Relativity in Fundamental Astronomy:
Dynamics, Reference Frames, and Data Analysis
Proceedings IAU Symposium No. 261, 2009
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DOI: 00.0000/X000000000000000X

Units of relativistic time scales and associated quantities

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Abstract. This note suggests nomenclature for dealing with the units of various astronomical quantities that are used with the relativistic time scales TT, TDB, TCB and TCG. It is suggested to avoid wordings like "TDB units" and "TT units" and avoid contrasting them to "SI units". The quantities intended for use with TCG, TCB, TT or TDB should be called "TCG-compatible", "TCB-compatible", "TT-compatible" or "TDB-compatible", respectively. The names of the units second and meter for numerical values of all these quantities should be used with out any adjectives. This suggestion comes from a special discussion forum created within IAU Commission 52 "Relativity in Fundamental Astronomy".

1. Introduction

In the current literature one can read different, sometimes contradictory and illogical statements about the units associated with the values of various astronomical parameters. One sees wording like "TDB units" and "TT units" that is often contrasted to "SI units". Such terminology is often a source of confusion: no serious discussion of how those "TDB units" and "TT units" are defined can be found in the literature. The present note puts forward the case for using clear and consistent wording concerning the units and values of various astronomical quantities to be used with all standard astronomical reference systems (BCRS and GCRS) and time scales (TT, TDB, TCB and TCG). It is the result of a special discussion forum created within IAU Commission 52 "Relativity in Fundamental Astronomy".

2. Quantities, values and units

For the purposes of the present note, it is important to distinguish clearly between quantities and their numerical values. According to ISO (1993, definition 1.1), *quantity is an attribute of a phenomenon, body or substance that may be distinguished qualitatively and determined quantitatively. A value (of a quantity)* is defined as *the magnitude of a particular quantity generally expressed as a unit of measurement multiplied by a number* (ISO 1993, definition 1.18). The numerical values of quantities are pure numbers that appear when quantities are expressed using some units. For any quantity A one has

$$A = \{A\}\,[A], \tag{2.1}$$

where $\{A\}$ is the numerical value (a pure number) of quantity A and $[A]$ is the corresponding unit. Notations $\{A\}$ and $[A]$ for the numerical value and unit of a quantity A, respectively, are recommended in ISO 31-0 (ISO 1992).

The official definition of the concept of "unit" is given by ISO (1993, definition 1.7): *a unit (of measurement) is a particular quantity, defined and adopted by convention, with which other quantities of the same kind are compared in order to express their magnitudes relative to that quantity.* Therefore, a unit is a sort of recipe of how an observer can realize a specific physical quantity. The observer can then express numerically all other quantities which have the same physical dimensionality by comparing them with that specific quantity called "unit".

3. SI second as the unit of proper time

The official definition of the SI second can be found in (BIPM 2006):

The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.

In the relativistic framework it is very important to realize that this definition does not contain any hints on how the observer realizing the second should move, or where (in what gravitational field) that observer should be situated. From the relativistic point of view this is the only correct approach to define a physical unit of time. A physical unit of time can be realized only by physical observations. One of the fundamental principles of General Relativity, the Einstein Equivalence Principle, in combination with the so-called locality hypothesis claims, in particular, that an observer using only its proper time (the reading of an ideal clock moving together with him) cannot judge how he is moving and how strong the gravitation field along his trajectory is. Therefore, the concrete second, called in the following the "SI second", can be realized by any observer: a clock on the surface of the Earth, or on Mars or on board a space vehicle far away from any planet. In this sense, the SI second is the same for any observer and represents a recipe (as for any unit of measurement: see Section 2 above) to be executed in order to realize unit time intervals locally. Thus, the SI second is the unit of proper time and, as for proper time itself, it can and should be realized only locally (but by any observer at an arbitrary location and in an arbitrary gravitational field). Hence, it is clear that wording "SI seconds on the geoid" used in the original definition of TAI does not mean that SI seconds are defined on the geoid or can be realized only on the geoid. Such a wording actually refers to the proper time of an observer on the geoid expressed in SI seconds.

