A DECOUPLED SYSTEM OF HYPERBOLIC EQUATIONS FOR LINEARIZED
COSMOLOGICAL PERTURBATIONS

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(Dated: February 7, 2008)

A decoupled system of hyperbolic partial differential equations for linear perturbations around any
spatially flat FRW universe is obtained for a wide class of perturbations. The considered perturbing
energy momentum-tensors can be expressed as the sum of the perturbation of a minimally coupled
calar field plus an arbitrary (weak) energy-momentum tensor which is covariantly conserved with
respect to the background. The key ingredient in obtaining the decoupling of the equations is the
introduction of a new covariant gauge which plays a similar role as harmonic gauge does for
perturbations around Minkowski space-time. The case of universes satisfying a linear equation of
state is discussed in particular, and closed analytic expressions for the retarded Green’s functions
solving the de Sitter, dust and radiation dominated cases are given.

PACS numbers: PACS number(s) 04.25.Nx, 98.80.Hw

1. Introduction

The theory of linearized perturbations around cosmo-
logical backgrounds is a cornerstone in the understanding
of structure formation in the universe, because it provides
the connection between what we see today, and the ini-
tial conditions in the early universe [1]. This theory has
a long history, beginning (in the context of general rel-
itivity) with the pioneering work of E. Lifshitz [2]. An
important step was the introduction by J. Bardeen of
gauge invariant potentials to describe the metric pertur-
bations [3]. This is the so called standard treatment of
cosmological perturbations, and many authors have elab-
orated this approach and used it to deal with the analysis of observations [4]. The standard treatment is based on the decomposition of the metric perturbations in scalar, vector and tensor parts according to their behaviour under the rotation group of isometries of the FRW background [5]. In the standard approach these three kinds of perturbations are treated independently, and most at-
tention has been devoted to the discussion of scalar per-
turbations in the particular case in which the two gauge
invariant scalar potentials coincide. This particular case
is well motivated because it arises naturally when dealing
with hydrodynamic or scalar field perturbations. In these
cases, the space-space components of the perturbing en-
ergy momentum-tensor take the form \( \delta T^i_j \propto S \delta^i_j \), where \( S \) is a scalar, and in turn this implies the equality \( \Phi = \Psi \), between the two gauge invariant Bardeen potentials for the scalar sector. An important technical consequence of this equality is that one has to deal only with a second order differential equation for \( \Phi \); in contrast with the general case of scalar perturbations, in which the stan-
dard formalism yields two coupled equations for \( \Phi \) and
\( \Psi \).

In this letter we generalize previous results that we first obtained for a dust filled universe [6], and present a new
approach to cosmological perturbations, which allows a unified treatment of all types of perturbations: scalar,
vector and tensor. The method follows a very similar
path to that used for computing the gravitational rad-
iation emitted by astrophysical sources in Minkowski
space-time, using harmonic gauge [7]. As a result of our
approach, we obtain a decoupled system of hyperbolic
partial differential equations for all components of the
metric perturbation \( h_{\mu\nu} \). This decoupling of equations
holds for a very wide class of perturbations, which we
define below, including the presence of seeds like cos-
nic strings or primordial black holes [8]. In addition the
equations for perturbations are formulated in a covari-
ant way, and in our opinion they are best fitted to study
linearized quantum gravity on spatially flat cosmological
backgrounds.

