# EXISTENCE AND CONSTRUCTION OF REAL-VALUED EQUIANGULAR TIGHT FRAMES

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

EXISTENCE AND CONSTRUCTION OF REAL-VALUED EQUIANGULAR TIGHT FRAMES

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a candidate for the degree of Doctor of Philosophy

and hereby certify that in their opinion it is worthy of acceptance.

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________________________________________ (David Retzloff)

# Dedication

For my grandparents, for my brother Brian, and for my daughter Marin Elizabeth: this work is dedicated to you.

# Acknowledgments

I would like to thank my family and friends, especially my parents, for their unconditional support. I was encouraged in the areas of math and computers by my father, and by a shared love of science fiction with my mother. I was inspired to learn about technology by my grandfather Robert Berger.

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# Nomenclature

$\aleph_0$ aleph-naught, the smallest infinite cardinal number, representing countable infinity

$\mathbb{Z}$ the set of integers: $\{\ldots,-2,-1,0,1,2,\ldots\}$

$\mathbb{H}$ a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product

$\mathbb{N}$ the set of natural numbers: $\{1,2,3,\ldots\}$

$\mathbb{R}$ the set of real numbers

$\mathbb{R}^N$ N-dimensional space with real coordinates

$\mathbb{S}^{N-1}$ the unit sphere in $\mathbb{R}^N$

Existence and construction of real-valued equiangular tight frames

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## ABSTRACT

This paper presents results on real-valued equiangular tight frames (ETFs) and related topics. Some geometric theorems are developed, and aspects of frame theory are used to gain insight into ETFs. We develop a projection method for analyzing equiangular tight frames that leads to new existence results, and that establishes a link between the geometry of the ETF and the spectrum of the associated Gramian and signature matrices. A new lower bound on the number of frame vectors improves on the best known necessary conditions. We recover the Holmes-Paulsen criterion two different ways, along with additional necessary conditions. We also show that ETFs can be rotated to match a standard position, and that this corresponds to a binary tree structure (partial ordering) of embedded sub-spheres of decreasing dimension. This leads to a new canonical form and an enumerative algorithm to algebraically construct or prove the non-existence of equiangular tight frames.

# Chapter 1

# Introduction

## 1.1 Outline

This paper presents original and significant results of research into open problems surrounding the study of equiangular lines and equiangular tight frames. This was done under the guidance of, and in cooperation with, my adviser Dr. Peter Casazza, and the some of the shared results are published in [9] and an upcoming paper [10].

We use frame theory tools to analyze equiangular lines and equiangular tight frames, and give a new projection method to analyze equiangular tight frames that uses both equiangularity and tightness, yields new theoretical existence results, leads to a new canonical form, and results in a novel construction and existence-testing algorithm.

In Chapter 1, Introduction, we give some basic definitions, review the results and the structure of the paper, discuss the motivation for studying equiangular lines and equiangular tight frames, and define the problems that will be addressed. We also provide some of the important existing results as background information, and then introduce some theorems as initial observations which are built upon in subsequent chapters.

In Chapter 2, Properties of Unit-Norm Equiangular Tight Frames, we apply techniques from frame theory to the problem of the existence of equiangular tight frames (ETFs). We derive a new lower bound on the number of lines in a given space. We then develop a projection method that recovers the Holmes-Paulsen Theorem and establishes a link between the geometry of equiangular tight frames and the spectrum of their associated Gramian and signature matrices. The projection method also yields new insights into the existence conditions for equiangular tight frames, and allows for a type of counting that leads to a discrete enumerative construction and existence-testing algorithm in the last chapter.

In Chapter 3, Constructing Equiangular Tight Frames, we expand again on the projection method from Chapters 1 and 2. This is done in an intuitive way, where we build up to it using examples of constructing the equiangular tight frames in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^7$. We outline the decomposition of equiangular tight frames into a dyadic partial ordering of embedded sub-spheres with prescribed size, location, and internal angular conditions (corresponding to equiangularity in the full space), and define a new standard position orientation of equiangular tight frames using rotations and inversions to facilitate determining existence theoretically and with the construction algorithm. This standard position is essentially a new canonical form for the matrix of points, and leads to a discrete, finite, algebraic, enumerative, combinatoric construction algorithm.

The major results are Theorem 12 on page 15, Theorem 20 on page 24 and the subsequent corollaries and lemmata, and the canonical form and construction algorithm outlined in Chapter 3.

Results and tables are summarized in several areas. A summary of the major results given is in the next section. Major formulas and relationships summarized in Section 2.2.2.3. A summary of existence conditions on equiangular tight frames is in Section 2.3. A List of Tables and List of Figures can be found at the beginning of the document. The appendices contain computer outputs in the form of matrices and tables, and some additional proofs.

## 1.2 Overview

### 1.2.1 Motivation

We are interested in equiangular lines, equiangular frames, and equiangular tight frames. These are closely related to Grassmannian frames, optimal Grassmannian frames, and strongly regular graphs. They overlap other concepts such as spherical codes, spherical t-designs, algebraic geometry, and sphere packing.

The idea of studying equiangular lines was first introduced by Haantjes in 1948 [14], and the first major results come from the seminal paper by van Lint and Seidel in 1966 [29]. In 1973, Lemmens and Seidel [17] made a comprehensive study of real equiangular line sets which is still a fundamental contribution. Little progress was made until recently, but with the development of frame theory, and other new tools, several recent advances, such as [25, 16], have reinvigorated the topic. For our purposes, the area can be divided into three related topics: equiangular lines, equiangular frames, and tight equiangular frames.

The study of equiangular lines is interesting in its own right as a geometric construction problem, and relates to many other areas of mathematics. We are interested in determining the maximum number of equiangular lines (through the origin) in a given space, and constructing them. Still almost nothing is known about sufficient conditions or methods of construction, and the maximum number of lines for $\mathbb{R}^N$ is only known for some of the dimensions $N < 50$.

We discuss the real-valued case from the point of view of frame theory and linear algebra, as frames are a generalization of bases. Real-valued equiangular frames correspond to equiangular lines that span their space. In particular we are interested in the existence and construction of real-valued equiangular tight frames.

Equiangular tight frames are particularly interesting and useful. Tightness, in the setting of ETFs, is a spectral and a geometric condition. Equiangular tight frames correspond to optimal grassmannian packings of 1-dimensional spaces [25], and to strongly regular graphs [25, 26]. In signal processing, equiangular tight frames meet the Welch bound for optimal codes [25]. In coding theory they correspond to optimal Gaussian channels [9, 26, 20, 18], and are thus ideal for certain information transmission scenarios. Complex-valued equiangular tight frames are useful in quantum information theory [23, 22, 21]. Tight frames are applicable to CDMA cellular telephone network communication [19, 20], and are optimal for reconstruction in some cases of erasures [16], or loss of coefficients. "Due to their rich theoretical properties and their numerous practical applications, equiangular tight frames are arguably the most important class of finite-dimensional frames, and they are the natural choice when one tries to combine the advantages of orthonormal bases with the concept of redundancy provided by frames" [24]. Equiangular tight frames were introduced by van Lint and Seidel in the setting of discrete geometry [29].

For an introduction to frame theory, see [11]. A good source for more information about abstract frame theory is The Art of Frame Theory by Peter Casazza [7].

### 1.2.2 Background definitions

A sequence of vectors $F = \{f_k\}_{k\in I}$, with $|I| \leq \aleph_0$, belonging to a (separable) Hilbert space $\mathbb{H}$, is a **frame** for $\mathbb{H}$ if there exist $0 < A, B$ such that

$$A\,\|f\|_2^2 \leq \sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B\,\|f\|_2^2 \tag{1.1}$$

for each $f \in \mathbb{H}$. A frame $\{f_k\}_{k\in I}$ is **tight** if there exists $0 < A$ such that

$$\sum_{k\in I} |\langle f, f_k\rangle|^2 = A \|f\|_2^2 \tag{1.2}$$

for each $f \in \mathbb{H}$. If $A = B = 1$, $F$ is a **Parseval frame**. If $\|f_i\| = \|f_j\|$ for all $i, j \in I$, then $F$ is an **equal-norm** frame, and if $\|f_i\| = 1$ for all $i \in I$ then $F$ is a **unit-norm** or **norm-1** frame. If we have the right hand inequality in inequality 1.1, then we call $\{f_i\}_{i\in I}$ a $B$-**Bessel** sequence.

If $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ with frame bounds $A$, $B$ we define the **analysis operator** $T : \mathbb{H} \to l_2(I)$ to be

$$T(f) = \sum_{i\in I} < f, f_i > e_i, \text{ for all } f \in \mathbb{H},$$

where $\{e_i\}_{i\in I}$ is the natural orthonormal basis of $\ell_2(I)$. The adjoint of the analysis operator is the **synthesis operator**, which is given by

$$T^*(e_i) = f_i.$$

It follows that $T$ is a bounded operator if and only if $\{f_i\}_{i\in I}$ is a Bessel sequence. We also have that

$$\|T(f)\|^2 = \sum_{i\in I} |< f, f_i >|^2 .$$

The **frame operator** for the frame is $S = T^*T : \mathbb{H} \to \mathbb{H}$ given by

$$Sf = T^*Tf = T^*\left(\sum_{i\in I} < f, f_i > e_i\right) = \sum_{i\in I} < f, f_i > T^*e_i = \sum_{i\in I} < f, f_i > f_i,$$

which implies that

$$< Sf, f >= \sum_{i\in I} |< f, f_i >|^2 .$$

Therefore, the frame operator is a **positive**, **self-adjoint**, and **invertible operator** on $\mathbb{H}$, with

$$A \cdot I \leq S \leq B \cdot I.$$

We can reconstruct vectors in the space using

$$\begin{aligned} f &= SS^{-1}f \\ &= \sum_{i\in I} < S^{-1}f, f_i > f_i \\ &= \sum_{i\in I} < f, S^{-1}f_i > f_i \\ &= \sum_{i\in I} < f, S^{-\frac{1}{2}}f_i > S^{-\frac{1}{2}}f_i. \end{aligned}$$

The sequence $\{< f, f_i >\}$ is called the **frame coefficients** of the vector $f \in \mathbb{H}$. Since $S$ is invertible, the family $\{S^{-1}f_i\}_{i\in I}$ is also a frame for $\mathbb{H}$ called the **canonical dual frame**.

In the case where $F$ is a tight frame, we have that $S = AI$, where $I$ is the identity map. This allows for decomposition (or reconstruction) with

$$f = \frac{1}{A}\sum_{m=1}^{M} < f, f_m > f_m, \; for\ all\ f \in \mathbb{H},$$

which is a big advantage in applications, in that it acts like a basis up to a scaling constant. We use this fact in the proof of Theorem 20 on page 24.

The **Gramian** matrix of a set of vectors $f_1, \ldots, f_M$ is the symmetric matrix $G$ of inner products, defined by $G_{i,j} =< f_i, f_j >$. In other words,

$$G = F^*F. \tag{1.3}$$

A frame $\{f_k\}_{k=1}^{M}$ in $H$ is called a **Grassmannian frame** if it is a solution to

$$\min\{\mathcal{M}(\{f_k\}_{k=1}^{M})\}$$

where min is taken over all unit-norm frames and

$$\mathcal{M}(\{f_m\}_{m=1}^M) =: \max\{|\langle f_m, f_n\rangle| : 1 \leq m \neq n \leq M\}.$$

It is known that

$$\mathcal{M}(\{f_k\}_{k=1}^M) \geq \sqrt{\frac{M-N}{N(M-1)}},$$

and if we have equality then the Grassmannian frame is **optimal**. For more on optimal Grassmannian frames, see [25].

For any family of vectors $\{f_m\}_{m=1}^M$, the **frame potential** is

$$FP(\{f_m\}_{m=1}^M) = \sum_{m,k=1}^M |\langle f_m, f_k\rangle|^2.$$

In this paper we will always have $\mathbb{H} = \mathbb{R}^N$ for some $N \in \mathbb{N}$.

### 1.2.3 Definition of the problem

We are interested in equiangular lines, equiangular frames, and, in particular, equiangular tight frames.

A set $L = \{l_i\}_{i\in I}$, with $|I| \leq \aleph_0$, of lines through the origin, is called a set of **equiangular lines** if the acute angles between any two of them are the same. That is, there exists some $\theta \in (0,1)$, so that the acute angle between $l_i$ and $l_j$ is $\arccos\theta$, for all $i, j \in I$ with $i \neq j$.

If we have a frame, $F = \{f_i\}_{i=1}^M$ in $\mathbb{R}^N$ and $\theta \in (0,1)$, such that

$$|\langle f_i, f_j\rangle| = \theta, \text{ for all } 1 \leq i \neq j \leq M, \tag{1.4}$$

then $F$ is called an **equiangular frame**, where $\theta$ is actually the *cosine* of the acute angle of the equiangular lines corresponding to the frame vectors. In the same spirit as “equiangular”, for this paper we will call $\theta$ the **system angle** of $F$ (we could say that $\theta$ is the **inner product of** $F$, instead). If $F$ is also tight, then it is an **equiangular tight frame** or ETF. If $F$ is a frame that is equiangular, tight, and unit-norm, we call it a **unit-norm equiangular tight frame** or UNETF.

In a recent paper by Bodmann, Paulsen, and Tomforde [3], it is shown that if a frame is tight and equiangular, it is automatically equal-norm. However, for clarity, we will work with unit-norm equiangular tight frames in this paper.

**Definition 1.** For convenience we will now assign

$$\alpha = \frac{1}{\theta}$$

for the rest of this paper. That is, if $F$ is a UNETF, then $\alpha$ is the reciprocal of the system angle of $F$.

Also, define the **signature matrix** of $F$:

$$A = \frac{1}{\theta}(F^T F - I) = \frac{1}{\theta}(G - I),$$

where $G$ is the Gramian of $F$. Analysis of the eigenvalues of the signature matrix $A$ has led to some fundamental results such as Theorem 4 on page 10 and Theorem 7 on page 11.

We are specifically interested in unit-norm tight equiangular frames for their interesting theoretical properties, their intersection with other areas of research, and their usefulness in applications. Tightness is generally thought of as as spectral property, but we show it also has geometric implications in settings of equiangular frames.

We are primarily interested in unit-norm tight equiangular frames, so we are looking for a set $F$ of vectors $\{f_k\}_{k=1}^M$ in $\mathbb{R}^N$, a constant $\theta \in (0, 1)$, and $A > 0$ such that

$$|\langle f_i, f_j \rangle| = \theta \ and \ \|f_i\| = 1$$

and

$$\sum_{k \in I} |\langle f, f_k \rangle|^2 = A \|f\|_2^2$$

for each $f \in \mathbb{R}^{\mathbb{N}}$. It is also known that the tight frame bound is $A = \frac{M}{N}$ (Theorem 10). If all of these conditions are all satisfied, then $F$ is a UNETF.

## 1.3 Summary of existing results

Most of the known information about equiangular lines is summarized in [17]. What results there have been since then are included in this paper and summarized in later sections.

Let $M$ be the number of elements in a real-valued unit-norm equiangular tight frame in $\mathbb{R}^N$. It is known (see [17] or Theorem 2 below) that (1.4) can only hold if $M \leq \frac{N(N+1)}{2}$. Also, we must have $M \geq N$ since if $0 \leq M < N$, then $F$ cannot be a frame because there are not enough vectors to span the space, and the lower frame bound in equation (1.1) will fail if we choose a non-zero $f$ outside of the span of the vectors of $F$. If $M = N$, then $F$ corresponds to a basis for $\mathbb{R}^N$. Therefore, we want to construct sets $F$ of norm-1 vectors which satisfy (1.4) with $N < M \leq \frac{N(N+1)}{2}$. It is known (see Theorem 3 on the next page and 10 on page 14) that

$$\theta \geq \sqrt{\frac{M-N}{N(M-1)}} \tag{1.5}$$

with equality in (1.5) if and only if $F$ is equiangular and tight (also see Theorem 2.3 from [25], or [10]). In this case $F$ is an **optimal Grassmannian frame** as in Section 1.2.2.

An important result is the upper bound by Gerzon [17] on the number of equiangular lines (and therefore on the number of elements of an equiangular frame) with respect to the number of dimensions.

**Theorem 2.** *[Gerzon] If there exist $M$ equiangular lines in $\mathbb{R}^N$, then*

$$M \leq \frac{N(N+1)}{2} \tag{1.6}$$

*is an upper bound on the number of lines.*

The next inequality was first discovered by Welch [30] in the context of coding theory.

**Theorem 3.** *Let $\{f_m\}_{m=1}^M$ be a unit-norm frame for $\mathbb{H}^N$. Then*

$$\mathcal{M}(\{f_m\}_{m=1}^M) \geq \sqrt{\frac{M-N}{N(M-1)}},$$

*with equality if and only if $\{f_m\}_{m=1}^M$ is an equiangular tight frame. In this case the tight frame bound is $\frac{M}{N}$.*

We give a proof of this in the next section. Also, if $F$ is a UNETF, we have that

$$\theta = \sqrt{\frac{M-N}{N(M-1)}} \tag{1.7}$$

and will assume this whenever discussing UNETFs for the rest of the paper.

An important theorem on the structure of equiangular lines was given by Peter Neumann in [17], and its main idea was recently used to extend the result by the authors of [26]. See Theorem 7 below.

**Theorem 4.** *[Peter Neumann] If there exist $M$ equiangular lines in $\mathbb{R}^N$ with $M > 2N$ and system angle $\theta = \frac{1}{\alpha}$, then $\alpha$ is an odd integer.*

**Corollary 5.** *If $F$ is a UNETF with $M > 2N$ and system angle $\theta = \frac{1}{\alpha}$, then $\alpha$ is an odd integer.*

Also important is the Holmes-Paulsen criterion which we will later recover in two ways.

**Theorem 6.** *[Holmes-Paulsen] If $N < M$ and an ETF exists with $M$ vectors for $\mathbb{R}^N$, then*

$$(M-2N)\sqrt{\frac{M-1}{N(M-N)}} \text{ is an integer.}$$

The two previous results were improved upon in [26], where the authors prove the following:

**Theorem 7.** *[Theorem A] Suppose that $1 < N < M-1$. When $M \neq 2N$, a UNETF can exist only if*

$$\sqrt{\frac{N(M-1)}{M-N}} \quad \text{and} \quad \sqrt{\frac{(M-N)(M-1)}{N}} \quad \text{are odd integers.}$$

*In particular, $M$ is an even number. Furthermore, if $M = 2N$, a UNETF can exist only if $N$ is an odd number and $2N-1$ is the sum of two squares.*

Due to this, for the rest of the paper will we assign

$$\alpha = \sqrt{\frac{N(M-1)}{M-N}} = \frac{1}{\theta}$$

and

$$\beta = \sqrt{\frac{(M-N)(M-1)}{N}}.$$

It is also known, see [26], that the above numbers $\alpha$ and $\beta$ closely relate to the eigenvalues of the signature matrix. That is, if $F$ is a UNETF, then the associated signature matrix has two distinct eigenvalues, $\lambda_1$ and $\lambda_2$, given by:

$$\begin{aligned}\lambda_1 &= -\alpha \\ \lambda_2 &= \beta.\end{aligned}$$

### 1.3.1 Equiangular line sets

We are interested in constructing the maximal number of equiangular lines for a given number of dimensions $N$ for $\mathbb{R}^N$. Not much is known about this for large $N$. What is known [17, 27, 9, 10] is summarized in Table 1.1:

Table 1.1: Known maximal equiangular line sets

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| N | = | 2 | 3 | 4 | 5 | 6 | 7 | ... | 13 | 14 | 15 | 16 | 17 | 18 |
| M | = | 3 | 6 | 6 | 10 | 16 | 28 | ... | 28 | 28–30 | 36 | $\geq 40$ | $\geq 48$ | $\geq 48$ |
| $\alpha$ | = | 2 | $\sqrt{5}$ | 3 | 3 | 3 | 3 | ... | 3 | 5 | 5 | 5 | 5 | 5 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| N | = | 19 | 20 | 21 | 22 | 23 | ... | 41 | 42 | 43 |
| M | = | 72–76$^\dagger$ | 92–96$^\dagger$ | 126 | 176 | 276 | ... | 276 | $\geq 276$ | 344 |
| $\alpha$ | = | 5 | 5 | 5 | 5 | 5 | ... | 5 | 5 | 7 |

$^\dagger$reported to be solved but actually still open

### 1.3.2 Equiangular tight frames

It is well known that orthonormal bases are UNETFs, and also that the standard $N$-simplex of $N+1$ points in $\mathbb{R}^{N+1}$ is equivalent to a $(N+1)$-element UNETF for $\mathbb{R}^N$. Therefore, there are always $N$ and $(N+1)$-element UNETFs for $\mathbb{R}^N$, and so we will focus on the case $M > N+1$.

Equiangular lines are not necessarily equiangular frames, as they may not span the full space. A table of equiangular tight frames with $M \leq 100$ is given in [26]. For dimension $N$ less than 50, and $M > N+1$ we list the equiangular tight frames in Table 1.2.

