DEVELOPING SECONDARY STUDENTS' UNDERSTANDING OF THE GENERALITY AND PURPOSE OF PROOF

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by

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The undersigned, appointed by the dean of the Graduate School, have examined the dissertation entitled

DEVELOPING SECONDARY STUDENTS' UNDERSTANDING OF THE GENERALITY AND PURPOSE OF PROOF

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ABSTRACT

In spite of their fundamental role in mathematics, prior studies have found that high school students struggle to develop conceptual understanding of proof, and in particular, understand the generality requirement for proofs. Using design research methodology, I investigated an alternative method of introducing ten advanced 9^{th} grade students to proof using problems and instructional strategies that emphasized the generality and purpose of proofs. The design study consisted of 14 sessions, each lasting approximately 30 minutes, that were held twice a week for seven weeks. Semi-structured interviews were also conducted with students at the beginning and end of the study to track changes in students' understanding of proof.

Findings from this dissertation study are presented as three stand-alone articles. In the first article, I present a framework for assessing students' understanding of different proof components based on Stylianides' (2007) definition of proof. I argue that the proposed framework provides a more nuanced perspective of students' understanding of proof and allows the researcher or teacher to identify aspects of proof that students understand as well as ways that they can improve their argument. In the second article, I analyze my design conjecture that using universal claims, or claims involving the universal quantifier "all" or "any", would afford opportunities for students to 1) engage in the reasoning-and-proving process, 2) talk *about* reasoning-and-proving, and 3) develop an intellectual necessity for proof. In the third article, I trace one student's successes and challenges in constructing proofs that adhered to the generality requirement over the course of the study. In this article, I demonstrate how her understanding of the purpose of proof supported her transition from empirical to deductive arguments on proof tasks. I conclude in chapter 5 with implications for research and teaching.

CHAPTER 1

Motivating an Alternate Approach to Teaching Proof

Proofs are fundamental to mathematics as they provide a method for conclusively demonstrating the validity of a mathematical claim by using a logical sequence of assumed or previously proven mathematical statements. Despite their importance in the discipline, proofs received minimal attention in K–12 mathematics standards prior to the introduction of the Common Core Standards for Mathematics in 2010 (CCSSM)(Kosko & Herbst, 2011). In an analysis of United States state mathematics standards, Kosko and Herbst (2011) found that proofs were featured in an average of 11.7% of the high school geometry standards; in contrast, roughly 24% of the Common Core Geometry standards pertain to proof. In addition to increasing the focus of proofs in geometry, the National Council of Teachers of Mathematics (NCTM, 2000) has called for reasoning-and-proving to be present throughout all K–12 instruction. Given this increase in attention to reasoning-and-proving, it is important that teachers develop students' understanding of proof and their purpose in mathematics. At its very core, proofs are important in mathematics because they provide a method for demonstrating that a statement or conjecture is true for *all* possible cases without having to individually check every case. This generality requirement of proofs has found to be challenging for some students to understand, including some students who were able to successfully construct a proof on their own (Chazan, 1993; Martin, McCrone, Bower, & Dindyal, 2005). For example, two students in Chazan's (1993) study asserted that they would have to construct a proof for each type of triangle (e.g., acute, obtuse, right) in order to prove that a statement about all triangles was true. In this example, the students' response indicates a lack of awareness that the triangle used in the diagram was arbitrary and represented all possible triangles;

thus, a single deductive argument was sufficient to prove the statement was true. One possible explanation for students' difficulty in understanding the generality requirement is that they do not receive sufficient opportunities to explore and prove universal claims, where the generality requirement is most evident, in their mathematics classes. This explanation is supported by analyses of proof opportunities in U.S. geometry textbooks, which found that more than half of the proof exercises asked students to prove a particular statement (e.g., a conjecture about a specific triangle) rather than a universal statement (e.g., a conjecture about all rectangles) (Otten, Gilbertson, Males, & Clark, 2014). Without frequent opportunities to prove universal statements, it is possible that some students may fail to see proof as a necessary component of mathematics and, instead, view them as an arbitrary requirement set forth by their teacher or textbook (McCrone & Martin, 2004).

Students' understanding of the proving process is not only shaped by the kinds of reasoning-and-proving opportunities they are provided in textbooks, but also by the way in which proofs are taught in the classroom. Traditionally, proofs have been taught in high school geometry classrooms using the two-column format where every statement, written in the left-hand column, must be accompanied by a reason justifying the statement, written in the right-hand column (Herbst, 2002). Requiring all proofs to be written in this format has the potential to result in an over-emphasis on the form of proofs in the classroom and, in some cases, could result in a de-emphsis on what the proof is trying to accomplish (Schoenfeld, 1988). In an effort to support students in the learning process, some teachers and textbooks initially pose proof tasks involving only a few statements and then gradually increase the complexity of the statements being proven as students develop their proof construction skills and mathematical content knowledge. This method of scaffolding proof instruction can result in initially asking students to prove statements that have trivial or obvious conclusions (Harel & Sowder, 1998). As a result of these instructional approaches, “students do not learn that proofs are first and foremost *convincing* arguments, that proofs (and theorems) are a product of human activity, in which they can and should participate; that they are an essential part of doing mathematics” (Harel & Sowder, 1998, p. 237).

Even though twenty years have passed, Harel and Sowder’s evaluation of students’ learning in regards to proof is still applicable to the types of learning that occurs in present day classrooms. Their statement, however, can also be viewed as a goal for the types of understanding that students *should* develop. In light of the desired student understanding described by Harel and Sowder, the overarching purpose of this study was to investigate students' learning process as they developed their initial understanding of proof through engaging in an instructional sequence that focused on the generality and purpose of proofs. The first iteration of the design study described in this dissertation was conducted with ten advanced 9^{th} graders who had no prior exposure to formal deductive proof. By using design research methodology, I was able to study both students’ learning process and the specific features of the tasks used in the instructional sequence. Findings from this study provide multiple contributions to both mathematics research and practice. Specifically, my analysis of the use of universal claims in the instructional sequence in chapter three contributes greater understanding of the ways in which this design decision influenced students’ learning opportunities, including the potential challenges some students faced in understanding the scope of the claim. Through studying how one student developed understanding of the generality requirement over the course of the design study in chapter four, I contribute more nuanced understanding of this learning process, including a description of the possible intermediate stages students may experience. Finally, my proposed framework for assessing students' understanding of proof components, which I describe in chapter two, offers a way for researchers to assess multiple aspects of students' work on proof tasks instead of assigning a single score or category. As a result, findings from using the framework have the potential to shift the conversation about what students can and cannot do when constructing proofs and provide additional information that can inform future instruction.

In the following sections, I provide a foundation for the articles in chapters two through four by describing my positionality, key terminology, and literature central to all three articles. In describing my positionality, I aim to illuminate the ways in which my prior experiences informed my motivation for this study and the perspective I took in both the teacher and researcher roles. I chose to include selected literature review sections in this chapter in an acknowledgement that the three articles all draw on data from a single study and in an effort to reduce the amount of duplicated information across the three articles. I then provide a brief summary of this information in each article alongside more thorough sections describing literature that is only relevant to the given article.

Positionality

My experiences as a mathematics learner and teacher significantly shaped all facets of this study, from the aspects of proof I chose to emphasize in the instruction and the way that I approached teaching the students, to the lens I used when analyzing students' arguments on proof tasks and their learning progression during the study. In taking on the dual roles of teacher and researcher in this study, it is important that I acknowledge and describe my own background with proofs and the subsequent ways it influenced the teaching, analyzing, and reporting components of this study.

My introduction to proof in high school was both minimal and superficial. Specifically, proofs were introduced as something we "had" to learn how to do and taught in a procedural manner, without any broader explanation of what we accomplished by proving the statement or why the deductive process was necessary. As a result of the traditional proof instruction I received in high school, I entered college as a mathematics major who believed she "hated proofs". Although I was able to develop understanding and appreciation of the role of proofs in my college courses, my experiences with proof as a high school student significantly shaped my interest and approach to proof in this study. In particular, my belief in the importance of establishing an intellectual necessity for proofs stem from the negative feelings I initially had towards proof as a result of lacking this understanding in my first proof experiences. Although I prioritize establishing proofs as a way of developing certainty and causality in mathematics in this study, I recognize that some students come to enjoy and experience success in constructing proofs for other reasons, such as enjoying the problem-solving aspect of proof or appreciating the multiple ways to prove a single claim. Furthermore, I recognize that not all students will equally value or express the need for certainty and causality in mathematics. In choosing to privilege these purposes of proof, I recognize that I am minimizing other equally valid purposes of proof.

My rior experiences with proof also shaped my decision to radically depart from the traditional method of introducing proof in the instructional sequence and to breach many of the sociomathematical norms present in secondary classrooms. Although I chose to explore an alternate approach to proof, I recognize that not all of the norms and proof practices present in secondary classrooms are inherently bad and that choosing to do things differently did not necessarily mean that they would automatically be better. For example, I chose to only use tasks involving universal claims, instead of particular claims as is commonly used in textbooks (Otten, Gilbertson, et al., 2014), in order to support students understanding of the generality requirement. As I describe in greater detail in chapter three, this instructional decision did produce additional opportunities for students to engage in reasoning-and-proving and talk about the reasoning-and-proving practice; however, it also resulted in additional challenges in helping students understand the scope of the claim (something that is not an issue when proving particular statements). Although I concluded this trade-off was worthwhile given the students I was working with and my instructional goals, other teachers or researchers might make the opposite conclusion given different circumstances. Finally, I acknowledge that departing from many of the established classroom norms reduces the likelihood that the approach I took in this study can be easily adopted and implemented by other classroom teachers.

While teaching the design experiment sessions, I was cognizant that my desire to have students construct their own knowledge individually was at odds with the direct-instruction approach the students were accustomed to experiencing in their regular mathematics classrooms. I was additionally aware of my status in the classroom, both as a former mathematics teacher and graduate student, and as someone who was new to their school and not familiar with the norms and experiences of their community. As a teacher-researcher, I tried to balance my desire to meet the needs of the students as their "teacher" with my goal to conduct systematic inqury into their thinking and learning as a researcher. These two goals were at odds with each other at times, especially in relation to instructional choices I had to make regarding when and how to respond to students' misconceptions about proofs that arose during the sessions.

During the analysis and writing process, I was aware of the importance that I represent students' learning and my own teaching in a way that was neither overly generous nor too critical. In order to help achieve this objectivity, I frequently checked my analysis with the two outside observers as well as with researchers not directly involved in the study. With that said, my analysis of students' understanding of proofs and work on proof tasks was shaped by my understanding of what is considered normative or correct by mathematicians and mathematics educators. Although students' work was not evaluated based on the extent to which they produced "correct" proofs, my analysis of their work inherently privileged normative components of proof, such as whether their argument contained mathematically correct justifications or contained a logical flow. Privileging aspects of proof that are considered normative by the broader mathematics community provides an additional layer of objectivity in my analysis, but does so by choosing not to press against normative practices that may or may not be in the best interest of beginning proof learners.

Definitions of Key Mathematical Terms

There is a lack of consensus within the mathematics and mathematics education communities regarding the definition of proof and other related terms, such as argument. Indeed, Weber (2015) has questioned whether it is possible to construct a single definition of proof that encompasses the many different ways in which the term is used. Instead of trying to gain consensus in the mathematics and mathematics education communities around a single definition of proof, Weber proposed thinking of proof as a cluster concept and recommended minimizing the emphasis, particularly at the K-12 level, on whether something should or should not be considered a proof. Articulating the different definitions of proof and their implications for research falls beyond the purpose of this study and can be found elsewhere (e.g., Reid & Knipping, 2010; G. J. Stylianides, Stylianides, & Weber, 2017). In this section, I describe the definition of the terms proof and argument that I use in the present study as well as the specific contexts in which I use each term. By doing so, I respond to calls for greater clarity in the ways that researchers use the term proof in their studies (Balacheff, 2002; Cai & Cirillo, 2014).

My study draws on Andreas Stylianides' (2007) definition of proof: "a mathematical argument, a connected sequence of assertions for or against a mathematical claim" that uses acceptable justifications, valid modes of argumentation, and representations that are appropriate and understood by the classroom community (p. 291). Note that Stylianides intentionally chose not to explicitly define the group(s) of people that should be considered when interpreting the terms "valid", "acceptable", and "appropriate" so that his definition could be applied to work written by both elementary and secondary students. In this study, I interpret these terms according to both our classroom community and the broader mathematics community. Following the recommendations made by Weber (2015), I have chosen not to draw distinctions between students' work that should or should not be considered a proof; instead, I use the term *argument* to refer to any verbal or written work that a student made in response to a proof task without making any judgment in regards to the quality of the work.

Stylianides' (2007) definition of proof served as a guide when developing and enacting instruction but was not formally presented to students or used to evaluate student's arguments as being a proof or not a proof[1]. I chose not to explicitly define proof during the sessions so that I could develop students' understanding of a proof over the course of the study by unpacking the different components of the definition to varying extents. For example, we discussed what counts as an acceptable justification throughout the study; in contrast, students' exposure to valid modes of argumentation was limited to proof by deduction (without focusing on the specific format of the argument) and proof by counterexample. Additionally, I chose not to use Stylianides' definition during the analysis process to distinguish between student work that was and was not a proof since the proof tasks were complex, requiring students to construct multi-step arguments and make connections between multiple mathematical ideas. Given the difficulty level of the proof tasks, especially for an introduction-to-proof unit, I contend that it was unreasonable to expect students to construct flawless proofs on their first attempt. Since the focus of this dissertation study was to develop students' understanding of the generality requirement and purpose of proofs, I further contend it was neither necessary nor worthwhile to make claims about whether their work on the different proof tasks should be considered a proof.

I use the term *proof* when a) referring to tasks where students were expected to construct a proof and b) describing students' conceptions of proofs. Proof tasks included

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[1] I do, however, use Stylianides' (2007) definition as a foundation for the framework I present in chapter 2. Whereas Stylianides' definition suggests that students' work should contain all four components of the definition in order to be considered a proof, my proposed framework assesses students' work for evidence of the individual components of the definition (e.g., uses acceptable justifications) without making broad claims about the argument overall.

both formal statements, such as “prove that the sum of two consecutive numbers is an odd number”, and more informal statements, such as “is this a coincidence?” where the instruction to construct a proof was verbally given once students made a conjecture about the validity of the given mathematical relationship. The second way that I use the term proof is when describing a student’s conception of proof, or my interpretation of what the student appears to believe or know about proof. This use of the term proof only occurs in chapter four when describing the criteria a case study student, Lexi, evoked when evaluating provided responses for a proof task. For example, when Lexi stated that a provided student’s argument was not a proof because “she never said why it worked and why it didn’t work”, I stated that this justification revealed her conception that proofs should explain why the statement was true. All uses of the term *proof* in the second context are my inferences about the student's knowledge of proofs and may or may not align with the definition of proof provided earlier.

Literature Review

I begin this section by describing the different purposes that proofs serve in mathematics. Next, I describe the traditional way that proof has been taught in the classroom in order to motivate a need for alternate approaches, followed by some of the other proof intervention studies that have been conducted with secondary and undergraduate students. I conclude the literature review section by describing findings from prior studies on students’ approaches to proofs, especially within the context of their understanding of the generality requirement.

Purpose of Proofs

There are a range of purposes that proofs can take on in mathematics. Drawing on the works of other scholars, such as Hanna (1990), Balacheff (1988), Bell (1976), and

Hersh (1993), de Villiers (1990) suggested six roles that mathematical proofs can fulfill: verification, explanation, systemization (organization of results into a deductive system), discovery, communication, and intellectual challenge. Although proofs in secondary classrooms are often used as a means of verifying mathematical statements, de Villiers (1990) argued that, instead, teachers should promote "the more fundamental function of explanation" in order to present the proving process as a meaningful activity (p. 23). Similarly, Hersh (1993) argued that proof at the high school and undergraduate levels should "provide insight into why the theorem is true" (p. 396). While Weber (2002) rightly contends that not all worthwhile proofs are explanatory or convincing, those that achieve one of these two goals are particularly useful when first introducing and motivating a need for proof. Moreover, when selecting proof tasks that are designed to convince or explain, teachers should be aware that what they find convincing or explanatory may not be similarly convincing or explanatory for students (McCrone & Martin, 2009).

Students, to varying degrees, have expressed understanding of the role and purpose of proofs in mathematics. In a study by Healy and Hoyles (2000), half of the high achieving 14–15 year old student participants stated that proofs are used to "establish the truth of a mathematical statement" (p. 418). An additional 35% of the students discussed using proofs as a means of explanation and communication with others; however, follow-up interviews with select students suggested that recognizing proofs for their explanatory capacity was potentially more widespread than the initial analysis suggested (Healy & Hoyles, 2000). On the other hand, 28% of the students interviewed had no idea what proof was or their purpose in mathematics. A lack of understanding in regards to the purpose of proofs in mathematics was similarly expressed by a student in Chazan's (1993) study, who said that she had "never found any real reason to do it" (p. 382). Given the increased focus on reasoning and proving in secondary mathematics (Kosko & Herbst, 2011), more research is needed to develop and assess instructional approaches that help all students recognize the roles of proof in mathematics and establish an intellectual need for proofs (Harel, 2008).

Proof Instruction

Traditional approaches to teaching proof. There are many norms that have been established in Geometry classrooms regarding proof, such as the way they are formatted and the types of information that is expected to be provided by the textbook or teacher. For more than a century, high school Geometry proofs have been written in the two-column format, which was developed in an effort to help all students learn how to produce proofs in a more standardized fashion (Herbst, 2002). In this format, each sentence in the proof is separated into numbered "statements" and their corresponding "reasons", beginning with the given information and concluding with the result being proven (Schoenfeld, 1988). An illustration of the way that proof problems are traditionally formatted is shown in Figure 1.1. Notice that the problem is presented to students in separate "given" and "prove" statements instead of writing it as a single sentence ("If ROMP is a square, then the diagonals are congruent" or alternatively, "prove that the diagonals of ROMP are congruent"). Researchers such as Chazan (1993) have argued that writing the proof claim in the separate "given" and "prove" format "obscures the generality of the claim and gives students no indication that the argument presented is not an argument for a single case" (p. 384).

Given: ROMP is a square

Prove: $\overline{RM} \cong \overline{PO}$

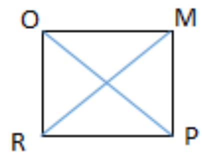

Figure 1.1. Proof problem written in the diagrammatic register commonly used in traditional textbooks.

Another normative feature of the proof claim presented in Figure 1.1 is the inclusion of the two auxiliary lines $\overline{RM}$ and $\overline{PO}$ in the original diagram, even though they were not part of the given information. The inclusion of the two lines in the diagram are in line with the findings from a study conducted by Herbst and Brach (2006), who found that the students they surveyed expected their teacher or the textbook to include the associated diagram with all of the key elements labeled and provide hints if any auxiliary lines were needed to complete the proof. Finally, the claim being proven in Figure 1.1 pertains only to square ROMP even though the statement is true for all squares. Otten, Gilbertson and colleagues (2014) questioned whether proving claims about single diagrams, which were found in the majority of proof opportunities in Geometry textbooks, might reduce students' understanding of the intellectual need for proofs since claims about single diagrams can be verified using non-deductive methods such as visual inspection or measurement. In contrast, proving the universal claim, "the diagonals of a square are congruent" *must* be proven deductively since it is not possible to visually inspect the diagonals of all squares.

While the aforementioned proof norms were established as a way of scaffolding the proving process, they reduce the opportunities for students to engage in all aspects of reasoning-and-proving, such as considering what information they can use to begin the proof or what features of the diagram can be labeled and assumed to be true (Cirillo &

Herbst, 2011). For instance, in order to construct a diagram to accompany a mathematical claim, students would need to translate between the mathematical claim and their corresponding visual representations, a task that could help them develop understanding of the relationship between the two components and potentially facilitate recognition of the generality of the claim. If the proof claim in Figure 1.1 had instead be given to students in the form of a single statement, “If ROMP is a square, then $\overline{RM} \cong \overline{PO}$”, drawing the diagram would require students to label the vertices in consecutive order going either clockwise or counterclockwise around the square and then realize that $\overline{RM}$ and $\overline{PO}$ are the diagonals rather than the sides of the shape. Additionally, if the problem had instead been stated as a universal claim and no diagram had been provided, then students would ideally need to consider all possible squares when depicting the mathematical claim as a visual representation.

Multiple studies have found that teachers spend a significant amount of time in their Geometry classes focusing on the details of proofs, such as whether each “step” in the proof contained a mathematically correct justification and logically flowed from the previous statements (Martin & McCrone, 2003; Otten, Bleiler-Baxter, & Engledowl, 2017; Schoenfeld, 1988). “This appeared to be important to the teachers in the proof-writing process, and became an important part of what the students focused on in their proofs and proofs written by others” (Martin & McCrone, 2003, p. 27). In an analysis of one classroom video, 22 of the 37 minutes the teacher spent reviewing students’ proofs focused on discussing the form of the proof instead of whether the proof was mathematically correct (Schoenfeld, 1988). When focusing on the form of proofs in the classroom, there is a potential for teachers to convey the idea that there is only one mathematically correct way of constructing the proof, even though this is not always the case. For example, the teacher in Otten and colleagues' (2017) study conveyed this message to her students through the use of phrases such as "you have to" or "the only way" when talking about the proving process (p. 122). While developing students' understanding of the logical flow and the different components of proof is important, an overemphasis on these aspects can potentially mask the purpose of proofs or the mathematical ideas being conveyed through the proof. More research is needed to identify ways of engaging students in conversations around proof that attend to these broader ideas.

Most studies that have documented naturalistic (non-interventionist) teaching and learning of proofs in the secondary classroom have done so towards the middle or end of the school year; as a result, there is very little known about how students are first introduced to proofs in a traditional classroom. One exception to this is the study conducted by Cirillo (2014), who reported that all five teachers she observed introduced proof in their Geometry classrooms through a show-and-tell approach. During the teachers' proof demonstrations, Cirillo noted that the teachers did not explicitly unpack the many different components of the proof for students, such as how they were using definitions to draw conclusions or what can and cannot be assumed from a diagram. Otten, Males, and Gilbertson (2014) refer to statements that explicitly unpack the different components of proof or broadly focus on the reasoning-and-proving process itself as statements *about* reasoning-and-proving. In an analysis of the six most commonly used U.S. Geometry textbooks, Otten and colleagues (2014) found that statements or questions *about* reasoning-and-proving were more commonly located in the lesson's expositional text than in the student exercises and that very few statements or questions *about* reasoning-and-proving occurred beyond the introduction-to-proof chapter. These findings suggest that while textbooks do provide some supports in understanding the reasoning-and-proving process, students are only likely to come across these supports if they intentionally go looking for them by reading the lesson text. This decision by textbooks authors to place statements *about* reasoning-and-proving primarily in the lesson exposition places additional burden on the teacher to incorporate these statements into classroom conversations.

Proof interventions. There has been a recent increase in the number of intervention-based studies (e.g., teaching experiments, design studies, and classroom interventions) focused on improving students' understanding of different aspects of proof, such as developing their understanding of various forms or approaches to proof (Harel, 2002; Miyazaki, Fujita, & Jones, 2015; Yopp, 2017), the need for proof (G. J. Stylianides & Stylianides, 2009), the axiomatic structure (Jahnke & Wambach, 2013; Mariotti, 2000), and the connections between proof and other topics, such as algebra (Martinez, Brizuela, & Superfine, 2011) and mathematical definitions (Larsen & Zandieh, 2005). For example, Yopp (2017) introduced proof to 8^{th} graders as a process of "eliminating counterexamples", or demonstrating that it is not possible for a counterexample to exist for a given claim. This indirect method of proof was motivated by the idea of pivotal intermediate conceptions (Lobato, Hohensee, Rhodehamel, & Diamond, 2012), or initial conceptions and approaches to proof that may be useful for students in the beginning stages of their learning process, which they later discard once they have developed more sophisticated methods. Although the student in Yopp's (2017) single case study initially struggled to articulate counterexamples she did not believe existed, Yopp concluded that the eliminating counterexamples approach produced promising results in helping to facilitate students' initial approaches to constructing mathematical arguments.

Instead of focusing on specific approaches to constructing arguments, other studies, such as the ones by Jahnke and Wambach (2013) and Stylianides and Styliandies (2009), targeted students' overall motivation for proof. In particular, Jahnke and Wambach (2013) aimed to develop students' understanding of the role of hypotheses in proof through looking at how ancient astronomer's revised their hypotheses regarding the orbit of the sun around the earth. At the end of the student exploration phase, the researchers analyzed the extent to which the students seemed to recognize that the validity of a statement is dependent upon the initial assumptions, or hypotheses, one makes. Jahnke and Wambash (2013) concluded that the secondary students demonstrated an increased understanding of the impact of hypotheses in arguments by the conclusion of the study; however, by motivating a need for proof through a scientific rather than mathematical context, the researchers and students had to contend with the different uses of the word "hypothesis" in the two disciplines as well as students' prior misconceptions about hypotheses in science. Gabriel and Andreas Stylianides (2009) aimed to develop elementary preservice teachers' understanding of the need for deductive reasoning through exposing the limitations of assuming that a pattern found from a few examples will also be true for every possible case. Their instructional sequence was based on the assumption that "pivotal counterexamples" (p. 317) produce the cognitive conflict needed to help students make the transition from empirical justifications to deductive reasoning. While pivotal counterexamples have the potential to shift students' conceptions, they do not necessarily provide motivation for *why* proofs are needed in mathematics (Harel, 2008) or convey the purpose of mathematical proofs (de Villiers, 1990). In other words, this intervention sought to transition preservice teachers to deductive arguments solely through casting doubt on examples-based arguments without *also* discussing the value or purpose of deductive arguments.

Collectively, the diverse methods researchers have used to develop students' understanding of proof and their ability to construct proofs highlights the complexities of the topic and reveals differences in the components of proof that researchers prioritize in their interventions. Given the relatively recent focus on developing and testing proof interventions, it is not yet clear what the advantages and disadvantages are regarding introducing students to proof through focusing on developing their ability to construct proofs (e.g., Harel, 2001; Miyazaki, Fukita, & Jones, 2016; Yopp, 2017) or on developing an overall understanding of proof (e.g., Stylianides & Styliandies, 2009). The present study takes the latter approach by developing students' overall understanding of the generality and purpose of proof and providing minimal instruction on the specific form and components of proof.

Students' Approaches to Proof in Relation to the Generality Requirement

The generality requirement is a crucial component of proof that impacts the ways in which mathematical statements are determined (proven) to be true or false. Specifically, the generality requirement necessitates that in order for a statement in mathematics to be considered true, it must be true for all possible cases; thus, only a single counterexample is needed to disprove a mathematical claim. Particular statements, or statements about a finite number of cases, can be proven by checking every case, whereas universal statements, or statements containing the stated or implied universal quantifiers “all” or “any”, must be proven deductively. The scope of a mathematical claim is often implicitly stated or masked by the language used in proof statements and the way in which they are presented to students. For example, students might interpret the theorem “the diagonals of a rectangle are congruent” to mean, “there is a rectangle that has congruent diagonals” even though mathematicians intend for the statement to be interpreted as “the diagonals of all rectangles are congruent.” Researchers such as Chazan (1993) have contended that presenting proof claims to students in the form of separate “given” and “prove” statements further mask the scope of the claim, as this translation tends to result in statements that directly refer to the accompanied diagram instead of all possible diagrams. In addition to masking the generality of the mathematical claim, textbooks also commonly ask students to construct deductive arguments for particular claims (Otten, Gilbertson, Males & Clark, 2014), which conceals the fact that deductive arguments are only required when proving universal claims. In the following sections, I describe students’ understanding of the generality requirement in relation to their approaches to proof tasks for mathematical conjectures that are both true and false.

Proving and evaluating true mathematical claims. Synthesis of findings from prior studies have found that secondary students tend to construct empirical arguments for proof tasks and are seemingly more convinced by empirical justifications than they are by deductive arguments (Reid & Knipping, 2010; G. J. Stylianides & Stylianides, 2017) One possible explanation for these findings is that students are already convinced of the validity of the statement prior to constructing the requested proof; as a result, proofs are no longer used as a way of convincing oneself of the statement’s validity. A second possibility is that students realize the limitation of using examples as justification but construct example-based justifications because they are unsure of how to construct a more general argument. This section will explore the research supporting each of these explanations as well as possible classroom implications.

One possible explanation for students' reliance on empirical evidence when constructing and evaluating proofs is that they view the process primarily as a way of verifying the validity of statements previously established to be true rather than conclusively determining that a mathematical statement is true. Geometry students interviewed by Herbst and Brach (2006) claimed that while they might be asked to conjecture what should be proven, "they would not take responsibility for proving their conjecture unless and until it would be sanctioned, and perhaps restated, by the teacher or text" (p. 89). Through confirming the conjecture prior to constructing the proof, students no longer questioned the validity of the statement being proven; instead, the process of constructing a proof becomes an exercise in confirming what has been previously established as true (Schoenfeld, 1988). Asking students to solely or predominantly prove statements they already know to be true has the potential to minimize their perceived need for constructing a deductive argument since the validity of the statement has already been established. For example, Knuth and colleagues (2009) found that 8th grade participants produced fewer arguments that attempted to address the general case than 7^{th} grade participants for the task, "show that when you add any two even numbers, your answer is always even." The researchers hypothesized that the 8^{th} grade students may not have produced deductive arguments because they had not seen the need to justify a mathematical fact that had been previously established to be true in elementary school. Instead of having students prove statements that are seemingly obvious or have already been sanctioned as true, teachers should provide students with problems where they are given the dual directions of constructing a proof or writing a counterexample in order to promote the verification/conviction role of proofs (de Villiers, 1990).

