Post-Newtonian limitations on measurement of the PPN parameters caused by motion of gravitating bodies

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Accepted 2009 July 8. Received 2009 July 8; in original form 2009 June 9

ABSTRACT

We derive an explicit Lorentz-invariant solution of the Einstein and null geodesic equations for data processing of the time delay and ranging experiments in the gravitational field of moving gravitating bodies of the Solar system – the Sun and major planets. We discuss the general-relativistic interpretation of these experiments and the limitations imposed by motion of the massive bodies on measurement of the parameters \( \gamma_{\text{PPN}}, \beta_{\text{PPN}} \) and \( \delta_{\text{PPN}} \) of the parametrized post-Newtonian (PPN) formalism.

Key words: gravitation – methods: analytical – techniques: interferometric – techniques: radar astronomy.

1 INTRODUCTION

Theoretical speculations beyond the standard model suggest that gravity must be naturally accompanied by a partner – one or more scalar fields, which contribute to the hybrid metric of space–time through a system of equations of a scalar–tensor gravity theory (Damour & Esposito-Farèse 1992). Such scalar partners generically arise in all extra-dimensional theories, and notably in string theory. Scalar fields also play an important role in modern cosmological scenarios with the inflationary stage (Mukhanov 2005). Therefore, the unambiguous experimental verification of existence of the scalar fields is among the primary goals of gravitational physics.

The phenomenological presence of the scalar field in the metric tensor is parametrized by three parameters – \( \gamma_{\text{PPN}}, \beta_{\text{PPN}} \) and \( \delta_{\text{PPN}} \) – of the parametrized post-Newtonian (PPN) formalism. These parameters enter the metric tensor of a static and spherically symmetric gravitating body in the following form (Brumberg 1992; Will 1993, 2001; Damour & Esposito-Farèse 1996):

\[
g_{00} = -1 + 2\frac{GM}{c^2 R} - 2(1 + \beta_{\text{PPN}}) \left( \frac{GM}{c^2 R} \right)^2 + O \left( c^{-6} \right),
\]

\[
g_{ij} = \delta_{ij} \left[ 2(1 + \gamma_{\text{PPN}}) \frac{GM}{c^2 R} + \frac{3}{2} (1 + \beta_{\text{PPN}}) \left( \frac{GM}{c^2 R} \right)^2 \right] + O \left( c^{-4} \right),
\]

where we have used the isotropic coordinates \( X^a = (cT, X), R = |X| \), and denoted deviation from general relativity with the comparative PPN parameters \( \gamma_{\text{PPN}} \equiv \gamma_{\text{PPN}} - 1, \beta_{\text{PPN}} \equiv \beta_{\text{PPN}} - 1, \delta_{\text{PPN}} = \delta_{\text{PPN}} - 1 \). Parameter \( \delta_{\text{PPN}} \) generalizes the standard PPN formalism (Will 1993) to the second post-Newtonian approximation (Brumberg 1992). One notes that \( \delta_{\text{PPN}} \) is actually related to \( \beta_{\text{PPN}} \) and \( \gamma_{\text{PPN}} \) in a generic scalar–tensor theory of gravity (Damour & Esposito-Farèse 1996). In particular, this theory predicts that the \( \beta_{\text{PPN}} \) cancels in the combination \( -\beta_{\text{PPN}} + 3/4 \delta_{\text{PPN}} \) in equation (4) of the present paper, which should depend, theoretically, only on \( \gamma_{\text{PPN}} \) and its square. Thus, high-precision missions will have a very clean access to \( \gamma_{\text{PPN}} \). However, we prefer to keep the combination \( -\beta_{\text{PPN}} + 3/4 \delta_{\text{PPN}} \) explicitly in our equations in order to separate parametrization of the second post-Newtonian effects associated with \( \delta_{\text{PPN}} \) from the linearized Shapiro time delay, which is parametrized by \( \gamma_{\text{PPN}} \) alone. Moreover, parameter \( \delta_{\text{PPN}} \) is independent from \( \beta_{\text{PPN}} \) and \( \gamma_{\text{PPN}} \) in vector–tensor theories of gravity (Deng, Xie & Huang 2009). In general relativity, \( \beta_{\text{PPN}} = \gamma_{\text{PPN}} = \delta_{\text{PPN}} = 0 \).

The best experimental bound on \( \gamma_{\text{PPN}} = (2.1 \pm 2.3) \times 10^{-5} \) has been obtained [under a certain implicit assumption (Kopeikin et al. 2007)] in the Cassini experiment (Bertotti, Less & Tortora 2003). Limits on the parameter \( \beta_{\text{PPN}} \) depend on the precision in measuring \( \gamma_{\text{PPN}} \), and are derived from a linear combination \( 2 \gamma_{\text{PPN}} - \beta_{\text{PPN}} < 3 \times 10^{-3} \) by observing Mercury’s perihelion shift and from \( 4 \beta_{\text{PPN}} - \gamma_{\text{PPN}} = (4.5 \pm 4.5) \times 10^{-4} \) imposed by lunar laser ranging (Williams, Turyashev & Boggs 2004). Parameter \( \delta_{\text{PPN}} \) has not yet been measured.

The most precise measurement of \( \gamma_{\text{PPN}} \) and \( \delta_{\text{PPN}} \) can be achieved in near-future gravitational experiments with light propagating in the field of the Sun or a major planet. The post-Newtonian equation of the relativistic time delay in a static gravitational field is obtained from the

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metric (1) and (2). It was derived by a number of authors (Richter & Matzner 1982, 1983; Brumberg 1992; Teyssandier & Le Poncin-Lafitte 2008) and reads (in the isotropic coordinates) as follows:

$$T_2 - T_1 = \frac{R}{c} + \Delta T + O(G^3),$$

(3)

where $T_1$ and $T_2$ are coordinate times of emission and observation of the photon, $R = |X_2 - X_1|$ is the coordinate distance between the point of emission, $X_1$, and observation, $X_2$, of the photon, and

$$\Delta T = (2 + \gamma_{\text{PPN}}) \frac{GM}{c^5} \ln \left( \frac{R_1 + R_2 + R}{R_1 + R_2 - R} \right) + \frac{G^2 M^2}{c^5} \frac{R}{R_1 R_2} \left[ \left( \frac{15}{4} + 2\gamma_{\text{PPN}} - \beta_{\text{PPN}} + \frac{3 \gamma_{\text{PPN}}^2}{4} \right) \frac{\arccos(N_1 \cdot N_2)}{|N_1 \times N_2|} - \frac{(2 + \gamma_{\text{PPN}})^2}{1 + N_1 \cdot N_2} \right]$$

(4)

is the extra time delay caused by the gravitational field, $N_1 = X_1/R_1$ and $N_2 = X_2/R_2$ are the unit vectors directed outward of the gravitating body, $R_1 = |X_1|$ and $R_2 = |X_2|$ are radial distances to the points of emission and observation, respectively.

The Sun and planets are not at rest in the Solar system because they are moving with respect to the barycentre of the Solar system as well as with respect to the observer. Motion of the light-ray deflecting body (the Sun, a major planet) affects propagation of light bringing the post-Newtonian corrections of the order of $(GM/c^3)(v/c)$, $(GM/c^3)(v/c)^2$, etc. to equation (4), where $v$ is a characteristic speed of the massive body with respect to the reference frame used for data processing, which can be chosen as either the barycentric frame of the Solar system or the geocentric frame of the observer. These motion-induced post-Newtonian corrections to the static time delay $\Delta T$ correlate with the PPN parameters making their observed numerical value biased. Therefore, it is important to disentangle the genuine effects associated with the presence of the scalar field from the special relativistic effects in equation (4) imparted by the motion of the body.

This problem has not been addressed until recently because the accuracy of astronomical observations was not high enough. However, very long baseline interferometry (VLBI) measurement of the null-cone gravity retardation effect (Kopeikin 2001, 2004; Fomalont & Kopeikin 2003; Fomalont et al. 2009) and frequency-shift measurement of $\gamma_{\text{PPN}}$ in the Cassini experiment (Bertotti, Iess & Tortora 2003; Anderson, Lau & Giampieri 2004) made it evident that modern technology has achieved the level at which relativistic effects caused by the dependence of the gravitational field on time can be no longer ignored. Future gravitational light-ray deflection experiments (Kopeikin & Mashhoon 2002), the radio ranging BepiColombo experiment (Milani et al. 2002), laser ranging experiments ASTROD (NI 2007) and LATOR (Turyshhev, Shao & Nordtvedt 2004) will definitely reach the precision in measuring $\gamma_{\text{PPN}}$, $\beta_{\text{PPN}}$ and $\delta_{\text{PPN}}$ that is comparable with the post-Newtonian corrections to the static time delay and to the deflection angle caused by the motion of the massive bodies in the Solar system (Plowman & Hellings 2006). Therefore, it is worthwhile to undertake a scrutiny theoretical study of the time-dependent relativistic corrections to the static Shapiro time delay.

