Gravitational Radiation Theory and Light Propagation*

Luc Blanchet[1], Sergei Kopeikin[2], and Gerhard Schäfer[3]
[1] Département d'Astrophysique Relativiste et de Cosmologie (CNRS), Observatoire de Paris, 92195 Meudon Cedex, France
[2] Department of Physics & Astronomy, University of Missouri-Columbia, Physics Building 223, Columbia, MO 65211, USA
[3] Theoretisch–Physikalisches Institut, Friedrich–Schiller–Universität, Max–Wien–Platz 1, 07743 Jena, Germany

February 7, 2008

Abstract

The paper gives an introduction to the gravitational radiation theory of isolated sources and to the propagation properties of light rays in radiative gravitational fields. It presents a theoretical study of the generation, propagation, back–reaction, and detection of gravitational waves from astrophysical sources. After reviewing the various quadrupole–moment laws for gravitational radiation in the Newtonian approximation, we show how to incorporate post–Newtonian corrections into the source multipole moments, the radiative multipole moments at infinity, and the back-reaction potentials. We further treat the light propagation in the linearized gravitational field outside a gravitational wave emitting source. The effects of time delay, bending of light, and moving source frequency shift are presented in terms of the gravitational lens potential. Time delay results are applied in the description of the procedure of the detection of gravitational waves.

Pacs Numbers : 04.25.-g, 04.25.Nx

1 Introduction

It was only in the late fifties of the twentieth century that by the work of Hermann Bondi and Joseph Weber gravitational radiation entered the domain of physics. Before that time gravitational radiation was not considered to be of observational relevance and the gravitational radiation theory was not developed very deeply.

The supposed detection of gravitational radiation by J. Weber in the late sixties triggered strong and still on-going efforts both in the building of gravitational wave detectors and in the elaboration of the gravitational radiation theory, including investigations of the most reliable sources of detectable gravitational waves, calculations of wave forms, and analysis of data from detectors (cf. [1]). It turned out that coalescing neutron stars and/or stellar-mass black holes, together with gravitationally collapsing objects (type II supernovae), are the most relevant sources for detectable gravitational waves on Earth because they are strong and fit well to the frequency band of the Earth-based detectors which ranges from 10 Hz to 10 kHz. The strength of these sources is such high that several detection events per year might be expected in future fully developed detectors. The most sensitive Earth-based detectors will go into operation in the first few years of the new millenium. These are the laser-interferometric detectors in Germany, GEO600, built by a German/British consortium, in Italy, VIRGO, built by a Italian/French consortium, the two LIGO detectors in the United States, and the TAMA300 detector in Japan. There are several bar detectors already operating on

*to appear in: C. Lämmerzahl, C.W.F. Everitt, F.W. Hehl (eds.): *Gyros, Clocks, and Interferometers: Testing Relativistic Gravity in Space*, Springer–Verlag, Berlin 2000

Earth (ALLEGRO in the United States, AURIGA and NAUTILUS in Italy, EXPLORER at CERN, NIOBE in Australia). These detectors are being permanently upgraded and will be supplementing the measurements of the interferometric detectors later. For the measurement of gravitational waves in the frequency range between 0.1 Hz and 0.1 mHz the space-borne laser-interferometric detector LISA is devised which is expected to be flown around 2010 by NASA/ESA. The astrophysical sources of the gravitational waves to be detected by this detector are a variety of orbiting stars (interacting white dwarf binaries, compact binaries), orbiting massive black holes, as well as the formation and coalescence of supermassive black holes. Stochastic gravitational waves from the early universe are expected to exist in the whole measurable frequency range from 10^4 Hz down to 10^{-18} Hz. The tools to possibly measure the primordial waves are the mentioned Earth-based and space-borne detectors, Doppler tracking, pulsar timing, very long baseline interferometry, as well as the cosmic microwave background.

On the theoretical side there are essentially two approaches which permit to investigate the properties of and to make predictions about gravitational waves from various sources. The first approach, that we can qualify as "exact", stays within the exact theory, solving or establishing theorems about the complete non-linear Einstein field equations. Within this approach one can distinguish the work dealing with exact solutions of the field equations in the form of plane gravitational waves, and especially colliding plane waves. Since the waves are never planar in nature, this work is not very relevant to real astrophysics, but its academic interest is important in that it permits notably the study of the appearance of singularities triggered by collisions of waves. Also within the exact approach, but more important for applications in astrophysics, is all the work concerned with the study of the asymptotic structure of the gravitational field of isolated radiating systems. The work on asymptotics started with the papers of Bondi *et al.* [2] and Penrose [3]. The second approach is much more general, in the sense that it is not restricted to any particular symmetry of the system, nor it is applicable only in the far region of the system. However, the drawback of this approach is that it is only *approximate* and essentially looks for the solutions of the Einstein field equations in the form of formal expansions when $c \to \infty$ (post-Newtonian approximation). This approximate post-Newtonian method can be applied to the study of all theoretical aspects of gravitational radiation: the equations of motion of the source including the gravitational radiation reaction (works of Einstein, Infeld, and Hoffmann [4], Chandrasekhar and Esposito [5], Burke and Thorne [6, 7, 8], Ehlers [9], Papapetrou and Linet [10], Damour and Deruelle [11, 12], Schäfer [13], Kopejkin [14]); the structure of the radiation field (work of Bonnor [15], Thorne [16], Blanchet and Damour [17]), and, more recently, accurate post-Newtonian wave generation formalisms [18, 19, 20, 21, 22].

To the lowest, Newtonian order, the wave-generation formalism is called the quadrupole formalism, because as a consequence of the equality of the inertial and gravitational mass of all bodies the dominant radiating moment of any system is the (mass-type) quadrupole, which simply is at this approximation the standard Newtonian quadrupole moment. We are very much confident in using the theoretical framework of the post-Newtonian approximation because, marvellously enough, the framework of the Newtonian, quadrupole formalism has been checked by astronomical observations.

In fact, there are two observational tests of the validity of the quadrupole formalism. The first test concerns the famous Hulse–Taylor binary pulsar whose decrease of the orbital period P_b by gravitational radiation is predicted from the quadrupole formula to be [23, 24, 25, 26]

$$\dot{P}_b = -\frac{192\pi}{5c^5}\left(\frac{2\pi G}{P_b}\right)^{5/3}\frac{M_p M_c}{(M_p+M_c)^{1/3}}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}\,, \tag{1}$$

where M_p and M_c are the pulsar and companion masses, e is the orbit eccentricity, and G, c are the universal gravitational constant and the speed of light. Numerically, one finds $\dot{P}_b = -2.4 \times 10^{-12}$ sec/sec, in excellent agreement (0.35% precision) with the observations by Taylor *et al.* [27, 28]. The second test concerns the so-called cataclysmic variables. There we have binary systems in which a star filling its Roche lobe (the "secondary" with mass M_2) transfers mass onto a more massive white dwarf (the "primary" with mass $M_1 > M_2$). From the formula for the angular momentum in Newtonian theory $J = GM_1M_2(a/GM)^{1/2}$ (where $M = M_1 + M_2$), we deduce the secular evolution of the orbital semi-major radius a (whatever may be the mechanism for the variation of J),

$$\frac{\dot{a}}{a} = \frac{2\dot{J}}{J} - \frac{2\dot{M}_2}{M_2}\left(1 - \frac{M_2}{M_1}\right) , \tag{2}$$

where $\dot{M}_2$ is the rate at which the secondary transfers mass to the primary ($\dot{M}_2 < 0$). Since $M_1 > M_2$, the mass transfer tends to increase the radius a of the orbit, hence to increase the radius of the secondary's Roche lobe, and, thus, to stop the mass transfer. Therefore a long lived mass transfer is possible only if the system looses angular momentum to compensate for the increase of a. For cataclismic binaries with periods longer than about two hours, the loss of angular momentum is explained by standard astrophysical theory (interaction between the magnetic field and the stellar wind of the secondary). But for short-period binaries, with period less than about two hours, the only way to explain the loss of angular momentum is to invoke gravitational radiation. Now, from the quadrupole formula, we have

$$\left(\frac{\dot{J}}{J}\right)_{\rm GW} = -\frac{32G^3}{5c^5}\frac{M_1 M_2 M}{a^4} . \tag{3}$$

Inserting this into (2) one can then predict what should be $\dot{M}_2$ in order that $\dot{a}/a \sim 0$, and the result is in good agreement with the mass transfer measured from the X-rays observations of cataclysmic binaries.

Another important aspect of the theory of gravitational radiation, with obvious implications in astronomy, is the interaction of the gravitational wave field with photons. In this article we present the results of a thorough investigation of light propagation in the gravitational wave field generated by some isolated system. Our motivation is that electromagnetic waves are still the main carrier of astrophysically important information from very remote domains of our universe. Also, the operation of interferometric gravitational wave detectors and other techniques used for making experiments in gravitational physics (lunar laser ranging, very long baseline interferometry, pulsar timing, Doppler tracking, etc.) are fully based on the degree of our understanding of how light propagates in variable, time-dependent gravitational fields generated by various celestial bodies. Although quite a lot of work has been done on this subject (see, for example, [29]-[36]) a real progress and much deeper insight into the nature of the problem has been achieved only recently [37]-[39]. The main advantage of the integration technique which has been developed for finding the light-ray trajectory perturbed by the gravitational field is its account for the important physical property of gravitational radiation, namely, its retardation character. Previous authors, apart from Damour and Esposito-Farèse [40], accounted for the retardation of the gravitational field only in form of plane gravitational waves. Hence, effects produced in the near and induction zones of isolated astronomical sources emitting waves could not be treated in full detail. As a particular example of importance of such effects we note the problem of detection of gravitational waves created by g-modes of the Sun. The space interferometer LISA will be able to detect those waves. However, the problem is that LISA will fly in the induction zone of the emission process of these gravitational waves and, hence, a much more complete theoretical analysis of the working of the detector is needed. The approximation of a plane gravitational wave for the description of the detection procedure is definitely not sufficient. The other example could be effects caused by the time-dependent gravitational field of the ensemble of binary stars in our galaxy. Timing of high-stable millisecond pulsars might be a tool for the detection of stochastic effects produced by that field [41].