Table 1.2: All equiangular tight frames for dimension $M > N+1$ and $N \leq 50$.

| N | M | $\theta$ | | N | M | $\theta$ | | N | M | $\theta$ | | N | M | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 6 | $\frac{1}{\sqrt{5}}$ | | 19 | 38 | $\frac{1}{\sqrt{37}}$ | | 25 | 50 | $\frac{1}{7}$ | | 41 | 246 | $\frac{1}{9}$ |
| 5 | 10 | $\frac{1}{3}$ | | 19 | $76^{\dagger\dagger}$ | $\frac{1}{5}$ | | 27 | 54 | $\frac{1}{\sqrt{53}}$ | | 42 | $288^{\dagger\dagger}$ | $\frac{1}{7}$ |
| 6 | 16 | $\frac{1}{3}$ | | 20 | $96^{\dagger\dagger}$ | $\frac{1}{5}$ | | 28 | 64 | $\frac{1}{7}$ | | 43 | 86 | $\frac{1}{\sqrt{85}}$ |
| 7 | 14 | $\frac{1}{\sqrt{13}}$ | | 21 | 28 | $\frac{1}{9}$ | | 31 | 62 | $\frac{1}{\sqrt{61}}$ | | 43 | 344 | $\frac{1}{7}$ |
| 7 | 28 | $\frac{1}{3}$ | | 21 | 36 | $\frac{1}{7}$ | | 33 | 66 | $\frac{1}{\sqrt{65}}$ | | 45 | 90 | $\frac{1}{\sqrt{89}}$ |
| 9 | 18 | $\frac{1}{\sqrt{17}}$ | | 21 | 42 | $\frac{1}{\sqrt{41}}$ | | 35 | $120^{\dagger\dagger}$ | $\frac{1}{7}$ | | 45 | 100 | $\frac{1}{9}$ |
| 10 | 16 | $\frac{1}{5}$ | | 21 | 126 | $\frac{1}{5}$ | | 36 | 64 | $\frac{1}{9}$ | | 45 | $540^{\dagger\dagger}$ | $\frac{1}{7}$ |
| 13 | 26 | $\frac{1}{5}$ | | 22 | 176 | $\frac{1}{5}$ | | 37 | 74 | $\frac{1}{\sqrt{73}}$ | | 46 | $736^{\dagger\dagger}$ | $\frac{1}{7}$ |
| 15 | 30 | $\frac{1}{\sqrt{29}}$ | | 23 | 46 | $\frac{1}{\sqrt{45}}$ | | 37 | $148^{\dagger\dagger}$ | $\frac{1}{7}$ | | 49 | 98 | $\frac{1}{\sqrt{97}}$ |
| 15 | 36 | $\frac{1}{5}$ | | 23 | 276 | $\frac{1}{5}$ | | 41 | 82 | $\frac{1}{9}$ | | | | |

$^{\dagger\dagger}$ means the existence is unknown.

### 1.3.3 Background

For completeness, some background theorems are given without proof, which can be found in [10].

**Theorem 8.** *If $\{f_i\}_{i=1}^M$ is a family of vectors in $\mathbb{H}^N$ with frame operator $S$ having eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Then*

*(1) We have:*

$$\sum_{m=1}^{M} \|f_m\|^2 = \sum_{n=1}^{N} \lambda_n.$$

*(2) We have:*

$$\sum_{n,m=1}^{M} |\langle f_n, f_m \rangle|^2 = \sum_{n=1}^{N} \lambda_n^2.$$

*Proof.* The proof is detailed in [10]. □

**Proposition 9.** *For any family of vectors $\{f_m\}_{m=1}^M$ in $\mathbb{H}^N$,*

$$FP(\{f_m\}_{m=1}^M) \geq \frac{1}{N} \left( \sum_{m=1}^{M} \|f_m\|^2 \right)^2.$$

*We have equality if and only if the frame is tight. If this is a unit-norm frame then*

$$FP(\{f_m\}_{m=1}^M) \geq \frac{M^2}{N},$$

*with equality if and only if the frame is tight.*

*Proof.* The proof is detailed in [10]. □

**Theorem 10.** *Let $\{f_m\}_{m=1}^M$ be a unit-norm frame for $\mathbb{H}^N$. Then*

$$\mathcal{M}(\{f_m\}_{m=1}^M) \geq \sqrt{\frac{M-N}{N(M-1)}},$$

*with equality if and only if $\{f_m\}_{m=1}^M$ is an equiangular tight frame. In this case the tight frame bound is $\frac{M}{N}$.*

*Proof.* Let $S$ be the frame operator for the frame with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Then applying Theorem 8 and Proposition 9 we have:

$$\begin{aligned} M + (M^2 - M)\mathcal{M}(\{f_m\}_{m=1}^M)^2 &\geq \sum_{m,k=1}^M |\langle f_m, f_k \rangle|^2 \\ &\geq \frac{M^2}{N}. \end{aligned}$$

Hence,

$$\mathcal{M}(\{f_m\}_{m=1}^M)^2 \geq \frac{\frac{M^2}{N} - M}{M^2 - M} = \frac{M-N}{N(M-1)}.$$

Since we have equality, we have:

$$\sum_{m,k=1}^M |\langle f_m, f_k \rangle|^2 = \frac{M^2}{N},$$

and so the frame is tight. We also have

$$\sum_{m,k=1}^M |\langle f_m, f_k \rangle|^2 = M + (M^2 - M)\mathcal{M}(\{f_m\}_{m=1}^M),$$

and so the frame is equiangular. ☐

**Corollary 11.** *If* $\{f_m\}_{m=1}^M$ *is a unit-norm equiangular tight frame for* $\mathbb{H}^N$*, then no subset is a tight frame for* $\mathbb{H}^N$*.*

*Proof.* The function

$$f(x) = \frac{N(x-1)}{x-N}$$

is strictly decreasing and so we can have equality only once in Theorem 10. ☐

## 1.4 Fundamental observation

The following is a useful observation about equiangular lines and equiangular frames.

**Theorem 12.** *Let* $\{f_i\}_{i=1}^M$ *be a UNETF in* $\mathbb{R}^N$*. By applying a change of phase, and taking inverses as necessary, we can assume that*

$$< f_M, f_m >= \theta \ for\ all\ m = 1, 2, \ldots, M-1.$$

*Let* $I - P$ *be the orthogonal projection of* $\mathbb{R}^N$ *onto the span of* $f_M$*. Let*

$$\psi_m = \frac{Pf_m}{\|Pf_m\|},\ for\ all\ m = 1, 2, \ldots, M-1.$$

*The following are equivalent:*

*(1)* $|\langle f_i, f_j \rangle| = \theta$ *for all* $1 \leq i \neq j \leq M-1$*.*

*(2)* $\langle P(f_i), P(f_j) \rangle = \pm\theta - \theta^2$ *for all* $1 \leq i \neq j \leq M-1$*.*

*(3)* $\langle \psi_i, \psi_j \rangle = \frac{\pm\theta-\theta^2}{1-\theta^2} = \frac{\theta}{1+\theta}$ *or* $\frac{-\theta}{1-\theta}$ *for all* $1 \leq i \neq j \leq M-1$*.*

*Proof.* (1) $\Leftrightarrow$ (2): Let $f_i, f_j$ be elements of $F$ with $i \neq j$. By our assumptions, there is an orthonormal basis for $\mathbb{R}^N$ whose last element is $\psi_M$, so we can write:

$$\begin{aligned} f_i &= (P(f_i), \theta) \\ f_j &= (P(f_j), \theta) \end{aligned}$$

where $\theta \in (0,1) \subset \mathbb{R}^1$. Now we have that

$$\begin{aligned} \theta &= |\langle f_i, f_j \rangle| \\ &= |\langle (P(f_i), \theta), (P(f_j), \theta) \rangle| \\ &= |\langle (P(f_i), 0), (P(f_j), 0) \rangle + \langle (0, \dots, 0, \theta), (0, \dots, 0, \theta) \rangle| \\ &= \left| \langle P(f_i), P(f_j) \rangle + \theta^2 \right|. \end{aligned}$$

Remove the absolute value to get that $|\langle f_i, f_j \rangle| = \theta$ if and only if

$$\langle P(f_i), P(f_j) \rangle = \pm\theta - \theta^2.$$

(2) $\Leftrightarrow$ (3): Since $|\langle f_M, f_j \rangle| = \theta$ and $\|f_i\| = 1$ for all $1 \leq i \leq M$, we have that

$$\|P(f_i)\| = \sqrt{1 - \theta^2} \text{ for all } 1 \leq i \leq M - 1.$$

Therefore,

$$\psi_i = \frac{P(\phi_i)}{\sqrt{1 - \theta^2}},$$

and we get that

$$\langle P(\phi_i), P(\phi_j) \rangle = \pm\theta - \theta^2$$

if and only if

$$\langle \psi_j, \psi_k \rangle = \left\langle \frac{P(\phi_j)}{\sqrt{1 - \theta^2}}, \frac{P(\phi_k)}{\sqrt{1 - \theta^2}} \right\rangle = \frac{\pm\theta - \theta^2}{1 - \theta^2}.$$

$\square$

This theorem is used to prove some of our main results, such as Theorem 20 in Section 2.2.2.

The rational functions in (3) are used repeatedly and so we will label them. These functions map the system angle $\theta$ for dimension $N$ to the two values $\frac{\theta}{1+\theta}$ and $\frac{-\theta}{1-\theta}$ for the inner products between the relative north pole of the reduced $(N-1)$-dimensional space and points in the space, a necessary condition if the points in the full $N$-dimensional space are mutually equiangular with system angle $\theta$ on $\mathbb{S}^{N-1} \subset \mathbb{R}^N$.

**Definition 13.** Let

$$f_+(x) = \frac{x}{1+x}$$

with domain $(-\frac{1}{2}, 1)$ and range $(-1, \frac{1}{2})$. Next, let

$$f_-(x) = \frac{-x}{1-x},$$

with domain $(-1, \frac{1}{2})$ and range $(-1, \frac{1}{2})$. Note that $f_+^{-1}(x) = -f_-(x)$, and also that $f_-^{-1}(x) = -f_+(x)$.

The graphs of these functions are given in Figure 1.1 and Figure 1.2. Theorem 12 can be useful for constructing UNETFs. Examples of the construction of the UNETFs with 3 vectors in $\mathbb{R}^2$ and 6 vectors in $\mathbb{R}^3$ are given in Section 3.2.1 on page 48, and this technique is developed further in subsequent sections.

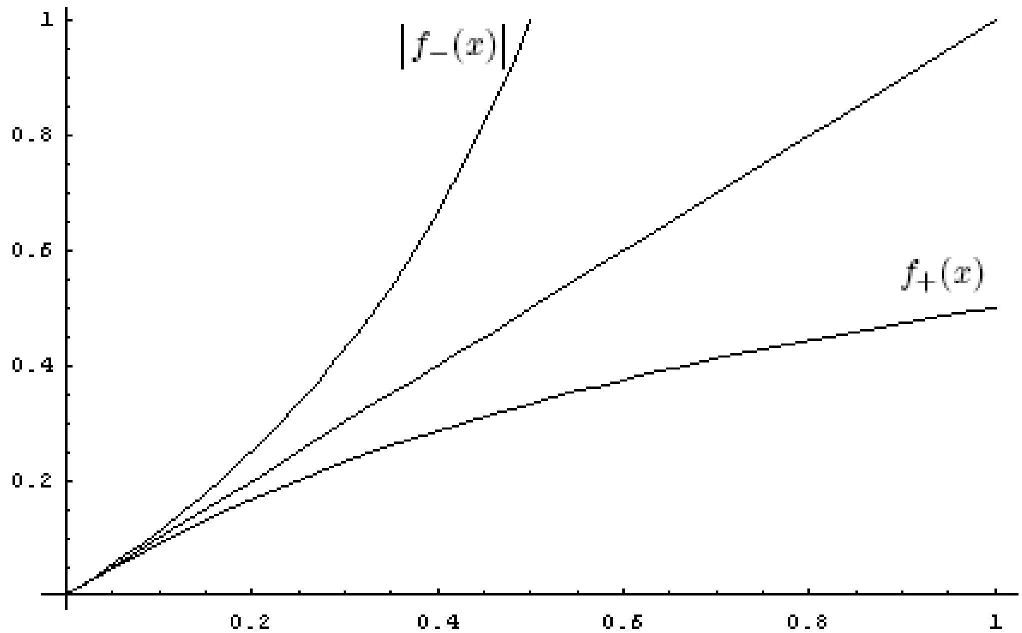



Figure 1.1: The graphs of $f_+$ and $|f_-|$

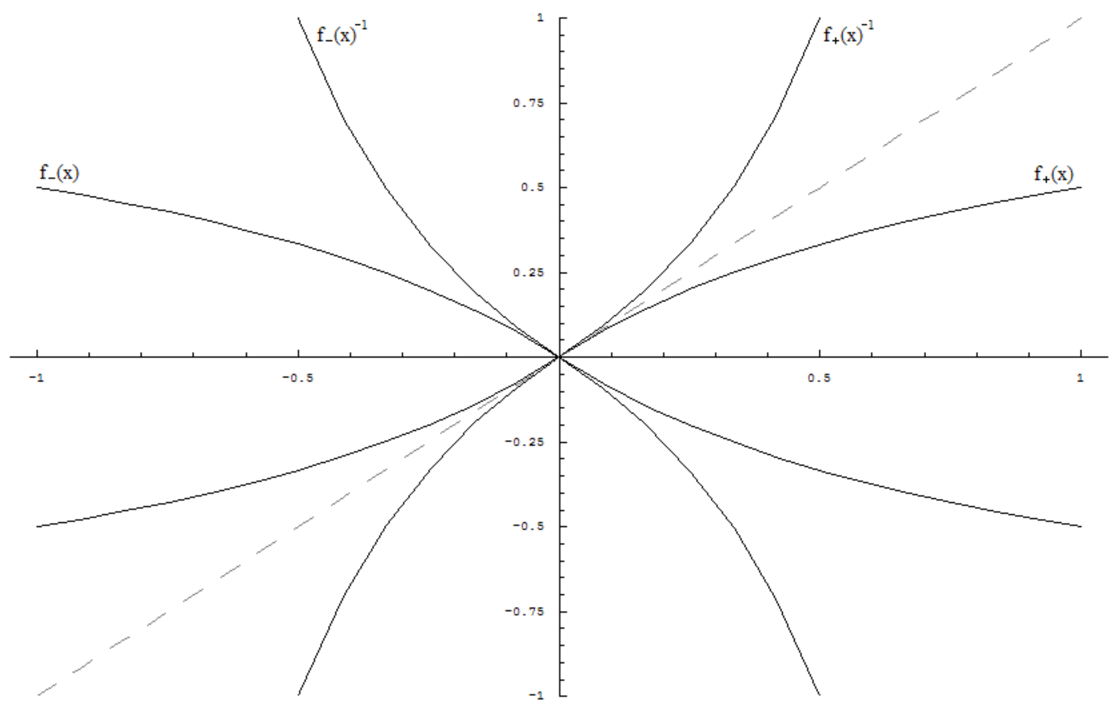



Figure 1.2: The graphs of $f_+$, $f_-$, $f_+^{-1}$, and $f_-^{-1}$

# Chapter 2

# Properties of Unit-Norm Equiangular Tight Frames

## 2.1 Initial ideas

In this section we will derive a number of relationships between the system angle of an equiangular tight frame and the values of $M$ and $N$.

First we use Naimark's Theorem, a tool from frame theory, to develop the concept of the complementary UNETF, which leads to a new lower bound on the number of vectors in a UNETF for a given dimension. This is an improvement on the best known necessary conditions, and a counterexample to the conjecture in [26], that their Theorem A (given on page 11 above) is a sufficient condition for existence. We list all such counterexamples up to $N = 1,000,000$ in Table 2.2 on page 24.

Next, we use Theorem 12 on page 15, and apply projection techniques to analyze an arbitrary UNETF. Using both tightness and equiangularity, we are able to divide $M - 2$ of the $M$ points into two sets, and count how many vectors are in each set. This establishes a link between the geometry of the UNETF and the spectrum of the associated Gramian and signature matrices. It also provides a number of insights into the relationship between $N$, $M$, and $\alpha$ that help to classify UNETFs. We recover the Holmes-Paulsen criterion and improve upon it. This technique leads to the constructing and existence testing algorithm of Chapter 3.

## 2.2 Frame theoretic analysis

### 2.2.1 Complementary equiangular tight frame

A theorem by Naimark [8, 15] says that:

**Theorem 14.** *[Naimark] The family* $\{f_m\}_{m=1}^M$ *is a Parseval frame for* $\mathbb{R}^N$ *if and only if there is an orthogonal projection* $P$ *on* $\mathbb{R}^M$ *satisfying* $Pe_m = f_m$ *for all* $m = 1, 2, \ldots, M$ *where* $\{e_m\}_{m=1}^M$ *is an orthonormal basis for* $\mathbb{R}^N$.

**Lemma 15.** *Let* $P$ *be an orthogonal projection on* $\mathbb{R}^M$ *and let* $\{e_m\}_{m=1}^M$ *be an orthonormal basis for* $\mathbb{R}^M$.

1. $\{Pe_m\}_{m=1}^M$ *is equal norm if and only if* $\{(I-P)e_m\}_{m=1}^M$ *is equal norm.*
2. $\{Pe_m\}_{m=1}^M$ *is equiangular if and only if* $\{(I-P)e_m\}_{m=1}^M$ *is equiangular.*

*Proof.* (1) is obvious.

(2) For all $1 \leq i \neq m \leq M$ we have

$$\begin{aligned} |\langle (I-P)e_i, (I-P)e_m \rangle| &= |\langle e_i, e_i \rangle - \langle Pe_i, e_m \rangle - \langle e_i, Pe_m \rangle + \langle Pe_i, Pe_m \rangle| \\ &= |\langle Pe_i, Pe_m \rangle|. \end{aligned}$$

□

Now, since we can renormalize any tight frame into a Parseval frame, we get that:

**Theorem 16.** *If* $F = \{f_m\}_{m=1}^M$ *is an equiangular tight frame for* $\mathbb{R}^N$ *with* $Pe_m = \sqrt{\frac{N}{M}} f_m$*, then* $F^c = \{\sqrt{\frac{M}{M-N}}(I-P)e_m\}_{m=1}^M$ *is an equiangular tight frame for* $\mathbb{R}^{(M-N)}$.

We call this the **complementary equiangular tight frame**, and will denote it as $F^c$.

This shows that UNETFs come in pairs, one with $M$ elements in $\mathbb{R}^N$ and the complementary one with $M$ elements in $\mathbb{R}^{(M-N)}$. Applying the known upper bound to the complementary equiangular tight frame gives a new lower bound on the existence of UNETFs. Essentially, if the number of vectors in the original frame is below the new lower bound, then the number of vectors in the complementary equiangular tight frame will be above the known upper bound.

**Definition 17.** If $F$ is a UNETF with $M$ vectors for $\mathbb{R}^N$, then let $F^c$ denote it's complementary UNETF with $M$ vectors in $\mathbb{R}^{M-N}$.

**Theorem 18.** [*New Lower Bound*] *If $F$ is an* UNETF *with $M$ vectors in $\mathbb{R}^N$, then we have*

$$M \leq \min\left\{\frac{N(N+1)}{2}, \frac{(M-N)(M-N+1)}{2}\right\}, \tag{2.1}$$

*except for when $M = N$ or $M = N+1$. Solving the 2nd value for $M$ leads to the bound:*

$$LB(N) = \left\lceil \frac{2N+1+\sqrt{8N+1}}{2} \right\rceil \leq M \leq \frac{N(N+1)}{2}. \tag{2.2}$$

*Proof.* Note that we need the conditions on (2.1), because the theorem fails if $M =$

$N+1$. Rearranging the 2nd value in (2.1), and solving for $M$, we get

$$\begin{aligned} M &\leq \frac{(M-N)(M-N+1)}{2} \\ &\Longleftrightarrow \\ 2M &\leq M^2 - 2NM + M + N^2 - N \\ &\Longleftrightarrow \\ 0 &\leq M^2 - 2NM - M + N^2 - N \\ &\Longleftrightarrow \\ 0 &\leq M^2 + (-2N-1)M + (N^2 - N). \end{aligned} \tag{2.3}$$

The zeros of (2.3) (i.e. equality) occur when

$$\begin{aligned} M &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2N+1 \pm \sqrt{(2N+1)^2 - 4(N^2-N)}}{2a} \\ & \frac{2N+1 \pm \sqrt{4N^2+4N+1-4N^2+4N}}{2a} \\ & \frac{2N+1 \pm \sqrt{8N+1}}{2}. \end{aligned}$$

We can discard the negative option since $M < \frac{2N+1 \pm \sqrt{8N+1}}{2} = N + \frac{1}{2} \pm \sqrt{2N + \frac{1}{4}} < N$, a contradiction by assumption since we started with $N+1 < M \leq \frac{N(N+1)}{2}$. The function in (2.3) is increasing in terms of $M$, so we get that

$$M \geq \left\lceil \frac{2N+1+\sqrt{8N+1}}{2} \right\rceil = LB(N).$$

This means that $F$ must not have too few vectors in order to guarantee that $F^c$, the complementary UNETF of $F$, does not have too many vectors. A list of the lower bound for each $2 \leq N \leq 50$ is given in Table 2.1. □

Table 2.1: $N$ and $LB(N)$ for $2 \leq N \leq 50$

| $N$ | $LB(N)$ | $N$ | $LB(N)$ | $N$ | $LB(N)$ | $N$ | $LB(N)$ | $N$ | $LB(N)$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 5 | 11 | 17 | 21 | 28 | 31 | 40 | 41 | 51 |
| 3 | 6 | 12 | 18 | 22 | 30 | 32 | 41 | 42 | 52 |
| 4 | 8 | 13 | 19 | 23 | 31 | 33 | 42 | 43 | 53 |
| 5 | 9 | 14 | 20 | 24 | 32 | 34 | 43 | 44 | 54 |
| 6 | 10 | 15 | 21 | 25 | 33 | 35 | 44 | 45 | 55 |
| 7 | 12 | 16 | 23 | 26 | 34 | 36 | 45 | 46 | 57 |
| 8 | 13 | 17 | 24 | 27 | 35 | 37 | 47 | 47 | 58 |
| 9 | 14 | 18 | 25 | 28 | 36 | 38 | 48 | 48 | 59 |
| 10 | 15 | 19 | 26 | 29 | 38 | 39 | 49 | 49 | 60 |
| | | 20 | 27 | 30 | 39 | 40 | 50 | 50 | 61 |

**Example 19.** Theorem 7 (Theorem A from [26]) allows that there could exist 64 vectors forming a UNETF in $\mathbb{R}^{56}$, since

$$\alpha = \sqrt{\frac{N(M-1)}{M-N}} = \sqrt{\frac{56(64-1)}{64-56}} = 21$$

is an odd integer and so is

$$\beta = \sqrt{\frac{(M-N)(M-1)}{N}} = \sqrt{\frac{(64-56)(64-1)}{56}} = 3$$

However, this is not possible since

$$M = 64 < 68 = \left\lceil \frac{2(56)+1+\sqrt{8(56)+1}}{2} \right\rceil = LB(56) = LB(N),$$

a contradiction of Theorem 18.

Table 2.2 gives a list of computed $N$, $M$ values that satisfy the conditions of Theorem A, but which do not exist because its complementary UNETF does not exist (it has too many vectors) by the lower bound of Theorem 2.2.