A second possibility is that students recognize that examples-based arguments are not sufficient to be considered proof, but still write empirical arguments because they lack the mathematical skills to be able to construct a more general argument. Healy and Hoyles (2000) drew this conclusion when interpreting the findings that students were better at selecting mathematically correct (deductive) arguments than constructing them. "These differences not only showed that students were more likely to construct empirical arguments than to choose them but also supported the suggestion that they were the best arguments available to the students, and not necessarily that they were satisfied with them as proofs" (p. 412). Even when students construct empirical arguments, their careful selection of examples that represent a range of situations (e.g., acute, right, obtuse, and equilateral triangles) provide evidence to suggest recognition that a few examples are not sufficient evidence to conclude the statement is always true (Chazan, 1993; S.S. McCrone & Martin, 2004). One implication of this conclusion is that teachers should recognize the added cognitive difficulty students face when asked to prove statements using mathematical knowledge recently learned. However, asking students to prove statements using previously mastered content will not necessarily guarantee successful proof construction; students will still need continued support in understanding the purpose of proofs and practicing the process of constructing a proof.

Given the possible disconnect between what students believe to be a proof or convincing argument and the type of argument they are able to produce, some researchers have chosen to assess students' conceptions of proof through analyzing their evaluation of provided responses to proof tasks (e.g., Bieda & Lepak, 2009; Healy & Hoyles, 2000). For example, Bieda and Lepak (2009) presented 22 middle school students with a deductive argument and empirical argument for the claim, "when you add any two consecutive numbers, the answer is always odd", and then asked them to select which argument they thought was more convincing and explain why. Fifteen students thought the examples-based argument was more convincing, because it either a) enhanced their understanding of the claim, b) provided more information, or because c) examples are essential in an argument (p. 170). The four students who thought the deductive argument was more convincing stated either that a) examples do not prove the general case or b) the deductive argument provided more explanation. Healy and Hoyles (2000) asked advanced algebra students to select the provided solution that was most similar to what they would produce for the task as well as the solution they thought would be scored the highest by their teacher. Analyzing students' responses to both of these questions revealed that while students consistently rated the algebraic arguments as most likely to be scored the highest by their teacher, they tended to select non-algebraic arguments (such as the empirical or narrative arguments) as ones they would most likely produce. Students explained this disconnect by stating the algebraic arguments "offered little in terms of communicating and explaining the mathematics involved" (p. 425). Findings from these studies reveal the benefits of asking students to evaluate provided arguments as well as construct their own arguments for proof tasks; namely, evaluating provided arguments allows the researcher or classroom teacher to assess students' understanding of proof without their answers being potentially influenced by their mathematical content knowledge or proof construction skills.

Proving false mathematical claims. In order to fully grasp the generality requirement, it is important for students to have repeated opportunities to prove both true and false mathematical statements; however, an analysis of student exercises in U.S. Geometry textbooks found that students were significantly more likely to be asked to construct a mathematical proof for a true claim than to state a counterexample for a false claim (Otten, Gilbertson, et al., 2014). After assessing students' understanding of proof through their work on proof tasks for both true and false mathematical claims, Buchbinder and Zaslavsky (2013) found that just because a student recognized that a single counterexample was sufficient to disprove a mathematical claim did not necessarily imply recognition that a true statement must be demonstrated to be true for all possible cases. While researchers have primarily used tasks involving true mathematical claims to assess students' understanding of proof, a few have also asked students questions designed to target their understanding of whether a counterexample and deductive argument could co-exist for a single mathematical claim. For example, two students in Chazan's (1993) study viewed proofs as a form of evidence and did not think that the existence of a deductive argument eliminated the possibility for a counterexample to exist. Similarly, students' understanding of counterexamples was also evident in two students' dialog when analyzing two provided responses that made different conclusions about the validity of the statement (Buchbinder & Zaslavsky, 2013). The two students thought that both of the provided responses were correct, but at different moments in time, suggesting that they were operating under a non-mathematical understanding of valid statements. Findings from these studies highlight the importance of discussing the difference between mathematical and non-mathematical uses of terms such as “prove” as well as engaging students in discussion about the relationship between proofs and counterexamples.

Understanding the boundaries of a claim. In addition to understanding how to construct arguments that adhere to the generality requirement, students must also develop recognition of the boundaries of what a proof demonstrates in order to avoid over- or under-generalizing a deductive argument. The majority (62%) of students (high achieving 14–15 yr. olds) in Healy and Hoyles’ (2000) study recognized that if you prove the sum of two even numbers is always even, then no further work was needed to demonstrate that the statement, “when you add 2 even numbers that are square, your answer is always even” was also true (p. 403). While this question highlights students’ understanding that the argument proved the statement was true for subcases of the original statement, it does not assess whether students would overextend the statement to believe it also proved that, for example, the sum of two odd numbers is always even. Knuth and Sutherland (2004) found varied results in middle-schoolers’ understanding of the scope of the claim after they had produced a proof-by-exhaustion for a particular claim. Specifically, some students thought their proof-by-exhaustion for ten numbers proved the statement was true for all possible numbers whereas others recognized that their initial response only proved the statement true for the specific cases. More research is needed to understand whether and how often students over-generalize proofs as well as the extent to which the mathematical context (i.e., numeral vs. geometry) impacts their tendency to over- or under-generalize.

Students' understanding of the boundaries of what a proof demonstrates has also been studied within the context of proofs accompanied by a diagram. Studies have found that many students do not recognize the generality of a proof when a diagram was involved, but instead believed that the proof only demonstrated the validity for the single given diagram (Chazan, 1993; Hoyles & Healy, 1999; Martin et al., 2005). In other words, the students did not perceive the proof diagram as representing a broader class of objects. For example, students' difficulty in understanding what a diagram represented was observed by Martin and colleagues (2005) during a classroom lesson where students asked their teacher to re-prove a statement for different types of triangles immediately following the discussion of the original proof. During this process of re-proving the statement for three different triangles, some students appeared unable to recognize the fact that the structure and content of the proof did not change with each new diagram. The researchers concluded that even though the students all appeared proficient in their ability to construct deductive arguments, some of them were still "operating at a conceptual level with an empirical proof scheme" (2005, p. 117). Prior studies related to students' understanding of the generality requirement collectively highlight the importance of making this apect of proof an explicit object of focus and not assuming that students will automatically develop conceptual understanding of proofs, and specifically of the generality requirement, through engaging in the proving process.

Overview of the Dissertation

This dissertation is organized into a series of three standalone articles, located in chapters 2 – 4, that all draw on data from a single design study. As a result, there is both information that is repeated across the three articles as well as information that is only included in a subset of the articles. For example, although I describe the participants and data in all three articles, the section in chapters two and three focus on the participants as a group whereas the section in chapter four primarily describes the focus student, Lexi. Given the scope of the design study, I had multiple conjectures that informed my instructional decisions. Instead of describing all of my conjectures in each article, I focus on my design decision to use only universal claims in the chapter three article and then discuss more specific design conjectures related to the sequence and selection of specific tasks used in the instructional sequence in the chapter 4 article. A comprehensive description of the instructional sequence and methods are located in Appendices A and B respectively. The three articles also contain a summary of the key ideas presented in the literature review sections detailed in this chapter. By providing a condensed summary of selected literature review, I acknowledge that the articles are located within a broader dissertation while also maintain the possibility for them to be read as standalone articles.

In chapter two, I propose a framework for assessing students' understanding of proof that draws on Andreas Stylianides' (2007) definition of proof and then illustrate the advantages of using the framework by analyzing students' work on proof tasks from the conclusion of the design study. In chapter three, I analyze my design decision to use universal claims, or claims involving the universal quantifier "any" or "all", in terms of the opportunities it afforded to motivate an intellectual need for proof, engage in reasoning-and-proving, and talk *about* the reasoning-and-proving process. In chapter four, I present a case study analysis of one student, Lexi, and describe her learning process as she developed understanding of the generality requirement. I conclude the dissertation by briefly summarizing implications of each study and posing suggestions for future research in chapter five.

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CHAPTER 2

An Analytic Framework for Assessing Students' Understanding of Proof Components

The ways in which classroom teachers and researchers assess students' work should ideally align with the way that students are taught and the aspects of the content that are prioritized in the lessons. Studies have found that proofs tend to be taught with a emphasis on the form, and specifically on making sure that the "correct" statements and reasons are placed in the "proper" order (Martin & McCrone, 2003; Otten et al., 2017; Schoenfeld, 1988). For example, the two teachers in Martin and McCrone's (2003) study both spent considerable time in the classroom helping students correct their proofs so that they contained all of the necessary steps and justifications. When leading students through the construction of a proof in class, the teacher in Otten and colleagues' (2017) study directed students to follow her mathematical conventions (e.g., writing the proof in the two-column format) and used language that suggested limited possible ways of correctly constructing the proof. Given teachers' focus on aspects of the form of proofs during instruction, it is likely that they similarly emphasize the form of proofs when assessing students' work on proof tasks rather than on the overall characteristics of the argument (e.g., whether the statements adhere to the generality requirement or the argument is convincing).

Researchers have historically chosen to evaluate students' work using a variety of frameworks (Bell, 1976; Healy & Hoyles, 2000; Knuth et al., 2009; Lee, 2016; D. Stylianou, Chae, & Blanton, 2006). Although these frameworks differ in foci, and in particular whether they assess students' ability to construct proofs or their understanding of proof broadly, they are similar in the fact that they all yield a single score, level, or classification. By assessing students' work holistically, none of the frameworks are able to differentiate between instances whether an empirical solution was produced due to the student's content knowledge or to their understanding of proof (e.g., Healy & Hoyles, 2000; Knuth et al, 2009). For students who are still in the beginning stages of learning to construct proofs, attention to multiple characteristics of their solution is especially important as it can serve to not only characterize students' current understanding, as evident in their work on a specific task, but also suggest possible areas for growth in their mathematical and proof understanding. For example, if a student included correct mathematical justifications on some of their statements and did not provide a justification for other key statements, the teacher could respond by acknowledging that all of the provided justifications were mathematically correct and then encourage the student to revise specific statements so that they also included a mathematical justification. Additionally, in situations where students were being interviewed or were allowed flexibility in the way that their proof is formatted, a student could make written or verbal statements indicating an awareness of the generality requirement even if they produced an empirical argument (e.g., "I know that these two examples don't prove the statement is always true"). By attending to multiple aspects of the student's work, the researcher or teacher is able to acknowledge the student's awareness of the generality requirement instead of solely focusing on the fact that they produced an empirical argument. For this reason, I chose to develop a new framework that attends to multiple aspects of students' work, including categories that assess whether students demonstrated an awareness of a proof component and categories that assess students' ability to correctly apply the given component.

This framework builds on Andreas Stylianides (2007)'s definition of proof: “a mathematical argument, a connected sequence of assertions for or against a mathematical claim” that uses statements with acceptable justifications, valid modes of argumentation, and representations that are appropriate and understood by the classroom community (p. 291). For the purposes of this study, I am interpreting the terms “valid”, “acceptable” and “appropriate” according to both the classroom community and the broader mathematics community. Even though Stylianides’ (2007) definition of proof forms the basis of the proposed framework, I will use the term *argument* to refer to students’ work on proof tasks so as not to make any claims as to whether their work should be considered a proof by an outside authority. This article extends the definition of proof articulated by Andreas Stylianides (2007) by:

1) proposing an multi-faceted analytic framework that can be used by mathematics education researchers and classroom teachers to assess students’ written and/or verbal responses to proof tasks; and
2) illustrating the utility of the analytic framework in terms of identifying evidence in student work related to their understanding of specific aspects of proof and ways in which the argument could be improved.

In the final section, I describe possible implications for classroom teachers in terms of using the framework to assess students’ understanding of proof and how the feedback it provides could be used to support their learning process.

The student work presented in this paper was collected as part of a larger, introduction-to-proof design research study that focused on developing students' understanding of the generality and purpose of proof. There were ten students who participated in this study – seven females (Amanda, Arin, Heather, Lauren, Lexi, Megan, and Sadie) and three males (Brian, Clay, and Wilson; all pseudonyms). They were the only students enrolled in the accelerated 9th grade mathematics class at a rural, public school in the Midwest United States. The accelerated 9th grade mathematics class was the first in a two-course sequence that covered Algebra 1, Geometry, and Algebra 2, with the first course focusing on Algebra 1 and Algebra 2 concepts. This meant that the students had not received any high school Geometry instruction in their regular math class prior to the study. See appendices A and B for more details about the instructional sequence and study methods, respectively.

The data used in the illustrations consist of students' written and verbal work on three proof tasks—the *sum of three odd numbers* task, the exterior angle theorem, and the similar rhombuses task. Students completed the *sum of three odd numbers* task during both the initial and final interviews to track possible changes in their approach. This task was presented to students using the following written prompt: "Sarah said, 'If you add any three odd numbers together, your answer will be odd.' Is she right? Explain your answer." I intentionally chose not to include the word "always" in the original prompt to see if students would produce a deductive argument on their own. Then, as a follow-up question, students were asked whether they thought their answer proved that the statement was always true.

The exterior angle theorem task was initially posed to students during Session 13 of the design experiment (Figure 2.1). After deciding in their small groups that the angle relationship was not a coincidence, I asked students to individually prove that this relationship would *always* occur. I intentionally chose to phrase the task as a question to provoke curiosity and doubt regarding the validity of the statement (Buchbinder & Zaslavsky, 2011); students were not given the actual proof claim when writing the proof until the final interview, after they wrote their original argument. During the final interview, I reminded each student of the proof statement using angle notation on their paper (e.g., "Prove that ∠A + ∠B = ∠D") in lieu of using the formal vocabulary found in the theorem.

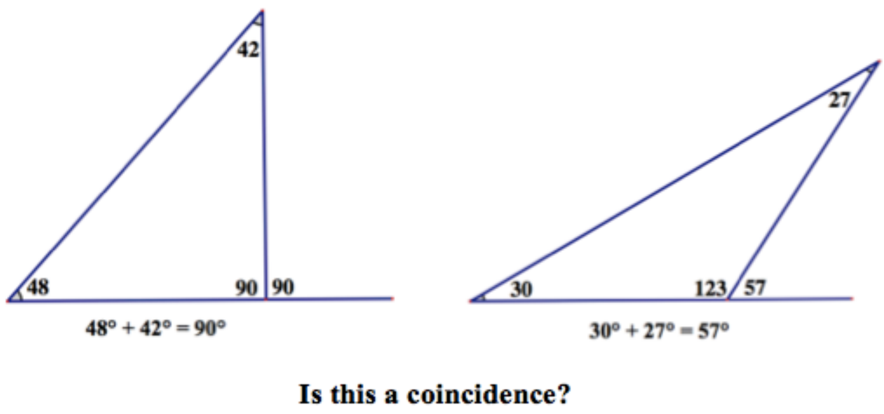


Figure 2.1. Prompt for the exterior angle theorem proof task.

The third and final task asked students to consider the false conjecture, "all rhombuses are similar". This conjecture was initially posed by one of the students during Session 11; however, none of the students worked on this task prior to the final interview. During the final interview, I asked students to decide whether they thought the conjecture was true or false and then prove their response. Students were given an iPad with a Geometer's Sketchpad (GSP) document to use during this task. The GSP document contained two rhombuses that could be manipulated, with all of the angle and side measurements listed on the screen.

Analytic Framework

The analytic framework described in this paper is separated into the four components of Stylianides' (2007) definition of proof, with each component containing two sub-categories (see Table 2.1). Before describing the different components of the analytic framework and illustrating its advantages using student work, I make five comments about the framework. First, this framework is designed to analyze different elements of students' arguments and is not intended to make any overarching claims about the validity of their work overall. Specifically, it is not designed to make claims about whether a student's work should be considered a proof, even in instances where the student's work is coded as demonstrating evidence for every aspect in the framework. While it may be useful to categorize students' work as "proofs" or "not-proofs" in certain circumstances, I contend that these labels are not particularly helpful when analyzing students' initial attempts at constructing arguments for proof tasks, as I will do in this paper. Furthermore, this distinction also seems counterproductive for researchers given the lack of agreement within the community regarding the definition of proof (Reid & Knipping, 2010; Weber, 2015) and an unnecessary distinction for proof at the secondary level (Weber, 2015).

Second, analysis of students' work using this framework allows the researcher or teacher to make claims about the students' demonstrated understanding *for the particular proof task* and is not intended to make broader claims about students' overall understanding of proof. This echoes the caution Dawkins and Karunakaran (2016) assert that researchers should take when making content-general claims about students' proof activity, as doing so neglects attention to the role of mathematical content within the proof task.

Third, a student's work being coded as *not* demonstrating evidence of a particular category does not necessarily indicate an overall lack of understanding, just that their work did not explicitly provide evidence demonstrating that understanding.

Fourth, the framework assumes that students were held responsible for producing all elements of the proof and were only provided with the minimal information needed for the task, as was the case in this study. For example, the proof conjectures used in this study were all phrased as a single statement or question and were not broken down into separate "given" and "prove" statements. Additionally, students were not introduced to the two-column format commonly used in the U.S. (Herbst, 2002), but instead were encouraged to write their arguments using whatever format made sense to them. Use of this framework in contexts where students were provided additional supports might make some of the categories less appropriate or useful.

Fifth, this framework is designed to assess students' understanding of proof while they are still learning how to construct formal proofs. Consequently, components of the framework that may seem obvious or trivial to mathematicians or mathematics educators are included because they are *not* necessarily obvious to those who are just beginning to learn how to construct a proof. By separating out the different components of a mathematical proof, it is my hope that the framework can also serve as a scaffold to support teachers and students in the teaching and learning process.

Table 2.1

An analytic framework for assessing students' understanding of proof

Stylianides' (2007) definition of proof	**Framework Categories**[2]	
A connected sequence of assertions	Assertions follow a logical flow	Identifies and uses the correct hypothesis
	• Yes • No	• Yes • No
Uses set of accepted statements	Attention to the justification requirement	Uses mathematically accurate justifications
	• All key statements • Some key statements • None	• All key statements • Some key statements • None
Modes of argumentation	Attention to the generality requirement	Overall type of argument
	• Evident • Not evident	• Deductive (e.g., paragraph, two-column, flow) • Counterexample • Empirical
Modes of argument representation	Forms of expression used	Relationship between written text and other forms of expression, if applicable
	• Written • Diagrammatic/pictorial • Symbolic/algebraic	• Integrated • Aligned • Not aligned

I will now unpack each of the codes by providing a more detailed description of how to assess students' understanding of the given category. I also provide examples of student work to further illustrate the codes and highlight the distinctions between students' work that it illuminates.

[2] The listed codes are not meant to be exhaustive. There are other codes that may be appropriate for different proof tasks, especially for the overall type of argument and forms of expression used categories.

Connected Sequence of Assertions

Assertions follow a logical flow. Stylianides' (2007) definition begins by stating that a proof is "a connected sequence of assertions for or against a mathematical claim" (p. 291). In other words, this portion of the definition relates to the logical structure of proofs and the idea that each assertion in the proof logically builds on or connects to other ideas to form a broader argument stating that the given claim (conjecture) is either true or false. I chose to assess whether students' arguments contained a connected sequence of assertions holistically in in order to acknowledge the fact that students were still in the beginning stages of learning how to construct proofs. Thus, I coded an argument "yes" if the assertions were generally connected and demonstrated logical flow. Students' work that was coded as *not* having a logical flow were those that repeated a single idea over multiple sentences or arguments where the sentences were disjoint and contained significant gaps between statements. This is not to say, however, that an argument coded as having logical flow was complete, just that the ideas presented in the argument generally flowed from one sentence to another. As students develop their understanding of how to structure their arguments, it might be useful to add a third code between the two included codes for arguments that contain only minor flaws in the logic flow.

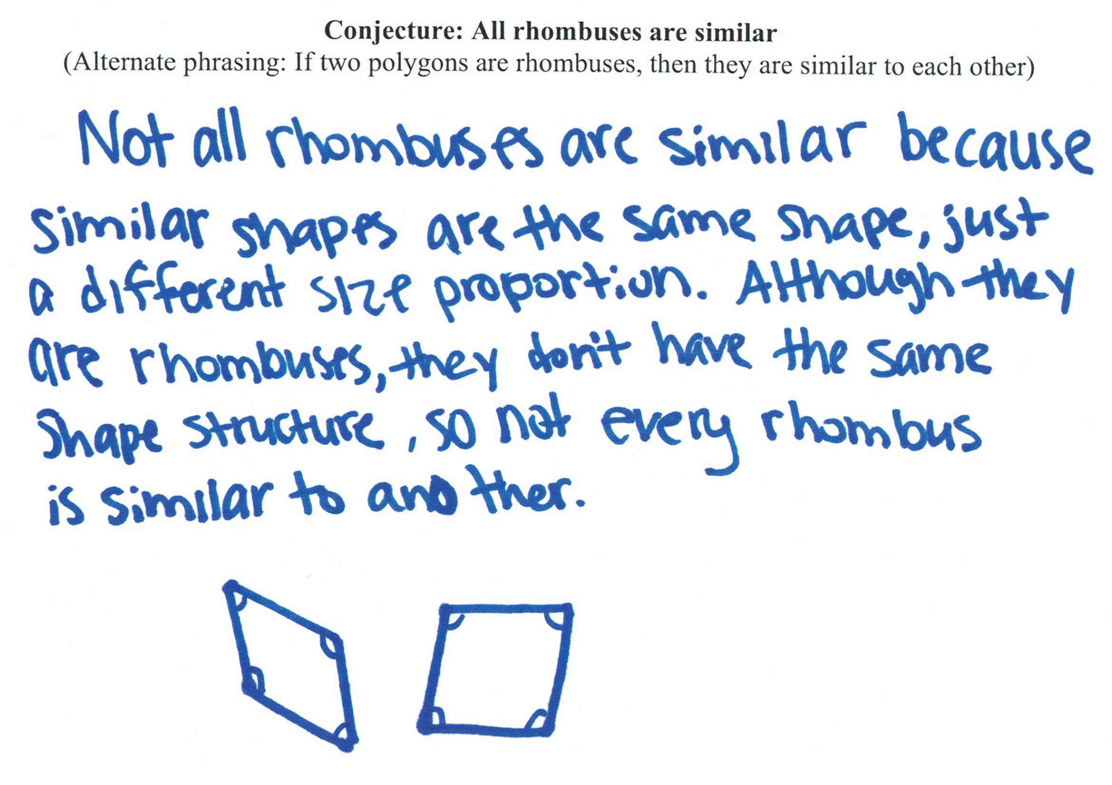
Megan's argument for the exterior angle theorem is an example of an argument that contained clear logical flow.

> Start with two triangles. By the definition of a triangle, we know that all of the angles are different and it has 3 sides. Label the angles of a triangle A, B, and C. Label the outside angle by C, D. A and B add to D. By the definition of supplementary angles two angles that touch equal a straight line, or 180º. So C

and D added add up to 180. By the definition of supplementary angles and triangles, this is not a coincidence.

In her argument, Megan carefully described how to label the different angles in her constructed diagram before using the labels in her statements. Additionally, she alternated between making broad claims, which applied to all triangles, and statements that applied those definitions to the specific angle measurements in her drawing, which logically connected the ideas in each statement. Although Megan did not directly talk about the sum of the angles in a triangle, resulting in a small gap between her final statement and the remainder of the argument, her work overall demonstrates logical flow.

On the similar rhombus task, both Lauren and Lexi constructed written arguments stating that the conjecture was false; however, only Lauren constructed an argument that contained logical flow. Lauren's argument, shown in Figure 2.2, began by stating the definition of similar polygons and then used that definition to argue why all rhombuses were not similar to one another. Note that when reading Lauren's argument, the word "they" used in the second sentence refers to the diagram drawn below the written text.



Conjecture: All rhombuses are similar
(Alternate phrasing: If two polygons are rhombuses, then they are similar to each other)

Not all rhombuses are similar because similar shapes are the same shape, just a different size proportion. Although they are rhombuses, they don't have the same shape structure, so not every rhombus is similar to another.

Figure 2.2. Lauren's argument for the similar rhombuses proof task.

Whereas Lauren used her definition of similar polygons to help justify her conclusion that all rhombuses were not similar to each other, thereby drawing a logical connection between the first and second sentences, Lexi's argument, shown below, lacked this explicit connection.

> This is not a true statement. By the definition of a rhombus, we know that there are two sets of parallel sides and the diagonal angles are the same. Also, all side lengths are the same. The reason why this is not true is because a square is a rhombus but a rhombus is not a square.

Lexi's final statement can be interpreted as a class of counterexamples—that is, the relationship between squares and non-square rhombuses. While the class of counterexamples correctly disproves the mathematical conjecture, her answer requires the reader to interpret how her final statement connects to the definition of a rhombus stated in the second sentence. Specifically, it is up to the reader to make the connection that squares are a subset of rhombuses and the differences in angle measurements between squares and non-square rhombuses implies that they are not similar, thereby proving the conjecture false. As a result of a lack of logical connection between the information in the second and third sentences and her final statement, her argument was coded as *not* containing logical flow.

Identifies and uses the correct hypothesis. When considering the overall structure of mathematical proofs, students must also learn that a mathematical claim can be broken down into the hypothesis, or given information assumed to be true, and the conclusion, or statement to be proven. Understanding this distinction is important so that students recognize what information they can use as a foundation for their proof (i.e., hypothesis) and so they avoid circular reasoning in their argument. Although textbooks traditionally identify the given information for students, allowing students to take ownership of this step provides an opportunity for them to make sense of the broader claim and the ways in which conditional statements are used in mathematics (Cirillo, 2017). For example, in order to prove theorems such as “if the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus”, a student must recognize that they cannot assume the shape is a rhombus or reference specific properties of rhombuses in their argument. When coding students’ work for this category, the emphasis was placed on whether each student identified and used the correct hypothesis instead of whether they began their proof by stating the hypothesis.

In the *sum of three odd numbers* task, students either identified the hypothesis as odd numbers or even numbers. Students who referenced odd numbers as their hypothesis either did so by stating a definition of an odd number (e.g., “an odd number is an even number +/- 1”) or by referencing a previously established fact about sums of odd numbers (e.g., “if you add any 2 odd numbers together, your answer will be even”). Notably, students referenced the given information in a variety of ways, demonstrating their understanding of this aspect of proof in an authentic rather than proceduralized manner[3]. The remaining students who referenced even numbers as their hypothesis used a mathematically accurate approach to the task but did not explicitly link their statements about even numbers to the original proof task about odd numbers. For example, Lexi’s argument stated: “Sarah is correct. I know this because anytime you add two even numbers together it equals an even number. Therefore, I know that an odd number is just an even number plus one.” While it is possible that Lexi recognized that the given information was three odd numbers, her use of the word “therefore” in the second sentence suggests that it was building off of the first sentence and was not meant to be the hypothesis. Thus, I coded the hypothesis in Lexi’s argument as two even numbers, which is not the correct hypothesis for the *sum of three odd numbers* task.

In the exterior angle theorem task, all students began their argument by either saying “Start with one triangle” (four students) or “Start with two triangles” (six students). Students’ decision to begin with two triangles instead of one was likely due to them copying an earlier proof we had constructed as a class proving that all squares are similar. When asked during the final interview why she began her argument by starting

[3] The hypothesis for the claim, “If you add three odd numbers together, your answer will be odd” is adding three odd numbers; however, a student could also choose to take a step back from the hypothesis by first considering the sum of two odd numbers or the definition of a single odd number and then working up to the official hypothesis of adding three odd numbers. Thus, I considered any hypothesis that involved adding one, two, or three odd numbers to be correct.

with two triangles, Amanda, for example, stated that she had been copying off of our earlier proof about similar squares, but “I remember when we were leaving I thought about that and I realized that we really didn’t need that.” Amanda's reflection highlights the potential value of asking students to identify the given information—namely, that it may encourage them to think through each proof task instead of copying sentences from one proof to another without thinking. On the other hand, Clay and Sadie stated during the final interview that they thought the proof should begin with *at least* two triangles. Sadie justified her initial statement saying, “You could have more so… should we just put like, we shoulda put ‘start with at least two triangles.’” Sadie's justification reveals an awareness that the angle relationship should hold for all possible triangles; however, it is unclear from her response whether she recognizes the one triangle used in the hypothesis *represents* all possible triangles.

Set of Accepted Statements

The first numbered criterion in Stylianides’ (2007) definition is that proofs should contain “statements accepted by the classroom community [. . .] that are true and available without further justification” (p. 291). While Stylianides included definitions, theorems, and axioms as examples of accepted statements, he also acknowledged that statements such as “the sum of two odd numbers is even” may be considered an axiom that does not need further justification in some instances and a statement that needs to be proven in other instances, depending on the context and intended learning goal. Since my proposed framework is intended to be used in instances where students are learning how to construct formal proofs, I have shifted the focus from statements to the justifications that are included (or not) for statements in the students’ argument. In particular, the two sub-categories are attention to the justification requirement and using mathematically accurate justifications.