2 Wave generation from isolated sources

2.1 Einstein field equations

The gravitational field is described in general relativity solely by the metric tensor $g_{\mu\nu}$ (and its inverse $g^{\mu\nu}$). It is generated by the stress-energy tensor of the matter fields $T^{\mu\nu}$ via the second-order differential equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu} \,, \tag{4}$$

where $R^{\mu\nu}$ and $R = g_{\rho\sigma}R^{\rho\sigma}$ denote, respectively, the Ricci tensor and the Ricci scalar. We assume that the matter tensor $T^{\mu\nu}$ corresponds to an isolated source, i.e. $T^{\mu\nu}$ has a spatially compact support with maximal radius a, and that the internal gravity of the source is weak in the sense that its mass M satisfies $GM \ll ac^2$. Within these conditions it is appropriate to write the metric $g^{\mu\nu}$ in the form of a small deformation of the flat metric $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. We pose $h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$ (g = determinant of $g_{\mu\nu}$) and assume that each component of $h^{\mu\nu}$ is numerically small: $|h^{\mu\nu}| \ll 1$. We lower and raise all indices of our metric perturbation $h^{\mu\nu}$ with the flat metric; for instance, $h_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}h^{\rho\sigma}$ and $h = \eta^{\rho\sigma}h_{\rho\sigma}$. Then the field equations (4) can be re-written in terms of the metric perturbation $h^{\mu\nu}$ by separating out a second-order linear operator acting on $h^{\mu\nu}$, and the remaining part of the equations, which is at least quadratic in $h^{\mu\nu}$ and its first and second derivatives, we conventionally set to the right side of the equations together with the matter tensor. This yields

$$\Box\, h^{\mu\nu} - \partial^\mu H^\nu - \partial^\nu H^\mu + \eta^{\mu\nu}\partial_\rho H^\rho = \frac{16\pi G}{c^4} \tau^{\mu\nu} \,, \tag{5}$$

where $\Box = \Box_\eta$ denotes the flat d'Alembertian operator and where $H^\mu \equiv \partial_\nu h^{\mu\nu}$; on the right side of the equation we have put

$$\tau^{\mu\nu} = (-g)\, T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu} \,, \tag{6}$$

which represents the total stress-energy distribution of both the matter fields – first term in (6) – and the gravitational field itself – second term involving the non-linear gravitational source $\Lambda^{\mu\nu} = O(h^2)$ (note that $\tau^{\mu\nu}$ transforms as a Minkowskian tensor under Lorentz transformations). The divergence of the left side of (6) is identically zero by virtue of the Bianchi identities, therefore the pseudo-tensor $\tau^{\mu\nu}$ is conserved in the ordinary sense,

$$\partial_\nu \tau^{\mu\nu} = 0 \,, \tag{7}$$

which is equivalent to the covariant conservation of the matter tensor, $\nabla_\nu T^{\mu\nu} = 0$. A gauge transformation $h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu - \eta^{\mu\nu}\partial_\lambda\xi^\lambda$ does not affect the left side of (5), and consequently by solving for a vector ξ^μ the wave equation $\Box\xi^\mu = -H^\mu$ one can arrange that $h^{\mu\nu}$ satisfies the *harmonic-gauge* condition $\partial_\nu h^{\mu\nu} = 0$. In this gauge the field equations (5) simply become

$$\Box\, h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu} \,. \tag{8}$$

We want now to formulate the condition that the source is really isolated, i.e. it does not receive any radiation from other sources located far away, at infinity. Recall that we can express any homogeneous regular solution of the wave equation $\Box h_{\mathrm{hom}} = 0$ at a given field point in terms of the values of h_{hom} at some source points forming a surrounding surface at retarded times. This is the Kirchhoff formula (see e.g. [42]), which reads in the case where the surrounding surface is a sphere,

$$h_{\mathrm{hom}}(\mathbf{x}', t') = \int\!\!\int \frac{d\Omega}{4\pi} \left[\frac{\partial}{\partial\rho}(\rho h_{\mathrm{hom}}) + \frac{\partial}{c\partial t}(\rho h_{\mathrm{hom}})\right](\mathbf{x}, t), \tag{9}$$

where $\rho = |\mathbf{x} - \mathbf{x}'|$ and $t = t' - \rho/c$. To formulate the no-incoming radiation condition we say that there should be no such homogeneous regular solutions h_{hom} (since they correspond physically to waves propagating from sources at infinity). Taking the limit $r \to +\infty$ with $t + r/c =$ const. in Kirchhoff's formula, we then arrive at the physical conditions that

$$\lim_{\substack{r\to+\infty \\ t+\frac{r}{c}=\mathrm{const.}}} \left[\frac{\partial}{\partial r}(r h^{\mu\nu}) + \frac{\partial}{c\partial t}(r h^{\mu\nu})\right](\mathbf{x}, t) = 0 \,, \tag{10}$$

and that $r\partial_\lambda h^{\mu\nu}$ should be bounded in this limit. The no-incoming radiation condition (10) is thus imposed at (Minkowskian) past null infinity $\mathcal{J}^-$ in a conformally rescaled space-time diagram.

2.2 Multipole expansion in linearized gravity

For the rest of this Section (and also in Section 5) we shall restrict ourselves to the case of linearized gravity, defined in particular by the neglect of the non-linear gravitational source term $\Lambda^{\mu\nu}$, for which the field equations in harmonic gauge $\partial_\nu h^{\mu\nu} = 0$ read

$$\Box\, h^{\mu\nu} = \frac{16\pi G}{c^4}\, T^{\mu\nu} \,. \tag{11}$$

Within the linearized approximation the matter stress-energy tensor is divergenceless: $\partial_\nu T^{\mu\nu} = 0$. Therefore the linearized approximation is inconsistent as regards the motion of the matter source, which in this approximation stays unaffected by the gravitational field. However this approximation is quite adequate for describing the generation of waves by a given source (for instance acted on by non-gravitational forces). From the no-incoming radiation condition (10), we find that the unique solution of (11) is the retarded one:

$$h^{\mu\nu}(\mathbf{x},t) = -\frac{4G}{c^4}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\, T^{\mu\nu}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|) \,. \tag{12}$$

Since we are being interested in the wave-generation problem, we choose the field point outside the source, that is $r = |\mathbf{x}| > a$ (with the origin of the spatial coordinates at the center of the ball with radius a, so $a > |\mathbf{x}'|$), and we decompose in that region $h^{\mu\nu}$ into "multipole moments". The straightforward way to do this is to employ the standard Taylor formula for the formal limit $\mathbf{x}' \to 0$,

$$\frac{T(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} x'_L \partial_L \left[\frac{T(\mathbf{x}', t - r/c)}{r}\right] \,. \tag{13}$$

Notice the short-hand notation $L = i_1 i_2 \cdots i_l$ for a multi-index with l indices, as well as $x'_L = x'^L = x'^{i_1} x'^{i_2} \cdots x'^{i_l}$, $\partial_L = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}$ where $\partial_i = \partial/\partial x^i$. From this Taylor expansion we immediately arrive at the following expression for the multipole decomposition of the metric perturbation,

$$\mathcal{M}(h^{\mu\nu})(\mathbf{x},t) = -\frac{4G}{c^4}\sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left[\frac{1}{r}\mathcal{H}_L^{\mu\nu}(t - \frac{r}{c})\right] \,, \tag{14}$$

where the "multipole moments" depend on the retarded time $u \equiv t - r/c$ and are given by

$$\mathcal{H}_L^{\mu\nu}(u) = \int d^3\mathbf{x}'\, x'_L T^{\mu\nu}(\mathbf{x}', u) \,, \tag{15}$$

In (14) we employ the notation $\mathcal{M}$ to distinguish the multipole expansion $\mathcal{M}(h)$ from h itself. Of course, we have numerically $\mathcal{M}(h) = h$ outside the source, however inside the source $\mathcal{M}(h)$ and h will differ from each other; indeed h is a smooth solution of the equations (12) while $\mathcal{M}(h)$ satisfies $\Box\mathcal{M}(h) = 0$ and becomes infinite when $r \to 0$ [as it is clear from (14)]. In Section 4, dealing with the general case of the non-linear theory, we shall prefer to use the multipole expansion in terms of symmetric and trace-free (STF) multipole moments. In the present case it is simpler to use the non-STF moments $\mathcal{H}_L^{\mu\nu}$. A systematic investigation of the STF multipole expansion in linearized gravity can be found in [43].

Applying the (linearized) conservation law $\partial_\nu T^{\mu\nu} = 0$ we easily find certain physical evolution equations (and conservation laws) to be satisfied by the multipole moments (15). Making use of the Gauss theorem to discard spatial divergences of compact-support terms we successively obtain (with (n) referring to n time-derivatives)

$$\frac{1}{c}\overset{(1)}{\mathcal{H}}{}_L^{\mu 0} = l\mathcal{H}_{L-1)}^{\mu(i_l} \,, \tag{16}$$

$$\frac{1}{c}\overset{(2)}{\mathcal{H}}{}_L^{00} = l(l-1)\mathcal{H}_{L-2)}^{(i_l i_{l-1}} \,, \tag{17}$$

where the round brackets around spatial indices denote the symmetrization (and where $L-1=i_1\cdots i_{l-1}$; $L-2=i_1\cdots i_{l-2}$). As a consequence of (16) we see that the anti-symmetric part of $\mathcal{H}^{i0}_j$ in the indices ij is constant. A more general consequence is

$$\frac{1}{c}\epsilon_{ijk}\overset{(1)}{\mathcal{H}^{j0}_{kL-1}}=(l-1)\epsilon_{ijk}\mathcal{H}^{j(i_{l-1}}_{L-2)k}\ . \tag{18}$$

In the case of the lowest-order ($l=0$ and $l=1$) multipole moments the right sides vanish, and therefore these equations represent the conservation laws for the corresponding moments. Over all we find ten conservation laws, one for the mass-type monopole or total mass M, three for the mass-type dipole or center of mass position X_i (times M), three for the time derivative of the mass dipole or linear momentum P_i, and three for the current-type dipole or total angular momentum S_i. Specifically, we define

$$M\equiv\ \frac{1}{c^2}\mathcal{H}^{00}=\int d^3\mathbf{x}\,\frac{T^{00}}{c^2}\ , \tag{19}$$

$$P_i\equiv\ \frac{1}{c}\mathcal{H}^{0i}=\int d^3\mathbf{x}\,\frac{T^{0i}}{c}\ , \tag{20}$$

$$S_i\equiv\frac{1}{c}\,\epsilon_{ijk}\mathcal{H}^{0k}_j=\epsilon_{ijk}\int d^3\mathbf{x}\,x_j\frac{T^{0k}}{c}\ , \tag{21}$$

$$MX_i\equiv\ \frac{1}{c^2}\mathcal{H}^{00}_i=\int d^3\mathbf{x}\,x_i\frac{T^{00}}{c^2}\ . \tag{22}$$