In this paper, we focus on deriving two apparently different forms of the Lorentz-invariant solution of the light-ray equations (see equations 31 and 43) in the linearized (with respect to the universal gravitational constant $G$) approximation of general relativity by making use of the technique of the Liénard–Wiechert potentials (Kopeikin & Schäfer 1999) and algebraic transformations of the retarded quantities. In particular, equation (43) of the present paper significantly generalizes the result of Bertotti, Ashby & Iess (2008) for the gravitational time delay. We expand this retarded-time solution in the post-Newtonian series in three various ways (see equations 63, 68 and 86) and analyse the impact of the velocity-dependent corrections on measuring values of the PPN parameters in the gravitational time-delay experiments. Section 8 discusses a correspondence between the Lorentz symmetry group for gravity and light as revealed by the time-delay experiments. Section 9 gives a justification that the Orbit Determination Program (ODP) code of NASA must be revamped for doing adequate processing of high-precision data in ranging gravitational experiments.

2 NOTATIONS

In what follows the Greek indices $\alpha, \beta, \ldots$ run from 0 to 3, the Roman indices $i, j, \ldots$ run from 1 to 3, repeated Greek indices mean Einstein’s summation from 0 to 3, and bold letters $a = (a^1, a^2, a^3), b = (b^1, b^2, b^3)$, etc. denote spatial (three-dimensional) vectors. A dot between two spatial vectors, for example $a \cdot b = a^1 b^1 + a^2 b^2 + a^3 b^3$, means the Euclidean dot product, and a cross between two vectors, for example $a \times b$, means the Euclidean cross product. We also use a shorthand notation for partial derivatives $\partial_\alpha = \partial/\partial x^\alpha$. Greek indices are raised and lowered with full metric $g_{\alpha\beta}$. The Minkowski (flat) space–time metric $\eta_{\alpha\beta} = \text{diag} (-1, +1, +1, +1)$. This metric is used to raise and lower indices of the unperturbed wave vector $k^\alpha$ of light, and the gravitational perturbation $h_{\alpha\beta}$.

3 THE LIÉNAUD–WIECHERT GRAVITATIONAL POTENTIALS

We introduce the post-Minkowski decomposition of the metric tensor

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta},$$

(5)

where $h_{\alpha\beta}$ is the post-Minkowskian perturbation of the Minkowski metric tensor $\eta_{\alpha\beta}$. We impose the harmonic gauge condition (Misner, Thorne & Wheeler 1973) on the metric tensor

$$\partial_\alpha h^{\alpha\beta} - \frac{1}{2} \partial^\alpha h_\alpha^\beta = 0.$$

(6)
In arbitrary harmonic coordinates $x^\alpha = (ct, \mathbf{x})$, and in the first post-Minkowskian approximation the Einstein equations read
\[
\left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2\right) h^{\mu\nu} = -\frac{16\pi G}{c^4} \left( T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right),
\]
where $T^{\mu\nu}$ is the stress–energy tensor of a light-ray deflecting body. In the linearized approximation, this tensor is given by the following equation:
\[
T^{\mu\nu}(t, \mathbf{x}) = M u^\mu u^\nu \sqrt{1 - \beta^2} \delta(0)(\mathbf{x} - z(t)),
\]
where $M$ is the (constant) rest mass of the body, $z(t)$ is the time-dependent spatial coordinate of the body, $\beta = c^{-1} dz/dt$ is velocity of the body normalized to the fundamental speed $c$,
\[
u = (1 - \beta^2)^{-1/2}, \quad u^\nu = \beta'(1 - \beta^2)^{-1/2},
\]
is the four-velocity of the body normalized such that $u_\alpha u^\alpha = -1$, and $\delta(0)(\mathbf{x})$ is the three-dimensional Dirac’s delta function. We have neglected $\sqrt{-g}$ in equation (8) because in the linearized approximation $\sqrt{-g} = 1 + O(G)$, and the quadratic terms proportional to $G^2$ are irrelevant in $T^{\mu\nu}$ since they will give time-dependent terms of the second post-Minkowskian order of magnitude, which are currently negligible for measurement in the Solar system. For the same reason, we do not use the metric derived by Blanchet, Faye & Ponsot (1998) as it goes beyond the approximation used in the present paper. We have also used a standard notation $\beta$ for the dimensionless velocity of the body. This notation should not be confused with the PPN parameter $\beta_{\text{PPN}}$.

Because the Einstein equations (7) are linear, we can consider their solution as a linear superposition of the solutions for each body. It allows us to focus on the relativistic effects caused by only one body (the Sun, a planet). Solving Einstein’s equations (7) by making use of the retarded Liénard–Wiechert tensor potentials (Bel et al. 1981), one obtains the post-Minkowski metric tensor perturbation (Bel et al. 1981; Kopeikin & Schäfer 1999):
\[
h^{\mu\nu}(t, \mathbf{x}) = \frac{4GM}{c^2} u^\mu u^\nu + \frac{\eta^{\mu\nu}}{\rho_G},
\]
where
\[
\rho_G = -u_\alpha u^\alpha, \quad \rho^\nu = x^\nu - z^\nu(s),
\]
in equation (10), all time-dependent quantities are taken at a retarded time $s$ defined by the null-cone equation (13) given below, $u^\alpha \equiv u^\alpha(s) = c^{-1} dz^\alpha(s)/ds$ is its four-velocity, with $s$ being the retarded time (see below), $\beta(s) = c^{-1} dz(s)/ds$ is the body’s coordinate velocity normalized to the fundamental speed $c$. Notice that the metric tensor perturbation (10) is valid for accelerated motion of the gravitating body as well, and is not restricted by the approximation of a body moving on a straight line (see Bel et al. (1981) for more detail). In other words, the four-velocity $u^\alpha$ in equation (10) is not a constant, taken at one, particular event on the world line of the body.

Because we solved the Einstein equations (7) in terms of the retarded Liénard–Wiechert potentials, the distance $\rho^\nu = x^\nu - z^\nu(s)$, the body’s worldline $z^\nu(s) = [cs, z(s)]$, and the four-velocity $u^\alpha(s)$ are all functions of the retarded time $s$ (Bel et al. 1981). The retarded time $s$ is found in the first post-Minkowski approximation as a solution of the null-cone equation
\[
\eta_{\mu\nu} \rho^\mu \rho^\nu \equiv \eta_{\mu\nu}[x^\mu - z^\mu(s)][x^\nu - z^\nu(s)] = 0,
\]
that is
\[
s = t - \frac{1}{c} |x - z(s)|,
\]
where the constant $c$ in equation (14) denotes the fundamental speed in the Minkowski space–time, which in terms of the physical meaning in equation (14) is the speed of propagation of gravity as it originates from the gravity field equations (7). It is important to note that equation (14) is a complicated function of the retarded time $s = s(t, x)$, which has an analytical solution only in case of a uniform motion of the gravitating body along a straight line (Kopeikin 2004). Geometrically, equation (14) connects the point of observation $x$ and the retarded position of the gravitating body $z(s)$ by a null characteristic of the linearized Einstein field equations (7). Radio waves (light) are also propagating along a null characteristic connecting the observer and the radio emitter. However, the null characteristic of the linearized Einstein equations (14) is well separated on the space–time manifold (and in the sky) from the null characteristic associated with the propagation of the radio wave in any kind of ranging and time-delay experiments. Hence, they should not be confused in relativistic experiments involving light propagation in the field of a moving gravitating body in which the gravitational field depends on time (Will 2001; Kopeikin & Fomalont 2006).

All components of the time-dependent gravitational field (the metric tensor perturbation $h_{\mu\nu}$) of the Solar system bodies interact with radio (light) waves moving from a radio (light) source to the Earth, and perturb each element of the phase of the electromagnetic wave with the retardation given by equation (14). The use of the retarded Liénard–Wiechert gravitational potentials, rather than the advanced potentials, is consistent with the principle of causality (Kopeikin & Fomalont 2007), and the observation of the orbital decay of the relativistic binary pulsar B1913+16 caused by the emission of gravitational radiation, according to general relativity (Weisberg & Taylor 2005).
4 THE ELECTROMAGNETIC PHASE

Any ranging or time-delay experiment measures the phase \( \psi \) of an electromagnetic wave coming from a spacecraft or a radio (light) source outside the Solar system. The phase is a scalar function being invariant with respect to coordinate transformations. It is determined in the approximation of geometric optics from the Eikonal equation (Landau & Lifshitz 1971; Misner et al. 1973)

\[
g^{\mu\nu}\partial_\mu \psi \partial_\nu \psi = 0,\tag{15}\]

where \( g^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} \). The Eikonal equation (15) is a direct consequence of Maxwell’s equations (Misner et al. 1973; Kopeikin & Mashhoon 2002) and its solution describes localization of the front of an electromagnetic wave propagating on a curved space–time manifold in which geometric properties are defined by the metric tensor (5) and (10) that is a solution of the Einstein equations. We emphasize that the electromagnetic wave in equation (15) has no back-action on the properties of the metric tensor \( g_{\mu\nu} \), and does not change the curvature of the space–time caused by the presence of the gravitating body. Thus, the experimental study of the propagation of the electromagnetic wave allows us to measure the important properties of the background gravitational field and space–time manifold.