The official definition of the SI meter reads BIPM (2006):

The metre is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.

The SI meter is, therefore, based on the SI second and on the specific defining value of the speed of light $c = 299\,792\,458$ m/s (assumed here to be constant according to Special Relativity).

In the framework of General Relativity, one should distinguish between observable (or measurable) and coordinate quantities. A measurable quantity has dimension, a unit, and gets a numerical value after comparison with its unit. Its value is independent of the choice of theory and reference systems.

A coordinate quantity has dimension, cannot be measured directly but can get a numerical value after computation from observables with proper theoretical (relativistic) relations. Its numerical value is usually followed by "second", "meter" or some combination according to its dimension and the system of units used for the observables. Its value depends on the choice of theory (General Relativity in present IAU Resolutions) and reference systems.

In practice, all quantities not resulting directly from measurements are coordinate quantities, such as time and space coordinates, orbital elements, distances between remote points, and so on.

4. Unit time intervals of different observers

It is a common mistake to believe that intervals of proper time $\Delta\tau_1$ and $\Delta\tau_2$ measured by different observers can be "uniquely" and "naturally" compared to each other. The only way to do so in General Relativity is to define a 4-dimensional relativistic reference system having coordinate time t, establish a relativistic procedure of coordinate synchronization of clocks with respect to t, and convert the intervals of proper time $\Delta\tau_1$ and $\Delta\tau_2$ of each observer into corresponding intervals of coordinate time Δt_1 and Δt_2. These two intervals of coordinate time can indeed be compared directly. If both observers use SI seconds to measure their proper times, and $\{\Delta\tau_1\} = 1$ and $\{\Delta\tau_2\} = 1$ (i.e. both proper time intervals have length of 1 SI second as realized by the corresponding observer), the values of coordinate time intervals $\{\Delta t_1\}$ and $\{\Delta t_2\}$ are in general different. It does not mean, however, that the observers use different units of time. Only if the same units are used for $\Delta\tau$ and Δt, numerical values $\{\Delta t_1\}$ and $\{\Delta t_2\}$ are related to each other according to the standard formulas of special- and general-relativistic time dilations.

5. Proper time, and coordinate times TCB and TCG

Along with the proper times of individual observers, coordinate times are indispensable for relativistic modelling of physical processes. Coordinate times together with 3 spatial coordinates constitute relativistic 4-dimensional reference systems. Full definition of a reference system can be achieved only by fixing its metric tensor, as is done by the IAU (IAU 2006; Rickman 2001) for the standard reference systems BCRS and GCRS. Physical and mathematical details of this definition can be found in Soffel *et al.* (2003).

A relativistic reference system can be associated with a set of rules allowing one to label any phenomena or physical events with 4 real numbers. One of these numbers is called coordinate time and the other three are called spatial coordinates. Any coordinate time is a coordinate and, therefore, cannot be measured directly. They can only be *computed,* from the readings of real clocks together with additional parameters and information. For this computation one should use the theoretical relation between the proper time of an observer and coordinate time in General Relativity:

$$\frac{d\tau}{dt} = \left(-g_{00}(t,\boldsymbol{x}_{\rm obs}(t)) - \frac{2}{c}\,g_{0i}(t,\boldsymbol{x}_{\rm obs}(t))\,\dot{x}^i_{\rm obs}(t) - \frac{1}{c^2}\,g_{ij}(t,\boldsymbol{x}_{\rm obs}(t))\,\dot{x}^i_{\rm obs}(t)\,\dot{x}^j_{\rm obs}(t)\right)^{1/2}, \tag{5.1}$$

where t is the coordinate time of a reference system having a metric tensor with components g_{00}, g_{0i} and g_{ij} (i and j running from 1 to 3, and each component of the metric tensor being a function of coordinate time t and spatial position $\boldsymbol{x}$), and τ is the proper time of an observer having position $\boldsymbol{x}_{\rm obs}(t)$ and velocity $\dot{\boldsymbol{x}}_{\rm obs}(t)$ with respect to this reference system. Einstein's implicit summation is used in the above formula. In order to be useful this formula needs an initial condition of the form

$$\tau(t_0) = \tau_0, \tag{5.2}$$

where t_0 and τ_0 are constants to be determined from the procedure of clock synchronization (if there is only one observer these two constants can be taken to be zero). In the form written above, Eq. (5.1) allows one to compute τ if t, $\boldsymbol{x}_{\rm obs}(t)$, $\dot{\boldsymbol{x}}_{\rm obs}(t)$ and the components of the metric tensor $g_{\alpha\beta}$ are given. This formula can be inverted (e.g. numerically) in order to compute t for a given τ.