Table 2.2: List of all possible UNETFs for $2 \leq N \leq 1,000,000$ that satisfy Theorem A, but do not exist by Theorem 18.

| $N$ | $M$ *d.n.e* | $LB(N)$ |
|---|---|---|
| 56 | 64 | 68 |
| 552 | 576 | 586 |
| 2256 | 2304 | 2324 |
| 6320 | 6400 | 6433 |
| 14280 | 14400 | 14450 |
| 28056 | 28224 | 28294 |
| 49952 | 50176 | 50269 |
| 82656 | 82944 | 83064 |
| 129240 | 129600 | 129749 |
| 193160 | 193600 | 193783 |
| 278256 | 278784 | 279003 |
| 388752 | 389376 | 389635 |
| 529256 | 529984 | 530286 |
| 704760 | 705600 | 705948 |
| 920640 | 921600 | 921998 |

### 2.2.2 Projection method

The following important theorem lets us count the number of points on the parts of a projected ETF corresponding to the two possible inner product values as given in Theorem 12. We also get a new relationship between the eigenvalues of the associated Gramian and signature matrices, and the geometric distribution of the points. The power of this next theorem comes from that it uses both equiangularity and tightness, while reducing the dimension of the problem.

**Theorem 20.** *Let $\{f_m\}_{m=1}^M$ be a unit-norm equiangular tight frame for $\mathbb{R}^N$. Then the tight frame bound is $\frac{M}{N}$ and the system angle is*

$$\theta = \sqrt{\frac{M-N}{N(M-1)}}.$$

*By applying a change of phase and taking inverses, we assume*

$$\langle f_M, f_m \rangle = \theta, \ for \ all \ m = 1, 2, \ldots, M-1.$$

*Let $I-P$ be the orthogonal projection of $\mathbb{R}^N$ onto the span of $f_M$. Let*

$$\psi_m = \frac{Pf_m}{\|Pf_m\|}, \quad \textit{for all } m = 1, 2, \ldots, M-1.$$

*The following hold:*

*(1) $\{\psi_m\}_{m=1}^{M-1}$ is a unit-norm tight frame with tight frame bound*

$$\frac{M}{N(1-\theta^2)} = \frac{M-1}{N-1}.$$

*(2) We have*

$$\sum_{m=1}^{M-1} \psi_m = 0.$$

*(3) We have*

$$-1 = x\frac{\theta}{\theta+1} + y\frac{\theta}{\theta-1},$$

*where*

$$x = \left|\{1 \le m \le M-2 : \langle \psi_{M-1}, \psi_m \rangle = \frac{\theta}{\theta+1}\}\right|,$$

*and*

$$y = \left|\{1 \le m \le M-2 : \langle \psi_{M-1}, \psi_m \rangle = \frac{\theta}{\theta-1}\}\right|.$$

*(4) We have*

$$\begin{aligned} 1 &= (M-1)\theta^2 + (x-y)\theta \\ &= \frac{M-N}{N} + (x-y)\sqrt{\frac{M-N}{N(M-1)}}. \end{aligned}$$

*Hence,*

$$\frac{M-2N}{N} = (y-x)\sqrt{\frac{M-N}{N(M-1)}}.$$

*Proof.* (1) Since $\{f_m\}_{m=1}^{M}$ is a tight frame with tight frame bound $\frac{M}{N}$, and $P$ is an orthogonal projection, we have that $\{Pf_m\}_{m=1}^{M-1}$ is a tight frame with bound $\frac{M}{N}$. Since

$\|Pf_m\|^2 = 1-\theta^2$, we get that $\{\psi_m\}_{m=1}^M$ is a tight frame with the new tight frame bound as stated.

(2) Since

$$f_M = \frac{M}{N}\sum_{m=1}^{M}\langle f_M, f_m\rangle f_m = \frac{M}{N}\|f_M\|^2 f_M + \frac{M}{N}\sum_{m=1}^{M-1}\langle f_M, f_m\rangle f_m.$$

Hence,

$$(1-\frac{M}{N}\|f_M\|^2)f_M = \frac{M}{N}\sum_{m=1}^{M-1}\langle f_M, f_m\rangle f_m.$$

Now,

$$0 = P(1-\frac{M}{N}\|f_M\|^2)f_M = \frac{M}{N}\sum_{m=1}^{M-1}\langle f_M, f_m\rangle Pf_m = \theta\frac{M}{N}\sum_{m=1}^{M-1}Pf_m.$$

Finally,

$$0 = \sum_{m=1}^{M-1}Pf_m = \sqrt{(1-\theta^2)}\sum_{m=1}^{M-1}\psi_m.$$

(3) By (2),

$$-\psi_{M-1} = \sum_{m=1}^{M-2}\psi_m.$$

Now we compute,

$$\begin{aligned} -1 &= \langle\psi_{M-1}, -\psi_{M-1}\rangle \\ &= \sum_{m=1}^{M-2}\langle\psi_{M-1}, \psi_m\rangle \\ &= x\frac{\theta}{\theta+1} + y\frac{\theta}{\theta-1}. \end{aligned}$$

(4) Multiply through the equation in (3) by $(\theta+1)(\theta-1)$ to get

$$-(\theta+1)(\theta-1) = x\theta(\theta-1) + y\theta(\theta+1).$$

Hence,

$$1-\theta^2 = x\theta^2 - x\theta + y\theta^2 + y\theta.$$

We now have

$$1 = (x + y + 1)\theta^2 + (y - x)\theta.$$

Since $x + y + 1 = M - 1$ and

$$\theta^2 = \frac{M - N}{N(M - 1)},$$

we have (4). □

Part (2) is interesting. It is known that if $F$ is a tight frame for $\mathbb{R}^2$ then its elements must sum to zero. In general, this is not true for tight frames for $\mathbb{R}^N$ when $N > 2$. Part (2) tells us that for any $N$, if $F$ is a tight equiangular frame, then all the points sum to zero after a projection along one of them, after taking inverses if necessary.

The following is immediate from Theorem 20, Part (4).

**Corollary 21.** *If $\{f_m\}_{m=1}^M$ is an equal norm equiangular tight frame for $\mathbb{R}^N$, then one of the following must hold:*

*(1) $x = y$ and so $M = 2N$.*

*(2) $x \neq y$ and*

$$\sqrt{\frac{M - N}{N(M - 1)}} \quad \textit{is rational.}$$

Another important consequence is that if we substitute $\theta = \frac{1}{\alpha}$ into the equation in part (4) of Theorem 20, we get that

$$\begin{aligned} 1 &= (M-1)\theta^2 + (y - x)\theta \\ &\Leftrightarrow \\ 0 &= \alpha^2 - (y - x)\alpha - (M - 1). \end{aligned} \tag{2.4}$$

**Lemma 22.** *We have that $\alpha\beta = M - 1$.*

*Proof.* By direct computation

$$\alpha\beta = \sqrt{\frac{N(M-1)}{M-N}}\sqrt{\frac{(M-N)(M-1)}{N}} = \sqrt{(M-1)^2} = M-1. \tag{2.5}$$

□

Now recall the Rational Root Theorem, which can be proven using Gauss's lemma about primitive polynomials.

**Theorem 23.** *[Rational Root Theorem] If $f(x) = a_n x^n + \cdots + a_0 x^0$ is a polynomial with integer coefficients, and $a_0 \neq 0$, then if $f(x)$ has any rational root $r = \pm\frac{p}{q}$, with $p, q$ relatively prime positive integers, then $p$ is a divisor of $a_0$ and $q$ is a divisor of $a_n$.*

The Integral Root Theorem is a special case of the Rational Root Theorem.

**Theorem 24.** *[Integral Root Theorem] Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0$ be a monic polynomial with integer coefficients and $a_0 \neq 0$, then any rational root of $f(x)$ must be an integer.*

**Corollary 25.** *If $F$ is a UNETF, the numbers $\alpha$ and $\beta$ are both integers, and so are the eigenvalues of the signature matrix.*

*Proof.* By Corollary 21, we know that $\alpha$ is a rational number. By the Integral Root Theorem 24, it must be an integer. By the Rational Root Theorem, $\alpha$ is a divisor of $(M-1)$, so that $\frac{M-1}{\alpha}$ is an integer. But now by Lemma 22,

$$\beta = \frac{M-1}{\alpha} \text{ is an integer.}$$

This shows that $\alpha$ and $\beta$ are both positive integers. □

**Lemma 26.** $y - x = \alpha - \beta$.

*Proof.* Equation (2.4) says that $0 = \alpha^2 - (y-x)\alpha - (M-1)$. Rearrange and divide by $\alpha$ to get

$$y - x = \alpha - \frac{M-1}{\alpha} = \alpha - \beta.$$

□

In this notation we can now rewrite the Holmes-Paulsen condition.

**Lemma 27.** *The Holmes-Paulsen criterion is equivalent to saying that $F$ is an ETF only if the sum of the eigenvalues of the signature matrix is an integer. That is, $(M-2N)\sqrt{\frac{M-1}{N(M-N)}}$ is an integer if and only if $-\alpha + \beta = \lambda_1 + \lambda_2$ is an integer, or equivalently*

$$\textit{Holmes-Paulsen} \Longleftrightarrow \alpha - \beta \in \mathbb{Z}.$$

*Proof.*

$$\begin{aligned} (M-2N)\sqrt{\frac{M-1}{N(M-N)}} &= \frac{(M-2N)}{N}\sqrt{\frac{N(M-1)}{(M-N)}} \\ &= x - y \\ &= \beta - \alpha \\ &= \lambda_1 + \lambda_2 \end{aligned}$$

where the second step is from Theorem 20, part (4). □

Now we recover the Holmes-Paulsen criterion in two ways:

If $F$ is a UNETF, then by Corollary 25, $\alpha$ and $\beta$ are both positive integers, and therefore $\beta - \alpha$ is an integer. By Lemma 27, the Holmes-Paulsen criterion is satisfied.

Lemma 26 also recovers the Holmes-Paulsen criterion, since if $F$ is a UNETF, $\alpha - \beta = y - x$ must be an integer, since $x$ and $y$ are integers, being cardinalities of finite sets as defined in Theorem 20. Once $\alpha - \beta$ is an integer, then we are done after applying Lemma 27. To summarize:

$$(M-2N)\sqrt{\frac{M-1}{N(M-N)}} = \beta - \alpha = y - x$$

is an integer, since $y-x$ is an integer.

#### 2.2.2.1 Some resulting equations

**Lemma 28.** *Formula for $x$ and $y$ (from part (3) of Theorem 20) in terms of $\alpha$ and $\beta$:*

$$x = \frac{(\alpha+1)(\beta-1)}{2}$$
$$y = \frac{(\alpha-1)(\beta+1)}{2}.$$

*Proof.* We have that shown in Lemma 26 that $y-x=\alpha-\beta$, and by definition, from Theorem 20, we have that $x+y+2=M$. Starting with the first equation, and then substituting for $y$ using the second, we get that

$$\begin{aligned} x &= y-\alpha+\beta \\ &= (M-x-2)-\alpha+\beta \\ &\Leftrightarrow \\ 2x &= (M-1)-1-\alpha+\beta \\ &= \alpha\beta-1-\alpha+\beta \end{aligned}$$

and so $x=\frac{(\alpha+1)(\beta-1)}{2}$, where the substitution using $(M-1)=\alpha\beta$ comes from Lemma 22. The proof for $y$ is similar. □

**Lemma 29.** *With no restrictions on $x$ and $y$, we have that*

$$y = x\left(\frac{\alpha-1}{\alpha+1}\right) + \alpha - 1 \tag{2.6}$$

*and*

$$y - x \leq \alpha - 1. \tag{2.7}$$

*Proof.* If there are no restrictions on $x$ and $y$, then (2.10) is equivalent to (2.6) by algebraic manipulation. Since $0 < \frac{\alpha-1}{\alpha+1} < 1$ we can write

$$x \geq x\left(\frac{\alpha - 1}{\alpha + 1}\right) = y - \alpha + 1$$

Hence

$$y - x \leq \alpha - 1$$

or

$$x \geq y - \alpha + 1.$$

□

Lemma 29 tells us that $y$ cannot be much larger than $x$ in a sense.

#### 2.2.2.2 Conditions on M, N, and $\alpha$

The following describes when the maximal number of vectors can be achieved.

**Lemma 30.** *We can have* $M = \frac{N(N+1)}{2}$ *if and only if* $N = \alpha^2 - 2$ *where* $\alpha \in \{3, 5, 7, \ldots\}$.

*Proof.* If $N = \alpha^2 - 2$, then it is easy to check using equation (2.15):

$$\begin{aligned} M &= \frac{(\alpha^2 - 1)N}{\alpha^2 - N} \\ &= \frac{(\alpha^2 - 1)(\alpha^2 - 2)}{\alpha^2 - (\alpha^2 - 2)} \\ &= \frac{(N + 1)N}{2}. \end{aligned}$$

Conversely, if $M = \frac{N(N+1)}{2}$, then by equation (2.15) again

$$\frac{N(N+1)}{2} = \frac{(\alpha^2 - 1)N}{\alpha^2 - N}$$

if and only if

$$(N+1)(\alpha^2 - N) = 2\alpha^2 - 2$$

if and only if

$$N^2 - \alpha^2 N + N - a^2 + 2\alpha^2 - 2 = 0$$

if and only if

$$N^2 - (\alpha^2 - 1)N + (\alpha^2 - 2) = 0$$

if and only if

$$\left[N - (\alpha^2 - 2)\right](N - 1) = 0$$

if and only if (since $N \geq 2$)

$$N = (\alpha^2 - 2).$$

□

**Lemma 31.** *$N|\alpha M$.*

*Proof.* From equations (2.9) and (2.14) (which comes from part (4) of Theorem (20), we get that

$$\frac{\alpha M}{N} - 2\alpha = y - x,$$

so that $\frac{\alpha M}{N}$ is an integer. □

**Lemma 32.** *If $N$ is even and $\mathbb{R}^N$ has an equiangular tight frame with $M$ elements then 4 divides $M$.*

*Proof.* We have that

$$\frac{\alpha M}{N} = 2\alpha + x - y, \quad \text{is even.}$$

i.e.

$$\frac{\alpha M}{2N} = \alpha + \frac{x-y}{2}, \quad \text{an integer.}$$

Since $\alpha$ is odd and $N$ is even, we have that 4 divides $M$. □

**Lemma 33.** *$(M-N)|N(N-1)$ and $\alpha^2 > N \geq \alpha$ (for both $x = y$ and $x \neq y$) and*

$$\alpha^2 = N + \frac{N(N-1)}{M-N}. \tag{2.8}$$

*Also, $\alpha = N$ if and only if $M = N+1$.*

*Proof.* Using equations (2.9) and (2.15), we get that

$$\begin{aligned} \alpha^2 &= \frac{N(M-1)}{M-N} \\ &= \frac{(M-N)N + N^2 - N}{M-N} \\ &= N + \frac{N(N-1)}{M-N} \end{aligned}$$

which gives (2.8) and shows that $(M-N)|N(N-1)$.

From equation (2.8) we have immediately that $\alpha^2 > N$. To see that $N > \alpha$. Set $f(x) = \sqrt{\frac{N(x-1)}{x-N}}$ on $[N+1, \infty)$. Since $f(x)$ is strictly increasing, $\alpha = f(x) < f(N+1) = N$ for all $x \in [N+1, \infty)$. We have that $N \geq \alpha$ when $M \geq N+1$.

Next, $M = N+1$ implies $\alpha = N$ by equation (2.8). The other direction, that $\alpha = N$ implies $M = N+1$ is clear because $f(x)$ is strictly increasing on $[N+1, \infty)$. □

#### 2.2.2.3 Summary of relationships

If $F$ is an $M$ vector UNETF for $\mathbb{R}^N$ with system angle $\theta = \frac{1}{\alpha}$, and $F^c$ is the complementary UNETF with $M$ vectors for $\mathbb{R}^{M-N}$ with system angle $\theta^c = \frac{1}{\beta}$ then we have the following:

$$\frac{1}{\alpha} = \theta = \sqrt{\frac{M-N}{N(M-1)}} \tag{2.9}$$

$$-1 = x\frac{\theta}{\theta+1} + y\frac{\theta}{\theta-1} \tag{2.10}$$

$$1 \quad = \quad (M-1)\theta^2 + (y-x)\theta \tag{2.11}$$

$$\alpha^2 - (y-x)\alpha - (M-1) = 0 \tag{2.12}$$

$$\frac{M}{N} - 2 = (y-x)\sqrt{\frac{M-N}{N(M-1)}} \tag{2.13}$$

$$\frac{M-2N}{N(y-x)} = \sqrt{\frac{M-N}{N(M-1)}} \tag{2.14}$$

$$M = \frac{(\alpha^2-1)N}{\alpha^2-N} \tag{2.15}$$

$$N = \frac{\alpha^2 M}{\alpha^2+M-1} \tag{2.16}$$

$$x = \frac{(\alpha+1)(\beta-1)}{2} \tag{2.17}$$

$$y = \frac{(\alpha-1)(\beta+1)}{2} \tag{2.18}$$

$$M = \alpha\beta + 1 \tag{2.19}$$

$$N = \frac{1+\alpha\beta}{1+\frac{\beta}{\alpha}} = \frac{\alpha(\alpha\beta+1)}{\alpha+\beta} \tag{2.20}$$

$$\frac{M-N}{N} = \frac{M-1}{\alpha^2}. \tag{2.21}$$

## 2.3 Existence conditions

### 2.3.1 Connection to tight spherical 5-designs

In a recent paper [1], Bannai, Munemasa, and Venkov showed, in the setting of tight spherical t-designs, that a UNETF does not exist for $N = 47$ and $M = 1128$ (which also means that no ETF exists for $N = 47$) and infinitely many other values of $N$.

In their setting, tight 5-designs (with tight having a different meaning than that of a tight frame) correspond to ETFs that attain the maximal number of points, that is when $N = \alpha^2 - 2$, and $M = \frac{N(N+1)}{2}$ for odd $\alpha$.

**Theorem 34.** *[Bannai, Munemasa, and Venkov] [1] Suppose that $m = 2k$ is even, $k \equiv 2 \mod 3$, and that both $k$ and $2k+1$ are square-free. Then no tight spherical 5-design exists in $\mathbb{R}^N$ with $N = (2m+1)^2 - 2$, and there are infinitely many such occurrences. In addition, no tight spherical 5-design exists when $m = 3$.*

The theorem rules out existence of a tight spherical 5-design for infinitely many values of $m$, starting with $m = \{4, 10, 22, 28, 34, 46, 52, 58, \ldots\}$. Therefore, infinitely many (previously thought to be possibly valid) values of $M$ and $N$ for UNETFs are ruled out, with $N = (2m+1)^2 - 2$, $M = \frac{N(N+1)}{2}$, and $\alpha = 2m+1$.

**Example 35.** If we choose $m = 4$, then $N = (2 \cdot 4 + 1)^2 - 2 = 79$ cannot have a UNETF with the maximal number of points, which would be $M = 3160$ by equation (2.15). This rules out the case of $\alpha = 9$, $N = 79$, $M = 3160$.

### 2.3.2 Summary of best known necessary conditions

If $N + 1 < M < \frac{N(N+1)}{2}$, then we must have that:

1. If $M \neq 2N$, then $\alpha$ and $\beta$ must be odd (Theorem 7).
2. If $M = 2N$, then $N$ is an odd number and $2N - 1$ is the sum of two squares (Theorem 7).
3. We must have that $\left\lceil \frac{2N+1+\sqrt{8N+1}}{2} \right\rceil \leq M \leq \frac{N(N+1)}{2}$ (by equation 2.2 of Theorem 18.
4. If $M = \frac{N(N+1)}{2}$, and there exists $m, k \in \mathbb{N}$ such that $N = (2m+1)^2 - 2$ with $m = 2k$, $k \equiv 2 \mod 3$, and both $k$ and $2k+1$ are square-free, then there is

no UNETF with $M$ vectors for $\mathbb{R}^N$. Additionally, we cannot have $N = 47$ and $M = 1128$ ($m = 3$), by [1] and Theorem 34.

For a table of all possible $M$ and $N$ values, up to $N = 1000$, remaining after applying all of the known necessary conditions above, see Appendix C.

# Chapter 3

# Constructing Equiangular Tight Frames

Very little is known about methods to construct or prove the existence of equiangular lines, equiangular frames, or tight equiangular frames.

Most constructions for equiangular frames come from graph theory, and there is a 1–1 correspondence between UNETFs and strongly regular graphs [26]. For an introduction to strongly regular graphs, see [4, 29, 5, 6]; for the related concept of two-graphs, see [13]. A construction given by de Caen [12] yields $\frac{2}{9}(N+1)^2$ equiangular lines in $\mathbb{R}^N$ when $N = 3 \cdot 2^{2t-1}$ for any $t \in \mathbb{N}$. This compares well with the upper bound of $\frac{N(N+1)}{2}$, but in general does not give tight equiangular frames.

A large number of exact-valued UNETFs and equiangular line sets have been constructed by Janet Tremain [27]. These are usually sparse and are interesting, in that they show patterns that might lead to generalization or other insights.

A search algorithm was developed in [28], which uses projections alternating between geometric and spectral conditions to solve a variety of matrix nearness problems. It converges numerically, and the authors are able to find the equiangular tight frames for $\mathbb{R}^N$ up to dimension $N = 6$.

We give examples of manually constructing UNETFs using the ideas established in the previous chapters. These lead naturally to defining a new standard position orientation of ETFs using rotations and inversions to facilitate determining existence and methods of construction. This standard position is essentially a new canonical form for the matrix representation of the frame, and leads to a discrete, finite, enumerative, and combinatoric construction algorithm that finds the UNETFs if and only if they exist, has deterministic runtime, is guaranteed to terminate, and gives exact algebraic coordinates.