Then, from (16)-(18), we have

$$\dot{M}=0,\qquad \dot{P}_i=0,\qquad \dot{S}_i=0,\qquad \text{and}\qquad \dot{X}_i=\frac{P_i}{M}\ . \tag{23}$$

3 The quadrupole moment formalism

3.1 Multipole expansion in the far region

We analyze the gravitational field in the far-zone of the source, in which we perform the expansion of the multipolar expansion $\mathcal{M}(h^{\mu\nu})$ when $r\to+\infty$ with $t-r/c=$const (Minkowskian future null infinity $\mathcal{J}^+$). To leading order $1/r$ the formula (14) yields

$$\mathcal{M}(h^{\mu\nu})=-\frac{4G}{c^4r}\sum_{l=0}^{+\infty}\frac{n_L}{c^l l!}\overset{(l)}{\mathcal{H}^{\mu\nu}_L}(u)+O\left(\frac{1}{r^2}\right)\ , \tag{24}$$

[where (l) represents the lth time-derivative and where $n_L\equiv n^L=n^{i_1}n^{i_2}\cdots n^{i_l}$ with $n^i=x^i/r$]. Because of the powers of $1/c$ in front of each multipolar piece, it is clear that the far-zone expansion of $\mathcal{M}(h)$ is especially useful when the numerical value of each term of the formula (24) really scales with the factor $1/c^l$ in front of it. This will be the case when the typical velocities of the particles composing the system are small with respect to the speed of light, $v/c\equiv\varepsilon\ll1$, or, equivalently, when the maximal radius a of the system is much smaller than the wavelength λ of the emitted gravitational radiation ($\lambda=cP$ where P is the typical period of the internal motion). In particular, this "slow motion" assumption is always realized in the case of a self-gravitating system with weak internal gravity, for which we have

$$\varepsilon\equiv\frac{v}{c}\sim\frac{a}{\lambda}\sim\sqrt{\frac{GM}{c^2a}}\ll1\ . \tag{25}$$

Thus, for slowly-moving systems we can retain only the first few terms in the multipolar-post-Newtonian expansion (24). Let us restrict ourselves to the terms

$$h^{00} = -\frac{4G}{c^4 r}\left\{\mathcal{H}^{00} + \frac{n_a}{c} \overset{(1)}{\mathcal{H}^{00}_a} + \frac{n_{ab}}{2c^2} \overset{(2)}{\mathcal{H}^{00}_{ab}}\right\}, \tag{26}$$

$$h^{0i} = -\frac{4G}{c^4 r}\left\{\mathcal{H}^{0i} + \frac{n_a}{c} \overset{(1)}{\mathcal{H}^{0i}_a}\right\}, \tag{27}$$

$$h^{ij} = -\frac{4G}{c^4 r}\left\{\mathcal{H}^{ij}\right\}. \tag{28}$$

For easier notation we do not indicate the multipole expansion $\mathcal{M}$, nor the neglected $O(1/r^2)$ in the distance to the source. Using the conservation laws (16)-(17) we can easily re-express the latter expressions in terms of the total mass M, total linear-momentum P_i, and the Newtonian quadrupole moment

$$Q_{ij} \equiv \frac{1}{c^2}\mathcal{H}^{00}_{ij} = \int d^3\mathbf{x}\frac{T^{00}}{c^2}x_i x_j . \tag{29}$$

Since M and P_i are conserved they do not participate to the radiation field which is therefore dominantly quadrupolar. Restoring the neglected post-Newtonian error terms we obtain

$$h^{00} = -\frac{4G}{c^2 r}\left\{M + \frac{n_a}{c}P_a + \frac{n_{ab}}{2c^2} \overset{(2)}{Q}_{ab}(u) + O\left(\varepsilon^3\right)\right\}, \tag{30}$$

$$h^{0i} = -\frac{4G}{c^3 r}\left\{P_i + \frac{n_a}{2c} \overset{(2)}{Q}_{ai}(u) + O\left(\varepsilon^2\right)\right\}, \tag{31}$$

$$h^{ij} = -\frac{4G}{c^4 r}\left\{\frac{1}{2} \overset{(2)}{Q}_{ij}(u) + O\left(\varepsilon\right)\right\}. \tag{32}$$

Note that the contribution of the angular momentum S_i appears only at the sub-dominant order $O(1/r^2)$ [see for instance (78) below]. When acting on terms of order $1/r$ in the distance like in (30)-(32) the derivative ∂_ν is proportional to the (Minkowskian) null vector $k_\nu = (-1, \mathbf{n})$; namely $\partial_\nu = -k_\nu \partial_0 + O(1/r^2)$. Using this, the harmonic gauge condition $\partial_\nu h^{\mu\nu} = 0$ reads

$$k_\nu \overset{(1)}{h^{\mu\nu}} = O\left(\frac{1}{r^2}\right) \tag{33}$$

which is checked directly to be satisfied by the expressions (30)-(32).

3.2 The far-field quadrupole formula

Using the gauge freedom $h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu - \eta^{\mu\nu}\partial_\lambda\xi^\lambda$, we apply a gauge transformation to what is called the transverse-traceless (TT) gauge. Namely we pose

$$\xi^0 = \frac{G}{2rc^3}\left[-n_{ab} \overset{(2)}{Q}_{ab} - \overset{(2)}{Q}_{aa}\right], \tag{34}$$

$$\xi^i = \frac{G}{2rc^3}\left[n_{iab} \overset{(2)}{Q}_{ab} + n_i \overset{(2)}{Q}_{aa} - 4n_a \overset{(2)}{Q}_{ia}\right]. \tag{35}$$

The new metric in TT coordinates, say $h^{\mathrm{TT}}_{\mu\nu}$ (where we lower indices with the flat metric), is straightforwardly seen to involve in its 00 and $0i$ components only the static contributions of the mass monopole and dipoles, namely

$$h^{\mathrm{TT}}_{00} = -\frac{4GM}{c^2 r}\left(1 + \frac{n_a}{c}\dot{X}_a\right), \qquad h^{\mathrm{TT}}_{0i} = -\frac{4GP_i}{c^3 r}. \tag{36}$$

In the TT gauge the only radiating components of the field are the spatial ones, ij, and we obtain

$$h_{ij}^{\rm TT} = -\frac{2G}{c^4 r} P_{ijab}(\mathbf{n}) \left\{ \overset{(2)}{I}_{ab}(u) + O(\varepsilon) \right\} + O\left(\frac{1}{r^2}\right) . \tag{37}$$

The latter equation is known as the far-field quadrupole equation. The TT projection operator is defined by $P_{ijab} = P_{ia}P_{jb} - \frac{1}{2}P_{ij}P_{ab}$, where $P_{ij} = \delta_{ij} - n_i n_j$. This is a projector: namely $P_{ijkl}P_{klab} = P_{ijab}$, onto the plane orthogonal to $\mathbf{n}$: thus for instance, $n_i P_{ijab} = 0$; and it is trace free: $P_{ijab}\delta_{ab} = 0$, so we have substituted in (37) the trace free part of the quadrupole moment Q_{ab}, i.e.

$$I_{ij} = Q_{ij} - \frac{\delta_{ij}}{3} Q_{aa} + O\left(\varepsilon^2\right) . \tag{38}$$

We have added a remainder $O(\varepsilon^2)$ to indicate the post-Newtonian corrections in the source moment I_L computed in Section 4 [note that the remainder in (37) is only $O(\varepsilon)$].

As we see all the physical properties of the gravitational wave are contained into the TT projection (37). As a consequence, the effects of the wave on matter fields are transverse: the motion of matter induced by the wave takes place only in the plane orthogonal to the propagation of the wave. Furthermore, from the trace-less character of the wave we see there can be only two independent components or polarization states. We introduce two polarization vectors, $\mathbf{p}$ and $\mathbf{q}$, in the plane orthogonal to the direction of propagation $\mathbf{n}$, forming an orthonormal right-handed triad. In terms of these polarization vectors the projector onto the transverse plane reads $P_{ij} = p_i p_j + q_i q_j$. The two polarization states (customarily referred to as the "plus" and "cross" polarizations) are defined by

$$h_+ = \frac{p_i p_j - q_i q_j}{2}\, h_{ij}^{\rm TT} , \qquad h_\times = \frac{p_i q_j + p_j q_i}{2}\, h_{ij}^{\rm TT} . \tag{39}$$

Until very recently all expectations was that any astrophysical (slowly-moving) source would emit gravitational radiation according to (at least dominantly) the quadrupole formula (37), involving the *mass-type* quadrupole moment I_{ij}. For instance the waves from the binary pulsar obey this formula. However, it has been realized by Andersson [44] and Friedman and Morsink [45] that in the case of the secular instability of the r-modes (rotation, or Rossby modes) of isolated newly-born neutron stars, the gravitational radiation is dominated by the variation of the *current-type* quadrupole moment J_{ij}. Here we give, without proof, the formula analogous to (37) but for the current quadrupole:

$$h_{ij|\rm current}^{\rm TT} = \frac{8G}{3c^5 r}\, P_{ijab}(\mathbf{n})\, \epsilon_{acd}\, n_c\, \overset{(2)}{J}_{bd}(u) , \tag{40}$$

where the (trace-free) current quadrupole moment is given by

$$J_{ij} = \epsilon_{ab(i} \int d^3\mathbf{x}\, x_{j)} x_a \frac{T^{0b}}{c} + O\left(\varepsilon^2\right) . \tag{41}$$

It could be, rather ironically, that the first direct detection of gravitational waves would follow the formula (40) rather than the classic formula (37) appearing in all text-books such as [8].