We should note that Eq. (5.1) is a relation between quantities τ and t. By analogy with the rules of quantity calculus, the values of τ and t can be also related by this formula if and only if the same units are used for both τ and t. This implies that if proper time τ is expressed in SI seconds, then values of t computed from numerical inversion of Eq. (5.1) should be also expressed in SI seconds. A semantic difficulty here is that SI seconds can only be realized for physically measurable proper time, whereas t is a non-measurable coordinate quantity related to the measurable τ by Eqs. (5.1)–(5.2). To accommodate this objection, one can agree to call the unit of time t "SI-induced second". These "SI-induced seconds" can be realized only through the proper time of an observer. In the following we will call both "SI seconds" (used for proper times) and "SI-induced seconds" (used for coordinate times) simply "seconds".

All the comments and arguments given above are equally correct for both TCG and TCB.

Spatial coordinates $\boldsymbol{x}$ and $\boldsymbol{X}$ of BCRS and GCRS, respectively, are also defined by the metric tensors of these reference systems. The standard formulas of Special and General Relativity assume that the locally measured light speed in vacuum is equal to a constant quantity c that enters equations in many ways. In practical calculations the specific value of c from the definition of the SI meter is always used. Therefore, if t is expressed in seconds, x is expressed in meters.

Applying the same arguments of inheritance of the units to the equations of motion for celestial bodies (Newtonian equations of motion or post-Newtonian EIH equations) we conclude that mass parameters $\mu = GM$ of celestial bodies should also be expressed in the units of the SI.

We note that the views expressed above are closely related to the idea of symbolic or abstract quantities and units developed in metrology (see, de Boer (1994) and Emerson (2008)). In particular the "SI-induced second" discussed above appears as a symbolic second (see e.g., de Boer 1994). A detailed discussion of these concepts in the relativistic framework can be found in Guinot (1997).

6. Scaled time scales TDB and TT

The reasons for and the mathematical details of the relativistic scaling of BCRS and GCRS, and in particular their coordinate times, are summarized by Klioner (2008). TT and TDB are conventional linear functions of TCG and TCB, respectively. The definition of TT (given by IAU 2000 Resolution B1.9 and IAU 1991 Resolution A4) can be written as

$$TT = F_G\, TCG \tag{6.1}$$

where $F_G = 1 - L_G$, and $L_G = 6.969290134 \times 10^{-10}$ is a defining constant. TDB defined in IAU 2006 Resolution 3 is related to TCB as

$$TDB = F_B\, TCB + TDB_0 \tag{6.2}$$

where $F_B = 1 - L_B$, and $L_B = 1.550519768 \times 10^{-8}$ and TDB_0 are defining constants. Let us note here that TAI is a physical realization of TT with a shift of -32.184 s for historical reasons. Therefore, TAI is a realization of coordinate time "TT−32.184 s". The difference between TAI and an ideal realization of "TT−32.184 s" is only due to imperfections of the participating clocks and the clock synchronization procedures.

The slopes F_G and F_B of both linear functions have the same purpose: both TT and TDB should show no linear drift with respect to proper times of observers situated on the rotating geoid, i.e. on the surface of the Earth. Since this requirement depends on our model of the solar system (i.e. on a planetary ephemeris and on a number of astronomical and geodetic constants), the latter requirement cannot be satisfied exactly. Therefore, some conventional constants have been chosen in the definitions of TT and TDB so that the requirement is satisfied approximately, but with an accuracy totally sufficient for practical purposes. Similarly, the constant TDB_0 was chosen merely to keep $TDB-TT$ approximately centered on zero.