Attention to the justification requirement. When coding students' work for evidence of attention to the justification requirement, the emphasis was placed on whether statements were justified instead of whether those justifications were mathematically accurate. For this category, students' work was coded as demonstrating attention to the justification requirement if they provided a justification or, if justifications were absent, they critiqued their own argument by saying that it *should* include a justification. This category acknowledges the fact that most students are not accustomed to justifying every statement they make in mathematics prior to learning how to construct proofs. Traditionally, teachers have accommodated for this shift in requirements through the use of the two-column format, which visually reminds students that every statement should be accompanied by a justification. While this structure emphasizes the need for every statement to be justified, it also results in the use of trivial justifications such as "simplify". Thus, the codes for this category focus on whether students justified all key statements, or statements central to the argument, instead of whether they justified every single statement. Evidence that students were attending to the justification requirement include the use of words such as "because", "by", or "since" or by students directly referring to a definition, axiom, theorem, etc. within their argument. The use of the "justified some key statements" code was assigned when a student justified at least one key idea <u>and</u> stated at least one additional key idea without justification.

In the *sum of three odd numbers* task, students either explicitly justified all key ideas or utilized previously established facts. For example, Lauren's argument shown below was based on her informal definition of odd numbers and the previously established property of even and odd number sums.

> She is right because every number will have a set of even numbers within it, but it will always have one left over. If you add them, there will be 3 of the numbers left over. This makes it odd because the 3 left over will be odd and that makes the sum odd.

Although Lauren could have clarified her second sentence to remind the reader that the task involved adding three odd numbers, the first and third sentences, which contained ideas central to her argument, both included justifications. Specifically, Lauren justified her claim in the first sentence that "she is right" by informally defining odd numbers as even numbers plus one. In the third sentence, she justified her claim that the sum of three odds is odd by stating that adding three and the implied even numbers from the first sentence will result in an odd number. In this sentence, the justification was taken to be the previously established fact that the sum of an odd number (3) and an even number is odd. For the purposes of this study, I considered Lauren's final statement to include a justification; however, there may be different contexts where the researcher or teacher would expect students to provide a more explicit justification.

The sole empirical argument produced on the *sum of three odd numbers* task, written by Sadie, was also coded as demonstrating attention to the justification requirement for all main ideas. Sadie's argument began with the following two sentences: "Sarah is right, because when you add different examples [of] three odd numbers together it equals an odd number. Examples of this are 1+3+7 =11 and 11+1+5 = 27." In this argument, the examples serve as a justification for Sadie's assertion in the first sentence that adding different sums of three odd numbers yields an odd number. Recall that this component of the framework is not focused on the correctness or appropriateness of the justification, just on the fact that the student realized a justification was expected.

The exterior angle theorem task required students to justify multiple statements involving different mathematical ideas. Heather's argument, shown below, demonstrated "some" attention to the justification requirement as she referenced "the definition of a triangle" when stating that "all of the angles add to be 180°" but did not provide a justification for her written equation "D + C = 180°" without being explicitly prompted to do so in the interview. Note that the third sentence ("If the angles, for example…") was interpreted as being a further explanation of the second sentence; as a result, it did not need an additional justification.

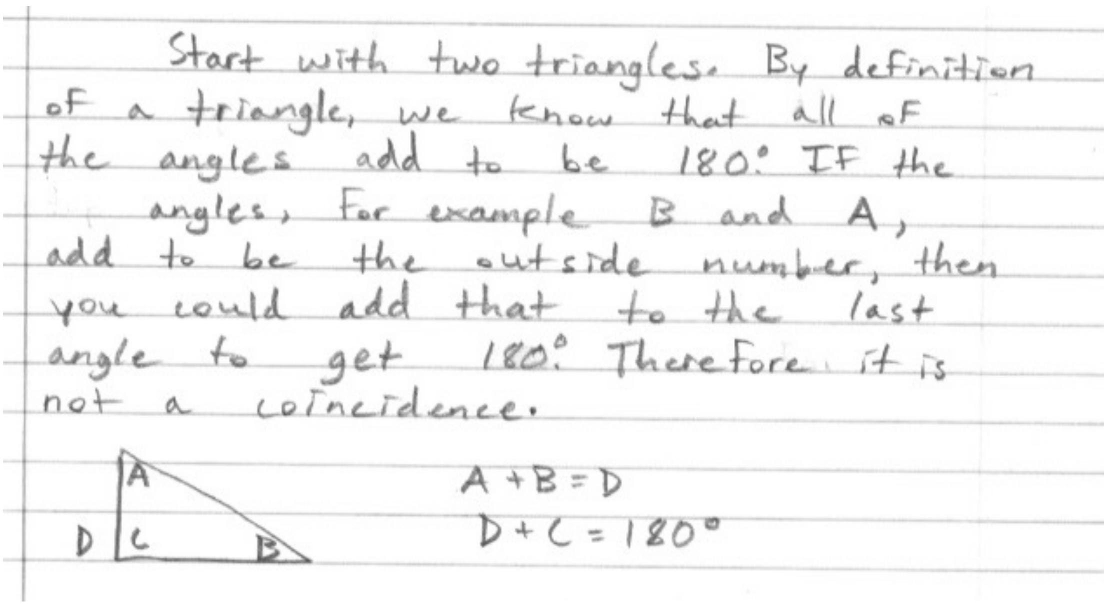


Figure 2.3. Heather's argument for the exterior angle theorem.

On the similar rhombuses task, seven students justified all main ideas and the remaining three students, who all thought the statement was true, justified at least one main idea in their argument. The students who thought the conjecture was false justified their conclusion by either stating a specific example or by making a broader statement referencing a class of counterexamples. For example, Wilson justified his conclusion that the conjecture was false by defining rhombuses and squares and then stating, "a rhombus doesn't have to have 90º angles and a square does. Because of this 2 rhombuses don't necessarily have to be similar." In these sentences, Wilson not only provided a class of counterexamples—the relationship between squares and non-square rhombuses—but also stated that the two shapes were not similar due to differences in the possible angle measurements for each shape. Three of the four students who wrote an argument stating that the similar rhombus conjecture was *true* demonstrated some attention to the justification requirement, as their argument contained at least one claim that was not justified. Specifically, they claimed that connecting equal side lengths would result in equal angle measurements, but they did not justify how the sides determined the angle measures. An example argument written by Brian follows:

> By the definition of a rhombus, all of the sides will be of equal length. Connecting equal length sides will result in equal angle measures. Since all of the sides are the same length, they will be proportional. All of the angles will also be equal to the original shape since the side lengths are the same.

While Brain demonstrated attention to the justification requirement in the first sentence through referencing the definition of a rhombus, his claim about the angles in the second and third sentences were left unjustified. It is likely that the second statement was left unjustified since no mathematical justification exists for the claim. With that said, had Brian been paying closer attention to the justification requirement, he might have been able to recognize the error in his argument.

Table 2.2

Summary of students' work and associated codes for attention to the justification requirement and uses mathematically accurate justifications.

Task	Student	Written Argument	Attention to Justification Requirement	Uses Math. Correct Justifications
Sum of Three Odds	Lauren	"She is right because every number will have a set of even numbers within it, but it will always have one left over. If you add them, there will be 3 of the numbers left over. This makes it odd because the 3 left over will be odd and that makes the sum odd."	All key statements	All key statements
Sum of Three Odds	Sadie	"Sarah is right, because when you add different examples [of] three odd numbers together it equals an odd number. Examples of this are 1+3+7 =11 and 11+1+5 = 27. Both of these examples end up being odd. Just having these two examples doesn't make it completely true though. Trying all these examples would take forever, so based on the time we have, we can tell that three odds added together will be odd."	All key statements	None
Exterior Angle Theorem	Heather	"Start with two triangles. By the definition of a triangle, we know that all of the angles add to be 180º. If the angles, for example B and A, add to be the outside number, then you could add that to the last angle to get 180º. Therefore it is not a coincidence."	Some key statements	All key statements
Similar Rhombus Conjecture	Wilson	"By definition a rhombus is a polygon that has 4 equal sides that the angles add up to 360º. By definition a square has 4 90 degree angles with sides that are equal. A rhombus doesn't have to have 90º angles and a square does. Because of this 2 rhombuses don't necessarily have to be similar."	All key statements	All key statements
Similar Rhombus Conjecture	Brian	"By the definition of a rhombus, all of the sides will be of equal length. Connecting equal length sides will result in equal angle measures. Since all of the sides are the same length, they will be proportional. All of the angles will also be equal to the original shape since the side lengths are the same."	Some key statements	Some key statements

Uses mathematically accurate justifications. Whereas the last category, attention to the justification requirement, assessed whether students included *any* justification, this category assesses whether the provided justifications are mathematically correct. In alignment with Stylianides (2007), this category is flexible and can be adjusted according to students' mathematical knowledge. When assessing whether students used mathematically correct justifications in their arguments, I placed more emphasis on whether students' justifications were used in acceptable ways than on whether they cited the correct formal name. As students are introduced to more formal terminology, the researcher or teacher can adjust what is and is not considered to be a mathematically accurate justification.

In this section, I revisit the students' arguments described in the attention to the justification requirement section to assess whether the justifications provided in each argument were mathematically correct. See Table 2.2 for a summary of students' written arguments and their associated codes for the attention to the justification requirement and uses mathematically accurate justifications categories. Lauren's written argument for the *sum of three odd numbers* task not only contained justifications for all key statements, but also contained justifications that were mathematically correct. Notice that both of her justifications were descriptive and did not explicitly reference a definition or property. Nonetheless, her justifications were mathematically accurate and likely ones that could be understood by her peers. In contrast, Sadie's use of examples to justify her conclusion in the sum of three odds task was coded as having no mathematically correct justifications since examples are not a mathematically correct justification for a universal claim.

Heather's argument for the exterior angle theorem was coded as containing mathematically correct justifications for all main statements that included a justification because she referenced the “definition of a triangle” when stating that the sum of the angles in the triangle add to 180º. Since the students had not yet been formally introduced to theorems, I considered her justification as being mathematically correct since it was clear that she was referring to a property of triangles that she had learned in a prior math class.

Although Wilson's justification on the false similar rhombuses task was non-normative in the sense that he provided a class of counterexamples instead of a single counterexample, his justification was coded as being mathematically accurate since the justification correctly identified a reason why the statement was false. Specifically, his justification referenced the fact that the corresponding angles in squares and non-square rhombuses will not be congruent. Brian's argument, in contrast, was coded as including only some mathematically correct justifications since he correctly justified his first sentence (“by the definition of a rhombus…”) but did not provide a mathematically correct justification for his fourth sentence (“All of the angles will also be equal to the original shape since the side lengths are the same”).

Modes of Argumentation

The next criterion in Stylianides’ (2007) definition is that proofs should be communicated using appropriate modes of argumentation, or assertions that are mathematically valid and able to be understood by the given community. Implicit within this portion of Stylianides’ definition is the idea that the statements within the argument, in whatever form it takes, should adhere to the generality requirement. For example, proof by exhaustion is an appropriate mode of argumentation when the statement applies to a finite number of cases, but is *not* an appropriate method for proving a universal claim. Thus, the two codes associated with modes of argumentation are attention to the generality requirement and overall type of argument.

Attention to the generality requirement. The first category, attention to the generality requirement, assesses whether each student demonstrates explicit evidence acknowledging that their argument should apply to all possible cases. Evidence of attention to the generality requirement includes language indicating generality (e.g., “any”), the use of variables to represent all possible numbers, or the inclusion of written or verbal statements directly indicating an awareness that their argument *should* apply to all possible cases. When coding for attention to the generality requirement, inclusion of the aforementioned evidence should be interpreted as indicators that the student is attending to the generality requirement, but should not be interpreted as definitive evidence. For example, it is possible that a student might use variables because they have learned that specific numbers should not be used to represent side lengths and not because they recognize that the variable represents all possible side lengths. Note that it is also possible for students to construct a general argument without explicitly acknowledging the generality requirement.

Nearly all of the students’ arguments during the final interview demonstrated attention to the generality requirement, possibly in part because this was a major focus of the design study. In order to illustrate how a student could construct a deductive argument that does not necessarily demonstrate attention to the generality requirement, I present Brian’s argument for the sum of three odds task that he constructed during the initial interview. “Yes, because an odd + an odd is even, and even + odd would be odd.”

Although his argument applied to all possible cases, he did not explicitly use language indicating awareness that his assertions were true for any possible odd number. His verbal explanation also lacked specific attention to the generality requirement.

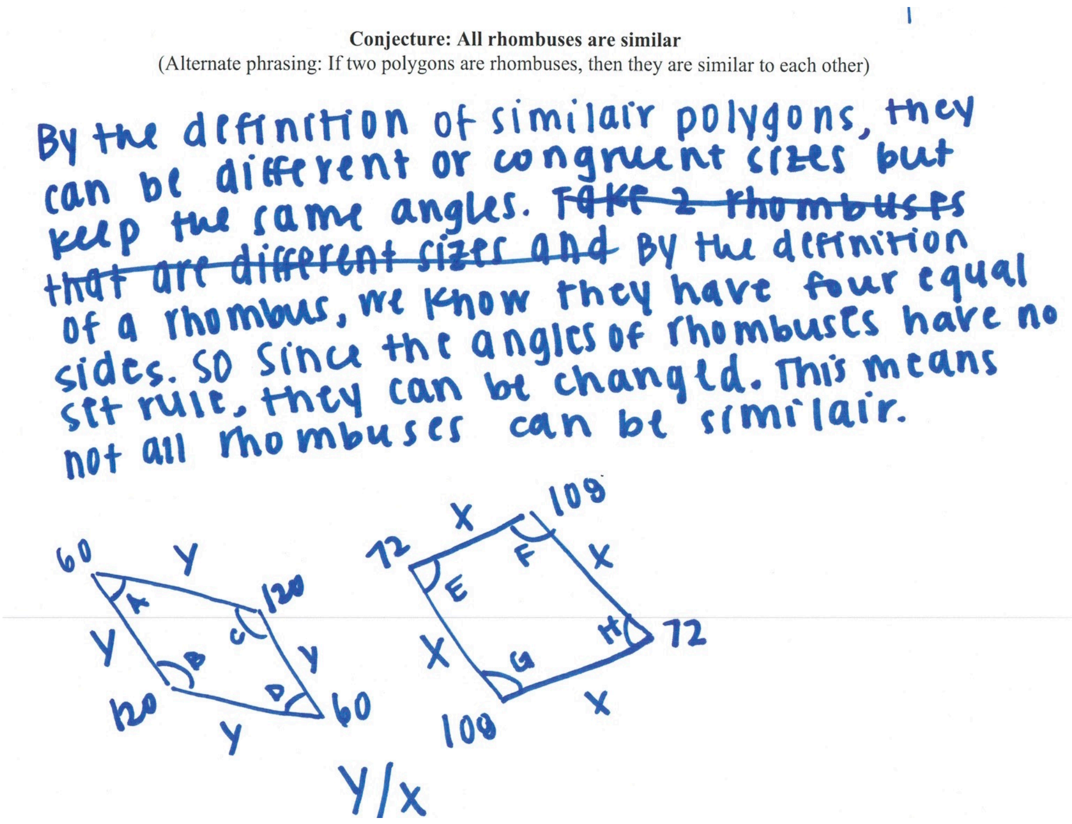
> If you add two odd numbers together, 3 and 7 for example, that's 10, which is even. If you did like 7 and 9, that's 16 and so on, basically you get an even number. And an even number plus an odd number is going to be odd.

While it is certainly possible that Brian recognized that his argument should apply to all possible sums of three odd numbers, his verbal explanation and written argument did not *directly* acknowledge this requirement. Thus, his argument was coded as not demonstrating attention to the generality requirement.

In contrast to students' work during the initial interview, nearly all of the arguments produced in the final interview demonstrated attention to the generality requirement. For example, students expressed their understanding of the generality requirement on the sum of three odds task by using variables to represent odd numbers, through the use of a generic example, or by using language indicating that they were talking about all possible even and odd numbers (e.g., "every odd number will have a set of even numbers within it, but it will always have one left over."). The sole empirical argument constructed during the final interview, written by Sadie, also demonstrated attention to the generality requirement. After showing two specific examples, Sadie wrote, "Just having these two examples doesn't make it completely true though. Trying all of these examples would take forever, so based on the time we have, we can tell that three odds added together will be odd." Although Sadie ultimately chose to generalize her two examples to conclude the statement was always true, her first sentence provides evidence to support her recognition that the argument did not sufficiently meet the generality requirement. By assessing students' attention to the generality requirement separate from their overall argument (and specifically, whether it adhered to the generality requirement), the researcher or teacher is able to differentiate between students' understanding of the component of proof and the type of argument they are mathematically able to produce.

In the exterior angle theorem task, all students demonstrated attention to the generality requirement through their inclusion of at least one diagram where the angles were labeled using variables. Students also emphasized the generality of definitions by adding the words "any" and "all" to the definition, even when those words are typically implied. For example, Arin wrote the definition of supplementary angles as "any two angles that make up a straight line, add to 180º." The use of the unnecessary word, "any" places additional emphasis on the fact that the statement applied to all possible angles that form a straight line. Similarly, half of the students stated the triangle angle sum theorem as the following: "By definition of a triangle, we know that all of the angles add up to be 180º." Students' use of the word "all" in their "definition" of a triangle could be a reference to the three angles in a triangle or could be a reference to all of the possible sums of the three angles in a triangle; in either situation, the use of the word "all" instead of saying "the sum of the three angles" places additional emphasis on the generality requirement.

All students demonstrated understanding of the generality requirement through their work on the similar rhombus task, regardless of whether they initially thought the conjecture was true or false. Specifically, six students thought the conjecture was false, while the remaining four students initially wrote deductive arguments proving the statement was true and then changed their answer during the interview. Of the six students who initially thought the similar rhombuses conjecture was false, half wrote non-normative arguments, or arguments written in the form of a deductive proof that referenced a class of counterexamples as evidence that the conjecture was false. For example, Megan’s argument began by citing the definition of similar polygons and rhombuses and then stated that “not all rhombuses can be similar” since the angles of rhombuses “have no set rule” (see Figure 2.4). When constructing her response, Megan initially labeled the angles with the variable labels and then decided later in the interview to label them with specific angle measurements. For this reason, her argument was coded as deductive (to establish falsity) instead of proof by (single) counterexample. Even though her argument was non-normative, her final statement that “not all rhombuses can be similar” provides evidence that she was attending to the generality requirement.



Conjecture: All rhombuses are similar

(Alternate phrasing: If two polygons are rhombuses, then they are similar to each other)

By the definition of similair polygons, they can be different or congruent sizes but keep the same angles. ~~Take 2 rhombuses that are different sizes and~~ By the definition of a rhombus, we know they have four equal sides. So since the angles of rhombuses have no set rule, they can be changed. This means not all rhombuses can be similair.

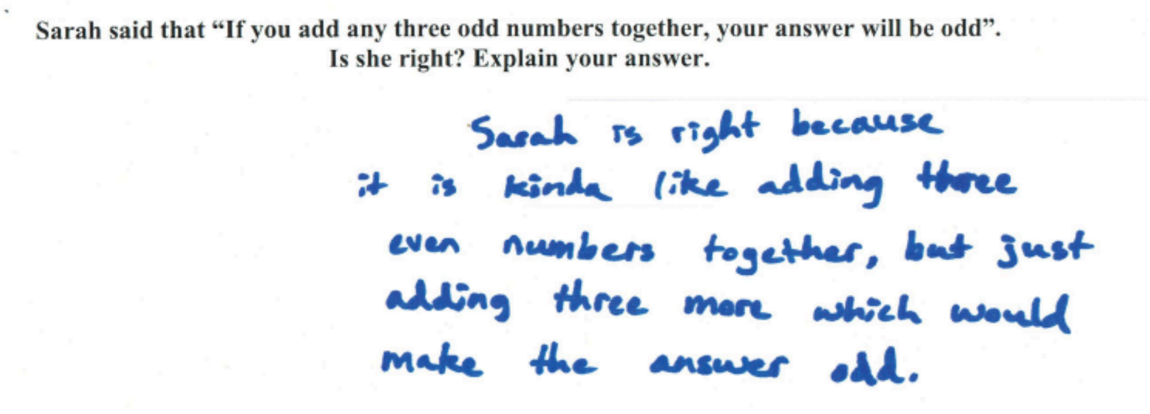
Figure 2.4. Megan's argument for the similar rhombuses conjecture.

Four students initially wrote arguments proving that the similar rhombuses conjecture was true; however, all of them changed their mind after I asked them targeted questioned during the interview. For these students, I analyzed their verbal responses once they had decided the statement was false to determine whether they demonstrated attention to the generality requirement for false conjectures. For example, after deciding the statement was false, Clay responded, "all rhombuses aren't the same [similar] because not every time the angle measurements will be the same and the sides proportional." Although Clay did not point to a specific example that disproved the conjecture, his response that the angle measurements would not be congruent "every time" demonstrated awareness of the generality within the conjecture.

Overall type of argument. The second category, overall type of argument, identifies students' overall approach to the proof task (e.g., deductive, counterexample, empirical). This category can be useful in a number of situations, especially when viewed in conjunction with the attention to the generality code or the expected response for the given proof task. For example, it is possible for a student to construct an empirical response and demonstrate attention to the generality requirement; without this category, one might assume that all arguments that demonstrated attention to the generality requirement were deductive. Additionally, this category is useful in instances where the validity of the conjecture was unknown by allowing the researcher to distinguish between students who constructed deductive arguments stating the statement was true and those who provided a proof by counterexample to show that the statement was false.

Students overwhelmingly produced deductive arguments during the final interview across the three tasks. Subsequently, I revisit students' work during the initial interview in order to illustrate empirical arguments for the sum of three odds task. In the initial interview, eight students constructed empirical arguments and two students constructed deductive arguments, although neither deductive argument demonstrated attention to the generality requirement (see Brian's response in the attention to the generality requirement section). The remaining eight empirical arguments included a total of 14 written examples. However, only three examples contained at least one number greater than 10 and six examples were special cases where a student added the same number three times (e.g., 5 + 5 + 5). In addition to the written examples, five students also included a written statement justifying their answer. Clay, for example, wrote a statement that went beyond referencing his written examples. "Yes she is right because you will never get the extra number or amount you need to make an even number.” The use of the word ‘never’ suggests that Clay was attempting to make a general argument about any sum of three odd numbers; however, his argument lacks justification for how he knew this would never be the case. On its own, the statement was too vague to understand what he meant by “the extra amount you need to make an even number” or to determine whether his line of thinking truly proved the statement was always true. Subsequently, this statement was not enough to warrant coding it as a deductive argument.

Although students were not directly taught different ways to format a proof (e.g., two-column, paragraph, flow) or different proof techniques (e.g., proof by induction), they still used a variety of deductive approaches to prove the sum of three odds conjecture. For example, Heather (Figure 2.5) utilized her understanding of the relationship between even and odd numbers, whereas Amanda (Figure 2.7) used a generic example to support her argument.


Sarah said that “If you add any three odd numbers together, your answer will be odd”. Is she right? Explain your answer.

Sarah is right because it is kinda like adding three even numbers together, but just adding three more which would make the answer odd.

Figure 2.5. Heather’s argument for the *sum of three odd numbers* task.

On the false similar rhombus task, four students constructed deductive arguments to establish truth, three students constructive deductive arguments to establish *falsity* (e.g., Megan's argument in Figure 2.4), and the remaining three students provided a single counterexample. Most of the counterexamples students provided depicted the relationship between a square and a (non-square) rhombus. Students who initially thought the conjecture was true either assumed rhombuses had fixed angle measurements (e.g., Brian's argument in Table 2.2) or only focused on the sides in their argument.

Modes of Argument Representation

Stylianides' (2007) final criterion is that proofs should be communicated using forms of expression that are familiar and can be understood by the classroom community. The first category, forms of expression used, replicates Stylianides' (2007) final criterion. As proofs become more complex, it is common for students to need to produce or use multiple forms of expression in their proof. In such instances, students must understand how the different representations work together to produce a single argument, or proof, for the given mathematical claim. Thus, the second sub-category analyzes the relationship between students' written text and other forms of expression, if additional forms were constructed or provided for students to use.

Forms of expression used. This category in the framework is purely descriptive and does not carry a judgment on the preferred form used. Additionally, this is the only category where students' work can receive multiple codes. For example, an argument proving the triangle exterior angle theorem could be coded as written, diagrammatic, and algebraic if a student included all three forms.

Nearly all of the arguments students constructed on the three tasks contained written text. Students' work on the sum of three odds task also included the following representations: number line (1), pictorial representation (2), algebraic (4), and symbolic examples (4). Amanda's argument shown in Figure 2.5 is an example of a pictorial representation that was included alongside written text for this task. For the exterior angle theorem, all of the students included at least one diagram and nine of the students included at least one algebraic equation. In the argument below, notice that Arin included two diagrams: the top diagram that labeled the angle measurements using variables and the second diagram, possibly drawn to help think through the angle relationship, that labeled the angle measurements with specific numbers. Arin's argument also contained two equations: A+D = 180 and B+C = D, the angle relationship she was trying to prove.

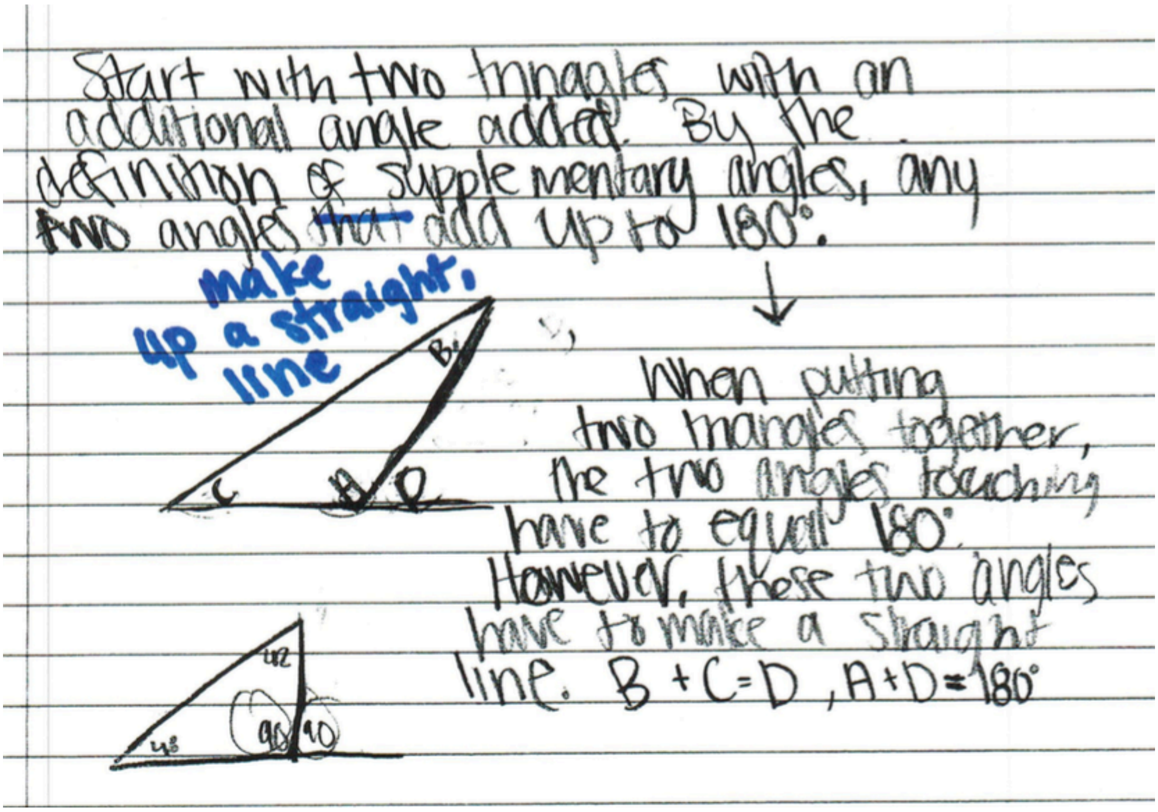


Figure 2.6. Arin's argument for the exterior angle theorem. Note that the blue text was added during the final interview.

Finally, only six students included a diagram in their argument for the similar rhombus conjecture. For examples, see Lauren's argument in Figure 2.2 and Megan's argument in Figure 2.4; Wilson's argument, reproduced in Table 2.2, did not include a diagram. There are two notable takeaways from analysis of students' use of representations other than written statements in their arguments. First, students tended to include multiple representations in their arguments; and second, not all students included a diagram for the similar rhombus task despite the geometric context of the task.

Relationship between written text and other forms of expression. The second category assesses the relationship between the given forms of expression students used in their arguments whenever multiple forms of expression were used. A student's work was coded as "integrated" if they made explicit connections between at the written text and other form(s) of expression. These connections include referring to specific angles or side lengths or circling and labeling different parts of their pictorial representation to link it to the written statements. If the student included multiple forms of expression that all contribute to making the same argument, but were not explicitly connected to one another, then the argument was coded as "aligned." This code allows the researcher or teacher to differentiate between instances where the student appears to understand how the different components of the proof work together and instances where the student might have included multiple components without providing evidence of fully understanding how to use each component to support their overall argument. The third code, "not aligned", was used in instances where the different forms of expression either contradict each other or make different arguments. This code was not present in the data collected during my study, but could occur in instances where the student produces a response containing multiple attempts at proving the statement instead of editing their answer to contain a single approach.