2. The background model

We choose a spatially flat FRW universe as back-
ground. This simplifies matters a little, at not a very
high cost concerning physics, because the recent analysis
of observations of supernova type Ia [9] and of acous-
tic peaks in the distribution of CMB temperature versus
angular momentum [10], favour a flat universe. In addi-
tion, inflationary models of the very early universe also
favour flat universes [11]. The metric of such background
model written in conformal time \( \eta \) and spatial cartesian
coordinates \( \vec{x} \), reads [12]

\[
\begin{aligned}
    ds^2 &= \gamma_{\mu\nu} \, dx^\mu \, dx^\nu = a(\eta)^2 \left( -d\eta^2 + d\vec{x}^2 \right), \\
\end{aligned}
\]

where the scale factor \( a(\eta) \) is an arbitrary function of

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time. Thus, the equations for perturbations that we are going to derive will apply to any flat FRW background, and in particular to multifluid universes, like our own universe. To describe the matter producing this background, we shall use a scalar field $\phi$ minimally coupled to gravity, i.e. we shall consider the action

$$S[g_{\alpha\beta}, \phi] = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \int d^4x \sqrt{-\bar{g}} \left( \frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + V(\phi) \right) \tag{2}$$

The full metric $g_{\alpha\beta}$ and the scalar field $\phi$ are to be developed up to first order in perturbations, in the form $g_{\alpha\beta} = \gamma_{\alpha\beta} + h_{\alpha\beta}$, and $\phi = \bar{\phi} + \delta \phi$. The zero order parts describe the background. It is important to emphasize that this description of the background, by means of the scalar field $\phi$, does not place any restriction on the scale factor $a(\eta)$. Any desired function $a(\eta)$ can be obtained by an appropriate choice of the potential $V(\phi)$. In this sense, the scalar field $\phi$ is not necessarily to be thought of as a fundamental field filling the universe, but as a convenient way of parametrizing the matter content of the universe at cosmological scales; as for example cold matter plus a cosmological constant, or whatever model future observations will support.

Using this scalar field parametrization of the matter, the zeroth order energy-momentum tensor can be written in the perfect fluid covariant form

$$\bar{T}^\nu_{\mu} = \left[ \frac{1}{2} \bar{u}^\alpha \bar{u}^\beta \bar{\phi}_{,\alpha} \bar{\phi}_{,\beta} - V(\bar{\phi}) \right] \delta^\nu_{\mu} + \bar{u}^\alpha \bar{u}^\beta \bar{\phi}_{,\alpha} \bar{\phi}_{,\beta} \bar{u}_\mu \bar{u}^\nu , \tag{3}$$

where $\bar{u}^\nu$ is the background velocity field of the fluid. The values of the background pressure and density can be derived from (3), and the covariant conservation of $\bar{T}^\nu_{\mu}$, with respect to the background metric, yields the zeroth order equation of motion for the field $\bar{\phi}$

$$\bar{\phi}_{,\alpha}^{[\alpha} = \frac{\partial V(\bar{\phi})}{\partial \bar{\phi}} , \tag{4}$$

where the vertical bar $|$ denotes the covariant derivative with respect to the background metric.

In coordinates $(\eta, \vec{x})$, and using a dot for derivatives with respect to conformal time $\eta$, the background values of pressure, density, and velocity take the form:

$$\bar{p} = \frac{1}{2} a^{-2} \dot{\bar{\phi}}^2 - V(\bar{\phi}) , \tag{5}$$

$$\bar{\rho} = \frac{1}{2} a^{-2} \dot{\bar{\phi}}^2 + V(\bar{\phi}) , \tag{6}$$

and

$$\bar{u}^\nu = a^{-1} \delta^\nu_0 . \tag{7}$$

Writing $a = \exp \Omega$, the Hubble parameter is given by

$$H = \frac{\dot{a}}{a^2} = \Omega \exp(-\Omega) , \tag{8}$$

the Einstein equations for the background are

$$\ddot{\Omega} - \dot{\Omega}^2 = -4\pi G \frac{\dot{\phi}^2}{\phi} , \tag{9}$$

and the integrability condition [3] reads

$$\ddot{\phi} + 2\Omega \dot{\phi} = a^2 \frac{\partial V(\bar{\phi})}{\partial \bar{\phi}} = 0 . \tag{10}$$