## 3.1 Projecting and spherical decomposition

Let $F$ be a UNETF. If we apply a Givens rotation, then $F$ is still a UNETF, since rotations are angle preserving conformal mappings. Also, if we take inverses of points, the equiangularity condition is preserved since

$$|< f_i, f_j >| = |< -f_i, f_j >| \text{ for all } 1 \le i,\ j \le M,$$

and tightness is preserved since

$$\sum_{k \in I \setminus \{i\}} |\langle f, f_k \rangle|^2 + |\langle f, f_i \rangle|^2 = \sum_{k \in I \setminus \{i\}} |\langle f, f_k \rangle|^2 + |\langle f, -f_i \rangle|^2 .$$

Therefore, we may rotate and take inverses of $F$ without loss of generality. If $F$ is a UNETF with $M$ vectors for $\mathbb{R}^N$, then apply Givens rotations until one of the elements lies on the first coordinate axis. For instance $f_M = (1, 0, \ldots, 0) \in \mathbb{R}^{\mathbb{N}}$. Similarly, we may take the inverses of the remaining points, as needed, so that

$$< f_i, f_M >= +\theta, \text{ for all } 1 \le i \le M - 1.$$

That is, we can assume that $f_1, \ldots, f_{M-1}$ all lie in the same upper half space as $f_M$. We will call this orientation the **standard position**, but a more general version will be given in a later section, see Definition 40 on page 57.

We saw in Lemma 28, that we can write:

$$x = \frac{(\alpha+1)(\beta-1)}{2}$$

$$y = \frac{(\alpha-1)(\beta+1)}{2}.$$

Similarly, $x$ and $y$ are determined exactly by $M$ and $N$, since these determine $\alpha$ and $\beta$. Geometrically, this means that, if we assume without loss of generality that $F$ is in standard position, the points $f_{1,} \ldots, f_{M-1}$ have the property that you may pick any one of them, and out of the remaining $M-2$ points, $x$ of them are "near", with inner product $\theta$, and $y$ of them are "far" with inner product $-\theta$.

### 3.1.1 Projecting and normalizing to get a similar problem

In general, given $\mathbb{S}^{N-1}$, the unit sphere in $\mathbb{R}^N$, with $N$ arbitrary and finite, if we fix the desired system angle $\theta$ ahead of time, and without loss of generality choose $\phi_0$ to be the north pole $(0, 0, \ldots, 0, 1)$ of $\mathbb{S}^{n-1}$, then any further points must lie on a sphere of one lower dimension with center at $(0, 0, \ldots, 0, \theta)$ and radius $\sqrt{1-\theta^2}$ (see Figure 3.6). We then want to use a projection to map this sub-sphere down to its first $N-1$ dimensions.

To start the construction of the $M$ vectors in $\mathbb{R}^N$, we will use rotations, Theorem 12, and Theorem 20. Let $\theta \in (0, 1) \subset \mathbb{R}$. We are looking for sets $F = \{\phi_i\}_{i \in I \subset \mathbb{N}}$ of points on $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ satisfying condition (1.4) that also form a tight frame for

$\mathbb{R}^N$. Define

$$\begin{aligned} M_{0,0} = & \quad \mathbb{S}^{N-1} \subset \mathbb{R}^N & (3.1) \\ \theta_{0,0} = & \quad \theta \\ \phi_0 = & \quad (0, \ldots, 0, 1) \in M_0 \\ M_{1,0}^* = & \quad \{f \in \mathbb{R}^N |\ \langle \phi_0, f \rangle = \theta_{0,0}\ and\ \|f\| = 1\} & (3.2) \end{aligned}$$

We can also define

$$C_i^+ \quad = \quad \{x \in \mathbb{R}^n \,|\, \langle \phi_i, x \rangle = \theta\,\}$$

as the **positive cone of equiangularity** about $\phi_i$.

Geometrically, $M_{1,0}^* = C_0^+ \bigcap \mathbb{S}^{N-1}$ where $C_0^+ = \{x \in \mathbb{R}^N \,|\, \langle \phi_0, x \rangle = \theta_0\,\}$ or the positive cone of equiangularity about $\phi_0$, and $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N \,|\, \|x\| = 1\,\}$ is the unit sphere in $N$-dimensional space. Without loss of generality, since $\phi_0$ is the **north pole** of $M_{0,0}$, we use the positive cone only. The absolute value in (1) makes the negative cone redundant, because for all $x$ we have that

$$|\langle \phi_0, -x \rangle| = \quad |- \langle \phi_0, x \rangle| \quad = |\langle \phi_0, x \rangle|\,.$$

Consider $P(M_{1,0}^*)$ where $I - P$ is the orthogonal projection of $\mathbb{R}^N$ onto the span of $\phi_0$. As shown in Theorem 12, the equiangularity condition corresponds to a similar condition on the projected sub-sphere, that is that:

$$|< P\phi_i, P\phi_j >| = a \text{ or } b,$$

where, in this case, $a = \theta - \theta^2$ and $b = -\theta - \theta^2$.

This means that, for an arbitrary point $P(\phi_1)$ on $P(M_{1,0}^*)$, the remaining points

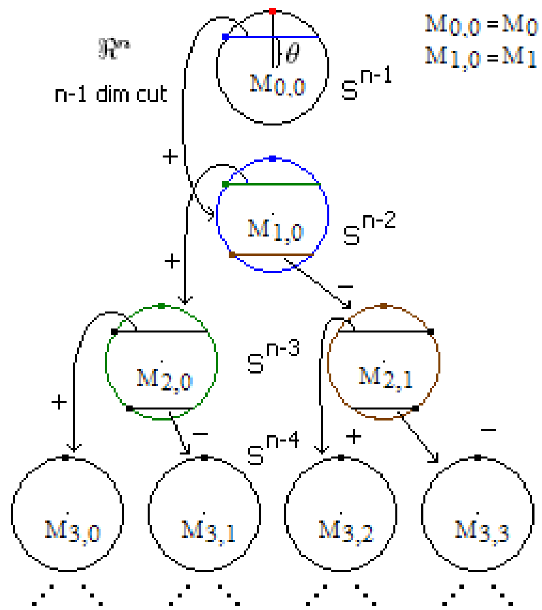



Figure 3.1: Simplified structure of possible points on the spherical decomposition

must lie on one of two spheres (of one lower dimension) corresponding to the two new inner product values $a$ and $b$. Suppose these sub-spheres are projected along the arbitrary point $P(\phi_1)$ (that is, the orthogonal projection that reduces dimension by 1 and maps $P(\phi_1)$ to the origin), and then similarly normalized to be unit spheres. We label these in a dyadic fashion, so that for a normalized sphere $M_{i,j}$, the normalized projected sub-spheres are $M_{i+1,2j}$ and $M_{i+1,2j+1}$. See Figure 3.1.

The idea is to fix some of the points correctly without loss of generality, and then lower the dimensionality of the problem and repeat until the potential for placing more points is exhausted. The downside is that constraints between multiple "cuts" must be explicitly constructed and are complex algebraically; nevertheless, this method can establish and construct norm-1 equiangular frames with the smallest possible system angle $\theta$.

### 3.1.2 Decomposing into sub-spheres of one lower dimension

As shown previously, we have reduced the main problem of finding points on the sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ to a related problem (not involving absolute values) in $\mathbb{R}^{N-1}$ where we want to find points on $\mathbb{S}^{N-2}$ satisfying

$$\langle \psi_j, \psi_k \rangle = a \ or\ b,\ for\ all\ 1 \leq j \neq k \leq M-1$$

where

$$a = \frac{\theta}{1+\theta} = f_+(\theta) \qquad b = \frac{-\theta}{1-\theta} = f_-(\theta)$$

We would like to keep reducing the dimension, and, repeating the process, we have that at each level the two new possible angles are given by a function of two variables:

**Definition 36.** For $x, y \in \mathbb{R}$, let

$$g_y(x) = \frac{x - y^2}{1 - y^2}.$$

Suppose we consider a sub-sphere $M_{k,l}$ as in Figure 3.1, and suppose we want that $\langle \phi_i, \phi_j \rangle = \theta^+_{k,l}\ or\ \theta^-_{k,l}$, for all appropriate $i \neq j$ with $\phi_i, \phi_j \in \mathbb{S}^{n-k-1} \subset \mathbb{R}^{n-k}$, where $\theta^+_{k,l}$ and $\theta^-_{k,l}$ are arbitrary in $(-1, 1)$. Looking for points on $M_{k+1,2l}$ (for example), call them $\{\psi_i\}$ with $\langle \psi_i, \psi_j \rangle = \theta^+_{k+1,2l}\ or\ \theta^-_{k+1,2l}$, does not work the same way as starting from $M_0$, because we do not have that $\left|\theta^+_{k,l}\right| = \left|\theta^-_{k,l}\right|$.

Reducing dimension the first time $M_{0,0} \mapsto M_{1,0}$, we get that the functions mapping the allowable angles $\theta^+_{k,l}$ and $\theta^-_{k,l}$, because of the absolute value in (1.4), are special cases of the more general function for mapping points $M_{k,l} \mapsto M_{k+1,2l},\ M_{k+1,2l+1}$:

$$g_y(x) = \frac{x - y^2}{1 - y^2}$$

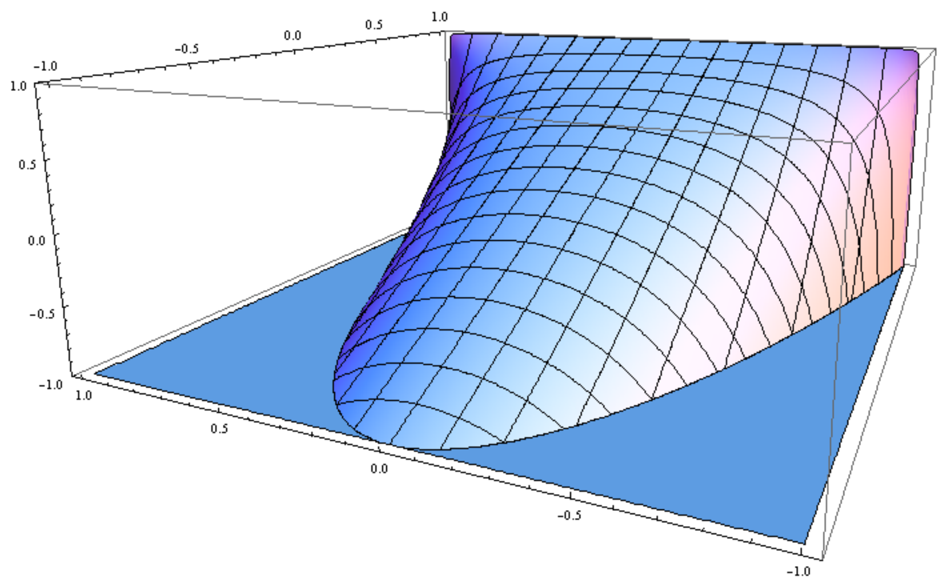



Figure 3.2: Graph of the inner product transition function $g_y(x) = \frac{x-y^2}{1-y^2}$

where $x$ is the positive or negative preimage of the inner product and y is which cut we are picking to project onto, (x and y are $\theta^+_{k,l}$ and $\theta^-_{k,l}$ of the previous $M_{k,l}$ for some $k$ and $l$.)

This can be viewed as $x$ determining the allowable angles in the new sphere, and $y$ as the amount of normalization. Dropping the $l$ for convenience (localize notation to one sphere), if we project into the positive cut (moving left on Figure 3.1), then

$$\theta^+_{k+1} = g_{\theta^+_k}(\theta^+_k)$$
$$\theta^-_{k+1} = g_{\theta^+_k}(\theta^-_k)$$

If we project onto the negative cut (move down and right on Figure 3.1), then

$$\theta^+_{k+1} = g_{\theta^-_k}(\theta^+_k)$$
$$\theta^-_{k+1} = g_{\theta^-_k}(\theta^-_k)$$

Letting $w_{i,2j}$ and $w_{i,2j+1}$ be the inner-product values corresponding to the equiangularity conditions mapped down to the $M_{i,j}$, we can find a recursive definition for the inner-product values corresponding to a lower-dimensional sphere as depicted in the tree from Figure 3.1.

**Definition 37.** Let $w_{0,0} = +\theta$, $w_{0,1} = -\theta$, and

$$w_{i,j} = g_{w_{i-1,\lfloor \frac{j}{2} \rfloor}}(w_{i-1,2\lfloor \frac{j}{4} \rfloor + j \, mod \, 2}) \tag{3.3}$$

Using this formula we can compute the tree of rational functions associated with the tree of sub-spheres[1]. See Figure 3.3[2]:

$$\begin{array}{ccccc} & & \theta & & -\theta \leftarrow unused \\ & g_\theta(\theta) & or & g_\theta(-\theta) & \\ g_{g_\theta(\theta)}(g_\theta(\theta)) \;\; or \;\; g_{g_\theta(\theta)}(g_\theta(-\theta)) & & & g_{g_\theta(-\theta)}(g_\theta(\theta)) \;\; or \;\; g_{g_\theta(-\theta)}(g_\theta(-\theta)) & \\ g_{g_{g_\theta(\theta)}(g_\theta(\theta))}(g_{g_\theta(\theta)}(g_\theta(\theta))) \;\; or \;\; g_{g_{g_\theta(\theta)}(g_\theta(\theta))}(g_{g_\theta(\theta)}(g_\theta(-\theta))) \;\; \ldots \iddots & & & & \ddots \end{array}$$

or

$$\begin{array}{cccc} & w_{0,0} = +\theta & & w_{0,1} = -\theta \leftarrow unused \\ w_{1,0} = g_{w_{0,0}}(w_{0,0}) & & w_{1,1} = g_{w_{0,0}}(w_{0,1}) & \\ w_{2,0} = g_{w_{1,0}}(w_{1,0}) \;\; w_{2,1} = g_{w_{1,0}}(w_{1,1}) & & w_{2,2} = g_{w_{1,1}}(w_{1,0}) & w_{2,3} = g_{w_{1,1}}(w_{1,1}) \end{array}$$

with[3]:

$$g_{w_{0,0}}(w_{0,0}) = \tfrac{\theta}{\theta+1}$$

$$g_{w_{0,0}}(w_{0,1}) = \tfrac{\theta}{\theta-1}$$

[1] recall that, without loss of generality, $-\theta$ has no branches below it

[2] computed by Mathematica

[3] computed by hand

$$g_{w_{1,0}}(w_{1,0}) = \tfrac{\theta}{2\theta+1}$$

$$g_{w_{1,0}}(w_{1,1}) = \tfrac{\theta(1+3\theta)}{(\theta-1)(1+2\theta)}$$

$$g_{w_{1,1}}(w_{1,0}) = \tfrac{\theta(1-3\theta)}{(\theta+1)(1-2\theta)}$$

$$g_{w_{1,1}}(w_{1,1}) = \tfrac{\theta}{2\theta-1}$$

$$\begin{array}{cccccc} & & +\theta & & -\theta \leftarrow \textit{unused} & \\ & \dfrac{\theta}{\theta+1} & & \dfrac{\theta}{\theta-1} & & \\ \dfrac{\theta}{2\theta+1} & \dfrac{\theta(1+3\theta)}{(\theta-1)(1+2\theta)} & & \dfrac{\theta(1-3\theta)}{(\theta+1)(1-2\theta)} & & \dfrac{\theta}{2\theta-1}. \end{array}$$

Each pair of two produces four at the next level down. Equivalently, each problem statement in $M_{k,l}$ produces two new problem statements, one each in $M_{k+1,2l}$ and $M_{k+1,2l+1}$.

$$\begin{array}{c} \theta \\ \left\{\frac{\theta}{\theta+1}, \frac{\theta}{\theta-1}\right\} \\ \left\{\frac{\theta}{2\theta+1}, \frac{\theta(3\theta+1)}{(\theta-1)(2\theta+1)}, \frac{\theta(3\theta-1)}{(\theta+1)(2\theta-1)}, \frac{\theta}{2\theta-1}\right\} \\ \left\{\frac{\theta}{3\theta+1}, \frac{\theta(5\theta+1)}{(\theta-1)(3\theta+1)}, \frac{\theta\left(7\theta^2+2\theta-1\right)}{(\theta+1)(5\theta^2-1)}, \frac{\theta(3\theta+1)}{5\theta^2-1}, \frac{\theta(3\theta-1)}{5\theta^2-1}, \frac{\theta\left(7\theta^2-2\theta-1\right)}{(\theta-1)(5\theta^2-1)}, \frac{\theta(5\theta-1)}{(\theta+1)(3\theta-1)}, \frac{\theta}{3\theta-1}\right\} \\ \left\{\frac{\theta}{4\theta+1}, \frac{\theta(7\theta+1)}{(\theta-1)(4\theta+1)}, \frac{\theta\left(11\theta^2+2\theta-1\right)}{(\theta+1)(8\theta^2-\theta-1)}, \frac{\theta(5\theta+1)}{8\theta^2-\theta-1}, \frac{\theta\left(7\theta^2+2\theta-1\right)}{(3\theta+1)(4\theta^2+\theta-1)}, \frac{\theta\left(17\theta^3+5\theta^2-5\theta-1\right)}{(\theta-1)(3\theta+1)(4\theta^2+\theta-1)}, \frac{\theta\left(13\theta^2+4\theta-1\right)}{(\theta+1)(8\theta^2+\theta-1)}, \frac{\theta(3\theta+1)}{8\theta^2+\theta-1}, \right. \\ \left. \frac{\theta(3\theta-1)}{8\theta^2-\theta-1}, \frac{\theta\left(13\theta^2-4\theta-1\right)}{(\theta-1)(8\theta^2-\theta-1)}, \frac{\theta\left(17\theta^3-5\theta^2-5\theta+1\right)}{(\theta+1)(3\theta-1)(4\theta^2-\theta-1)}, \frac{\theta\left(7\theta^2-2\theta-1\right)}{(3\theta-1)(4\theta^2-\theta-1)}, \frac{\theta(5\theta-1)}{8\theta^2+\theta-1}, \frac{\theta\left(11\theta^2-2\theta-1\right)}{(\theta-1)(8\theta^2+\theta-1)}, \frac{\theta(7\theta-1)}{(\theta+1)(4\theta-1)}, \frac{\theta}{4\theta-1}\right\} \end{array}$$

Figure 3.3: Evaluation of $w_{i,j}|_\theta$ for $i \in \{0, \ldots 4\}$ and $j \in \{0, \ldots, 2^i - 1\}$

notice the pattern up and down the outer diagonals. The pattern of the left diagonal, for instance, is $\left\{\theta, \frac{\theta}{\theta+1}, \frac{\theta}{2\theta+1}, \frac{\theta}{3\theta+1}, \ldots\right\}$ and hints at more structure than we are currently using.

**Lemma 38.** $w_{k,0} = \frac{\theta}{k\theta+1}$

*Proof.* It is easy to prove the formula with induction. First, $w_{0,0} = \frac{\theta}{0\cdot\theta+1} = \theta$, which is true by definition. Next, if we assume $w_{k-1,0} = \frac{\theta}{(k-1)\theta+1}$, then

$$\begin{aligned}
w_{k,0} &= g_{w_{k-1,0}}(w_{k-1,0}) \quad (defn.\ of\ w_{i,j}) \\
&= \frac{w_{k-1,0} - w_{k-1,0}^2}{1 - w_{k-1,0}^2} \quad (defn.\ of\ g) \\
&= \frac{w_{k-1,0}}{w_{k-1,0} + 1} \\
&= \frac{\frac{\theta}{(k-1)\theta+1}}{\frac{\theta}{(k-1)\theta+1} + 1} \\
&= \frac{\frac{\theta}{(k-1)\theta+1}}{\frac{\theta}{(k-1)\theta+1} + \frac{(k-1)\theta+1}{(k-1)\theta+1}} \\
&= \frac{\frac{\theta}{(k-1)\theta+1}}{\frac{k\theta+1}{(k-1)\theta+1}} \\
&= \frac{\theta}{k\theta + 1}.
\end{aligned}$$

□

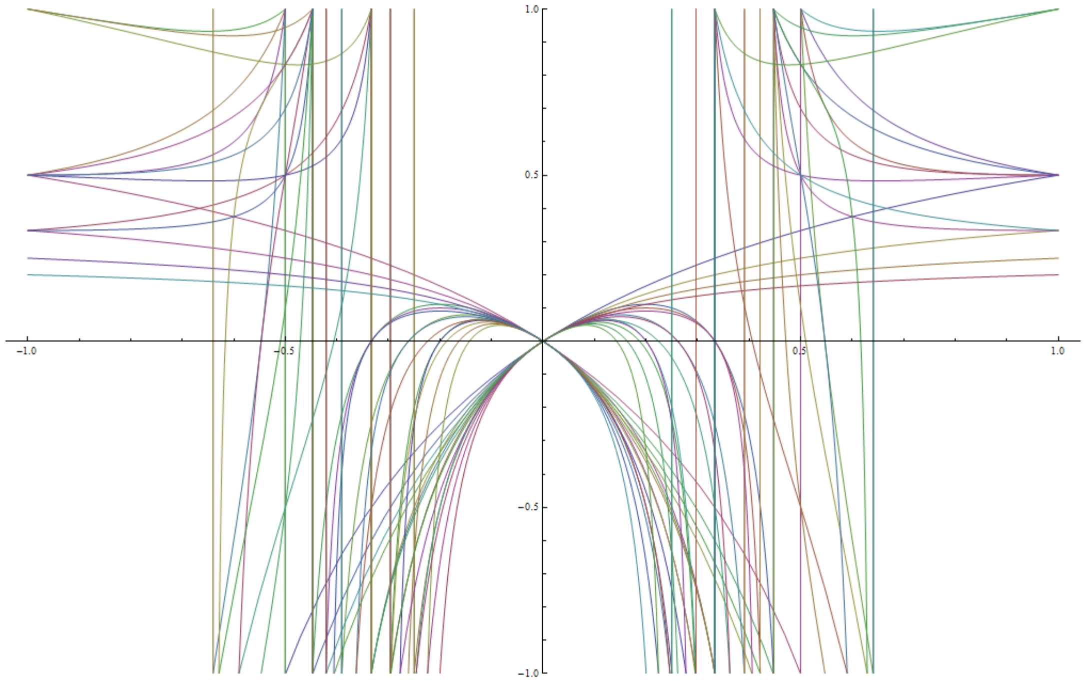



Figure 3.4: Plot of the rational functions from Figure 3.3

The tree structure of inner products is similar to Pascal's triangle, in that each entry depends on the two entries immediately above it (except the ones on the ends).