On the *sum of three odd numbers* task, only three students included a visual representation; additionally, two of the representations were constructed after I encouraged the student to do so during the interview to help explain their thinking. Amanda, the only student who included a visual representation without prompting, drew a generic example consisting of circles drawn in vertical columns of two (see Figure 2.6). The circles in her argument were interpreted as an generic example because she did not refer to the specific number of circles in her written or verbal for this task. Her argument was coded as “integrated” because she indicated in her picture that the final circle “makes it odd”, which repeated the final portion of her written text.

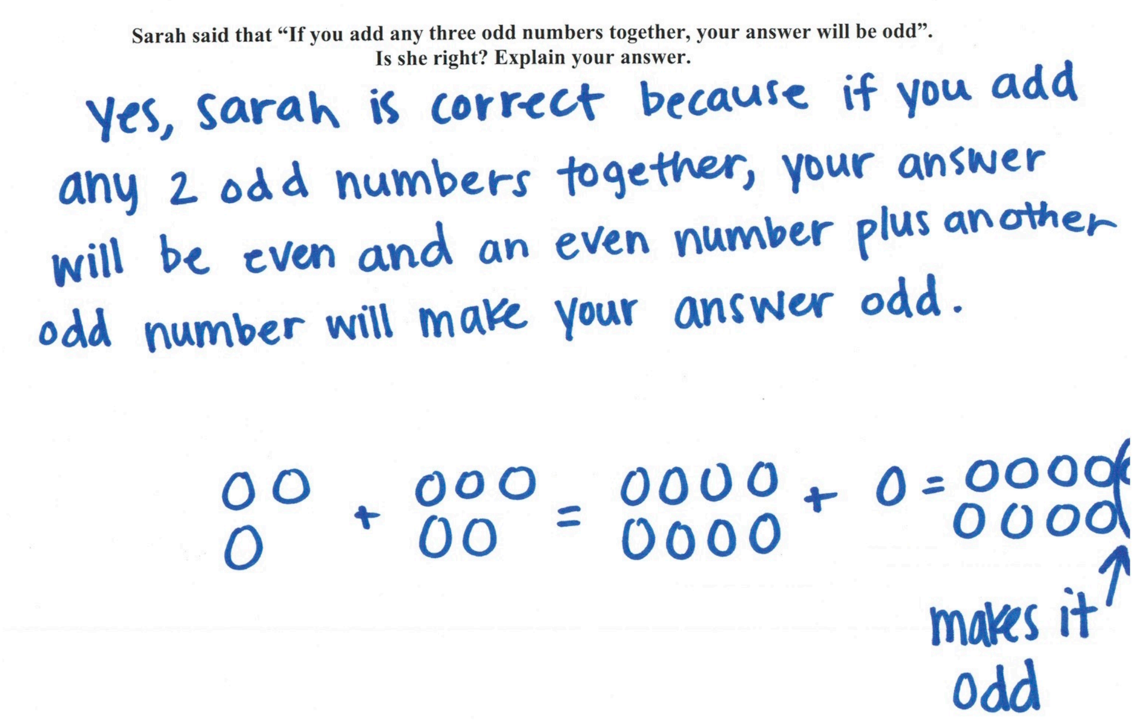


Figure 2.7. Amanda's argument for the sum of three odds task.

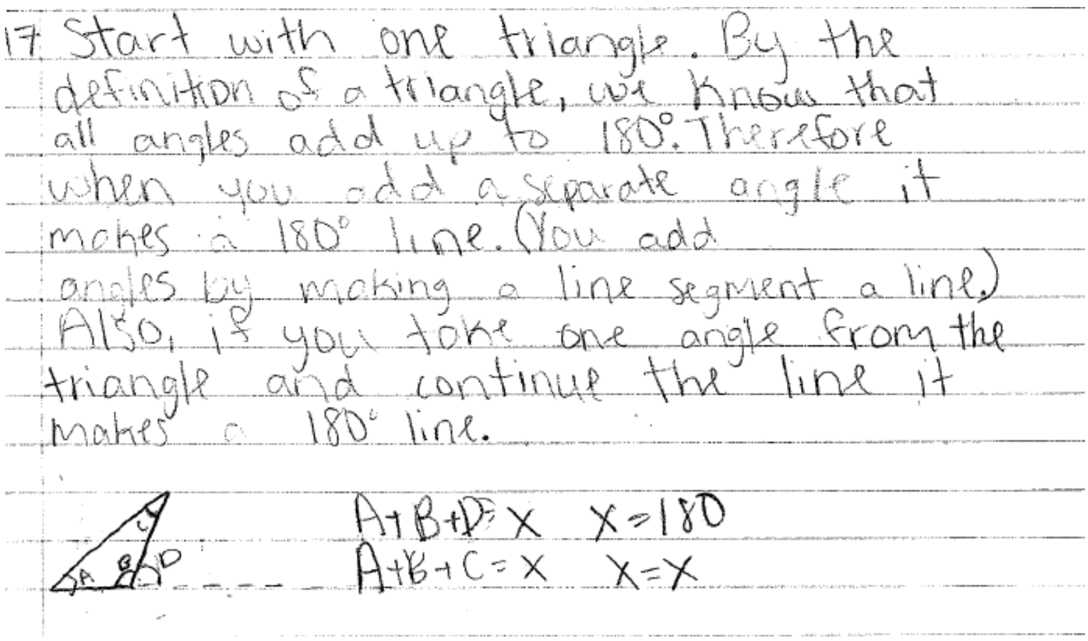
Of the four students who included algebraic expressions or equations in their sum of three odds argument, three students integrated the expressions/equations within their written statements. Wilson's argument shown below is representative of the way that all four students used variables in their argument.

> By definition an odd number is anything that is not divisible by 2. A+B = even number if a and b are odd. A+B = odd number when A and B are even and odd. So when adding A+B+C = odd A, B, C could all be odd numbers.

Wilson integrated the use of variables directly into the written text, demonstrating that they were central to his argument[4].

All ten students drew at least one diagram on the exterior angle theorem task and six students referenced a specific component of their diagram in their written text. In other words, six students wrote statements such as “angles A, B, and C add up to 180º”, while the remaining four students wrote sentences such as “the three angles in a triangle add up to 180º”. Eight of the nine students who included algebraic equations in their argument placed the equations next to their diagram instead of embedding them within the written text. Lexi's argument, shown in Figure 2.7, was coded as “aligned” because she did not directly reference specific angles or include the equations within her written text. Thus, it was not clear from her work that she recognized how the diagram and algebraic equations supported her written argument.

[4] Recall that this code assesses the connections between representations, not whether the use of additional reprentations strengthened the overall argument.


17 Start with one triangle. By the definition of a triangle, we know that all angles add up to 180°. Therefore when you add a separate angle it makes a 180° line. (You add angles by making a line segment a line.) Also, if you take one angle from the triangle and continue the line it makes a 180° line.

A+B+D=X X=180
A+B+C=X X=X

Figure 2.8. Lexi's argument for the exterior angle theorem.

For the similar rhombus task, seven students included both a diagram and written text, with all of the diagrams located at the middle or bottom of their page; of the seven included diagrams, three arguments made explicit references to the diagram in the written text (see Brian’s argument in Figure 2.9 and Megan's argument in Figure 2.4). The placement and lack of explicit references to the diagram suggest that while students recognized that geometric proofs typically include a diagram, it is unclear whether they understood how to use the diagram to assist in the proof construction process. Notice in the pictures below that Brian completed his written argument before constructing a diagram to accompany the argument. While it is possible that Brian and the other students did not feel the need to initially construct a diagram for the similar rhombus conjecture since they could look at the one on the iPad screen, their placement of the diagrams towards the bottom or middle of their arguments for the other two proof tasks as well (see e.g., Figures 2.3, 2.6, 2.7, and 2.8) suggests that the diagrams were not consistently being used as an aid to support the written arguments. Based on students' placement of the diagrams and when they chose to construct them, it is unclear whether they viewed the diagram as a crucial part of the proving process.

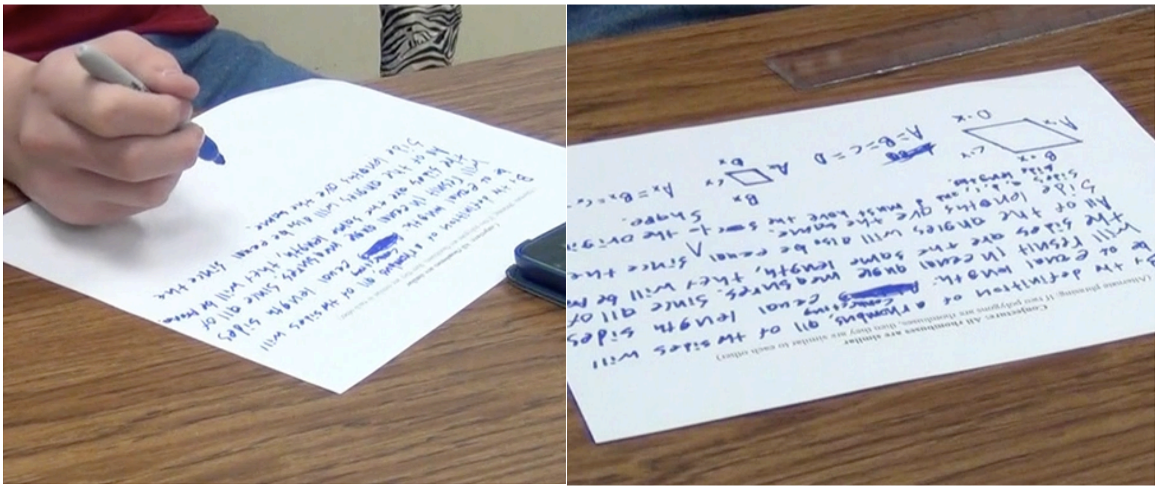


Figure 2.9. Brian's argument for the similar rhombuses conjecture in progress (left) and completed (right), demonstrating that he wrote the argument before he constructed a diagram. See Table 2.2 for the text of Brian's written argument.

Discussion

The analytic framework presented in this paper has the potential to support research studies in the area of proof by allowing the researcher to acknowledge and attend to the complexities of students' arguments instead of assigning their work a single score, category, or level. Indeed, this benefit was why I chose to develop the framework, as using one of the many previously developed frameworks would have resulted in the majority of students' arguments receiving the same code. Had I coded students' arguments with Harel and Sowder's (1998) proof scheme, for example, 23 of the 30 arguments would have been classified as "analytical", one argument classified as "empirical-perceptual", and the remaining six arguments not classified because they proved a statement was false. Although this classification suggests that students tended to produce analytical, or deductive arguments, it fails to capture the nuanced details within each argument in terms of the components of proof that students demonstrated understanding of as well as reveal areas for continued growth.

One feature of my framework that is particularly useful when analyzing arguments produced by students who are in the initial stage of learning how to construct proofs is that it allows the researcher to separate analysis of students' proof understanding and their content knowledge. This is achieved through the separate "attention to" and "mathematically correct" categories for generality and justifications. Although the researcher is still limited by the extent to which the student is able to communicate their ideas, the framework has the potential to support making claims about the source of difficulty that were beyond the scope of previous frameworks. For instance, looking across the coding of student work summarized in Table 2.2 reveals distinctions between those who did not provide a justification for all key statements (Heather), those who provided a justification that was not mathematically accurate (Sadie), and those whose work both contained at least one missing justification and one incorrect justification (Brian). If these arguments had been coded using the framework by Waring (2000), both Heather and Brian's arguments would have been classified at a level 3 ("pupils are aware of the need for a generalized proof, and, although unable to construct a valid proof unaided, are likely to be able to understand the creation of a proof at an appropriate level of difficulty." p. 6), even though Heather's argument contains fewer aspects that could benefit from revisions. Finally, analyzing the details of students' arguments affords the researcher more opportunities to focus on what students demonstrate they know and can do in the area of proof. In the case of the present study, all of the students demonstrated attention to the generality and justification requirements across the three proof tasks even though they did not always construct arguments that were deductive and contained mathematically correct justifications. As the field seeks to move the teaching and learning of proof in K-12 classrooms forward, it is important that we look for ways of identifying the skills and understanding students have developed, which can then be leveraged in future instruction.

In keeping with the flexibility built into Stylianides' (2007) definition, my framework allows for statements to be coded as having a mathematically correct justification when the justification is a previously established fact that students assume to be true. On the one hand, this decision allows for flexibility in the types of justifications that are considered acceptable and is in line with my goal of not rigidly requiring students to justify trivial statements (e.g., "simplify"). On the other hand, this flexibility has the potential to make it more difficult to compare findings across multiple studies since the statements that were assumed to be true could vary significantly between researchers. In order to minimize this limitation, researchers who use this framework to analyze students' work should be explicit about the types of statements they accepted as not needing an explicit justification due to the students referencing previously established facts. In the present study, I considered all statements involving sums of even and odd numbers (e.g., "an even plus an odd is odd") as having an implicit justification.

In the framework, I distinguished between attention to the generality requirement and overall type of argument (that is, whether it *adhered* to the generality requirement) in order to acknowledge students who stated that their empirical argument did not demonstrate that the statement was always true. The attention to the generality requirement category raises questions regarding the possible ways that students can demonstrate awareness of the generality requirement without constructing an argument that adhered to this requirement. Said another way, although it is easy for researchers to distinguish between the inclusion of a justification and referencing mathematically correct justifications, the difference between attention to the generality requirement and constructing arguments that adhere to the generality requirement is less clear. This is both a potential limitation of the framework as well as an opportunity for future research. Specifically, what are ways that students attempt to acknowledge the generality requirement in a way that is not mathematically correct?

Potential Implications for the Classroom

To this point, the article has focused primarily on the use of the framework for research purposes; however, it also has the potential to serve as a rubric in K-12 classes to assess students' arguments and then provide feedback to further their understanding of proofs. In order to illustrate this potential application, I am going to revisit Lexi's argument shown in Figure 2.5.

Lexi's argument for the exterior angle theorem demonstrated correct identification and use of the hypothesis, overall understanding of logical flow, some attention to the justification requirement, correct mathematical justifications, and attention to the generality requirement. Additionally, she included equations and diagrams in her argument, but they were only aligned (and not integrated) with her written text. After using the rubric to assess students' work, the teacher could ask students to revise their arguments individually or in small groups in order to address specific aspects of their argument. In Lexi's case, this would include revisiting her equation A+B+D = X so that it included a justification and thinking more broadly about how to incorporate her equations and diagrams into her written text. The teacher might also direct Lexi to think about her last statement and how that information could be useful in proving that the angle relationship will always occur. During this revision process, Lexi would hopefully revise the equation, A+B+D = X in the process of looking for a justification. Alternatively, the teacher might decide to engage the entire class in thinking through some aspect of proofs that students were struggling with, such as what it looks like to attend to the generality requirement and why this is important when constructing a proof.

Using the framework as a rubric in the classroom has a few important implications. First, it removes the expectation that students should be constructing proofs every time they work on a proof task and instead replaces it with the idea that students are expected to develop their understanding and skill in constructing proofs over time through constructing initial draft arguments and then making revisions to their work. Second, this framework removes the overemphasis on the form of proofs while still expecting that students will provide justifications for all main claims. Lessening the focus on the form of proofs allows for the emphasis to be placed on constructing proofs as a way of ascertaining that a statement is true, understanding *why* it is true, and communicating that understanding in a way that makes sense and is convincing to others. Third, the framework could be easily modified as students develop their proof skills to include other elements of proofs, such as whether the student's argument is complete or whether it contained unnecessary statements. While I ultimately decided these codes were not needed when evaluating arguments produced by students who were just beginning to learn how to construct proofs, they would be valuable to support students in refining their arguments until they could be considered proofs.

Conclusion

The framework for assessing components of students' arguments differs significantly from previous proof frameworks (e.g., Bell, 1976; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Lee, 2016; D. Stylianou, Chae, & Blanton, 2006) in that it reveals nuanced distinctions between students' arguments and has the ability to shed light on what students do know and can do in regards to proof without holding them to the standard of producing accurate and complete proofs. From a research perspective, focusing on the nuances of students' arguments has the potential to further the conversation about what a mathematical argument can and should look like at different stages of elementary and secondary school while continuing to provide a way to look at students' work across multiple proof tasks. This framework also provides a way for classroom teachers to provide students with feedback on their arguments that does not focus on whether they placed the statements and reasons in the "proper" order (Martin & McCrone, 2003; Otten et al., 2017; Schoenfeld, 1988). Although applying this framework to assess students' work in the classroom would likely require additional time and effort, use of the framework also has the potential to challenge teachers' notions of the types of arguments beginning proof learners are capable of producing and what it means to demonstrate understanding of different components of proof. For example, use of this framework has the potential to shift teachers' focus from the form of the argument to broader components of the argument, such as which statements referenced key mathematical ideas, thereby requiring a mathematical justification, or what it look like for an argument to contain logical flow. As a result, use of this framework in the classroom requires teachers to have strong mathematical and proof content knowledge as well as shift their expectations for the way that students' arguments should look or the extent to which students' argument should replicate the argument they or the textbook produce.

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CHAPTER 3

Engaging Students in Tasks Involving Universal Claims in an Introduction-to-Proof Unit

Proofs are fundamental to mathematics as they provide a method for conclusively demonstrating the validity of a mathematical statement, including universal statements about a class of objects, without having to check every possible case. The idea that proofs demonstrate the validity for all possible cases is frequently misunderstood by students, including some who are able to successfully construct a proof on their own (Chazan, 1993; Martin et al., 2005). For example, two students in Chazan's (1993) study asserted that they would have to construct a proof for each type of triangle (e.g., acute, obtuse, right) in order to prove that a statement about all triangles was true. However, these findings are not surprising when viewed alongside many of the proof customs present in traditional classrooms and textbooks that mask the generality of the claim, such as presenting proof claims to students in the form of separate "given" and "to prove" statements and providing a diagram for students to use in their proof (Herbst, 2002). Furthermore, more than half of the proof exercises in the six most commonly used geometry textbooks in the U.S. ask students to prove a particular statement (e.g., a conjecture about a specific triangle) rather than a universal statement (e.g., a conjecture about all rectangles) (Otten, Gilbertson, et al., 2014). Without frequent opportunities to prove universal claims, it is possible that some students may fail to internalize why proofs are a necessary component of mathematics and, instead, view them as an arbitrary requirement set forth by their teacher or textbook (McCrone & Martin, 2004; Otten, Gilbertson, et al., 2014).

Within the reasoning-and-proving opportunities found in introduction-to-proof textbook chapters, Otten, Males, and Gilbertson (2014) found that students were frequently asked to pose or investigate conjectures but were rarely asked to construct a proof. While it certainly makes sense that students would experience additional opportunities to construct proofs as they increase their knowledge of proof components and geometry content, “this [finding] implies one is beginning the road to proof with something other than proof” (p. 117). The present study extends Otten, Males, and Gilbertson’s (2014) work through analyzing an alternate introduction-to-proof unit, implemented as a part of a design research study conducted with ten ninth graders, that contained increased opportunities to engage in reasoning-and-proving tasks involving universal claims and reflect or talk *about* the reasoning-and-proving process. In this paper, I examine the impact of using tasks involving universal claims in terms of the opportunities they provided to engage students in reasoning-and-proving activity, talk *about* the reasoning-and-proving process, and develop an intellectual necessity for proof. I then use data from students’ work during the proof tasks to look at the impact of using tasks involving universal claims in terms of the challenges students faced in interpreting the language and scope of the claims.

Literature Review

Definition of Key Terms

This study draws on Andreas Stylianides (2007)'s definition of proof: “a mathematical argument, a connected sequence of assertions for or against a mathematical claim” that uses acceptable justifications, valid modes of argumentation, and representations that are appropriate and understood by the classroom community (p. 291). In particular, I am interpreting the terms “valid”, “acceptable” and “appropriate” according to both our classroom community and the broader mathematics community. This definition was used to inform my own instructional decisions but was not formally presented to students or used to evaluate students' work to determine whether it should be considered a proof. I use the term *proof* to refer to the tasks where students were expected to construct a proof. Similarly, I use *reasoning-and-proving* in this article to refer broadly to all of the activity that goes into establishing the truth-value of a claim, from proposing a conjecture and investigating the validity of the claim to constructing a proof or non-proof rationale (G. J. Stylianides, 2008). A non-proof rationale is when a student provides a justification for their assertion in a situation where they were not directly asked to construct a proof but instead was asked to "explain", "justify", or show "why" their statement is true. When using the terms *proof* and *reasoning-and-proving*, I am not making a judgment statement about the quality of response students produced or the extent to it aligned with Andreas Stylianides' (2007) definition of proof. Finally, I use the term *argument* to refer to any verbal or written work that students made in response to a proof task without making any judgment in regards to the quality of their work.

Students' Approaches to Proof in Relation to the Generality Requirement

Students tend to both construct empirical arguments, and to a lesser extent, evaluate provided arguments that are empirical as proofs. One possible explanation for students' preference for empirical arguments is that they are already convinced of the validity of the statement prior to constructing the requested proof; as a result, proofs are no longer used as a way of convincing oneself of the statement's validity. A second possibility is that students realize the limitation of using examples as justification but construct example-based justifications because they are unsure of how to construct a more general argument. Researchers who have asked students to both construct and evaluate provided arguments (e.g., Healy & Hoyles, 2000) have found that students are more likely to select deductive arguments as proofs than they are to construct deductive arguments. The ability for some students to identify deductive arguments as proofs, even in cases where they constructed empirical arguments, suggests a disconnect between students' understanding of what consistutes a proof and their ability to construct a proof that adheres to those requirements.

In an analysis of commonly used geometry textbooks in the U.S., Otten, Gilbertson and colleagues (2014) found that students were given fewer opportunities in the student exercises to find a counterexample or disprove a false mathematical claim than they were to prove a true mathematical claim. In addition to eliminating the uncertainty regarding the validity of the claim, thereby potentially decreasing students' understanding of the need for proof (Buchbinder & Zaslavsky, 2008), the minimal opportunities to produce counterexamples for false conjectures could potentially result in students' developing limited conceptions about the scope of mathematical claims, such as not accepting that a statement was false after being presented with a counterexample (Buchbinder & Zaslavsky, 2013), or believing the possibility of a counterexample after being shown a proof for the claim (Chazan, 1993). In order to develop students' understanding of the generality requirement, it is important that students have multiple opportunities to prove both true and false conjectures. Additionally, students also need opportunities to talk *about* the reasoning-and-proving process where the generality requirement is made an explicit item of focus (Otten, Males, & Gilbertson, 2014), as being able to construct a proof does not necessarily indicate an understanding of the generality requirement (e.g., Martin et al., 2005).

Students' understanding of the generality within diagrams. Multiple studies have found that many students do not recognize the generality of a proof involving diagrams but instead believe that the proof demonstrated the validity of the claim only for the given diagram (Chazan, 1993; Hoyles & Healy, 1999; Martin et al., 2005). For example, a few students asked their Geometry teacher to prove a statement for two different types of triangles immediately after he finished constructing the original proof (Martin et al., 2005). Similarly, some of the students Chazan (1993) interviewed believed that a proof only proved the statement for the provided diagram and that the existence of a proof did not guarantee that a counterexample did not also exist. For example, one of the students justified this belief saying, "This [deductive proof] could be true for this triangle [in the associated diagram or in diagrams of that type], but up here it [the statement] says in any triangle. I would have to think of all the other types of triangles, it would be true for. I couldn't just do it right now." (p. 372). One possible explanation for students' difficulty in interpreting diagrams as a specific case representing all possible cases could be their lack of experience in constructing diagrams to accompany a proof. Students are rarely, if ever, asked to produce diagrams to accompany their proof, but instead are provided a diagram by the teacher or textbook (Cirillo, 2017; Herbst & Brach, 2006). Although this norm increases the consistency and accuracy of the diagrams students use, it limits students' opportunities to reason about what the diagram represents or about the given information in the proof claim (Komatsu, Jones, Ikeda, & Narazaki, 2017). In an effort to minimize students making assumptions based off of visual inspection of the provided diagram, some teachers will tell students that the textbook diagrams are not always drawn to scale (Herbst & Branch, 2006). This tactic is problematic because it avoids engaging students in conversations about what can and cannot be interpreted from a diagram and provides a weak justification for why deductive arguments are necessary in mathematics.

Proof Instruction

Proof interventions. In response to findings revealing the conceptions of proofs students develop as a result of traditional instruction, there has been a recent increase in the number of intervention-based studies (e.g., teaching experiments, design studies, and classroom interventions) focused on improving students' understanding of proof. Interventions have focused on topics such as developing students' understanding of the different forms or approaches to proof (Harel, 2002; Miyazaki et al., 2015; Yopp, 2017), the need for proof (G. J. Stylianides & Stylianides, 2009), the axiomatic structure (Jahnke & Wambach, 2013; Mariotti, 2000), and the connections between proof and other topics, such as algebra (Martinez et al., 2011) and constructing mathematical definitions (Larsen & Zandieh, 2005). For example, Gabriel and Andreas Stylianides (2009) aimed to develop elementary preservice teachers' understanding of the need for deductive reasoning through exposing the limitations of assuming that a pattern found from a few examples will also be true for every possible case. Their instructional sequence was constructed based on the assumption that "pivotal counterexamples" produce the cognitive conflict needed to help students make the transition from empirical justifications to deductive reasoning (p. 317). Given the relatively recent focus on developing and testing proof interventions, it is not yet clear what the advantages and disadvantages are regarding introducing students to proof through focusing on developing their ability to construct proofs (e.g., Harel, 2002; Miyazaki, Fukita, & Jones, 2016; Yopp, 2017) or on developing an overall understanding of proof (e.g., Stylianides &

Styliandies, 2009). The present study takes the latter approach by developing students' overall understanding of the generality and purpose of proof while providing minimal instruction on the specific form and components of proof.

Traditional approaches to teaching proof. As discussed in chapter one, proofs in secondary mathematics classes are typically written in the two column-format, with the teacher placing significant emphasis on whether students place the statements and reasons in the "proper" form (Martin & McCrone, 2003; Schoenfeld, 1988). Although very few studies have documented the ways that teachers first introduce proofs in the secondary classroom, Cirillo (2014) found that the five Geometry teachers she observed all introduced proofs through a show-and-tell method and struggled to find ways to scaffold the introduction to proof for students. Given prior research's findings on the ways that proofs are often taught in the classroom, it is unsurprising that students struggle to understand the purpose of proofs and experience a range of challenges in learning how to construct proofs.

Reasoning and Proving Opportunities in Textbooks

Although research thus far has only provided minimal insights into the ways in which teachers introduce students to formal deductive arguments, there is a greater understanding about the kinds of reasoning-and-proving opportunities provided in textbooks (Otten, Males, & Gilbertson, 2014). Otten and colleagues (2014) analyzed the six most common geometry textbooks, used in roughly 90% of U.S. classrooms, for the frequency and types of reasoning-and-proving opportunities and statements/questions *about* reasoning-and-proving found in the exposition and student exercises of the introduction to proof chapter. They found that textbooks varied significantly in the percentage of student exercises in the introduction-to-proof chapter that engaged students in reasoning-and-proving activity, ranging from 27% to 65% of the student exercises. Within these reasoning-and-proving questions, students were primarily asked to develop non-proof rationales, pose conjectures, or investigate conjectures, but were rarely asked to construct a proof. In particular, only CME, Key Curriculum Press, and Glencoe contained more than 5% of the exercises asking students to construct a proof. As Otten and colleagues (2014) concluded, these findings indicate that students are likely developing their initial understanding of proof without actually engaging in the proving process.

In addition to analyzing the types of opportunities textbooks provided for students to engage in reasoning-and-proving, Otten and colleagues (2014) also analyzed the amount of statements and exercises that focused on the reasoning-and-proving process itself (*about* reasoning-and-proving). Statements *about* reasoning-and-proving were primarily located within the lesson exposition; these statements talked about ideas such as the deductive reasoning process, conditional statements, format of proofs, and the axiomatic structure. Within the student exercises, 1% to 20% of the questions encouraged students to think or respond to questions related specifically to the reasoning-and-proving process. Finally, Otten and colleagues (2014) found that statements or questions *about* reasoning-and-proving tended to largely disappear from the textbook exposition or student exercises beyond the introduction-to-proof chapter.

Analysis of the types of statements being made in the lesson exposition and student exercises revealed a contrast in the use of universal claims, or claims involving the universal quantifiers “all” or “any”, versus particular claims about a single or finite number of cases (Otten, Males, & Gilbertson, 2014). Specifically, all of the textbooks except UCSMP contained more particular statements than universal statements within the student exercises, but primarily used universal claims in the lesson exposition. The authors questioned whether the use of primarily particular statements, instead of universal statements, in the student exercises could fail to motivate an intellectual need for proofs and instead, position proof as an arbitrary process imposed by teachers and textbooks. Additionally, they wondered whether increased statements and exercises *about* the reasoning-and-proving process, both within and beyond the introduction-to-proof unit, could help to develop an intellectual necessity for proof (Harel, 2008). Based on the findings from this study, what would it look like for an introduction-to-proof unit that developed students' understanding of proof through engagement in the proving process? What are some ways that the instructional sequence could establish an intellectual necessity for proof? What kinds of tasks involving universal claims are appropriate and within conceptual reach of students in an introduction-to-proof unit? The purpose of this study was to develop, enact, and study using design research methodology one possible instructional sequence that aimed to address these questions.

