name with the third task presented in the first interview because they both asked the same claim and required almost the same proof work. That is, the third task in the first interview was the Right Triangle Task in truth-unknown and inaccurate diagram features; however, this section focuses on the Right Triangle Task in truth-known and accurate diagram features. It is worth reminding here that the two tasks were both in the first interview and students worked on the same tasks sequentially. Below, the task is discussed similarly with the aforementioned diagram-given tasks in general, and in comparison with the previous Right Triangle Task in particular.

Semiotic Resources Used in the Right Triangle Task

Gesture

Although gesture resources were prevalent for this task as they were in general, one type of gesture in particular, pointing at an angle, was conspicuously high in the accurate, truth-known Right Triangle Task (Table 4.8). In fact, when compared to the truth-unknown Right Triangle Task, the frequency of pointing at a side segment was lower but pointing at an angle was higher in the last task. The reason why pointing at an angle occurred most frequently might be because of the accurate diagram and truth-known features in the latter task. Due to these features, students probably did not need to think about the validity of the claim and instead, focused on how to prove the given claim which required them to engage in angles because the claim was about one of the angles of the given triangle. Moreover, using pronouns and accompanying them with gestures to explain the geometric reasoning was a pervasive approach that students employed in this task as they did in other diagram-given tasks discussed above.

Table 4.8

Semiotic resources that nine students employed while proving the truth-known Right Triangle Task in the first interview

Semiotic Resources	**Frequencies of the Codes Total**

Visual	VS: visual-draw side/segment	18
	VH: visual-draw hatch/tick marks	12
	VL: visual-draw line	4
	VF: visual-draw a new figure/diagram	7
	VP: visual-draw point	4
	VA: visual-draw angle marking	24
	VSC: visual-draw side congruency	4
	Total	73
Gesture	GW: gesture-pointing something in the work	2
	GS: gesture-pointing at side/segment	90
	GV: gesture-pointing at vertex	2
	GT: gesture-turning the paper	6
	GG: gesture-pointing at a given in the task	10
	GP: gesture-pointing at a point	7
	GF: gesture-pointing at figure/diagram	38
	GA: gesture-pointing at angle	142
	Total	**297**
Symbolic	SV: symbolic-label vertex	10
	SP: symbolic-label point	2
	SS: symbolic-label side/segment	5
	SG: symbolic-use geometric symbols	2
	SA: symbolic-label angle	29
	SAS: symbolic-writing algebraic symbols	2
	SE: symbolic-writing an equation	**6**
	Total	56

Symbols and Visuals

Drawing angle marking and labeling angles were not prominent resources in the truth-unknown Right Triangle Task, but in the case of the accurate diagram and truth-known version, symbolic and visual resources were used relatively frequently in relation to angles (Table 4.8). As an example, Figure 4.12 shows the proof work produced by Ben that includes angle markings and labeling on the given diagram in the accurate, truth-known Right Triangle Task. In his proving, Ben labeled the base angles of the inner isosceles triangles in the given diagram as 45-degrees and concluded that ACB is a right angle. In fact, Ben seemed to be using visual cues to determine these angles and he did write them symbolically on the diagram.

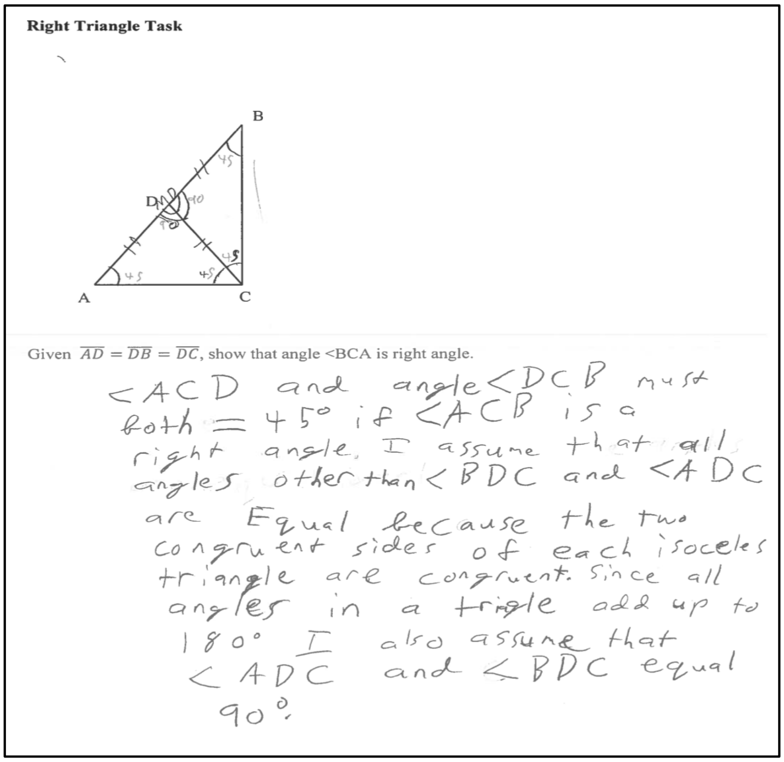


Figure 4.12. Ben's proof work on the Right Triangle Task.

Another symbolic resource used in the Right Triangle Task, but not appearing frequently in the other diagram-given tasks, was writing an equation as part of the argument. One student, Samantha, wrote algebraic equations to calculate the value of the angle BCA in the Right Triangle Task (Figure 4.13).

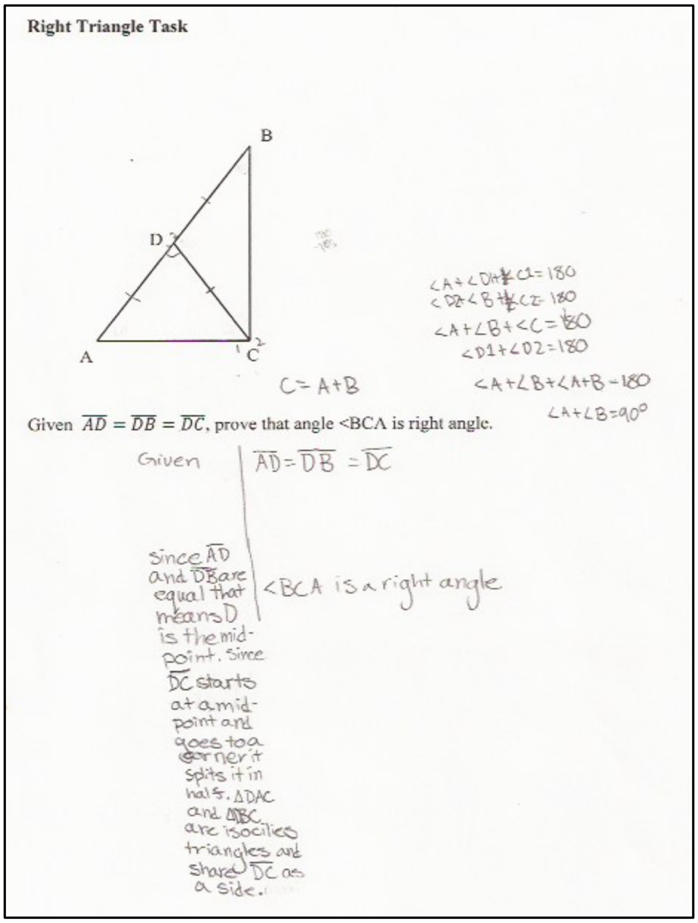


Figure 4.13. Samantha's proof work on the Right Triangle Task.

As shown in Figure 4.13, Samantha considered the values of angles A, D (D1 and D2), C (C1 and C2), and B as unknowns. Then calculated them in algebraic equations to find the sum of the values of angles A and B, which was also equal to angle C.

Students' Use of Diagrams

The students showed three salient types of diagram use, two of which (drawing a variety of diagrams and applying the figural features of the diagram) also appeared in the truth-unknown Right Triangle Task. However, using the diagram as a basis for algebraic determinations was the new type of diagram use that occurred in the truth-known Right Triangle Task. The first prevalent type was drawing a variety of figures in the proof work. Three students preferred drawing additional diagrams to prove the claim. For instance, Lila drew several figures in the Right Triangle Task to show that the angle of BCA would be 90-degrees for all different cases of the diagram. Figure 4.14 illustrates examples of various variations of the diagram that Lila drew to prove that the claimed angle would be 90-degrees for each of them.

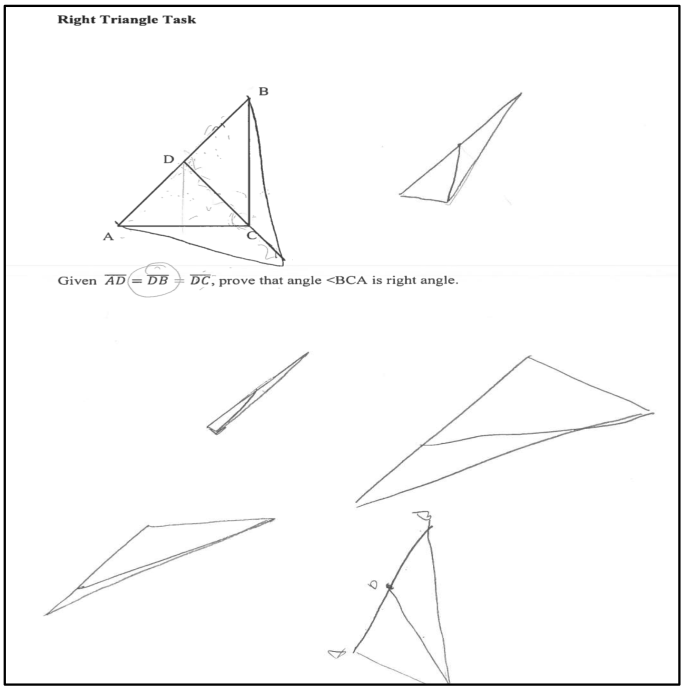


Figure 4.14. Lila's proof work on the Right Triangle Task.

By drawing several diagrams, Lila tried to demonstrate that the claim would be true even if the given diagram would be drawn differently. Below is an excerpt from her reasoning in the task.

> *Lila:* Like these two sides don't have to be the same [pointing to two triangles ADC and DCB], like the two triangles don't have to look the exact same, like one of them could be really narrow and the other one could be really wide [pointing to two inner triangles in her top drawing], but at the end it's still gonna come down to a right angle [pointing to the angle that is opposite to the midpoint in her top right drawing].

Interviewer OK, is it because of the midpoints?

Lila: Because of the midpoint because the midpoint splits it exactly in half. So, these two sides [pointing to sides AD and DB] are gonna have to be the same so they are gonna somehow have to meet down there [pointing to angle C in the given triangle], and they are not gonna since they [pointing to sides AD and DB] are already the same there is no way there is a slant. So, that's gonna make it a right angle.

Interviewer: And it is, even though this diagram is like that [pointing to students top right drawing], will it make it a right angle?

Lila: Yes. *I think so because you could set up differently possibly make it like that [drawing another triangle below the given triangle].*

Lila's approach to the last Right Triangle Task was similar to the previous one regarding drawing more than one diagram to prove the claim. That proving strategy was apparent in other students who drew more than one diagram in the former Right Triangle Task. Students probably realized the similarities between the two tasks and gave the same reasoning they provided in the previous Right Triangle Task even though students did not call two tasks the same tasks openly.

The second prominent type of diagram use was applying the figural properties of the given figure in the Right Triangle Task. This kind of approach was shared in the previous Right Triangle Task as well. In general, five students gave absolute values to inner angles of the given triangle based on the visual characteristics of the inner triangles and the given triangle. Then they concluded that the angle BCA in the given triangle had to be 90-degrees. For a typical example, Ben said the following.

Ben: Uh, which means that these two angles [pointing to angles DBC and BCD] also have to be the same. So, if we are going to say this is a right triangle [pointing to angle

BCA], then these angles would add up to 45 [pointing to angles DBC, BCD, DCA, and DAC] and this will be a 90 degree. Or, this ADC would be a 90-degree angle and so is the CDB [pointing to angles ADC and CDB] (pause). *So, it would work if that is a right triangle. I didn't necessarily prove that this has to be a right triangle*.

Interviewer: So, how did you give the angles 45-degrees then?

Ben: Umm because, well, because if this [pointing to angle BCA] is a right angle, then 45 45 [pointing to angles DCB and DCA] add up to be 90 and I know that this, the angle DCB has to be the same as the angle DCA [pointing to angles DCB and DCA].

Interviewer: How do you, why do you think that?

Ben: Well, because, I think at least since these are isosceles triangles [pointing to triangles BDC and ADC], the angles on the bottom have to be the same for the triangle and since all of these sides *for both isosceles triangles are the exact same length* [pointing to sides BD, AD, and DC]. I think *that means that they will have the same angle, as well*.

Interviewer: Hmm, because they have the same length?

Ben: Yeah and also that because they lead to the same point here [pointing to vertex C].

As mentioned in the above excerpt and shown in Figure 4.12, Ben already accepts that the given diagram is a right triangle based on its figural characteristics. He provides equal values to the inner angles of the given triangle by reasoning with the length values of the isosceles triangles.

Using diagram as a basis for algebraic determinations was the third type of diagram use employed by two students with my guidance. Apparently, in that kind of diagram use, students applied both figural and conceptual features of the diagram. That kind of approach did not occur in the previous Right Triangle Task, although I gave the same guidance on that task. As shown in the Figure 4.13, Samantha used algebraic notations for angle values in the given triangle and calculated the sum of two angles that constructed the angle BCA in the given triangle. Similarly, Flynn proved the task by making algebraic calculations with the angle values although he did not present them on his written work. The following excerpt shows Flynn's reasoning with relations between the angle values in the given triangle after having some directions from me.

Flynn: I am assuming that these two angles are the same [pointing to the base angles of the triangle ADC] and this angle [pointing to the top angle of the triangle ADC] is different, and same as this triangle's [pointing to the base and top angles of the triangle BDC].

Interviewer: Hmm hmm.

Flynn: But I don't know how to prove it is a right angle.

Interviewer: Can you, what is the angle here like? How much is this?

Flynn: This one [pointing to the angle A]?

Interviewer: Hmm hmm, or any other angles in the triangle?

Flynn: Well, this is 180 right here [pointing to the 180 angle on D].

Interviewer: Hmm hmm.

Flynn: So, umm (pause).

Interviewer Can you say anything about that angle [pointing to the top angle of triangle ADC] in relation to these two [pointing to the base angles of the triangle ADC]?

Flynn: Which one, this one [pointing to the angle D in triangle ADC]?

Interviewer: Yes, this one.

Flynn: Uh, both of them together [pointing to the base angles of the triangle ADC] is same as this [pointing to the outside angle of D in triangle ADC].

Interviewer: Hmm hmm.

Flynn: OK so, uh, *so then these two* [pointing to the base angles of the triangle BDC] add *together is the same as this one* [pointing to the outside angle of D in triangle BDC].

Interviewer: Hmm hmm.

Flynn: *So that means this and this* [pointing to two angles on angle C in triangle ACB] *together is half of 180 which makes it 90*.

As evident from his argument, Flynn did consider the values of the base angles of two inner triangles as unknowns and used the theoretical fact that angles on a straight line add to 180-degrees.

In general, using the diagram with algebraic methods was the new diagram use appeared in the truth-known Right Triangle Task. However, when previous diagram-given and truth-known tasks were considered, by drawing additional diagrams in the last Right Triangle Task, students showed an unusual diagram use because in other diagram-given and truth-known tasks students accepted the accuracy of the figure and did not produce new diagrams. That is, some students checked their arguments by drawing other diagrams in the Right Triangle Task even though they made the arguments depending on the figural properties of the given figure. That might be because of the structure of the previous Right Triangle Task with truth-unknown and inaccurate diagram features. Students probably realized that the claims in both Right Triangle Tasks are parallel and pursued the same proof approach they employed in the former Right Triangle Task. As discussed above, especially students who drew more than one diagram in the truth-unknown Right Triangle Task were more likely to produce various diagrams in the last task. Furthermore, other students who relied on the figural properties of the given diagram in the truth-unknown Right Triangle Task showed the same kind of reasoning in the last task as well.

Lastly, seven students used the diagrammatic register in their proof work either in their written explanation or on the diagram by labeling specific parts of the diagram.

Proving Actions of the Students

Table 4.9 shows that types of JUST and STEP were noticeable proving actions in the Right Triangle Task with accurate diagram and truth known feature. Students were more successful in proving overall than they were in the previous Right Triangle Task with inaccurate diagram and truth-unknown feature. Moreover, as discussed above in students' use of diagrams, some students applied figural properties of the figure which in most cases led to wrong or limited assumptions about the relations between angle values. Hence, wrong and limited proving actions were also salient in the last Right Triangle Task.

Table 4.9

Proving actions that nine students employed while proving the accurate diagram, truth-known Right Triangle Task

Proving Actions	Frequencies of the Codes
CONF-JUST	1
CRIT-STEP	2
CRIT-STEP-L	4
CRIT-INVEST	3
CLAR-ABOUT	2
CLAR-STEP-W	1
CLAR-REFINE	1
CLAR-STEP	1
CLAR-JUST-W	1
CLAR-INVEST	4
ABOUT	10
INVEST	**16**
JUST	10
JUST-L	4
JUST-W	4
JUST-G	**34**
JUST-G-L	20

JUST-G-W	8
JUST-S-L	2
JUST-V-L	1
STEP	**30**
STEP-L	17
STEP-W	10
ACCEP	3
ACCEP-L	2
REFINE	1
Repetition	6

Moreover, INVEST proving action was regularly applied by students because students employed several new drawings in the proof work to investigate if their arguments were valid. That proving action frequently appeared in the previous Right Triangle Task as well. Lastly, various proving interactions were performed in the Right Triangle Task as they were in the previous task. In particular, students were uncertain about the proof work they produced in both Right Triangle Tasks even though they used the figural features of the given diagram. Therefore, they performed proving interactions when they critiqued or clarified actions such as steps and investigations in their proof work.

Non-diagrammatic Register Task

Bisector Ray Task

The Bisector Ray Task is a diagram-given task in the second interview. It includes an accurate diagram and the truth is known but it was not presented in the diagrammatic register. The task was unique in this study being a non-diagrammatic register task but given with an unlabeled diagram. As usual, below the findings are presented with attention given to the semiotic resources, types of diagram use, and proving actions occurring in the students' proof production.

Semiotic Resources Used in the Bisector Ray Task

Gesture

In the Bisector Ray task, pointing at sides, angles, and figures were the most frequently applied semiotic resources (Table 4.10). All students reasoned with the diagrams, either the one provided in the task or ones they drew, while explaining their proof work.

Table 4.10

Semiotic resources that four students employed while proving the diagram-given Bisector Ray Task

	Semiotic Resources	**Frequencies of the Codes**
Visual	VY: visual-draw ray	1
	VH: visual-draw hatch/tick marks	**8**
	VR: visual-redraw figure	1
	VF: visual-draw a new figure/diagram	**6**
	VA: visual-draw angle marking	**12**
	VSC: visual-draw side congruency	2
	Total	30
Gesture	Gesture\GS: gesture-pointing at side/segment	68
	Gesture\GM: gesture-referring to movement	2
	Gesture\GY: gesture-pointing at ray	7
	Gesture\GV: gesture-pointing at vertex	6
	Gesture\GH: gesture-pointing at hatch/tick marks	2
	Gesture\GG: gesture-pointing at a given in the task	8
	Gesture\GP: gesture-pointing at a point	3
	Gesture\GF: gesture-pointing at figure/diagram	23
	\GA: gesture-pointing at angle	78
	Total	**197**
Symbolic	SV: symbolic-label vertex	3
	SP: symbolic-label point	1
	SA: symbolic-label angle	6
	Total	10

In general, students, applied gestures accompanying pronouns while explaining their proofs. The following excerpt by Deborah exemplifies the use of gestures with pronouns in students' reasoning.

Deborah: So, if it's dividing, I said that if, if *it's* [*pointing to the ray*] bisecting the angle [pointing to top angle in the given diagram] then it's also bisecting the opposite side [pointing to the opposite side] because, umm, if it's gonna hit it at a 90-degree angle. Because if it didn't, then it wouldn't be bisecting the angle, umm, if it, if *it weren't to hit at a 90-degree angle* [*pointing to the 90-degree angle marking on the diagram*]. It wouldn't be bisecting *this angle* [*pointing to the angle that the bisector ray emanates in the given diagram*] like *this example here, umm* [*pointing to her drawing in the explanation*].

Deborah's gestures here supplement her reasoning since she does not include precise terms for which parts of the diagram she refers to in her argument. However, one student, Megan, used gesture resources when stating precise geometric terms in her explanation, as follows:

Megan: So, um, I, basically, you can get two congruent triangles out of *triangle ABC* [*pointing to triangle ABC*]. Then with, uh, line BD and they'll be congruent because they have two equal sides and two equal or two congruent angles, uh*, angle ABD angle CBD also angle BDC and angle BDA* [*pointing to these angles in the given diagram*]. And then, side-lengths-wise, *they would have similar sides of DA and side DC* [*pointing to sides DA and DC on the given triangle*].