As an example, take the case of the maximal number of lines for $\mathbb{R}^6$ (which do not actually exist), so that $N = 6$, and $M = 21$. For the largest $M = \frac{N(N+1)}{2}$ the optimal angle is $\theta = \sqrt{\frac{M-N}{N(M-1)}} = \sqrt{\frac{\frac{N(N+1)}{2}-N}{N(\frac{N(N+1)}{2}-1)}} = \sqrt{\frac{\frac{N^2+N}{2}-\frac{2N}{2}}{\frac{N^2(N+1)}{2}-\frac{2N}{2}}} = \sqrt{\frac{N^2-N}{N^3+N^2-2N}} = \sqrt{\frac{N(N-1)}{N(N^2+N-2)}} = \sqrt{\frac{N(N-1)}{N(N+2)(N-1)}} = \sqrt{\frac{1}{(N+2)}}$. We get that $\theta = \frac{1}{2\sqrt{2}}$. Substituting into the above, we get the tree of inner products:

$$\{0.261204, -0.546918\}$$
$$\{0.207107, -0.660189, -0.0540971, -1.20711\}$$
$$\{0.171573, -0.734591, -0.405463, -1.94281, -0.057191, -1.21358, 3.30602, 5.82843\}$$
$$\{0.146447, -0.787201, -0.799456, -2.76777, -0.681981, -2.52179, 1.50656, 2.06066,$$
$$-0.0606602, -1.22085, 3.23608, 5.68198, 0.767767, 0.513742, 0.930058, 0.853553\},$$

and the same, but in angular degrees:

$$\{74.8585, 123.156\}$$
$$\{78.0471, 131.314, 93.101, 180. - 36.2668i\}$$
$$\{80.1207, 137.273, 113.92, 180. - 73.5271i, 93.2786, 180. - 36.8111i, 0. + 106.868i, 0. + 140.286i\}$$
$$\{81.5789, 141.925, 143.078, 180. - 96.0752i, 132.999, 180. - 90.3131i, 0. + 55.4778i, 0. + 77.4234i,$$
$$93.4777, 180. - 37.4109i, 0. + 105.581i, 0. + 138.805i, 39.8462, 59.0866, 21.5561, 31.3997\}$$

Notice that many of the values in the first listing are outside of $(-1, 1)$. If a value is outside of $(-1, 1)$, then that also eliminates all of its sub-spheres (in the sense of Figure 3.1) from consideration, even if their required inner product values are inside of the interval $(-1, 1)$.

## 3.2 Examples

### 3.2.1 3 equiangular lines in $\mathbb{R}^2$

We will construct the 3 vectors for the UNETF in $\mathbb{R}^2$. By Gerzon's Theorem, we know that 3 is the maximal number of lines or vectors in $\mathbb{R}^2$. Without loss of generality, let $\phi_0 = (0, 1)$ in $\mathbb{R}^2$, let $\phi_1$ and $\phi_2$ be in the same half-space as $\phi_0$, and suppose we want the system angle to be equal to an arbitrary value $\theta \in (0, 1) \subset \mathbb{R}$. Then we have that the last coordinate for $\phi_1$ and $\phi_2$ must be $\theta$. That is, set

$$\begin{aligned}\phi_1 &= (\phi_{1,1}, \theta) \\ \phi_2 &= (\phi_{2,1}, \theta),\end{aligned}$$

so that $\phi_1$ and $\phi_2$ are both equiangular to $\phi_0$.

Now, since $\phi_1$ and $\phi_2$ are unit-norm, we have that $\phi_{1,1}^2 + \theta^2 = 1$ and $\phi_{2,1}^2 + \theta^2 = 1$. This implies that

$$\begin{aligned}\phi_{1,1} &= \pm\sqrt{1-\theta^2} \\ \phi_{2,1} &= \pm\sqrt{1-\theta^2}.\end{aligned}$$

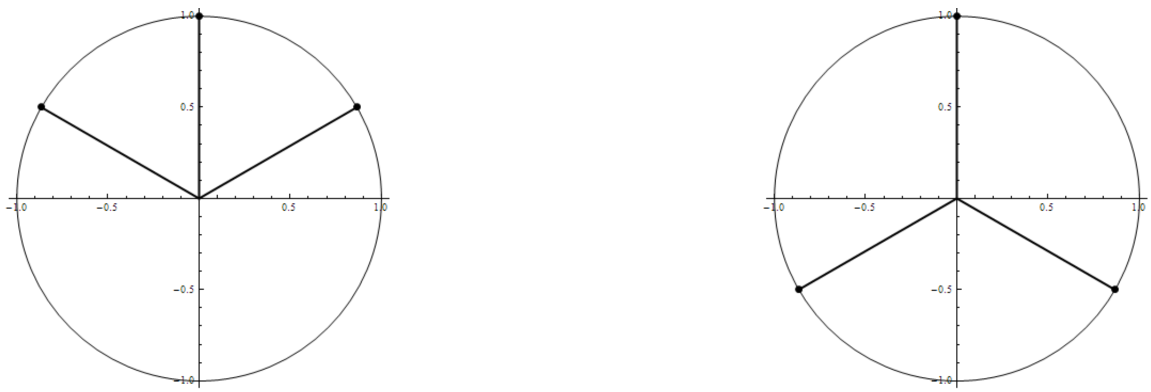
Since, $\phi_1$ and $\phi_2$ must be distinct, we choose without loss of generality that

$$\begin{aligned}\phi_{1,1} &= +\sqrt{1-\theta^2} \\ \phi_{2,1} &= -\sqrt{1-\theta^2}.\end{aligned}$$

Next, for the three vectors to be equiangular, we must have that $<\phi_1, \phi_2>= \theta$ or $-\theta$. However, this means that

$$\begin{aligned}\phi_{1,1} \cdot \phi_{2,1} + \theta^2 &= \left(\sqrt{1-\theta^2}\right)\left(-\sqrt{1-\theta^2}\right) + \theta^2 \\ &= -1 + 2\theta^2 \\ &= \theta \text{ or } -\theta.\end{aligned}$$

First, consider the case $-1 + 2\theta^2 = \theta$. This is impossible, because this quadratic equation has two real solutions 1 and $-\frac{1}{2}$, both of which are out of bounds, as $\theta \in (0, 1)$. Second, consider the case $-1 + 2\theta^2 = -\theta$. This quadratic equation has two real solutions $-1$ and $\frac{1}{2}$. Only the latter is allowable, and so it is the only



(a) Our constructed vectors

(b) Traditional Mercedes-Benz symbol representation

Figure 3.5: 3 vectors forming the equiangular tight frame for $\mathbb{R}^2$

choice that remains. That is, by applying the unit-norm and equiangular conditions, we have eliminated the possibilities so that we are left with the solution $\theta = \frac{1}{2}$ as the system angle, and this gives the first coordinates $\phi_{1,1} = \sqrt{1-\theta^2} = \frac{\sqrt{3}}{2}$ and $\phi_{2,1} = -\sqrt{1-\theta^2} = -\frac{\sqrt{3}}{2}$. The 3 vectors are:

$$\begin{aligned} \phi_0 &= (0,1) \\ \phi_1 &= (\frac{\sqrt{3}}{2}, \theta) \\ \phi_2 &= (-\frac{\sqrt{3}}{2}, \theta). \end{aligned}$$

The set must also form a tight frame by Theorem 10 on page 14, and so we have constructed the 3 vectors forming a unit-norm equiangular tight frame for $\mathbb{R}^2$. In matrix form we have:

$$\begin{pmatrix} 0 & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

We can also take inverses of the last two points, to recover the traditional Mercedes-Benz symbol as a representative of this equivalence class (under rotations and inversions) of equiangular tight frames. See Figure 3.5 on the preceding page.

### 3.2.2 6 equiangular lines in $\mathbb{R}^3$

To start the construction, of the 6 equiangular lines in $\mathbb{R}^3$ using rotations and Theorem 12, let $\theta \in (0,1) \subset \mathbb{R}$. We are looking for sets $F = \{\phi_i\}_{i \in I \subset \mathbb{N}}$ of norm-1 vectors in $\mathbb{R}^N$ satisfying condition (1.4).

**Example 39.** 6 in $\mathbb{R}^3$. Lemma 12 can be used to analytically find the six vectors in $\mathbb{R}^3$ that satisfy (1.4), i.e. they are mutually equiangular in the sense that the absolute values of all pairwise inner products are equal to the same $\theta \in (0,1)$. We find the correct $\theta$ analytically.

Let $\phi_0 = (0,0,1)$ and $\theta \in (0,1)$. Next consider the points on $M^*_{1,0}$ (see 3.1.1 on page 39) and set $\phi_1 = (0, \sqrt{1-\theta^2}, \theta)$ and set $P$ such that $I - P$ is the orthogonal projection onto the span of $\phi_0$. Now consider the projected sphere $P(M^*_{1,0})$ where $P(\phi_1)$ is the relative north pole of the sub-sphere, a circle (see Figure 3.6). Define $\theta_1$ as the inner product of the elements of $M_{1,0}$. By Lemma 12,

$$\theta_1 \quad = \quad \pm\theta - \theta^2.$$

Without loss of generality, let $\phi_1, \ldots, \phi_5$ be oriented counter-clockwise starting with $\phi_1$ at the north pole in $M_{1,0}$. Next define $\psi_i = P(\phi_i)$ for $i \in \{1, \ldots, 5\}$ and corresponding $\phi_1, \ldots, \phi_5$ on $M^*_{1,0}$. It is also required that the distances between adjacent elements on the circle $P(M^*_{1,0})$ must be the same, because under rotation any other point on $P(M^*_{1,0})$ can be considered the relative north pole for purposes of computing the correct inner product between elements, and "angle (not inner product) between adjacent elements being equal" is a *sufficient condition* for this. If $\alpha_{i,j}$ is the angle

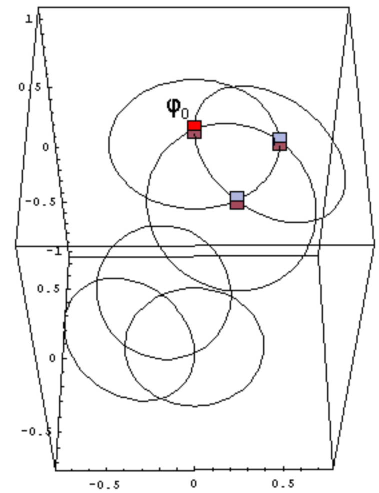



Figure 3.6: Three equiangular points on $\mathbb{S}^2 \subset \mathbb{R}^3$ with $\phi_0 = (0, 0, 1)$

between $\psi_i$ and $\psi_j$, the equidistance condition is equivalent to saying that

$$2\alpha_{1,2} \quad = \quad \alpha_{1,3}.$$

Using the formula

$$\cos(\alpha_{i,j}) = \frac{< \psi_i, \psi_j >}{\|\psi_i\|_2 \, \|\psi_j\|_2}, \tag{3.4}$$

we get that

$$\begin{aligned} \alpha_{1,2} &= \arccos\left(\frac{< \psi_1, \psi_2 >}{\|\psi_1\|_2 \, \|\psi_2\|_2}\right) \\ &= \arccos\left(\frac{\theta - \theta^2}{1 - \theta^2}\right) \\ &= \arccos\left(\frac{\theta}{1 - \theta}\right), \end{aligned}$$

and similarly,

$$\alpha_{1,3} \quad = \quad \arccos\left(\frac{-\theta}{1 - \theta}\right).$$

Combining we get an equation for $\theta$ which constrains the vectors so that they are all equiangular to $\phi_0$ , and each other, by the Lemma 12:

$$\arccos\left(\frac{-\theta}{1-\theta}\right) \quad = \quad 2\arccos\left(\frac{\theta}{1+\theta}\right)$$

Compare the graphs of these functions to see that a numerical algorithm will converge quickly.

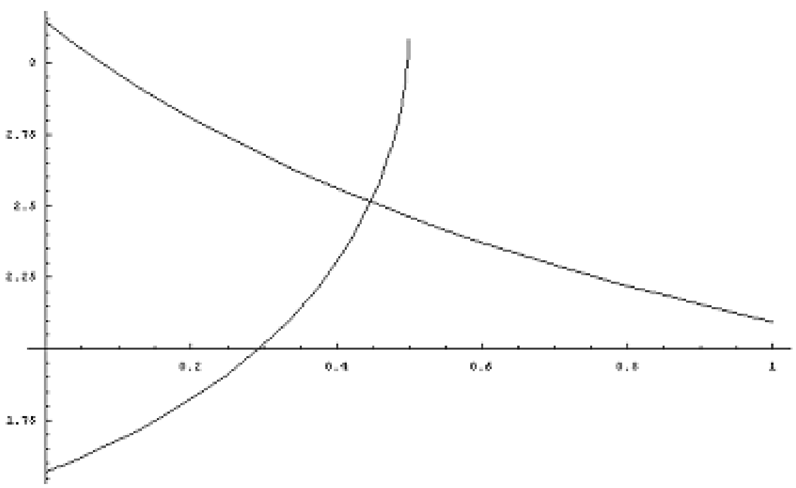



Figure 3.7: Graph of $\arccos\left(\frac{-\theta}{1-\theta}\right)$ and $2\arccos\left(\frac{\theta}{1+\theta}\right)$

Using the identity $\cos(2x) = 2\cos(x) - 1$, we can solve for $\theta$ analytically:

$$\begin{aligned}
\left(\frac{-\theta}{1-\theta}\right) &= \cos\left(2 \cdot \arccos\left(\frac{\theta}{1+\theta}\right)\right) \\
&= 2\cos^2\left(\arccos\left(\frac{\theta}{1+\theta}\right)\right) - 1 \\
&= 2\left(\frac{\theta}{1+\theta}\right)^2 - 1 \\
&= \frac{2\theta^2}{(1+\theta)^2} - \frac{(1+\theta)^2}{(1+\theta)^2} \\
&= \frac{\theta^2 - 2\theta - 1}{(1+\theta)^2}.
\end{aligned}$$

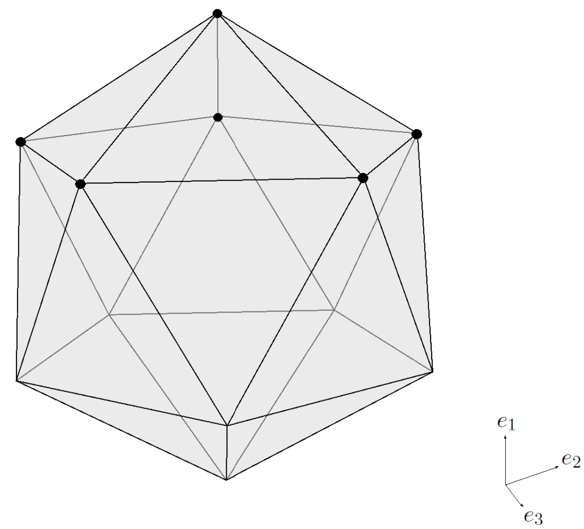



Figure 3.8: Six equiangular points on the icosahedron

Equivalently,

$$\frac{-\theta\,(1+\theta)^2}{(1-\theta)\,(1+\theta)^2} = \frac{(1-\theta)\,(\theta^2-2\theta-1)}{(1-\theta)\,(1+\theta)^2}$$

$$\Longleftrightarrow$$

$$-\theta - 2\theta^2 - \theta^3 = -\theta^3 + 3\theta^2 - \theta - 1$$

$$\Longleftrightarrow$$

$$5\theta^2 - 1 = 0.$$

Solving for $\theta$ yields a unique positive value $\frac{1}{\sqrt{5}} \approx 0.447214$, in which case $F$ is tight and equiangular by condition (1.5). The points given by $F \bigcup \{-f | f \in F\}$ correspond to the points of an icosahedron inscribed in the unit sphere, as shown in Figure 3.8.

The exact coordinates, with respect to the standard basis, are:

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{10}\left(5-\sqrt{5}\right) & \sqrt{\frac{1}{10}\left(5+\sqrt{5}\right)} \\ \frac{1}{\sqrt{5}} & \frac{1}{10}\left(5-\sqrt{5}\right) & -\sqrt{\frac{1}{10}\left(5+\sqrt{5}\right)} \\ \frac{1}{\sqrt{5}} & \frac{1}{10}\left(-5-\sqrt{5}\right) & \sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} \\ \frac{1}{\sqrt{5}} & \frac{1}{10}\left(-5-\sqrt{5}\right) & -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} \end{pmatrix}$$

This demonstrates the construction of 3 and 6 equiangular lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, which are the maximum possible as given by equation (1.6). A paper by Benedetto and Kolesar [2] does this in another way. The maximum number of equiangular lines for $\mathbb{R}^2$ and $\mathbb{R}^3$ was first determined by Haantjes [14] in the setting of elliptic geometry.

#### 3.2.2.1 A simpler method

Another way, using the methods outlined above, is to start off by using the fact that if $F$ is a UNETF with 6 vectors in $\mathbb{R}^3$ we know that $\theta = \sqrt{\frac{M-N}{N(M-1)}} = \frac{1}{\sqrt{5}}$. By Theorem 12 on page 15, we know the inner products on the projected normalized sub-sphere $M_{1,0}$. That is, if $\psi_i$ and $\psi_j$ are arbitrary points on $M_{1,0}$ corresponding to vectors of $F$, then we must have that

$$\langle \psi_i, \psi_j \rangle = \frac{\theta}{1+\theta} \quad \text{or} \quad \frac{-\theta}{1-\theta} = \frac{1}{1+\sqrt{5}} \quad \text{or} \quad \frac{1}{1-\sqrt{5}},$$

which correspond to 72 and 144 degrees respectively. Since $2N = M$, we know by Corollary 21 on page 27 that $\alpha = \beta = \sqrt{5}$. By Lemma 28 on page 30, we can determine that $x$ and $y$ are both 2. Now since $M_{1,0}$ is 2-dimensional, the sub-spheres corresponding to each inner-product value, $M_{2,0}$ and $M_{2,1}$ are both 1-dimensional, containing a maximum of two possible points. This gives the possible layout of the 6 points: the first one chosen arbitrarily in $M_{0,0}$, the second one chosen arbitrarily in $M_{1,0}$, and two sets of two possible points on $M_{2,0}$ and $M_{2,1}$.

Because we know the radius and inner product of the highest dimensional sphere, we can determine the radius, inner product, and location of the sub-spheres, in an iterative fashion. We can then check that the given points are all mutually equiangular and form a tight frame, which they do.

### 3.2.3 28 equiangular lines in $\mathbb{R}^7$

The UNETF consisting of 28 vectors in $\mathbb{R}^N$ has the nice property that for most of the sub-spheres in its decomposition, the points lying on that (projected) sub-sphere all sum to zero. This lets us characterize the UNETF and give the exact layout up to the rotations between the sub-spheres. Later we use a more advanced technique to give the exact coordinates. See Figure 3.9 on the following page to see the completed diagram, with the placement of all 28 points given up to scaling and rotations. For all the details, see Appendix A on page 71.

This does not give the relative orientation (via rotations) of sibling nodes (spheres). For example, with $M_{5,0}$ and $M_{5,1}$ we have to apply Lemma 44, and check that the lengths of the sums are equal (have to undo normalization to pull back up into $M_{4,0}$), and that the directions of the sums are opposite, which gives the rotations.

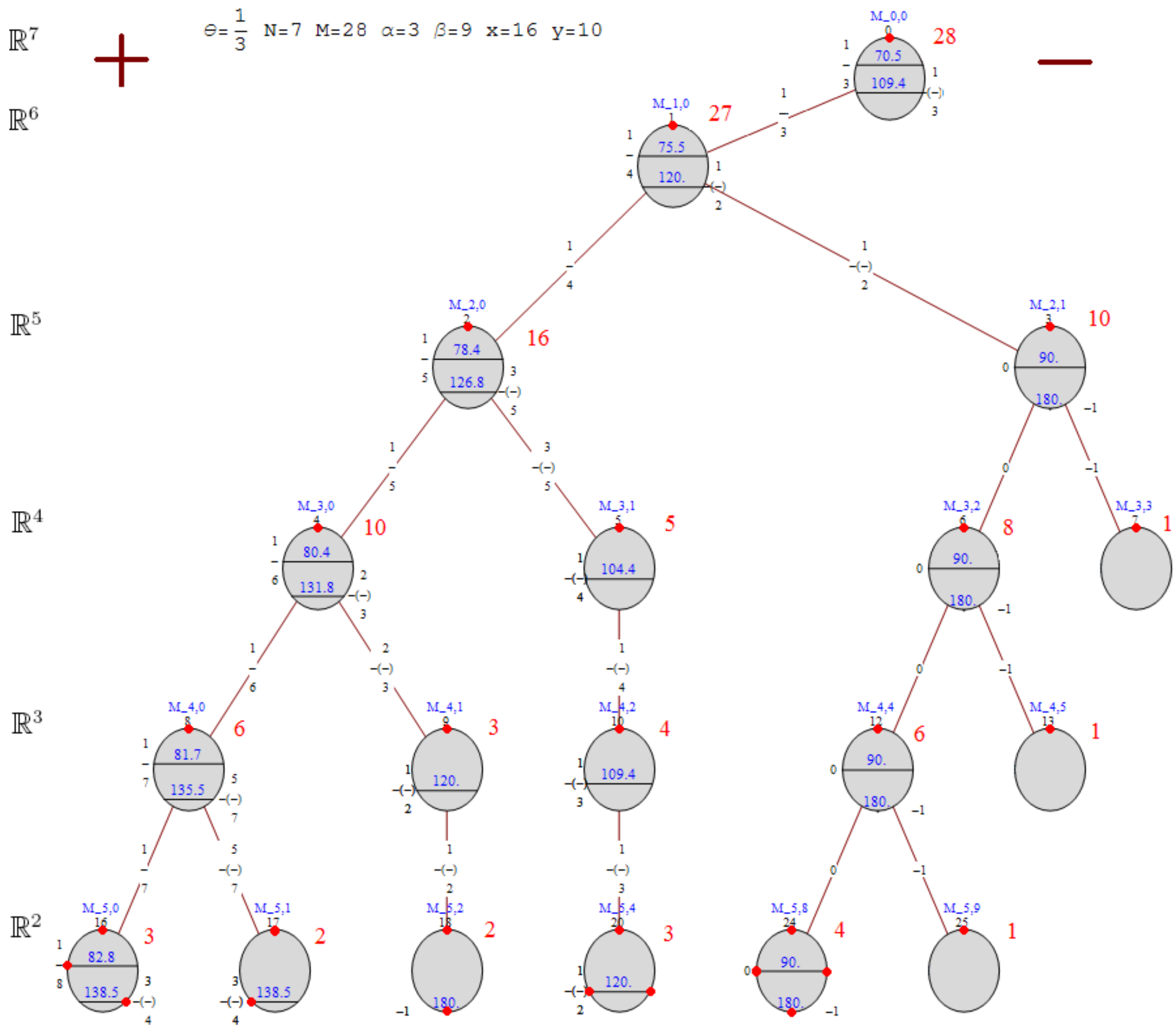



Figure 3.9: Completed diagram of normalized sub-spheres showing relative layout (up to rotations) of the 28 points forming a UNETF for $\mathbb{R}^7$

## 3.3 A type of canonical form

We continue the process of applying rotations to align the UNETF into a favorable position.