Let us introduce a co-vector of the electromagnetic wave, \( K_\alpha = \partial_\alpha \psi \). Let \( \lambda \) be an affine parameter along a light ray being orthogonal to the electromagnetic wave front \( \psi \). Vector \( K^\alpha = dx^\alpha/d\lambda = g^{\alpha\beta}\partial_\beta \psi \) is tangent to the light ray. Equation (15) expresses a simple fact that vector \( K^\alpha \) is null, that is \( g_{\mu\nu} K^\mu K^\nu = 0 \). Thus, the light rays are null geodesics (Landau & Lifshitz 1971) defined by the equation

\[
d K_\alpha/d\lambda = g_{\alpha\mu} K^\mu K^\nu,\tag{16}\]

The Eikonal equation (15) and light-ray equation (16) have equivalent physical content in general relativity since equation (15) is a first integral of equation (16).

Regarding propagation of the electromagnetic wave, it is more straightforward to find a solution of equation (15). To this end, we expand the Eikonal \( \psi \) in the post-Minkowskian series with respect to the universal gravitational constant \( G \) assuming that the unperturbed solution of equation (15) is a plane electromagnetic wave (i.e. the parallax of the radio source is neglected). The expansion reads

\[
\psi = \psi_0 + \frac{v}{c} [k_\alpha x^\alpha + \varphi(x^\nu)] + O(G^2),\tag{17}\]

where \( \psi_0 \) is a constant of integration, \( k^\alpha = (1, k) \) is a constant null vector directed along the trajectory of propagation of the unperturbed electromagnetic wave such that \( \eta_{\alpha\beta} k^\alpha k^\beta = 0, v \) is the constant frequency of the unperturbed electromagnetic wave, and \( \varphi \) is the first post-Minkowskian perturbation of the Eikonal, which is Lorentz-invariant. Substituting expansions (5) and (17) into equation (15), and leaving only terms of the order of \( G \), one obtains an ordinary differential equation for the post-Minkowskian perturbation of the Eikonal,

\[
\frac{d\varphi}{d\lambda} = \frac{1}{2}\hbar^{\alpha\beta} k_\alpha k_\beta = \frac{2GM}{c^2} \left( \frac{u_\alpha k^\alpha}{\rho_R} \right),\tag{18}\]

which can also be obtained as a first integral of the null geodesic equation (16). Equation (18) can be readily integrated, if one employs an exact relationship

\[
\frac{d\lambda}{\rho_R} = -\frac{1}{k_\alpha u^\alpha} = \frac{1}{k_\alpha u^\alpha} d \ln \left( -k_\alpha \rho^\alpha \right),\tag{19}\]

which makes the integration straightforward. Indeed, if the body’s acceleration is neglected, a plane-wave solution of equation (18) is

\[
\varphi(x^\nu) = \frac{2GMv}{c^4} (k_\alpha u^\alpha) \ln \left( -k_\alpha \rho^\alpha \right),\tag{20}\]

where all quantities in the right-hand side are taken at the retarded instant of time \( s \) in compliance with the null-cone equation (14). One notes that the time \( t \) of the closest approach of the light ray to the moving body does not play any role in calculation of the gravitational perturbation of the electromagnetic phase. The time \( t \) is a good approximation of the retarded time \( s \) (Kopeikin 2001), and can be used in practical calculations of light propagation in the gravitational field of moving bodies (Klioner & Kopeikin 1992; Klioner 2003b). However, it does not properly reflect the Lorentz-invariant nature of the gravitational time delay and makes the post-Newtonian expansion look more entangled and complicated. Further discussion on this issue is given in Section 7.2.

One can easily check that equation (20) is a particular solution of equation (15). Indeed, observing that

\[
\partial_\alpha \rho^\alpha = \delta_\alpha^\nu - u^\nu \partial_\nu s,\tag{21}\]

one obtains from the null-cone equation (13)

\[
\partial_\alpha s = -\frac{\rho_0}{\rho_R}.\tag{22}\]

Differentiation of equation (20) using equations (21) and (22) shows that equation (15) is satisfied.

Equation (20) for the electromagnetic phase is clearly Lorentz-invariant and valid in an arbitrary coordinate system. It tells us that a massive body (the Sun, a planet) interacts with the electromagnetic wave by means of its gravitational field, which originates at the retarded position \( z(s) \) of the body and propagates on the hypersurface of null-cone (14). The gravitational field perturbs the phase front of the electromagnetic wave at the field point \( x^\alpha \) regardless to the direction of motion of the incoming photon or the magnitude of its impact parameter with respect to the body. This consideration indicates a remarkable experimental opportunity to observe the retardation effect of the gravitational field by measuring the shape of the ranging (Shapiro) time delay and comparing it with the Jet Propulsion Laboratory (JPL) ephemeris position of the body (Standish & Williams 2006) obtained independently from direct radio/optical observations of the body,
conducted in preceding epochs. This idea was executed in a VLBI experiment with Jupiter (Kopeikin 2001; Fomalont & Kopeikin 2003). Section 5 explains the null-cone relationship between the characteristics of the Maxwell and Einstein equations.

5 THE RANGING TIME DELAY

The Lorentz-invariant, general-relativistic time-delay equation, generalizing the static Shapiro delay (Shapiro 1964), can be obtained directly from equation (20). We consider a ranging time-delay experiment in which an electromagnetic wave (a photon) is emitted at the event with four-dimensional coordinates \( x^\alpha_i = (ct_i, x_i) \), passes near the moving gravitating body, and is received by an observer at the event with coordinates \( x^\alpha_j = (ct_j, x_j) \). In the most general case, the emitter and observer can move, which means that coordinates \( x_i \) and \( x_j \) must be understood as functions depending on time \( t_i \) and \( t_j \), respectively, that is \( x_i = x(t_i) \) and \( x_j = x(t_j) \), where \( x(t) \) is a spatial coordinate of the photon taken at time \( t \). The gravitating body is also moving during the time of propagation of the electromagnetic wave from the emitter to the observer. In the approximation of a uniform and rectilinear motion, which is sufficient for our purpose, the spatial coordinate of the body is given by a straight line

\[
z(t) = z_0 + vt,
\]

where \( z_0 \) is the position of the body taken at time \( t = 0 \). One notes that the spatial coordinate of the body entering the Liénard–Wiechert solution of the gravity field equations depends on the retarded time \( s \). It means that the time argument \( t \) in equation (23) must be replaced with the retarded time \( s \) without changing the form of this equation. In other words,

\[
z(s) = z_0 + vs,
\]

where the retarded time \( s \) is given by the solution of the gravity null-cone equation (14). In the case of rectilinear and uniform motion of the gravitating body,

\[
s = t - \frac{\mathbf{R} \cdot \mathbf{v} + \sqrt{\mathbf{R}^2 - (\mathbf{R} \cdot \mathbf{v})^2}}{c(1 - \beta^2)},
\]

and \( \mathbf{R} = x - z(t) \) with \( z(t) \) defined in equation (23).

The unperturbed spatial components \( (k')^\alpha = k^\alpha \) of the wave vector \( k^\alpha \) are expressed in terms of the coordinates of the emitting and observing points

\[
k = \frac{x^\alpha_j - x^\alpha_i}{|x^\alpha_j - x^\alpha_i|},
\]

This vector is a constant for a single passage of the electromagnetic wave from the emitter to the observer. However, in the case when the emitter and/or observer are in motion, the direction of vector \( k \) will change as time progresses. This remark is important for calculation of the Doppler shift of frequency, where one has to take the time derivative of the vector \( k \) (Kopeikin & Schäfer 1999; Kopeikin et al. 2007).

The perturbed wave vector, \( K^\alpha = dx^\alpha /d\lambda \), is obtained from the Eikonal equation (20) by making use of identification \( K^\alpha = \partial \psi /\partial x^\alpha \), which is a consequence of the Hamiltonian theory of light rays and can be used for further integration in order to determine the trajectory of propagation of the electromagnetic wave in the curved space–time. The explicit integration has been performed in the paper by Kopeikin & Fomalont (2006) and could be used for calculation of the ranging time delay. However, in the present paper we shall rely upon a different method.