As seen from above excerpt, she used a precise language with the gesturing. In fact, the task was outside the diagrammatic register since the diagram was not labeled and the statement of the task did not include any geometric symbols or labels that refer to particular sides of the given diagram. However, Megan labeled the vertices of the given diagram and used them to call the sides or the angles of the given diagram in a precise way when she explained her proof.

Symbols and Visuals

Table 4.10 shows that both visual and symbolic resources also occurred in the Bisector Ray task. The most salient visual and symbolic resources that appeared in students' works were drawing a new diagram, drawing hatch/tick marks, and labeling the angle. In fact, in their proof attempts, two students drew more than one diagram that included hatch/tick marks and angle labeling on it. Figure 4.15 illustrates the works that Deborah and Flynn produced by drawing multiple diagrams in the Bisector Ray Task.

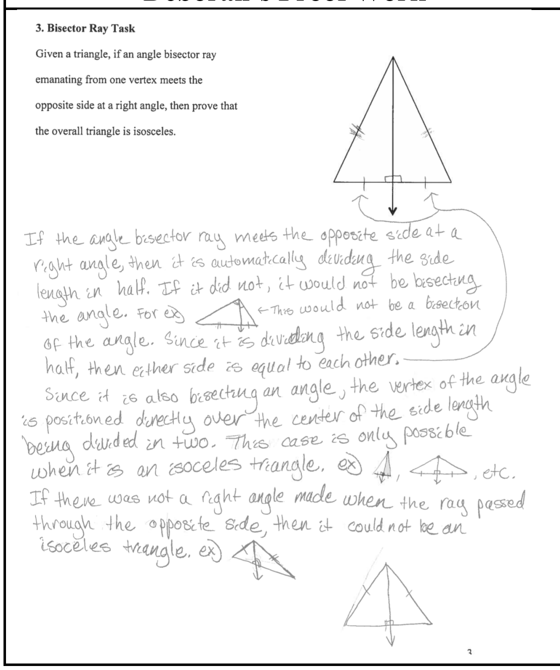
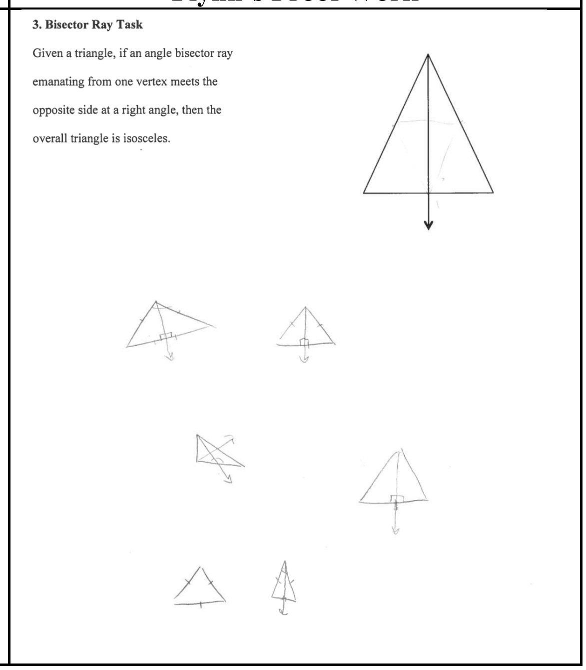
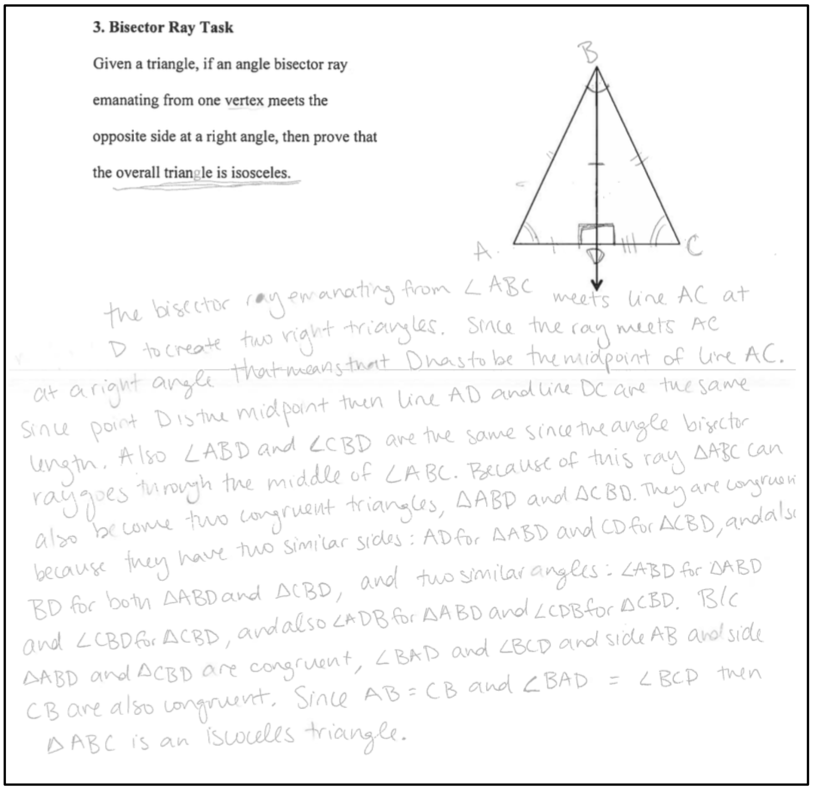
Deborah's Proof Work	**Flynn's Proof Work**
	

Figure 4.15. Deborah and Flynn's proof work on the Bisector Ray Task.

The nature of the two students' proofs seem to have a consideration of various types of triangles on which they tested the givens in the task. As seen in their work, neither student included the diagrammatic register in their written work.

In contrast, two other students preferred labeling vertexes of the diagram. Especially, one student, Megan, specifically included geometric terms in her written explanation (Figure 4.16).


3. Bisector Ray Task

Given a triangle, if an angle bisector ray emanating from one vertex meets the opposite side at a right angle, then prove that the overall triangle is isosceles.

the bisector ray emanating from ∠ABC meets line AC at D to create two right triangles. Since the ray meets AC at a right angle that means that D has to be the midpoint of line AC. Since point D is the midpoint then line AD and line DC are the same length. Also ∠ABD and ∠CBD are the same since the angle bisector ray goes through the middle of ∠ABC. Because of this ray ΔABC can also become two congruent triangles, ΔABD and ΔCBD. They are congruent because they have two similar sides: AD for ΔABD and CD for ΔCBD, and also BD for both ΔABD and ΔCBD, and two similar angles: ∠ABD for ΔABD and ∠CBD for ΔCBD, and also ∠ADB for ΔABD and ∠CDB for ΔCBD. B/c ΔABD and ΔCBD are congruent, ∠BAD and ∠BCD and side AB and side CB are also congruent. Since AB = CB and ∠BAD = ∠BCD then ΔABC is an isocelles triangle.

Figure 4.16. Megan’s proof work on the Bisector Ray Task.

As seen from above figure, Megan labeled the vertices of the given triangle and used geometric symbols such as angle and triangle markings accompanying to vertex labels to name precisely the particular angles, sides, and triangles inside the given triangle in her written explanation.

Students’ Use of Diagrams

Students used the diagram(s) in five obvious ways when proving the Bisector Ray task. First, sketching and working with more than one diagram, as described above, was present in two students’ proving (see Figure 4.15). By drawing more than one diagram, Flynn examined if the givens would be valid in different types of triangles such as scalene, equilateral, or obtuse triangles. He argued as follows.

Flynn: This [pointing to the task] has to be true cause, uh, *if this is a scalene triangle and if you take an angle and go down, it doesn't make any right angles. If you go down to the middle* [drawing a ray in one of his drawings] *with a different angle than 90, and, uh, it can be an equilateral, too. But I think I remember in geometry we were saying that equilateral triangles are also isosceles, but isosceles triangles aren't necessarily equilateral.*

Flynn considered two particular types of triangles and explored the givens with them to conclude his proof.

Second, two students merely worked with the given diagram in the Bisector Ray task (see Figure 4.16). Lee, for example, explained the way he used the givens in the diagram and concluded that two side lengths in the given triangle were equal.

Lee: So, because, umm, the ray splits it in half [pointing to point where the ray and side intersects] and *creates two 90 degree angles* [pointing to 90 degree angles in the diagram] on *either side, umm, making these two triangles identical* [pointing to two inner triangles inside the given triangle]. Uh, meaning that the hypotenuses [pointing to two hypotenuses in the diagram] are the exact same and their bases [pointing to two bases of the inner triangles] are the exact same.

Lee's reasoning, based on the unjustified assumption that the ray bisects the opposite side, also exemplifies the third type of diagram use—that is, using the physical properties of a figure. Lee, notably, inferred the equality of two sides in the given triangle by considering the figural properties of the diagram.

Interviewer: How do you say this bigger triangle is isosceles?

Lee : Um, because the hypotenuse [pointing to two hypotenuse] are the exact same and then, by doing that you already started the definition of isosceles where two side lengths are the same, but then, *by doing that you rule out, um, that it can't be an equilateral because since this is the hypotenuse of this and this is hypotenuse of this* [pointing to the sides that are opposite to 90 degree angles in two inner triangles and their corresponding triangles], it forces this to be smaller [pointing to the base sides on the side where bisector ray intersects] because hypotenuse has to be the biggest.

Interviewe: Hmm hmm.

Lee: So, um, it can't be an equilateral because, uh, this [pointing to the side that bisector ray meets] *would have to be the same exact size as these* [pointing to the other two sides of the big triangle, the two hypotenuses*]. But it can't because these are the hypotenuse on both sides. And then it can't be a right, it can't, the whole thing can't be a right triangle because, uh, all the side, it has to have a, a right triangle has to have a hypotenuse, but it doesn't. This doesn't have a hypotenuse* [pointing to the given triangle] *because these two side lengths are the exact same* [pointing to two sides, two hypotenuses, of the given triangle].

At the beginning of his reasoning, Lee claimed that two hypotenuses—two sides of the given triangle—are the same. He then used the figural properties of the given triangle (i.e., the appearance of the side lengths of the given triangle) and tried to eliminate the options of being an Equilateral or Right Triangle for the given triangle. Consequently, Lee argued that the given triangle could not be any type of triangle but an isosceles triangle.

The fourth type of diagram use combined figural and theoretical properties of the given figure. One student, Megan, reasonably used both properties in her proof (see Figure 4.16). As an illustration, in the following excerpt, she discusses the congruency of two inner triangles in the given triangle by providing logical arguments.

> *Megan:* So, um, I, basically, you can get two congruent triangles out of triangle ABC [pointing to triangle ABC]. Then with, uh, *line BD and they'll be congruent because they have two equal sides and two equal or two congruent angles, uh, angle ABD angle CBD also angle BDC and angle BDA [pointing to these angles in the given diagram]*.
>
> *Interviewer:* Hmm hmm.
>
> *Megan:* And then side, side lengths wise, *they would have similar sides of DA and side DC* [pointing to sides DA and DC on the given triangle] *and then also, side BD* [pointing to side BD]. And if, if there, um, similar or congruent triangles then all of the sides [pointing to sides each three sides of the big triangle] and angles would be the same which means that angle BAD and angle BCD [pointing to these angles] and then (inaudible) side BA and side BC [pointing to these sides] would be the same. And if that's true then triangle ABC [pointing to that triangle] would be an isosceles triangle.

Although Megan does not support her claim that the point where the bisector ray and the side of the triangle meets is the midpoint, she later argues that same two angles and same two sides satisfy the congruency of two inner triangles.

> *Megan:* And *you get the congruency from the same two angles and the same shared side* [pointing to bisector and right angles and equal sides AD and BC, and shared side BD].

Note that Megan mentions about the same angles and same sides in two inner triangles in the given triangle. As shown previously in Figure 4.16, she drew same angle markings for the angles that bisector ray constructed in the given triangle and used hatch/tick marks to show the shared side and the sides that were not proven as congruent sides yet. By using the given information with the angle-side-angle postulate, she concludes the congruency of two inner triangles. Apparently, labeling and drawing the given information in the triangle helped Megan built on the previously known facts and hence, she provided a logical reasoning in her proof.

Finally, the last type of diagram use includes both figural and theoretical properties of diagrams but relies mostly on the figural aspects. In other words, as reviewed above, Deborah and Flynn sketched several diagrams in their proof work and examined the claim in various triangles, yet their conclusions were based on figural features of particular types of diagrams.

Interviewer: So in that case, this [pointing to one of his drawing] is not an isosceles.

Flynn: No, that's a scalene triangle, and it can't make any right angles with the bisector ray. It just can't be scalene. It has to have at least two equal sides.

Interviewer: So, if this [pointing to one of his top drawings] has two equal sides, you say this will always be right, a right angle.

Flynn: Yes, because the bisector ray will be perpendicular to the base side

Here, Flynn concluded by checking the givens on a scalene triangle and considering the visual aspects of the diagram. Thus, he did not find any conceptual reasoning with the scalene triangle to have a logical relationship between angles in the diagram.

Proving Actions of the Students

In the Bisector Ray task, there were significant proving actions in students' proof works. For example, as mentioned above, gesturing frequently appeared in students' reasoning which led to high frequencies of JUST-G codes (Table 4.11).

Table 4.11

Proving actions that four students employed while proving the diagram-given Bisector Ray Task

Proving Actions	Frequencies of the Codes
CONF-INVEST	2

CRIT-STEP-L	1
CRIT-INVEST	1
CRIT-CLAIM	1
CLAR-ABOUT	4
CLAR-CLAIM	1
CLAR-STEP-L	7
CLAR-STEP	1
STRUC	2
ABOUT	9
END	1
INVEST	**6**
JUST	8
JUST-L	**19**
JUST-W	5
JUST-G	16
JUST-G-L	**16**
JUST-G-W	5
JUST-V	1
JUST-V-L	2
STEP	9
STEP-L	**24**
STEP-W	8
ACCEP	11
ACCEP-L	4
CLAIM	1
Repetition	6

Moreover, two students drew multiple diagrams and hence, had several INVEST proving actions in their proofs. For example, Deborah discussed different possibilities of diagrams in her proof by drawing various types of triangles (see Figure 4.15). The following figure shows INVEST proving actions in Deborah's proof work.

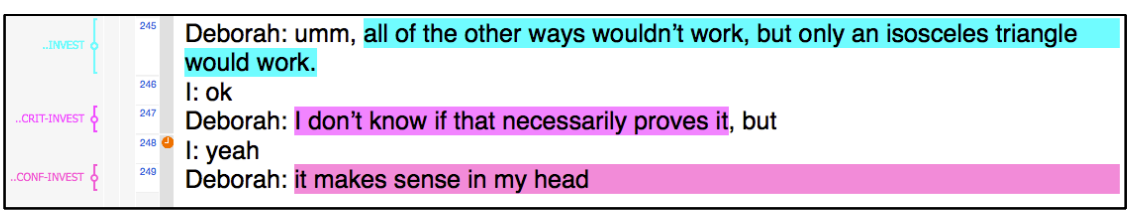


Figure 4.17. INVEST proving actions in Deborah's proof work.

As mentioned above, there was only one student who produced a valid proof on this task. Three other students either were stuck on the figural properties of the diagram or could not combine both figural and theoretical features of a diagram inevitably in their proofs. Therefore, limited and wrong JUST and STEP proving actions were salient in the Bisector Ray task. A sample excerpt with proving actions is presented in Figure 4.18.

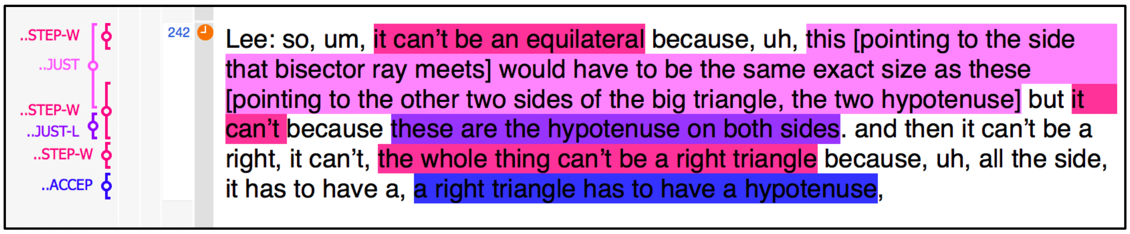


Figure 4.18. Some proving actions in Lee's proof for the Bisector Ray Task.

As shown in the above figure, Lee presents a wrong step when he argued that the given triangle could not be an equilateral. In fact, the given triangle might be an equilateral triangle because it was not apparent from the given information. Moreover, he argued that with two sides of the given triangle being the hypotenuses undercuts the possibility for the given triangle become an equilateral triangle. Because that argument was built on the previous wrong step, it was coded as a limited justification. Likewise, his argument with the given triangle not being a right triangle had similar proving actions in his reasoning.

In the Bisector Ray task, only Megan provided a valid proof through combining both figural and conceptual features on the given diagram that she labelled and referred to via the diagrammatic register. Deborah and Flynn, who created various figures in the non-diagrammatic register, reasoned with figural and conceptual properties of the diagram, but they proved the claim by using specific types of figure. Hence, they produced reasonable but limited proofs. One student, Lee, who worked with the given diagram partially in the diagrammatic register, did reason mostly with the figural properties of the diagram and failed to produce a valid proof.

Summary

Four diagram-given tasks in the first interview and one diagram-given task in the second interview were discussed regarding truth-unknown, diagrammatic register, and non-diagrammatic register features of the tasks. Under each task, I presented the semiotic resources, diagram use, and proving actions engaged by students. Several patterns of semiotic resources occurred under each diagram-given task. For example, gesturing was frequent during students' oral explanations to their proof work and was used to clarify what the pronouns were referring to in students' reasoning. Thus, gesture resources were especially active in students' justifications to the arguments and hence, of the proving actions, JUST-G were common in all tasks. However, visual and symbolic resources varied for some tasks. In particular, drawing a new figure was peculiar to the truth-unknown Right Triangle Task and non-diagrammatic register Bisector Ray Task. Moreover, students' proving of the Right Triangle Task and Bisector Ray task included more INVEST proving action.

Regarding the types of diagram use, one diagram use, reliance on the figural properties, was notable among five diagram-given tasks; in contrast, having multiple drawings to conclude the validity of the claim or considering the mobility of some parts of the given diagram occurred in the truth-unknown and non-diagrammatic register tasks (Right Triangle Task and Bisector Ray Task). Students used the diagrammatic register in their works when proving diagrammatic register tasks (e.g, Isosceles Task and Midpoints Tasks). Whereas, three students did not implement the diagrammatic register in their proofs in the non-diagrammatic register Bisector Ray Task.

In three diagram-given and truth-known tasks, some students produced valid arguments as a proof, but in the Right Triangle Task with an inaccurate diagram, no student presented a valid proof. Students who used the given figure with symbolic resources and integrated the conceptual properties of the diagram into their reasoning were more successful on strong proof production (e.g., Deborah in the Midpoints Task and Megan in the Bisector Ray Task).

Diagram-Free Tasks

In this section, I present the findings related to students' use of diagrams in four diagram-free tasks from the interviews. Diagram-free tasks were presented as the first task in the first interview. As mentioned before, there were five students who worked on the diagram-free Isosceles Task first and four students who worked on the diagram-free Midpoints Task first. Unlike the first interview, where only one diagram-free task was administered, the second interview involved two—the Perimeter Task and the Pentagon Task. As described in Chapter 3, the Perimeter Task was given at the beginning, and the Pentagon Task was given at the end of the second interview. Table 4.12 shows the features of each task and the number of students who worked on them.

Table 4.12

Diagram-free tasks from the first and second interviews and number of students worked on them

Interview	Task Name	Task Features	Number of Students
1st Interview	Isosceles Task	with no diagram truth-known in diagrammatic register	5
1st Interview	Midpoints Task	with no diagram truth-known in non-diagrammatic register	4
2nd Interview	Perimeter Task	with no diagram truth-known in non-diagrammatic register	4
2nd Interview	Pentagon Task	with no diagram truth-unknown in non-diagrammatic register	4

Truth-Unknown Task

Pentagon Task

The Pentagon Task is the diagram-free task presented as the fourth task in the second interview. It includes non-diagrammatic register and truth-unknown features.

Semiotic Resources Used in the Pentagon Task

Gesture

Gesture resources were frequently used in the Pentagon Task, as usual. As shown in the Table 4.13, pointing at figure/diagram and pointing at an angle were the most common gesture resources applied in the work on the task.

Table 4.13

Semiotic resources that four students employed while proving the diagram-free Pentagon Task

	Semiotic Resources	Frequencies of the Codes
Visual	VS: visual-draw side/segment	25
	VAC: visual-draw angle congruency	7
	VL: visual-draw line	4
	VR: visual-redraw figure	1
	VF: visual-draw a new figure/diagram	**20**
	VP: visual-draw point	2
	VA: visual-draw angle marking	11
	Total	70
Gesture	**Total**	**143**
Symbolic	SSE: symbolic-solving an equation	2
	SV: symbolic-label vertex	4
	SG: symbolic-use geometric symbols	3
	SE: symbolic-writing an equation	**5**
	Total	14

The following excerpt shows Flynn's reasoning by pointing to angles of a pentagon he drew.