3.3 Energy balance equation and radiation reaction

The stress-energy pseudo-tensor of all matter and gravitational fields (in harmonic coordinates) $\tau^{\mu\nu}$ is defined by (6). Now, for gravitational waves propagating in vacuum at large distances from their sources (in regions where the waves are almost planar), it is appropriate to define the stress-energy tensor of the waves as the gravitational source term [involving $\Lambda^{\mu\nu}(h)$] in the definition of $\tau^{\mu\nu}$, in which $h_{\mu\nu}$ is replaced by the far-field metric (37). Since the expression of the metric is valid up to fractional terms $O(1/r^2)$ in the distance, and since $\Lambda^{\mu\nu}$ is at least quadratic in h, the stress-energy tensor of gravitational waves will be valid up to $O(1/r^3)$. Thus, we define, in the far-zone,

$$T^{\mu\nu}_{\rm GW} = \frac{c^4}{16\pi G}\, \Lambda^{\mu\nu} + O\left(\frac{1}{r^3}\right) . \tag{42}$$

Now to quadratic order $\Lambda^{\mu\nu}$ is a complicated sum of terms like $h\partial\partial h + \partial h \partial h$. But when using (33) together with the fact that $k^2 = 0$, this sum simplifies drastically and we end up with [still neglecting $O(1/r^3)$]

$$T^{\mu\nu}_{\rm GW} = \frac{c^2}{32\pi G}\, k^\mu k^\nu \, \overset{(1)}{h}{}^{\rm TT}_{ij} \overset{(1)}{h}{}^{\rm TT}_{ij} = \frac{c^2}{16\pi G}\, k^\mu k^\nu \left[(\overset{(1)}{h}_+)^2 + (\overset{(1)}{h}_\times)^2\right] . \tag{43}$$

The second form is obtained from the inverse of (39): $h^{\rm TT}_{ij} = (p_i p_j - q_i q_j) h_+ + (p_i q_j + p_j q_i) h_\times$. The expression (43) takes the classic form $\sigma \; k^\mu k^\nu$ of the stress-energy tensor for a swarm of massless particles (gravitons) moving with the speed of light. Notice from (43) that the energy density of waves is positive definite. In the general case where we do not neglect the terms $O(1/r^3)$ the previous expressions of $T^{\mu\nu}_{\rm GW}$ are still valid, but provided that one performs a suitable average over several gravitational wavelengths (see [8]). For quadrupole waves, substituting the quadrupole formula (37), we get

$$T^{\mu\nu}_{\rm GW} = \frac{G}{8\pi r^2 c^4}\, k^\mu k^\nu P_{ijkl}(\mathbf{n}) \; \overset{(3)}{I}_{ij} \overset{(3)}{I}_{kl} . \tag{44}$$

We can integrate the conservation law $\partial_\nu \tau^{\mu\nu} = 0$ over the usual three-dimensional space (volume element d^3x), and use the Gauss theorem to obtain a flux of $T^{\mu\nu}_{\rm GW}$ through a surface at infinity (exterior surface element dS_i), so that

$$\frac{d}{dt} \int d^3\mathbf{x}\; \tau^{\mu 0} = -c \int dS_i \, T^{\mu i}_{\rm GW} . \tag{45}$$

We consider the $\mu = 0$ component of this law, substitute for $T^{\mu\nu}_{\rm GW}$ the expression (44) at the quadrupole approximation, perform the angular integration assuming for simplicity a coordinate sphere at infinity (i.e. $dS_i = r^2 n_i d\Omega$), and obtain the famous Einstein (mass-type) quadrupole formula

$$\frac{dE}{dt} = -\frac{G}{5c^5} \left\{ \overset{(3)}{I}_{ij} \overset{(3)}{I}_{ij} + O\left(\varepsilon^2\right) \right\} , \tag{46}$$

where $E = \int d^3x \; \tau^{00}$ represents the energy (matter + gravitation) of the source, and where we re-installed the correct post-Newtonian remainder $O(\varepsilon^2)$. Without proof we give also the formula corresponding to the current-type quadrupole moment,

$$\left(\frac{dE}{dt}\right)_{|\rm current} = -\frac{16 G}{45 c^7} \, \overset{(3)}{J}_{ij} \overset{(3)}{J}_{ij} , \tag{47}$$

where the current quadrupole moment is defined by (41).

Interestingly, we can treat the decrease of the energy as the result of the back-action of a radiative force (cf. [46]). We operate by parts the time-derivatives in (46) so as to obtain

$$\frac{d}{dt}\left(E + \frac{\delta E_5}{c^5}\right) = -\frac{G}{5c^5} \, \overset{(1)}{I}_{ij} \overset{(5)}{I}_{ij} \tag{48}$$

where we put on the left side a term in the form of a total time-derivative, representing a correction of order $1/c^5$ to the energy E, given by

$$\frac{\delta E_5}{c^5} = \frac{G}{5c^5} \left[\overset{(3)}{I}_{ij} \overset{(2)}{I}_{ij} - \overset{(4)}{I}_{ij} \overset{(1)}{I}_{ij} \right] . \tag{49}$$

Now, after a time much longer than the characteristic period of the source (for definiteness one can consider a quasi-periodic source or perform a suitable average; see e.g. [47]), the contribution due to the correcting term (49) will become negligible as compared to the right hand side of (48). Therefore, *in the long term*, we can ignore this term and finally, the equation (48) can be re-written equivalently in a form where the energy loss in the source is the result of the work of a radiation reaction force $\mathbf{F}_{\rm reac}$, namely

$$\frac{dE}{dt} = -\int d^3x \; \mathbf{F}_{\rm reac} \cdot \mathbf{v} \tag{50}$$

where

$$F^i_{\rm reac}(t,\mathbf{x}) = \frac{2G}{5c^5} \, \rho \, x_j \overset{(5)}{I}_{ij}(t) \; , \qquad \rho \equiv T^{00}/c^2 \; . \tag{51}$$

The equation (50)-(51) is called the radiation-reaction quadrupole formula; the specific expression (51) of the radiation reaction force is called after Burke and Thorne [6, 7, 8]. This force is to be interpreted as a small Newtonian-like force superposed to the usual Newtonian force at the 2.5PN order (or ε^5). Actually, the Burke-Thorne radiation reaction force is valid only in a special gauge. That is, only in a special gauge, differing for instance from the harmonic or ADM gauges, does the source equation of motion involve at 2.5PN order the correcting force (50)-(51). (See [48] for a discussion of various expressions of the radiation-reaction force in different gauges.) Notice that the reaction force (51) contains time derivatives up to the fifth order inclusively. In practice, for implementation in numerical codes, high time-derivatives have the tendency of decreasing the precision of a numerical computation, and therefore it is advantageous to choose other expressions of the reaction force for implementation in numerical codes [49, 50]. On the other hand, the order of second, third, and higher time-derivatives can be reduced by making use of the Newtonian equations of motion of the matter source. Subsequent implication of such a form of the radiation reaction in binary systems leads, for example, to the theoretical prediction of the rate of orbital decay shown in (1).

4 Post-Newtonian gravitational radiation

4.1 The multipole moments in the post-Newtonian approximation

In Section 2 we presented the formula for the multipole expansion of the field outside the source in linearized gravity. In the present section let us present, without proof, the corresponding formula in the full *non-linear* theory, i.e. when the Einstein field equations (8) are solved taking into account the gravitational source term $\Lambda^{\mu\nu}$. The formula will be valid whenever the post-Newtonian expansion is valid, i.e. when (25) holds. Under this assumption the field in the near-zone of a slowly-moving source can be expanded in non-analytic (involving logarithms) series of $1/c$ [17]. The general structure of the expansion is

$$\overline{h}^{\mu\nu}(t,\mathbf{x},c) = \sum_{p,q} \frac{(\ln c)^q}{c^p} \, h^{\mu\nu}_{pq}(t,\mathbf{x}) \; , \tag{52}$$

where $h^{\mu\nu}_{pq}$ are the functional coefficients of the expansion ($p, q =$ integers, including the zero). The general multipole expansion of the metric field $\mathcal{M}(h)$ is found by requiring that when re-developed in the near-zone in the limit of the parameter $r/c \to 0$ (which is equivalent with the formal re-expansion in the limit $c \to \infty$) it *matches* with the previous post-Newtonian expansion (52) in the sense of the mathematical techniques of matched asymptotic expansions, i.e.

$$\overline{\mathcal{M}(h)} = \mathcal{M}(\overline{h}) \; . \tag{53}$$

It is worthwhile noting that the equality (53) should be true in the sense of formal series, i.e. term by term in each coefficient after both sides of the equation are re-arranged with respect to the same expansion parameter.

We find [21, 51] that the multipole expansion generalizing (14) to the full theory is composed of two terms,

$$\mathcal{M}(h^{\mu\nu}) = \text{finite part}\, \Box_R^{-1}[\mathcal{M}(\Lambda^{\mu\nu})] - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{H}_L^{\mu\nu}(t-r/c)\right\} , \tag{54}$$

where $\Box_R^{-1}$ is the inverse flat space-time retarded operator. Herein, the first term is a particular solution of the Einstein field equations outside the matter compact support, i.e. it satisfies $\Box h^{\mu\nu}_{\text{part}} = \Lambda^{\mu\nu}$, and the second term is a retarded solution of the source-free (homogeneous) wave equation, i.e. $\Box h^{\mu\nu}_{\text{hom}} = 0$. The "multipole moments" parametrizing this homogeneous solution are given explicitly by an expression similar to (15),

$$\mathcal{H}_L^{\mu\nu}(u) = \text{finite part} \int d^3\mathbf{x}\; x_L\, \overline{\tau}^{\mu\nu}(\mathbf{x}, u) , \tag{55}$$

but involving in place of the matter stress-energy tensor $T^{\mu\nu}$ the *post-Newtonian* expansion $\overline{\tau}^{\mu\nu}$, in the sense of (52), of the *total* (matter+gravitation) pseudo-tensor $\tau^{\mu\nu}$ defined by (6). Both terms in (54) involve an operation of taking the finite part. This finite part can be defined precisely by means of an analytic continuation (see [51] for details), but in fact it is basically equivalent to taking the finite part of a divergent integral in the sense of Hadamard [52]. Notice in particular that the finite part in the expression of the multipole moments (55) deals with the behaviour of the integral *at infinity*: $r \to \infty$ (without the finite part the integral would be divergent because of the factor $x_L \sim r^l$ in the integrand and the fact that the pseudo-tensor $\overline{\tau}^{\mu\nu}$ is not of compact support). One can show that the multipole expansion (54)-(55) is equivalent with a different one proposed recently by Will and Wiseman [22].