We note that phase \( \psi \) of the electromagnetic wave, emitted at the point \( x^\alpha_i = (ct_i, x_i) \) and received at the point \( x^\alpha_j = (ct_j, x_j) \), remains constant along the wave’s path (Landau & Lifshitz 1971; Misner et al. 1973; Kopeikin & Mashhoon 2002). Indeed, since \( \lambda \) is an affine parameter along the path, one has for the phase’s derivative

\[
\frac{d\psi}{d\lambda} = \frac{\partial \psi}{\partial x^\alpha} \frac{dx^\alpha}{d\lambda} = K_\alpha K^\alpha = 0,
\]

which means that \( \psi[x^{\alpha}(\lambda)] = \) constant, in accordance with our assertion. Equating the two values of phase \( \psi \) at the point of emission of the electromagnetic wave, \( x^\alpha_i \), and at the point of its receptions, \( x^\alpha_j \), and separating time from space coordinates, one obtains from equations (17) and (20)

\[
t_2 - t_1 = \frac{1}{c} \mathbf{k} \cdot (x_2 - x_1) - \frac{2GM}{c^2} (k_\alpha u^\alpha) \ln \left( \frac{k_\beta \rho_2^\beta}{k_\beta \rho_1^\beta} \right),
\]

where the retarded distances \( \rho_2^\alpha = x_2^\alpha - z^\alpha(s_2), \rho_1^\alpha = x_1^\alpha - z^\alpha(s_1) \), and the retarded times \( s_2, s_1 \) are defined by the null-cone equations

\[
s_2 = t_2 - \frac{1}{c} |x_2 - z(s_2)|,
\]

\[
s_1 = t_1 - \frac{1}{c} |x_1 - z(s_1)|,
\]

which are inferred from equation (14). Expanding all Lorentz-invariant scalar products, and replacing relationship (26) in equation (28) yields the ranging delay

\[
t_2 - t_1 = \frac{1}{c} |x_2 - x_1| + \Delta t,
\]
\[ \Delta t = \frac{-2GM}{c^3} \frac{1 - k \cdot \beta}{\sqrt{1 - \beta^2}} \ln \left( \frac{\rho_2 - k \cdot \rho_2}{\rho_1 - k \cdot \rho_1} \right) \]

where the retarded, null-cone distances \( \rho_2 = x_2 - z \cdot (s_2), \rho_1 = x_1 - z \cdot (s_1), \rho_2 = |\rho_2|, \rho_1 = |\rho_1| \).

The Lorentz-invariant expression for ranging delay (32) was derived first by Kopeikin & Schäfer (1999) by solving equations for light geodesics in the gravitational field of moving bodies with the Liénard–Wiechert gravitational potentials. Later on, Klioner (2003a) obtained this expression by making use of the Lorentz transformation of the Shapiro time delay (which is equivalent to a simultaneous transformation of the solutions of both the Einstein and Maxwell equations) from a static frame of the body to a moving frame of the observer. Notice that in general relativity, equation (31) describes a hypersurface of the null-cone along which both electromagnetic and gravitational field are propagating. The electromagnetic characteristic of the null-cone is given by the null vector \( k \) of the photon, while the null characteristic of the gravity field enters the time-delay equation (32) in the form of the retarded time \( s \), which is the time argument of the coordinate \( z \) of the moving body under consideration.

In the present paper, we derive another useful form of the Lorentz-invariant expression for the ranging delay, which can be directly compared with and generalizes the approximate ranging delay formula currently used in the NASA ODPR. This derivation comes about from the following exact relationships:

\[ \rho_2 - k \cdot \rho_2 = \frac{|\rho_1 - z(s_2) + z(s_1)|^2 - (r - \rho_2)^2}{2r}, \]

\[ \rho_1 - k \cdot \rho_1 = \frac{|\rho_2 + z(s_2) - z(s_1)|^2 - (r + \rho_1)^2}{2r}, \]

where \( r = |r|, r = x_2 - x_1, \) so that

\[ r^o = r^k \equiv (r, r) \]

is a null vector in the flat space–time connecting coordinates of the point of emission and reception of the electromagnetic wave: \( \eta_{\alpha\beta} r^\alpha r^\beta = 0 \). Because the gravitating body moves uniformly with constant speed \( v \), its coordinate \( z(s) \) is not constant and can be expanded as follows (see equation 24):

\[ z(s_2) = z(s_1) + v (s_2 - s_1), \]

where the time interval \( s_2 - s_1 \) can be expressed in terms of the null-cone distances by making use of the retarded time equations (29) and (30), and the ranging equation (31). One has

\[ s_2 - s_1 \equiv (s_2 - t_2) + (t_2 - t_1) + (t_1 - s_1) = \frac{1}{c} (r + \rho_1 - \rho_2) + O(c^{-3}). \]

Plugging equation (37) into (36), and replacing it in equations (33) and (34) allows us to transform the ranging time-delay logarithm to the following form:

\[ \ln \left( \frac{\rho_2 - k \cdot \rho_2}{\rho_1 - k \cdot \rho_1} \right) = - \ln \left[ \frac{\rho_2 + \rho_1 + r - 2(\rho_2 \cdot \beta) - \beta^2 (r + \rho_1 - \rho_2)}{\rho_2 + \rho_1 - r - 2(\rho_1 \cdot \beta) + \beta^2 (r + \rho_1 - \rho_2)} \right]. \]

Let us now make use of definition (11) of the Lorentz-invariant distances

\[ \rho_{2R} = -u^a \rho_2^a = \frac{\rho_2 - \beta \cdot \rho_2}{\sqrt{1 - \beta^2}}, \]

\[ \rho_{1K} = -u^a \rho_1^a = \frac{\rho_1 - \beta \cdot \rho_1}{\sqrt{1 - \beta^2}}. \]

Tidious but straightforward calculations reveal that

\[ \rho_2 + \rho_1 + r - 2(\rho_2 \cdot \beta) - \beta^2 (r + \rho_1 - \rho_2) = \sqrt{1 - \beta^2} \left( \rho_{2R} + \rho_{1R} - rk_{u} u^a \right), \]

\[ \rho_2 + \rho_1 - r - 2(\rho_1 \cdot \beta) + \beta^2 (r + \rho_1 - \rho_2) = \sqrt{1 - \beta^2} \left( \rho_{2R} + \rho_{1R} + rk_{u} u^a \right). \]

These equations taken along with equation (35) allows us to reduce the time-delay logarithm in equation (38) to another Lorentz-invariant form

\[ \ln \left( \frac{\rho_2 - k \cdot \rho_2}{\rho_1 - k \cdot \rho_1} \right) = - \ln \left( \frac{\rho_{2R} + \rho_{1K} - \rho_{22}}{\rho_{2R} + \rho_{1K} + \rho_{22}} \right), \]

where the ranging distance \( \rho_{12} = rk_{u} u^a = u^a r^u \) is invariant with respect to the Lorentz transformation. It represents contraction of the null vector \( r^u \) defined in equation (35) with four-velocity \( u^a \) of the gravitating body. The null vector \( r^u \) determines (unperturbed) propagation of the electromagnetic signal. Distances \( \rho_{1R}, \rho_{2R} \) are defined in equations (39) and (40), and they also represent contraction of the null vectors \( \rho_{1K}, \rho_{2K} \) with four-velocity \( u^a \) of the gravitating body. However, contrary to vector \( r^u \), vectors \( \rho_{1K}, \rho_{2K} \) describe the null characteristics of the gravitational field.

Accounting for equation (43), the Lorentz-invariant expression for the time delay assumes the following form:

\[ \Delta t = \frac{2GM}{c^3} \frac{1 - k \cdot \beta}{\sqrt{1 - \beta^2}} \ln \left( \frac{\rho_{2R} + \rho_{1R} - \rho_{22}}{\rho_{2R} + \rho_{1R} + \rho_{22}} \right). \]
This equation is apparently Lorentz-invariant, valid for any value of the velocity of the light-ray deflecting body, and essentially generalizes the result of the paper by Bertotti et al. (2008).

6 POST-NEWTONIAN EXPANSION OF THE RANGING DELAY

Let us introduce the auxiliary vectors (Bel et al. 1981)

\[ n^\alpha_{2R} = \frac{\partial^\alpha \rho_{2R}}{\rho_{2R}} - u^\alpha, \quad n^\alpha_{1R} = \frac{\partial^\alpha \rho_{1R}}{\rho_{1R}} - u^\alpha. \]  

(Vectors \( \rho^\alpha_{2R} \) and \( \rho^\alpha_{1R} \) are null as defined by the (gravity field) null-cone equations (29) and (30). The four-velocity of the body, \( u^\alpha \), is a time-like vector, \( \rho_{1R} u^\alpha = -1 \). The difference between the null and time-like vector yields the space-like vectors \( n^\alpha_{2R}, n^\alpha_{1R} \), because \( n_{1R} n^\alpha_{1R} = n_{2R} n^\alpha_{2R} = +1 \).