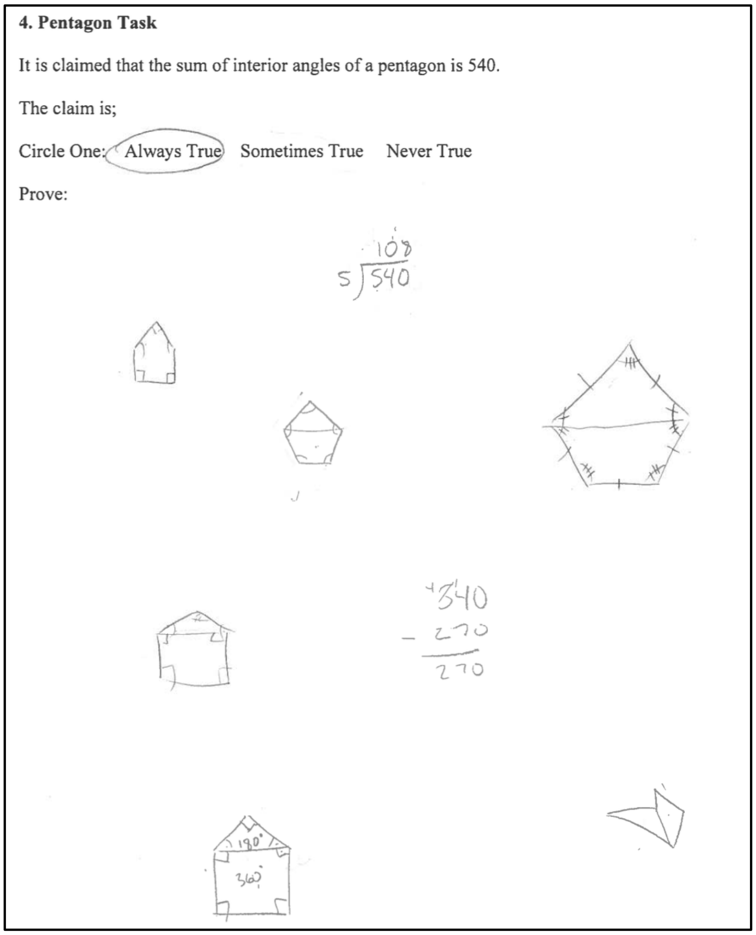
Flynn: Those [*pointing to inner angles of a pentagon*] all have to make up 540, but I don't know if that will equal up to 540, so I am not sure.

> …
>
> *Flynn:* But if *this one is 90, this one is 90, and this one is 90* [*pointing to three inner angles of one of the top pentagons he drew*]. That's, uh, *270*, so these [*pointing to other two angles in the pentagon*] would each have to be [student makes calculations], uh, 270, makes sense, uh, so this has to be *135, this has to be 135* [*pointing to other two angles in the pentagon*] and it looks like 135.

Flynn gestured to the angles where he used pronouns to signify them. Likewise, two other students showed similarities on the use of gesture resources. It is noteworthy that these students did not introduce labels to place them in the diagrammatic register either on the diagrams they produced or written explanation they had in their proofs.

Symbols and Visuals

Various symbolic and visual resources occurred in the Pentagon Task. One of the visual resources that frequently appeared in the Pentagon Task was drawing a diagram. In fact, three students drew more than one diagram in their proof works. As an example, Figure 4.17 shows Flynn's work on the Pentagon Task.


4. Pentagon Task

It is claimed that the sum of interior angles of a pentagon is 540.

The claim is;

Circle One: Always True Sometimes True Never True

Prove:

108
5)540

540
− 270
270

180°
360°

Figure 4.19. Flynn's proof work with visual and symbolic resources.

Moreover, in Flynn's proof work, symbolic resources such as numerical calculations were apparent.

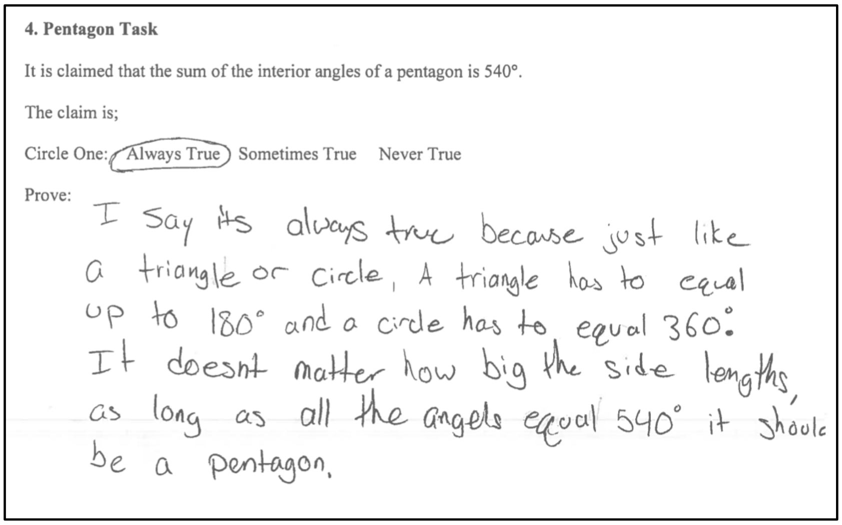
However, one student, Lee, did not draw any diagram in his proof work (Figure 4.20). Instead, he provided a general statement about pentagons by indicating a relationship between pentagons and triangles.

Interviewer: And you did not draw any pentagon.

Lee: Uh, no because really, *it's kinda like the triangle situation, um, you can have a right triangle, an isosceles, an equilateral and all look different, but they are still triangles*.

Interviewer: Hmm hmm.

Lee: So, umm, I don't know how many different types or like figures of a pentagon you can draw, but there is probably more than just one.



4. Pentagon Task

It is claimed that the sum of the interior angles of a pentagon is 540°.

The claim is;

Circle One: Always True Sometimes True Never True

Prove:

I say its always true because just like a triangle or circle, A triangle has to equal up to 180° and a circle has to equal 360°. It doesnt matter how big the side lengths, as long as all the angels equal 540° it should be a pentagon.

Figure 4.20. Lee's proof work with no diagram on the Pentagon Task.

Students' Use of Diagrams

Before discussing the types of diagram use, it is noteworthy to state again that one student, Lee, did not produce any figure in his proof work. As shown in the above excerpt and figure, Lee argued that the claim would always be true by making a generalization based on the sum of the interior angle of the triangles.

There were, however, three notable types of diagram use in three students' proving of the Pentagon Task. The first one is that two students, Flynn and Deborah, drew more than one pentagon in their work (see Figure 4.19). With each drawing, they considered a different type of pentagon and tried to reason with the inner angles of the pentagons to determine if their sum would always, sometimes, or never be true. As an example, Flynn discussed the angles of a pentagon by constructing a square and a triangle in it.

Flynn: Well, I mean like this one [pointing to his drawing of the pentagon with right angles], *if I have a square and then* [student draws another pentagon and constructs a square and a triangle in it by drawing a segment], *if this is a right angle* [pointing to one of the angles in the square] all these are right angles [pointing to the other three angles in the square].

Interviewer: Hmm hmm.

Flynn: So this [one of the inner angles of a square] is gonna be 90, so this has to be, I can't remember what I said, uh, well this has to be 180 [pointing to the triangle], so these have to be 45 45, so this [pointing to the angles that share an angle from the square and one angle from the triangle] would have to be 135 and that would make this [pointing to the pentagon] equal up to 540.

Interviewer: Hmm hmm

Flynn: So for this one, *I can use the square and the triangle to figure out that the pentagon is 540*.

In fact, the use of several special kinds of pentagons showed that both Flynn and Deborah somehow drew conclusions based on the figural properties of the diagram and both argued that the task would be *sometimes true*. In the following, Deborah mentioned why she thinks the claim would be sometimes true.

Deborah: I was thinking maybe the same thing [as happens with triangles] apply to pentagons, but at the same time triangles are kind of special, at least I think they are, but I'm not quite sure. So, umm, *I think, I don't know if, if the properties would be different for a pentagon, if they were different, then I would say that it would be sometimes true*.

Although Deborah considered several diagrams and stated that the variability of the sides and angles in the pentagon would not change the sum of the interior angles of the pentagon, she was still not certain it was always true.

The second prominent type of diagram use was using the conceptual properties of the diagram. In particular, one student, Megan, used the conceptual properties of the diagram (Figure 4.21).

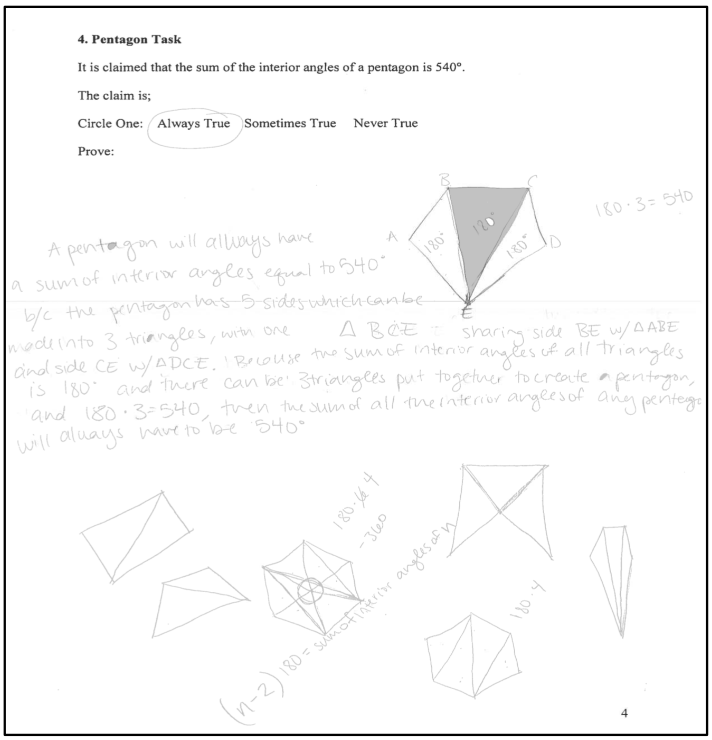


Figure 4.21. Megan’s proof work on the Pentagon Task.

Note that Megan drew one pentagon and constructed three triangles inside the pentagon. She then reasoned with the sum of the interior angles of the triangles and added them to prove the sum of the interior angles of the pentagon. Moreover, she labeled the vertices of the Pentagon and applied diagrammatic register in her written explanation which clarified how she reasoned with the triangles to prove the claim of the task. Similar to Megan, two other students, Flynn and

Deborah, also applied to the conceptual features of the diagram after I guided them to use the previously known facts such as the sum of the interior angles of a quadrilateral or triangle.

The third type of diagram use was the mobility of a diagram; that is, variation in the diagrams by considering some sort of dynamic changes in the figures. That approach was applied by only one student, Deborah, in the proof work (Figure 4.22).

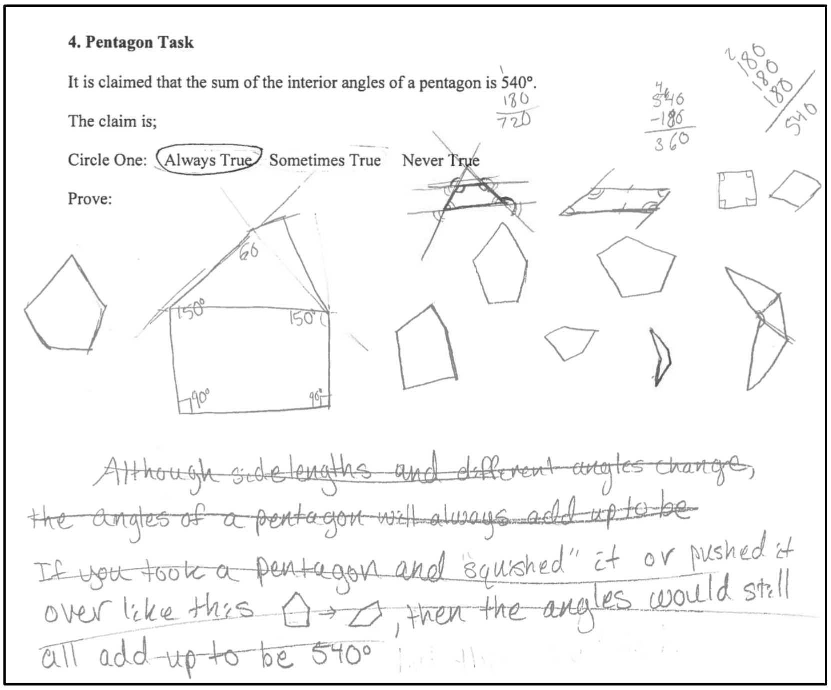


Figure 4.22. Deborah's proof work on the Pentagon Task.

Essentially, Deborah discussed the modifications that would happen on a pentagon when applying physical changes to it.

> *Deborah:* What I was saying like right here [pointing to her drawings in the explanation] was *if you, like, pushed, if you had a, you had like a, a model of a pentagon and you like pushed that over and the angles would still be the same because they kind of, some angles become bigger, some angles become smaller and they kind of balance each other of to make it the same.*

Interviewer: So when you say it's pushed over it is more like changing the shape of the pentagon in which these sides may …

Deborah: The side lengths will stay the same. Well, the side lengths will stay the same but the angles wouldn't.

Here, Deborah discusses that even with the variation of the sides of a pentagon, the sum of the interior angles of the pentagon will not change.

Proving Actions of the Students

A wide range of proving actions appeared in the students' work on the Pentagon Task (Table 4.14). As discussed in the semiotic resources used in the Pentagon Task, three students sketched various diagrams during the investigation of the truth of the claim which led to the INVEST proving action as they checked whether the claim held for their diagrams. Moreover, students clarified or critiqued their proving actions which resulted in codes such as CRIT-INVEST or CLAR-INVEST.

Table 4.14

Proving actions that four students employed while proving the diagram-free Pentagon Task

Proving Actions	Frequencies of the Codes
CRIT-STEP	2
CRIT-ABOUT	2
CRIT-INVEST	**7**
CRIT-CLAIM	1
CLAR-ABOUT	3
CLAR-STRUC	1
CLAR-STEP-L	1
CLAR-STEP	2
CLAR-INVEST	**3**
STRUC	1
ABOUT	10
INVEST	**27**
JUST	8
JUST-L	6

JUST-G	11
JUST-G-L	2
JUST-S	1
JUST-V	1
STEP	15
STEP-L	9
STEP-W	1
ACCEP	7
ACCEP-L	5
REFINE	1
Repetition	4

In the following figure, an excerpt of Deborah's proving actions and interactions are presented.

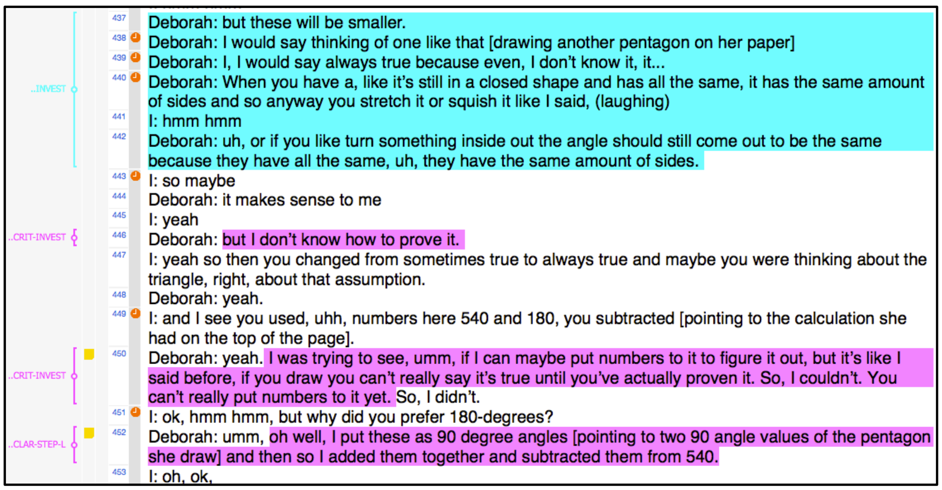

..INVEST
437 Deborah: but these will be smaller.
438 Deborah: I would say thinking of one like that [drawing another pentagon on her paper]
439 Deborah: I, I would say always true because even, I don't know it, it...
440 Deborah: When you have a, like it's still in a closed shape and has all the same, it has the same amount of sides and so anyway you stretch it or squish it like I said, (laughing)
441 I: hmm hmm
442 Deborah: uh, or if you like turn something inside out the angle should still come out to be the same because they have all the same, uh, they have the same amount of sides.
443 I: so maybe
444 Deborah: it makes sense to me
445 I: yeah
..CRIT-INVEST
446 Deborah: but I don't know how to prove it.
447 I: yeah so then you changed from sometimes true to always true and maybe you were thinking about the triangle, right, about that assumption.
448 Deborah: yeah.
449 I: and I see you used, uhh, numbers here 540 and 180, you subtracted [pointing to the calculation she had on the top of the page].
..CRIT-INVEST
450 Deborah: yeah. I was trying to see, umm, if I can maybe put numbers to it to figure it out, but it's like I said before, if you draw you can't really say it's true until you've actually proven it. So, I couldn't. You can't really put numbers to it yet. So, I didn't.
451 I: ok, hmm hmm, but why did you prefer 180-degrees?
..CLAR-STEP-L
452 Deborah: umm, oh well, I put these as 90 degree angles [pointing to two 90 angle values of the pentagon she draw] and then so I added them together and subtracted them from 540.
453 I: oh, ok,

Figure 4.23. Some proving actions in Deborah's explanation to the proof of Pentagon Task.

Deborah drew a pentagon and reasoned with that diagram to predict the truth value of the claim. She argued that the variations of the sides for a pentagon would not change the sum of the interior angles of a pentagon, but she was still hesitant if she could prove it by assigning number values to the inner angles of the pentagon.

However, one of the students, Lee, who circled *always true*, did not base his arguments on logical premises and hence, did not support a valid conclusion. To clarify, as shown above in

Figure 4.20, Lee made a general assumption by using the sum of the interior angles of a triangle and assumed that the claim would be true for all pentagons without producing any valid reasoning. He was then asked the same question in another way.

> *Interviewer:* OK, *what if I asked you what is the interior, the sum of the interior angles of a pentagon? How would your solution be different than this one do you think?*
>
> *Lee:* Um, I am not a hundred percent sure but my guess is 540 [pointing to the number 540 in the last task].
>
> *Interviewer:* But I don't know a lot about pentagons. So, I don't know for sure what the exact number is or, or anything. *I don't know because we're kinda always told what it has to equal up to, like a triangle and never really specifically figured it out like exactly what it is, we're kinda just told. So, I don't know which way I would prefer to learn how to, how to do that.*

According to Lee, the sum of the interior angles of a diagram is something that should be already known. Thus, he was not even sure how to prove the claim.

Diagrammatic Register Task

Isosceles Task

The diagram-free Isosceles Task was given with the truth-known and in the diagrammatic register (see Table 4.12). Five students received the diagram-free Isosceles Task at the beginning of the first interview.

Semiotic Resources Used in the Isosceles Task

Gesture

All five students constructed a figure in their proofs and then used gesture resources on the diagram in their reasoning. Gesturing on the angles and sides of a diagram was the most frequent gesture resources used in the Isosceles Task (Table 4.15). For example, in the following excerpt, Megan explains the diagram she drew and gestures on the angles and vertexes of the diagram.

> *Megan*: When I drew this one, I did A, B, C [*pointing to the vertexes of the triangle she drew*] and I wanted A to be at the top since those two are the [*pointing to the angles CAE and DAB*] congruent.

As can be seen from the above excerpt, Megan gestured on specific parts of the diagram when calling the names of the vertexes or using pronouns to refer to those particular portions of the diagram. Megan's use of gesture resources on the diagram she drew represents a common use of gesture resources that the other students also employed on the figure they drew.