Principles of Instruction

The teaching that occurred in this design experiment was based on the idea that students are not blank slates or passive receptors of information (Fenstermacher & Richardson, 2005), but instead construct their own knowledge through engaging in rich tasks that allow them to make sense of mathematical ideas and provide opportunities to participate in the learning process, both individually and in collaboration with their peers (National Council of Teachers of Mathematics, 2014). From this perspective, the role of the teacher is to support students' learning process through enacting instructional tasks that allow the teacher to assess students' current understanding and then pose questions and facilitate mathematical discourse with the goal of moving their thinking forward. The work of teaching described above is both unnatural and intricate (Ball & Forzani, 2009), requiring the teacher to make many instructional decisions before and during a lesson based on their goals for the lesson and their current understanding of students' thinking.

In drawing on Stylianides' (2007) definition of proof, I consider both the classroom community and the broader mathematical community when determining what consistitues true mathematical statements, valid forms of reasoning, and appropriate forms of expression. Thus, one of my roles as the teacher was to serve as a representative of the broader mathematical community and enculturate students into the mathematical community's norms with respect to proof (Yackel & Cobb, 1996). One of the ways that I introduced students to the proof norms and practices of the mathematical community was through the use of statements and questions *about* reasoning-and-proving. Statements and questions *about* reasoning-and-proving are designed to support students in reflecting on the process of proving itself and can focus on topics such as the form of mathematical proofs, methods of proving a mathematical statement, or general strategies for approaching a mathematical proof (Otten, Males, & Gilbertson, 2014). Alongside developing students' understanding of the proving practice, I also engaged students in the negotiation of the sociomathematical norms for our community, with particular attention placed on developing shared understanding of what was considered an acceptable mathematical justification (Yackel & Cobb, 1996). Over the course of the study, students shifted from viewing examples or statements such as "it's worked for all of the ones I tried, so it must be always true" as acceptable justifications to explicitly referencing mathematical definitions and properties to justify their claims. This negotiation of what consistitues acceptable justification was central to the introduction-to-proof study as students not only had to transition from a non-mathematical understanding of proof as evidence (broadly conceived) to a mathematical definition, but also develop understanding in the differences between the types of justifications acceptable when solving an algebraic task versus when constructing a mathematical proof.

DNR Based Instruction

In addition to drawing on broad ideas of the role of the teacher in the mathematics classroom, I also drew on the DNR conceptual framework for guidance on *how* mathematics should be taught from a pedagogical perspective. Specifically, I drew on the *DNR-based instruction in mathematics* (*DNR)* conceptual framework developed by Harel (2008) when envisioning the instructional sequence for this study. The acronym *DNR* refers to the three main principles of the framework: *duality, necessity,* and *repeated reasoning*. The *duality* principle maintains that students' ways of thinking and ways of understanding are intertwined. When applied to the context of proofs, an argument that a student produces for a given proof task is a representation of their way of understanding for that specific task. Observing a student's way of understanding, represented by their arguments for multiple proof tasks, provides insight into their ways of thinking in regards to proofs, or the ways in which the student gains personal conviction and persuades others of the validity of the given statement (also referred to as their proof schemes) (Harel, 2010). At the same time, a student's proof scheme acts as a lens through which they approach constructing responses for future proof tasks. This bidirectional relationship between students' ways of thinking and understanding is facilitated by the remaining two components of the framework; that is, the *repeated reasoning* and the *necessity* principles.

The *repeated reasoning* principle states that students must be provided with multiple opportunities to practice mathematical reasoning in the classroom to develop the desired ways of thinking and understanding (Harel, 2008). Within the context of proofs, engaging students in repeated reasoning does not mean simply having them construct multiple proofs, but instead means providing them with repeated opportunities that reinforce the desired ways of understanding and thinking (Harel, 2010). Examples of this include having students reflect on their understanding at the conclusion of a task, which Otten and colleagues (2014) refer to as statements or exercises *about* reasoning-and-proving, or through expanding the opportunities students have to engage in different aspects of the proving process (e.g., Cirillo & Herbst, 2011).

The *necessity* principle asserts that establishing the intellectual need for a particular topic is crucial in the learning process. “For students to learn the mathematics we intend to teach them, they must have a need for it, where ‘need’ here refers to intellectual need” (Harel, 2008, p. 900). Said another way, the necessity principle describes the importance of providing students with the intellectual motivation to engage in the reasoning that facilitates shifts in their ways of understanding and thinking. Harel separates students’ intellectual need into five related categories: the need for certainty, the need for causality, the need for computation, the need for communication, and the need for connection and structure (p. 905). When developing tasks to help establish an intellectual need for proofs, I primarily focused on incorporating tasks and reflection questions that targeted the need for certainty, causality, and communication. These three were chosen because secondary-level proofs typically do not require computation and developing understanding of mathematical connections and structure requires experiences with proof that fell beyond the scope of this study.

The need for certainty is the need to remove doubts about the validity of a claim (Harel, 2008). At first glance, this intellectual need aligns well with proofs since deductive arguments are the only way to conclusively determine that a universal claim is true. However, this assumes that students do not gain certainty until the statement has been deductively proven, which contradicts the conclusion, based on synthesis of prior research, that students are often convinced by the validity of a statement after trying a few examples (Reid & Knipping, 2010; Stylianides, Stylianides, & Weber, 2017). Furthermore, the way in which the proof task is posed to students impacts the extent to which the statement is assumed to be true. For example, Buchbinder and Zaslavsky (2011) recommended the use of the question, “Is this a coincidence?” as a way to motivate doubt about the validity of the claim and foster a need for proof. Zaslavksy (2005) also asserted that uncertainty can be evoked in tasks that involve completing claims, where the outcome can not be easily verified, or when the path is unknown or the conclusion is questionable.

The need for causality is the need to explain *why* the phenomenon is true (Harel, 2008). Researchers such as de Villiers (1990), Hanna (2000), and Hersh (1993) have called for the explanatory feature of proofs to be emphasized in secondary classrooms as a way of motivating a need for proofs, since examples alone rarely explain *why* the statement is true. Using proofs as a way to explain why the statement is true can also support students’ continued engagement in the task even after they are fairly convinced of the statements’ validity (Mudaly & DeVilliers, 2000). With that said, it is not always possible to prove a mathematical statement using a mode of argumentation that explains why the statement is true (Hanna, 2000). Teachers should also be aware that what may seem explanatory to them may not be viewed as explanatory by students (McCrone & Martin, 2009).

The need for communication is the need to be able to present an argument in such a way that it convinces the reader of the statement's validity (Harel, 2008). The idea that a proof should be written for an intended audience is central to Andreas Stylianides' (2007) definition of proof, as each component of the definition is written in terms of what is accepted, valid, or appropriate for the given classroom community. In spite of this, the communication aspect of proofs remains a relatively unexplored feature of proofs, particularly at the secondary level. How might a shift in the intended reader, say from the teacher to a classmate, change the way in which a student writes their argument? How could the communication aspect of proofs be used in the classroom as a way of refining students' proof construction skills?

Focus of the Present Study

The development of the instructional sequence for this study was based on three main design principles. First, students should develop their understanding of proof through engaging in the reasoning-and-proving process (Otten, Males, & Gilbertson, 2014). With that said, not all reasoning-and-proving tasks provide equal opportunities for students; in this study, I was specifically looking for non-trivial proof tasks where the validity of the claim and/or the solution path was not immediately apparent. Second, students need opportunities to reflect or talk *about* the reasoning-and-proving practice itself in addition to engaging *in* the process (Otten, Males, & Gilbertson, 2014). Statements *about* reasoning-and-proving include conversation about what can and cannot be assumed from a diagram, the process of proving that a mathematical claim is true or false, or the way that proofs are structured. Overall, the purpose of these statements is to make explicit what is often left implicit within proofs. Third, the instructional tasks should provide opportunities for students to establish an intellectual necessity for proofs (Harel, 2008). In other words, students at the conclusion of the study should not only have a beginning understanding of how to construct a proof, but should also be able to explain why proofs are a fundamental component of mathematics.

After establishing the underlying design principles, I conjectured that tasks involving universal claims, or claims containing the universal quantifiers “all” or “any”, were capable of meeting these principles. In other words, I conjectured that tasks involving universal claims would afford opportunities for students to 1) engage in the reasoning-and-proving process, 2) develop understanding of proofs through talking *about* reasoning-and-proving, and 3) develop an intellectual necessity for proof. This is not to say that all tasks involving universal conjectures support students in achieving these design principles or that the use of universal conjectures was the only important design element. I also drew on other elements, such as the use of uncertainty to motivate a need for proof (Balacheff, 1988; Buchbinder & Zaslavsky, 2011) and the use of pivotal counterexamples as a means of creating cognitive conflict to support development in students’ understanding (G. J. Stylianides & Stylianides, 2009). The purpose of this article, however, is to specifically focus on the role of using universal claims in providing opportunities for the aforementioned design principles.

Method

Design Research Methodology

Design research studies, similar to teaching experiments, are conducted with the goal of investigating students' processes of learning specific content while simultaneously studying the activities, instructional moves, and classroom norms that support their learning (Brown, 1992; Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003; Cobb & Gravemeijer, 2008). One important feature of design experiments is that they allow for the study of innovative teaching, or teaching that incorporates elements such as tasks, topic sequences, or pedagogical techniques not likely to be seen in a typical classroom (The Design-Based Research Collective, 2003). The analysis process used in this study is in the spirit of design research in that it focuses on a particular design aspect of my study and investigates the impact of that design decision in terms of the learning opportunities students experienced. With that said, my findings differ from the findings of traditional design research in that I will not be presenting a learning trajectory or articulating significant shifts in students' learning and the means that supported those shifts. There were two factors that contributed to my decision to deviate from traditional design analysis. First, this study reports findings from the first iteration of a design study; subsequently, I do not yet have sufficient evidence to make claims that the learning that occurred could only be attributed to specific features of the instructional sequence, especially given that the students who participated in this study were in the "advanced" track at their school and thus not representative of "typical" Algebra 1 students. Second, the topic of proof is multi-faceted and requires that students develop a range of skills and understanding in order to be able to construct a proof. As a result, the instructional sequence did not focus on a single mathematical topic but instead introduced multiple ideas that all led towards students being able to construct their own arguments for proof tasks involving classes of similar polygons (e.g., squares). Given that I have not conducted multiple iterations of the instructional sequence, it is not yet clear whether the way that I sequenced the activities is the way that must be sequenced, or if re-ordering the tasks would produce better results. Instead, findings of this study will be able to speak to the use of tasks involving universal claims in terms of the learning opportunities it provided and the potential challenges students faced while engaging in such tasks.

Participants and Data Collection

There were ten students who participated in this study – seven females (Amanda, Arin, Heather, Lauren, Lexi, Megan, and Sadie) and three males (Brian, Clay, and Wilson; all pseudonyms). They were the only students enrolled in the accelerated 9th grade mathematics course at a rural, public school in the Midwest United States (see Appendix B for more information about the school context). The accelerated 9th grade mathematics course was the first in a two-course sequence that covered Algebra 1, Geometry, and Algebra 2, with the first course focusing on Algebra 1 and Algebra 2 concepts. This meant that the students had not received any high school Geometry instruction in their regular math class prior to the study. Students' participation in this study was voluntary and all sessions were held during their study hall. Their math teacher and I invited students to participate in this study by saying that it was an opportunity for them to get a head start on content they would learn the following year; additionally, all students received a graphing calculator for their participation.

The design experiment contained 14 sessions, held twice a week for seven weeks. I was the teacher for all of the sessions. Students typically worked in three small groups during the sessions. There were two outside observers who attended 10 of the 14 sessions — the primary observer who attended eight sessions and the secondary observer who attended two sessions. Whenever the primary outside observer was not available to attend, I would meet with her after the session and go through the collected data in order to review students' current understanding and plan for the next session. Each session ranged in length from 28 - 38 minutes. All of the sessions were video and audio recorded to ensure that students' gestures, expressions, manipulation of physical objects, and voices during small group discussions could be heard, with one audio and video recorder focused on each of the three groups. In addition, I collected all written work during the sessions, including students' individual responses to reflection prompts. In summary, I analyzed audio/video recordings as my primary data source and then referenced students' written work and students' journal reflections during the sessions in order to provide a more complete picture of what occurred during the sessions.

Analytic Process

I began the analytic process by narrowing the data to focus on geometric tasks that took at least one session, or roughly 30 minutes, to complete. This yielded four broad tasks: the tessellation tasks, the constructing diagrams task, the proving similar polygon conjectures task, and the exterior angle theorem task (see Figure 3.1). I describe each of these tasks in the findings section. Students worked on the tessellation tasks in Sessions 1–4 for a total of 103 minutes; the constructing diagrams task in Sessions 7–8 for a total of 68 minutes; the proving similar polygon conjectures task in Sessions 11–12 for a total of 64 minutes; and the exterior angle theorem task in Sessions 13–14 for a total of 34 minutes. In this paper, I will present findings for the tessellation tasks, constructing diagrams task, and proving similar polygon conjectures task; the findings from the exterior angle theorem paralleled findings from the other three tasks, so I chose not to include this task for the sake of space.

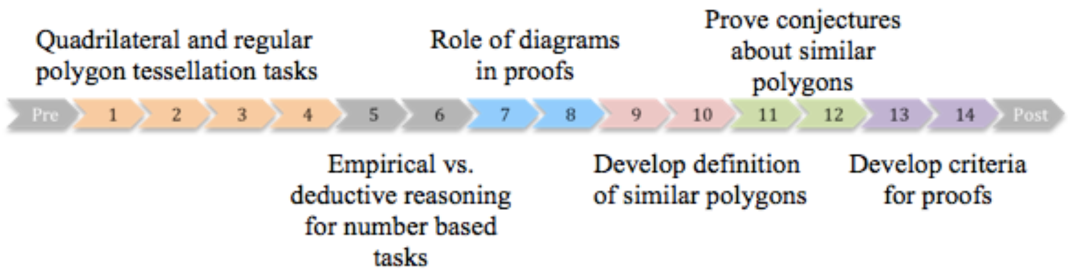

Figure 3.1. Overview of the instructional sequence.

Using the qualitative research software MAXQDA, I coded the session video data for all instances of the three design principles: instances where students engaged in reasoning-and-proving (Otten, Males, & Gilbertson, 2014), conversation *about* reasoning-and-proving (Otten, Males, & Gilbertson, 2014), and statements or questions that established an intellectual need for proof (Harel, 2008). My unit of analysis for reasoning-and-proving was the proof action, or the length of time that a student was engaging in a particular reasoning-and-proving action. For example, if a student was actively working on constructing a proof, then I coded the entire time as "constructing a proof" (see Table 3.1). Additionally, if a whole class conversation involved a student posing a conjecture, another student providing a counterexample to the first conjecture, and then a third student posing a refined conjecture, then each student's comments would be coded with a separate reasoning-and-proving code. The remaining two categories, conversation about reasoning-and-proving and establishing an intellectual necessity for proof, were both coded by theme. In other words, all consecutive statements or questions that pertained to the same theme (e.g., how to demonstrate that a statement in math is false or establishing a need for certainty) were assigned a single code.

Table 3.1

Reasoning-and-Proving activity codes

Related to Mathematical Claims	Related to Mathematical Arguments	Emergent Codes
Make a conjecture	Construct a proof	Revise an argument
Refine a statement or conjecture	Develop a rationale or non-proof argument	Construct a diagram
Draw a conclusion	Evaluate an argument	
Investigate a conjecture or statement	Find a counterexample	

To code for instances where students engaged in reasoning-and-proving, I adapted the expected student activity codes for mathematical claims and arguments from the analytic framework developed by Otten, Males, and Gilbertson (2014) so that they applied to the reasoning-or-proving activity that students *actually* engaged in during the sessions (see Table 3.1). As I coded the data, I looked for other instances of reasoning-and-proving that occurred in the sessions but were not captured by Otten and colleagues' (2014) codes. This resulted in two additional codes: revise an argument and construct a diagram for a proof conjecture/statement. When coding for reasoning-and-proving, I attended to both the reasoning-and-proving activity that was built into the initial task and the reasoning-and-proving opportunities that emerged organically through small group or whole class conversation.

Next, I coded all statements or questions that asked students to think or talk *about* the reasoning-and-proving process (Otten, Males, & Gilbertson, 2014). Specifically, statements and questions coded as *about* reasoning-and-proving focused students' attention on a particular aspect of the proving practice itself, including particular aspects of a proof (e.g., including mathematical justifications for statements), methods of proof (e.g., proof by contradiction), interpretation of diagrams (e.g., what can and cannot be assumed to be true), and general proof strategies (e.g., try tricky examples to gain sense of the validity of the claim). In order to be able to look at the statements across video clips, I coded all statements in this category *about* reasoning-and-proving and then wrote a brief description of the theme (e.g., how to demonstrate that a statement in math is false) as a comment attached to the code.

When coding for establishing an intellectual necessity for proof, I looked for a) instances where the task or my statements/questions targeted a specific necessity for proof and b) instances where the students demonstrated evidence of expressing an intellectual need for proof. These were coded at the theme level based on the necessity it addressed (certainty, causality, or communication)(Harel, 2008). For example, I coded the following question that I posed to students as a writing prompt as establishing a need for certainty and causality: "Why does talking about the properties of angles in quadrilaterals explain, or prove, why ALL quadrilaterals tessellate?" The question fosters a need for certainty by emphasizing the fact that the statement is true for all quadrilaterals and implied that the justification I provided students with during Session 4 (properties of angles) applied to all quadrilaterals, unlike some of the justifications they had provided previously that only applied to certain cases. Additionally, this question connects the idea that talking about the property of the angles not only established that it was always true, but also explained *why* it was true (need for causality). An example of a student expressing an intellectual need for certainty would be if they vocally questioned whether all quadrilaterals tessellate after finding out that their original hypothesis that all regular polygons tessellate was false. In this instance, the student would be expressing a need for certainty by questioning their original conclusion, based on trying examples, that all quadrilaterals tessellate.

After coding all of the relevant session video, I reviewed the coded sessions for patterns in the data and moments that particularly captured the intended design principle. I transcribed all of the clips that were selected to illustrate each design principle. During this process, I also referenced additional data sources, such as students' written work on the task and their responses to journal prompts, to triangulate the data and gain further insight into their current thinking. While analyzing students' written and verbal work on the four tasks, I noted any instance where a student appeared to be interpreting some aspect of the task in a non-normative way. Next, I constructed memos that described my perceived understanding of the student's interpretation of the task and then proposed possible explanations for how the student came to that interpretation (e.g., interpreting a word within the task prompt using a non-mathematical definition of the word). I then looked across my memos for themes in the possible explanations for the ways that students interpreted the tasks.

Findings

In presenting the results, I first describe the extent to which three tasks — the tessellation tasks, constructing diagrams task, and proving similar polygon conjectures tasks — accomplished the three design principles underlying my instructional sequence. Specifically, after providing an overview of each enacted task, I report the ways in which the universal claims involved in the task facilitated students' engagement in a variety of reasoning-and-proving activities and supported their understanding of proofs through talking *about* reasoning-and-proving. Overall, all three tasks provided students with multiple opportunities to engage in a variety of reasoning-and-proving activities, including some that arose spontaneously as a result of students' comments or questions in class. Additionally, all three tasks provided opportunities for me, as the teacher, to talk *about* reasoning-and-proving, especially in regards to the generality requirement. With regard to establishing an intellectual necessity for proof, the tessellation and proving similar polygon conjectures tasks provided opportunities for me to establish the need for certainty, causality, and communication through the structure of the tasks and my questions to the class; however, there were only minimial hints of students *expressing* each need. Finally, there were features of both the constructing diagrams and similar polygons tasks that students interpreted in a non-normative manner. I provide examples of students' interpretations for both tasks at the end of the findings section to illustrate some of the possible challenges students may face in understanding the scope of universal claims.

Tessellation Tasks

Task overview. During the first three sessions students explored the questions, "do all quadrilaterals tessellate?" and "do all regular polygons tessellate?" The purpose of the tessellation task was to establish an intellectual necessity for proof (Harel, 2008) by asking students to directly think whether and how it's possible to know that a universal claim is *always* true. I attempted to counteract students' tendency to make claims about the validity of a universal statement after trying a few examples in two ways: first, I chose to use a geometric task because I hypothesized that the cases (e.g., different types of shapes) would feel more distinct from one another than the cases would in a number-based task. Second, I tried to problematize students' tendency to over-generalize by asking them to explore the false question about tessellating regular polygons alongside the question about tessellating quadrilaterals. After realizing their assumption about the regular polygon question was incorrect, I wondered if some students might also begin to question their conclusion for the quadrilateral question. By having students consider the answer to these two questions, I aimed to establish proofs as a way of establishing *why* a statement in math is true (causality) as well as a way of knowing *that* it is true without checking all possible cases (certainty).

Data from the initial interviews, conducted prior to the beginning of the sessions, revealed that students varied greatly in their belief about whether it was possible to know if a statement in math was *always* true. Specifically, three students (Amanda, Megan, and Wilson) thought is was possible to know a statement in math was always true, three students (Clay, Arin, and Brian) thought it was *not* possible, and the remaining four students (Lauren, Lexi, Sadie, and Heather) were unsure. For example, Amanda justified her response that it was possible to know a mathematical statement was always true by stating, "if it works out and you can check your answer and it works out then it can be proven true." Brian, on the other hand, stated that it was not possible because "there's an infinite number of numbers, so it's kind of difficult to say that this will always be true with that many numbers in existence." Initial interview data also established that all students were familiar with the term "prove" in a non-mathematical context; however, none of them had been formally introduced to what it meant to prove a statement in mathematics. Instead, most students thought that proving in a mathematics class involved showing or checking your answer to "prove" it was correct.

In Session 1, I launched the quadrilateral tessellation task by eliciting students' prior knowledge of quadrilaterals and establishing what it means to create a tessellation using a single shape through giving each small group an example and non-example of a tessellation. Next, I posed the question, “do all quadrilaterals tessellate?” I gave each group seven sets of paper quadrilaterals, including both regular and irregular convex quadrilaterals, to aid in their exploration of the question. Towards the end of the first session, I asked students to record in their notebooks how confident they were that all quadrilaterals tessellated and explain their answer. After students finished recording their responses, I held up a paper concave quadrilateral and asked students whether they thought it would tessellate. All students thought that the concave quadrilateral would tessellate based on the idea that all of the shapes they had tried thus far had tessellated.

During Session 2, I asked students to work in their groups to create step-by-step “how to” directions describing how to tessellate *any* quadrilateral. The purpose of this task was to help students become more systematic when tessellating different quadrilaterals instead of relying on the guess-and-check approach most groups had used in the first session. In order to write the directions, students also had to consider the universality of the claim in a way that they had largely avoided in the first session due to the requirement that their directions should work for any quadrilateral. By the end of the second session, the three group’s directions all involved a variation of “flipping and mirroring” a quadrilateral so that the same side lengths, but different angles were matched. I concluded the session by asking each small group to share out their directions with the whole class and then compare and contrast the different groups’ directions.

Session 3 began by having the three students who were absent from Session 2 test the directions the remaining students had written in the previous session. During this process, I encouraged groups to revise their steps as needed to clear up any confusion or difficulties the absent student had when trying to follow the directions. Next, I asked students to pose conjectures of other polygons that might tessellate. After students posed a few conjectures, I introduced them to the term, regular polygon, and asked whether they thought all regular polygons tessellate. I selected regular polygons to serve as our pivotal counterexample (Stylianides & Stylianides, 2009) because it aligned with students' rule from Session 2 to "match up the sides", which meant that students would likely assume that all regular polygons tessellate. As ancitipated, every student initially agreed that the statement, "all regular polygons tessellate" was true. After collectively deciding that equilateral triangles and squares would always tessellate without checking, I handed each group a set of regular hexagons to try. Finally, I handed each group a different set of regular polygons (regular pentagons, septagons, and octagons) in order to "speed up the process". After each group concluded that their shape didn't tessellate, I asked students to respond to following reflection questions: "Do you still think that <u>all</u> quadrilaterals tessellate? If no, explain why. If yes, is there something special about quadrilaterals that make it so that they will always tessellate?" After students recorded their individual thinking on paper, they talked about their ideas in small groups and then shared their thinking with the entire class.

In Session 4, I wrapped up our exploration of the tessellation tasks by first summarizing students' key ideas from each of the earlier sessions and reminding them how their thinking and approach to the task had evolved over the three sessions. Next, I showed students an image of a irregular convex quadrilateral with the angles labeled A, B, C, D and asked students what they noticed in the image. After we talked about how the four angles of the quadrilateral always met to form 360º, I showed students a picture of regular hexagons, regular pentagons, and regular octagons with the specific angle measurements labeled and asked students to talk about why this additional information might help to explain why regular hexagons, but not regular pentagons or octagons, tessellated. I concluded the tessellation tasks by asking students to respond to the following reflection prompt:

> Suppose another student came into the classroom on Wednesday at the end of our discussion about quadrilaterals tessellating. She heard our discussion, but still thought there could be a quadrilateral that didn't tessellate. Why does talking about the properties of angles in quadrilaterals explain, or prove, why ALL quadrilaterals tessellate?

This prompt aimed to help students reflect on what they had learned over the past four sessions and specifically, on how it was possible to be certain that all quadrilaterals will tessellate.

The tessellation tasks facilitated meaningful and varied reasoning-and-proving activity. During the tessellation tasks, students investigated the validity of claims, developed non-proof rationales, produced conjectures/claims, posed counterexamples in response to a student's claim, and refined another student's conjecture. While the original tasks given to students only asked them to consider the validity of a claim, the remaining reasoning-and-proving opportunities surfaced in response to questions I posed to groups and organically through students' discussion with their peers.

In Session 1, students investigated the question, do all quadrilaterals tessellate?, using a guess and check approach and then generalized their findings to conclude that the statement was true. For example, Megan and Arin asserted that all quadrilaterals will tessellate and then provided the following responses when I asked them to justify their statement.

Megan: I think that since, like if you can get a couple to fit, they should all fit, just like. Cause it's like little patterns.

Arin: Yeah, like if you do tests, like so many different tests.

Megan: Yeah, if you just keep moving them around they end up fitting somewhere.

In this exchange, both Megan and Arin produced non-proof rationales for why they thought all quadrilaterals tessellated. Notice that the students' justifications relied entirely on the fact that all of the cases they had tried so far had worked and made no reference to specific features of quadrilaterals that might help to explain why *all* quadrilaterals tessellate.

Students' engagement in reasoning-and-proving activity increased significantly towards the end of Session 3 as they were trying to make sense of why quadrilaterals and regular hexagons tessellated but regular pentagons, septagons, and octagons did not. After recording their hypotheses in their notebooks and discussing their ideas in small groups, I asked students to share their current thoughts why only some shapes tessellated. In the following exchange, notice that each student's statement directly responds to the previous student's claim or counterexample.

Amanda: Um, I said that maybe after a shape gets like, like after they have four sides, like 5 and on, then maybe the angles become too wide, because they have too many sides

Kimberly: Okay. What do ya'll think about that?

Wilson: Well, hexagons work, but…

Kimberly: So hexagons work… (Lexi) can you speak up a little bit?

Lexi: Okay, well I said maybe. *(Arin quietly interrupts her)*

Kimberly: Go ahead and say what you were thinking.

Lexi: Okay, well I said maybe like after 4 sides the sides have to be even with the amount, cause 5 didn't work.

Kimberly: 5 didn't work, yeah; so that would be a reason for that…but what, hold on, will you talk a little bit louder?

Arin: The octagon didn't work.

In the discussion above, Amanda began by posing the claim that the angles become too wide to tessellate after 5 sides, which Wilson responded to by posing a counterexample that hexagons tessellate. Next, Lexi posed a revision to Amanda's claim by suggesting that the number of sides had to be even for the shape to tessellate, to which Arin countered with another counterexample by stating that (regular) octagons did not tessellate. The students' exchange of posing a claim – responding with a counterexample – refining a claim – responding with another counterexample is particularly notable given that they had minimal reasoning-and-proving opportunities prior to the start of this study, had not yet been formally introduced to counterexamples or the idea of revising a claim, and had engaged with each others' ideas with only minimal involvement from the teacher.

The tessellation tasks provided opportunities for the teacher to incorporate statements about reasoning-and-proving. The tessellation tasks, which were both phrased as questions involving universal quantifiers, allowed me to introduce students to the generality requirement, or the requirement that claims in mathematics must apply to all possible cases. I introduced this idea to students during Session 4 as I summarized some of students' key ideas that had surfaced in the first three sessions. In particular, I revisisted a claim made by Clay in Session 3 that two quadrilaterals put together would "make a square". During session 3, Heather had responded to his claim saying, "I know but that's not just for… I mean that's for just this shape [cross talk] it's not for all quadrilaterals." In my summary during Session 4, I used Heather's response to emphasize the importance that our statements and justifications applied to *all* possible cases.

> So one of the things that I really liked about [Heather]'s comment is that it's focusing on this idea that we have to think about ALL quadrilaterals, we can't say "well this one made a rectangle, or this one made a square, so therefore that's always going to happen" or we can talk about it like that. So we've got to attend to this fact that we can't say anything that isn't true about all of them.

The purpose of these comments was to encourage students to begin reflecting on the types of statements that can be made in mathematical arguments involving a universal claim. Specifically, I contrasted students' initial approaches of generalizing from a few examples with Heather's comment that the statements in the directions must apply to all cases. In order to continue emphasizing this idea, I asked students to respond to a journal prompt containing the question, "Why does talking about the properties of angles in quadrilaterals explain, or prove, why <u>ALL</u> quadrilaterals tessellate?" This reflection prompt intended to help students reflect on two ideas: first, that examples-based explanations are not sufficient because they do not demonstrate the statement is true for all possible cases; and second, that attending to the angles of the quadrilaterals not only demonstrated *that* all quadrilaterals tessellate, but *why* they will always tessellate.