**Definition 40.** [Standard Position] Let $F = \{\phi_i\}_{i=1}^{M}$ be a UNETF with $M$ vectors for $\mathbb{R}^N$ and system angle $\theta$. Without loss of generality, we may apply a series of planar Givens rotations (or other types) so that the first vector $\phi_1$ aligns with the first element of the standard basis for $\mathbb{R}^N$, $e_1$. We have shown that the remaining

$M-1$ elements must lie on a sphere of one lower dimension, with radius $\sqrt{1-\theta^2}$. The projected, normalized version of this sphere is labeled $M_{1,0}$. Now any rotations we apply in dimensions $2,\ldots,N$ will leave $\phi_0$ fixed. Therefore, we are free to choose one of the $M-1$ points, call it $\phi_2$ on $M_{1,0}$ and apply rotations so that it lines up with the second element of the standard basis, $e_2$. That is, on the projected sub-sphere $M_{1,0}$, we apply rotations so that $\phi_2$ lies on the axis corresponding to $e_2$.

Continuing, since we have chosen $\phi_2$, we get two new sub-spheres of again one lower dimension, see Figure 3.1 on page 41. In this case, $M_{2,0}$ and $M_{2,1}$ are the projected and normalized versions of these sub-spheres. Again we are free to choose an arbitrary point $\phi_3$, and we choose, in order of priority from the cut corresponding to the most positive valued inner product of the numerically lowest index $j$ out of all the $M_{2,j}$, from where there are actually points (there may not always be points on all sub-spheres). This process can always succeed, because since we have that $M > N$ and we can only reduce dimension $N-1$ times, there will always be points remaining on at least one of the sub-spheres. After choosing $\phi_3$, again apply rotations in dimensions $3,\ldots,N$ so that $\phi_3$ aligns with $e_3$.

Repeat this process $N-1$ times, each time choosing the first available point to be $\phi_i$ (with respect to positive inner product and numerically low valued $j$ out of the $M_{i,j}$), and applying rotations in the unused dimensions to align that point with the standard basis element $e_i$. We call this alignment the **standard position**.

The standard position yields a matrix representation for $F$ where the first $N$ rows of the $N$ by $M$ matrix form a lower triangular matrix. Using this idea, and the preceding sections, we can now recursively define a new canonical form for $F$:

Let $F = \{\phi_i\}_{i=1}^M$ be a UNETF for $\mathbb{R}^N$ with $|\langle \phi_i, \phi_j \rangle| = \theta$ for $1 \le i \ne j \le M$. Let $r_1 = 1$, and $s_{1,1} = \theta$. Define the block matrices $B_{i,j}$:

$$
\begin{aligned}
B_{0,1} &= \begin{bmatrix} r_1 & 0 \\ s_{1,1} & B_{1,1} \\ s_{1,2} & B_{1,2} \end{bmatrix} \\
B_{1,1} &= \begin{bmatrix} r_2 & 0 \\ s_{2,1} & B_{2,1} \\ s_{2,2} & B_{2,2} \end{bmatrix} \\
B_{2,1} &= \begin{bmatrix} r_3 & 0 \\ s_{3,1} & B_{3,1} \\ s_{3,2} & B_{3,2} \end{bmatrix} \\
B_{2,2} &= \begin{bmatrix} s_{3,3} & B_{3,3} \\ s_{3,4} & B_{3,4} \end{bmatrix} \\
B_{3,1} &= \begin{bmatrix} r_4 & 0 \\ s_{4,1} & B_{4,1} \\ s_{4,2} & B_{4,2} \end{bmatrix} \\
B_{3,2} &= \begin{bmatrix} s_{4,3} & B_{4,3} \\ s_{4,4} & B_{4,4} \end{bmatrix} \\
B_{3,3} &= \begin{bmatrix} s_{4,5} & B_{4,5} \\ s_{4,6} & B_{4,6} \end{bmatrix} \\
B_{3,4} &= \begin{bmatrix} s_{4,7} & B_{4,7} \\ s_{4,8} & B_{4,8} \end{bmatrix} \\
\vdots \quad & \vdots \quad \vdots \\
B_{N-1,2^{N-2}} &= \begin{bmatrix} s_{4,7} \\ s_{4,8} \end{bmatrix}
\end{aligned}
$$

$$r_1 = 1,\ s_{1,1} = s_{1,2} = \theta$$

$$r_2 = \sqrt{r_1 - s_{1,1}^2}, \qquad s_{2,1} = w_{2,1} \cdot r_2 = \frac{\theta_{2,1}}{r_2} \qquad s_{2,2} = w_{2,2} \cdot r_2 = \frac{\theta_{2,2}}{r_2}$$

$$r_{i+1} = \sqrt{r_i - s_{i,\star}^2}, \qquad s_{i+1,j} = w_{i+1,j} \cdot r_{i+1} = \frac{\theta_{i+1,j}}{r_{i+1}},$$

where $\star$ corresponds to the first non-empty sub-sphere on that level, with the ordering as given in the definition of standard position.

Identifying the elements $\phi_i$ of $F$ as row vectors of a matrix, we can write

$$F = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_M \end{bmatrix} = B_{0,1}$$

$$B_{i,1} = \begin{bmatrix} \underbrace{(r_{i+1}, 0, \dots, 0)}_{N-i} \\ (s_{i+1,1}) \times B_{i+1,1} \\ (s_{i+1,2}) \times B_{i+1,2} \end{bmatrix}$$

$$B_{i,j>1} = \begin{bmatrix} (s_{i+1,2j-1}) \times B_{i+1,2j-1} \\ (s_{i+1,2j}) \times B_{i+1,2j} \end{bmatrix}$$

Example for $N = 5$:

$$B_{0,1} = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ s_{1,1} & r_2 & 0 & 0 & 0 \\ s_{1,1} & s_{2,1} & r_3 & 0 & 0 \\ s_{1,1} & s_{2,1} & s_{3,1} & r_4 & 0 \\ s_{1,1} & s_{2,1} & s_{3,1} & s_{4,1} & B_{4,1} \\ s_{1,1} & s_{2,1} & s_{3,1} & s_{4,2} & B_{4,2} \\ s_{1,1} & s_{2,1} & s_{3,2} & s_{4,3} & B_{4,3} \\ s_{1,1} & s_{2,1} & s_{3,2} & s_{4,4} & B_{4,4} \\ s_{1,1} & s_{2,2} & s_{3,3} & s_{4,5} & B_{4,5} \\ s_{1,1} & s_{2,2} & s_{3,3} & s_{4,6} & B_{4,6} \\ s_{1,1} & s_{2,2} & s_{3,4} & s_{4,7} & B_{4,7} \\ s_{1,1} & s_{2,2} & s_{3,4} & s_{4,8} & B_{4,8} \end{bmatrix}$$

$$
B_{0,1} = \begin{bmatrix}
1 & \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} \\
\theta & \begin{matrix} r_2 & 0 & 0 & 0 \end{matrix} \\
\begin{matrix} \theta \\ \theta \\ \theta \\ \theta \\ \theta \\ \theta \end{matrix} & x\left\{ \begin{bmatrix}
s_{2,1} & r_3 & 0 & 0 \\
s_{2,1} & s_{3,1} & r_4 & 0 \\
s_{2,1} & s_{3,1} & s_{4,1} & B_{4,1} \\
s_{2,1} & s_{3,1} & s_{4,2} & B_{4,2} \\
s_{2,1} & s_{3,2} & s_{4,3} & B_{4,3} \\
s_{2,1} & s_{3,2} & s_{4,4} & B_{4,4}
\end{bmatrix} \right. \\
\begin{matrix} \theta \\ \theta \\ \theta \\ \theta \end{matrix} & y\left\{ \begin{bmatrix}
s_{2,2} & s_{3,3} & s_{4,5} & B_{4,5} \\
s_{2,2} & s_{3,3} & s_{4,6} & B_{4,6} \\
s_{2,2} & s_{3,4} & s_{4,7} & B_{4,7} \\
s_{2,2} & s_{3,4} & s_{4,8} & B_{4,8}
\end{bmatrix} \right.
\end{bmatrix}
$$

## 3.4 Construction algorithm

The algorithm presented in this section has the following characteristics:

- Discrete algebraic enumeration
- If the UNETF exists for a given $N$ and $M$, the algorithm is guaranteed to find it and can find all configurations matching the standard position and canonical form
- If the algorithm does not find a UNETF, then none exists for that given $N$ and $M$ with corresponding $\theta = \sqrt{\frac{M-N}{N(M-1)}}$

- Fixed runtime
- Guaranteed to terminate
- Combinatorial growth in runtime with respect to $N$

The basic idea is to suppose a UNETF exists for a given $N$ and $M$, and apply rotations to the standard position, while using dimension-reducing iterative projections which yield a tree structure of sub-spheres, the tightness property, and knowledge of the required inner-products at each sub-sphere in order to allow for equiangularity in the full space. It turns out that the points can only be distributed finitely many ways, which can be counted by solving Diophantine equations. By enumerating through the solutions at each level, and projecting and repeating the process for each solution, all possible distributions of points can be examined in recursive manner. When there is a solution for each level of dimension-reduction, with each solution relating to an equation given by the solution for one higher dimension, the points are tested to see if they form a UNETF. This method finds all solutions (distributions of points corresponding to a UNETF) fitting the standard form, up to rotations and inversions, and if it fails, it guarantees that the UNETF does not exist for that choice of $N$ and $M$. This method produces the exact algebraic coordinates, with respect to the standard basis, of the UNETF in our standard form.

### 3.4.1 Algorithm outline

Note: in this algorithm description, the index for $i$ in subscript for $r_{i,j}$ is offset by 1 from the index used to describe the canonical form: $r_{i,j}$ in the algorithm is equal to $r_{i+1,j}$ in the canonical form.

I. Let $N$ and $M$ be candidate values for some $F$ to be a UNETF with $M$ vectors in $\mathbb{R}^N$. This fixes $\theta$, $\alpha$, $x$, and $y$.

II. Let $\theta_{i,j}$ be the inner products on the unnormalized and non-projected sub-spheres, required for equiangularity in the full space. Note that the inner products on the normalized projected sub-spheres is given by $w_{i,j}$ from section 3.1.1. Let $r_{i,j}$ be the radius of the unnormalized sub-sphere corresponding to $M_{i,j}$, and let $r_i$ be the radius of the chosen (in the sense of the definition of standard position) sub-sphere having dimension $N-i$. Let $x_{i,j}$ be the number of points on $M_{i,j}$. Let $s_{i,j}$ be the coordinate of the unnormalized nonprojected sub-sphere corresponding to $M_{i,j}$, in the direction of $e_i$, inside of $M_{i-1,\lfloor \frac{j}{2} \rfloor}$. In general, all these constants are exactly determined by the constants from one-higher dimension, except the $x_{i,j}$ which are found by setting up and solving equations.

III. Starting at dimension $N$, rotate one point into standard position and setup the equations to determine the possible distributions of points for the next lower dimension.

1. Dimension $= N$. We have $r_{0,0} = 1$. Set $\theta_{0,0} = \theta$, and $\theta_{0,1} = -\theta$. Immediately $x_{1,0} = M - 1$, and $s_{1,0} = \theta$. This determines the first column of $F$.

2. Dimension $= N - 1$. We have $r_{1,0} = \sqrt{1 - \theta_{0,0}^2}$, $\theta_{1,0} = w_{1,0} \cdot r_{1,0}^2$, and $\theta_{1,1} = w_{1,1} \cdot r_{1,0}^2$. Also, we compute the coordinate values $s_{2,0} = w_{2,0} \cdot r_{1,0}$ and $s_{2,1} = w_{2,1} \cdot r_{1,0}$. Because the points sum to zero by Theorem 20, we get the equation $r_{1,0} + s_{2,0} \cdot x_{2,0} + s_{2,1} \cdot x_{2,1} = 0$, which corresponds to summing the 2nd column of $F$ to 0. However, we also have the equation $x_{2,0} + x_{2,1} = M - 2$. This is the same as $x + y = M - 2$ in the notation of Chapter 2. Now there are two equations and two unknowns so we can solve exactly for $x_{2,0}$ and $x_{2,1}$ and we now know the number of points on the sub-sphere corresponding to $M_{2,0}$ and $M_{2,1}$. Proceed to the next step; if we return to this step, the algorithm has failed.

3. Dimension $= N - 2$. Determine $r_{2,0,}, r_{2,1},\ \theta_{3,0}, \ldots, \theta_{3,3}$, and $s_{3,0}, \ldots, s_{3,3}$. Because the points of $M_{1,0}$ must sum to zero, the coordinates of the matrix must sum to zero column-wise. Because of the standard position, we can setup the equations[4]:

$$r_{2,0} + s_{3,0} \cdot x_{3,0} + \cdots + s_{3,3} \cdot x_{3,3} \quad = \quad 0$$
$$x_{3,0} + x_{3,1} = x_{2,0} - 1 \quad \text{and} \quad x_{3,2} + x_{3,3} = x_{2,1}.$$

Now there are more unknowns than equations and so we cannot solve explicitly. However, subject to the constraints that $0 \leq x_{3,0} \leq x_{2,0} - 1$, $0 \leq x_{3,1} \leq x_{2,0} - 1$, $0 \leq x_{3,2} \leq x_{2,1}$, $0 \leq x_{3,3} \leq x_{2,1}$, and that $x_{3,0}, \ldots, x_{3,3}$ must be integers, this forms a set of Diophantine equations, for which there are finitely many solutions $\{x_{3,0}, \ldots, x_{3,3}\}$. For each solution, assume that it is the correct solution and proceed to the next step. If that step fails, then try the next step again with the next solution. If all solutions fail, this step fails, and so return to the previous step.

$$\vdots$$

4. Dimension $= N-(i-1)$. Determine $r_{i-1,0}, \ldots, r_{i-1,2^{i-2}-1}, \theta_{i,0}, \ldots, \theta_{i,2^{i-1}-1}$, and $s_{i,0}, \ldots, s_{i,2^{i-1}-1}$. Let $r_{i-1} = r_{i-1,\star}$ where $\star$ is the index corresponding to the first nonempty sub-sphere in the sense of standard position. We get equations[5]:

$$r_{i-1} + \sum_{j=0}^{2^i-1} s_{i,j} \cdot x_{i,j} \quad = \quad 0$$
$$x_{i,j} + x_{i,j+1} = x_{i-1,\lfloor \frac{j}{2} \rfloor} - \delta(k, \left\lfloor \frac{j}{2} \right\rfloor) \quad \text{for all} \quad j \in \{0, 2, 4, \ldots, 2^{i-1} - 2\}.$$

[4]where the term $x_{2,0} - 1$ has the $-1$ because one of the points is represented by $r_{2,0}$ and was chosen without loss of generality in the sense of the standard position

[5]define $\delta(k, l) = 1$ if $k = l$, and 0 otherwise

Solve these equations subject to the constraints that $0 \leq x_{i,j} \leq x_{i-1,\lfloor \frac{j}{2} \rfloor} - \delta(k,j)$ for all $0 \leq j \leq 2^{i-1} - 1$, and that $x_{i,0}, \ldots, x_{i,2^{i-1}-1}$ are integers. For each solution, assume that it is the correct solution and proceed to the next step. If that step fails, then try the next step again with the next solution. If all solutions fail, this step fails, and so return to the previous step.

$\vdots$

5. Dimension = 2. Then $i - 1 = N - 2$ and $i = N - 1$. As above, determine $r_{i-1,0}, \ldots, r_{i-1,2^{i-2}-1}$, $\theta_{i,0}, \ldots, \theta_{i,2^{i-1}-1}$, and $s_{i,0}, \ldots, s_{i,2^{i-1}-1}$. Let $r_{i-1} = r_{i-1,k}$ where $k$ is the index corresponding to the first nonempty sub-sphere in the sense of standard position. We get equations[6]:

$$\begin{aligned} r_{i-1} + \sum_{j=0}^{2^i-1} s_{i,j} \cdot x_{i,j} &= 0 \\ x_{i,j} + x_{i,j+1} = x_{i-1,\lfloor \frac{j}{2} \rfloor} - \delta(k, \left\lfloor \frac{j}{2} \right\rfloor) &\text{ for all } \quad j \in \{0, 2, 4, \ldots, 2^{i-1} - 2\}. \end{aligned}$$

Solve these equations subject to the constraints that $0 \leq x_{i,j} \leq x_{i-1,\lfloor \frac{j}{2} \rfloor} - \delta(k,j)$ for all $0 \leq j \leq 2^{i-1} - 1$, and that $x_{i,0}, \ldots, x_{i,2^{i-1}-1}$ are integers. Heuristic optimization can happen here, since, for example, the next sub-spheres are 1-dimensional, and so they can have at most 2 points. Therefore, any solutions with more than 2 points on a sub-sphere can be discarded. Each solution at this level, along with the current solution for all the previous levels, forms a candidate solution, that may correspond to a UNETF. If all solutions fail, this step fails, and so return to the previous step.

IV. When a candidate solution is found, check that it is a UNETF by forming the

[6] define $\delta(k,l) = 1$ if $k = l$, and 0 otherwise

matrix for $F$ and checking the entries of the Gramian $G = FF^T$. If it is 1 on the diagonal, and $\pm\theta$ everywhere else, then $F$ is unit-norm and equiangular. By the conditions on $N$ and $M$, the set of vectors is also a tight frame.

V. If no candidates are found, or every candidate fails to be a UNETF, then there is no UNETF for the choice of $N$ and $M$.

There are many ways to optimize the above algorithm. For instance, solutions with $x_{i-1,j} > 1$ and $r_{i,j} = 0$ can be discarded because $r_{i,j} = 0$ indicates that $M_{i,j}$ is a 0-dimensional sphere, and contains at most 1 point. There are many other possible optimizations.

Some of the initial results for low dimensions are listed in matrix form in Appendix B.

### 3.4.2 Example of 10 points forming a UNETF for $\mathbb{R}^5$

Let $N = 5$ and $M = 10$. If $F$ is a UNETF for $\mathbb{R}^5$ with 10 vectors, then $\theta = \frac{1}{3}$.

1. Dimension = 5. $\theta_{0,0} = \frac{1}{3}$, $\theta_{0,1} = -\frac{1}{3}$, $x_{0,0} = M$. We get that $x_{1,0} = M - 1 = 9$.

2. Dimension = 4. $r_{1,0} = \frac{2\sqrt{2}}{3}$, $\theta_{1,0} = \frac{2}{9}$, $\theta_{1,1} = -\frac{4}{9}$, $x_{1,0} = M - 1$. Next $w_{2,0} = \frac{1}{4}$ and $w_{2,1} = -\frac{1}{2}$. $s_{2,0} = \frac{1}{3\sqrt{2}}$ and $s_{2,1} = -\frac{\sqrt{2}}{3}$. Solve

$$\begin{aligned} r_{1,0} + s_{2,0} \cdot x_{2,0} + s_{2,1} \cdot x_{2,1} &= 0 \\ &\text{or} \\ \frac{2\sqrt{2}}{3} + \frac{1}{3\sqrt{2}} x_{2,0} - \frac{\sqrt{2}}{3} x_{2,1} &= 0 \end{aligned}$$

along with $x_{2,0} + x_{2,1} = 9 - 1$ and the constraints on the domain of $x_{2,0}$ and $x_{2,1}$, to get that $x_{2,0} = x_{2,1} = 4$.

3. Dimension $= 3$. $r_{2,0} = \sqrt{\frac{5}{6}}$ and $r_{2,1} = \sqrt{\frac{2}{3}}$. Compute $\theta_{2,0} = \frac{1}{6}$, $\theta_{2,1} = -\frac{1}{2}$, $\theta_{2,2} = \frac{1}{3}$, $\theta_{2,3} = -\frac{1}{3}$, $s_{3,0} = \frac{1}{\sqrt{30}}$, $s_{3,1} = -\sqrt{\frac{3}{10}}$, $s_{3,2} = \sqrt{\frac{2}{15}}$, $s_{3,3} = -\sqrt{\frac{2}{15}}$. Solve

$$\begin{aligned} r_{2,0} + s_{3,0} \cdot x_{3,0} + s_{3,1} \cdot x_{3,1} + s_{3,2} \cdot x_{3,2} + s_{3,3} \cdot x_{3,3} &= 0 \\ &\text{or} \\ \sqrt{\frac{5}{6}} + \frac{1}{\sqrt{30}} x_{3,0} - \sqrt{\frac{3}{10}} x_{3,1} + \sqrt{\frac{2}{15}} x_{3,2} - \sqrt{\frac{2}{15}} x_{3,3} &= 0 \end{aligned}$$

along with the other constraints. There are four possible solutions, and we have that $(x_{3,0}, x_{3,1}, x_{3,2}, x_{3,3}) \in$

$$\{(0,3,3,1),\ (1,2,2,2),\ (2,1,1,3),\ (3,0,0,4)\}.$$

Trying the solutions in order, they all fail at subsequent levels, except the last one, so that $(x_{3,0}, x_{3,1}, x_{3,2}, x_{3,3}) = (3,0,0,4)$ is the only remaining solution.

4. Dimension $= 2$. Doing as above, we find that there are six possible solutions, with the only correct one being

$$(x_{4,1}, \ldots, x_{4,8}) = (0,0,0,2,0,2,2,0).$$

At this point, there are at most two points on the next sub-spheres, which correspond to spheres of dimension 1, such that there are only two possible points, which also agrees with the solution. We can now compute the next $s_{i,j}$ values, and write down the coordinates in the canonical form, since we know the coordinates at each level, and the number of points per level (the $x_{i,j}$):

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3\sqrt{2}} & \sqrt{\frac{5}{6}} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

5. Checking the Gramian of $F$, we see that it is a UNETF.

## 3.5 Future work

The research presented has revealed some new types of structure in equiangular tight frames. There are several directions of research arising from this work which should be explored further.