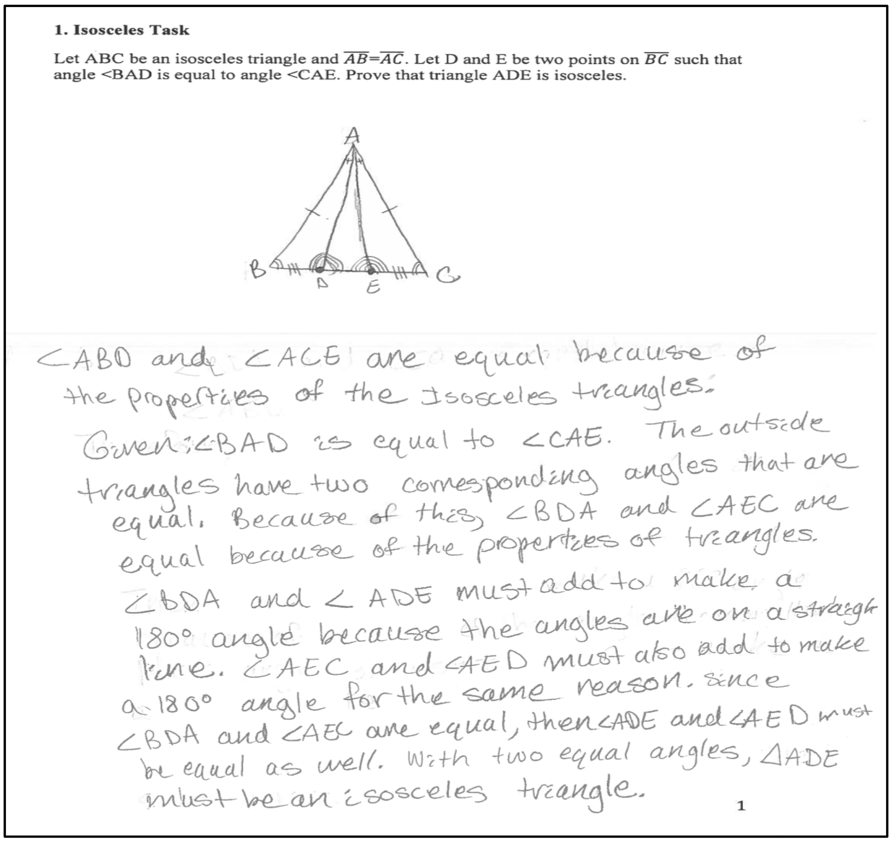
Table 4.15

Semiotic resources that five students employed while proving the diagram-free Isosceles Task

	Semiotic Resources	Frequencies of the Codes
Visual	VS: visual-draw side/segment	10
	VH: visual-draw hatch/tick marks	9
	VR: visual-redraw figure	**4**
	VF: visual-draw a new figure/diagram	**5**
	VP: visual-draw point	9
	VA: visual-draw angle marking	9
	VSC: visual-draw side congruency	4
	Total	50
Gesture	Total	**83**
Symbolic	SV: symbolic-label vertex	17
	SP: symbolic-label point	7
	Total	24

Symbols and Visuals

Each of the five students used visual and symbolic resources in addition to the gesture resources discussed above. In fact, all five students drew a diagram in the diagram-free Isosceles Task and labeled the vertex or points on the figure by using the givens in the task. Drawing a diagram and labeling particular parts in the diagram indicated that students employed the diagrammatic register in their geometrical reasoning, building upon the fact that the task was presented with labels and givens as is typical in the diagrammatic register. As an example, Deborah's proof work on the Isosceles task is shown in Figure 4.24 to illustrate a sample interpretation and use of a diagram with visual and symbol resources.



1. Isosceles Task

Let ABC be an isosceles triangle and $\overline{AB}=\overline{AC}$. Let D and E be two points on $\overline{BC}$ such that angle <BAD is equal to angle <CAE. Prove that triangle ADE is isosceles.

$\angle ABD$ and $\angle ACE$ are equal because of the properties of the Isosceles triangles. Given: $\angle BAD$ is equal to $\angle CAE$. The outside triangles have two corresponding angles that are equal. Because of this, $\angle BDA$ and $\angle AEC$ are equal because of the properties of triangles. $\angle BDA$ and $\angle ADE$ must add to make a 180° angle because the angles are on a straight line. $\angle AEC$ and $\angle AED$ must also add to make a 180° angle for the same reason. Since $\angle BDA$ and $\angle AEC$ are equal, then $\angle ADE$ and $\angle AED$ must be equal as well. With two equal angles, $\triangle ADE$ must be an isosceles triangle.

1

Figure 4.24. Deborah's proof work on the diagram-free Isosceles Task.

Students' Use of Diagrams

Each student who worked on the diagram-free Isosceles Task drew and used a diagram in their proof work. All students labeled the givens on the figure they drew and provided a written explanation with a diagrammatic register in their proofs (e.g., Figure 4.24). Only one student, Flynn, did not provide a written explanation accompanying the diagram he created. However, he used the diagrammatic register in his verbal report as can be seen in Flynn's verbal explanation to the proof of the Isosceles Task in the following excerpt.

> *Flynn:* OK so if, ah, *BAD* [*pointing to angle BAD*] *has to equal the same as CAE* [*pointing to angle CAE*]. Then that would mean that these have to be, *point E and point D have to be, point E has to be the same distance from point B as D does from C* [*pointing to points B, D, E, and C*].
>
> *Interviewer:* Hmm hmm.
>
> *Flynn*: Which would make this right here [drawing the top angle of the inner triangle] the same angle and these two are the same angle [drawing the base angles of the inner triangle] and so that this is an isosceles triangle.

Stating the names of the angles while pointing at them in the diagram was considered as Flynn's use of diagrammatic register in the proof work.

In addition to drawing a figure in the proof work and applying diagrammatic register in their study, students showed two prominent types of diagram use, using only the figural properties and using both figural and conceptual properties. First, three students used figural properties of the diagram without making a strong emphasis on the conceptual properties of the arguments they made. That is, students stated the visual features of the figure they drew by using symbolic and visual resources; however, they could not provide valid reasons for the conceptual properties they used on the diagram. As an example, Ben proved the Isosceles Task by sketching a diagram and writing an explanation for that (Figure 4.25). Although Ben drew a logical diagram, his argument in his written explanation "AD is spaced the same distance from ∠ACE as AE is from ∠ABD." was not supported by a valid justification.

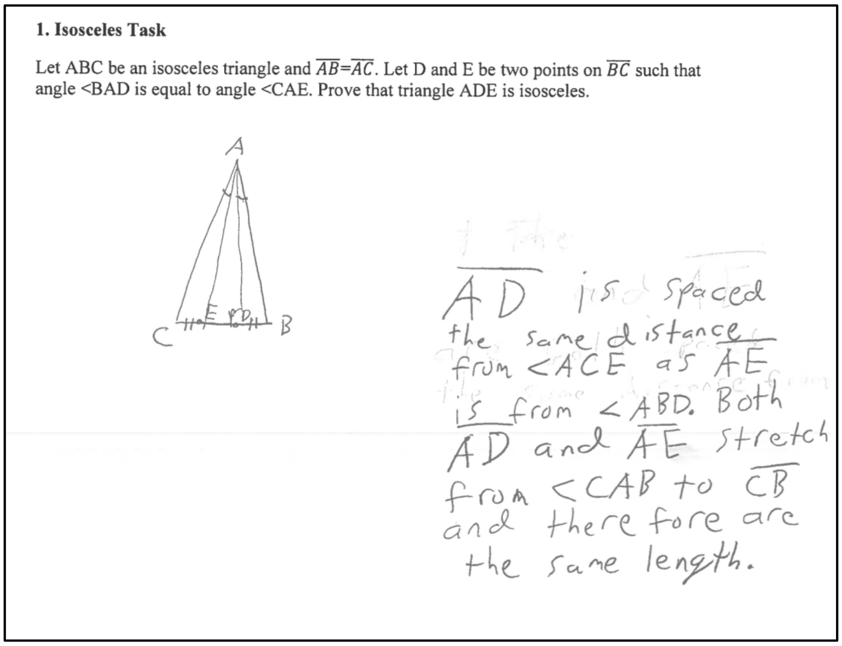


Figure 4.25. Ben's proof work on the diagram-free Isosceles Task.

In his oral explanation for his proof work, Ben provided the same answer as he did in the written one.

> *Ben:* So, *I drew the lines and I said that AD, uh line AD, is spaced like the same distance as line*. So, it is the same distance away from the angle ABC [pointing to angle ABC] as line AE is from the angle ACB [pointing to angle ACB].
>
> *Interviewer:* Hmm hmm.
>
> *Ben:* And, umm, *both lines stretch from A or from angle CAB* [pointing to angle CAB] that is down to line CB [pointing to the segment CB] and *since they are the same distance from either side, then they have to be the same length which makes it an isosceles triangle*.

Ben's explanation to his proof of isosceles triangle is grounded in reasoning that contains visual features. However, his arguments fail to provide conceptual reasoning because it is not clear how being the same distance apart from the corresponding sides guarantee that both AE and AD are the same length.

Second, two students, Deborah and Megan, applied both figural and conceptual properties on the diagram and gave valid reasoning in their proof work. As shown in Figure 4.24, Deborah drew a figure and used the givens on the diagram. She blended the figural properties with previously known facts and hence concluded a valid proof in the Isosceles Task (e.g., "∠BDA and ∠ADE must add to make a 180-degree angle because the angles are on a straight line"). Likewise, Megan based her reasoning on the physical and conceptual properties and constructed a valid proof (Figure 4.26). In fact, she argued about the congruency of two inner triangles (triangles ACE and ABD in the diagram) by using the angle-side-angle postulate, which led her to determine the congruency of two sides of the triangle ADE. Furthermore, it is noteworthy that in both Deborah's and Megan's proof work, it was noticeable that they actively used the diagram with symbolic and visual resources. In other words, they drew an isosceles triangle by using the given information in the task and labeled the equal sides and angles by using symbolic and visual resources. Apparently, applying both visual and symbolic resources helped them to move forward in their reasoning and employ previously known facts in their arguments such as using angle-side-angle postulate or the straight line has 180-degrees angle. Consequently, both Deborah and Megan produced valid proving actions in their reasoning.

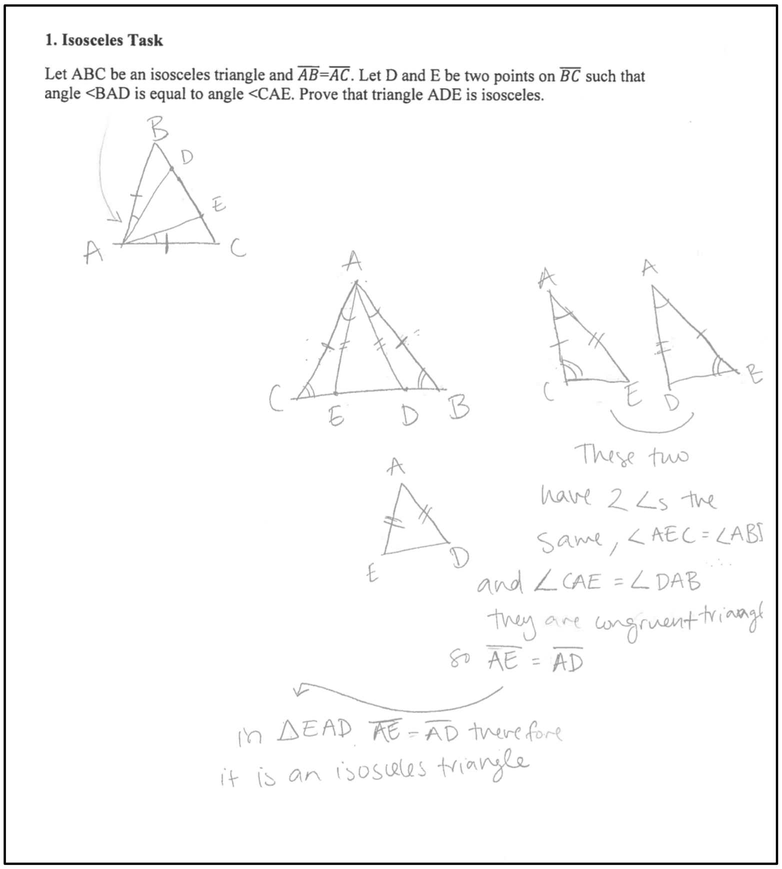


Figure 4.26. Megan's proof work on the Isosceles Task.

Proving Actions of the Student

Table 4.16 shows the proving actions that occurred in students' proof work on the Isosceles Task. As discussed above, gesture resources were the most common semiotic resources in the Isosceles Task. Thus, justifications by gesturing are familiar proving actions that arose in students' geometrical reasoning.

Table 4.16

Proving actions that five students employed while proving the diagram-free Isosceles Task

Proving Actions	Frequencies of the Codes
CLAR-REFINE	1
CLAR-STEP	2

CLAR-JUST-L	1
CLAR-INVEST	1
STRUC	1
ABOUT	2
INVEST	3
JUST	9
JUST-L	6
JUST-W	2
JUST-G	23
JUST-G-L	11
JUST-G-W	1
STEP	16
STEP-L	13
STEP-W	3
ACCEP	2
ACCEP-L	2
REFINE	1

Three students who only used figural properties in their proof work presented mostly limited or wrong justifications and steps in their proof work (JUST-L, JUST-W, STEP-L, STEP-W). As an example, Figure 4.27 illustrates proving actions that happened in Flynn's reasoning on the Isosceles Task.

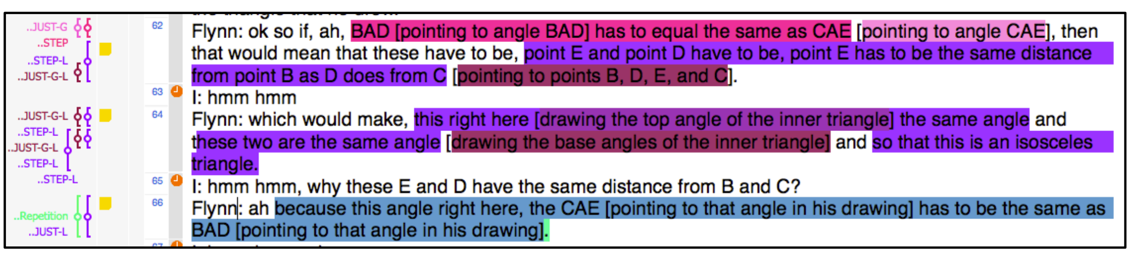


Figure 4.27. Proving actions in Flynn's proof work on the Isosceles Task.

Although Flynn's premises are correct, they were not supported by valid justifications. He claims that two inner points (i.e., point E and point D) on the base of the bigger triangle (i.e., Triangle ABC) are the same distance away from the corresponding vertices (i.e., vertex B and vertex C, respectively). Whereas his argument is reasonable, he did not sufficiently justify how the points he pointed at would be the same distance apart from the vertices of the outside triangle. Therefore, his justification and step actions were coded as Limited.

Students who used both figural and conceptual properties produced acceptable JUST and STEP proving actions in their proofs. As discussed above, Deborah employed both figural and conceptual properties of the diagram. She also provided valid proving actions in her work. Figure 4.28 shows an excerpt with proving action codes from Deborah's proof work.

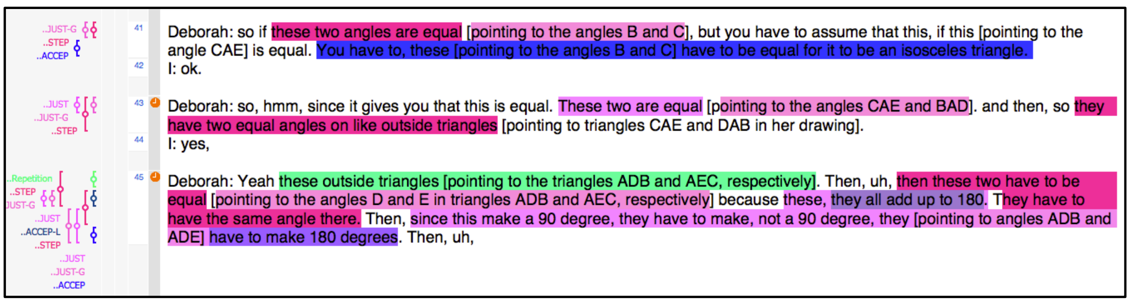


Figure 4.28. Proving actions in Deborah's proof work on the Isosceles Task.

As can be seen from her excerpt, she blends the givens in the task with previously known facts in geometry which helps her move forward in her proof with logical steps and justifications. Recall that Deborah's proof work on the Isosceles Task was presented in Figure 4.24. She drew an isosceles triangle (i.e., Triangle ABC) and identified the two points (i.e., points D and E) on the base of the isosceles triangle by using the given information. She then labeled and drew angle markings for the equal sides and angles on the triangle she constructed which helped her to figure out the congruency of the base angles of the inner triangle (i.e., triangle ADE) inside the triangle she drew.

Non-diagrammatic Register Tasks

Midpoints Task

The Midpoints Task was presented with no diagram, truth-known, and non-diagrammatic register features (see Table 4.12). The Midpoints Task was the first task offered to four students in the first interview.

Semiotic Resources Used in the Midpoints Task

Gesture

As usual, the most frequently used semiotic resources were gesture resources. Pointing at a side/segment, figure, or point was the most common gesture resources students employed in the diagram-free Midpoints Task as shown in Table 4.17, gesturing at diagrams that they produced themselves.

Table 4.17

Semiotic resources that four students employed while proving the diagram-free Midpoints Task

	Semiotic Resources	**Frequencies of the Codes**
Visual	VS: visual-draw side/segment	23
	VH: visual-draw hatch/tick marks	13
	VF: visual-draw a new figure/diagram	8
	VP: visual-draw point	18
	VA: visual-draw angle marking	2
	VSC: visual-draw side congruency	4
	Total	68
Gesture	GW: gesture-pointing something in the work	1
	GS: gesture-pointing at side/segment	66
	GV: gesture-pointing at vertex	1
	GG: gesture-pointing at a given in the task	1
	GP: gesture-pointing at a point	19
	GF: gesture-pointing at figure/diagram	44
	GA: gesture-pointing at angle	4
	Total	**136**
Symbolic	SS: symbolic-label side/segment	14
	SA: symbolic-label angle	3
	Total	17

The following excerpt from Lee's reasoning is an example of the implementation of some common gesture resources in proving.

Lee: Basically, what I was saying you like, if you did the, if you split it here and here [*pointing to the triangle midpoints*]

Interviewer: Hmm hmm.

Lee: Since *it's the same exact midpoint* [*pointing to two midpoints and the segment joining the midpoints*], that would be the same exact height [*pointing to the half of the height*].

Interviewer: Hmm hmm.

Lee: And, umm, since you split in the middle here [*pointing to one side of the triangle*] it would be, it cuts the normal length in half [*pointing to the base length*] and then, so these [*pointing to two segments of the base length*] would be the same.

In this excerpt, Lee gestures on certain parts of the diagram he drew to explain exact meanings in his argument. Moreover, he gestured to points and segments but not angles because points and segments were the focus of the claim in the task. That approach was evident in the other students' proof works as well.

Symbols and Visuals

In addition to gesture resources, visual resources frequently appeared in students' proof works (see Table 4.17). Each of the four students drew at least one diagram in their proofs. As shown in Figure 4.29, Nathan and Lila sketched various diagrams in the Midpoints Task. In particular, Nathan produced an equilateral, a right triangle, and an obtuse triangle in his proof to Midpoints Task.

Nathan's Proof	**Lila's Proof**

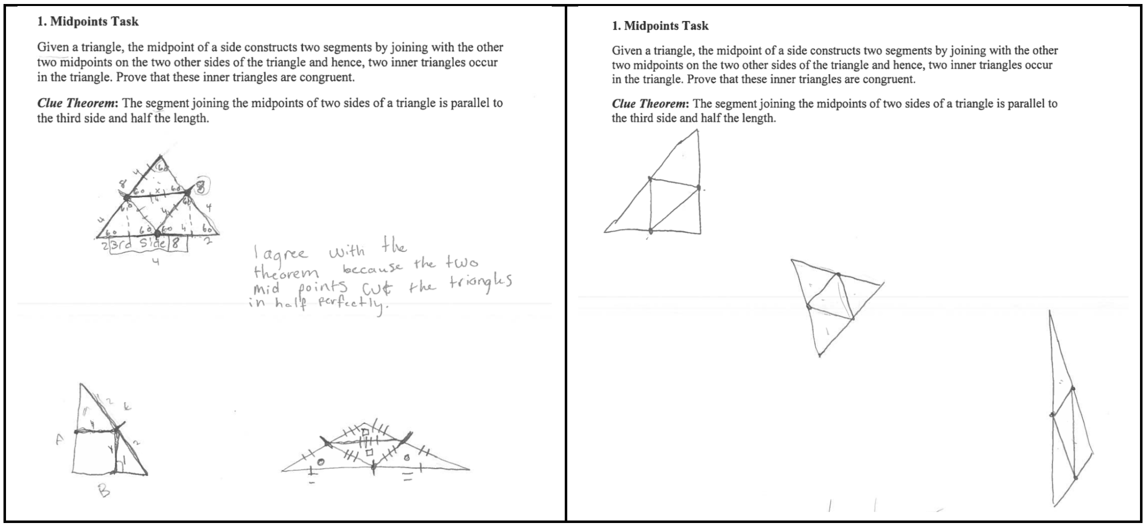


Figure 4.29. Nathan's and Lila's proof work on the diagram-free Midpoints Task.

Note that, similar to Nathan, Lila also did draw a right triangle and two other types of triangles in her proof work. As far as the symbolic resources employed in the Midpoints task, labeling side and angle, which can be seen in Nathan's proof work in above figure, were the only symbolic resources used in the diagram. Lila, in contrast, did not use any symbolic resources on the diagrams she drew. Likewise, other students did not apply symbolic resources frequently in the Task. Therefore, there was no diagrammatic register by labeling vertexes or points on the figure, and no student's written explanation included a diagrammatic register.