Generally, it is more useful (for applications) to express the multipole expansion not in terms of the moments (55), but in terms of symmetric trace-free (STF) moments. We denote the STF projection with a hat, $\hat{x}_L \equiv \text{STF}(x^L)$, so that, for instance, $\hat{x}_{ij} = x_i x_j - \frac{1}{3}\delta_{ij}\mathbf{x}^2$. Then it can be shown that the STF multipole expansion equivalent to (54)-(55) reads,

$$\mathcal{M}(h^{\mu\nu}) = \text{finite part}\, \Box_R^{-1}[\mathcal{M}(\Lambda^{\mu\nu})] - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{F}_L^{\mu\nu}(t-r/c)\right\} , \tag{56}$$

where the parametrizing multipole moments are a bit more complicated,

$$\mathcal{F}_L^{\mu\nu}(u) = \text{finite part} \int d^3\mathbf{x}\; \hat{x}_L \int_{-1}^{1} dz\; \delta_l(z)\overline{\tau}^{\mu\nu}(\mathbf{x}, u + z|\mathbf{x}|/c) . \tag{57}$$

With respect to the non-tracefree expression (55) this involves an extra integration over the variable z, with weighting function

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l , \qquad \int_{-1}^{1} dz\delta_l(z) = 1 , \qquad \lim_{l\to+\infty}\delta_l(z) = \delta(z) . \tag{58}$$

The results (56)-(58) permit us to define a very convenient notion of the *source multipole moments*. Quite naturally, these are constructed from the ten components of $\mathcal{F}_L^{\mu\nu}(u)$. First of all, we reduce the number of independent components to only six by using the four relations given by the harmonic gauge condition $\partial_\nu h^{\mu\nu} = 0$. Next we apply standard STF techniques (see [18, 43, 51] for details), and, in this way, we are able to define *six* STF-irreducible multipole moments, denoted $I_L, J_L, W_L, X_L, Y_L, Z_L$, which are given by *explicit* integrals extending over the post-Newtonian-expanded pseudo-tensor $\overline{\tau}^{\mu\nu}$ like in (57). All of the moments $I_L, J_L, \cdots, Z_L$ are referred to as the moments of the source; however notice that, among them, only the moments I_L (mass-type moment) and J_L (current-type) play a physical role at the *linearized* level. The other four moments W_L, X_L, Y_L, Z_L simply parametrize a linear gauge transformation and can often be omitted from the consideration. Only at the order 2.5PN or ε^5 do they start playing a physical role. The complete formulas for the moments I_L, J_L are [51]

$$\begin{aligned} I_L(u) &= \text{finite part} \int d^3\mathbf{x} \int_{-1}^{1} dz \Big\{ \delta_l \hat{x}_L \Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)} \delta_{l+1} \hat{x}_{iL} \partial_t \Sigma_i \\ &\qquad + \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)} \delta_{l+2} \hat{x}_{ijL} \partial_t^2 \Sigma_{ij} \Big\} (\mathbf{x}, u + z|\mathbf{x}|/c) \ , \end{aligned} \tag{59}$$

$$\begin{aligned} J_L(u) &= \text{finite part} \int d^3\mathbf{x} \int_{-1}^{1} dz \ \varepsilon_{ab<i_l} \Big\{ \delta_l \hat{x}_{L-1>a} \Sigma_b \\ &\qquad - \frac{2l+1}{c^2(l+2)(2l+3)} \delta_{l+1} \hat{x}_{L-1>ac} \partial_t \Sigma_{bc} \Big\} (\mathbf{x}, u + z|\mathbf{x}|/c) \ . \end{aligned} \tag{60}$$

In these expressions, $<>$ refers to the STF projection, and we have posed

$$\Sigma \equiv \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \quad (\text{where } \overline{\tau}^{ii} \equiv \delta_{ij} \overline{\tau}^{ij}) \ , \qquad \Sigma_i \equiv \frac{\overline{\tau}^{0i}}{c} \ , \qquad \Sigma_{ij} \equiv \overline{\tau}^{ij} \ . \tag{61}$$

The moments I_L, J_L given by these formulas are valid formally up to any post-Newtonian order. They constitute the generalization in the non-linear theory of the Newtonian moments introduced earlier in (38) and (41).

In order to apply usefully these moments to a given problem, one must find the explicit expressions of the moments at a given post-Newtonian order by inserting into them the components of the pseudo-tensor $\overline{\tau}^{\mu\nu}$ obtained from an explicit post-Newtonian algorithm. Without entering into details, we find for instance that at the 1PN order the mass-type source moment I_L is given (rather remarquably) by a simple compact-support formula [17, 21], on which we can, therefore, remove the finite part prescription:

$$I_L = \int d^3\mathbf{x} \Big\{ \hat{x}_L \sigma + \frac{|\mathbf{x}|^2 \hat{x}_L}{2c^2(2l+3)} \partial_t^2 \sigma - \frac{4(2l+1)\hat{x}_{iL}}{c^2(l+1)(2l+3)} \partial_t \sigma_i \Big\} + O\left(\varepsilon^4\right) \ . \tag{62}$$

We denote the compact-support parts of the source scalar and vector densities in (61) by

$$\sigma \equiv \frac{T^{00} + T^{ii}}{c^2} \ , \qquad \sigma_i \equiv \frac{T^{0i}}{c} \ . \tag{63}$$

See Blanchet and Schäfer [53] for application of the formula (62) to the computation of the relativistic correction in the $\dot{P}_b$ of a binary pulsar [given to lowest order by (1)]. On the other hand, Damour, Soffel, and Xu [54] (extending previous work of Brumberg and Kopeikin [55], [56]) used the formula in their study of the Solar-system dynamics at 1PN order. The property of being of compact support is a special feature of the 1PN mass-moment I_L. To higher-order (2PN and higher) the mass moment I_L is intrinsically of non-compact support (see its expression in [21]); hence the finite part prescription in the definition of the moment I_L plays a crucial role at 2PN. Similarly, starting already at 1PN order, the current-moment J_L is intrinsically of non-compact support [19, 21].

In a linear theory, the source multipole moments coincide evidently with the radiative multipole moments, defined as the coefficients of the multipole expansion of the $1/r$ term in the distance to the source at retarded times $t - r/c =$const. (this is evident from Section 2). However, in a non-linear theory like general relativity, the source multipole moments interact with each other in the exterior field through the non-linearities. This is clear from the presence of the first term in (56), containing the gravitational source $\Lambda^{\mu\nu}$, and which does contribute to the $1/r$ part of the metric at infinity. Therefore the source multipole moments must be related to the radiative ones, the latter constituting in this approach the actual observables of the field at infinity.

In the TT projection of the metric field one can define two sets of radiative moments U_L (mass-type) and V_L (current-type). The *definition* of these moments is that they parametrize the $1/r$-term of the ij components of the metric in TT gauge. Thus, extending the formula (37), the radiative moments are given by the decomposition

$$h_{ij}^{\mathrm{TT}} = -\frac{4G}{c^2 r} P_{ijab} \sum_{l\geq 2} \frac{1}{c^l l!} \left\{ n_{L-2} U_{abL-2} - \frac{2l}{c(l+1)} n_{cL-2} \varepsilon_{cd(a} V_{b)dL-2} \right\} + O\left(\frac{1}{r^2}\right) . \tag{64}$$

The radiative moments U_L, V_L are related to the l-th time-derivatives of the corresponding source moments. Let us give, without proof, the result of the connection of the radiative moments to the source moments (59)-(60) to the order ε^3 (or 1.5PN) inclusively. To this order some non-linear "monopole-radiative l-pole" interactions appear, which correspond physically to the scattering of the l-pole wave on the static curvature induced by the total mass of the source (i.e. the mass monopole $M \equiv I$) – an effect well known under the name of tail of gravitational waves. We find [20, 21]

$$U_L(u) = \overset{(2)}{I}_L(u) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau \, \overset{(4)}{I}_L(u-\tau) \left[\ln\left(\frac{\tau}{2b}\right) + \kappa_l \right] + O\left(\varepsilon^4\right) , \tag{65}$$

$$V_L(u) = \overset{(2)}{J}_L(u) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau \, \overset{(4)}{J}_L(u-\tau) \left[\ln\left(\frac{\tau}{2b}\right) + \pi_l \right] + O\left(\varepsilon^4\right) . \tag{66}$$

The same expressions come out from the Will and Wiseman formalism [22]. Here, b is a normalization constant (essentially irrelevant since it corresponds to a choice of the origin of time in the far zone), and κ_l, π_l are given by

$$\kappa_l = \frac{2l^2+5l+4}{l(l+1)(l+2)} + \sum_{k=1}^{l-2} \frac{1}{k} , \qquad \pi_l = \frac{l-1}{l(l+1)} + \sum_{k=1}^{l-1} \frac{1}{k} . \tag{67}$$

The first term in (65) relates essentially to the original quadrupole formula (see (37)). From (65)-(66) one sees that the first non-linearity in the propagation of the waves is at 1.5PN order with respect to the quadrupole formula. Thus, with enough precision, one can replace in (65) the mass-type moment I_L by its compact-support expression that we computed in (62) [since the post-Newtonian remainder in (62) is $O(\varepsilon^4)$].

4.2 Post-Newtonian radiation reaction

Emission of gravitational radiation affects the equations of motion of an isolated system dominantly at the 2.5PN order beyond the Newtonian acceleration. In a suitable gauge the radiation-reaction force density at the 2.5PN order is given by the quadrupole formula (51). In this Section we extend this quadrupole formula to include the relativistic corrections up to the relative 1.5PN order, which means the absolute 4PN order with respect to the Newtonian force. The method is to compute the radiation reaction by means of the matching [in the sense of (53)] of the post-Newtonian field to the exterior multipolar field. Indeed, recall that the post-Newtonian field is valid only in the near zone, and, thus, only via a matching can it incorporate information from the correct boundary condition, *viz* the no-incoming radiation condition imposed at infinity by equation (10), which specifies the braking character of gravitational radiation reaction.