For example, in the particular case of a flat FRW background with a linear equation of state $p = \alpha \rho$, the corresponding potential for the scalar field is

$$V(\phi) = \frac{3(1 - \alpha)}{16\pi G} \exp \left(-2\sqrt{6(1 + \alpha)\pi G} (\phi - \phi_0) \right) , \tag{12}$$

and the solutions of the Einstein equations for the Hubble parameter, the background scalar field $\bar{\phi}$, and the conformal Hubble parameter $\bar{\Omega}$, are

$$H(\eta) = H_0 \left( \frac{\eta_0}{\eta} \right)^{\frac{3 + 3\alpha}{1 + 3\alpha}} , \tag{13}$$

$$\bar{\phi} = \bar{\phi}_0 + \frac{1}{1 + 3\alpha} \sqrt{\frac{3 + 3\alpha}{2\pi G}} \ln \frac{\eta}{\eta_0} , \tag{14}$$

and

$$\bar{\Omega} = \frac{2}{(1 + 3\alpha) \eta} , \tag{15}$$

the 0 label refers, as usual, to the present epoch values of the corresponding quantities, and the evolution of the density follows the critical value law $\rho = 3H^2/8\pi G$.

3. Structure of perturbations and gauge invariance

Now we are going to specify the class of perturbations that we will allow on the background. The equations
for perturbations come out, as usual, by developing the Einstein tensor up to first order in perturbations \( G^\nu_\mu = \bar{G}^\nu_\mu + \delta G^\nu_\mu \), and splitting the Einstein equations into a zeroth order part

\[
\bar{G}^\nu_\mu = -8\pi G \bar{T}^\nu_\mu ,
\]

which describes how the background is created by \( \bar{T}^\nu_\mu \), and a first order part

\[
\delta G^\nu_\mu = -8\pi G \delta T^\nu_\mu ,
\]

which gives the equation governing perturbations.

Expansion of the Einstein tensor up to first order in perturbations yields the covariant expression

\[
2 \delta G^\nu_\mu = \psi^\nu_\mu |^\lambda - \psi^\nu_\mu |^\lambda \mu + \delta \psi^\nu_\mu |^\lambda \beta |^\alpha
+ 2 \bar{R}^{\alpha \beta} \psi^\mu_\alpha \beta + \bar{R}^\alpha_\mu \psi^\alpha - \bar{R}^\alpha_\mu \psi^\alpha_\nu + \tilde{\delta} \psi^\mu + \delta^\nu_{\rho \gamma} \left[ \bar{R}^{\alpha \beta} \psi^{\gamma \beta} - \frac{1}{2} \bar{R} \psi \right] ,
\]

where \( \psi^\nu_\mu = h^\nu_\mu - \frac{1}{2} \bar{h} h \) is the so called trace reversed graviton field. On the other hand the structure of the perturbing energy-momentum tensor \( \delta T^\nu_\mu \) is constrained by the Bianchi identity. Expanding the full Bianchi identity to first order in perturbations, and taking into account the covariant conservation of \( \bar{T}^\nu_\mu \) with respect to the background metric, one obtains the constraint

\[
\delta T^\nu_\mu |^\nu_{|\nu} \bar{T}^\alpha_\mu - \delta T^\gamma_\alpha \bar{T}^\nu_\gamma = 0 ,
\]

where \( \delta T^\gamma_\alpha \) is the first order part of the full metric connection. The inhomogeneous equation \((23)\) must be fulfilled by any perturbing energy momentum tensor. Thus, any perturbing \( \delta T^\nu_\mu \) can be decomposed in the form

\[
\delta T^\nu_\mu = \delta T^{(I)}_\mu + \delta T^{(F)}_\mu ,
\]

where \( \delta T^{(I)}_\mu \) is a particular solution of the constraint equation \((19)\), and \( \delta T^{(F)}_\mu \) is any solution of the homogeneous equation

\[
\delta T^{(F)}_\mu = 0 .
\]

In addition, we will assume that \( \delta T^{(F)}_\mu \) does not functionally depend on the metric perturbation \( h^\mu_\nu \). We shall call \( \delta T^{(I)}_\mu \) the intrinsic perturbation, and \( \delta T^{(F)}_\mu \) the free perturbation \((23)\).