The post-Newtonian expansion of \( z^\alpha(s_2) \) around time \( t_2 \), and the post-Newtonian expansion of \( z^\alpha(s_1) \) around time \( t_1 \) are obtained by making use of a Taylor expansion. Omitting acceleration, one gets

\[ \rho_{2R} = r^\alpha_{2} - (s_2 - t_2) \frac{\partial^\alpha \rho_{2R}}{\partial s_2} = r^\alpha_{2} + \rho_2 u^\alpha, \]  

\[ \rho_{1R} = r^\alpha_{1} - (s_1 - t_1) \frac{\partial^\alpha \rho_{1R}}{\partial s_1} = r^\alpha_{1} + \rho_1 u^\alpha \]  

and

\[ \rho_2 = \rho_{2R} + u^\beta r^\beta_{2}, \]  

\[ \rho_1 = \rho_{1R} + u^\beta r^\beta_{1}, \]

where the retarded time equations (29) and (30) have been used to replace time intervals \( s_2 - t_2 \) and \( s_1 - t_1 \). We have also introduced in previous equations the pure spatial vectors

\[ r^\alpha_{2} = x^\alpha_{2} - z^\alpha(t_2) = \{ r^\alpha_{0} = 0 , r^\alpha_{2} = x^\alpha_{2} - z^\alpha(t_2) \}, \]  

\[ r^\alpha_{1} = x^\alpha_{1} - z^\alpha(t_1) = \{ r^\alpha_{0} = 0 , r^\alpha_{1} = x^\alpha_{1} - z^\alpha(t_1) \}, \]

which are lying on the hypersurface of constant time \( t_2 \) and \( t_1 \), respectively.

Substituting equations (46)–(49) into equation (45) reveals that

\[ n^\alpha_{2R} \rho_{2R} = r^\alpha_{2} + u^\alpha \left( u^\beta r^\beta_{2} \right), \]  

\[ n^\alpha_{1R} \rho_{1R} = r^\alpha_{1} + u^\alpha \left( u^\beta r^\beta_{1} \right). \]

Taking into account that \( n^\alpha_{2R} \) and \( n^\alpha_{1R} \) are space-like unit vectors, one has

\[ \rho_{2R} = \sqrt{r_{\alpha\beta} r^\alpha_{2} + (u^\alpha r^\alpha_{2})^2} = \sqrt{r^2_{2} - (\vec{\beta} \times r_{2})^2 \over 1 - \beta^2}, \]  

\[ \rho_{1R} = \sqrt{r_{\alpha\beta} r^\alpha_{1} + (u^\alpha r^\alpha_{1})^2} = \sqrt{r^2_{1} - (\vec{\beta} \times r_{1})^2 \over 1 - \beta^2}. \]

We further note that, if acceleration is neglected,

\[ \rho_{12} = \frac{k \cdot \sigma}{\sqrt{1 - \beta^2}} r_{12}, \]  

where the unit vector

\[ \sigma = \frac{k - \beta}{|k - \beta|}, \]  

the relative distance

\[ r_{12} = |r_{2} - r_{1}|, \]  

and

\[ r_{2} = x_{2} - z(t_{2}), \]  

\[ r_{1} = x_{1} - z(t_{1}). \]

are spatial distances from the observer to the body and from the emitter to the body taken respectively at the time of reception and that of emission of the electromagnetic wave. It is worth observing that the post-Newtonian expansion of the Euclidean dot product \( k \cdot \sigma \) does not have a term which is linear with respect to the velocity

\[ k \cdot \sigma = 1 - \frac{1}{2} (k \times \beta)^2 + O(\beta^3). \]
Ranging time-delay experiment. Electromagnetic signal is emitted at a distance \( r_1 \) from the massive body, passes by it at the minimal distance \( d \), and is received by observer at distance \( r_2 \). The emitter, observer and the massive body move with respect to each other as the electromagnetic signal propagates. This makes the ranging delay experiment sensitive to the null-cone structure of space–time in general relativity and modifies the Shapiro time delay.

This expansion yields

\[
\rho_{12} = r_{12} + O(\beta^2),
\]

that is the distance \( r_{12} \) is a Lorentz-invariant function up to the second post-Newtonian corrections of the order of \( \beta^2 \). This justifies the replacement of the heliocentric coordinates of the massive bodies of the Solar system to their barycentric counterparts introduced by Moyer in the \( \text{OSF} \) manual (Moyer 2003) ad hoc (see Section 9 of the present paper for further details).

After preceding preparations, we are ready to write down the post-Newtonian expansion for the ranging time delay. We would like to emphasize that the post-Newtonian expansion of the ranging delay is not unique and can be represented in several different forms, which are physically and computationally equivalent. However, this non-uniqueness complicates things and has been debated in papers (Bertotti et al. 2008; Kopeikin 2009) regarding the nature of the relativistic time-delay effects associated with motion of the gravitating body. In what follows, we derive all possible forms of the post-Newtonian expansion of the ranging delay demonstrating that the relativistic effects associated with the motion of the light-ray deflecting body are induced by the gravitomagnetic field arising due to the translational motion of the body with respect to the observer (Kopeikin & Mashhoon 2002; Kopeikin 2004).

First of all, substituting equations (39) and (40) to (43) casts the ranging delay (32) in the following form:

\[
\Delta t = \frac{2GM}{c^3} \left( 1 - k \cdot \beta + \frac{1}{2} \beta^2 \right) \ln \left( \frac{\sqrt{\frac{r_1 \cdot r_2 - (\beta \times r_1)^2}{r_1^2}} + \sqrt{\frac{r_1 \cdot r_2 - (\beta \times r_1)^2}{r_1^2}} + (k \cdot \sigma)_{r_{12}}}{\sqrt{\frac{r_1 \cdot r_2 - (\beta \times r_1)^2}{r_1^2}} + \sqrt{\frac{r_1 \cdot r_2 - (\beta \times r_1)^2}{r_1^2}} - (k \cdot \sigma)_{r_{12}}} \right),
\]

which is most convenient for making its explicit post-Newtonian expansion with respect to the ratio of \( \beta = v/c \). Neglecting terms of the order of \( \beta^3 \) one has

\[
\Delta t = \left( 1 - k \cdot \beta + \frac{1}{2} \beta^2 \right) \frac{2GM}{c^3} \ln \left( \frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right) + \frac{GM}{c^3} \frac{r_{12} (n_1 \times \beta)^2 r_1 + (n_2 \times \beta)^2 r_2 - (k \times \beta)^2 (r_1 + r_2)}{1 + n_1 \cdot n_2} + O \left( \frac{GM}{c^3} \beta^3 \right),
\]

where the unit vectors \( n_1 = r_1/r_1, n_2 = r_2/r_2 \) with \( r_1, r_2 \) being defined in equations (59) and (60) (see Fig. 1).

Velocity-dependent corrections appear in this expression explicitly as the terms depending on \( \beta = v/c \), and implicitly in the argument of the logarithm, which depends on the two positions of the body taken at times \( t_1 \) and \( t_2 \), that is \( z(t_2) = z(t_1) + v(t_2 - t_1) = z(t_1) + \beta r \), so that \( r_2 \) and \( r_{12} \) are not independent of \( r_1 \). We discuss the impact of the velocity-dependent terms on the measured values of the PPN parameters in the next section.

It is also instructive to derive the time-delay equation in the linearized form as it is given by Will (2001). We make use of equations (45)–(55) to get the post-Newtonian expansion of functions entering the argument of the logarithm in the ranging delay (28)

\[
k_\alpha \rho_\alpha^\beta = k_\alpha r_\beta^\alpha + (k_\alpha u^\beta) \left[ u_\beta r_\alpha^\beta + \sqrt{r_\beta r_\alpha^\beta + (u_\beta r_\alpha^\beta)^2} \right],
\]

\[
k_\alpha \rho_1^\alpha = k_\alpha r_1^\alpha + (k_\alpha u^\alpha) \left[ u_\beta r_1^\beta + \sqrt{r_\beta r_1^\beta + (u_\beta r_1^\beta)^2} \right].
\]

Explicit expansion of these equations with respect to the powers of the velocity-tracking parameter \( \beta = v/c \) brings about the following result:

\[
\rho_2 - k \cdot \rho_2 = r_2 - k \cdot r_2 + \beta \cdot r_2 - r_2 (k \cdot \beta) + O \left( \beta^2 \right),
\]

\[
\rho_1 - k \cdot \rho_1 = r_1 - k \cdot r_1 + \beta \cdot r_1 - r_1 (k \cdot \beta) + O \left( \beta^2 \right).
\]

Applying these expansions to the argument of the logarithm in the ranging delay (32) yields the first term in the post-Newtonian expansion of the ranging delay in the form given in Will (2001):

\[
\Delta t = \left( 1 - k \cdot \beta \right) \frac{2GM}{c^3} \ln \left( \frac{r_2 - \sigma \cdot r_2}{r_1 - \sigma \cdot r_1} \right) + O \left( \frac{2GM}{c^3} \beta^2 \right),
\]
where the unit vector
\[ \sigma = k - k \times (\beta \times k) + O(\beta^3) \]  
(70)
is the same as that defined by equation (57).