To the relative 1.5PN order, and in a suitable gauge, it can be shown that the reaction force derives from some "electromagnetic-like" scalar and vector reaction potentials V^{reac} and V_i^{reac}. Explicitly we have [57]

$$\begin{aligned} V^{\mathrm{reac}}(\mathbf{x},t) &= -\frac{G}{5c^5} x_{ij} \left\{ \overset{(5)}{I}_{ij}(t) + \frac{4GM}{c^3} \int_0^{+\infty} d\tau \ln\left(\frac{\tau}{2b}\right) \overset{(7)}{I}_{ij}(t-\tau) \right\} \\ &+ \frac{G}{c^7} \left[\frac{1}{189} x_{ijk} \overset{(7)}{I}_{ijk}(t) - \frac{1}{70} \mathbf{x}^2 x_{ij} \overset{(7)}{I}_{ij}(t) \right] + O\left(\varepsilon^9\right) , \end{aligned} \tag{68}$$

$$V_i^{\mathrm{reac}}(\mathbf{x},t) = \frac{G}{c^5} \left[\frac{1}{21} \hat{x}_{ijk} \overset{(6)}{I}_{jk}(t) - \frac{4}{45} \epsilon_{ijk} \, x_{jm} \overset{(5)}{J}_{km}(t) \right] + O\left(\varepsilon^7\right) . \tag{69}$$

The dominant term in the formula (68)-(69) is the standard Burke-Thorne reactive *scalar* potential at 2.5PN order [compare (68) with (51)]. In this term, consistently with the approximation, one must insert the 1PN expression of the moment as given by (62). The Burke-Thorne term is of "odd"-parity-type as it corresponds to an odd power of $1/c$, and thus changes sign upon a time reversal (or more precisely when we replace the retarded potentials by advanced ones). Similarly is the next term in (68)-(69) is 3.5PN (i.e. ε^7), involving both the mass-quadrupole, mass-octupole and current-quadrupole moments (the term ε^5 in the vector potential V_i^{reac} corresponds really to ε^7 in the equations of motion). However, notice that the next term in the reaction scalar potential V^{reac}, at 4PN or ε^8 order, belongs to the "even"-parity-type. Nevertheless this term is really part of the radiation reaction for it is not invariant under a time reversal, as is involves an integration over the "past" history of the source, so that when changing the retarded potentials to advanced ones the integration range would change to the whole "future"; hence, this term does not stay invariant. It represents the contribution of tails in the radiation-reaction force, and is nicely consistent, in the sense of energy conservation, with the tails in the far zone [equation (65)]. For explicit computations of the back-reaction to 3.5PN order in the case of point-mass binary systems see Iyer and Will [58, 59], and Jaranowski and Schäfer [60].

Using the matching (53) one finds that the near-zone post-Newtonian metric (to 1.5PN relative order in both the "damping" and "conservative" effects) is parametrized in this gauge by some generalized potentials

$$\mathcal{V}_\mu = \Box^{-1}_{\text{sym}}\left[-4\pi G\sigma_\mu\right] + V_\mu^{\text{reac}} \ . \tag{70}$$

The first term represents, to this post-Newtonian order, the conservative part of the metric; it is of the normal "even"-parity-type and is given by the usual symmetric integral (half-retarded plus half-advanced) of the source densities $\sigma_\mu = (\sigma, \sigma_i)$ given by (63). The second term V_μ^{reac} denotes the radiation-reaction potentials (68)-(69). By inserting the metric parametrized by (70) into the equations of motion of the source (i.e. $\partial_\nu \tau^{\mu\nu} = 0 \Leftrightarrow \nabla_\nu T^{\mu\nu} = 0$), and considering the integral of energy, we obtain the balance equation

$$\frac{dE}{dt} = \int d^3\mathbf{x} \left\{ -\sigma \partial_t V^{\text{reac}} + \frac{4}{c^2}\sigma_j \partial_t V_j^{\text{reac}} \right\} + O\left(\varepsilon^9\right) \ . \tag{71}$$

Here E denotes the energy of the source at the 1PN (or even 1.5PN) order. Actually what we obtain is not E but some $E + \delta E_5/c^5 + \delta E_7/c^7$ like in (48). Arguing as before we neglect these δE_5 and δE_7. Substituting now the expressions (68)-(69) for the reactive potentials (and neglecting other δE's) we get

$$\begin{aligned} \frac{dE}{dt} = &- \frac{G}{5c^5}\left\{ \overset{(3)}{I}_{ij} + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \ln\left(\frac{\tau}{2b}\right) \overset{(5)}{I}_{ij}(t-\tau) \right\}^2 \\ &- \frac{G}{c^7}\left[\frac{1}{189} \overset{(4)}{I}_{ijk} \overset{(4)}{I}_{ijk} + \frac{16}{45} \overset{(3)}{J}_{ij} \overset{(3)}{J}_{ij} \right] + O\left(\varepsilon^9\right) \ , \end{aligned} \tag{72}$$

The right-side is exactly in agreement with the computation of the total flux energy emitted in gravitational waves at infinity, which is computed making use of the stress-energy tensor of gravitational waves (43). In particular we recover in the brackets of the first term of (72) the third time-derivative of the radiative moment U_{ij} including its tail contribution. The difference with the standard derivation of the flux is that instead of computing a surface integral at infinity we have performed the computation completely within the source, using the local source equations of motion.

5 Light propagation in gravitational fields of isolated sources

5.1 General solution of the light propagation equation

We are going now to calculate in linearized approximation the propagation of a light ray in the gravitational field of an isolated source at rest showing a mass-monopole, a spin-dipole and a time-dependent mass-quadrupole moment. In linearized approximation, the propagation equation for a particle (massless or with mass) with space coordinates $x^i(t)$ reads

$$\begin{aligned}\ddot{x}^i(t) &= \frac{1}{2}g_{00,i} - g_{0i,t} - \frac{1}{2}g_{00,t}\dot{x}^i - g_{ik,t}\dot{x}^k - \left(g_{0i,k} - g_{0k,i}\right)\dot{x}^k - \\ & g_{00,k}\dot{x}^k\dot{x}^i - \left(g_{ik,j} - \frac{1}{2}g_{kj,i}\right)\dot{x}^k\dot{x}^j + \left(\frac{1}{2}g_{kj,t} - g_{0k,j}\right)\dot{x}^k\dot{x}^j\dot{x}^i \,,\end{aligned} \tag{73}$$

where the metric coefficients $g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}$, in linear approximation, are related with $h^{\mu\nu}$ from Sections 2–4 through $f_{\mu\nu} = -h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h$. The dots denote differentiation with respect to time t and $c = 1$ has been put for simplicity. In the linear approximation scheme, the velocity $\dot{x}^i$ appearing on the right-hand-side of (73) can be treated as a constant vector. For massless particles (photons) it has unit length, i.e. $\dot{x}^i\dot{x}^i = 1$. In the following we use $\dot{x}^i = k^i$ in the right hand side of (73).

The unperturbed motion of photons reads

$$x^i(t) = x_0^i + k^i(t - t_0) \,, \tag{74}$$

where x_0^i denotes the position of the photon at time of emission t_0. For solving the light propagation equation (73) it is very convenient to introduce the new time parameter τ defined by $\tau = t - t^*$, where t^* denotes the time of the closest approach of the photon to the source of the gravitational field. Then it holds

$$x^i(\tau) = \xi^i + k^i\tau \,, \tag{75}$$

where ξ^i is the vector pointing from the position of the source to the position of the photon at the closest approach. Its length is the impact parameter $|\xi| = d$. Obviously, ξ^i and k^i are orthogonal to each other in the euclidean sense, i.e. $\xi^i k^i = 0$. Therefore, the length of x^i, $r = |x|$, takes the simple form $r = \sqrt{\tau^2 + d^2}$. Introducing the derivatives $\hat{\partial}_i = \hat{P}_{ij}\partial/\partial\xi^j$ and $\hat{\partial}_\tau = \partial/\partial\tau$, where $\hat{P}_{ij} = \delta_{ij} - k_i k_j$ is the projection operator onto the plane orthogonal to k^i, allows the light-propagation equation to be written as

$$\ddot{x}^i(\tau) = \frac{1}{2}k^\alpha k^\beta \hat{\partial}_i f_{\alpha\beta} - \hat{\partial}_\tau\left(k^\alpha f_{i\alpha} + \frac{1}{2}k^i f_{00} - \frac{1}{2}k^i k^j k^p f_{jp}\right) \,, \tag{76}$$

where the four-dimensional vector k^α reads $k^\alpha = (1, k^i)$.

To get a complete overview of the influence of a gravitational wave, emitted from an isolated source, on the propagation of light rays, we use the representation of the metric coefficients (14) which is valid all-over in the space outside a domain which includes the matter source. Splitting $f_{\mu\nu}$ into a canonical part $f_{\mu\nu}^{\rm can}$ which contains trace-free tensors only, and a gauge part, i.e. $f_{\mu\nu} = f_{\mu\nu}^{\rm can} + \partial_\mu w_\nu + \partial_\nu w_\mu$, we obtain in case of mass-monopole, spin-dipole, and mass-quadrupole source moments (remember the source being at rest)

$$f_{00}^{\rm can} = \frac{2M}{r} + \partial_{pq}\left[\frac{I_{pq}(t-r)}{r}\right] \,, \tag{77}$$

$$f_{0i}^{\rm can} = -\frac{2\varepsilon_{ipq}S_p n_q}{r^2} + 2\partial_j\left[\frac{\dot{I}_{ij}(t-r)}{r}\right] \,, \tag{78}$$

$$f_{ij}^{\rm can} = \delta_{ij} f_{00}^{\rm can} + \frac{2}{r}\ddot{I}_{ij}(t-r) \,. \tag{79}$$

Herein, for simplicity, we have put $G = 1$, and $\partial_i = \partial/\partial x^i$. The mass M, spin S^i, and the quadrupole moment I_{ij} of the source of gravitational waves are given (in the Newtonian approximation) by the expressions (19), (21), and (38). The explicit expressions for the gauge functions w^μ relating $f_{\mu\nu}^{\rm can}$ with $f_{\mu\nu}$ are important for general discussion of light propagation in the field of gravitational waves emitted by the isolated source. However, for the sake of simplicity they will be omitted. Their precise form can be found in [37].

The insertion of these expressions (77)-(79) into the equation (76) results in the equation

$$
\begin{aligned}
\ddot{x}^i(\tau) &= \left[2M\left(\hat{\partial}_i - k_i\hat{\partial}_\tau\right) - 2S^p\left(\varepsilon_{ipq}\hat{\partial}_{q\tau} - k_j\varepsilon_{jpq}\hat{\partial}_{iq}\right)\right]\left\{\frac{1}{r}\right\} \\
&\quad + \left(\hat{\partial}_{ipq} - k_i\hat{\partial}_{pq\tau} + 2k_p\hat{\partial}_{iq\tau}\right)\left\{\frac{I_{pq}(t-r)}{r}\right\} - 2\hat{P}_{ij}\hat{\partial}_{q\tau}\left\{\frac{\dot{I}_{jq}(t-r)}{r}\right\} \\
&\quad - \hat{\partial}_{\tau\tau}\left[w^i + \varphi^i - k^i\ \left(w^0 + \varphi^0\right)\right] ,
\end{aligned} \tag{80}
$$

where the vector φ^μ denotes terms which are of gauge type. The precise form of φ^μ is not important here and can be found in [37].