The second idea *about* reasoning-and-proving that surfaced as a result of using tasks involving universal claims was the process for disproving a mathematical claim. While it is possible to talk about how to disprove a mathematical claim with a particular statement, the significance that only *one* counterexample is needed to disprove a claim is minimized when the statement is about a single or finite number of cases. After talking about the definition of a counterexample, I asked students how many counterexamples they thought were needed to demonstrate that a universal claim was false. Amanda replied that only one counterexample was needed, "because that proves that it doesn't work all the time." I revoiced Amanda's justification, placing emphasis on the universal quantifier in our question. "All the time, yeah. So if it doesn't work once, because we're talking about a statement where it has to work ALL the time, we only need one counterexample, that's enough." The concept that only one counterexample is needed to disprove a mathematical claim is a central idea about proof and one that students do not consistently understand. Furthermore, the reason why only one counterexample is needed to disprove a mathematical claim is due to the requirement that statements in math be true for all possible cases. Thus, introducing the process for disproving mathematical claims within the context of universal statements emphasizes this relationship and motivates why only one counterexample is needed.

Features of the tessellation tasks aimed to establish an intellectual necessity for proof but did not result in students outwardly expressing this need. The use of a question involving a universal quanitifier allowed me to pose questions to students that aimed to challenge their certainty that *all* quadrilaterals tessellate. For example, one small group declared "all quadrilaterals tessellate" after trying different examples for less than seven minutes. After hearing their declaration, I asked the small group if they thought other quadrilaterals they hadn't yet tried would tessellate and pressed them to explain *how* they knew they would tessellate without trying every case. Although this group was not yet able to provide a justification beyond generalizing from the examples they had tried, the questions I raised aimed to challenge their conviction and encourage them to continue to seek out another justification for why all quadrilaterals tessellate.

Despite repeatedly questioning students how they knew that *all* quadrilaterals tessellate, all of the students reported that they were either pretty confident (2) or very confident (8) in their answer at the end of the first session. Once students recorded their responses in their notebooks, I held up a convave quadrilateral and asked the class whether they thought it would tessellate as well. All students stated it would tessellate; Wilson offered the justification, "cause it's a quadrilateral". I responded to his response by asking for other students to provide a justification saying, "you've gotta have a better reason than just 'because it's a quadrilateral', cause we don't know for sure that all quadrilaterals tessellate." Although my response may not have impacted students' conviction in the validity of the statement, it introduced the sociomathematical norm that all justifications were not equally valid. After another student attempted to justify why the convex quadrilateral would tessellate, I continued to press on students' conviction through emphasizing the universality of the claim.

Kimberly: Do you think you could draw a quadrilateral that would *not* tessellate?

Students: No

Kimberly: So y'all don't think there's a SINGLE quadrilateral out there that will not tessellate… even though we haven't checked them all… you still think it's going to ALWAYS work?

Students: *Nod heads*

Kimberly: How do you know?

Wilson: Cause these worked

Kimberly: Cause those worked…ah okay… But what if those are just special?

Wilson: … I don't know

In this exchange, I posed a series of questions that specifically focused students' attention on the universal quantifier to challenge the strength of their conviction. I also questioned

Wilson's and other students' assumption that because it worked for the examples they tried, it must work for all cases through bringing up the possibility that I had given them special examples to try. This possibility was reasonable since I had intentionally given students only one irregular convex quadrilateral to try in Session 1 and had not yet allowed them to physically tessellate a concave quadrilateral. While it is possible that my questions provoked some doubt for students in regards to their certainty of the answer, none of the students chose to voice this doubt during the first session.

Although students did not directly express doubt regarding whether all quadrilaterals and all regular polygons tessellate, a few students did express surprise during Session 3 upon discovering that their initial assumption that all regular polygons will tessellate was false. For example, after not being able to tessellate a regular pentagon, Megan questioned whether the sides all had the same length and then expressed surprise and confusion through her facial expressions upon hearing that the sides were all the same length. Heather reacted to her group's shape not tessellating by saying, "I mean it tessellates in some ways", even though she recognized that the shape did not tessellate according to our definition of using copies of a single shape to form the tessellation. Instead of questioning their initial assumption that all regular polygons tessellate, Megan and Heather reacted to the evidence by questioning some portion of the task itself – namely, whether the shapes were regular in the case of Megan and our definition of tessellation in the case of Heather. After concluding as a group that all regular polygons do not tessellate, Clay began to raise a question by saying, "wait, does that mean…nevermind". After posting the reflection prompt "Do you still think that all quadrilaterals tessellate?" Clay remarked, "that was my question!" Clay's statements provide some evidence to suggest that he may have begun to question our earlier conclusion that all quadrilaterals tessellate; however, he ultimately chose not to raise this question with the group. Additionally, he responded to the writing prompt by stating that he still thought all quadrilaterals tessellate. While it is possible that some students had developed an intellectual necessity for certainty/causality as a result of exploring the false conjecture about regular polygons, none of the students vocally expressed this need during our conversations.

In Session 3, I used the false conjecture that all regular polygons tessellated to transition our focus to understanding *why* the statement about quadrilaterals was true, instead of simply considering *whether* it was true. Although I did not directly ask students to confront *why* the statement was true until Session 3, retrospective analysis of the data suggested that students were generally confident and comfortable with their conclusion *that* the statement was true without any additional justification prior to engaging in the regular polygon tessellation question. Seeking to understand *why* the statement was true in Session 3 not only provided increased reasoning-and-proving opportunities, which I describe in the next section, but also allowed me to emphasize the causality purpose of proofs. In particular, I used the reflection prompt, "Do you still think that all quadrilaterals tessellate? If yes, is there something special about quadrilaterals that make it so that they will always tessellate?", as a way of motivating a need for understanding *why* the statement is always true. None of the students expressed uncertainty in whether all quadrilaterals tessellate; when answering the follow up question, students either proposed ideas of why it might be true or stated that they were not sure what made quadrilaterals special. For example, Brian wrote,

> Yes, I think they do all tessellate. I think this because even the really weird shapes we have tessellate. I wonder if having 4 sides and 4 angles have anything to do with it. Maybe it has something to do with the angle measurements too.

Brian's second statement indicates a continued belief in the validity of the statement based on the generalization of the examples he had previously tried as well as an approach of assuming that the statement is true until provided evidence to suggest otherwise. Although he expressed absolute certainty in the validity of the statement, his statements about why all quadrilaterals tessellate both contained hedges ("wonder" "maybe"), suggesting some uncertainty regarding his responses for why the statement was true. Seven of the ten students used at least one hedge in their journal prompt when explaining what made quadrilaterals special, including three students who directly stated they did not know what made the quadrilaterals special so that they would always tessellate. Students' use of hedges and directly stating that they did not know why quadrilaterals always tessellated could be interpreted as evidence that shifting the conversation to understanding why the statement was true had evoked a need for causality; alternatively, they could be attributed to the phrasing of the writing prompt. In other words, while students were not sure why all quadrilaterals tessellated, they may not have had an intellectual need for this information. By asking students to reflect on this question immediately after completing the regular polygon question, they were not given the opportunitity to raise the question of why only some polygons tessellate on their own.

Summary. A significant portion of the reasoning-and-proving activity that occurred during the tessellation tasks surfaced as a direct result of asking students to consider the validity of two similarly structured universal claims – do all quadrilaterals tessellate? and do all regular polygons tessellate? – that differed in their truth-value. The contrasting answers of the two claims provided an opportunity for students to develop a broader theory, or explanation, of the characteristics shapes must possess in order to tessellate. This opportunity to develop a broader theory only occurred because the two claims were about classes of shapes instead of a single shape and both claims had the same conclusion (the given shape will tessellate).

The tessellation tasks provided opportunities to make statements *about* reasoning-and-proving as I summarized students' progression of thinking during the first three sessions. In particular, I drew students' attention to the process of posing a conjecture and responding with a counterexample that multiple students had done in earlier sessions as one way of engaging in the reasoning-and-proving process. I also introduced the generality requirement by asking students how many counterexamples were needed to disprove a universal claim. These opportunities to talk about reasoning-and-proving only surfaced because of the use of universal claims. While the universal claims afforded the opportunities, it was up to me as the teacher to bring them to students' attention during our summary conversation in Session 4.

Finally, there was a mismatch between aspects of the tasks that aimed to establish an intellectual necessity for proof and students' expressed need. Specifically, whereas both tasks contained features that aimed to provoke a need for certainty and causality, the instructional sequence did not provide sufficient opportunities for students to express either intellectual need. As a result, it is not clear from the session data whether features of the tessellation tasks succeeded in establishing an intellectual necessity for proof.

Constructing Diagrams for Quadrilateral Theorems

Task overview. The purpose of the constructing diagrams task (Sessions 7–8) was to introduce diagrams as a specific case that represents all possible cases and to discuss ways of notating diagrams so that the notations attended to the generality requirement. Immediately prior to this session, I introduced students to the idea of generic examples within the context of number based proof tasks by showing students a set of visual images (generic example) representing each step in a number trick[5] and asking them to think about how the structure of the images could be used to explain why the number trick would *always* work. After our discussion on generic examples, I introduced students to the concept of using variables, representing all possible numbers, to demonstrate that a number relationship will always work. I conjectured that the constructing diagrams task would help students continue to think about generic examples, now within a geometric context, and would scaffold their understanding of diagrams in preparation for the similar polygons proof tasks in Sessions 11–12.

I began Session 7 by introducing conditional statements and then demonstrating how to use the components of a conditional statement in a proof by showing a proof for a conditional statement, "if a quadrilateral has 360º, then it will tessellate". This statement was initially posed by Sadie in Session 4; I intentionally chose to use her phrasing even though it was not mathematically precise. As I talked through the proof, I emphasized the way that the argument began by stating the hypothesis and ended by stating the conclusion. I also emphasized the fact that I had not used any specific characteristics of the provided diagram in my proof even though I had intentionally drawn the diagram in

[5] The number trick and generic example can be found at https://nrich.maths.org/2280.

such a way that one of the angles appeared to be 90º (see Appendix A for more information).

After concluding our discussion of the proof, I told students they would be working in their small groups to construct diagrams for two sets of three statements about quadrilaterals. “We are going to draw a diagram that could be used as a generic example, so one specific shape, but we could use it to represent all possible shapes for the statement I am going to give you.” All of the statements were universal claims, written as conditional statements, about properties of different quadrilaterals (see Figure 3.2). I rephrased some of the statements to accommodate for the fact that students had not yet taken high school Geometry and may not be familiar with terminology such as “congruent”, “consecutive angles” and “supplementary”.

Directions: As a group, draw a diagram that could be used as a generic example **if** we were proving the statement (we won’t be actually proving the statements!). Remember, the diagram represents <u>all</u> possible shapes that have the features included in the hypothesis portion of the statement.

1. If the polygon is a rectangle, then the diagonals have the same length.
2. If a quadrilateral is a parallelogram, then the measures of the angles on the same side of the shape add to 180 degrees.
3. If a quadrilateral is an isosceles trapezoid, then the diagonals have the same length.
4. If two sides of a parallelogram that intersect have the same length, then the parallelogram is a rhombus.
5. If the diagonals of a parallelogram form a 90-degree angle, then the parallelogram is a rhombus.
6. If one angle of a parallelogram is a right angle, then the parallelogram is a rectangle.

Figure 3.2. Directions and statements used in the constructing diagrams for quadrilateral theorems task.

Students worked in their small groups to draw diagrams for statements 1–3 during Session 7 and statements 4–6 during Session 8. At the end of each session, I posted each groups' diagrams on the board and asked students to compare and contrast features of the diagrams students had drawn, including the different ways that groups had notated their diagram. At the end of Session 8, students respond to the following two journal prompts: "1) Why is it okay that our diagrams weren't exactly the same? 2) Suppose a student was proving a statement about rectangles. Would it be okay for them to draw a square as their diagram? Why or why not?" The purpose of these reflection questions was to assess students' understanding that there can be multiple correct diagrams for a single statement and that does not necessarily have to look like the given shape so long as the diagram contains all of the properties of the given shape.

The constructing diagram task facilitated meaningful reasoning-and-proving activity. The main reasoning-and-proving activity that students engaged in during this task was constructing diagrams for provided statements, which required them to make sense of the claim in order to decide what quadrilateral to draw for their diagram. Although the constructing diagrams code was not explicitly included in Otten, Males, and Gilbertson's (2014) framework[6], Cirillo and Herbst (2011) suggested this task as way to support students' understanding of mathematical diagrams and scaffold their ability to complete this aspect of the proving process on their own.

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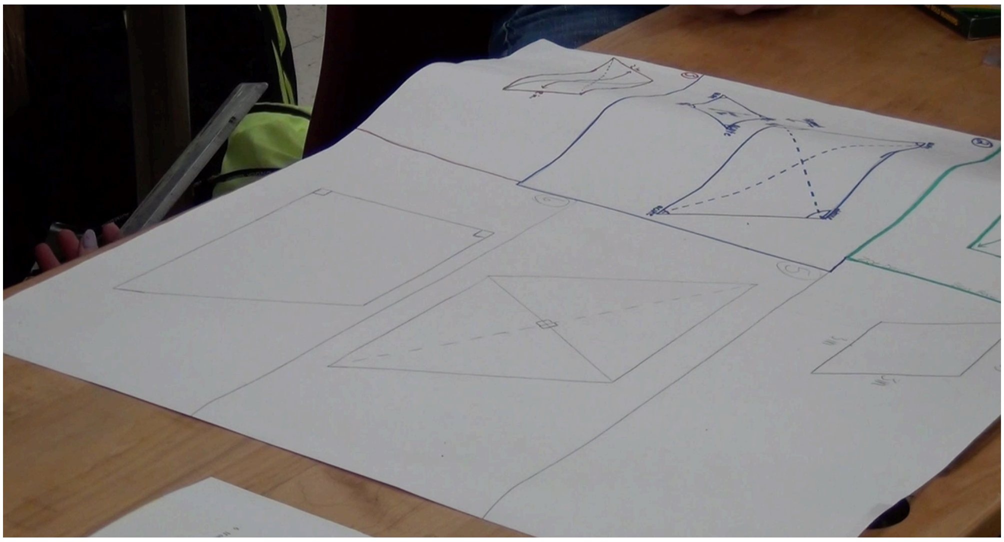
[6] I could have coded drawing a diagram using the "modify or revise a mathematical statement" code since students were adding a diagram to accompany the provided statements. However, I chose to add a new code to emphasize the fact that textbooks and teacher rarely, if ever, hold students responsible for producing a diagram for a proof task (Cirillo, 2017; Herbst & Brach, 2006).

The last three statements in the task (see Figure 3.2) provided increased opportunities for students to make sense of the claim since all three statements referenced a different quadrilateral in the hypothesis and the conclusion. To illustrate the reasoning-and-proving that occurred during this task, I am going to focus on a discussion that occurred as two groups were making sense of how to draw a diagram for the last statement: "if one angle of a parallelogram is a right angle, then the parallelogram is a rectangle." Arin, Sadie, and Brian approached this statement by focusing on drawing a shape that depicted the information in the hypothesis. This resulted in a right trapezoid, depicted at the bottom right side of their poster (see conversation below). During my conversation with the small group, the students were able to utilize their prior knowledge of parallelograms to reason why they could draw a rectangle to represent the claim even though the hypothesis mentioned a parallelogram.

Kimberly: How did you figure out what shape to draw?

Arin: We just started, like, made one right angle, she drew these two first to make one right angle and then just… made a slant because… parallelograms have slant.



Kimberly: Okay. So, kind of, start with a right angle and then try to finish it off with a parallelogram?

Sadie: Yeah

Brian: See if it was just parallelogram, couldn't you tactically just draw a rectangle? Cause they're...

Arin: Are rectangles parallelograms?

Kimberly: Well, what's the definition of a parallelogram, do you know?

Sadie: Two parallel sides

Kimberly: Two parallel sides

Arin: Oh, so it *could* just be a rectangle

Brian: Maybe we should just change the trapezoid into a rectangle

In order to construct a diagram for this statement, students had to reason about the relationship between parallelograms and rectangles and how they wanted to capture the two shapes within their diagram. Although students could have drawn a non-rectangle parallelogram and labeled one of the angles 90º, all three groups drew a rectangle for the final statement; additionally, two of the three groups labeled all four right angles instead of only one.

As students worked to make sense of the claims in order to determine how to construct a diagram for each statement, some students also began to verbally draw conclusions from the hypotheses by drawing on their knowledge of quadrilaterals. For example, as Megan and Wilson debated how to draw the diagram for statement six, Wilson provided a non-proof rationale in his final statement below for his conclusion that all of the angles must be 90 degrees.

Wilson: They have to be parallel lines, right

Megan: Yeah, okay

Wilson: And so since one's 90 degrees

Megan: So then the rest of them have to be 90 degrees?

Wilson: Right, this one has to be 90 degrees since...they're all 90 degrees yeah. …. Because these have to add to 180 and if one of them is 90 degrees, the other has to be 90 degrees.

In this conversation, Wilson combined his prior knowledge about the angles in a parallelogram with the given information that one of the angles was a right angle to conclude that the other angle on the same side of the parallelogram would also be 90 degrees. Although he did not justify his claim that the same side interior angles add to 180 degrees, the informal reasoning he used in this discussion represents a solid first step in constructing a proof for the given theorem.

The constructing diagrams task provided opportunities for the teacher to incorporate statements about reasoning-and-proving. The constructing diagrams task for universal claims provided multiple opportunities to talk about reasoning-and-proving, particularly with respect to the idea of representing all possible shapes within a single diagram. After establishing that you can represent congruent side lengths on a diagram by labeling them with the same variable, I showed the class the following diagram (Figure 3.3) and asked whether it could be used as a diagram for a statement about isosceles trapezoids. Arin responded, "I was like, it doesn't look like they're the same, but since you have C and C there… it probably means that they're the same like on theirs (another group's diagram). I don't know." This question targeted a possible misconception students might have when working with diagrams and supported their ability to think flexibly about diagrams instead of making claims that rely on information gleaned from visual inspection.

Figure 3.3. Isosceles trapezoid I drew for whole class conversation.

The diagrams for statements 4–6 (see Figure 3.2) in particular provided opportunities for students to reason about what is known from the hypothesis versus what is known from the conclusion. This distinction was challenging for some students to comprehend, especially when trying to decide what should be labeled on the diagram. For example, Megan and Wilson differed in how they wanted to label the sides of their diagram for the statement, "If two sides of a parallelogram that intersect have the same length, then the parallelogram is a rhombus". As you will see in the conversation below, Megan focused on only using what was given in the hypothesis in their labels whereas Wilson wanted to consider the information in the conclusion as well.

Megan: It's two sides, so then label the bottom two different, like B and B or B and C or…

Wilson: Alright, but in a rhombus they're all the same, they're all the same length, all sides are the same length, right?

Kimberly: That's true, but we're trying to prove that the shape ends up being a rhombus if you start off with two sides that are touching being the same length.

Megan: So just pick two and then say they're A and A and then the other two can be… B and C or B and B or

Wilson: K

Kimberly: Lauren, what do you think?

Lauren: I don't know. I think that since they're all the same, it confuses me that the other two are B, so I don't know.

Kimberly: Okay. Megan, want to say something?

Megan: I just said that just cause that's what that says, so. Since you have to prove it, you just have to change it

Kimberly: Yeah, so we're not saying that definitely they're NOT the same, we're saying we don't know *for sure* that they're the same

This conversation revealed the different understandings that Wilson, Megan, and Lauren had in terms of what should be included in their diagram. Specifically, whereas Megan wanted to focus only on the information provided in the hypothesis, Wilson and Laruen appeared to be considering the conclusion and his prior knowledge about rhombuses when labeling the diagram. Conversations about what should and should not be included in the diagram occurred in all three groups, especially when constructing the diagrams in Session 8 (see Figure 3.2). These conversations suggest that understanding the difference between what can be assumed at the beginning of the proof versus what will be demonstrated to be true at the end of the proof can be challenging for beginning learners, especially when the statements are more complex. At the same time, understanding this distinction is critical to avoiding arguments containing circular reasoning. Although only some of the students demonstrated evidence of understanding the distinction between what can be assumed at the beginning and end of a proof, the conversations that occurred as they constructed diagrams helped to reveal their current understanding and provide opportunities to talk about this important distinction.

Features of the constructing diagrams task did not establish an intellectual necessity for proof. The constructing diagrams task did not motivate an intellectual need for certainty or causality since students were not considering the validity of the claims or exploring why the statements were always true. At the end of sessions 7 and 8, students were asked to compare the different notation that groups included in their diagrams; however, this conversation was not directly focused on motivating a need for communication through the use of different types of notation.

Summary. The constructing diagrams task was included in the instructional sequence as a way to develop students' ability to construct their own diagrams when proving conjectures about similar polygons in Sessions 11–12. Although students did not have the mathematical background to prove the quadrilateral statements, the process of constructing a diagram for each statement provided important opportunities for students to engage in reasoning-and-proving and talk *about* the reasoning-and-proving process. These opportunities primarily occurred as students were discussing the universal claims that each involved two different quadrilaterals. In order to construct a diagram for these statements, students not only had to make sense of the claim itself, but also had to reason about what information to include in their diagram notations. Using universal claims to talk about different ways of notating a diagram provided a natural motivation for why students should use notations, such as variables, that apply to all possible cases.

Proving Similar Polygon Conjectures

Task overview. The primary goal for the similar polygons task was to begin developing students' understanding of how to write proofs. Prior to this lesson, students had been introduced to different components of proofs, such as the need for proofs to establish the statement was true for all possible cases (throughout), the use of mathematical properties and definitions to explain why a statement is always true (Session 4, 7), the use of variables to represent all possible numbers (Session 6), and the different ways of labeling diagrams (Session 7–8). In Sessions 9–10, students developed a definition for similar polygons, providing the group with a shared understanding of the key vocabulary needed for this proof task. Instead of directly telling students how to write a proof, I wanted to give them the opportunity to construct arguments for a proof conjecture in their small groups first so that I could build on their knowledge when adding in specific details about the content and format of proofs.

I introduced the similar polygons task by asking students to pose conjectures of specific polygons that they thought might be similar (i.e., "all ____ are similar"). I then posed the example conjecture that all polygons were similar in order to illustrate what it would mean for the statement to be true.

> For instance, if we were to say that all polygons are similar, what that would mean is that if I had a giant bag of all possible polygons, and I were to reach in and pick out two of them, no matter which two I picked out, they would be similar to each other. And I could keep doing this, for as long as I wanted to, and every time I picked out two they would be similar.

In response to my question regarding the validity of the conjecture, students quickly stated that it was false. I then asked students to pose a counterexample to support their evaluation of the conjecture. Brian suggested a square and a triangle explaining, “one of them has three sides and another one has four sides”. After revoicing his response, I added the additional justification that “the sides can’t be proportional if one [shape] has three sides and another has four” in order to connect back to the definition of similar polygons that students had constructed in Session 10.

Students posed a total of four conjectures for the class to investigate: all squares are similar, all equilateral triangles are similar, all right triangles are similar, and all rhombuses are similar. The class then decided that the first conjecture they wanted to prove in their small groups was the conjecture involving squares. Once each small group finished constructing their argument that the “all squares are similar” conjecture was true, I asked groups to exchange papers and provide feedback on each others’ arguments. In order to focus their feedback, I suggested that they evaluate the group’s work in light of the following questions: “Is it convincing? Does it convince you that no matter what two squares I draw, they’re going to be similar? And is there anything someone could say to poke a hole in the argument?” After explaining the new component of the task, I reminded students that they were all just beginning to learn how to write proofs and reiterated the norm that they were to support each other through their feedback. Once groups had provided written feedback to each other, I gave groups back their original paper and asked them to revise their argument in light of the provided feedback. During the remaining time in Sessions 11 and 12, students continued to investigate the classes' conjectures for right triangles and equilateral triangles and then prove that each statement was either true or false. At the beginning of Session 12, I led the entire class through the proof of the conjecture, "all squares are similar". During this time, I incorporated multiple discussions about different components of the proof and elicited students' ideas throughout the process.

The proving similar polygon conjectures task facilitated meaningful and varied reasoning-and-proving activity. The original structure of the proving similar polygon conjectures task provided all students with the opportunity to pose and investigate multiple conjectures, construct two proofs (arguments), find a counterexample, evaluate an argument[7], and refine their argument[7] in response to their peer's feedback. Throughout this process, students also provided non-proof rationales in their conversations when suggesting a statement for their argument or proposing a revision for another group's argument. In addition to the reasoning-and-proving opportunities that were built into the original task, some small groups engaged in additional reasoning-and-proving as they were discussing possible conjectures to pose. For example, Clay suggested the conjecture, "all octagons are similar" to the other members of his small group. Lexi replied saying that his conjecture was false and then

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[7] I refer to this as evaluating and revising an argument instead evaluating and revising a proof since these two codes refer to students' work on a proof task, which I call arguments so that I am not making an evaluation of whether their work should be considered a proof.

drew the counterexample shown below (Figure 3.4). After confirming that both of the shapes Lexi drew were octagons, Clay revised his conjecture to “all regular octagons are similar”. However, this final step occurred after the class had compiled their conjectures; as a result, the group did not end up investigating Clay’s revised claim. This brief exchange of posing a conjecture, replying with a counterexample, and refining a conjecture that occurred between Lexi and Clay suggests an awareness of the relationship between the different forms of reasoning-and-proving activity.

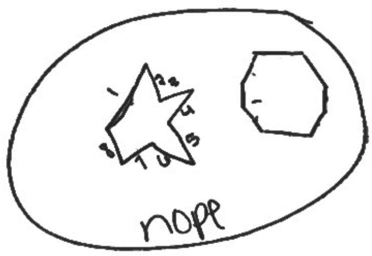

Figure 3.4. Lexi’s counterexample for the conjecture, “all octagons are similar”.

Recall that after proposing four conjectures, the class decided to start by investigating and proving the conjecture “all squares are similar” in their small groups. To illustrate the different reasoning-and-proving opportunities embedded within this task, I am going to describe the original argument constructed by Group 1 (Arin, Brian, and Sadie), the feedback Group 2 (Megan, Wilson, and Lauren) and Group 3 (Clay, Amanda, Heather, and Lexi) gave to Group 1, and then the revisions Group 1 made to their argument in response to the provided feedback. Specifically, I first share Group 1’s original argument for the task, prove that “all squares are similar”, followed by the results of the other two groups evaluating their argument, and then conclude with the revisions Group 1 made in response to their peers’ feedback.

Group 1's original argument included a diagram consisting of two different-sized squares with the angles of each square labeled using the variables A, B, C, and D and the sides unlabeled. Their written argument consisted of the following statements: "If all of the angles on a square are 90 degree angles, then they are the same. If all sides have the same measurements, then they will be proportional." Each sentence in their argument addressed one component of the definition for similar polygons; however, their argument did not justify how they knew the sides would be proportional or explicitly mention the definition of similar polygons or squares.

When evaluating the argument written by Group 1, Megan, Wilson, and Lauren (Group 2) specifically focused on the way that the students in Group 1 had chosen to label their diagram.

Megan: I think that they should put…like, if they're going to do like letters then there should be ones on the sides too because that's like what makes it a square

Wilson: I think they should all be (unclear), I don't think they should be A, B, C, D, I think it should be A, A, A, A cause they're all the same angle.

Notice that both Megan and Wilson justified their proposed revisions to the first group's diagram by referencing different properties of a square (non-proof rationale). Instead of writing their feedback, they chose to redraw the diagram. At this point, I checked in with Group 2 and asked them to explain their proposed revision.

Wilson: Cause when they put A, B, C, D it seems like they're different angles. I know they're just representing that, but they're all the same, they all equal (unclear)

Kimberly: Okay, do we know what the angles are going to be for all squares?

Wilson: 90 degrees

Kimberly: 90 degrees. So do we have to put a letter?

Wilson: No

Megan: We could put 90

Kimberly: We could put 90, because we know, no matter what square we draw, it's going to be 90 degrees.

In this exchange, Wilson began by reiterating his non-proof rationale for why he thought all four angles should be labeled with the same letter. My question to the group invited them to further reflect on the practice of labeling the angles with respect to their knowledge of squares and the generality requirement (*about* reasoning-and-proving). After this exchange, Group 2 finished writing their feedback by deciding to label the angles 90° in their revised diagram along with the statement, “If the angles are the same, the side measurements will be proportional.” Their written statement incorrectly assumed a relationship between congruent angles and proportional sides. With that said, we had not yet discussed as a class how to demonstrate that the sides of a polygon were proportional in a proof.

Group 3 provided feedback to Group 1 by highlighting their use of the words “same” and “proportional” at the end of each sentence and then stating, “We’re not trying to prove that they are proportional, but that they are similar.” This feedback suggests that Group 3 did not recognize that each sentence in the original argument referred to one of the components of the definition for similar polygons. When revising their argument, Group 1 decided to tweak the first sentence to clarify that the *angles* are the same in response to Group 3’s feedback. They also changed their angle notation in the diagram and added labels to the sides of the two squares in response to Group 2’s feedback (see Figure 3.5).