The explicit post-Newtonian dependence of the time delay on velocity of the gravitating body \( v \) enters the argument of the logarithm in the form of equation (70), which looks like the aberration of light for the unit vector \( k \). However, equation (69) approximates the exact time-delay equation (31), which demonstrates that the argument of the logarithmic function is a four-dimensional dot product \( k_\alpha \rho^\alpha \) of two null vectors \( k^\alpha \) and \( \rho^\alpha \). Vector \( k^\alpha \) points out the direction of propagation of light ray, while the null vector \( \rho^\alpha = x^\alpha - z^\alpha(s) \) points out the direction of the null characteristic of the gravity field equations. The Lorentz transformation, \( \Lambda^{\alpha'}_{\alpha} \), from one frame to another changes the null vector \( k^\alpha = \Lambda^{\alpha'}_{\alpha} k^{\alpha'} \), but in order to preserve the Lorentz-invariance of the gravitational time delay \( \Delta t \), the null vector \( \rho^\alpha \) directed along the body’s gravity field must change accordingly \( \rho^\alpha = \Lambda^{\alpha'}_{\alpha} \rho^{\alpha'} \), so that the dot product \( k_\alpha \rho^\alpha = k_\alpha \rho^{\alpha'} \rho^{\alpha'} \) remains the same. Hence, not only the light undergoes aberration, when one goes from one frame to another, but the null characteristics of the gravitational field in time delay \( \Delta t \) must change too in the same proportion, if general relativity is valid. In other words, equation (70) is not the ordinary equation of the aberration of light in flat space–time (without gravity field) but a more profound relationship for a curved space–time showing that even in the presence of the gravitational field of the moving body, affecting the light propagation, the aberration of light equation remains the same as in the flat space–time. This can be true if and only if both the gravitational field perturbation \( h_{\mu \nu} \) and the affine connection \( \Gamma^\alpha_{\beta \gamma} \) remain invariant under the Lorentz group transformation, which is parameterized with the same fundamental speed \( c \) as the Lorentz group of the underlying electromagnetic wave used in the ranging time-delay experiment. This interpretation is further discussed in more detail elsewhere (Kopeikin & Fomalont 2006; Kopeikin & Fomalont 2007; Kopeikin & Makarov 2007).

7 COUPLING OF THE PPN PARAMETERS WITH THE VELOCITY-DEPENDENT TERMS

7.1 Explicit coupling

Equation (64) describes the Lorentz transformation of the (static) Shapiro time delay from the rest frame of the massive body (Sun, planet) to the frame of reference in which the data processing is performed. For we have restricted ourselves with the post-Newtonian expansion of the linearized time delay up to the terms which are quadratic with respect to velocity of the moving gravitating body, equation (64) can be superimposed with the static terms of the second order with respect to the universal gravitational constant \( G \) in equation (4). This is because these terms have the same order of magnitude so that we do not need to develop the Lorentz-invariant expression for the terms which are quadratic with respect to \( G \). We shall also neglect for simplicity the terms which are products of \( \beta^2 \) with the PPN parameter \( \gamma_{\text{PPN}} \) because \( \gamma_{\text{PPN}} \) has been already limited by the Solar system experiments up to a value not exceeding \( 10^{-4} \). Thus, the product \( \gamma_{\text{PPN}} \beta^2 \) exceeds the accuracy of the post-post-Newtonian approximation.

Our calculation yields the following, Lorentz-invariant equation for the post-post-Newtonian time delay:

\[
\Delta t = \left( 1 + \frac{\gamma_{\text{PPN}}}{2} - \frac{k \cdot \beta - \gamma_{\text{PPN}}}{2} \right) \frac{2GM}{c^3} \ln \left( \frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right) \\
+ \left( 1 + \frac{\gamma_{\text{PPN}}}{2} \right) \frac{GM}{c^3} \left( \frac{r_{12}}{r_1 r_2} \left( 1 + \frac{3}{4} \gamma_{\text{PPN}} - \beta_{\text{PPN}} + \frac{3}{4} \beta_{\text{PPN}} \right) \arccos \left( \frac{n_1 \cdot n_2}{|n_1| |n_2|} \right) - \frac{2 + \gamma_{\text{PPN}}}{1 + n_1 \cdot n_2} \right) + O \left( \frac{GM}{c^3} \beta^2 \right). 
\]  
(71)

One can immediately observe that the PPN parameter \( \gamma_{\text{PPN}} \) couples with the velocity terms in front of the logarithmic term. This means that the amplitude of the Shapiro delay is effectively sensitive to the linear combination

\[
\Gamma = \gamma_{\text{PPN}} - 2\beta_R - 2\gamma_{\text{PPN}} \beta_b + \beta^2 - \beta^2_T
\]  
(72)

that will be measured in high-precision space-based experiments like BepiColombo, ASTROD, LATOR, etc. Here and elsewhere, we denote respectively \( \beta_R \equiv k \cdot \beta \) – the radial velocity, and \( \beta_T \equiv |k \times \beta| \) – the transverse velocity of the massive body that deflects the light ray.

Equation (72) elucidates that the measured value \( \Gamma \) of the parameter \( \gamma_{\text{PPN}} \) is affected by the velocity terms, which explicitly present in the post-Newtonian expansion of the Shapiro time delay. In case of a ranging gravitational experiment in the field of Sun with the light ray grazing the solar limb, one has \( d = R_\odot = 7 \times 10^{10} \text{ cm} \) – the solar radius, and \( r_s = 3 \times 10^5 \text{ cm} \) – the Schwarzschild radius of the Sun. The Sun, in moving in its orbit around the barycentre, has an average distance of 1.1 \( R_\odot \) from it but may be as far as 2.3 \( R_\odot \). The orbital path of the Sun about the barycentre traces out a curve that closely resembles an epicycloid – a three-lobe rosette, with three large and three small loops – with a loop period of 9–14 yr. Fifteen successive orbits comprise a 179-yr cycle of the solar motion around the barycentre (Jose 1965; Fairbridge & Shirley 1987) – the duration, which is also the time taken for the planets to occupy approximately the same positions again relative to each other and the Sun. The solar velocity \( v_\odot \) with respect to the barycentre of the Solar system can reach a maximal value of 15.8 \( \text{m s}^{-1} \) giving rise to \( \beta_\odot = v_\odot / c = 5.3 \times 10^{-8} \). Because space missions LATOR and ASTROD are going to measure the \( \gamma_{\text{PPN}} \) parameter with a precision approaching \( 10^{-9} \) (Turyshhev et al. 2004; Ni 2007), the explicit velocity-dependent correction to the Shapiro time delay in the solar gravitational field must be apparently taken into account. Current indeterminacy in the solar velocity vector is about 0.366 \( \text{m d}^{-1} \) (Pitjeva, private communication) which yields an error of \( \Delta \beta_\odot \approx 1.4 \times 10^{-14} \). This error is comparable with the contribution of the second-order velocity terms \( \beta^2_\odot \leq 2.8 \times 10^{-15} \). However, they are too small and can be neglected in the measurement of \( \gamma_{\text{PPN}} \).
Coupling of the velocity-dependent terms with parameters $\hat{\beta}_{\text{PPN}}$ and $\delta_{\text{PPN}}$ can be understood after making expansion of high-order terms in equation (71) with respect to the impact parameter of the light ray $d = |k \times r_1| = |k \times r_2|$ which is assumed to be small: $d \ll r_1, d \ll r_2$. The unit vectors $n_1$ and $n_2$ can be decomposed in the post-post-Newtonian terms as follows:

$$n_1 = -k \cos \theta_1 + n \sin \theta_1,$$

$$n_2 = k \cos \theta_2 + n \sin \theta_2,$$

where the unit vector $n$ is directed from the massive body to the light-ray trajectory along the impact parameter $r$.