From the physical point of view, the intrinsic perturbation \( \delta T^{(I)}_\mu \) corresponds to the irregularities in the matter that creates the background, i.e. to deviations of homogeneity and isotropy in the matter which produces the background metric. On the other hand, what we call the free perturbation corresponds to additional matter (more or less exotic) that can exist in the universe in addition to the main component, and which moves as test matter on the background, i.e. it fulfills equation \((24)\). This extra matter could be, for instance, topological defects as cosmic strings or domain walls produced by phase transitions in the early universe \((24)\), or any other seed perturbation as for example primordial black holes. But also, ordinary baryonic matter, amounting only to two or three percent of the total energy density in the present universe, can be considered as a free perturbation too. In the last case the intrinsic perturbation \( \delta T^{(I)}_\mu \) would correspond entirely to nonbaryonic dark matter or energy.

Now the expression for \( \delta T^{(I)}_\mu \) is obtained from the action \((25)\), and in turn, it can be decomposed as

\[
\delta T^{(I)}_\mu = \delta T^{(h)}_\mu + \delta T^{(\phi)}_\mu ,
\]

where \( \delta T^{(h)}_\mu \) and \( \delta T^{(\phi)}_\mu \) are the pieces linear in the metric perturbation \( h^\mu_\nu \), and in the scalar field perturbation \( \delta \phi \) respectively. The covariant form of these pieces is

\[
\delta T^{(h)}_\mu = (\bar{p} + \bar{p}) \left[ \frac{1}{2} \bar{u}_\alpha \bar{u}_\beta \bar{h}^{\alpha \beta} \delta^\nu_\mu - \bar{u}_\mu \bar{u}_\alpha \bar{h}^{\alpha \nu} \right] ,
\]

and

\[
\delta T^{(\phi)}_\mu = \gamma^{\nu \alpha} \left[ \bar{\phi}_\mu \delta \phi_\nu + \bar{\phi}_\nu \delta \phi_\mu \right] \\
- \delta^\nu_\mu \left[ \gamma^{\alpha \beta} \bar{\phi}_\mu \delta \phi_\nu \right] .
\]

Let us discuss now how the perturbations behave under gauge transformations induced by infinitesimal coordinate transformations. Given an infinitesimal coordinate transformation \( x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu (z) \), the corresponding gauge transformations of the metric and scalar field perturbations are

\[
h^\mu_\nu \rightarrow h'^\mu_\nu = h^\mu_\nu + \xi_\mu |^\nu_\nu + \xi_\nu |^\mu_\nu ,
\]

and

\[
\delta \phi \rightarrow \delta \phi' = \delta \phi + \xi^\mu \bar{\phi}_\mu .
\]

Then, the Einstein tensor perturbation and the intrinsic energy-momentum perturbation transform accordingly as

\[
\delta G^\nu_\mu \rightarrow \delta G'^\nu_\mu = \delta G^\nu_\mu + \mathcal{L}_\xi \bar{G}^\nu_\mu ,
\]

and

\[
\delta T^{(I)}_\mu \rightarrow \delta T'^{(I)}_\mu = \delta T^{(I)}_\mu + \mathcal{L}_\xi \bar{T}^{(I)}_\mu ,
\]
where $\mathcal{L}_\xi$ is the Lie derivative with respect to the infinitesimal vector field $\xi^\mu$. We have assumed, as usual, that the zeroth order parts of all tensors remain invariant in a fixed coordinate system on the background spacetime manifold (for example $(\eta, \vec{x})$, but not necessarily this one), and that the effect of infinitesimal coordinate transformations is fully charged to the perturbations of tensors. Also notice that the transformation law (27) as the intrinsic perturbation $\delta T^\mu_\nu$ the total perturbation $\delta T^\mu_\nu$ entails a combination of two things: choosing a convenient coordinate system on the background, and fixing the gauge. Although all gauges contain the same physics, most of them contain it in a complicated way, meaning that the independent functions which solve the equations that remain after the gauge fixing, have spurious complications related to the gauge, but not related to real complications in the physics. The situation here is similar to that of the simple system of gauge invariant Maxwell equations (16) for the photon field $\square A_\mu - \partial_\mu (\partial_\nu A^\nu) = J_\mu$. In Lorentz gauge ($\partial_\mu A^\mu = 0$), one has nice solutions for these equations, but one could complicate the functional form of the solutions by choosing an inappropriate (but possible) gauge fixing condition.