Students' Use of Diagrams

There were three salient types of diagram use in the diagram-free Midpoints Task: Creating a diagram, drawing a particular type of diagram, and working with more than one diagram. First and foremost, each student created a figure in their proof work. However, no student named the figure by labeling the vertexes or points on the diagram, which was a unique approach compared to other diagram-free task in the first interview (Isosceles Task). Nathan's proof work presented in Figure 4.29 is an example of this type of diagram use, which falls outside the diagrammatic register.

Second, three students started their proofs with a particular kind of diagram when proving the task. In fact, two students drew a right triangle, and one student produced an equilateral triangle to try to solve the Midpoints Task. The equilateral triangle was produced by Nathan, which was evident because of the angle markings in his diagram (see Figure 4.29). The other two students clarified what type of diagram they drew in their verbal explanation. As an illustration, Lee started his reasoning by stating, “Well, I made a right triangle,” and Lila said the following:

Lila: OK so first I got a triangle [pointing to the triangle she drew]. *It didn’t specify that it was a right triangle, but I just drew whatever type I thought*.

In contrast, Martin argued that the diagram he constructed was a random triangle.

Interviewer: Is this a specific triangle?

Martin: No, it’s just a big, it’s just a normal triangle.

Interviewer: OK, just any triangle?

Martin: Whatever I think when I see a triangle. yeah.

Third, three students worked with more than one diagram in their proofs. Nathan (see Figure 4.29) was one such student. He first produced a specific type of triangle (as described above) but then he produced the other triangles after he was asked about the validity of his reasoning on other types of diagrams. He concluded at the end that the claim would be valid for any kind of triangles.

Nathan: I think this one is the better diagram [pointing to the equilateral triangle he drew] and it shows that they are all congruent. *And I think with an equilateral, a right triangle and an obtuse, that if they all worked, it will be a general theorem*.

The Midpoints Task data did not yield much information about how students relied on the figural properties of the diagram, yet there were two students, Nathan and Lee, who gave absolute values to the sides of the figure and concluded the congruency of the inner triangles based on side lengths of the triangles. Nathan's proof work was already presented above in Figure 4.29 and Lee's proof work is shown in the following figure.

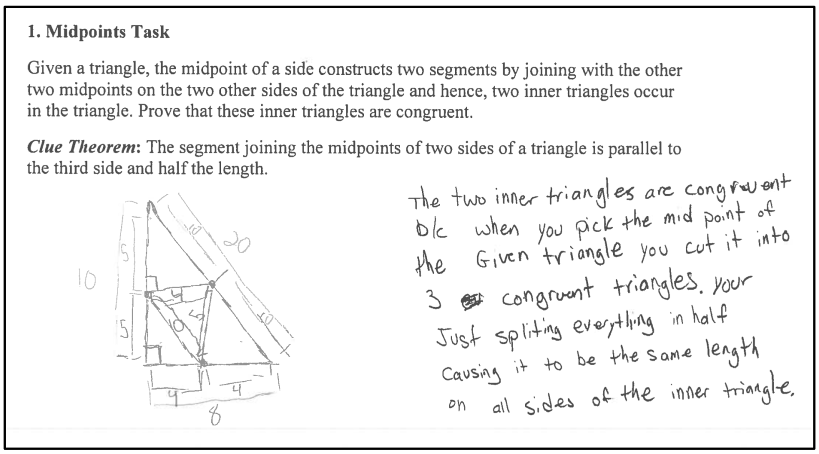


Figure 4.30. Lee's proof work on the diagram-free Midpoints Task.

Although Lee's proof work seems reasonable with the argument of side length values, it was not clear from his verbal explanation if he considered the conceptual properties such as using the side-side-side postulate to justify his argument or making a meaningful decision on the side length values because they did not satisfy Pythagorean Theorem. Therefore, it is not explicit from his work if his proof relied only on a particular type of diagram and individual side values on it.

Proving Actions of the Students

Different types of JUST and STEP proving actions (e.g., JUST-G-L, JUST-G, STEP-L) occurred regularly in student's proof work. Expectedly, justifications by gesturing were the most common proving actions employed in the Midpoints Task. Moreover, INVEST and ABOUT proving actions appeared in students' proving (Table 4.18). These proving actions occurred more often in the Midpoints Task than the Isosceles Task. That might be because of the multiple diagrams students created in the Midpoints Task while proving it or because of individual differences between students.

Concerning the ABOUT proving action in the Midpoints Task, the reason for its high frequency might be because the Midpoints Task was presented in the non-diagrammatic register and it was not similar to a typical diagram given task in geometry. Hence, students were likely to struggle with understanding what the task was asking and how to prove the claim. Hence, they made general statements about the task. For example, in the following excerpts, Martin and Lila stated why and for which parts of the claim they struggled.

> *Martin*: Yeah, it's been a long time since I have done stuff like that. So, I wasn't, I wasn't completely sure. *I didn't remember what congruency was completely* but doesn't that mean the angles are the same or is that the sides are the same of everything?

Here, Martin states his struggle with how to prove the task. He makes statements about the claim in the task and asks questions to clarify what the claim of the task actually demands. Likewise, Lila states her confusion with the task's description and how to set up the given information into the diagram she drew.

> *Lila*: *I was confused on how it would make two triangles in the center because I would think it makes like one* [pointing to the center triangle in her drawing]. *So, I was, that like throw me off.* And then, after that I kinda just, I forgot how to set them up and like, prove it and congruent and all these because I haven't done it so long.

Note that Lila also mentions about her confusion with the terms in the task and how to set up her proof.

Table 4.18

Proving actions that four students employed while proving the diagram-free Midpoints Task

Proving Actions	Frequencies of the Codes
CRIT-STEP-L	1
CRIT-STRUC	2
CRIT-INVEST	3
CLAR-ABOUT	3
CLAR-STEP-L	1
STRUC	1
ABOUT	**8**
INVEST	**7**
JUST	11
JUST-L	13
JUST-W	1
JUST-G, JUST-G-L, JUST-G-W	57
JUST-S, JUST-S-W	4
JUST-V, JUST-V-L, JUST-V-W	7
STEP	21
STEP-L	27
STEP-W	5
ACCEP	1
ACCEP-L	1

Regarding the production of valid proofs in the Midpoints Task, no student's proof work was considered valid because students provided limited arguments in their reasoning when they talked about the congruency of two inner triangles in the task. In fact, it was not obvious if each student used the congruency in a meaningful way. For example, in the above excerpt, Lila stated that she did not remember the term *congruent*.

Furthermore, even though some students produced strong arguments in their reasoning by using absolute values on the side lengths of the diagrams, it was not clear if their justifications were based on the particular triangle type and side lengths without thinking about the conceptual reasoning behind them. For instance, although Nathan's proof work by using equilateral and right triangle samples was valid, he did not provide an acceptable argument with the obtuse triangle because he drew different hatch/tick marks on congruent sides (see Figure 4.29). Moreover, the following excerpt is from Nathan's reasoning with the obtuse triangle he drew (Figure 4.31).

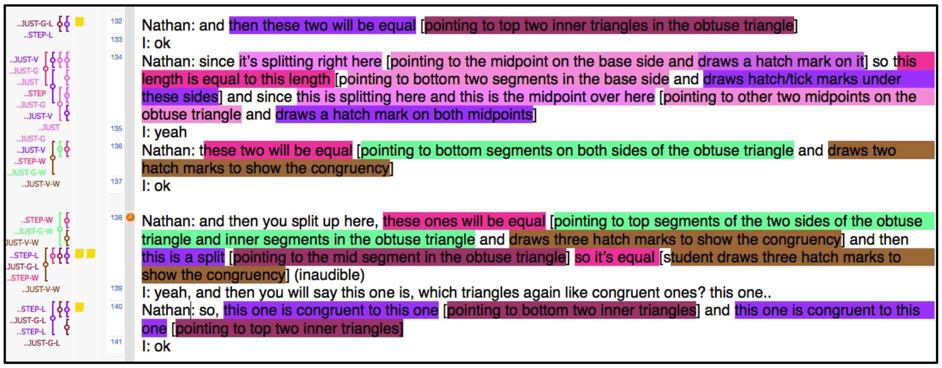


Figure 4.31. Nathan's proof work on the diagram-free Midpoints Task.

As it can be seen from his last sentence in the Figure 4.31, Nathan argued that two distinct groups of congruent triangles occur inside an obtuse triangle. Indeed, that is incorrect reasoning which leads to a different conclusion than his argument with the other two triangles (equilateral and right triangles) he used in the Midpoints Task.

Perimeter Task

The Perimeter Task was presented with truth-known and non-diagrammatic register features. The task was the first task given in the second interview and all four students who participated in the second interview worked on the task.

Semiotic Resources Used in the Perimeter Task

Gesture

Gesture resources were used by each student in the Perimeter Task and appeared more frequently than symbolic and visual resources (Table 4.19). As usual within my study, gesturing helped clarify the particular parts of the diagram that students pointed out in their reasoning because they used either pronouns or precise terms to address parts of the diagram.

Table 4.19

Semiotic resources that four students employed while proving the diagram-free Perimeter Task

	Semiotic Resources	**Frequencies of the Codes**
Visual	VS: visual-draw side/segment	18
	VH: visual-draw hatch/tick marks	9
	VR: visual-redraw figure	1
	VF: visual-draw a new figure/diagram	6
	VP: visual-draw point	12
	VA: visual-draw angle marking	1
	VSC: visual-draw side congruency	1
	Total	48
Gesture	GS: gesture-pointing at side/segment	64
	GG: gesture-pointing at a given in the task	7
	GP: gesture-pointing at a point	10
	GF: gesture-pointing at figure/diagram	10
	Total	**91**
Symbolic	SV: symbolic: label vertex	6
	SP: symbolic-label point	6
	SS: symbolic-label side/segment	4
	SA: symbolic-label angle	1
	Total	17

Following is a sample excerpt that shows Flynn's gesture accompanying the pronouns that were used in his verbal explanation.

> *Flynn*: Um, so, *this theorem* [*pointing to the clue theorem*] says that, uh, the segment joining the midpoints of two sides, so, we can at least meet *the midpoints of these lines* [*pointing to the two midpoints in his drawing*]
>
> *Interviewer:* Yes.

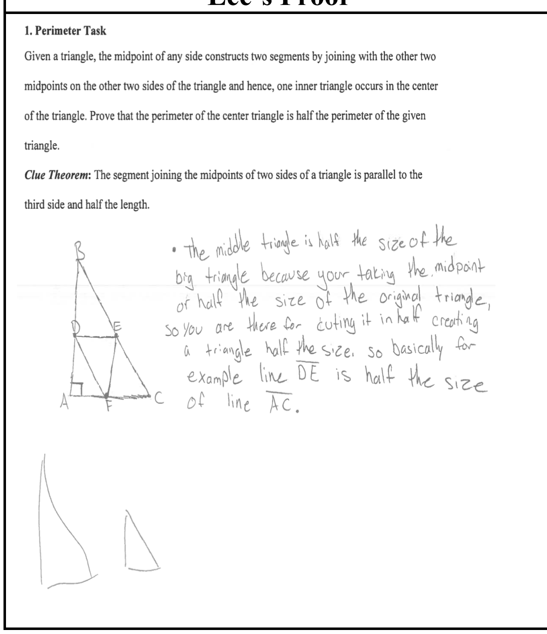
Flynn: … is parallel to the third side [reading the clue theorem], *this side* [*pointing to a side in his drawing*], and half the length [reads the theorem]. So that means that *this is half of this* [*pointing to a segment and the corresponding side*].

In his proof work, Flynn did not have any labels that specify which part of the diagram he meant. Instead, he gestured on particular sides or parts of the diagram to support his justifications. Note that the task was not presented in the diagrammatic register and Flynn remained outside the diagrammatic register, relying on his gestures instead of labels.

Symbols and Visuals

Symbolic and visual resources also occurred in students' proofs. Each student drew a diagram in the Perimeter Task and the types of diagrams they produced varied. For example, as shown in Figure 4.32, Lee sketched a right triangle in his work and Megan worked with an equilateral triangle to attempt to prove the task. However, two other students who took the task drew a generic triangle by stating that the triangles they drew were any triangle.

Lee's Proof



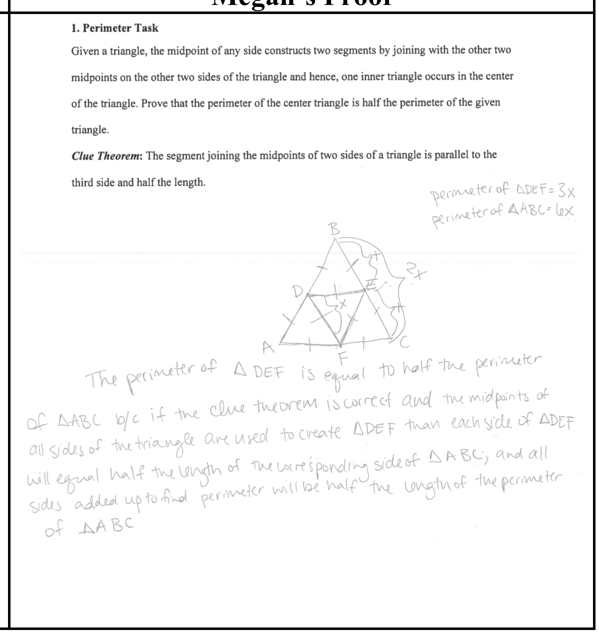
1. Perimeter Task

Given a triangle, the midpoint of any side constructs two segments by joining with the other two midpoints on the other two sides of the triangle and hence, one inner triangle occurs in the center of the triangle. Prove that the perimeter of the center triangle is half the perimeter of the given triangle.

Clue Theorem: The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

• The middle triangle is half the size of the big triangle because your taking the midpoint or half the size of the original triangle, so you are there for cuting it in half creating a triangle half the size. so basically for example line $\overline{DE}$ is half the size of line $\overline{AC}$.

Megan's Proof



1. Perimeter Task

Given a triangle, the midpoint of any side constructs two segments by joining with the other two midpoints on the other two sides of the triangle and hence, one inner triangle occurs in the center of the triangle. Prove that the perimeter of the center triangle is half the perimeter of the given triangle.

Clue Theorem: The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

perimeter of $\Delta DEF = 3x$
perimeter of $\Delta ABC = 6x$

The perimeter of ΔDEF is equal to half the perimeter of ΔABC b/c if the clue theorem is correct and the midpoints of all sides of the triangle are used to create ΔDEF than each side of ΔDEF will equal half the length of the corresponding side of ΔABC, and all sides added up to find perimeter will be half the length of the perimeter of ΔABC

Figure 4.32. Lee's and Megan's proof work on the Perimeter Task.

Although the task was given in the non-diagrammatic register, Lee and Megan labeled the vertexes of the triangle they had in the proofs and used these labels in their written explanation. Moreover, as seen in Figure 4.32, Megan used variables, x and $2x$, to label one side of the inner and bigger triangles, respectively. However, two other students did not mark any parts of the diagram they drew ad did not include any diagrammatic register in their written statement to the proof.

Students' Use of Diagrams

In the Perimeter Task, students showed three noticeable types of diagram use—drawing a particular or generic diagram, applying both figural and conceptual properties of the diagram, and applying only the figural properties of the diagram. In all, four students drew a triangle in the proof production as described above. Although some students preferred working with a particular kind of triangle such as a right triangle or an equilateral triangle (see Figure 4.32), they argued that the claim would be true for any triangle even if they used a specific triangle in the proof.

Interviewer: What if it was not a right triangle but an isosceles or equilateral?

Lee: Um, *I feel like I still might try and get to the same point, but it might, it might have taken a little bit a different row, or it might have taken me a bit longer to figure it out*.

Interviewer: Hmm hmm.

Lee: Maybe I don't, I am not sure, but that's what I am guessing, like, it'll just take me a little longer.

Interviewer: OK, but do you believe this will be true for other types of triangles?

Lee: Yeah*, I think it would be true for other types, so rather than just a right triangle*.

Here, Lee claimed that working with other types of triangles might have taken a little longer for him to prove the task, but the application of the theorem would still be valid for other types of triangles. Likewise, Megan worked with an equilateral triangle in the proof, but she used variables for the length values of the triangle and suggested that the triangle she drew represented any triangle.

> *Megan:* So, I drew that and then, uh, using this clue theorem [pointing to the clue theorem in the task], I kinda like, it was kinda hypothetical, but I made them all like the same length [pointing to the sides of the triangle DEF]. Uh, all the like, lines that make up the little diagram [pointing to the triangle DEF] because you could do that way or you could make them differently, but there would still, it would still work the same.
>
> …
>
> *Interviewer:* OK, can we say this is any triangle then, or a specific one?
>
> *Megan:* Yeah. *No, that's any triangle*.

The excerpts above also shed light on the second noticeable type of diagram use which is using both figural and conceptual features of the figure. In fact, each student used the clue theorem on the triangle, either a generic or a specific triangle, they drew. Flynn's excerpt presented above shows that he argued with the relations between side lengths of the inner triangle and the corresponding sides on the outside triangle, which he called "any triangle", by using the clue theorem given in the task. Similarly, in the following excerpt, by referring to the statement of the clue theorem in the task Megan stated that the side of the inner triangle would be x while the side of the outside triangle was $2x$.

> *Megan*: I decided that one side of triangle ABC [pointing to the side BC] *would equal 2x.*

...

Megan: And one side of triangle DEF cause, this would be triangle DEF [pointing to the inner triangle in her drawing], *would just equal x because using this theorem* [pointing to the clue theorem].

Interviewer: Hmm hmm

Megan: That proves that, um, like the midpoint [pointing to the segment DF] of two sides half the length of the, like corresponding side of the triangle [pointing to the side BC].

Megan assigned a variable to the side lengths of the triangle and applied the given theorem so that a blend of physical and theoretical features of the diagram ensued.

The third type of diagram use was making arguments based on the figural characteristics of a diagram. In particular, one student, Lee, showed that he mostly considered the visual aspects of the figure in his reasoning. At the beginning of his explanation, he seemed to argue with the figural and conceptual properties of the diagram as follows, but later he showed that he failed to use the conceptual properties correctly.

Lee: You are cutting it in half evenly and so basically if line DE is half the size of line AC [pointing to the sides DE and AC in the triangle] *and so on so forth* doing that, umm, it's half the size of the original triangle, and this is basically what I said.

Interviewer: But how about these other two sides [pointing to the sides BA and BC]?

Lee: Yeah, I think it will apply the same. so, if you take this line [pointing to side DF] and you move it over here [pointing to side DA] it should be half of this line [pointing to side BA] and this [pointing to side EF] should be half of this line [pointing to side BC], umm, just putting into a triangle.

Interviewer: OK, so, this one [pointing to the segment EF] then will be half of this side or this side [pointing to the sides AB and BC]?

Lee: Um, *half of this side I think* [pointing to the side BC].

Here, by matching inappropriate sides of the inner and outside triangle, Lee shows that he did not apply the clue theorem on the diagram correctly. Later, he also mentioned about his confusion on deciding the base of the inner triangle and made comparisons with the side lengths of the big triangle and the inner triangle based on the visual characteristics of the triangles.

Lee: What I originally was thinking was, uh, basically it's just this triangle [pointing to big triangle] half the size of upside down [pointing to triangle DEF] *just because the way you have to draw it, but it might be a different way.*

Interviewer: Which one do you refer? this one?

Lee: Yeah like this whole triangle [pointing to the inner triangle DEF] is half of this [pointing to the triangle ABC]. Just, uh, flipped in a different way than the original one.

Interviewer: Oh, OK, so do you mean like this one [pointing to triangle DEF] is half of this one [pointing to triangle ABC], but flipped?

Lee: Yeah, it's just in, uh, different position I guess you could say.

Interviewer: OK, yeah and how do you say that?

Lee: Um, well, because normally most people draw the base of the triangle at bottom like this one [pointing to the base of the big triangle], but in this case, *it looks like this* [pointing to the base DE] *is the base, but that might not be the base*.

Interviewer: Hmm hmm.

Lee And one of these sides [pointing to sides AF and FC] might be the base, umm, it's just positioned in a different way, um, yeah.

Interviewer: So, my question is, um, do you make that argument by looking at this drawing, the diagram, and saying like oh this looks like [pointing to triangle DEC] half of this one [pointing to triangle ABC]? Or do you have any …

Lee: Hmm hmm so, umm, it could still be half the size, uh, *I was just assuming for a minute that this was the base* [pointing to the side DF] *and that* [pointing to the side AC] *was the base and so this is half of this* [pointing to sides DF and AC], *but it might not be it could be this side or that side* [pointing to the sides EF and DF].

Lee seemed to use both figural and theoretical properties of the diagram initially when he mentioned about the inner triangle DEF being the half of the size of the bigger triangle BAC. However, when he was asked about a further explanation for his argument, he reasoned with the bases of the inner triangle and the bigger triangle but did not match the corresponding sides of the two triangles as shown in the following excerpt.

Interviewer: In the question, it's asking about the perimeter of the inner triangle is half of the perimeter of the big triangle. So, you mentioned about this one being half of this one [pointing to the sides DE and AC, respectively],

Lee: Hmm hmm

Interviewer: But how about these other two sides [pointing to the sides BA and BC]?