The solution of equation (80), using the boundary conditions $\dot{x}^i(-\infty) = k^i$ and $x^i(\tau_0) = x_0^i$ – emission point in space of the light ray at time τ_0, reads

$$
\dot{x}^i(\tau) = k^i + \dot{\Xi}^i(\tau) , \tag{81}
$$

$$
x^i(\tau) = x_N^i(\tau) + \Xi^i(\tau) - \Xi^i(\tau_0) , \tag{82}
$$

where $x_N^i(\tau)$ denotes the unperturbed trajectory (75). The relativistic perturbation of the trajectory is given by

$$
\begin{aligned}
\dot{\Xi}^i(\tau) &= \left(2M\hat{\partial}_i + 2S^p k_j\varepsilon_{jpq}\hat{\partial}_{iq}\right)A(\tau,\xi) + \hat{\partial}_{ipq}B_{pq}(\tau,\xi) - \\
&\quad \left(2Mk_i + 2S^p\varepsilon_{ipq}\hat{\partial}_q\right)\left\{\frac{1}{r}\right\} - \left(k_i\hat{\partial}_{pq} - 2k_p\hat{\partial}_{iq}\right)\left\{\frac{I_{pq}(t-r)}{r}\right\} \\
&\quad - 2\hat{P}_{ij}\hat{\partial}_q\left\{\frac{\dot{I}_{jq}(t-r)}{r}\right\} - \hat{\partial}_\tau\left[w^i + \varphi^i - k^i\ \left(w^0 + \varphi^0\right)\right] ,
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
\Xi^i(\tau) &= \left(2M\hat{\partial}_i + 2S^p k_j\varepsilon_{jpq}\hat{\partial}_{iq}\right)B(\tau,\xi) - \left(2Mk_i + 2S^p\varepsilon_{ipq}\hat{\partial}_q\right)A(\tau,\xi) \\
&\quad + \hat{\partial}_{ipq}D_{pq}(\tau,\xi) - \left(k_i\hat{\partial}_{pq} - 2k_p\hat{\partial}_{iq}\right)B_{pq}(\tau,\xi) - 2\hat{P}_{ij}\hat{\partial}_q C_{jq}(\tau,\xi) - \\
&\quad - w^i(\tau,\xi) - \varphi^i(\tau,\xi) + k^i\ \left[w^0(\tau,\xi) + \varphi^0(\tau,\xi)\right] ,
\end{aligned} \tag{84}
$$

whereby the scalar functions A and B and derivatives of the tensors B_{ij}, C_{ij}, D_{ij} are known fully explicitly. They are given by

$$
A(\tau,\xi) \equiv \int\frac{d\tau}{r} = \int\frac{d\tau}{\sqrt{d^2+\tau^2}} = -\ln\left(\sqrt{d^2+\tau^2} - \tau\right) , \tag{85}
$$

$$
B(\tau,\xi) \equiv \int A(\tau,\xi)d\tau = -\tau\ln\left(\sqrt{d^2+\tau^2} - \tau\right) - \sqrt{d^2+\tau^2} , \tag{86}
$$

$$
\hat{\partial}_k B_{ij}(\tau,\xi) = (yr)^{-1}\, I_{ij}(t-r)\xi^k , \tag{87}
$$

$$
\hat{\partial}_k C_{ij}(\tau,\xi) = (yr)^{-1}\, \dot{I}_{ij}(t-r)\xi^k , \tag{88}
$$

$$
\hat{\partial}_{ijk}D_{pq}(\tau,\xi) = \frac{1}{y}\left[\left(\hat{P}^{ij} + \frac{\xi^i\xi^j}{yr}\right)\hat{\partial}_k B_{pq}(\tau,\xi) + \hat{P}^{jk}\hat{\partial}_i B_{pq}(\tau,\xi) + \xi^j\hat{\partial}_{ik}B_{pq}(\tau,\xi)\right] , \tag{89}
$$

where the variable $y = \tau - \sqrt{\tau^2 + d^2}$ is the retarded time argument for the photon which passes through the point of closest approach to the source of the gravitational radiation at time $t^* = 0$. More details concerning the method of calculation of light ray trajectory in time dependent gravitational fields can be found in [37, 38].

5.2 Time delay and bending of light

The time delay results in the form

$$t - t_0 \;=\; |\mathbf{x} - \mathbf{x}_0| - \mathbf{k} \cdot \Xi(\tau) + \mathbf{k} \cdot \Xi(\tau_0) \;, \tag{90}$$

or

$$t - t_0 \;=\; |\mathbf{x} - \mathbf{x}_0| + \Delta_M(t, t_0) + \Delta_S(t, t_0) + \Delta_Q(t, t_0) \;, \tag{91}$$

where $|\mathbf{x} - \mathbf{x}_0|$ is the usual Euclidean distance between the points of emission, $\mathbf{x}_0$, and reception, $\mathbf{x}$, of the photon, Δ_M is the classical Shapiro delay produced by the (constant) spherically symmetric part of the gravitational field of the deflector (see, e.g. [8]), Δ_S is the Lense-Thirring or Kerr delay due to the (constant) spin of the localized source of gravitational waves [39], and Δ_Q describes an additional delay caused by the time dependent quadrupole moment of the source [37]. Specifically we obtain

$$\begin{aligned}
\Delta_M &= 2M \ln\left[\frac{r+\tau}{r_0+\tau_0}\right] \;, &(92)\\
\Delta_S &= -2\varepsilon_{ijk} k^j S^k \hat{\partial}_i \ln\left[\frac{r+\tau}{r_0+\tau_0}\right] \;, &(93)\\
\Delta_Q &= \hat{\partial}_{ij}\left[B_{ij}(\tau,\xi) - B_{ij}(\tau_0,\xi)\right] + \delta_Q(\tau,\xi) - \delta_Q(\tau_0,\xi) \;, &(94)
\end{aligned}$$

where

$$\delta_Q(\tau,\xi) = k^i\left(w^i + \varphi^i\right) - w^0 - \varphi^0 \;. \tag{95}$$

Let us now denote by α^i the dimensionless vector describing the total angle of deflection of the light ray measured at the point of observation and calculated with respect to vector k^i given at past null infinity. It is defined according to the relationship

$$\alpha^i(\tau,\xi) \;=\; k^i[\mathbf{k} \cdot \dot{\Xi}(\tau,\xi)] - \dot{\Xi}^i(\tau,\xi) = -\,\hat{P}_{ij}\,\dot{\Xi}^j(\tau,\xi) \;. \tag{96}$$

For observers being far away from the source of the gravitational wave the projection of the mass-quadrupole tensor of the source of gravitational radiation onto the plane orthogonal to the propagation direction of the gravitational wave is the crucial object which enters into the observable effects. It reads

$$I_{ij}^{\mathrm{TT}} = P_{ijpq} I_{pq} = I_{ij} + \frac{1}{2}\left(\delta_{ij} + n_i n_j\right) n_p n_q \; I_{pq} - \left(\delta_{ip} n_j n_q + \delta_{jp} n_i n_q\right)\; I_{pq} \;, \tag{97}$$

where again we denote $n_i = x^i/r$.

In the case of small impact parameter d ($d/r_0 \ll 1, d/r \ll 1$) we respectively obtain for the time delay and the angle of deflection

$$t - t_0 - |\mathbf{x} - \mathbf{x}_0| = -4\psi + 2M \ln(4 r r_0) \;, \tag{98}$$

$$\alpha_i = 4\hat{\partial}_i \psi \;, \tag{99}$$

where ψ is the gravitational lens potential having the form

$$\psi \;=\; \left[M + \varepsilon_{jpq} k^p S^q \hat{\partial}_j + \frac{1}{2}\, I_{pq}^{\mathrm{TT}}(t^*)\; \hat{\partial}_{pq}\right] \ln d \;. \tag{100}$$

(Notice that in this gravitational lens approximation $n^i = k^i$ holds.) Remarquably, the gravitational lens potential does depend on the gravitational source mass-quadrupole tensor only through its value at the time of closest approach. Furthermore, the gravitational lens potential decays like $1/d^2$, i.e. it is not being influenced by the wave part of the gravitational field [40, 37].

A direct consequence of the time delay formula is the frequency shift formula for a moving gravitational source

$$\frac{\delta\nu}{\nu} = 4\frac{\partial\psi}{\partial t^*} + v^i\alpha^i \ , \tag{101}$$

where v^i is the velocity of the observer. It is worthwhile noting that the expression (101) holds for the source of electromagnetic waves being at past null infinity. An exhaustive treatment of the gravitational frequency shift for arbitrary locations of observer, source of light, and the source of gravitational waves is rather complicated and has been done only recently in [38].