In the case of the equations for cosmological perturbations (30), the appropriate gauge simplifying the equations is not so obvious as in the case of electrodynamics. The strategy that we follow is inspired by the way in which harmonic gauge works to simplify the equations for perturbations around Minkowski space-time. First, we set the covariant gauge fixing condition

$$\psi^\mu_{\ |\nu} = B^\mu_{\ |\nu}, \quad (31)$$

where the field $B^\mu_{\ |\nu}$ is yet unspecified. Then, if $B^\mu_{\ |\nu}$ does not depend on the derivatives of the metric perturbation $\psi^\mu_{\ |\nu}$, this condition eliminates all terms in second derivatives coming from the second, third and fourth terms in the expression (18) for $\delta \phi^\mu_{\ |\nu}$. Thus, only the flat d’Alembertian remains as second order differential operator in the left hand side of (30). In addition, working in coordinates $(\eta, \vec{x})$, we fix the $B^\mu_{\ |\nu}$ field to simplify the equations (30) as much as possible. After the long but straightforward exercise of writing $\delta \hat{G}^\mu_{\ |\nu}$ in covariant form given by the covariant expressions (18), (23), (24) and (29) in coordinates $(\eta, \vec{x})$, the analysis of the resulting expression shows that a great simplification is achieved by the choice

$$B^\mu_{\ |\nu} = -2H \bar{\psi}^\mu_{\ |\nu} + 16\pi G \bar{\phi}_0 \delta \phi_{\ |\nu}, \quad (32)$$

which has also been written in covariant form. Then, using the choice (32) to specify the gauge (31), the cosmological perturbations equations (30) in this gauge, and in coordinates $(\eta, \vec{x})$, take the simple form

$$\square \psi^\mu_{\ |\nu} - 2\Omega \partial_\eta \psi^\mu_{\ |\nu} + 2\Omega \left[ \gamma^0 \partial'_0 \psi - \delta^0_\mu \psi_0 - \delta^0_\mu \psi_0 \right] + 32\pi G \bar{\phi}_0 \delta \phi = -16\pi G a^2 \delta T^{(F)}_{\mu\nu}, \quad (33)$$

where $\square = -\partial^2/\partial\eta^2 + \vec{\nabla}^2$ is the Minkowski d’Alembertian.

In addition the covariant conservation of $\delta T^{(F)}_{\mu\nu}$, when applied to equations (33) yields an equation for the perturbation $\delta \phi$, which can also be obtained by perturbing the full equation of motion for the field $\phi$. This last equation can be skipped since it is implicit in the equations (33), as can be checked by direct computation and tak-
gauge fixing condition has been imposed is given by the 
(32). The residual gauge invariance remaining under this 
d diabetic moving matter on the background. The key ingre-
can be decomposed as a scalar field perturbation plus an 
turbations: scalar, vector and tensor, and they have a 
high degree of generality since they are valid for any spa-
turbations: scalar field perturbation 
source in equation (37).

\[ \Box + \hat{\Omega}^2 - \hat{\Omega} + \left( \frac{\ddot{\psi}}{\psi} \right)^2 \partial_\eta \left( \frac{\ddot{\psi}}{\psi} \right) \left( \frac{a}{\phi} (2\psi_0^0 - \psi) \right) = -16\pi G a^3 \left( 2 \delta T^{(F)0}_0 - \delta T^{(F)} - 2 \frac{\ddot{\phi}}{\phi} a \partial_\eta \psi_0^0 \right)^{} \]

(37)

Equations (33), (34) and (37) are a decoupled system of 
hyperbolic partial differential equations for the ten com-
ponents of the metric perturbation. They can be exactly 
inTEGRATED and their solutions expressed as retarded poten-
tials using the retarded Green’s functions for the three 
differential operators appearing on the left hand sides.