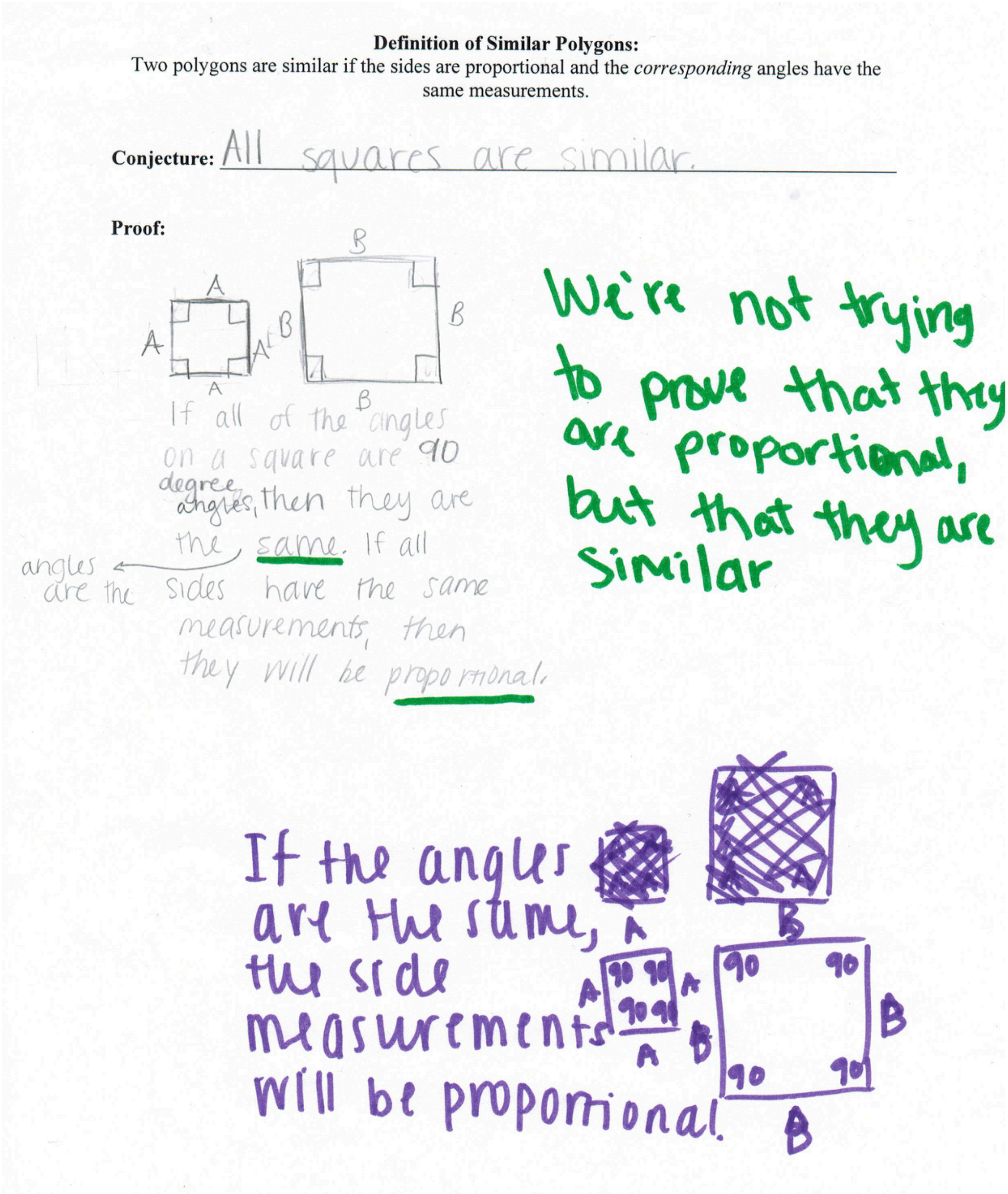

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Figure 3.5. Brian, Arin, and Sadie's (Group 1) revised argument for the conjecture, "all squares are similar", including the feedback given by Group 2 (shown at the bottom in purple) and Group 3 (shown at the top right in green).

In this task sequence, students had the opportunity to engage in a variety of reasoning-and-proving activities as they developed and honed their understanding of proof. Although none of the groups produced arguments that contained all of the elements and formatting of a traditional proof, their work in the task is especially notable given that this was their first experience constructing a proof. Additionally, the students were actively involved in all of the decision-making during this task; at no point in Session 11 did I confirm whether the classes' conjectures were correct, provide students with diagrams to accompany their arguments, or evaluate the feedback students provided their peers. Subsequently, students had to evaluate each other's feedback and decide whether they wanted to incorporate it into their revised argument.

The proving similar polygon conjectures task provided opportunities for the teacher to incorporate statements about reasoning-and-proving. Engaging in multiple reasoning-and-proving activity (e.g., pose a conjecture, construct an argument, evaluate arguments, and revise an argument) within the same set of proof tasks afforded multiple opportunities for the teacher to incorporate statements and questions *about* the reasoning-and-proving process. The first opportunity to think *about* reasoning-and-proving occurred immediately after Brian posed a counterexample for my conjecture that all polygons are similar. After revoicing Brian's justification, I asked the class whether we needed to check any other examples to know that the statement was false. Conversation about how to prove that a statement was false resurfaced with small groups over the course of Session 11 as they investigated the claim about all right triangles being similar.

In Session 12, I made statements and questions *about* the reasoning-and-proving process while constructing a proof for the conjecture, "all squares are similar" on the whiteboard. For example, I suggested the strategy that students might investigate the validity of a claim by trying out examples, including "tricky" shapes that might make the statement false, before beginning to construct their argument. After writing the statement, "By the definition of a square, we know that all of the angles are 90º and all of the sides have the same length" in our proof, I asked students to think about why I explicitly referenced the definition of a square in the statement. "Why might it be helpful to somebody reading our proof to start off by saying 'By the definition of a square'?" Arin replied, "It shows we didn't just make it up." Instead of directly telling students that all statements must be accompanied by a justification, as is custom in traditional proof instruction, I established this expectation through connecting it to the communication aspect of proofs. In addition to talking about the structure of proofs, I also posed multiple questions to the class to help them think about and attend to the generality requirement and its impact in how we label the diagram. At different times in the proof construction, I asked students why it was appropriate for us to label the angles of the two squares 90º and why we needed to use variables to represent the side lengths. Both of these questions helped me to assess students' understanding of the generality requirement as well as reiterate the idea we were proving the statement true for all possible pairs of squares.

Features of the proving similar polygon conjectures task aimed to establish an intellectual necessity for proof but did not result in students outwardly expressing this need. The similar polygons proof tasks contained multiple features that aimed to emphasize the need for certainty and communication. The need for certainty was built into the task by asking students to prove conjectures they had posed without confirming whether each conjecture was true. Indeed, I acknowledged during the first portion of the task that "some of your conjectures will be hopefully true, and some of them may not be, and that's fine, we're just making guesses." The purpose of this statement was to acknowledge that some of the conjectures would likely be false and to encourage students to pose multiple conjectures instead of just focusing on shapes they felt confident would be true. Although the small groups were encouraged to investigate the validity of the claim before constructing their argument, some of small groups were more successful than others in completing this first step of the proving process. For example, one group immediately began to construct an argument stating that all right triangles were similar based on their assumption that the angles of right triangles were always 45º, 45º, and 90º. After I asked them whether the acute angles would *always* be 45º, the group began to question their initial assumption and debate the validity of the conjecture. This suggests that the process of having students pose their own conjectures did not necessarily evoke greater uncertainty regarding the validity of the conjectures.

Once groups completed their argument for the conjecture about squares, I asked them to exchange papers and give each other feedback on their argument, specifically focusing on whether the argument was convincing and made sense. Afterwards, students were asked to revise their argument based on the feedback they received. These components of the task helped to introduce the need for communication and specifically, the idea that they should write their arguments so that they make sense to their peers. I did not directly ask students to reflect upon the process of giving their peers feedback or pose a question specifically designed to assess the extent to which the students viewed proofs as a way of establishing a need for communication. Subsequently, there is insufficient evidence to determine whether the features of this task established a need for certainty and communication from the students' perspectives.

Summary. The similar polygons task provided rich opportunities for students to engage in a sequence of reasoning-and-proving activity, where each component built off of their previous work. This sequence of tasks not only allowed students to develop their understanding of proofs through engaging in the reasoning-and-proving process, but did so in a way that was authentic to the work of mathematicians. While it is certainly possible for students to pose, investigate, and prove conjectures involving particular statements, the validity of such statements are rarely, if ever, a matter of serious consideration. In this task, much of the rich discussions small groups engaged in occurred as they were investigating the validity of the claims. Subsequently, the reasoning-and-proving process that occurred during this task served as a way for students to develop certainty about the validity of the conjecture instead of serving as an exercise to prove statements they had previously been told or believed to be true.

In addition to providing increased and authentic reasoning-and-proving opportunities, the use of universal statements also provided increased opportunities to talk *about* the reasoning-and-proving process. Over half of the statements or questions *about* reasoning-and-proving that occurred in the similar polygons task directly or indirectly attended to the generality requirement. The use of universal claims not only increased the quantity of statements *about* reasoning-and-proving, but also helped to motivate why proofs contain specific components (e.g., why we use definitions, but not examples, to justify statements in a proof). In traditional classrooms, some teachers choose to directly state the components of proofs to students and rely on mathematical convention as justification for the "rules." Indeed, some components of proof, such as the idea that proofs should begin by stating the given information, are primarily dictated by mathematical convention. Others, such as referencing definitions and theorems as justifications for statements in a proof or using variables to label side lengths, are impacted by the need for statements in proofs to adhere to the generality requirement. Using universal claims in an introduction-to-proof unit afforded the opportunity for students to see *why* many of the components of proofs were mathematically necessary. For example, the only way to justify that the corresponding angles of squares are always congruent is through citing the definition of a square; on the other hand, justifying the same statement for two specific squares only requires measuring the angles. By introducing students to key components of proofs, such as the use of definitions to justify mathematical claims, using proof tasks that demonstrate the necessity of those components, students are able to understand the components of proof and are less likely to view them as arbitrary requirements enforced by their teacher.

Students' Challenges in Understanding the Scope of Universal Claims

The use of universal claims in the reasoning-and-proving tasks afforded students rich and varied reasoning-and-proving activity and provided opportunities for the teacher to talk *about* reasoning-and-proving. On the other hand, proof tasks involving universal claims can be more challenging for students, in part because students have to imagine what it means to prove that a statement is true for all possible cases. Analysis of student thinking, evidenced by their conversations and written work, revealed that some students struggled to grasp the scope of the claim during the constructing diagrams and proving similar polygons tasks. In this section, I describe a few students' understanding of the universal claims and then propose opportunities for revisions in light of their understanding.

Diagram construction task. Students' discussions during the diagram construction task revealed two ways of thinking about the statements that differed from the ways in which the statements are typically interpreted by mathematicians. First, some students interpreted the directions to draw a diagram to represent all possible shapes using a non-mathematical definition of "represent" in which a specific shape is used as a symbol for all possible shapes. Drawing on this definition of represent, some students chose to label the side lengths with specific measurements even though they recognized that other shapes might have different side lengths. Second, some students interpreted phrases such as, "if two sides of a parallelogram that intersect have the same length" to mean "if *any* two sides" instead of the typical interpretation of "if *exactly one pair of* sides." Drawing on the first interpretation of the statement resulted in students deciding to label all of the sides of parallelogram congruent to one another. Students' thinking in both of these instances highlights the complexity of mathematical language and the ways in which the meaning of the terms are often left implicit.

During Session 7, two of the three groups initially labeled their side lengths and angles with specific measurements. When I asked one group whether their diagram with side lengths labeled 2in and 5in could be used to prove that the statement was true for *all* rectangles, Lauren replied, "I don't know […] because like it's only that one example, but it's still like true though." Lauren's response captures the limitations of geometric diagrams in terms of being able to represent all possible cases. In other words, there is no generic symbol or shape that we can use to represent all rectangles since every shape will necessarily include specific side lengths and angle measurements, even if those measurements are not labeled on the shape.

When drawing a diagram for the next statement, Wilson directed Lauren and Megan to label the angles of a parallelogram 100º and 80º "cause if it's 90 degrees, it's a square. We need a parallelogram." When asked if those were the *only* angle measurements for a parallelogram, he replied, "I don't know, those probably don't even, they add to 180 I know that, but those probably aren't the exact measurements you know, but…." Even though this group chose to label their angles with specific measurements, Wilson's response reveals that he had chosen the angle measurements "arbitrarily" and had not based them on the actual measurements in their diagram. Additionally Wilson was comfortable changing them to variables based on Megan's idea after they were asked if their labels were generic. It is unclear from this interaction whether Wilson understood why their notation needed to be changed or if he saw a qualitative difference between the two ways of labeling the angles. Students' interpretation of what it means for a diagram to represent all possible diagrams raises questions about whether, and how, to address their non-mathematical interpretation of this term. Is it better for students to begin by viewing diagrams as a specific shape that is representative of the larger group before transitioning to thinking about diagrams as only containing notations that are true of all cases, or should this non-mathematical understanding be avoided through explicitly telling students to only label and notate elements of the diagram that are true for all possible cases? What other verbal explanations or classroom experiences could support students' understanding of what a diagram represents and how they are used in proofs?

In addition to using a non-mathematical definition for the word, "represent", some students also added the word "any" to statements instead of "exactly one" or "at least one" that is typically implied by mathematicians. This led to students adding additional labels to their diagrams that were mathematically correct, but not known based off of the information in the hypothesis. For example, Amanda instructed her group to label all four sides of their diagram congruent for the statement, “If two sides of a parallelogram that intersect have the same length, then the parallelogram is a rhombus.” She justified this conclusion saying, “these two (sides) are intersecting, they have to be the same length” and then repeating this justification for each of the four pairs of sides of the shape. While it is true that all four sides of the shape will be the same length, her justification was based on her interpretation of the hypothesis to indicate that *any* or *every* two intersecting sides were congruent instead of justifying her conclusion using the properties of parallelograms. In other words, the ambiguity of the statement resulted in Amanda justifying her conclusion that all four sides were congruent using the hypothesis rather than properties of parallelograms.

Although the diagram construction task revealed challenges some students faced in deciding how to interpret the mathematical claims, this task also provides potential opportunities to help make these interpretations explicit. For example, what differences, if any, would students’ diagrams contain if they were asked to construct a diagram for the following three statements:

1) If two sides of a parallelogram that intersect have the same length, then the parallelogram is a rhombus

2) If exactly one pair of sides of a parallelogram that intersect have the same length, then the parallelogram is a rhombus

3) if any two sides of a parallelogram that intersect have the same length, then the parallelogram is a rhombus

Discussion of the diagrams for these three statements would reveal students' conceptions regarding what can be notated on diagrams prior to constructing the proof and allow the teacher/researcher the opportunity to talk *about* how to interpret the claim and what can be assumed to be true based on the hypothesis.

Similar polygons tasks. Students' understanding of scope of claim for the similar polygons tasks was particularly evident in their argument for the conjecture, "all rhombuses are similar", since rhombuses only meet one of the two criteria for similar polygons. Although a student posed the conjecture during Session 11, they did not complete an argument for the task until the final interview. After reminding each student of the conjecture, I asked them to consider the validity of the claim and then write a response proving that the statement was either true or false. I also provided students with a Geometer Sketchpad app containing two rhombuses that could be manipulated, along with the angle and side measurements listed for each rhombus, to use during the exploration phase.

Four of the ten students initially wrote arguments proving that the statement, all rhombuses are similar, was true. Three of those students assumed that the angle measurements would stay the same as the sides were proportionally changed, while the fourth student only mentioned proportional side lengths when stating the definition of similar polygons. Arin's argument, shown below, highlights the three students' misunderstanding about the relationship between the sides and the angles when determining if two polygons are similar.

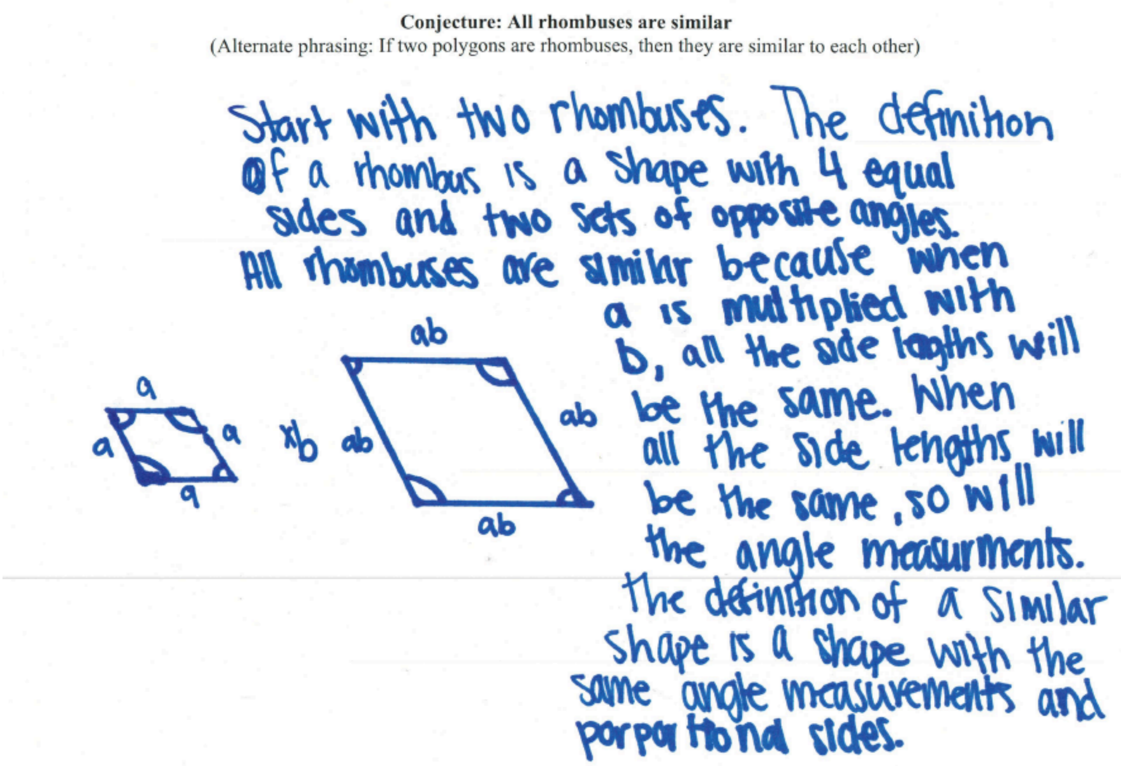


Figure 3.6. Arin's written argument for the similar rhombuses proof task.

In her argument, Arin appropriately defined a rhombus and stated the definition of similar polygons, but incorrectly claimed, "When all of the side lengths will be the same, so will the angle measurements." Although she did not indicate in her diagram that the angle were always a particular measurement (e.g., 60º and 120º as shown on the provided Geometer's Sketchpad app), the rhombuses were drawn in a way that aligned with this perceived misconception. After Arin read this claim out loud, I asked her how she knew that the angle measurements would stay the same. She replied saying:

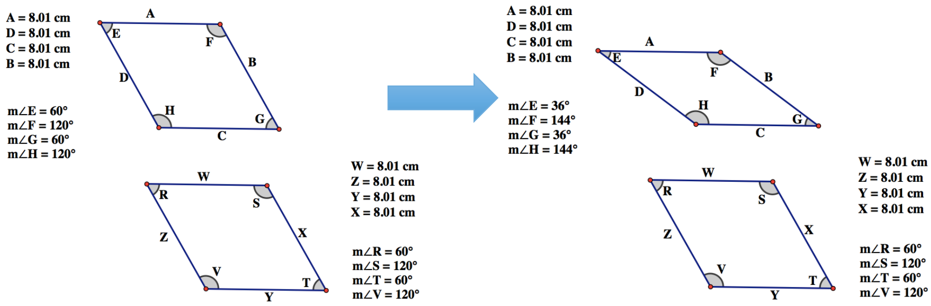
> Because if the shape's proportional, then it'll just… it'll like make the um, the shapes more bigger, but the angle measurements will stay the same because the shape isn't changing its shape, it's just changing its size. So like … and the ang-,

> like you can't gain or loose any angle measurements because it's just, it's the same shape.

Arin's response revealed the root of her misunderstanding about the angle measurements of the two rhombuses — that is, she was viewing one of the rhombuses as a dilation of the other. Instead of thinking about the conjecture as selecting two arbitrary rhombuses and then determining if they are similar, she appeared to be thinking about the task as selecting an arbitrary rhombus and then stretching or shrinking the sides to create the second rhombus. Additionally, she appeared to be conflating her understanding that the sum of the angles in a rhombus will always be 360º and that the opposite angles are congruent with the question of whether each individual angle measurement would remain the same. When probed whether rhombuses have to have particular angle measurements, she stated that the opposite angles "have to be the same, but other than that, they don't have to be specific." Her response further confirms the interpretation that her misunderstanding was not due to her understanding of the definition of a rhombus, but rather due to her understanding of similarity and the scope of the conjecture.

Notably, Arin had not chosen to utilize the dynamic nature of the Geometer's Sketchpad app, but instead chose to immediately begin constructing her argumentation once she was given the task. At this point in the interview, I chose to draw her attention to the Geometer's Sketchpad app by asking her what would happen if I moved one of rhombus' vertices while performing the associated action on the app (see image in the exchange below). When reading the exchange, notice that Arin did not immediately experience cognitive dissonance, but instead had to be asked a few follow up questions before she changed her thinking about the validity of the claim.

Kimberly: What happens if I just move that angle?



Arin: These two (angles) are going to get wider, and these two are going to get smaller.

Kimberly: Yeah, so are these shapes still similar?

Arin: Ummm… no, they're not.

Kimberly: No? Why not?

Arin: Because these two angles aren't the same, and for something to be similar their angle measurements have to be the same, right?

Kimberly: Okay, but are the sides still proportional?

Arin: Um… they are… *(gasps)* oh no!

Kimberly: Oh no! So what are you thinking now?

Arin: So, I don't think they are anymore, because this, like you just moved these two out. Man!

Kimberly: So if you had to, if we had time for you write some more, what would you say is your answer to this one?

Arin: Um, all rhombuses are NOT similar because um… the angle measurements, I don't know. Would you say like, um….

Kimberly: So what do you have to do to prove that something is NOT always true?

Arin: You have to find something, or an example that proves that it's not true, any example.

By the end of the exchange, Arin was able to recognize that not *all* rhombuses are similar to one another because the angles would not always be congruent. However, this change in belief about the validity of the conjecture did not immediately occur after being confronted with the image of two rhombuses with different angle measurements. Although she knew that the two rhombuses on the second iPad screen were not similar, she did not initially appear to view this image as a counterexample to the conjecture. This delay could have been because the image did not initially disrupt her idea of similar rhombuses as being dilations of each other; instead, the cognitive dissonance occurred once I focused her attention on the fact that the side lengths were still proportional.

The similar rhombuses proof task was particularly challenging for many of the students, including some like Arin who understood the definition of a rhombus and, at least to some degree, the definition of similar polygons. Instead, the challenge these students faced seemed to be rooted in their understanding of the scope of the claim and/or what it means for two shapes to be similar to one another. In other words, it is possible that the students had a surface level understanding of the definition of similar polygons (i.e., could state the definition), but did not fully grasp that in order to prove the conjecture, “all rhombuses are similar”, they must select two arbitrary (random) rhombuses and then determine whether they are similar. Interestingly, many of the students who concluded the statement was false did so after realizing that a square was a special type of rhombus. Students’ difficulties with this task highlight the challenge, especially for beginning learners, to understand what it means to prove the conjecture was true for all possible rhombuses and raises the question of how to support students in developing such understanding. Would, for example, the use of technology that could simulate selecting a “random” shape support students’ understanding of the scope of the claim? Additionally, to what extent can students’ difficulties in understanding the scope of claim be addressed through repeated experiences in investigating and proving universal claims?

Discussion

Studies that have analyzed students’ understanding of proof and ability to construct proofs have routinely used tasks involving universal claims (e.g., Buchbinder &

Zaslavsky, 2013; Chazan, 1993; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009), even though they are not representative of the types of proof tasks students are asked to complete in U.S. textbooks (Otten, Gilbertson, et al., 2014). The consistent use of universal claims instead of particular claims in research studies suggests a potential value in this type of proof task; this article sheds additional light on some values, and potential challenges, of using universal claims in reasoning-and-proof tasks given to secondary students. Specifically, I analyzed the impact of using universal claims in an introduction-to-proof unit in terms of the range of reasoning-and-proving activities afforded by the task, the opportunities to talk *about* reasoning-and-proving, and the features of the task that support the development of an intellectual need for proof. Although my analysis focused on the decision to use universal claims in an introduction-to-proof unit, there were other factors that contributed to the quality of the task, such as the use of uncertainty to motivate a need for proof (Buchbinder & Zaslavsky, 2008) and the use of pivotal counterexamples to further students' understanding (G. J. Stylianides & Stylianides, 2009).

The use of universal claims provided students with meaningful and varied reasoning-and-proving activity, including reasoning-and-proving that arose organically as students engaged in the original task. For example, students had opportunities to revise their own or a peer's conjecture in response to a student-provided counterexample during both the tessellation and similar polygon conjectures tasks. The multi-layered reasoning-and-proving that occurred in all three tasks, but especially in the tessellation and similar polygon conjectures tasks, reflects the central idea behind the hyphenated term reasoning-and-proving: "to describe the overarching activity that encompasses the following major activities that are frequently involved in the process of making sense of and establishing mathematical knowledge" (G. J. Stylianides, 2009, pp. 258–259). In other words, students in this study developed their initial understanding of proof through engaging in the complex process of determining and demonstrating that a mathematical statement is true. This differs sharply from traditional proof instruction, where teachers tend to introduce proof through a show-and-tell approach (Cirillo, 2014), and place more emphasis on the form of the proof than on its meaning (Martin & McCrone, 2003; Schoenfeld, 1988). Furthermore, students in the present study engaged in comparable amounts of the different types of reasoning-and-proving, compared to Otten and colleague's (2014) analysis of student exercises in the introduction-to-proof textbook chapters, which tended to ask students to pose/investigate conjectures or develop non-proof rationales but not to construct a proof or counterexample.

The use of universal claims also afforded multiple opportunities to talk *about* the reasoning-and-proving process. While universal claims provide more opportunities to talk about the reasoning-and-proving process than particular claims, especially with respect to talking about aspects of proofs related to the generality requirement, these conversations occurred because of my intentional decision as the teacher to incorporate them into our conversations. Statements *about* the reasoning-and-proving process focus explicitly on the proving process itself (Otten, Males, & Gilbertson, 2014), which can support students in understanding proof at a conceptual level and developing their strategic knowledge of proof, such as how to approach different types of proof tasks (Weber, 2001). Given the instructional focus on the introduction-to-proof, I chose to center many of our conversations *about* the reasoning-and-proving process around fundamental aspects of proof, such as how to prove a statement is false or strategies for constructing a proof. This paper contributes examples of the types of statements *about* reasoning-and-proving that teachers could incorporate into an introduction-to-proof unit alongside tasks that provided the opportunities for such conversations to occur. More research is needed to continue investigating the types of statements *about* reasoning-and-proving that are appropriate for beginning proof students as well as their potential impact in developing students' ability to understand and construct proofs. What are, for example, some of the key aspects of proofs (either the construction or the overall understanding) that are important for teachers to bring up as an explicit item of discussion in their classroom? How can these statements be incorporated into classroom conversation in a way that does not encourage students to view proving as a process to be memorized?

Developing an intellectual necessity for proof was one of the three underlying principles that influenced the planning of the instructional sequence. Although my decision to use universal claims in this study arose from my desire to develop an intellectual necessity for proof, since deductive reasoning is only truly required when proving universal claims, I was unable to report the extent to which the universal claims motivated an intellectual need for proof from the students' perspective. One explanation for this finding is that students did not have sufficient opportunities to express an intellectual need for proof since the features of the tasks that aimed to foster this need were either built directly into the tasks or were emphasized by me during conversations. Specifically, I emphasized the need for certainty and causality during the tessellation task through posing questions that probed students' confidence that all quadrilaterals tessellate and then asking students to reflect on why talking about the angles of quadrilaterals help to explain why the statement is always true. In the proving similar polygon conjectures task, establishing a need for certainty and communication was built directly into the task sequence through having students investigate their own conjectures without confirming whether they were true (need for certainty) and then having students exchange papers and give each other feedback (need for communication). In other words, the way that I sequenced the tasks and the questions I posed to the group moved students to consider the different intellectual needs for proof without them having to vocalize this need. Alternatively, the students may not have felt an intellectual need for proof, but instead just went along with the different instructional activities as they were presented in the sessions.

During the final interview, I posed the following hypothetical question: “Suppose some math teachers and policy makers were thinking about removing proofs from the high school curriculum. Would you think this is a good idea? Why or why not?” Eight of the ten students were able to provide specific reasons for why proofs are an important part of mathematics that each mapped to one of the purposes of proof documented in the literature (de Villiers, 1990). Their responses suggest that at least some component of the instructional sequence contributed to students developing an understanding of the intellectual need for proof; however, it is not clear from the data which aspect(s) of the sequence produced this understanding. The lack of data necessary to make direct links between aspects of the instructional sequence that aimed to establish a need for proof and students expressing this need raises the question of whether students must be provided the space and opportunity to vocalize a need for proof or whether it is sufficient to simply build this directly into features of the tasks. More research is needed to investigate the ways that students might express an intellectual need for proof as well as the extent to which students would vocalize this need.