Lee: Yeah, I think it will apply the same. So, *if you take this line* [pointing to side DF] *and you move it over here* [pointing to side DA] *it should be half of this line* [pointing to side BA], and *this* [pointing to side EF] *should be half of this line* [pointing to side BC].

Note that Lee matched the sides that are close to each other instead of reasoning with the clue theorem to identify corresponding sides of the two triangles. Hence, he apparently did not apply the conceptual properties of the diagram and reasoned with the visual appearance of the diagram.

Proving Actions of the Students

In the Perimeter Task, there were several proving actions and interactions that students engaged in their proof work. The most frequent proving actions were types of STEP and JUST proving actions (Table 4.20). They generally had success in their reasoning (as indicated by the small number of STEP-L, STEP-W, JUST-G-L, and JUST-G-W codes). In fact, only one student, Lee, as shown in the above excerpts, did not give reasonable arguments in his proof work because his assumptions were based on the wrong assumptions (i.e., false mismatch of the side lengths of two triangles).

Table 4.20

Proving actions that four students employed while proving the diagram-free Perimeter Task

Proving Actions	Frequencies of the Codes
CLAR-CLAIM	2
CLAR-STEP-W	2
CLAR-STEP	2
CLAR-JUST-S	1
CLAR-INVEST	1
ABOUT	5
INVEST	6
JUST	9
JUST-L	2
JUST-G	14
JUST-G-L	2
JUST-G-W	4
STEP	13
STEP-L	2
STEP-W	2
ACCEP-L	1
REFINE	1
CLAIM	1
Repetition	3

It is reasonable that students produced valid STEP, JUST, and JUST-G proving actions because three students provided logical proofs for the Perimeter Task. By using gesture resources, students created JUST-G proving actions in their proofs. As an illustration, Figure 4.33 shows a sample proof work with proving action codes.

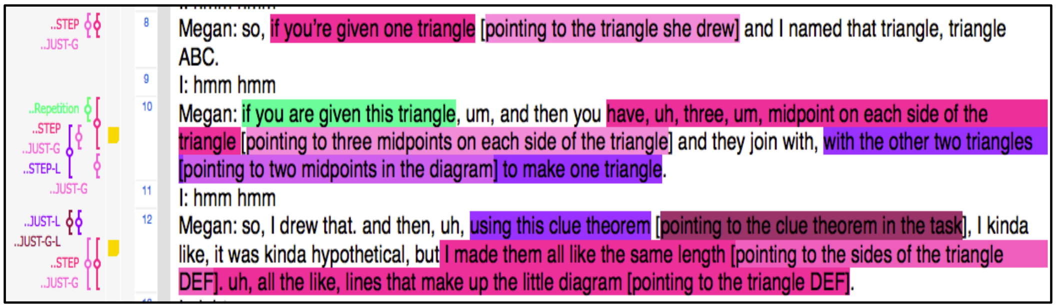


Figure 4.33. Megan's proof work with several proving actions.

Megan's proof included reasoning with the clue theorem and the side length values of the triangle she drew. Some steps and justifications she produced were coded as limited since it was not obvious at the beginning if she only considered an equilateral triangle to apply the clue theorem.

The Perimeter Task was mathematically similar to the Midpoints Task from the first interview because both involved a certain kind of setup, which is identifying the relationships between triangles constructed by joining the midpoints on each side of a triangle. There was no valid proof produced for the Midpoints Task, whereas three students provided acceptable proof work in the Perimeter Task. Although there is no apparent reason for the difference between students' performances on both tasks, it is worth noting that two students showed diagrammatic register in their proofs for Perimeter Task, but no student performed diagrammatic register in the proof of Midpoints Task. Furthermore, three students who were invited to the second task were successful in proving diagram-given Midpoints Task in the first interview. Hence, it might not be surprising that they were also successful in producing valid proofs in the Perimeter Task in the second interview since they might have realized the similar structure of the both tasks.

Summary

There were four diagram-free tasks presented in the first and second interviews. Although the tasks shared one feature in common (i.e., no diagram), they were distinct with respect to diagrammatic register and truth-known features. Students' proof work on each task feature varied regarding employing the semiotic resources, the types of diagram use, and using proving actions. Although gesture resources showed similarities among four tasks, the visual and symbolic resources were employed differently. That is, drawing only one diagram and naming the figure by labeling vertices were specific to the Isosceles Task (diagrammatic register); in contrast, drawing a new figure or more than one diagram with no vertex labels occurred in the non-diagrammatic register and truth-unknown tasks (Midpoints Task, Perimeter Task, and Pentagon Task).

Moreover, although justifications with gesturing were present in all four tasks, INVEST, ABOUT proving actions, and proving interactions were more prominent in the non-diagrammatic register and truth-unknown tasks (Midpoints Task, Perimeter Task, and Pentagon Task).

Lastly, there were valid proof works produced in the diagrammatic register task (Isosceles Task) by using figural and conceptual properties of the diagram. Whereas, in the non-diagrammatic register tasks (Midpoints Task and Perimeter Task), it was not apparent whether students merely applied the figural properties of the diagram or also considered the conceptual properties of the diagram in their reasoning when they used particular diagrams and side length values in their proofs. Hence, there was less valid proof work constructed in the non-diagrammatic register tasks. Most of the students' proof works in non-diagrammatic register tasks were presented outside the diagrammatic register.

Summary of Diagram-Given and Diagram-Free Tasks from the First and Second Interviews

In the first and second interviews, both diagram-given and diagram-free tasks showed similarities and differences regarding semiotic resources, diagram use, and proving actions students employed. All semiotic resources appeared in each task type, yet certain semiotic resources were more frequent in one type of task than the other. As shown in Table 4.21, gesture resources are the most common semiotic resources among all the tasks. In both diagram-given and diagram-free tasks, students gestured on the particular parts of the diagram when they used pronouns or precise terms in their explanation. Hence, gesturing helped students clarify the place they pointed out in the proof work.

Furthermore, visual and symbolic resources differed within the diagram-given tasks and diagram-free tasks. Several visual resources such as drawing a new figure were more prominent in truth-unknown and non-diagrammatic register tasks, as shown in Table 4.21. For example, on the one hand, drawing a new figure was more noticeable in the truth-unknown (e.g., Right Triangle Task) and non-diagrammatic register tasks (e.g, Midpoints Task); on the other hand, symbolic resources, in general, were more prominent in the non-diagrammatic register tasks (see Table 4.21).

Table 4.21

Semiotic resources that students employed while proving the diagram-free and diagram-given tasks in the first and second interviews.

Semiotic Resources		**Frequencies of the Codes**									
		Truth-Unknown Tasks		**Diagrammatic Register Tasks**					**Non-diagrammatic Registe Tasks**		
		Diagram-given	Diagram-free	Diagram-given				Diagram-free	Diagram-given	Diagram-free	
		Right Triangle Task	Pentagon Task	Isosceles Task	Midpoints Task	Right Triangle Task	Right Triangle Task	Isosceles Task	Bisector Ray Task	Midpoints Task	Perim Tas
Visual	VS: draw side/segment	14	25	1	4	18	0	10	0	23	18
	VY: draw ray	0	0	0	0	0	0	0	1	0	0
	VAC: draw angle congruency	0	7	0	0	0	0	0	0	0	0
	VH: draw hatch/tick marks	19	0	0	21	12	5	9	8	13	9
	VL: draw line	2	4	0	10	4	4	0	0	0	0
	VR: redraw figure	0	1	0	2	0	2	4	1	0	1
	VF: draw a new figure/diagram	**26**	**20**	0	0	7	2	5	**6**	**8**	**6**
	VP: draw point	5	2	3	0	4	2	9	0	18	12
	VA: draw angle marking	12	11	8	6	24	10	9	12	2	1
	VSC: draw side	6	0	0	7	4	3	4	2	4	1

	congruency										
	Total	84	70	12	50	73	28	50	30	68	48
Gesture	GW: pointing something in the work	0	3	1	1	2	0	0	0	1	0
	GS: pointing at side/segment	137	7	57	74	90	38	15	68	66	64
	GPA: acting to show perpendicular	1	0	0	0	0	0	0	0	0	0
	GM: referring to movement	1	1	1	0	0	0	0	2	0	0
	GY: pointing at ray	0	0	0	0	0	0	0	7	0	0
	GV: pointing at vertex	5	5	0	3	2	1	3	6	1	0
	GH: pointing at hatch/tick marks	2	0	0	2	0	2	0	2	0	0
	GC: pointing at a calculation in the work.	0	1	0	0	0	1	0	0	0	0
	GT: turning the paper	4	0	2	1	6	2	0	0	0	0
	GG: pointing at a given in the task	9	16	4	2	10	0	1	8	1	7
	GP: pointing at a point	4	0	3	9	7	5	8	3	19	10
	GF: pointing at figure/diagram	40	33	4	23	38	25	13	23	44	10
	GA: pointing at angle	105	77	36	27	142	70	43	78	4	0

	Total	**308**	**143**	**108**	**142**	**297**	**144**	**83**	**197**	**136**	**91**
	SSE: solving an equation	0	2	0	0	0	0	0	0	0	0
	SV: label vertex	22	4	0	0	10	6	17	3	0	6
	SF: label figure	3	0	0	0	0	0	0	0	0	0
	SP: label point	5	0	0	3	2	2	7	1	0	6
	SS: label side/segment	11	0	9	11	5	0	0	0	14	4
Symbolic	SG: use geometric symbols	1	3	1	0	2	1	0	0	0	0
	SA: label angle	7	0	2	0	29	12	0	6	3	1
	SAS: writing algebraic symbols	0	0	1	0	2	2	0	0	0	0
	SE: writing an equation	1	5	3	0	6	1	0	0	0	0
	Total	**50**	14	**16**	**14**	**56**	**24**	**24**	10	17	17

Concerning the proving actions students employed in the two types of task, as shown in table 4.22, justification by gesturing (JUST-G) was a common approach in all tasks because gesturing had a significant role in justifying the steps while proving both diagram-free and diagram-given tasks. However, some proving actions appeared more frequently in particular tasks. For instance, investigating the truth value of the problem (INVEST) was more prominent in the truth-unknown and non-diagrammatic register tasks. As far as the valid proof production, students were more likely to produce valid proofs in diagrammatic register tasks, though this may be related to the actual mathematical claims in the tasks and not just the diagram format.

Table 4.22

Proving actions that students employed while proving the diagram-free and diagram-given tasks in the first and second interviews.

Proving Actions	Frequencies of the Codes									
	Truth-Unknown Tasks		**Diagrammatic Register Tasks**					**Non-diagrammatic Register Task**		
	Diagram-given	Diagram-free	Diagram-given				Diagram-free	Diagram-given	Diagram-free	
	Right Triangle Task	**Pentagon Task**	**Isosceles Task**	**Midpoints Task**	**Right Triangle Task**	**Right Triangle Task**	**Isosceles Task**	**Bisector Ray Task**	**Midpoints Task**	**Perime Task**
CONF-STEP	0	0	1	0	0	0	0	0	0	0
CONF-JUST	0	0	0	0	1	0	0	0	0	0
CONF-INVEST	0	0	0	0	0	0	0	2	0	0
CRIT-STEP	1	2	0	0	2	0	0	0	0	0
CRIT-ABOUT	1	2	0	0	0	0	0	0	0	0
CRIT-STEP-W	0	0	0	0	0	2	0	0	0	0
CRIT-STEP-L	0	0	0	0	4	1	0	1	1	0
CRIT-STRUC	0	0	0	0	0	0	0	0	2	0
CRIT-JUST-W	0	0	4	0	0	1	0	0	0	0
CRIT-INVEST	1	7	0	0	3	2	0	1	3	0
CRIT-CLAR	2	0	0	0	0	1	0	0	0	0
CRIT-CLAIM	1	1	0	0	0	1	0	1	0	0
CRIT-JUST-L	0	0	0	1	0	0	0	0	0	0
CLAR-ABOUT	4	3	1	0	2	0	0	4	3	0
CLAR-CLAIM	0	0	0	0	0	1	0	1	0	2
CLAR-STRUC	0	1	0	0	0	1	0	0	0	0
CLAR-STEP-L	4	1	0	0	0	4	0	7	1	0
CLAR-STEP-W	1	0	0	0	1	1	0	0	0	2
CLAR-REFINE	0	0	0	0	1	1	1	0	0	0

CLAR-STEP	1	2	0	0	1	1	2	1	0	2
CLAR-JUST-L	0	0	0	0	0	0	1	0	0	0
CLAR-JUST-S	0	0	0	0	0	0	0	0	0	1
CLAR-JUST-W	0	0	2	0	1	0	0	0	0	0
CLAR-INVEST	3	3	0	0	4	0	1	0	0	1
STRUC	1	1	1	0	0	1	1	2	1	0
SUM	1	0	0	0	0	1	0	0	0	0
ABOUT	**15**	**10**	2	2	**10**	5	2	**9**	**8**	**5**
END	1	0	0	3	0	0	0	1	0	0
END-L	0	0	1	0	0	0	0	0	0	0
END-W	1	0	0	0	0	0	0	0	0	0
INVEST	**20**	**27**	1	2	16	1	3	**6**	**7**	**6**
JUST	7	8	5	17	10	18	9	8	11	9
JUST-L	8	6	4	4	4	1	6	19	13	2
JUST-W	15	0	7	2	4	2	2	5	1	0
JUST-G	29	11	15	39	34	37	23	16	25	14
JUST-G-L	20	2	14	9	20	2	11	16	28	2
JUST-G-W	26	0	7	2	8	5	1	5	4	4
JUST-S	1	1	2	0	0	0	0	0	2	0
JUST-S-L	3	0	0	0	2	0	0	0	0	0
JUST-S-W	0	0	0	0	0	0	0	0	2	0
JUST-V	4	1	0	6	0	1	0	1	4	0
JUST-V-L	6	0	1	0	1	0	0	2	2	0
JUST-V-W	5	0	1	0	0	0	0	0	3	0
STEP	29	15	11	41	30	30	16	9	21	13
STEP-L	21	9	15	10	17	9	13	24	27	2
STEP-W	32	1	5	1	10	11	3	8	5	2
ACCEP	2	7	1	4	3	6	2	11	1	0
ACCEP-L	1	5	2	0	2	5	2	4	1	1
REFINE	2	1	0	0	1	1	1	0	0	1
CLAIM	0	0	0	1	0	0	0	1	0	1

In sum, students applied various semiotic resources and proving actions in both diagram-given and diagram-free tasks in general; however, particular tasks showed differences regarding students use of diagrams and semiotic resources in the proofs. As discussed earlier, non-diagrammatic register and truth-unknown tasks had notable differences from other tasks. The findings showed that, in non-diagrammatic register tasks, students considered a diagram as a generic diagram since they called it "any triangle" or they worked with various types of figures in their reasoning. Hence, students applied conceptual properties of the diagrams more frequently in non-diagrammatic register tasks; however, students were more likely to rely on the figural properties of the diagrams when the task was given in the diagrammatic register. Likewise, in the truth-unknown task (Pentagon Task), sketching multiple diagrams and reflecting on the variability of some parts of the diagram were noticeable approaches that appeared in students' proof works. Consequently, INVEST proving action and proving interactions were salient in the truth-unknown tasks.

Chapter 4 presented findings of the study in a detailed manner. Although the results of this study provided insights into ideas that merge from the literature, this chapter did not include discussion of the findings. Hence, the following chapter discussed the findings as well as limitations and significance of the study.

CHAPTER 5: DISCUSSION, SIGINIFICANCE, AND RECOMMENDATIONS

In this chapter, I present the discussion, significance, and recommendations of the dissertation research. Specifically, this chapter has six major sections as follows: Summary of the study and findings; discussion of diagram-given and diagram-free tasks regarding accompanied accurate/inaccurate diagrams, truth-known/truth-unknown features, and diagrammatic/non-diagrammatic register features; limitations of a small-scale interview study; a focus on the significance and implications of the semiotics and proving actions in geometry proofs; and concluding remarks.

Summary of the Study and Findings

Study Motivation and Design

In high school geometry, proving theorems and applying them to geometry problems is an expectation from high school students (CCSSI, 2010). Diagrams are considered as an essential part of the geometry proofs because diagrams are included in a typical geometric statement such as a claim or problem (Manders, 2008; Shin et al., 2001). Geometry diagrams contain various meanings regarding the relations between the diagram, text, and the creator of the proof and hence are multi-semiotic in nature (Alshwaikh, 2010; Miller, 2007).

Although several studies have adopted a semiotic approach to understanding the role of a diagram in geometry proofs and the various meanings they convey (Dimmel & Herbst, 2015; Fischbein & Nachlieli, 1998), a particular focus on students' interpretation and use of diagrams in geometry proofs have required more attention. The purpose of this study was to investigate high school students' interpretation and use of diagrams in geometry proof tasks with a consideration of visual, gesture, and symbol semiotic resources, as well as proving actions (Otten et al., 2017) enacted during students' proving processes. The research questions that guide this investigation were as follows:

1. *What semiotic resources (in diagrams) do high school students use to prove geometric claims? How do the semiotic resources relate to the quality of reasoning students provide?*
2. *How do high school students interpret and use geometric diagrams to prove* ***diagram-given*** *geometric claims, and what is the semiotic structure of their proving process?*
3. *How do high school students produce and use (if at all) diagrams to prove* ***diagram-free*** *geometric claims, and what is the semiotic structure of their proving process (whether or not they produced a diagram)?*

In this study, I employed a qualitative research design to understand students' interpretation and meaning making in a particular experience (Merriam, 2009), in particular, working with diagrams in geometry proofs. I conducted one–on–one task-based interview with several high school students to understand the way they used diagrams while proving geometry theorems. Because the nature of the interviews was a form of the clinical interview, students had opportunities to present their natural thinking during the interviews (Ginsburg, 1981). The task-based interview design allowed a better understanding of participant students' meaning making (Goldin, 1997) and use of diagrams while producing geometry proofs.

In order to answer the research questions, data were gathered from video recordings of two task-based interviews, transcriptions of the video recordings, and the written work that students produced. Participating students received a sequence of four geometry tasks that some included diagrams and diagrammatic register feature, some not in each interview and students provided a verbal explanation to their proof work after each task. In the analysis of the data, I used a semiotic framework with symbolic, visual, and gesture components to investigate students' use of diagrams with these semiotic resources and a proving actions framework developed by Otten and colleagues (2017) to identify the procedures performed during students' proof processes. Furthermore, I drew upon the results from both frameworks to determine the relationship between particular semiotic resources and proving actions students employed in their proofs.

Findings

The summary of the findings is organized by interpretation and use of diagrams in diagram-given tasks and production and use of diagrams in diagram-free tasks. The discussion of the way students used diagrams concerning semiotic resources in the diagram-given and diagram-free tasks provides answers to the Research Questions 2 and 3, respectively. Furthermore, the relationship between semiotic resources and proving actions in both diagram-given and diagram-free tasks is presented to address the Research Question 1.

Interpretation and Use of Diagrams in Diagram-given Tasks

In all diagram-given tasks, working with the given diagram and relying on the figural properties of the given diagram were the most common form of diagram use. In some diagram-given tasks, especially in the Right Triangle Task with an inaccurate diagram from the first interview and the Bisector Ray Task from the second interview, students drew a new diagram or multiple diagrams in their proofs. It is worth noting here that these two tasks were the only tasks out of the six diagram-given tasks that had a truth-unknown format (Right Triangle Task with an inaccurate diagram from the first interview) or a non-diagrammatic register presentation (Bisector Ray Task).

In other tasks (non-diagrammatic register, truth unknown), students drew more diagrams. Moreover, they seemed to use the diagrams to think about the claim (INVEST), not just as an organization for the argument (STEP, JUST).

In general, students focused on the figural properties of the diagrams more frequently than the conceptual properties of the diagrams in their proofs even when they produced a new diagram or multiple diagrams. Figural properties refer to the visual appearance of the figure that are used as a foundation for the proof production even when the properties are not assured by the claim or rules of deduction. When the given figure is straightforward (accurate, truth known, diagrammatic register), the students used the given figure almost exclusively and had relatively better success integrating figural and conceptual properties (e.g., Midpoints Task and Isosceles Task). However, when the given diagram was abnormal in some way (inaccurate, truth unknown, non-diagrammatic register), students produced new figures (e.g., Right Triangle Task with an inaccurate diagram in the first interview and Bisector Ray Task).

Regarding the semiotic structure of students' proving process, gesture resources were prominent in the semiotic structure of students' proving process in diagram-given tasks. Students gestured to the particular parts of the diagram to clarify their proof explanation. In fact, in most cases, students did not verbalize the precise terms of the particular sides they were referring to, instead used pronouns accompanying their gestures (e.g., this [pointing to a segment inside a triangle] is half the length of this [pointing to a side of the triangle]).