One can easily observe that $\theta = \theta_1 + \theta_2$. Practically all gravitational ranging experiments are done in the small-angle approximation, when $\theta \ll 1, \theta_1 \ll 1, \theta_2 \ll 1$. In this approximation, one has

$$1 + n_1 \cdot n_2 = \frac{\theta^2}{2} + O(\theta^4),$$

$$n_1 \times β_1^2 r_1 + (n_2 \times β_2^2 r_2 - (k \times β)^2) (r_1 + r_2) = \theta d (\beta^2_k - \beta^2_r) + O(\theta^3).$$

Substituting equations (72), (75)–(77) to equation (71) yields

$$\Delta t = (2 + \Gamma) \frac{GM}{c^3} \ln \left( \frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right) + \frac{\gamma^2 \alpha^2}{c^3} \frac{\delta_{\text{PPN}}^2}{2 \pi \beta} \left( \frac{15}{4} + 2 \hat{\beta}_{\text{PPN}} - \hat{\beta}_{\text{PPN}} + \frac{3}{4} \hat{\beta}_{\text{PPN}} \right) \frac{\pi}{\theta} - \frac{2(2 + \hat{\beta}_{\text{PPN}})^2}{\theta^2} + O \left( \frac{GM}{c^3} \beta^3 \right),$$

where we have introduced a new notation $\delta_{\text{PPN}} = \hat{\delta}_{\text{PPN}} + \left( 1 + \frac{\delta_{\text{PPN}}}{2} \right) \frac{16}{3\pi} \frac{d}{\beta^2_k} \left( \beta^2_k - \beta^2_r \right)$.

and denoted $r_1 \equiv 2GM/c^2$ – the Schwarzschild radius of the massive body deflecting the light ray. The explicit contribution of the solar velocity terms to the parameter $\hat{\beta}_{\text{PPN}}$ can achieve $1.1 \times 10^{-9}$ which is much less than the precision of measurement of the PPN parameter $\hat{\delta}_{\text{PPN}}$ in the LATOR and ASTROD missions (Plowman & Hellings 2006) and can be currently neglected.

We recall to the reader that in the scalar–tensor theory of gravity parameter $\hat{\beta}_{\text{PPN}}$ cannot be determined separately from $\delta_{\text{PPN}}$ as they appear in the linear combination $-\hat{\beta}_{\text{PPN}} + 3/4\delta_{\text{PPN}}$. Following Plowman & Hellings (2006), we assume that $\hat{\beta}_{\text{PPN}}$ is determined from other kinds of gravitational experiments, and eliminate it from the fitting procedure.

### 7.2 Implicit coupling

In the previous section, we have made an explicit post-Newtonian expansion of the ranging time delay in powers of the velocity-tracking parameter $\beta = v/c$. This post-Newtonian expansion is shown in equation (71). It looks like the only place where the linear velocity correction to the Shapiro delay appears is in front of the logarithmic term. However, a scrutiny analysis reveals that the linear velocity-dependent correction is also present implicitly in the argument of the logarithmic function. Indeed, distances $r_1 = |x_1 - z(t_1)|$ and $r_2 = |x_1 - z(t_2)|$ depend on two positions of the massive body taken at two different instants of time, $t_1$ and $t_2$. The body moves as light propagates from the point of emission $x_1$ to the point of observation $x_2$, so that the coordinates of the body are not arbitrary but connected through a relationship

$$z(t_2) = z(t_1) + v(t_2 - t_1),$$

which, indeed, shows that the velocity of the body is involved in calculation of the numerical value of the argument of the time-delay logarithm.

Though this dependence on the velocity of the massive body is implicit, it definitely affects the measured values of the PPN parameters and makes their values biased either if general relativity is invalid or if the numerical code used for data processing of the ranging experiment does not incorporate the Solar system ephemeris properly (Kopeikin et al. 2007). Let us show how this impact on the PPN parameters can happen.

To this end, we shall assume that the light ray passes at a minimal distance $d$ from the body at the time of the closest approach $t_0$, which is defined in the approximation of the unperturbed light-ray trajectory, $x(t) = x_1 + k(t - t_1)$ for $t \geq t_1$, or $x(t) = x_2 + k(t - t_2)$ for $t \leq t_2$, from the condition (Klioner & Kopeikin 1992)

$$\left[ \frac{d}{dt} (x(t) - z(t)) \right]_{t=t_0} = 0,$$

where $x(t) = x_1 + k(t - t_1)$ is the (unperturbed) light-ray trajectory, and $z(t) = z(t_1) + v(t - t_1)$ is the body’s world line in the approximation of a straight line, uniform motion. Taking the time derivative and solving the equation yield

$$t_0 = t_1 - \frac{\sigma \cdot r_1}{c|k - \beta|} = t_2 - \frac{\sigma \cdot r_2}{c|k - \beta|},$$

where $\sigma \cdot r_1$ and $\sigma \cdot r_2$ are the components of the body’s velocity vector $\sigma$ along the body’s position vectors $r_1$ and $r_2$, respectively.
where the unit vector $\sigma$ has been defined in equation (57). The post-Newtonian expansion of various distances near the time of the closest approach gives us

$$r_1 = r_{1s} \left[ 1 - (\beta \cdot n_{1s}) \frac{l_1}{r_{1s}} + \frac{(\beta \times n_{1s})^2}{2} \left( \frac{l_1}{r_{1s}} \right)^2 \right],$$

(83)

$$r_2 = r_{2s} \left[ 1 - (\beta \cdot n_{2s}) \frac{l_2}{r_{2s}} + \frac{(\beta \times n_{2s})^2}{2} \left( \frac{l_2}{r_{2s}} \right)^2 \right],$$

(84)

$$r_{12} = r \left[ 1 - \beta \cdot k + \frac{(\beta \times k)^2}{2} \right],$$

(85)

where $l_1 = c(t_1 - t_0)$, $l_2 = c(t_2 - t_0)$, the unit vectors $n_{1s} = r_{1s}/r_{1s}$, $n_{2s} = r_{2s}/r_{2s}$, and distances $r_{1s} = x_1 - z(t_1)$, $r_{2s} = x_2 - z(t_2)$.

We substitute now the post-Newtonian expansions (83)–(85) to the logarithmic function of the Shapiro time delay and apply the small-angle approximation. It will yield

$$\ln \left( \frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right) = \ln \left( \frac{r_{1s} + r_{2s} + r}{r_{1s} + r_{2s} - r} \right) - \frac{2rd_1 \cdot \beta}{r_{1s}r_{2s}} \left[ 1 + O(\beta) + O(\theta_\star) \right],$$

(86)

where $\theta_\star$ is the angle between two vectors $n_{1s}$ and $n_{2s}$, defined as $n_{1s} \cdot n_{2s} = \cos(\pi - \theta_\star)$.

The post-Newtonian expansion of the ranging delay in the vicinity of the time of the closest approach of the light ray to the massive body reveals that the parameter $\delta_{\text{PPN}}$ is affected by the first-order velocity terms from equation (86). Specifically, taking into account equation (86) allows us to write down the ranging delay in the following form:

$$\Delta t = (2 + \Gamma) \frac{GM}{c^3} \ln \left( \frac{r_{1s} + r_{2s} + r}{r_{1s} + r_{2s} - r} \right) + \frac{G^2M^2}{c^5} \frac{r}{r_{1s}r_{2s}} \left[ \left( \frac{15}{4} + 2\beta_{\text{PPN}} - \beta_{\text{PPN}} \right) \frac{\pi}{\theta_\star} - \frac{2(\delta_{\text{PPN}})^2}{\theta_\star^2} + O \left( \frac{GM}{c^3} \beta^3 \right) \right],$$

(87)

where

$$\delta_{\text{PPN}} = \left( 1 + \frac{\delta_{\text{PPN}}}{2} \right) \frac{16d}{3\pi r_f} \beta_R.$$

(88)

The last term in equation (88) can amount to 0.02, which exceeds the expected accuracy of measuring the PPN parameter $\delta_{\text{PPN}}$ with the LATOR/ASTROD missions by a factor of 10 as follows from Plowman & Hellings (2006). This clearly indicates the necessity of inclusion of the velocity-dependent post-Newtonian corrections to the data analysis of the high-precision time delay and ranging gravitational experiments.

**8 RANGING EXPERIMENTS AND LORENTZ INVARIANCE OF GRAVITY**

In the case of special relativity, where the Minkowski geometry represents a flat space–time, the Lorentz symmetry is a global symmetry consisting of rotations and boosts. However, in curved space–time, in the most general case, the Lorentz symmetry is a local symmetry that transforms local vectors and tensors in the tangent (co-tangent) space at each space–time point. None the less, general relativity admits the Lorentz symmetry of the gravitational field can be traced in the invariant nature of the gravitational Liénard–Wiechert potentials given by equation (10), which are solutions of the linearized Einstein equations. The asymptotic Minkowskian space–time for isolated systems defines the background manifold for gravitational field perturbations, $h_{ab}$, and must have the same null-cone structure as the local tangent space–time, which is defined by motion of light particles (photons). However, this theoretical argument is a matter of experimental study (Kopeikin 2001; Fomalont & Kopeikin 2003).