Eqs. (6.1) and (6.2) define two new quantities: coordinate time scales TT and TDB. As discussed in Klioner (2008), for practical reasons (keeping equations of motion of celestial bodies and photons invariant) these scalings of coordinate time are accompanied by the corresponding scalings of spatial coordinates $\boldsymbol{x}_{TDB} = F_B\,\boldsymbol{x}$ and $\boldsymbol{X}_{TT} = F_G\,\boldsymbol{X}$ and mass parameters μ of celestial bodies $\mu_{TDB} = F_B\,\mu$ and $\mu_{TT} = F_G\,\mu$.

The scaled coordinate times and spatial coordinates can be thought of as defining two new reference systems: those with coordinates $(TT, \boldsymbol{x}_{TT})$ and $(TDB, \boldsymbol{x}_{TDB})$. These new reference systems can be characterized by their own metric tensors, different from those of the BCRS and GCRS. Formulas $\mu_{TDB} = F_B\,\mu$ and $\mu_{TT} = F_G\,\mu$ for the mass parameters follow from the requirement to have the same form of the equations of motion with both unscaled and scaled coordinates. This is an additional requirement that does not immediately follow from the scaling of time and spatial coordinates.

In combination with (5.1), Eqs. (6.1) and (6.2) define how TT and TDB are related to the proper time of any observer. The proper time can be considered a function of TT and TDB in a similar way to when we considered it as functions of TCG and TCB above. Therefore, the same arguments as in Section 5 can be used to demonstrate that if proper times are expressed in SI seconds, both TT and TDB are by inheritance expressed in SI-induced seconds or simply in seconds. Here "by inheritance" simply means that the formulas linking TT and TDB to the other timescales, and ultimately to proper times, provide a formal connection back to SI seconds.

Similarly, scaled spatial coordinates are expressed in SI-induced meters or simply in meters. The same arguments allow us to conclude that the scaled mass parameters are also expressed in the units of the SI.

7. Suggested terminology

All these arguments allow us to suggest the following nomenclature:

– Avoid using the wording "TDB units" ("TDB seconds/meters"), "TT units" ("TT seconds/meters") and avoid contrasting these terms with "SI units" ("SI seconds/meters").

– All quantities intended for use with TDB should be called "TDB-compatible quantities" and corresponding values "TDB-compatible values".

– All quantities intended for use with TT should be called "TT-compatible quantities" and corresponding values "TT-compatible values".

– All quantities intended for use with TCB or TCG should be called "TCB-compatible quantities" or "TCG-compatible quantities" and the corresponding values "TCB-compatible values" or "TCG-compatible values", respectively. In the case of constants having the same value in BCRS and GCRS (e.g. mass parameters $\mu = GM$ of celestial bodies) the value can be called "unscaled". Note that it is misleading to describe these values as

"SI-compatible" or "in SI units" since this does not distinguish unscaled values from TT- and TDB-compatible values. Such wording should be avoided.

– Consider that the numerical values of all above-mentioned quantities (TT-compatible, TCG-compatible, TDB-compatible and TCB-compatible) are expressed in the usual units of the SI. Avoid attaching any adjectives to the names of the units second and meter for numerical values of these quantities † ‡ .

Acknowledgement

The authors gratefully acknowledge discussions with George Kaplan, Gérard Petit, and Patrick Wallace who also contributed to the text of this paper and to the proposed terminology.

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† So, for example, it would be improper to say "The interval is xx seconds of TDB"; the correct wording is "The TDB interval is xx seconds".

‡ An example of numerical values for a quantity is as follows: (i) the TCB/TCG-compatible value for GM_E (the mass parameter of the Earth) is: $3.986004418 \times 10^{14}$ m^3s^{-2}; (ii) the TT–compatible value for GM_E is: $3.986004415 \times 10^{14}$ m^3s^{-2}; (iii) the TDB-compatible value for GM_E is: $3.986004356 \times 10^{14}$ m^3s^{-2}.