First, the projection method for analyzing equiangular tight frames in lower dimensions has established a link between the geometry of the frame and the spectrum of the associated Gramian and signature matrices. Several important results, such as Theorem 4 and Theorem 7, have come from analyzing the eigenvalues of the signature matrix. With this additional structure, some new results may be reached.

Second, the structure elucidated in both Chapter 2 and Chapter 3 hints at techniques for determining the existence of equiangular tight frames, and alternative methods for constructing them explicitly. In fact, we have already developed several more efficient construction algorithms, based on quite different techniques, which may be presented in a later paper.

Finally, equiangular tight frames can be studied in geometric setting. If $F$ is a UNETF for $\mathbb{R}^N$, then $F \cup -F$ forms a polytope for $\mathbb{R}^N$. We have made some progress in determining the generalized $f$-vector, which counts the $i$-faces of the polytope. Combining the research presented with standard geometric techniques may yield new insights into the existence of equiangular tight frames.

# Appendix A

# Layout of the UNETF with 28 Points in $\mathbb{R}^7$

## A.1 Equiangular tight frames with $\theta = \frac{1}{3}$

Equiangular tight frames correspond to a binary tree decomposition (partial ordering via subsets) of the sphere into embedded sub-spheres, with all but the lowest dimension containing two possible sub-spheres of one less dimension and associated angular criteria.

### A.1.1 Visualizing 28 points forming a UNETF for $\mathbb{R}^7$

The diagram in Figure A.1 shows a breakdown of $\mathbb{S}^6$ into sub-spheres, containing the 28 points of the UNETF at $\theta = \frac{1}{3}$ spanning $\mathbb{R}^7$. The topmost sphere at node 0, labeled $M_{0,0}$ corresponds to the unit sphere in $\mathbb{R}^7$ containing the 28 points, with one point $\phi_0$ oriented at the relative north-pole[1], and the remainder oriented in the upper-half space[2] (all without loss of generality via rotations and inversions). Therefore, the remaining points $\phi_i$ lie on a smaller sphere of radius $\sqrt{1-\theta^2}$

[1]relative to the last dimension, i.e. (0,0,0,0,0,0,1)
[2]again, relative to the last dimension, i.e. $\{x \in \mathbb{S}^7 | \langle x, \phi_0 \rangle \geq 0\}$

embedded at *latitude* $\theta$ (with respect to $\phi_0$ being in the north-pole position), as their 7th coordinates all must be $\theta$ since $|\langle \phi_i, \phi_0 \rangle| = \theta$.

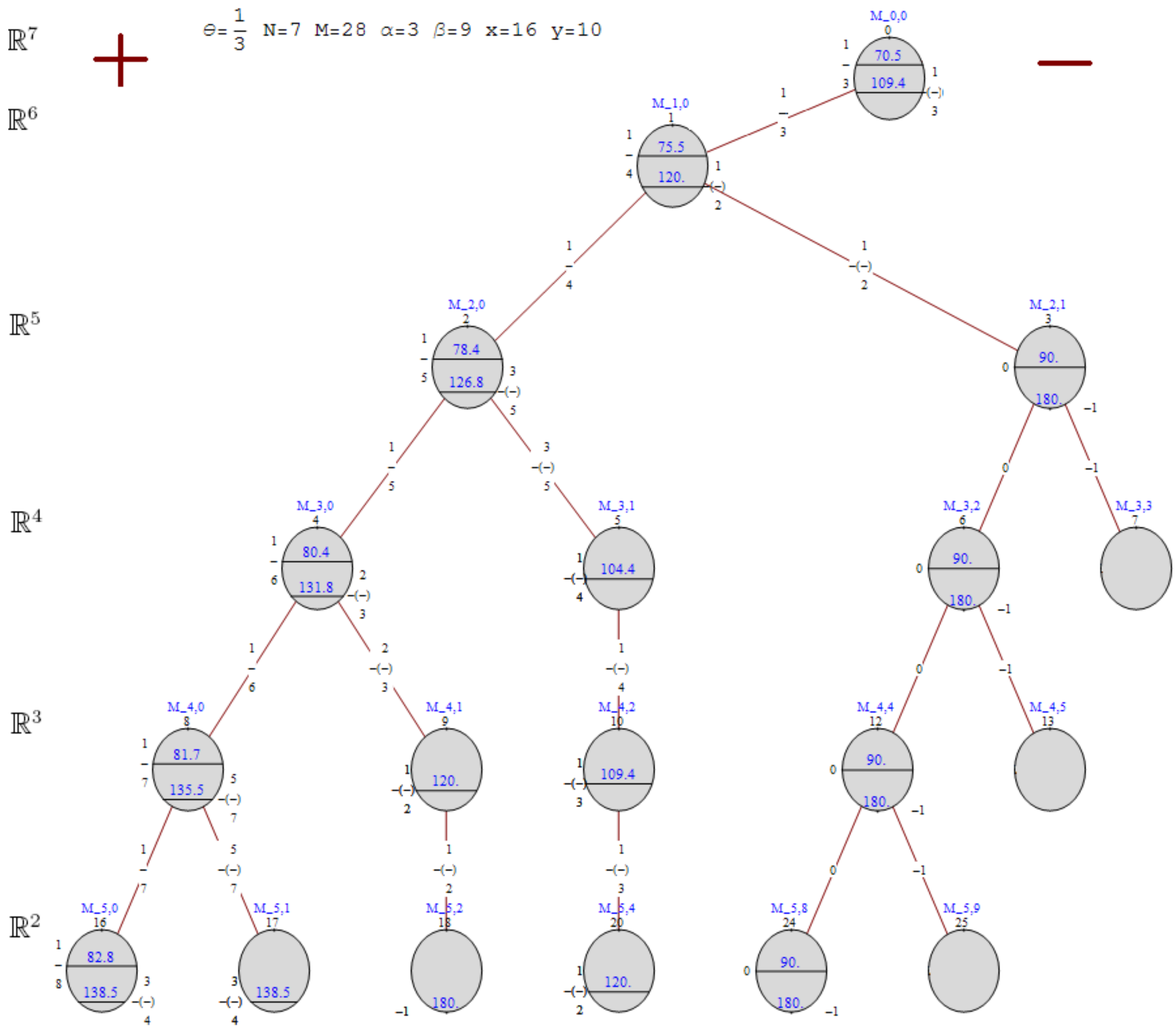


Figure A.1: Angular decomposition into a tree of normalized projected spheres

Consider the projected and normalized version of this sphere, spanning one-less dimension and that is node 1 in Figure A.1, also labeled $M_{1,0}$. Inside this there are two possible values for the inner product, they are labeled, along with the corresponding angles relative to an arbitrary point (one of the $\phi_i \in F$ rotated to a relative[3] north pole).

[3]in the sense that multiple nodes at the same level may be out of rotation with each other, but points inside any particular node (and its child nodes) are equiangular to the north pole of that node

Now considering $M_{1,0}$, once we pick a point $\phi_1$ and assume (without loss of equiangularity) it to be the relative north pole, we immediately have two angles and two sub-spheres upon which all of the remaining points must lie; that is, if $\psi_i = \frac{P(\phi_i)}{\|P(\phi_i)\|}$, then for $j \in \{2, .., M\}$ we have that $\langle \psi_j, \psi_i \rangle = a$ or $b$. After projecting along the natural projection given by $\phi_1$, and normalizing, we get two spheres $M_{2,0}$ and $M_{2,1}$ with some unknown orientation between the two.

Repeating the process gives the rest of the tree. It ends when the sphere is $\mathbb{S}^0$ or when the inner products go out of bounds via the transition functions

$$(a, b) \overset{+}{\longmapsto} (g_a(a),\ g_a(b))$$

$$(a, b) \overset{-}{\longmapsto} (g_b(a),\ g_b(b))$$

with

$$g_x(y) = \frac{x - y^2}{1 - y^2}$$

The + mapping corresponds to a transition down and to the left in the diagram, and the − mapping corresponds to a transition down and to the right (the more negative value).

### A.1.2 Explanation and filling-in of the diagram

We want to use the idea that the points of $M_{1,0}$ sum to zero. We know already that we can count the points on the sub-sphere since, by applying part (3) of Theorem 20 on page 24:

$$\begin{aligned} 0 &= 1 + x\left(\frac{1}{4}\right) + y\left(-\frac{1}{2}\right) \\ 28 &= 2 + x + y \end{aligned}$$

which gives

$$x = 16$$
$$y = 10$$

**Proposition 41.** *In addition to node $M_{1,0}$ having the property that its points sum to zero, the following sub-spheres also have that property: $M_{2,1}$, $M_{3,2}$, $M_{4,4}$, $M_{5,8}$, $M_{2,0}$, $M_{3,1}$, $M_{4,2}$, $M_{5,4}$, $M_{3,0}$, $M_{4,1}$, $M_{5,2}$, and $M_{4,0}$. The nodes $M_{3,3}$, $M_{4,5}$, and $M_{5,9}$ also sum to zero (they are a single point at zero). The nodes $M_{5,0}$ and $M_{5,1}$ do not sum to zero, but are contained in $M_{4,0}$ which does.*

*Proof.* First note that $M_{3,3}$, $M_{4,5}$, and $M_{5,9}$ are spheres of radius zero, a single point. They each correspond to the sub-sphere of the parent sphere which is at 180 degrees (inner product -1) from the relative north pole of that parent, which shows radius zero. The other inner product value for $M_{3,3}$, $M_{4,5}$, and $M_{5,9}$ is $\frac{0-1}{1-1} = \frac{-1}{0}$, which is why there is only the one point, on a sphere of radius 0. The rest of the proof will be with a series of lemmas. □

**Lemma 42.** *The 10 points on $M_{2,1}$ are exactly specified (up to rotation) and they add up to zero in $M_{2,1}$.*

*Proof.* Next consider that there are exactly 10 possible points on $M_{2,1}$, by looking at the sub-spheres. One way is to start at the bottom. Since $M_{5,8}$ is $\mathbb{S}^1$ in $\mathbb{R}^2$, there are a maximum of four possible points, and they can all four exist mutually satisfying the required relative angles regardless of which of the 4 is situated as the relative north pole. These four add to zero. Its sibling $M_{5,9}$ is only one point, so now in $M_{4,4}$ there is: the relative north pole, the *south pole* at 180 degrees from it in the diagram in Figure A.1, and the four points of $M_{5,9}$ lie along the equator. Six total points, the four points add to zero, and the two new points also add to zero; the points on $M_{4,4}$ add to zero. Similarly the points on $M_{3,2}$ and $M_{2,1}$ add to zero, and continuing to add points we see there are 10 total points possible, and only 10, so this must be all of them. □

**Lemma 43.** *Since the points on $M_{2,1}$ add to zero, so do the points on $M_{2,0}$. Since the points on $M_{1,0}$ add to zero, as shown in part (2) of Theorem 20.*

A generalization of this is:

**Lemma 44.** *If the points on $M_{i,j}$ add to zero for some $i$ and $j$, then the sum vector of the points on $M_{i+1,2j}$ has the same length (except they may have been normalized by different constants) and opposite orientation (about the origin) when compared to the sum vector of the points on $M_{i+1,2j+1}$.*

Now that we have that the points on $M_{2,0}$ adding to zero, we can repeat the idea of counting the number of points on the sub-spheres.

If $M_{i,j}$ is one of the sub-spheres with possible inner products $a$ and $b$ with $a > b$. Let $x_{i,j}$ be the number of points (if it is well defined) on the sub-sphere corresponding to $a$, and $y_{i,j}$ the number of points on the sub-sphere corresponding to b. This gives our normal $x = x_{1,0}$ and $y = y_{1,0}$.

For $M_{2,0}$ we solve:

$$0 = 1 + x_{2,0}\left(\frac{1}{5}\right) + y_{2,0}\left(-\frac{3}{5}\right)$$

$$16 = 1 + x_{2,0} + y_{2,0}$$

to get

$$x_{2,0} = 10 \text{ and } y_{2,0} = 5.$$

**Lemma 45.** *$M_{3,0}$ and $M_{3,1}$ add up to zero.*

*Proof.* Similar to Lemma 43, we start at the bottom of the right-hand side, at $M_{5,4}$. Because the angle is 120 degrees, all three possible points can exist, since it looks the same under rotation by 120 degrees, and they sum to zero. We can also check that it adds to zero in the direction of the north pole (vertical direction). Ex: $0 = 1 + \frac{-1}{2} + \frac{-1}{2}$, adding the y-coordinates of each point. Similarly for $M_{4,2}$ the three points on the *cut* at angle 109.4 degrees means that the three points that contribute $\frac{-1}{3}$ cancels out the one point at north-pole that contributes +1; they add up to zero. Similarly, $M_{3,1}$ has one point at the north-pole, four points on the sub-sphere, and they add to zero. But this means $M_{3,0}$ must also add to zero. □

For $M_{3,0}$ we solve:

$$0 = 1 + x_{3,0}\left(\frac{1}{6}\right) + y_{3,0}\left(-\frac{2}{3}\right)$$

$$10 = 1 + x_{3,0} + y_{3,0}$$

to get

$$x_{3,0} = 6 \text{ and } y_{3,0} = 3.$$

**Lemma 46.** *The points on $M_{4,0}$, $M_{4,1}$, and $M_{5,2}$ all add to zero.*

*Proof.* As previously we see $M_{5,2}$ has two possible points, both allowable by symmetry, and adding to zero. This gives that the points on $M_{4,1}$ adds to zero, and therefore the points on $M_{4,0}$ add to zero. □

For $M_{4,0}$ we solve:

$$0 = 1 + x_{4,0}\left(\frac{1}{7}\right) + y_{4,0}\left(-\frac{5}{7}\right)$$

$$6 = 1 + x_{4,0} + y_{4,0}$$

to get

$$x_{4,0} = 3 \text{ and } y_{4,0} = 2.$$

At most two points can fit on $M_{5,1}$ (the third does not fit because it is not at angle 138.5 degrees to the first two). Three points can fit on $M_{5,0}$ (see Figure 3.9). Partitioning the circle into degrees, $360 = 138.59 + 138.59 + 82.2$ with $138.59 \approx \arccos(\frac{-3}{4})$ and $82.2 \approx \arccos(\frac{1}{8})$, that is: the angle between points on *opposite sides* of $M_{5,0}$ works out to be rotationally symmetric. It does not work if they are on the same side (left-to-right).

**Exercise 47.** Try to visualize or compute the relative rotation between $M_{5,0}$ and $M_{5,1}$ as they sit inside of $M_{4,0}$ so that the six points on $M_{4,0}$ can add to zero.

### A.1.3 Conclusion

We have seen how the UNETF at $\alpha = 3$, $N = 7$, and $M = 28$ has a corresponding decomposition diagram with the property that for most of the projected normalized sub-spheres $M_{i,j}$ (with exception of some of the *leaf nodes* — those with no further possible points or copies of $\mathbb{S}^1$), the points on that sphere sum to zero. In general this property does not seem to hold as nicely for other angles associated with $\alpha = 5, 7, 9, \ldots$ but still is useful, and the same ideas and equations can tell where to look. It is a separate, but related, problem to find the relative rotations and merge siblings upwards into the final construction of points. This is done in the section 3.4 on page 62, and we give the matrix of coordinates determined algebraically, for the 28 vectors in standard position that are an equiangular tight frame for $\mathbb{R}^7$ (see Appendix B on the next page).

# Appendix B

# UNETF Algorithm Result Printouts

Table B.1: UNETF algorithm result printouts

$$6 \text{ in } \mathbb{R}^3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{10}\left(5-\sqrt{5}\right) & \sqrt{\frac{1}{10}\left(5+\sqrt{5}\right)} \\ \frac{1}{\sqrt{5}} & \frac{1}{10}\left(5-\sqrt{5}\right) & -\sqrt{\frac{1}{10}\left(5+\sqrt{5}\right)} \\ \frac{1}{\sqrt{5}} & \frac{1}{10}\left(-5-\sqrt{5}\right) & \sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} \\ \frac{1}{\sqrt{5}} & \frac{1}{10}\left(-5-\sqrt{5}\right) & -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} \end{pmatrix}$$

10 in $\mathbb{R}^5$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \sqrt{\frac{5}{6}} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{3}}
\end{pmatrix}$$

# 16 in $\mathbb{R}^6$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \sqrt{\frac{5}{6}} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{1}{3\sqrt{5}} & \frac{\sqrt{7}}{3} & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{1}{3\sqrt{5}} & \frac{1}{3\sqrt{7}} & \frac{4}{\sqrt{21}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & -\frac{4}{3\sqrt{5}} & -\frac{4}{3\sqrt{7}} & -\frac{2}{\sqrt{21}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & \frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{7}} & -\frac{2}{\sqrt{21}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{7}} & -\frac{2}{\sqrt{21}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{7}} & \frac{2}{\sqrt{21}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & \frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{7}} & -\frac{2}{\sqrt{21}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{7}} & -\frac{2}{\sqrt{21}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{7}} & \frac{2}{\sqrt{21}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{7}} & -\frac{2}{\sqrt{21}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{7}} & \frac{2}{\sqrt{21}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & -\frac{2}{3\sqrt{5}} & \frac{4}{3\sqrt{7}} & \frac{2}{\sqrt{21}}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \sqrt{\frac{5}{6}} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3} & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & -\frac{4}{3\sqrt{5}} & -\frac{1}{3} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & -\frac{4}{3\sqrt{5}} & -\frac{1}{3} & -\frac{1}{\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & \frac{2}{3\sqrt{5}} & \frac{2}{3} & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & \frac{2}{3\sqrt{5}} & -\frac{1}{3} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & \frac{2}{3\sqrt{5}} & -\frac{1}{3} & -\frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & \frac{2}{3\sqrt{5}} & \frac{2}{3} & 0 \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & \frac{2}{3\sqrt{5}} & -\frac{1}{3} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & \frac{2}{3\sqrt{5}} & -\frac{1}{3} & -\frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3} & -\frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & -\frac{2}{3\sqrt{5}} & -\frac{2}{3} & 0
\end{pmatrix}
$$

# 28 in $\mathbb{R}^7$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \sqrt{\frac{5}{6}} & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{1}{3\sqrt{5}} & \frac{\sqrt{7}}{3} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{1}{3\sqrt{5}} & \frac{1}{3\sqrt{7}} & \frac{4}{\sqrt{21}} & 0 \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3\sqrt{7}} & \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{1}{3\sqrt{5}} & \frac{1}{3\sqrt{7}} & \frac{1}{2\sqrt{21}} & -\frac{\sqrt{3}}{2} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{1}{3\sqrt{5}} & \frac{1}{3\sqrt{7}} & -\sqrt{\frac{3}{7}} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{1}{3\sqrt{5}} & -\frac{5}{3\sqrt{7}} & \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & \frac{1}{3\sqrt{5}} & -\frac{5}{3\sqrt{7}} & -\frac{5}{2\sqrt{21}} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3\sqrt{7}} & -\frac{5}{2\sqrt{21}} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{30}} & -\frac{4}{3\sqrt{5}} & -\frac{4}{3\sqrt{7}} & \frac{\sqrt{\frac{3}{7}}}{2} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & \frac{2}{3\sqrt{5}} & \frac{2}{3\sqrt{7}} & \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & \frac{2}{3\sqrt{5}} & \frac{2}{3\sqrt{7}} & -\frac{5}{2\sqrt{21}} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & \frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{7}} & \frac{\sqrt{\frac{3}{7}}}{2} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{7}} & \frac{\sqrt{\frac{3}{7}}}{2} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{7}} & -\frac{\sqrt{\frac{3}{7}}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & \frac{2}{3\sqrt{5}} & \frac{2}{3\sqrt{7}} & \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & \frac{2}{3\sqrt{5}} & \frac{2}{3\sqrt{7}} & -\frac{5}{2\sqrt{21}} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & \frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{7}} & \frac{\sqrt{\frac{3}{7}}}{2} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{7}} & \frac{\sqrt{\frac{3}{7}}}{2} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{7}} & -\frac{\sqrt{\frac{3}{7}}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{7}} & \frac{\sqrt{\frac{3}{7}}}{2} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{7}} & -\frac{\sqrt{\frac{3}{7}}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & -\frac{2}{3\sqrt{5}} & \frac{4}{3\sqrt{7}} & -\frac{\sqrt{\frac{3}{7}}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & -\frac{2}{3\sqrt{5}} & -\frac{2}{3\sqrt{7}} & \frac{5}{2\sqrt{21}} & \frac{1}{2\sqrt{3}} \\
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\sqrt{\frac{2}{15}} & -\frac{2}{3\sqrt{5}} & -\frac{2}{3\sqrt{7}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{3}}
\end{pmatrix}
$$

# Appendix C

# Table of Allowable $N$ and $M$ Values for UNETFs with $N \leq 1000$

This table lists values of $N$ and $M$ with $M > N+1$ for which UNETFs are allowed to exist under the best known necessary conditions. Some of them have been shown to exist (see Table 1.2), however most of them are unknown. If $N+1 < M < \frac{N(N+1)}{2}$, then we must have that:

1. If $M \neq 2N$, then $\alpha$ and $\beta$ must be odd (Theorem 7).

2. If $M = 2N$, then $N$ is an odd number and $2N-1$ is the sum of two squares (Theorem 7).

3. We must have that $\left\lceil \frac{2N+1+\sqrt{8N+1}}{2} \right\rceil \leq M \leq \frac{N(N+1)}{2}$ (by equation 2.2 of Theorem 18).

4. If $M = \frac{N(N+1)}{2}$, and there exists $m, k \in \mathbb{N}$ such that $N = (2m+1)^2 - 2$ with $m = 2k$, $k \equiv 2 \mod 3$, and both $k$ and $2k+1$ are square-free, then there is no UNETF with $M$ vectors for $\mathbb{R}^N$. Additionally, we cannot have $N = 47$ and $M = 1128$ ($m = 3$) [1] and Theorem 34.