6 Detection of gravitational waves

In the asymptotic regime of a gravitational wave field, the time delay reads

$$\Delta_Q(\mathbf{k}; t, t_0) = \frac{k^i k^j}{1-\cos\theta}\left[\frac{\dot{I}^{\mathrm{TT}}_{ij}(t-r)}{r} - \frac{\dot{I}^{\mathrm{TT}}_{ij}(t_0-r_0)}{r_0}\right] \ , \tag{102}$$

where θ is the angle between the receiver - (light) emitter and receiver - (gravitational wave) source direction ($\cos\theta = -N^i k^i$) and where the assumption $|\mathbf{x} - \mathbf{x}_0| \ll r$ has been made. Let us now apply the above formula to the time delay in a Michelson interferometer. Obviously, in this case, $r = r_0$ holds. For simplicity we assume that the interferometer device is oriented orthogonal to the propagation direction of the gravitational wave, i.e. $N^i k^i_1 = N^i k^i_2$, where k^i_1 and k^i_2 denote the directions of the two interferometer arms which are taken to be orthogonal ($k^i_1 k^i_2 = 0$) and of equal length L. We also assume that the light, emitted from the beam-splitter, is reflected once at the end mirrors. Furthermore, the interferometer arms are to be oriented such that they coincide with the main axes of the plus-polarization. Then the relative time delay of the reflected light beams reads

$$\Delta_Q(\mathbf{k}_1; t, t-2L) = \frac{k^i_1 k^j_1}{r_0}[\dot{I}^{\mathrm{TT}}_{ij}(t-r_0) - \dot{I}^{\mathrm{TT}}_{ij}(t-r_0-2L)] \ . \tag{103}$$

The multiplication of the relative time delay by the angular frequency of the laser light, ω, which is treated as constant in the approximation under consideration, results in the measurable phase shift at time t of $\Delta\Phi(t) = 2\omega\Delta_Q(\mathbf{k}_1; t, t-2L)$. If we assume for the plus-component of the gravitational wave the expression $h_+(t-r_0) = A_+\cos(\omega_g t)$, where ω_g is the constant frequency of the wave and A_+ its constant amplitude, we obtain for the phase shift (cf. [61])

$$\Delta\Phi(t) = 2A_+\frac{\omega}{\omega_g}\sin(\omega_g L)\cos[\omega_g(t-L)] \ . \tag{104}$$

The maximal amplitude is achieved if the condition $\omega_g L = \pi/2$ holds. This yields

$$\Delta_{\max}\Phi(t) = 2A_+\frac{\omega}{\omega_g}\sin(\omega_g t) \ . \tag{105}$$

At the photo-diode the following photo-current results [62]

$$I_{\mathrm{ph}}(t) = I_{\min} + \frac{I_{\max} - I_{\min}}{2}\left[1 - \cos\phi(t)\right] \ , \tag{106}$$

where the phase $\phi(t)$ is composed out of the signal $\Delta\Phi$ plus a modulation term from Pockels cells, $\phi_m\sin(\omega_m t)$, i.e.

$$\phi(t) = \Delta\Phi(t) + \phi_m\sin(\omega_m t) \ . \tag{107}$$

The approximate decomposition of the expression (106) into a *dc* ("direct current" or non-alternate) part and a signal part reads $I_{\rm ph}(t) = I_{\rm dc} + I_{\omega_m}$, where

$$I_{\rm dc} = I_{\rm min} + \frac{I_{\rm eff}}{2}\left[1 - J_0(\phi_m)\cos\Delta\Phi(t)\right] \ , \tag{108}$$

$$I_{\omega_m} = I_{\rm eff} J_1(\phi_m)\sin\Delta\Phi(t)\sin(\omega_m t) \ , \tag{109}$$

with Bessel functions J_0 and J_1; $I_{\rm eff} = I_{\rm max} - I_{\rm min}$. Because of the smallness of the gravitational phase shift we get

$$I_{\rm dc} = I_{\rm min} + \frac{I_{\rm eff}}{2}\left[1 - J_0(\phi_m)\right] \ , \tag{110}$$

$$I_{\omega_m} = I_{\rm eff} J_1(\phi_m)\Delta\Phi(t)\sin(\omega_m t) \ . \tag{111}$$

Using the equation (105), the latter equation can be written

$$I_{\omega_m} = I_{\rm eff} J_1(\phi_m) A_+ \frac{\omega}{\omega_g}\left\{\cos\left[(\omega_m - \omega_g)t\right] - \cos\left[(\omega_m + \omega_g)t\right]\right\} \ . \tag{112}$$

In this side-band form, the signal from the gravitational wave is being detected. For more details we refer to [61].

References

[1] D.G. Blair (ed.): *The Detection of Gravitational Waves* (Cambridge University Press, Cambridge 1991).

[2] H. Bondi, M.G.J. van der Burg, and A.W.K. Metzner, *Proc. R. Soc. London* **A 269**, 21 (1962).

[3] R. Penrose: *Proc. R. Soc.* (London) **A 284**, 159 (1965).

[4] A. Einstein, L. Infeld, and B. Hoffmann: *Ann. Math.* **39**, 65 (1938).

[5] S. Chandrasekhar and F.P. Esposito: *Astrophys. J.* **160**, 153 (1970).

[6] W.L. Burke: *J. Math. Phys.* **12**, 401 (1971).

[7] W.L. Burke and K.S. Thorne: in *Relativity*, M. Carmeli et al. (eds.), Plenum Press: New York (1970) p. 208.

[8] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, Freeman: San Francisco (1973).

[9] J. Ehlers: *Ann. N.Y. Acad. Sci.* **336**, 279 (1980).

[10] A. Papapetrou and B. Linet: *Gen. Relat. Grav.* **13**, 335 (1981).

[11] T. Damour and N. Deruelle: *Phys. Lett.* **87A**, 81 (1981).

[12] T. Damour, in *Gravitational Radiation*, N. Deruelle and T. Piran (eds.), North-Holland: Amsderdam (1983), p. 59.

[13] G. Schäfer: *Ann. Phys.* (N.Y.) **161**, 81 (1985).

[14] S.M. Kopejkin (Kopeikin): *Astron. Zh.* **62**, 889 (1985), (In Russian).

[15] W.B. Bonnor: *Philos. Trans. R. Soc. London* **A 251**, 233 (1959).

[16] K.S. Thorne: *Rev. Mod. Phys.* **52**, 299 (1980).

[17] L. Blanchet and T. Damour: *Philos. Trans. R. Soc. London* **A 320**, 379 (1986).

[18] L. Blanchet and T. Damour: *Ann. Inst. H. Poincaré* (Phys. Théorique) **50**, 377 (1989).

[19] T. Damour and B.R. Iyer: *Ann. Inst. H. Poincaré* (Phys. Théorique) **54**, 115 (1991).

[20] L. Blanchet and T. Damour: *Phys. Rev.* **D 46**, 4304 (1992).

[21] L. Blanchet: *Phys. Rev.* **D 51**, 2559 (1995).

[22] C.M. Will and A.G. Wiseman: *Phys. Rev.* **D 54**, 4813 (1996).

[23] P.C. Peters and J. Mathews: *Phys. Rev.* **131**, 435 (1963).

[24] L.W. Esposito and E.R. Harrison: *Astrophys. J.* **196**, L1 (1975).

[25] R.V. Wagoner: *Astrophys. J.* **196**, L63 (1975).

[26] T. Damour: *Phys. Rev. Lett.* **51**, 1019 (1983).

[27] J.H. Taylor, L.A. Fowler, and P.M. Mc Culloch: *Nature* **277**, 437 (1979).

[28] J.H. Taylor: *Class. Quantum Grav.* **10**, S167 (1993).

[29] V.A. Brumberg, *Relativistic Celestial Mechanics*, Nauka: Moscow (1972). In Russian.

[30] C.H. Brans: *Astrophys. J.* **197**, 1 (1975).

[31] B. Mashhoon and L.P. Grishchuk: *Astrophys. J.* **236**, 990 (1980).

[32] A.G. Polnarev: *Sov. Astronomy* **29**, 607 (1985).

[33] R.W. Hellings: *Astron. J.* **91**, 650; Erratum **92**, 1446 (1986).

[34] V.A. Brumberg, *Essential Relativistic Celestial Mechanics*, Adam Hilger: Bristol (1991).

[35] S.A. Klioner: *Astron. Zh.* **68**, 1046 (1991). In Russian.

[36] S.A. Klioner and S.M. Kopeikin: *Astron. J.* **104**, 897 (1992).

[37] S.M. Kopeikin, G. Schäfer, C.R. Gwinn, and T.M. Eubanks: *Phys. Rev.* **D 59**, 084023 (1999).

[38] S.M. Kopeikin and G. Schäfer: *Phys. Rev.* **D 60**, 124002 (1999).

[39] S.M. Kopeikin: *J. Math. Phys.* **38**, 2587 (1997).

[40] T. Damour and G. Esposito–Farèse: *Phys. Rev.* **D 58**, 042001 (1998).

[41] S.M. Kopeikin, *Timing Effects of Gravitational Waves from Localized Sources*, a talk given at the XXXIVth Rencontres de Moriond on “Gravitational Waves and Experimental Gravity”, Les Arcs, 23-30 January 1999; e-print gr-qc/9903070.

[42] V.A. Fock: *Theory of Space, Time and Gravitation* (Pergamon, London 1959).

[43] T. Damour and B.R. Iyer: *Phys. Rev.* **D 43**, 3259 (1991).

[44] N. Andersson: *Astrophys. J.* **502**, 708 (1998).

[45] J.F. Friedman and S.M. Morsink: *Astrophys. J.* **502**, 714 (1998).

[46] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York 1975).

[47] R. Breuer and E. Rudolph: *Gen. Relat. Grav.* **13**, 777 (1981).

[48] G. Schäfer: *Lett. Nuovo Cimento* **36**, 105 (1983).

[49] L. Blanchet, T. Damour, and G. Schäfer: *Mon. Not. R. Astr. Soc.* **242**, 289 (1990).

[50] L. Rezzolla, M. Shibata, H. Asada, T.W. Baumgarte, and S.L. Shapiro: *Astrophys. J.* **525**, 935 (1999).

[51] L. Blanchet: *Class. Quantum Grav.* **15**, 1971 (1998).

[52] J. Hadamard: *Le problème de Cauchy et les équations aux dérivées partielles hyperboliques* (Hermann, Paris 1932).

[53] L. Blanchet and G. Schäfer: *Mon. Not. R. Astr. Soc.* **239**, 845 (1989); Erratum **242**, 704 (1990).

[54] T. Damour, M. Soffel, and C. Xu: *Phys. Rev.* **D 43**, 3273 (1991).

[55] S.M. Kopeikin: *Cel. Mechanics* **44**, 87 (1988).

[56] V.A. Brumberg and S.M. Kopeikin: *Nuovo Cimento* **B 103**, 63 (1989).

[57] L. Blanchet: *Phys. Rev.* **D 55**, 714 (1997).

[58] B.R. Iyer and C.M. Will: *Phys. Rev. Lett.* **70**, 113 (1993).

[59] B.R. Iyer and C.M. Will: *Phys. Rev.* **D 52**, 6882 (1995).

[60] P. Jaranowski and G. Schäfer: *Phys. Rev.* **D 55**, 4712 (1997).

[61] P.R. Saulson: *Fundamentals of Interferometric Gravitational Wave Detection* (World Scientific, Singapore 1994).

[62] D. Shoemaker, R. Schilling, L. Schnupp, W. Winkler, K. Maischberger, and A. Rüdiger: *Phys. Rev.* **D 38**, 423 (1988).