Ranging time-delay experiments are, perhaps, the best experimental technique for making such a test. This is because they operate with the gauge-invariant fundamental field of the Maxwell theory having well established and unambiguous physical properties. Propagation of radio (light) signals traces the local structure of the null-cone hypersurface all the way from the point of emission down to the point of its observation. Now, if the massive body, which deflects radio (light) signals, is static with respect to the observer, one cannot draw any conclusion on the asymptotic structure of the space–time manifold and on whether its Lorentz symmetry is compatible with the Lorentz symmetry of the light cone. This is because the gravitational interaction of the body with the radio (light) signal is realized in the form of the instantaneous Coulomb-like gravitational force with no time derivatives of the gravitational potentials having been involved. However, if the massive body is moving with respect to the observer as light propagates, its gravitational force is not instantaneous and must propagate on the hypersurface of the null-cone of the asymptotic Minkowskian space–time as it is described by the Liénard–Wiechert gravitational potentials (10). The terms in the ranging time-delay (32), depending on both the translational velocity $\beta = v/c$ of the massive body and the retarded time $s$, originate from the time derivatives of the gravitational potentials and characterize the global Lorentz symmetry of the gravitational field. Therefore, measurement of these terms in the ranging time-delay experiments has a fundamental significance (Kopeikin & Fomalont 2006).

Currently, there is a growing interest of theoretical physicists in gravitational theories where the global Lorentz symmetry of gravitational field can be spontaneously violated (Bluhm 2008). This is motivated by the need of unification of the gravity field with other fundamental
interactions. These theories introduce additional long-range fields to the gravitational Lagrangian, which destroy the symmetry between the so-called observer and particle invariance (Kostelecký & Potting 1995; Colladay & Kostelecký 1997, 1998). Interaction terms involving these fields also appear in the equations of motion of test particles. It is the interaction with these fields that can lead to physical effects of the broken Lorentz symmetry that can be tested in experiments. The outcome of these experiments depends crucially on the assumptions made about the structure of the additional terms in the gravitational Lagrangian and the numerical value of the coupling constants of these fields with matter. On the other hand, the measurement of the post-Newtonian velocity-dependent and/or retarded-time corrections in the ranging time-delay experiments does not depend on any additional assumptions and relies solely on general-relativistic prediction of how the radio (light) signals propagate in time-dependent gravitational fields.

It is remarkable that current technology already allows us to measure the velocity-dependent and/or retarded-time post-Newtonian corrections in the ranging time-delay experiments conducted in the Solar system. The most notable experiment had been done in 2002 with the VLBI technique (Fomalont & Kopeikin 2003). It measured the retarded component of the near-zone gravitational field of Jupiter via its impact on the magnitude of the deflection angle of light from a quasar (Kopeikin 2001, 2004). Fomalont et al. (2009) have repeated this retardation of gravity experiment in 2009 by making use of the close encounter of Jupiter and Saturn with quasars in the plane of the sky.

The Cassini experiment (Bertotti et al. 2003; Anderson et al. 2004) is also sensitive to the time-dependent perturbation of gravitational field of the Sun caused by its orbital motion around the barycentre of the Solar system (Kopeikin et al. 2007; Bertotti et al. 2008; Kopeikin 2009). However, its detection requires re-processing of the Cassini data in order to separate the Cassini measurement of PPN parameter $\gamma_{\text{PPN}}$ from the gravitomagnetic deflection of light by the moving Sun (Kopeikin et al. 2007; Kopeikin 2009).

9 RANGING DELAY IN THE NASA ORBIT DETERMINATION PROGRAM

Relativistic ranging time delay, incorporated into the NASA ODP code, was originally calculated by Moyer (2003) under the assumption that the gravitating body that deflects light does not move. Regarding the Sun, it means that the ODP code derives the ranging delay in the heliocentric frame. Let us introduce the heliocentric coordinates $X^α = (cT, X')$, and use notation $x^α = (ct, x')$ for the barycentric coordinates of the Solar system, the origin of which is at the centre of mass of the Solar system. The Sun moves with respect to the barycentric frame with velocity $v_\odot = dx_\odot/dt$ amounting to $\sim 15 \text{ m s}^{-1}$. Though this velocity looks small, it cannot be neglected in such high-precision relativity experiments as, for example, Cassini (Kopeikin et al. 2007). A legitimate question arises whether the ODP code accounts for the solar motion or not. We demonstrate in this paper that the ranging time delay in the ODP code is consistent with general relativity in the linear-velocity approximation, but it fails to take into account the quadratic velocity terms properly. Thus, more advanced theoretical development of the ODP code is required.

The ranging time delay in the heliocentric coordinates with the Sun located at the origin of this frame follows directly from equation (44) after making use of the heliocentric coordinates. It reads

$$T_2 - T_1 = \frac{1}{c} |X_2 - X_1| + \Delta T,$$

$$\Delta T = \frac{2GM_\odot}{c^3} \ln \left( \frac{R_2 + R_1 + R_{12}}{R_2 + R_1 - R_{12}} \right),$$

where $X_2$ and $X_1$ are the heliocentric coordinates of the observer and emitter respectively, the distance of the emitter from the Sun is $R_2 = |X_2|$, the distance of the observer from the Sun is $R_1 = X_1$, and $R_{12} = |X_2 - X_1|$ is the null heliocentric distance between the emitter and observer. This equation coincides exactly (after reconciling our and Moyer’s notations for distances) with the ODP time-delay equation (8)–(38) given in Section 8 of the ODP manual (Moyer 2003) on pages 8–19. Moyer (2003) had transformed the argument of the logarithm in the heliocentric ranging delay (90) to the barycentric frame by making use of substitutions

$$X_2 \Rightarrow r_2 = x_2 - x_\odot(t_2), \quad X_1 \Rightarrow r_2 = x_1 - x_\odot(t_1).$$

The ODP manual (Moyer 2003) does not provide any evidence that these substitutions in the ranging time delay (90) are consistent with general relativity and do not violate the Lorentz symmetry. None the less, comparison of equations (90) and (91) with the post-Newtonian expression (64) for the ranging delay demonstrates that equations (90) and (91) are legitimate transformations from the heliocentric to the barycentric frame in the sense that they take into account the velocity of the Sun in the ranging time delay in the linearized, post-Newtonian term following the static Shapiro time delay.

Equation (64) also shows that the ODP code is missing the velocity-dependent term in front of the logarithmic function in equation (90).

The ranging time delay in the heliocentric and barycentric frames must be related by the simple equation

$$\Delta t = (1 - k \cdot \beta_\odot) \Delta T,$$

which is a linearized version of equation (64) that was derived by Kopeikin & Schäfer (1999). We conclude that the ODP code used by NASA for navigation of spacecraft in deep space is missing a high-order velocity-dependent corrections to the Shapiro time delay and cannot be used for processing and unambiguous interpretation of near-future ranging experiments in the Solar system. A corresponding relativistic modification and re-parametrization of the ODP code based on equations of the present paper is highly required.

Equation (92) has been also derived by Bertotti et al. (2008) who claimed that the velocity-dependent terms appear in the time delay only in front of the logarithmic function in equation (92). As we have shown in Section 7.2, the argument of the logarithm in equation (63)
also contains terms depending on the velocity $v$ of the gravitating body, which are implicitly present in the definition of the distance $r_{12}$. This distance is calculated between two spatial points separated by the time interval required by light to travel between the point of emission and observation, respectively (see equations 56–58). Coordinates $z(t_1)$ and $z(t_2)$ are not the same because the gravitating body is moving. These coordinates are related by means of the equation (72), which demonstrates that the velocity $v$ of the gravitating body must be known in order to calculate the distance $r_{12}$. Because one has to rely upon equation (72) in the odp data processing algorithm, the post-Newtonian expansion of distance $r_{12}$ yields

$$ r_{12} = r - r \cdot \beta + O(\beta^2), $$

(93)

where the null distance $r = |r|$ is defined in equation (35). It follows that the distances $r_{12}$ and $r$ entering equation (93) are not the same quantities as they differ by terms of the order of $v/c$. Equation (93) reduces the ranging delay (92) to the following form:

$$ \Delta t = (1 - k \cdot \beta) \frac{2GM}{c^3} \ln \left( \frac{r_2 + r_1 + r - r \cdot \beta}{r_2 + r_1 - r + r \cdot \beta} \right) + O \left( \frac{2GM\beta^2}{c^3} \right), $$

(94)

which has been derived in our paper (Kopeikin et al. 2007). Bertotti et al. (2008) claimed that the expression (94) for the ranging time delay does not appear in the odp manual (Moyer 2003) and is not allowed for theoretical analysis of the Cassini experiment as we did in Kopeikin et al. (2007). However, expression (94) is exactly the same function, $\Delta t$, given in the odp manual but expressed, instead of distance $r$ and velocity of the Sun, $v$, via self-consistent mathematical transformation (93). For this reason, the two expressions are mathematically equivalent and either of them can be used in the data processing of the ranging observations of the Cassini experiment (Kopeikin 2009).

**ACKNOWLEDGMENTS**

This work was supported by the Research Council Grant No. C1669103 of the University of Missouri-Columbia. I am grateful to Dr Slava Turyshev (JPL) for valuable conversations and critical comments, which helped to improve this paper.

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