In contrast, visual and symbolic resources surfaced differently based on the features of the tasks. As an illustration, visual resources such as drawing angle markings and hatch/tick marks to show congruent sides and symbolic resources such as labeling an angle and side were more common on the tasks that were given in the diagrammatic register with an accurate diagram. On these tasks, students mostly worked with the given diagram and applied visual and symbolic resources on the given diagram. Furthermore, students' written explanations in the proofs of these tasks were more likely to adhere to the diagrammatic register (i.e., having geometric terms in the written statement such as “side BD is congruent to side DA”). Interestingly, the symbolic resources employed in the non-diagrammatic register task (Bisector Ray Task) had the lowest frequency compared to other diagram-given tasks. Thus, the written explanations of the students were generally outside of the diagrammatic register in diagram-given tasks that had truth-unknown and non-diagrammatic register features.

Production and Use of Diagrams in Diagram-free Tasks

All students produced at least one diagram in their proofs, although the type of diagram they produced or the way they used the diagrams differed by student and task. In particular, the diagram-free tasks had different features. One task (Isosceles Task) was presented in the diagrammatic register, whereas the other three diagram-free tasks were outside the diagrammatic register. There was also one task (Pentagon Task) given with the truth-unknown. Concerning the differences in the features of the tasks, the Isosceles Task involved students producing only a typical isosceles triangle. However, non-diagrammatic register tasks involved students drawing different kinds of triangles such as an equilateral triangle, a right triangle, or an isosceles triangle, or they produced a generic triangle, which they called “any triangle.” Furthermore, students were more likely to produce and work with more than one diagram in non-diagrammatic register tasks. Particularly, in the Pentagon Task, students drew various types of pentagons to investigate the truth value of the problem.

Although most students utilized both figural and conceptual properties of the diagrams they produced in the diagram-free task that was in diagrammatic register (Isosceles Task), there were differences with the diagram-free tasks outside the diagrammatic register. Specifically, there was a lack of evidence of students utilizing both figural and conceptual properties and some evidence that they relied predominantly on the figural appearance of their diagrams. Most students drew conclusions based on the particular type of diagram they drew (e.g., a right triangle), although some students considered the triangle they produced as a representation for all kinds of triangles. Furthermore, when students created and worked with various pentagons to prove the claim in the Pentagon Task, they were still hesitant about whether the claim would be valid for all types of pentagons.

Regarding the semiotic structure of the students' proving process in diagram-free tasks, the most common semiotic resource was gesture resources, as it was in diagram-given tasks. In general, students gestured to the diagram when they justified their arguments. Again, similar to diagram-given tasks, students mostly employed gesture resources to accompany the pronouns they used in their verbal explanation. Yet some differences between diagram-free and diagram-given tasks surfaced as well. For example, concerning the visual resources, students sketched only one diagram in the diagrammatic-register task (Isosceles Task); however, they drew more than one diagram in the non-diagrammatic register tasks. Furthermore, symbolic resources were employed more frequently in the diagram-free task with diagrammatic register than all three diagram-free tasks with the non-diagrammatic register. Therefore, in the analysis of students' written explanation to the proof for diagram-free tasks, the diagrammatic register was more evident in the Isosceles Task, which was given in diagrammatic register, than other three diagram-free tasks with the non-diagrammatic register.

Relationship Between Semiotic Resources and Proving Actions

Across both interviews and with both diagram-given and diagram-free tasks, gesture resources were the most prominent semiotic resource that students used in their proving. In particular, the variations of justifications students made by gesturing (JUST-G) in their proof statements were the most common proving actions students employed. However, it was challenging to document which gesture resources specifically led to valid, limited, or wrong justifications in the proofs.

The findings also suggested that, in general, some visual resources such as drawing a new figure or multiple figures occurred regularly in particular tasks such as diagram-free tasks with non-diagrammatic register or truth-unknown features. Students seemed to use the diagrams to think about the claim. Therefore, these types of visual resources corresponded to the highest frequency of INVEST (investigating or guessing the truth value of the problem) and ABOUT (making a general statement about the proof) proving actions in those particular tasks.

Symbolic resources were the least frequently used semiotic resources in this study; however, they were more noticeable in tasks with the diagrammatic register and truth-known features than tasks with the non-diagrammatic register or truth-unknown features. That is, if the labels were already provided, students would use them in their proving, albeit rarely (because gesturing was more common), but students tended not to introduce symbols themselves. Furthermore, students were more likely to produce valid arguments in diagrammatic register tasks through an integration of figural and conceptual properties of the diagrams (e.g., Midpoints and Isosceles Tasks with diagrams in the first interview). However, it is worth noting that accurate diagrams and truth-known features in diagrammatic register tasks also played an important part in students' valid proof production. The diagrammatic register tasks which were most typical tasks in curriculum materials were related to the narrowest range of semiotics/proving actions. Moreover, it was not easy to identify the connections between symbolic resources and valid proving actions produced in the diagrammatic register tasks.

Discussion of Findings

This study examined the nature of students' use of diagrams in geometry proofs. Students use of diagrams were investigated through semiotic resources they produced and proving actions they employed in the proving process. In this section, the findings of this study are discussed under three major topics: (1) Use of diagrams in geometry proofs, (2) The role of semiotic resources in geometry proofs, and (3) Proving actions in geometry proofs.

Use of Diagrams in Geometry Proofs

Prior research addressed the use of diagrams as an essential step or method employed in the solution of geometry problems even if they were not constructed properly (Herbst et al., 2016; Poincare, 1963). In this study, the findings are consistent with this research because students tried to perform logical reasoning with the diagrams even if the given diagram in the task was not an accurate diagram with respect to the claim. In other words, all students produced or used the diagrams in the proof of the geometry tasks; however, in general, the students did not successfully generate strong reasoning in the tasks that were given with an inaccurate diagram.

Furthermore, in this study, some students used only diagrams in their proofs which align with the claims made in earlier studies that a diagram alone can be employed in the construction of a mathematical proof (Brown, 1997; Larkin & Simon,1987). Although the existence of student proofs with only diagrams was not surprising, the logical and meaningful reasoning that some students provided when providing verbal explanations to the proofs were striking. Similar to past studies (Laborde, 2005), the findings of this study indicated that students sometimes only considered the figural properties of the diagrams and did not utilize theoretical properties. However, one of the most important aspects of this study was the potential role of task features. Students showed differences regarding applying figural and conceptual properties of the diagrams in the tasks with the diagrammatic register, non-diagrammatic register, truth-known, and truth-unknown features. Laborde (2005) argued that geometry problems that involve either figural or theoretical property included the same properties in their solutions. Moreover, Laborde (2005) stated that some geometry problems require using both figural and theoretical properties such as the definition of a geometrical object. The results of this study specify that what kind of geometry problems lead implications of both figural and theoretical properties. In fact, the findings imply that students were more likely to integrate figural and theoretical properties of the diagrams when the task was presented in the diagrammatic register. However, this study suggests further research to better support that claim.

Another possible influence of different task features on students use of diagrams is the production of various diagrams in particular tasks. In fact, Avigad (2008) and Manders (2008) discussed that making conclusions based on one diagram in geometry proofs may limit the validity and generalizability of the geometrical arguments. Moreover, Otten and colleagues (2014) investigated the reasoning and proving opportunities in the secondary geometry textbooks, and they pointed out that, in U.S. geometry textbooks, students would very rarely have to explore and then prove a general (truth-unknown) claim. This investigation relates to the statements of some participating students in this study when they mention about the truth unknown and non-diagrammatic register tasks. Students' use of diagrams and semiotic resources in non-diagrammatic register and truth-unknown tasks provide students with a full range of proving opportunities.

As an illustration, in this study, it was evident that in non-diagrammatic register and truth-unknown tasks students created multiple diagrams in different types. In particular, some students called the truth-unknown tasks some of the hardest tasks because they considered it as a theorem rather than a mere problem. For example, Deborah stated that "I know how to prove like that two things are equal or congruent, but proving like, proving a theorem is a little harder because this is kinda like a theorem."

Moreover, in the first interview, Lee worked on the Midpoints Task in the non-diagrammatic register. After working on all four tasks in the first interview, Lee commented on the Midpoints tasks and argued that working more with non-diagrammatic register tasks in his geometry class could help him develop his geometry knowledge. He then related working in a non-diagrammatic register task with developing the writing skill. In particular, he stated the following:

> If I had to do more stuff like that when I was in class, I feel like I would have learned more, I had to like, you know, start from a sketch and do all by myself rather than given things. I feel like I would have kept the knowledge in my head longer and know how to do my own diagrams and stuff like that. I feel like that would help me learn more, would help me figure everything out because it's your own work and your own style. Once you figure it out, it's a lot easier to remember because, you know, like they say, if you write stuff a lot, then you will never have to spell that the earlier stuff. It's kinda like the same thing like this. If you do it yourself and you learn it, then it will be easier to remember because you did it yourself, you made the diagram and everything.

Similar to Lee, several students expressed that they enjoyed working with the tasks even though the tasks challenged them and were not always in a familiar form. Yet, students who worked on non-diagrammatic register and truth-unknown tasks were not generally successful in the production of valid proofs because students expect proof problems to be presented in a certain way (diagrammatic register) and they perform in certain, limited ways when they are in that expected situation. Students' particular comments about the truth unknown and non-diagrammatic register tasks are reasonable when the rareness of the types of truth unknown and non-diagrammatic register tasks in the U.S. geometry textbooks (Otten et al., 2014) is considered. However, if the features of typical tasks are violated somehow (e.g., with non-diagrammatic register, truth unknown, inaccurate diagram featuresz), then students are likely to think in more varied ways. Apparently, variations in task features provides more opportunities for thinking and learning.

The Role of Semiotic Resources in Geometry Proofs

The previous research focused on the meanings that geometry diagrams convey in students works and geometry textbooks (Alshwaikh, 2008; Dimmel & Herbst, 2015), but they did not investigate what meanings students draw from the diagrams. That is, although a semiotic approach on the geometric diagrams has a particular attention recently in the mathematics education field with a focus on the meanings of the diagrams, the semiotic approach students employed on the diagrams remained unclear. This study adds to previous research by providing insights into the semiotic resources that students used while working with diagrams in geometry proofs.

The gesture resources in students' proof work reported in this study support studies that have demonstrated the supportive role that gesture play in students' reasoning and the importance of gesturing in reflecting students' thinking (e.g., Nathan et al., 2014; Williams-Pierce et al., 2017). In all types of geometry tasks given in this study, students performed gestures prominently together with language when proving the geometric claim. In some tasks, especially truth-unknown tasks, students' gestures with diagrams represented the dynamic relationship between some sides of the diagram which was not easy to illustrate verbally, akin to the work of Williams-Pierce and colleagues (2017). Furthermore, the present study also found that tasks given with truth-unknown features had potential to capture gestures on the diagrams. That is, in truth-unknown tasks, students performed gestures to show the movement of some sides or angles of the diagram and they also sketched various possibilities of these sides or angles on the diagram to check the validity of the claim. Hence, the diagrams student drew apprehended their gestures. For example, in the Right Triangle Task with an inaccurate diagram in the first interview, students performed gestures on one particular side of the given triangle and considered the mobility of that side by redrawing it inside and outside of the triangle. This suggested that students were considering different variations of the given diagram, rather than a single static diagram, through gesture and diagram interaction (e.g., Châtelet, 2000; Sinclair et al., 2016). In fact, de Freitas and Sinclair (2012) draw attention to Châtelet's consideration of the interplay between gesture and diagrams. Figure 5.1 was given in the study of de Freitas and Sinclair (2012) to point out that the dotted lines in the figure show the mobility of the diagram. Hence, the paper captured the interplay between gestures and diagrams.

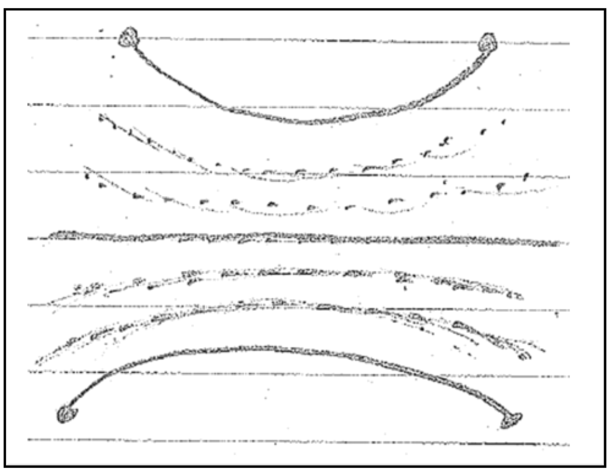


Figure 5.1. Student's diagram with dotted and solid arcs in de Freitas and Sinclair (2012, p. 135).

Likewise, in this study, students showed a similar gesture and diagram interaction when they worked on the truth-unknown Right Triangle Task in the first interview (Figure 5.2).

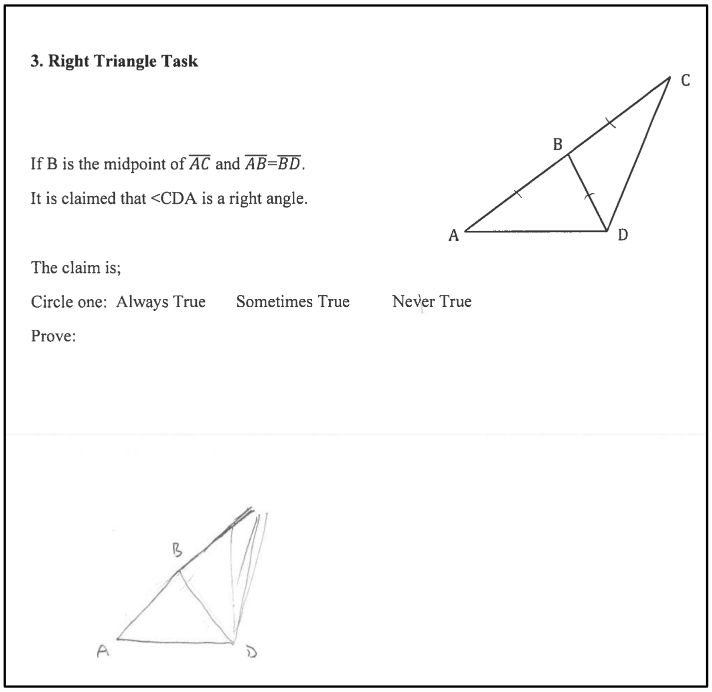


Figure 5.2. Student's diagram with sketched lines in the truth unknown Right Triangle Task in the first interview.

The associations reported in this paper between gestures and the language students used such as pronouns in the verbal explanation supported the study that have demonstrated the following: "Although a student may not be able to specifically articulate a valid mathematical proof, his or her gestures can shed light on potential key mathematical insights that he or she possesses but is not yet able to verbalize" (Pier et al., 2014, p. 655). Notably, in this study students' verbal explanations to geometry proofs included pronouns accompanied by gestures to justify the arguments they constructed in the reasoning.

Previous studies also demonstrated the role of visuals and symbols in supporting meaning making in mathematics and suggested future research to find a relationship between problem type and student-generated drawing type (Rellensmann et al., 2017). Given that, this study found that in the geometry tasks that were given with non-diagrammatic register and truth-unknown features, students drew various types of diagrams and reasoned with them in their proofs. Therefore, there was a relationship identified between these problem types and the higher frequency of visual resources students employed. On the other hand, this study also found that students performed symbolic resources consistently in the tasks that were given in diagrammatic register. The differing frequency patterns in symbolic and visual resources in different tasks suggests that the task features provide a predictor of particular visual and symbolic resources.

Furthermore, the previous study of Herbst and colleagues (2016) described *diagrammatic register* in geometry proof problems and discussed teachers' reactions to problems and presentation of proofs outside the diagrammatic register. In this study, the findings showed that students who worked with tasks in non-diagrammatic register regularly produced their proofs in the non-diagrammatic register, as well. Therefore, the non-diagrammatic feature of the tasks influenced not only the visual and symbolic resources students employed in the diagrams, but also the representation of their proof in both written and diagram format.

This study found that most students did not use symbolic resources and they stayed in non-diagrammatic register if the task was presented in non-diagrammatic register. However, one student, Megan, who most consistently produced valid proofs across the interviews, utilized symbolic resources in every task even when no labels or symbols were provided in the task. In other words, Megan actively engaged in symbolic resources such as labeling the vertex and angles in the tasks both given with diagrammatic register and non-diagrammatic register. In fact, she was the only student who utilized symbolic resources in all non-diagrammatic register tasks in the second interview and also, she was the only student who produced valid proofs for all tasks in the second interview without any guidance by the researcher. Specifically, in the second interview, when she was asked about the reason why she preferred labeling the vertexes and sides of the diagram she created in the Perimeter Task, which was outside the diagrammatic register, Megan responded as follows:

> I can see it in my head, but when I write it down on paper, um, it helps because then I can like, edit it or like, add things to it. Like, I added the, the lengths and stuff after I drew it out because number one, I just wanted to see what that question was talking about since there wasn't already a picture provided. And then number two, just helps for to keep me organized, like, in my thoughts organized.

Thus, by labeling the angle she created in the non-diagrammatic register task, Megan gave rise to the diagram she reasoned with in the justification of proof.

Proving Actions in Geometry Proofs

This study adopted a framework developed by Otten and colleagues (2017) to investigate the actions students employed while proving geometric claims. Applying this framework showed that types of JUST-G (Justifying a step in the argument by gesturing) proving actions such as JUST-G, JUST-G-L, and JUST-G-W were the most prominent proving actions students employed in the first and second interviews, regardless of what type of task students worked with. This suggests that gesturing has a significant role in justifying arguments and production of proofs (e.g., Nathan et al., 2014; Pier et al., 2014).

On the other hand, some aspects of the framework developed by Otten and colleagues (2017) are interactions such as clarification and critique within proving processes. Although Otten and colleagues (2017), studying collective proving in a classroom, identified proving interactions that were predominantly initiated by the teacher, this study found that students do sometimes perform proving interactions even when working individually. These interactions, however, tended to occur with tasks that have particular features such as truth-unknown or non-diagrammatic register features. In other words, more typical tasks such as those with truth-known and diagrammatic register features involved proving actions (STEP, JUST) but rarely proving interactions (CLAR, CRIT). This suggests that drawing multiple diagrams or reducing the given symbolic resources in these tasks might relate to students' opportunities to clarify or critique their arguments.

In the study of Harel and Sowder (1998), they suggested that "[a] person can be certain about the truth of an observation in one situation, but seek additional or different evidence for the same observation in another situation" (p. 243). The findings of this study showed that students' proof productions changed for various formats of the same claim. As an illustration, there were two Right Triangle Task in the first interview presented in two different formats: truth unknown with an inaccurate diagram and truth known with an accurate diagram. Except for one student (Megan), all students treated the Right Triangle task given with an accurate diagram and truth-known features as a new task even though it was given immediately after the other Right Triangle task. Not surprisingly, the proving actions students employed on the two Right Triangle Tasks also differed. For example, INVEST (investigating or guessing the truth value of a claim), ABOUT (Making a general statement about proof), and JUST-V (Justifying a step in the argument by drawing) proving actions were more prevalent in students' proofs of the Right Triangle task with the truth-unknown feature. Likewise, the Midpoints Task in the first interview was given in two formats: diagrammatic register and non-diagrammatic register. The analysis of students' proving actions showed that INVEST, ABOUT, and JUST-V proving actions as well as proving interactions were more prominent in the proof of the non-diagrammatic register Midpoints Task. The varying patterns of proving actions between two similar tasks with different features indicate that the structure of geometric claims likely has an influence on the type of proving students engage in.

Limitations of the Study

Although multiple data sources and the reliability check of the frameworks used in the analysis ensure the credibility of the study findings, there are several limitations to this study. First, although this study includes more than a few students and involved second interviews with half of them, I had students from a specific cultural background because they were selected through personal connections and most of them identified themselves as successful students in geometry. Hence, this study is not sufficient to make generalizations across high school students. That is, I have answered my research questions for the students in my study, but that should not be misconstrued as claims about students' use of diagrams universally.

Furthermore, I chose to interview students who already completed their high school geometry class. Hence, this study showed how students used diagrams after they had been enculturated in some way to proof in geometry. Also, I used tasks that are from the official geometry curriculum. Therefore, students may use diagrams differently with different tasks (e.g., ones that are on content outside the curriculum) or differently of their own accord (before being taught how to write proofs).

Second, regarding the features of the tasks in this study, I considered various features including diagrammatic register and whether the truth of the claim was known or unknown. However, there are other characteristics of the tasks that might have been considered in this study. For example, some tasks included a holistic diagram in the givens; that is, the whole diagram was included in the givens, and the claim of the theorem was about parts of the diagram such as showing congruency of two sides or two inner triangles (e.g., Isosceles Task and Midpoints Task). Whereas, other tasks provided information about parts of a figure in the givens such as congruency of segments or angles, and the claim referred to the whole diagram; that is, showing that the whole diagram is a right triangle or isosceles triangle (e.g., Right Triangle

Tasks and Bisector Ray Task). Thus, the interpretation of the results of this study may have been enhanced by the consideration of the other features of the tasks. Moreover, I only considered four tasks in each interview, and some of them were modified versions of each other. Although students provided rich data on each task, the number of tasks might still be limited in terms of drawing general conclusions about the diagram-given and diagram-free geometry proof tasks.