Equations (33) and (34), for the \( \psi_i \) and \( \psi_0 \) components, 
must be solved first because some combinations of the 
spatial components of the metric perturbation enter as a 
source in equation (37).

The decoupled system of hyperbolic equations (33), 
(34) and (37) is the central result of this paper. They 
give a unified description of all three types of metric 
perturbations: scalar, vector and tensor, and they have a 
high degree of generality since they are valid for any spa-
tially flat FRW background and any perturbation that 
can be decomposed as a scalar field perturbation plus an 
arbitrary (weak) perturbation corresponding to geodesi-
cally moving matter on the background. The key ingre-
dient in obtaining this system has been the appropriate 
selection of the gauge fixing condition given by (33) and 
(34). The residual gauge invariance remaining under this 
gauge fixing condition has been imposed is given by the 
vector fields \( \xi^\mu \) fulfilling the equation 

\[ \Box \xi_\mu + 2 \hat{\Omega} \partial_\eta \xi_\mu + 2 \ddot{\hat{\Omega}} \left( \xi_\mu - \delta_\mu^0 \xi_0 \right) = 0 \] 

(38)

that can be decomposed in the two simple equations 

\[ \Box + \hat{\Omega}^2 - \hat{\Omega} \left( a \xi^0 \right) = 0 \] 

(39)

and 

\[ \Box + \hat{\Omega}^2 + \hat{\Omega} \left( a \xi^i \right) = 0 \] 

(40)

involving the same differential operators (and hence the 
same mode solutions) as (34) and (33).

In the case in which the free perturbation \( \delta T^{(F)\nu}_\mu \) van-
ishes, the gauge fixing condition plus the residual gauge 
invariance reduce the number of physical degrees of free-
dom contained in the metric perturbation \( \psi_\mu^\nu \) plus the 
scalar field perturbation \( \delta \phi \), down two three: a scalar per-
turbation plus two polarizations for gravitational waves 
on the background.

5. Universes with a linear equation of state

To obtain a definite expression for the retarded Green’s 
functions solving equations (33), (34) and (37), a partic-
ular background model has to be chosen. In the case of 
universes obeying a linear equation of state \( p = \alpha \rho \), these 
equations take a particular simple form, and in addition 
the operators on the left hand side of equations (33) and 
(37) coincide. Thus, in this case only two Green’s func-
tions for the two differential operators 

\[ \Box + \hat{\Omega}^2 + \hat{\Omega} \equiv \Box + \frac{2 - 6\alpha}{(1 + 3\alpha)^2} \frac{1}{\eta^2} \] 

(41)

and 

\[ \Box + \hat{\Omega}^2 - \hat{\Omega} \equiv \Box + \frac{6 + 6\alpha}{(1 + 3\alpha)^2} \frac{1}{\eta^2} \] 

(42)

are required.