Table C.1: Table of allowable $N$ and $M$ values for UNETFs for $N < 1000$

If $M \neq 2N$, then the value of $\alpha = \frac{1}{\theta}$ is given in subscript after $M$ in the following table. If $M = 2N$, then by equation 1.7 we always have that

$$\alpha = \frac{1}{\theta} = \sqrt{\frac{N(M-1)}{M-N}} = \sqrt{\frac{N(2N-1)}{2N-N}} = \sqrt{2N-1},$$

and so no value is given.

| $N$ | $M$ | | | | $N$ | $M$ | | |
|---|---|---|---|---|---|---|---|---|
| 3 | 6 | | | | 46 | $736_7$ | | |
| 5 | 10 | | | | 49 | 98 | | |
| 6 | $16_3$ | | | | 51 | 102 | $136_9$ | |
| 7 | 14 | $28_3$ | | | 55 | $100_{11}$ | 110 | |
| 9 | 18 | | | | 57 | $76_{15}$ | 114 | $190_9$ |
| 10 | $16_5$ | | | | 59 | 118 | | |
| 13 | 26 | | | | 61 | 122 | $244_9$ | |
| 15 | 30 | $36_5$ | | | 63 | 126 | $280_9$ | |
| 19 | 38 | $76_5$ | | | 66 | $144_{11}$ | $352_9$ | |
| 20 | $96_5$ | | | | 69 | 138 | $460_9$ | |
| 21 | $28_9$ | $36_7$ | 42 | $126_5$ | 71 | $568_9$ | | |
| 22 | $176_5$ | | | | 72 | $640_9$ | | |
| 23 | 46 | $276_5$ | | | 73 | 146 | $730_9$ | |
| 25 | 50 | | | | 75 | 150 | $1000_9$ | |
| 27 | 54 | | | | 76 | $96_{19}$ | $1216_9$ | |
| 28 | $64_7$ | | | | 77 | 154 | $210_{11}$ | $1540_9$ |
| 31 | 62 | | | | 78 | $144_{13}$ | $2080_9$ | |
| 33 | 66 | | | | 79 | 158 | | |
| 35 | $120_7$ | | | | 85 | $120_{17}$ | $136_{15}$ | 170 |
| 36 | $64_9$ | | | | 87 | 174 | | |
| 37 | 74 | $148_7$ | | | 88 | $320_{11}$ | | |
| 41 | 82 | $246_7$ | | | 91 | 182 | $196_{13}$ | $364_{11}$ |
| 42 | $288_7$ | | | | 93 | 186 | | |
| 43 | 86 | $344_7$ | | | 97 | 194 | | |
| 45 | 90 | $100_9$ | $540_7$ | | 99 | 198 | $540_{11}$ | |

| $N$ | $M$ | | | $N$ | $M$ | | |
|---|---|---|---|---|---|---|---|
| 101 | $606_{11}$ | | | 143 | $924_{13}$ | | |
| 103 | 206 | | | 145 | 290 | $406_{15}$ | |
| 105 | $126_{25}$ | $196_{15}$ | | 147 | 294 | | |
| 106 | $848_{11}$ | | | 148 | $1184_{13}$ | | |
| 109 | $1090_{11}$ | | | 153 | 306 | $324_{17}$ | |
| 110 | $1200_{11}$ | | | 154 | $176_{35}$ | | |
| 111 | $148_{21}$ | 222 | $1332_{11}$ | 155 | $496_{15}$ | $1860_{13}$ | |
| 113 | 226 | | | 156 | $2016_{13}$ | | |
| 115 | 230 | $2300_{11}$ | | 157 | 314 | $2198_{13}$ | |
| 116 | $2784_{11}$ | | | 159 | 318 | | |
| 117 | 234 | $378_{13}$ | $3510_{11}$ | 162 | $3888_{13}$ | | |
| 118 | $4720_{11}$ | | | 163 | 326 | $4564_{13}$ | |
| 119 | $7140_{11}$ | | | 165 | $616_{15}$ | $6930_{13}$ | |
| 120 | $256_{15}$ | | | 166 | $9296_{13}$ | | |
| 121 | 242 | | | 167 | 334 | $14028_{13}$ | |
| 123 | 246 | | | 169 | 338 | $676_{15}$ | |
| 127 | $508_{13}$ | | | 171 | $324_{19}$ | | |
| 129 | 258 | | | 175 | 350 | | |
| 130 | $560_{13}$ | | | 177 | 354 | $826_{15}$ | |
| 131 | 262 | | | 181 | 362 | | |
| 133 | $190_{21}$ | $210_{19}$ | 266 | 183 | $244_{27}$ | 366 | $976_{15}$ |
| 135 | 270 | | | 185 | 370 | $1036_{15}$ | |
| 136 | $256_{17}$ | | | 187 | 374 | $528_{17}$ | |
| 139 | 278 | | | 189 | 378 | | |
| 141 | 282 | $376_{15}$ | $846_{13}$ | 190 | $400_{19}$ | $1216_{15}$ | |

| $N$ | $M$ | | | $N$ | $M$ | | | |
|---|---|---|---|---|---|---|---|---|
| 195 | $924_{13}$ | | | 239 | 478 | | | |
| 197 | 290 | $406_{15}$ | | 241 | 482 | $1446_{17}$ | | |
| 199 | 294 | | | 243 | 486 | | | |
| 201 | $1184_{13}$ | | | 246 | $288_{41}$ | | | |
| 203 | 306 | $324_{17}$ | | 247 | 494 | $780_{19}$ | | |
| 204 | $176_{35}$ | | | 253 | $276_{55}$ | $484_{23}$ | 506 | $2024_{17}$ |
| 205 | $496_{15}$ | $1860_{13}$ | | 255 | 510 | $2160_{17}$ | | |
| 209 | $2016_{13}$ | | | 261 | $378_{29}$ | $406_{27}$ | 522 | |
| 210 | 314 | $2198_{13}$ | | 265 | 530 | $3180_{17}$ | | |
| 211 | 318 | | | 266 | $1008_{19}$ | | | |
| 213 | $3888_{13}$ | | | 267 | 534 | | | |
| 215 | 326 | $4564_{13}$ | | 271 | 542 | $1084_{19}$ | $4336_{17}$ | |
| 217 | $616_{15}$ | $6930_{13}$ | | 272 | $4608_{17}$ | | | |
| 218 | $9296_{13}$ | | | 273 | $364_{33}$ | 546 | $4914_{17}$ | |
| 219 | 334 | $14028_{13}$ | | 275 | 550 | | | |
| 220 | 338 | $676_{15}$ | | 276 | $576_{23}$ | $736_{21}$ | | |
| 221 | $324_{19}$ | | | 277 | $6648_{17}$ | | | |
| 222 | 350 | | | 279 | 558 | | | |
| 223 | 354 | $826_{15}$ | | 280 | $8960_{17}$ | | | |
| 225 | 362 | | | 281 | $10116_{17}$ | | | |
| 229 | $244_{27}$ | 366 | $976_{15}$ | 283 | 566 | $13584_{17}$ | | |
| 231 | 370 | $1036_{15}$ | | 285 | 570 | $1350_{19}$ | $20520_{17}$ | |
| 232 | 374 | $528_{17}$ | | 286 | $352_{39}$ | $27456_{17}$ | | |
| 235 | 378 | | | 287 | $820_{21}$ | $41328_{17}$ | | |
| 238 | $400_{19}$ | $1216_{15}$ | | 289 | 578 | | | |

| $N$ | $M$ | | | | $N$ | $M$ | | |
|---|---|---|---|---|---|---|---|---|
| 293 | 586 | | | | 346 | $8304_{19}$ | | |
| 297 | 594 | | | | 349 | 698 | $10470_{19}$ | |
| 300 | $576_{25}$ | | | | 351 | $676_{27}$ | 702 | $12636_{19}$ |
| 301 | $344_{49}$ | 602 | $946_{21}$ | $1806_{19}$ | 352 | $14080_{19}$ | | |
| 303 | 606 | | | | 355 | 710 | $21300_{19}$ | |
| 304 | $1920_{19}$ | | | | 356 | $25632_{19}$ | | |
| 307 | 614 | | | | 357 | $1870_{21}$ | $32130_{19}$ | |
| 309 | 618 | $1030_{21}$ | | | 358 | $42960_{19}$ | | |
| 313 | 626 | | | | 359 | $64620_{19}$ | | |
| 315 | 630 | | | | 363 | 726 | | |
| 316 | $2528_{19}$ | | | | 364 | $2080_{21}$ | | |
| 319 | 638 | | | | 365 | 730 | $876_{25}$ | |
| 321 | 642 | | | | 367 | 734 | | |
| 323 | $3060_{19}$ | | | | 371 | $2332_{21}$ | | |
| 325 | $676_{25}$ | $3250_{19}$ | | | 373 | 746 | | |
| 327 | 654 | | | | 375 | $2500_{21}$ | | |
| 329 | 658 | | | | 378 | $784_{27}$ | | |
| 331 | 662 | $1324_{21}$ | $3972_{19}$ | | 379 | 758 | | |
| 336 | $1408_{21}$ | | | | 381 | $508_{39}$ | 762 | $2794_{21}$ |
| 337 | 674 | | | | 383 | 766 | | |
| 339 | 678 | | | | 385 | 770 | | |
| 341 | $496_{33}$ | $528_{31}$ | $6138_{19}$ | | 386 | $3088_{21}$ | | |
| 342 | $6480_{19}$ | | | | 387 | 774 | | |
| 343 | 686 | $6860_{19}$ | | | 391 | $460_{51}$ | $1496_{23}$ | |
| 345 | 690 | $990_{23}$ | | | 393 | 786 | | |

| $N$ | $M$ | | | $N$ | $M$ | | | |
|---|---|---|---|---|---|---|---|---|
| 397 | 794 | $1588_{23}$ | $3970_{21}$ | 438 | $64240_{21}$ | | | |
| 399 | 798 | $4180_{21}$ | | 439 | 878 | | | |
| 401 | 802 | | | 441 | $540_{49}$ | 882 | $2646_{23}$ | |
| 405 | 810 | | | 451 | $616_{41}$ | 902 | | |
| 406 | $784_{29}$ | $5104_{21}$ | | 453 | 906 | | | |
| 408 | $5440_{21}$ | | | 455 | 910 | | | |
| 411 | 822 | $6028_{21}$ | | 456 | $1216_{27}$ | | | |
| 413 | $6490_{21}$ | | | 460 | $736_{35}$ | $3520_{23}$ | | |
| 415 | 830 | | | 463 | 926 | $3704_{23}$ | | |
| 417 | 834 | | | 465 | $900_{31}$ | 930 | | |
| 419 | $8380_{21}$ | | | 469 | 938 | $1876_{25}$ | | |
| 420 | $8800_{21}$ | | | 471 | 942 | | | |
| 421 | 842 | $9262_{21}$ | | 475 | 950 | $1976_{25}$ | | |
| 423 | 846 | | | 477 | 954 | $1378_{27}$ | | |
| 425 | $1326_{25}$ | | | 481 | 962 | | | |
| 426 | $12496_{21}$ | | | 483 | 966 | $5544_{23}$ | | |
| 427 | 854 | $13420_{21}$ | | 485 | $5820_{23}$ | | | |
| 429 | 858 | $15730_{21}$ | | 489 | 978 | | | |
| 430 | $560_{43}$ | $1376_{25}$ | $17200_{21}$ | 491 | 982 | | | |
| 431 | $18964_{21}$ | | | 493 | 986 | $1190_{29}$ | | |
| 433 | 866 | | | 495 | $540_{77}$ | $1540_{27}$ | $2376_{25}$ | |
| 434 | $27280_{21}$ | | | 496 | $1024_{31}$ | $7936_{23}$ | | |
| 435 | $900_{29}$ | $31900_{21}$ | | 497 | $568_{63}$ | | | |
| 436 | $38368_{21}$ | | | 499 | 998 | | | |
| 437 | 874 | $2508_{23}$ | $48070_{21}$ | 505 | $606_{55}$ | 1010 | $2626_{25}$ | $11110_{23}$ |

| $N$ | $M$ | | | | $N$ | $M$ | | |
|---|---|---|---|---|---|---|---|---|
| 506 | $64240_{21}$ | | | | 555 | 1110 | | |
| 507 | 878 | | | | 559 | 1118 | | |
| 509 | $540_{49}$ | 882 | $2646_{23}$ | | 560 | $5376_{25}$ | | |
| 511 | $616_{41}$ | 902 | | | 561 | $1156_{33}$ | | |
| 513 | 906 | | | | 563 | 1126 | | |
| 517 | 910 | | | | 565 | 1130 | $5876_{25}$ | |
| 518 | $1216_{27}$ | | | | 568 | $640_{71}$ | | |
| 519 | $736_{35}$ | $3520_{23}$ | | | 573 | 1146 | $2674_{27}$ | $6876_{25}$ |
| 521 | 926 | $3704_{23}$ | | | 575 | $7176_{25}$ | | |
| 523 | $900_{31}$ | 930 | | | 577 | 1154 | | |
| 525 | 938 | $1876_{25}$ | | | 579 | 1158 | | |
| 526 | 942 | | | | 583 | 1166 | | |
| 527 | 950 | $1976_{25}$ | | | 585 | $9126_{25}$ | | |
| 528 | 954 | $1378_{27}$ | | | 586 | $9376_{25}$ | | |
| 531 | 962 | | | | 589 | $1520_{31}$ | | |
| 533 | 966 | $5544_{23}$ | | | 591 | 1182 | | |
| 535 | $5820_{23}$ | | | | 595 | $1156_{35}$ | 1190 | $12376_{25}$ |
| 537 | 978 | | | | 597 | 1194 | | |
| 540 | 982 | | | | 599 | $14376_{25}$ | | |
| 545 | 986 | $1190_{29}$ | | | 600 | $14976_{25}$ | | |
| 547 | $540_{77}$ | $1540_{27}$ | $2376_{25}$ | | 601 | 1202 | $15626_{25}$ | |
| 549 | $1024_{31}$ | $7936_{23}$ | | | 603 | 1206 | $3484_{27}$ | |
| 550 | $568_{63}$ | | | | 605 | $18876_{25}$ | | |
| 551 | 998 | | | | 607 | 1214 | | |
| 553 | $606_{55}$ | 1010 | $2626_{25}$ | $11110_{23}$ | 609 | 1218 | | |

| $N$ | $M$ | $N$ | $M$ |
|---|---|---|---|
| 610 | $25376_{25}$ | 671 | 1342 |
| 612 | $3808_{27}$ $29376_{25}$ | 673 | 1346 |
| 613 | 1226 $31876_{25}$ | 675 | $9100_{27}$ |
| 615 | 1230 $38376_{25}$ | 677 | $9478_{27}$ |
| 617 | 1234 $48126_{25}$ | 681 | 1362 $1816_{33}$ |
| 619 | 1238 $64376_{25}$ | 685 | 1370 |
| 620 | $77376_{25}$ | 687 | 1374 $11908_{27}$ |
| 621 | 1242 $4186_{27}$ $96876_{25}$ | 689 | 1378 |
| 622 | $129376_{25}$ | 690 | $736_{105}$ $12880_{27}$ |
| 623 | $194376_{25}$ | 691 | 1382 |
| 625 | 1250 | 693 | 1386 $14014_{27}$ |
| 630 | $1296_{35}$ | 696 | $4032_{29}$ |
| 631 | 1262 $2524_{29}$ | 697 | $1190_{41}$ |
| 638 | $2640_{29}$ $5104_{27}$ | 701 | $4206_{29}$ $18226_{27}$ |
| 639 | 1278 | 702 | $18928_{27}$ |
| 643 | 1286 | 703 | 1406 $1444_{37}$ $19684_{27}$ |
| 645 | $946_{45}$ $990_{43}$ 1290 $5590_{27}$ | 705 | $846_{65}$ 1410 |
| 649 | $826_{55}$ 1298 | 707 | 1414 |
| 651 | $868_{51}$ 1302 $2016_{31}$ $6076_{27}$ | 708 | $24544_{27}$ |
| 653 | 1306 | 709 | 1418 |
| 657 | $730_{81}$ 1314 | 711 | 1422 $28756_{27}$ |
| 661 | 1322 | 713 | $2760_{31}$ |
| 663 | 1326 | 715 | $936_{55}$ 1430 $1716_{35}$ |
| 666 | $1296_{37}$ $7696_{27}$ | | *and* $2080_{33}$ $37180_{27}$ |
| 667 | $3220_{29}$ | 716 | $40096_{27}$ |

| $N$ | $M$ | | $N$ | $M$ | | |
|---|---|---|---|---|---|---|
| 717 | 1434 | $43498_{27}$ | 775 | 1550 | $4000_{31}$ | |
| 720 | $58240_{27}$ | | 777 | 1554 | | |
| 721 | $1030_{49}$ | $2884_{31}$ | 779 | 1558 | | |
| 722 | $75088_{27}$ | | 780 | $1600_{39}$ | | |
| 723 | 1446 | $87724_{27}$ | 781 | $924_{71}$ | $10934_{29}$ | |
| 725 | $5250_{29}$ | $131950_{27}$ | 783 | 1566 | $11340_{29}$ | |
| 726 | $176176_{27}$ | | 787 | 1574 | | |
| 727 | 1454 | $264628_{27}$ | 793 | $976_{65}$ | 1586 | |
| 733 | 1466 | | 799 | 1598 | $15980_{29}$ | |
| 735 | 1470 | | 801 | 1602 | $4806_{31}$ | |
| 736 | $5888_{29}$ | | 805 | 1610 | $2346_{35}$ | |
| 737 | $2278_{33}$ | | 806 | $4992_{31}$ | $19344_{29}$ | |
| 741 | $1444_{39}$ | 1482 | 807 | 1614 | | |
| 742 | $848_{77}$ | $1008_{53}$ | 811 | 1622 | $22708_{29}$ | |
| 745 | 1490 | | 812 | $23520_{29}$ | | |
| 747 | 1494 | | 813 | $1084_{57}$ | 1626 | $24390_{29}$ |
| 749 | $1926_{35}$ | | 815 | 1630 | | |
| 754 | $7280_{29}$ | | 817 | $3268_{33}$ | | |
| 757 | 1514 | $7570_{29}$ | 819 | 1638 | | |
| 759 | 1518 | | 820 | $1600_{41}$ | $32800_{29}$ | |
| 760 | $1216_{45}$ | | 821 | $34482_{29}$ | | |
| 761 | 1522 | | 825 | 1650 | $3400_{33}$ | |
| 763 | 1526 | | 826 | $46256_{29}$ | | |
| 769 | 1538 | | 827 | $49620_{29}$ | | |
| 771 | $9252_{29}$ | | 829 | 1658 | $58030_{29}$ | |

| $N$ | $M$ | | | $N$ | $M$ | | |
|---|---|---|---|---|---|---|---|
| 831 | $69804_{29}$ | | | 885 | 1770 | $3186_{35}$ | $4720_{33}$ |
| 833 | 1666 | | | 887 | 1666 | | |
| 834 | $100080_{29}$ | | | 889 | $100080_{29}$ | | |
| 835 | 1670 | $116900_{29}$ | | 891 | 1670 | $116900_{29}$ | |
| 836 | $140448_{29}$ | | | 895 | $140448_{29}$ | | |
| 837 | $6480_{31}$ | $175770_{29}$ | | 899 | $6480_{31}$ | $175770_{29}$ | |
| 838 | $234640_{29}$ | | | 901 | $234640_{29}$ | | |
| 839 | $352380_{29}$ | | | 902 | $352380_{29}$ | | |
| 841 | 1682 | $6728_{31}$ | | 903 | 1682 | $6728_{31}$ | |
| 843 | 1686 | | | 907 | 1686 | | |
| 847 | 1694 | | | 910 | 1694 | | |
| 849 | 1698 | | | 913 | 1698 | | |
| 851 | $1036_{69}$ | | | 919 | $1036_{69}$ | | |
| 855 | 1710 | | | 921 | 1710 | | |
| 859 | 1718 | | | 923 | 1718 | | |
| 861 | 1722 | $1764_{41}$ | | 925 | 1722 | $1764_{41}$ | |
| 865 | $8650_{31}$ | | | 927 | $8650_{31}$ | | |
| 867 | 1734 | | | 929 | 1734 | | |
| 868 | $2976_{35}$ | $8960_{31}$ | | 930 | $2976_{35}$ | $8960_{31}$ | |
| 869 | 1738 | | | 931 | 1738 | | |
| 871 | 1742 | | | 933 | 1742 | | |
| 873 | 1746 | | | 937 | 1746 | | |
| 877 | 1754 | | | 939 | 1754 | | |
| 881 | $10572_{31}$ | | | 941 | $10572_{31}$ | | |
| 883 | 1766 | | | 943 | 1766 | | |

| $N$ | $M$ | | |
|---|---|---|---|
| 945 | 1770 | $3186_{35}$ | $4720_{33}$ |
| 946 | $1376_{55}$ | $1936_{43}$ | $60544_{31}$ |
| 949 | $75920_{31}$ | | |
| 951 | 1902 | $2536_{39}$ | $91296_{31}$ |
| 953 | $7624_{33}$ | $114360_{31}$ | |
| 955 | $152800_{31}$ | | |
| 956 | $183552_{31}$ | | |
| 957 | 1914 | $7888_{33}$ | $229680_{31}$ |
| 958 | $306560_{31}$ | | |
| 959 | $460320_{31}$ | | |
| 961 | 1922 | | |
| 967 | 1934 | | |
| 969 | 1938 | | |
| 970 | $4656_{35}$ | | |
| 973 | 1946 | $4726_{35}$ | |
| 975 | 1950 | | |
| 981 | $1090_{99}$ | 1962 | |
| 987 | 1974 | $5076_{35}$ | $10528_{33}$ |
| 990 | $1936_{45}$ | | |
| 993 | $1324_{63}$ | 1986 | $11254_{33}$ |
| 995 | 1990 | | |
| 997 | 1994 | | |
| 999 | 1998 | | |

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# Vita

Daniel Redmond was born in Kansas City, Missouri, USA to John and Marquita Redmond. He attended St. Bernadette's elementary school and Rockhurst High School, a Jesuit preparatory school, in Kansas City, Missouri. His undergraduate education was at the University of Missouri in Rolla, where he completed a Bachelor of Science degree in computer science in December of 1999, with a minor in mathematics. During this time he worked for AT&T Global Information Solutions, Imagination-Engines, Inc., and Los Alamos National Labs T-Division. He then attended graduate school at the University of Missouri in Columbia, where he worked with Dr. Peter Casazza in the area of frame theory, and is expected to receive the Doctor of Philosophy degree in mathematics in December of 2009.