Finally, in order to ensure that the tasks provided to students in this study were understandable to them, the only criteria for the participant selection was the completion of geometry class in high school. In fact, some other criteria such as curriculum sources used in the students' high schools or the geometry teachers they had might be a potential influence on students' approaches to the tasks. However, the participant selection criteria in this study is still sufficient, because the tasks were aligned with Common Core State Standards (CCSSI, 2016) and students were selected from two public high schools that adopted the Common Core State Standards in 2010. Furthermore, no student indicated that the tasks were not clear or familiar to them during the interviews.

Unfortunately, the current study does not address what types of symbolic or visual resources relate to valid proofs or particular proving actions. Likewise, it is not apparent that what specific gesture types lead to strong geometrical arguments in students' proofs. Hence, the future research should observe this point in more detail by considering proving actions concerning particular symbols, visuals, and gestures. Furthermore, this study identified some common gestures in certain tasks. In particular, some students employed dynamic gestures in the truth-unknown tasks and stated that they need a physical object to manipulate in order to prove the task better. Thus, future research should investigate the interaction between a physical tool or a touchscreen technology and gesture, and its influence on students' geometrical reasoning.

Significance and Implications of the Study

This section addresses the significance and three important implications of this study. In particular, the results from this study provide implications for research, professional development, and teacher education in the mathematics education field.

Significance of the Study

Previous research addressed the role that diagrams play in geometry either by considering the diagrams in geometry textbooks (Dimmel & Herbst, 2015) or discussing them in geometry theorems in general (Dove, 2002; Miller, 2007). In particular, Dimmel and Herbst (2015) and Miller (2007) called further research to understand the way students use diagrams in the geometrical contexts. In other words, to support students in their efficient use of diagrams in geometry proofs, the field of mathematics education needs to know more about what meanings students endow on the diagrams and, hence, how they use them in the geometry proofs. This study contributes to the field of mathematics education because by providing insights on what meanings students draw from diagrams makes an addition to the study of Dimmel and Herbst (2015) on the meanings that diagrams convey.

One of these insights relates to the conceptual and figural properties of the diagrams. Previous research investigated the implementation of conceptual/theoretical and figural/spatial properties of the diagrams under the consideration of different age groups in high school (Fischbein & Nachlieli, 1998; Laborde, 2005). Their investigation revealed that although geometry problems require using both figural/spatial and conceptual/ theoretical domains, students mostly rely on the figural properties. However, in this study, the interplay of figural and conceptual properties of diagrams was discussed in relation to different task features. That is, various features of the tasks in addition to the provision or lack of diagrams with the claims were an extension of the previous research because it valued the impact of the task as well as the targeted student group on the interplay of figural and conceptual properties of diagrams. Therefore, this study might establish a link between task features and students use of figural and conceptual properties of the diagrams.

In particular, concerning the diagrammatic register feature that several tasks included in this study, although Herbst and colleagues (2016) investigated high school mathematics teachers' interpretation of diagrammatic register or its lack thereof in the presentation of proof problems, they did not discuss the feature in students' proving. They argued that the diagrammatic register is the expected format for school-type proof opportunities, and that we generally teach students how to work in the diagrammatic register but not necessarily flexibly for a variety of proving scenarios. In fact, this study extends that research by a detailed analysis of students' approaches on proving tasks that included diagrammatic register or non-diagrammatic register features. Hence, results from this study add to the literature by providing insights into how students performed on the tasks with the diagrammatic register and non-diagrammatic register. The patterns of the use of diagrams in non-diagrammatic register tasks explored in this study, which is not employing symbolic resources such as labeling the angles or vertices of the diagrams, shed lights on the impact of the presentation of geometry tasks on students' proof production. The requirement we reinforce on students to show diagrammatic register probably undercuts the opportunity to think flexibly with diagrams. However, students in this study demonstrated a flexible diagram use such as drawing more than one diagram and considering various possibilities of particular parts of the diagram when the task was presented in the non-diagrammatic register. Still, most students struggled with producing valid proofs, and they stated that they were not exposed to the non-diagrammatic register tasks regularly in geometry classes (e.g., Bisector Ray Task and Pentagon Task).

Implications of the Study

Implications for Research

This study suggests some research implications and recommendations for future research based on the limitations discussed above and the results of the study. The design of this study was the clinical interview, and the data used in this study did not account for school-level factors such as curriculum materials and instructional practices in the geometry classrooms. However, future studies should investigate the curriculum materials and instructional implications that influence students' use of diagrams.

The findings suggest that students drew various diagrams in the tasks that had non-diagrammatic register and truth-unknown features. That is, there was a pattern of visual resources in geometry tasks that have these features. The diagrams students drew were sophisticated as they demanded the recognition of both conceptual and figural properties of the diagrams. However, further exploration is needed to understand the interaction of these properties in these kinds of tasks. Furthermore, the findings showed that students who frequently provided symbolic resources during proving were more likely to provide valid proofs of the claims. However, further research is required to corroborate these small-scale findings and to better understand if there is a direct relationship between them and what encourages the students to constitute symbolic resources on the proofs.

Another implication for research is that this study showed that particular proving actions appeared more frequently in certain types of the task features. For example, truth-known tasks corresponded with more STEP and JUST proving actions; whereas, non-diagrammatic register or truth-unknown tasks corresponded with more INVEST proving action and CLAR and CRIT proving interactions. Therefore, the results of this study suggest that if students encounter a wider variety of task types, they are more likely to engage in a wider variety of proving actions.

Future research may address how different task features in geometry influence students proving actions in more detail.

The non-diagrammatic register and truth-unknown tasks used in this study can be considered atypical types of geometry tasks for students because they require creating a diagram or drawing multiple diagrams to test the validity of the geometric claim instead of working with a referent diagram or given diagram only. Although the results of this study did not address that students performed the conceptual properties of the diagrams more frequently in these tasks, I hypothesize that the representations of these tasks support students to consider the theoretical features of the diagrams more than the typical diagram-given tasks. Future research may address this issue in a larger sample with a presentation of the non-diagrammatic register or truth-unknown tasks in addition to standard diagram given tasks. Thus, future research may look more closely at students who try to prove the same claim presented in various formats.

Implications for Professional Development

The results presented in this study demonstrate the importance of developing teacher knowledge about issues of representation of geometry tasks and the role of gestures in geometrical reasoning. This study found that students proved differently and used diagrams differently depending on the task features, even if the tasks addressed similar claims from a content perspective. As an illustration, in truth-unknown tasks, students were more likely to consider diagrams generally since they did not label the diagram in a particular way. Likewise, in non-diagrammatic register tasks, some students drew triangles and claimed that the diagram was "any triangle" even though it was drawn as a particular triangle. These results suggest that students might appreciate the generic feature of the diagrams when they engaged in these tasks. Therefore, to help students understand the generalization of the diagram, efforts could be made to improve teachers' knowledge of employing geometry tasks in different formats and making mindful instructional decisions through students' solutions to these tasks to improve the students' geometrical thinking.

Additionally, in this study, all students frequently gestured when explaining their proofs. Although some students did not produce a written explanation in the proofs or did not use precise geometrical terms in the verbal description, their gestures were clarifying the particular sides in the diagram and justifying the argument they made. Professional development programs may focus on improving teachers' knowledge on understanding the gestures as a way to get into students' thinking. It is important that professional development programs may provide teachers an appreciation of gestures in students' reasoning and when to accept them as a signal for students' understanding. Then, teachers may help students move beyond gestures to the diagrammatic register because it is a common expectation for success in school math.

Implications for Teacher Education

An important implication for teacher education is the need to address the figural and conceptual properties of the diagrams in geometry proofs because the results of this study showed that some students concluded their proofs merely based on the figural properties of the diagrams. For example, the arguments Lee made on the Isosceles Task in the first interview were based on the visual appearance of the given diagram. He, in fact, stated that if the diagram was given in another format, different than an isosceles triangle appearance, he might not be able to figure out and explain his proof. Therefore, teacher education programs may support prospective teachers to improve their skills on noticing and responding to students who base their geometrical reasoning merely on the figural properties of the geometry diagrams.

The results of this study showed that gesture resources were the most salient semiotic resources that students employed in their proofs. Students often relied on gestures with the diagrams as they were forming and explaining their argument; whereas, teachers should help students shift from talking/gesturing to writing formal arguments. Although diagrammatic register tasks were basically designed to help students make this transition, students only learn formal writing in these tasks. This study suggests that variations in task features (e.g., non-diagrammatic register and truth-unknown) have potential to support transitioning from talking, exploring, and gesturing to formal writing. Particularly, in this study, Megan exemplified an ideal case because she flexibly used semiotic resources and engaged in proving, regardless of how the claim was presented.

Moreover, existing research addressed the role of gestures in instructional communication and claimed that gesture accompanying to speech has a high impact in supporting students' mathematical thinking (Nathan, 2008; Williams-Pierce et al., 2017). Given that, teacher education programs may help teachers understand the importance of gesturing in the instruction and supporting students' thinking. However, teachers should make a careful decision on the types of gesture that might be helpful for students.

Regarding the gesture types, the results of this study identified some familiar gestures on certain tasks. For instance, in two truth-unknown tasks (The Right Triangle and Pentagon Task), most students considered the mobility of the sides of the diagrams by sketching different variations of the diagram on the paper. These actions showed that students employed dynamic gestures in some occasions depending on the tasks. In fact, Pier and colleagues (2014) suggested that teachers should support students to use dynamic gestures in the classroom. Therefore, the teacher education programs may inform prospective teachers to make instructional decisions such as using a particular task or physical tools that will foster dynamic gestures in the classrooms.

Concluding Remarks

The current analysis of high school students' interpretation and use of diagrams in geometry proofs is an extension of some existing research in mathematics education field. Close attention on the semiotic resources and proving actions students applied, and their interplay in students' proof works provide insights into an in-depth understanding of students' reasoning with diagrams. Moreover, the frameworks used in this study showed how important it is to consider the mathematics as multi semiotic, understanding the role of gestures in students' geometrical reasoning. The results of this study reveal striking relations between particular geometry proof tasks and the pattern of semiotic resources students employed on these tasks. This study focused on diagram-given and diagram-free geometry tasks in general. However, addressing other features of the tasks such as non-diagrammatic register and truth unknown features and identifying the students' use of diagrams under these tasks helped explore students' different approaches when the tasks were presented differently. The research on students' interactions with diagrams in geometry proofs through a semiotic perspective should continue to explore supportive ideas for learning and teaching geometry in particular and to make useful contributions to mathematics education field in general.

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APPENDIX A: STUDENT BACKGROUND INFROMATION

Student Interview Background

Interview Date:

Time: Start: End:

Part A. Basic Descriptive Information

1. Student Gender: ___Male ___Female ___Other

2. Student Ethnicity: ___American Indian or Alaskan Native

___Asian

___Hispanic or Latino

___Black or African-American

___Native Hawaiian or Other Pacific Islander

___White

3. School:

4. Grade Level:

5. When did the student take the Geometry Class?

APPENDIX B: PARENT AND CHILD CONSENT FORM

PARENT CONSENT FORM FOR A RESEARCH STUDY

Researcher's and Researcher Supervisor's Names: Ruveyda Karaman, Ph.D. Candidate, and Samuel Otten, Ph.D.
Project Number: 2005907

Project Title: High School Students' Interpretation and Use of Diagrams in Proof Exercises

Please ask the investigator or the study staff to explain any words or information that you do not clearly understand.

INTRODUCTION

Your child is being asked to participate in a research study. This research is being conducted to learn about how students use the given information in the proof exercises. When invited to participate in research, you and your child have the right to be informed about the study procedures so that you can decide whether you want to consent to participation.

Your child has the right to know what they will be asked to do so that they can decide whether or not to be in the study. Participation is voluntary. Your child does not have to be in the study. Your child may refuse to be in the study without penalty. If your child does not want to continue to be in the study, they may stop at any time without penalty or loss of benefits to which they are otherwise entitled.

WHY IS THIS STUDY BEING DONE?

This study is being conducted because the field of mathematics education currently considers how students make meaning with the information given in the proof tasks. Because this is a new phenomenon in mathematics education, we are hoping to learn about what it looks like when students work on geometrical proof tasks.

HOW MANY PEOPLE WILL BE IN THE STUDY?

Ten students will take part in this study, which is situated in Missouri. Your child has been selected as a possible participant in this study because they have taken the geometry class in the high school and can work with the tasks.

WHAT IS MY CHILD BEING ASKED TO DO? HOW LONG WILL IT TAKE?

Participation in this study involves task-based interviews with your child. During the interview, researcher will be observing your child's works on three geometrical proof exercises and their work will be video recorded. At the end of the interview, copies of your child's written work may be requested. The task-based interview should take 45-60 minutes to complete. Your child may be invited to the second interview after the first one.

WHAT ARE THE BENEFITS AND RISKS OF BEING IN THE STUDY?

There are only minimal foreseeable benefits and risks for participation in the study. The potential risks involve discomfort that your child may feel on the task performance or behavior while being observed by the researcher, but the researcher is not focused on assessing your child's performance. The benefits from the study will be advancement in the field of mathematics education in our knowledge of how students use the given information in the geometric proof tasks.

WHAT ARE THE COSTS OF BEING IN THE STUDY? WHAT OTHER OPTIONS ARE THERE?

There is no cost to you. Your child may opt out of this study and will not be penalized for that decision.

CONFIDENTIALITY

Information from this study will be stored in the investigators' secured files and hard drive. Outside of the research team, your child will be referred to with a pseudonym only. Identifiable information about your child will not be given to anyone outside the study except as required by law. Recordings of your child may be shown to others for the purposes of sharing the results of the study with other researchers, but in these cases your child's pseudonym will be used and your child's face won't be visible in the dissemination to other researchers. The recordings will not be given to anyone else nor will they be made available online. The recordings will not be shown publicly within the state of Missouri. If your child chooses not to participate, his or her image in video recording will not be used.

WILL MY CHILD BE COMPENSATED FOR PARTICIPATING IN THE STUDY?

Your child will receive a $20 gift card at the end of the first interview.

WHAT ARE MY CHILD'S RIGHTS AS A PARTICIPANT?

Participation in this research project is completely voluntary. Your child has the right to say no, and if they agree to participate, your child has the right at any time to change their mind and withdraw. At any time, your child may refuse to participate in certain procedures or answer certain questions without penalty or loss of benefits.

WHO DO I CONTACT IF I HAVE QUESTIONS, CONCERNS, OR COMPLAINTS?

If you have concerns or questions, please contact the researchers:

Researcher Supervisor
Samuel Otten, Ph.D.
121B Townsend Hall
University of Missouri
Columbia, MO 65211
ottensa@missouri.edu
(573) 882-6231

Researcher
Ruveyda Karaman
119 Townsend Hall
University of Missouri
Columbia, MO 65211
rk5f7@mail.missouri.edu
(573) 529-7771

If you have any questions regarding your child's rights as a participant in this research and/or concerns about the study, or if you feel under any pressure to enroll your child or have them continue to participate in this study, you may contact the University of Missouri Campus Institutional Review Board (which is a group of people who review the research studies to protect participants' rights) at (573) 882-9585 or umcresearchcirb@missouri.edu.

A copy of this consent form will be provided to you.

SIGNATURES

We have read this consent form and our questions have been answered. Our signature below means that we do want to be in the study. We know that we can remove ourselves from the study at any time without any problems.

____ ____________________
Parent/Guardian Signature Date

____
Student's Name (printed)

APPENDIX C: TWO DIFFERENT SEQUENCES OF TASKS IN THE FIRST INTERVIEW

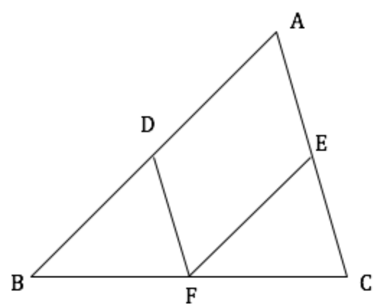
The First Sequence of Tasks

1. Isosceles Task

Let ABC be an isosceles triangle and $\overline{AB}=\overline{AC}$. Let D and E be two points on $\overline{BC}$ such that angle <BAD is equal to angle <CAE. Prove that triangle ADE is isosceles.

1

2. Midpoints Task



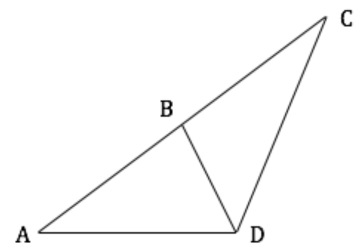
Given: D, E, F midpoints.

Prove: Triangle BDF is congruent to triangle FEC. (Δ BDF $\cong$ Δ FEC)

***Clue Theorem*:** The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

2

3. Right Triangle Task



If B is the midpoint of $\overline{AC}$ and $\overline{AB}$=$\overline{BD}$.

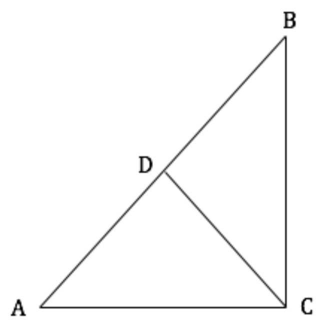
It is claimed that <CDA is a right angle.

The claim is;

Circle one: Always True Sometimes True Never True

Prove:

4. Right Triangle Task



Given $\overline{AD} = \overline{DB} = \overline{DC}$, prove that angle <BCA is right angle.

4

The Second Sequence of Tasks

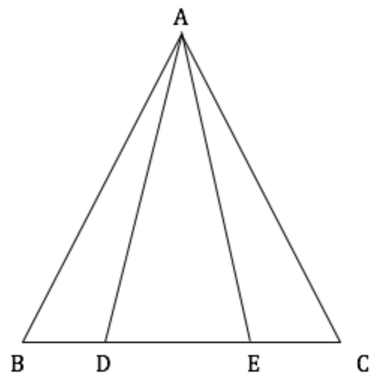
1. Midpoints Task

Given a triangle, the midpoint of a side constructs two segments by joining with the other two midpoints on the two other sides of the triangle and hence, two inner triangles occur in the triangle. Prove that these inner triangles are congruent.

***Clue Theorem*:** The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

1

2. Isosceles Task

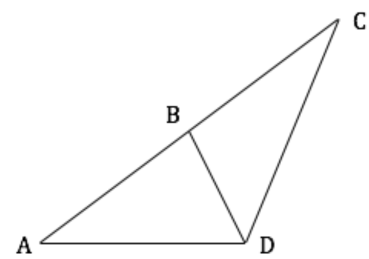


Given: $\overline{AB}$=$\overline{AC}$ $< BAD \cong < CAE$

Prove: Triangle ADE is isosceles.

2

3. Right Triangle Task



B is the midpoint of $\overline{AC}$ and $\overline{AB}=\overline{BD}$.

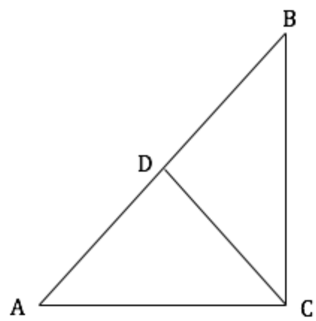
It is claimed that <CDA is a right angle.

The claim is;

Circle one: Always True Sometimes True Never True

Prove:

4. Right Triangle Task



Given $\overline{AD} = \overline{DB} = \overline{DC}$, prove that angle <BCA is right angle.

APPENDIX D: SEQUENCES OF TASKS IN THE SECOND INTERVIEW

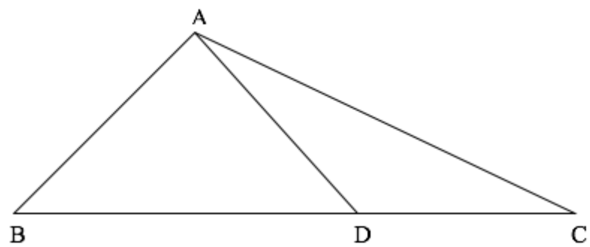
1. Perimeter Task

Given a triangle, the midpoint of any side constructs two segments by joining with the other two midpoints on the other two sides of the triangle and hence, one inner triangle occurs in the center of the triangle. Prove that the perimeter of the center triangle is half the perimeter of the given triangle.

Clue Theorem**:** The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.

1

2. Right Triangle Task

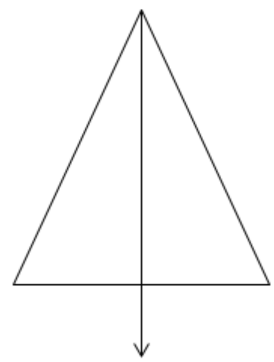


Δ ABD is equilateral and D is the midpoint of BC. Prove that Δ ABC is a right triangle.

2

3. Bisector Ray Task

Given a triangle, if an angle bisector ray emanating from one vertex meets the opposite side at a right angle, then prove that the overall triangle is isosceles.



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4. Pentagon Task

It is claimed that the sum of the interior angles of a pentagon is 540°.

The claim is;

Circle One: Always True Sometimes True Never True

Prove:

VITA

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