The Green’s functions for these operators can be ex-
pressed as superposition of homogeneous mode solutions, 
which are given by Bessel functions of indices \( \nu_1 = \pm(3-3\alpha)/(2+6\alpha) \) and \( \nu_2 = \pm(3\alpha+5)/(2+6\alpha) \). Moreover 
in the particular cases of de Sitter (\( \alpha = -1 \) [5]), dust 
(\( \alpha = 0 \) [6], and radiation (\( \alpha = 1/3 \)) dominated cosmo-
logical models, the Bessel indices are the simplest half in-
tegers (\( \nu_1 = \pm3/2, \nu_2 = \pm1/2 \)), (\( \nu_1 = \pm3/2, \nu_2 = \pm5/2 \)), 
and (\( \nu_1 = \pm1/2, \nu_2 = \pm3/2 \)) respectively.
Thus, to integrate the perturbations equations for de Sitter, dust, and radiation backgrounds we only require the three retarded Green’s functions solving
\[
\left( \Box + \frac{A}{\eta^2} \right) G^{(A)}_R(x, x') = -\delta^{(4)}(x - x') ,
\]
(43)
for \( A = 0, 2, 6 \) and where \( x = (\eta, \vec{x}) \).

Then \( G^{(0)}_R(x, x') \) is the well known retarded Green’s function for Minkowski, and \( G^{(2)}_R(x, x') \) and \( G^{(6)}_R(x, x') \) can be obtained by elementary QFT methods as explained in [15], with the result
\[
G^{(2)}_R(x, x') = \frac{1}{4\pi |\vec{x} - \vec{x}'|} \delta(\eta - \eta' - |\vec{x} - \vec{x}'|)
+ \frac{1}{4\pi \eta \eta'} \theta(\eta - \eta' - |\vec{x} - \vec{x}'|) ,
\]
(44)
and
\[
G^{(6)}_R(x, x') = \frac{1}{4\pi |\vec{x} - \vec{x}'|} \delta(\eta - \eta' - |\vec{x} - \vec{x}'|)
+ \frac{3}{8\pi} \left[ \frac{1}{\eta'^2} + \frac{1}{\eta^2} - \frac{|\vec{x} - \vec{x}'|^2}{\eta^2 \eta'^2} \right]
\times \theta(\eta - \eta' - |\vec{x} - \vec{x}'|) .
\]
(45)

Thus, \( G^{(2)}_R(x, x') \) and \( G^{(6)}_R(x, x') \) are given by the Minkowski retarded Green’s function with support on the past light cone plus an additional term with support in the interior of the past light cone. Then the metric perturbation \( \psi_i^{\mu \nu} \) can be expressed as retarded potentials by means of these Green’s functions.

6. Concluding remarks

We have analyzed cosmological perturbations on a spatially flat FRW background given by a scalar field perturbation plus an arbitrary (weak) perturbation corresponding to test matter moving geodesically on this background. We have shown that by an appropriate choice of the gauge the cosmological perturbations obey decoupled hyperbolic equations. The class of perturbations considered have a high degree of generality, although it is not fully general since a covariantly conserved tensor, linearly dependent in the metric perturbation, could still be added to the perturbing energy-momentum tensor.

On the other hand since the treatment and the gauge fixing condition that we have introduced is covariant with respect to finite coordinate transformations on the background, it provides a very suitable framework to develop linear quantum gravity around a spatially flat FRW background.

It is important to fully elucidate the relationships of the present treatment with the standard Lifshitz-Bardeen treatment in terms of gauge invariant perturbations and the decomposition of the metric perturbation in scalar, vector and tensor parts. This task together with more details of calculations and applications are being worked out and will be the subject of a forthcoming paper.

7. Acknowledgements

S. Kopeikin thanks the Departamento de Física Teórica I, Universidad Complutense de Madrid, for hospitality. J. Ramírez thanks the Department of Physics and Astronomy, University of Missouri-Columbia for hospitality. We are grateful to B. Mashhoon and A. López Maroto for valuable discussions as well as A. Corman for helpful conversations.

[5] Since we are dealing with a flat FRW background, the six dimensional isometry group of the background is generated by three independent rotations, and three independent translations which are profitable to do Fourier analysis of the perturbations.
[12] We use the same conventions for the signs of the metric and curvature tensors as in Weinberg’s book [8].
[13] In [8] we used the names bare and complementary for the free and intrinsic perturbations, respectively.