In this study, I chose to motivate a need for proof from an intellectual, rather than a practical or "real-world", perspective. While some might argue the value in trying to connect proofs to contexts in their everyday lives that students may be familiar with, I purposefully decided *not* to motivate proof through one of the other perspectives since these analogies can potentially provide students with an incorrect understanding of proof, and in particular, of the generality requirement. Discovering Geometry, for example, uses the analogy of a trial lawyer when introducing the lesson on deductive reasoning. "In a trial, lawyers use deductive arguments to show how the evidence that they present proves their case. A lawyer might make a very good argument. But first, the court must believe the evidence and accept it as true" (Serra, 2008, p. 119). While the author(s) are correct in highlighting the importance of using evidence in an argument that others accept to be true, their analogy glosses over the fact that evidence in a courtroom is often subject to debate and different interpretations from each side. Furthermore, what is considered evidence in a courtroom tends to be a collection of distinct, empirical data points, which contrasts sharply with the types of evidence considered appropriate in a mathematical argument. When students are introduced to proof through this type of analogy, there is a potential that they may view examples to be an acceptable form of reasoning or may not fully appreciate that statements in math are justified using definitions, theorems, and so forth because they hold true for *all* possible cases. All real-world analogies ultimately fall short of making an accurate comparison because no other context has the ability to claim that a statement is always true *except* the discipline of mathematics. Given the challenges students face in understanding the generality requirement (see e.g., Reid & Knipping, 2010; G. J. Stylianides et al., 2017), I did not want to motivate a need for proof in a way that could further perpetuate this difficulty.

Although the use of universal claims in the introduction-to-proof instructional sequence allowed students to develop their understanding of proof through engaging in meaningful reasoning-and-proving activity, the universal nature of the claims also posed some challenges for students in comprehending and interpreting the scope of the mathematical claim. These added difficulties are important for researchers to acknowledge, especially if they are trying to assess students' understanding of proof through tasks involving universal claims, as they have been done in the past. More research is needed to further explore the different ways that students interpret universal claims as well as the ways in which teachers can support students in understanding the scope of universal claims. Although the scope of universal claims may be more difficult than particular claims for students to understand, the affordances universal claims provide in terms of increased opportunities for reasoning-and-proving and discussion *about* reasoning-and-proving provide evidence to suggest their value in proof instruction at the secondary level.

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CHAPTER 4

Developing Understanding of the Generality Requirement of Proof: The Case of Lexi

The field of mathematics is unique in its ability to conclusively demonstrate, or prove, that a mathematical statement is *always* true. Mathematics not only allows for general statements to be proven always true, but requires it. In other words, in order for a mathematical statement to be considered true, it must be demonstrated true for all cases for whatever domain the statement implies. Fulfilling this generality requirement necessitates different demands depending on whether it is particular statement (e.g., a conjecture about a single triangle) or a universal statement (e.g., a conjecture about all rectangles). For example, proving that a triangle *ABC* is congruent to the triangle *DEF* only requires verification that the corresponding sides and angles are congruent to one another, whereas proving that, for any rectangle, its diagonals are congruent requires a deductive approach and cannot be verified by checking specific examples. It is unclear whether most students recognize this distinction, especially since they are frequently asked to construct deductive arguments for particular claims and are rarely asked to prove universal claims (Otten, Gilbertson, et al., 2014).

Synthesis of findings from prior studies have consistently highlighted the difficulties students face in recognizing and adhering to the generality requirement when constructing and evaluating arguments for proof tasks (Reid & Knipping, 2010; G. J. Stylianides et al., 2017). Specifically, students tend to be convinced that the given statement is always true after checking a few examples (e.g., Buchbinder & Zaslavsky, 2007; Healy & Hoyles, 2000), do not always recognize that a proof demonstrates the statement's validity for all possible cases (e.g., Martin, McCrone, Bower, & Dindyal, 2005), and remain open to the possibility of the existence of a counterexample even after being presented with a proof of the statement (Chazan, 1993). When constructing responses to proof tasks, 62% of eighth graders in Knuth, Chopin, and Bieda's (2009) study and 34% of advanced 14–15 year olds in Healy and Hoyles' (2000) study produced examples-based responses when asked to prove that the sum of two odd numbers is even. However, researchers in both studies suggested that their findings might reflect what the students were mathematically capable of producing more than their understanding of the generality requirement. In other words, it may be that the students realized examples were insufficient for proving the given claim but used them anyway because they could not come up with a deductive argument in the given moment. All of the previously mentioned studies, with the exception of the one conducted by Martin and colleagues (2005), drew on data collected from an interview or written assessment with students at a single point in time. Subsequently, while their findings reveal insights into some students' difficulty with the generality requirement, they do not examine students' learning over time or describe instructional contexts that allow students to overcome this difficulty. The present study addresses these limitations by documenting how one student, Lexi, developed understanding of the generality requirement while participating in a design research study that aimed to facilitate understanding of the generality and purpose of proof.

Theoretical Perspective

This study views mathematics learning as the process of students developing their own understanding, with the support of their teacher, through engaging in mathematical tasks both individually and collectively with their peers (Prediger, Gravemeijer, & Confrey, 2015). In particular, I draw on the constructivist perspective, which states that learning occurs when individual students assimilate new information into their current schema as a result of perturbations (von Glaserfeld, 1993). Within this perspective, students are treated as "epistemic agents of their own who bring to bear their own experience and resources" (Prediger, Gravemeijer, & Confrey, 2015, p. 881). In other words, students are not treated as passive receptors of information, but rather active participants who are capable of making sense of mathematics through drawing on their prior knowledge and experiences both within and outside of the classroom. Thus, it is the responsibility of the teacher-researcher to "make a *conscious* attempt to "see" both their own and the children's actions from the children's points of view" (Cobb & Steffe, 1983, p. 85). Within design research, Prediger, Gravemeijer, and Confrey (2015) contend that theories of learning tend to be used as background theories; once the aforementioned conditions for learning are met, the researcher then focuses on identifying and tracking changes in students' thinking over time, identifying both moments of growth and moments of failure with the purpose of refining the instructional sequence based on students' thinking.

DNR-Based Instruction

Since the constructivist perspective describes learning broadly without prescribing a particular method of teaching from a pedagogical perspective (Simon, 1995), I also drew on the *DNR-based instruction in mathematics* (*DNR)* conceptual framework developed by Harel (2008) when developing the instructional sequence. The *DNR* framework consists of three main principles: *duality, necessity,* and *repeated reasoning*. The *duality* principle maintains that students' ways of thinking and understanding are intertwined; the *repeated reasoning* principle states that students must be provided with multiple opportunities to practice mathematical reasoning in the classroom to develop the desired ways of thinking and understanding; and the *necessity* principle asserts that establishing the intellectual need for a particular topic is crucial in the learning process. Harel describes five different intellectual needs; however, this study primarily drew on three of them: the need for certainty, causality (to understand why), and communication. When viewing constructivism through the lens of the DNR-framework, learning occurs as a result of perturbations, which are “manifested by (a) intellectual and psychological needs that instigate or result from these phases and (b) ways of understanding or ways of thinking that are utilized and newly constructed during these phases” (Harel, 2008).

Within the context of developing understanding of the generality requirement, the duality principle states that Lexi’s understanding of the generality requirement is evidenced by her work on proof tasks (both in constructing and evaluating provided arguments); as she sees and constructs arguments that adhere to the generality requirement for specific mathematical claims, her ways of thinking about the generality requirement as a broad construct begin to shift. These shifts in thinking and understanding about the generality requirement occur as a result of repeated opportunities to engage in reasoning that adheres to the generality requirement and reflect (both individually and in conversation with others) on the generality requirement itself. Finally, the generality requirement was used as a means of establishing an intellectual necessity for proof through emphasizing the need for certainty and causality. In particular, I aimed to establish a need for certainty through emphasizing that a statement must be true for *all* possible cases and a need for causality through positioning the proving process as a way of developing understanding *why* a statement is always true.

Literature Review

Definition of Proof

This study draws on Andreas Stylianides' (2007) definition of proof: "a mathematical argument, a connected sequence of assertions for or against a mathematical claim" that uses acceptable justifications, valid modes of argumentation, and representations that are appropriate and understood by the classroom community (p. 291). In the present study, I interpret the terms "valid", "acceptable" and "appropriate" according to both our classroom community and the broader mathematics community. During our classroom conversations, I served as a representative of the broader mathematics community and brought this perspective into dialogue with the students' expectations and ideas. Note that Stylianides' definition was used to inform my own instructional decisions but was not formally presented to students. In this article, I use the term *proof* in two specific ways: as an adjective to describe tasks where Lexi was expected to construct a proof and as a noun encapsulating the process of proving when describing Lexi's conception of proof, or my interpretation of what the student appears to believe or know about proof. For example, when Lexi stated that a provided student's argument was not a proof because "she never said why it worked and why it didn't work", I stated that this justification revealed her conception that proofs should explain why the statement was true. When referring to Lexi's verbal or written work on a proof task, I use the term *argument* to avoid making any judgment as to whether her work should be considered a formal proof.

Proof Instruction

Traditional approaches to teaching proof. In U.S. secondary mathematics classes, proofs are commonly written in the two column-format (Herbst, 2002), with the teacher placing significant emphasis on whether students place the statements and reasons in the "proper" form (Martin & McCrone, 2003; Schoenfeld, 1988). Although few studies have documented the ways that teachers first introduce proof in the secondary classroom, Cirillo (2014) found that all five Geometry teachers she observed introduced proof through a show-and-tell method and struggled to find ways to scaffold the proving process for students. During the teachers' proof demonstrations, Cirillo noted that the teachers did not explicitly unpack the many different components of the proof for students, such as how they were using definitions to draw conclusions or what can and cannot be assumed from a diagram. Instead, students were expected to learn how to construct proofs through watching the teacher model the process and figuring it out as they practiced constructing proofs on their own. Given prior research findings on the ways that proofs are often taught in the classroom, it is unsurprising that students struggle to understand the purpose of proofs and experience a range of challenges in learning how to construct proofs.

Proof interventions. There has been a recent increase in intervention studies aimed at improving secondary and undergraduate students' understanding of proofs. In lieu of summarizing the different approaches researchers have taken in the interventions, I focus on the design experiment conducted by Gabriel and Andreas Stylianides (2009) with elementary preservice teachers as it greatly informed the approach taken in this study. The overarching goal of their three-hour intervention was to help participants transition from constructing example-based arguments to deductive arguments through highlighting the limitation of assuming that a pattern found from a few examples will also be true for every possible case. After asking students to determine the total number of different sized squares (e.g., 1x1, 2x2, etc.) in a 60-by-60 square, which students found by generalizing a pattern from smaller sized examples, preservice teachers were presented with two counterexample tasks, called the Circle and Spots task and the Monster Counterexample, where the pattern found from small-number cases did not hold true for larger cases. The intervention concluded by having the preservice teachers, assisted by the instructor, construct a proof to show that the pattern found in the original task held true for all cases. The Stylianides and Stylianides (2009) instructional sequence was constructed based on the assumption that “pivotal counterexamples” (p. 317) bring about the cognitive conflict needed to help students make the transition from empirical justifications to proofs. While pivotal counterexamples do have the potential to shift students’ conceptions, they do not provide motivation for *why* proofs are needed in mathematics (Harel, 2008) or clarify the purpose of mathematical proofs (de Villiers, 1990). In other words, their intervention sought to transition preservice teachers to deductive arguments solely through casting doubt on examples-based arguments, without also discussing the value or purpose of deductive arguments.

Purpose of Proofs

Drawing on the works of other scholars, such as Hanna (1990), Balacheff (1988), Bell (1976), and Hersh (1993), de Villiers (1990) suggested six roles that mathematical proofs can take: verification, explanation, systemization (organization of results into a deductive system), discovery, communication, and intellectual challenge. Proofs are often used in secondary classrooms as a means of *verifying* mathematical statements; however, de Villiers (1990) argued that emphasizing the *explanatory* feature of proofs could help students view proving as a meaningful activity in mathematics. While some students recognize that proofs should convince or explain why a statement is true, many have indicated that proofs were only convincing to teachers or others who understood the customs of mathematicians (McCrone & Martin, 2009). More work is needed to understand the features of proofs that students find explanatory and explore ways that teachers might promote this feature in the classroom.

Generality of Proofs

Students tend to both construct empirical arguments, and to a lesser extent, evaluate provided arguments that are empirical as proofs (e.g., Reid & Knipping, 2010; G. J. Stylianides et al., 2017). One possible explanation for students' preference for empirical arguments is that they are already convinced of the validity of the statement prior to constructing the requested proof; as a result, proofs are no longer used as a way of convincing oneself of the statement's validity. A second possibility is that students realize the limitation of using examples as justification, but construct example-based justifications because they are unsure of how to construct a more general argument. Researchers who have asked students to both construct and evaluate provided arguments (e.g., Healy & Hoyles, 2000) have found that students are more likely to select deductive arguments as proofs than they are to construct deductive arguments.

In an analysis of commonly used U.S. geometry textbooks, Otten and colleagues (2014) found that students were given significantly fewer opportunities in the exercises to find a counterexample for a false mathematical claims than they were to construct a proof for true mathematical claims. The gap between the number of opportunities to construct a proof and find a counterexample implies that the validity of each statement was known for a significant portion of the student exercises, as questions that asked students to either construct a proof or find a counterexample would be assigned both reasoning-and-proving activities. Constructing arguments for statements were the validity is known minimizes the opportunity for students to view the proving process as developing certainty about the validity of the conjecture (Buchbinder & Zaslavsky, 2008) and can potentially result in a lack of awareness that a deductive argument and a counterexample cannot coexist for a single mathematical claim (Buchbinder & Zaslavsky, 2013; Chazan, 1993). In order to develop students' understanding of the generality requirement, it is important that students have multiple opportunities to both prove and disprove mathematical conjectures. With that said, it is possible that engaging in the proving process will not necessarily develop students' understanding of the generality requirement (e.g., Martin et al., 2005); additionally, students need opportunities to talk *about* the reasoning-and-proving process where the generality requirement is made an explicit item of focus (Otten, Males, et al., 2014).

Variables. Generality is not only fundamental to proofs but also is at the core of algebra and mathematics as a discipline (Mason, 1996). Within algebra, generality is often represented through the use of variables that encompass all possible numbers for the given situation. Although the term "variable" broadly refers to a letter that represents a number (or numbers) in mathematics, they implicitly take on different roles in different contexts (Ely & Adams, 2012; Küchemann, 1978; Philipp, 1992; Schoenfeld & Arcavi, 1988; Usiskin, 1999). Students are initially exposed to the idea of variables as unknowns when solving algebraic equations such as $x + 3 = 5$ or $x^2 + 5x + 6 = 0$. In each equation, the variable x represents one or a finite number of unknown values that can be determined from the given information (Ely & Adams, 2012; Philipp, 1992). Variables can also represent an infinite number of values. These include instances where variables are used as placeholders (e.g., m in the equation $y = mx$), generalized numbers (e.g., t in $2t + 3t = 5t$), and varying quantities (e.g., x and y in the equation $y = mx$). In order to represent two consecutive whole numbers, for example, as n and $n + 1$, students need to recognize the relationship between the two consecutive whole numbers—namely, that they are one apart— and then represent that relationship using a single variable. Subsequently, students who represent two consecutive numbers as m and n appear to be using variables as generalized numbers instead of varying quantities.

Students' understanding of variables as varying quantities can not only impact their ability to comprehend advanced algebraic topics but can also impact their understanding of variables when used in the context of proofs. Although proof is largely contained in U.S. curricula within the context of geometry (Herbst, 2002), variables are used in geometric proofs to represent, for example, the length of a line segment or the measurement of an angle. In both of these contexts, variables are intended to be used as varying quantities even though some students may interpret them as placeholders or generalized numbers. Additionally, some textbooks such as Glencoe McGraw-Hill ask students to complete proof tasks that involve solving an algebraic equation and providing a mathematical property as justification for each step (Otten, Males, & Gilbertson, 2014). It is unclear how students interpret variables within the context of proofs and whether, for example, they recognize that the variable represents all possible values instead of a single, or finite number of values.

One possible way that teachers can develop students' understanding of variables within the context of proof is through engaging students in proving different properties of numbers. However, simply asking students to prove number-based conjectures does not ensure that they will utilize their algebraic knowledge to construct their response. For example, Healy and Hoyles (2000) found that advanced 14–15 year olds were highly unlikely to construct an algebraic proof for the task, "prove that when you add any 2 odd numbers, your answer is always even," even though they overwhelmingly evaluated the provided algebraic responses as most likely to receive the highest scores by their teachers. Of the 281 students (11% of study participants) who attempted to use variables to prove the sum-of-two-odds conjecture, only 35% of them constructed a partial or complete proof. Overall, this research suggests that while number-based proof tasks offer students an opportunity to consider the generality of mathematical claims within a non-geometric context, more work is needed to understand how to support students' understanding of using variables within the context of proof.

In summary, the generality requirement dictates many different aspects of proof, from the ways in which numbers are represented to the types of evidence needed to prove that a mathematical statement is true or false. Subsequently, understanding the generality requirement is essential to developing rich conceptual understanding of proof. The research question guiding this study is the following: How did one high school student's understanding of the generality requirement develop, if at all, over the course of a design study that introduced proofs through focusing on the generality and purpose of proofs? Given that the instructional sequence was intentionally designed to target students' understanding of the generality requirement, the learning process for the student in this study may not be reflective of the ways in which students develop understanding of the generality requirement in traditional classroom instruction. Nonetheless, her learning process reveals possible intermediate steps students may reach as they are developing understanding of the generality requirement and its impact on the proving process.

Introduction to Proof Project

Design research methodology is used to study interventionist, rather than naturalistic, teaching and learning with the goal of contributing to both theory and practice (Brown, 1992; Cobb et al., 2003; Cobb, Jackson, & Sharpe, 2017). In particular, the teaching that occurs in design studies should be innovative in some way through the use of tools, tasks, and/or instructional techniques not typically found in the classroom. One key feature of design studies is the testing and refinement of the researchers' conjectures for learning. Although the conjectures for learning are not a specific object of focus in this article, I describe some of my initial conjectures in this section in order to orient the reader to the overall instructional goals and sequence of tasks in the design study. In this section, I describe two conjectures for learning followed by descriptions of how each conjecture informed selection of tasks used in the instructional sequence.

Conjecture 1: Developing an Intellectual Need for Proofs

Before learning how to construct proofs, I conjectured that students should have an intellectual necessity for proofs (Harel, 2008). In particular, I conjectured that proofs needed to be established as a way of gaining certainty and causality about the given statement. In order to establish a need for certainty, I conjectured that students might initially have different perspectives on whether it was possible to know if a statement in mathematics was *always* true for an infinite number of cases; additionally, I predicted that some of the students might not have previously considered this question prior to the study. Thus, part of establishing an intellectual need for proof was also establishing the generality within mathematical statements, or highlighting the fact that knowing something is usually true or true for most cases is not sufficient in mathematics. When selecting tasks to help position proof as a way of developing certainty, I looked for universal claims where the conjecture's validity would not be immediately apparent. I intentionally chose to only use proof tasks involving universal claims so that it was not possible to prove the statement by checking every possible example. I also looked for tasks where students could initially explore specific examples but that those examples would feel different from one another. This characteristic was important to sustain students' engagement in the problem and to help motivate a lack of satisfaction in their answer based on examples alone.

In addition to positioning proof as a way of gaining certainty about a mathematical statement, I wanted the solution for each task to explain *why* the statement was true instead of simply stating *that* it was true. I conjectured that the explanatory feature of some proofs could help students transition away from examples-based arguments to deductive arguments since examples on their own do not tend to explain why a statement is true. While I agreed with the approach taken by Gabriel and Andreas Stylianides (2009) to highlight the limitations of generalizing from examples, I conjectured that this alone would not be sufficient to dissuade students from constructing empirical arguments. Instead, I wanted to highlight the explanatory power of deductive arguments alongside exposing the limitations of examples. In other words, I sought to help students realize that not only were examples insufficient to prove that a mathematical statement was always true, but they also did not provide a reason for *why* the statement was true.

The first four sessions in the design study focused on establishing an intellectual need for proof through problematizing whether it was possible to know if *all* quadrilaterals tessellate the plane (see Figure 4.1). In order to motivate a need for understanding *why* all quadrilaterals tessellated, students engaged in exploring the follow-up task of determing whether all regular polygons tessellated. This task served as a pivotal counterexample (Stylianides, G. & Stylianides, 2009) for students because it aligned with their current procedure for tessellating quadrilaterals but differed in the statement's validity, thereby necessitating additional explanation for why all quadrilaterals, but not all regular polygons, tessellate. In the fifth session, students engaged in the Circle and Spots and the Monster Counterexample tasks in order to further establish the limitations of using examples to demonstrate the validity of a mathematical claim (Stylianides, G. & Stylianides, A., 2009; tasks are also listed in Appendix A).

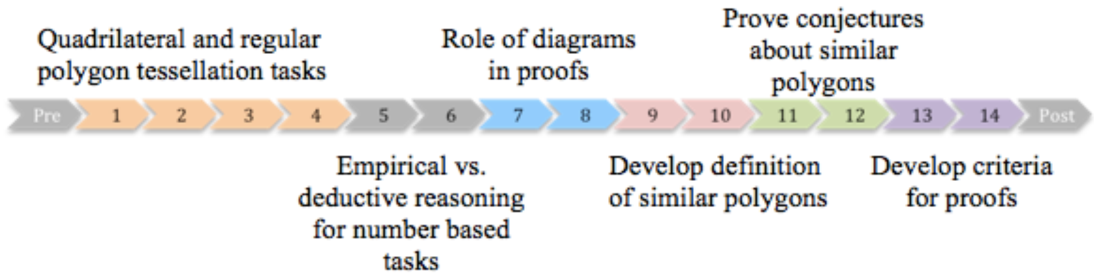


Figure 4.1. Overview of the instructional sequence.

Conjecture 2: Developing Understanding of Proof as a Mathematical Practice

Once students recognized an intellectual need for proof (evidenced, at least in part, by their journal entries during Session 3[8]) and were aware of the generality within mathematical statements (evidenced by their journal entries during Sessions 3 and 4[1]), I conjectured that they were ready for a more intentional focus on proof as a mathematical practice. Thus, my second main conjecture for learning was the following: before instruction develops and hones students' understanding of the form of proof, they need to understand proof as a mathematical practice. By proof as a practice, I refer to the Standard for Mathematical Practice #3 (SMP 3): Construct viable arguments and critique the reasoning of others. Specifically, SMP 3 states:

[8] See Appendix A for a description of the writing prompts.

> Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. […] Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. (National Governors Association Center for Best Practices & Officers, 2010)

Note that the description of SMP 3 aligns with Stylianides' (2007) definition of proof, but excludes the components of his definition that focus on the form of proof. By not devoting significant time to instruction on the form of proofs in my instructional sequence, I am not claiming that the form is unimportant, but rather am proposing that it should be developed after students have developed a broad understanding of other components of proof (i.e., generality, purpose).

In order to help students grasp proof as a mathematical practice, the remaining sessions focused on developing their understanding of different elements of proof in order to prepare them to be able to construct their own proofs in Sessions 11–13 (see Figure 4.1). In particular, students were introduced to using variables in proofs as a way to represent all possible numbers in Session 6, practiced constructing diagrams for theorems about quadrilaterals in Sessions 7 and 8 (Cirillo & Herbst, 2011), and developed a definition for similar polygons in Sessions 9 and 10 (Kobiela & Lehrer, 2015). Each task provided students with ways of developing understanding of a particular component of proof while also attending to the generality requirement. For example, in Sessions 9 and 10, students had to consider all possible similar polygons when developing a definition for similar polygons; thus, this task helped to establish the idea that definitions can be used in proofs to justify a mathematical claim since they apply to all possible cases. See appendix A for more details about the instruction in each session.

The first set of tasks in which students were asked to construct proofs occurred in Sessions 11 and 12, where they were asked, in their small groups, to conjecture and then prove statements about similar polygons (e.g., “all squares are similar”). Once all of the small groups completed their first draft of a proof for the conjecture, “all squares are similar,” I discussed the proof with the whole group. Students completed their first individual proof in Session 13 by proving the exterior angle theorem, which was posed using two specific examples and the question, “Is this a coincidence?” (see Appendix A) The exterior angle theorem was posed in this format to minimize the mathematical vocabulary in the task and to evoke uncertainty regarding the validity of the mathematical claim being proven (Buchbinder & Zaslavsky, 2008). After students finished writing their responses to the proof task, they exchanged work with their peers and provided feedback to each other. We did not review the proof for this task as a whole class so that I could follow up with students during the final interview. In the final session, students generated a set of criteria for “good” proofs and then used the criteria to evaluate four provided responses for the exterior angle theorem (Boyle, Bleiler, Yee, & Ko, 2015). This task served as a summative assessment of students’ current conceptions of proof and a way of assessing the extent to which students achieved the intended learning goal of understanding proof as a mathematical practice.

Method

Setting and Participant

The case of Lexi was situated within the larger, introduction-to-proof design research study described in the previous section. Ten 9th graders participated in the study (7 females, 3 males; all approximately 15 years old); they were the only students enrolled in the first of two courses in the accelerated math program at their rural, public school in the Midwestern United States at the time of the study. Students were enrolled in the accelerated math program based on their middle school mathematics grades and recomemndations from their teachers. During the first two years of high school (ages 14-16), accelerated math students learned Algebra 1, Algebra 2, and Geometry, with the first year focusing on Algebra 1 and parts of Algebra 2 and the second year focusing on the remaining topics in Algebra 2 and Geometry. As a result, students were learning Algebra 2 content during the study but had not yet been exposed to formal, deductive proofs. The design study sessions occurred during the students' study hall, with each session lasting between 28 and 38 minutes. All students received a graphing calculator for participating in the study. As a part of the larger study, I selected four focus students to interview an additional two times in order to further probe their current understanding of proof at different points in the instructional sequence. I selected four students—Lexi, Amanda, Heather, and Clay—as they expressed a range of initial ideas about proof in their initial interview. The four focus students worked together as a small group from Session 5 onward.

Lexi represents the case of a student who was beginning to develop an understanding of mathematical proofs, and in particular, of the generality requirement of proofs (Yin, 2014). I selected Lexi as the focus of this article because her understanding of proof over the course of the study was typical of the broader group. Additionally, her math teacher reported that she was a solid “A” mathematics student and considered her to be one of the strongest mathematics students in her grade. Her teacher also described Lexi as someone who regularly completed her work, actively engaged in class, sought clarification when needed, and generally learned new mathematical topics easily. As a result, Lexi represents a “best case scenario”; that is, a student who had a strong mathematical foundation, enjoyed learning mathematics, and experienced continued success in the mathematics classroom. During the initial interview, Lexi reported liking the faster pace of the accelerated math courses and wanting to be in the accelerated program so that she could get ahead in her math studies. She also demonstrated a willingness early on in the study to share her thinking with the whole group and was consistently vocal during small-group discussions. This resulted in additional information about her learning across the sessions that was not available for some of the other, less vocal focus students.

Data Collection and Analysis

The design study took place over seven weeks in the spring semester. I was the teacher for all sessions; additionally, there were two outside observers[9] who attended 10 of the 14 sessions. Data for this study included audio and video recordings of the sessions and the interviews as well as all student work and journal reflections produced during the study. Lexi participated in four semi-structured interviews (Roulston, 2010): an initial interview prior to the start of the sessions, two focus interviews that occurred after Sessions 5 and 11, and a final interview after the conclusion of the study. I conducted the

[9] The primary observer was a graduate student who had previously taught high school Geometry and the secondary observer was a professor in Mathematics Education.

first focus interview and the final interview and one of the outside observers conducted the remaining two interviews. All interviews were transcribed prior to analysis.

Design research is necessarily iterative and involves multiple layers of analysis. During the design study, I debriefed and conducted ongoing analysis after each session in collaboration with at least one outside observer and then made subsequent revisions to the instructional sequence (Cobb et al., 2017). After the conclusion of the study, I conducted retrospective analysis using an open and axial coding approach that focused on Lexi's understanding over the course of the study (Strauss & Corbin, 1998). Specifically, I first systematically reviewed all data collected in the study in chronological order and identified all instances where Lexi's understanding of the generality requirement was evident. Next, I categorized the data based on key features of the tasks that provided a context for revealing a particular aspect of her understanding of the generality requirement. These categories included instances where she was exploring and proving a true mathematical claim, exploring and disproving a false mathematical claim, evaluating the provided "student" arguments for a given mathematical claim, and using variables to represent generality. I chose to focus on her understanding of the generality requirement as viewed through her use of variables instead of other components of proofs (e.g., use of diagrams), as this component was in the proof tasks across the entire study.

After compiling the relevant data, I wrote memos describing Lexi's apparent understanding for each task (written work and spoken word) or clip of session data (transcribed conversation and/or journal reflection) within each of the four categories (Strauss & Corbin, 1998). When analyzing Lexi's arguments for proof tasks, I attended to the language she used in order to gain insights into her understanding of proof. For example, I analyzed Lexi's use of the phrase "so like" when describing her response to a proof task to indicate that she was about to provide an example to help illustrate her mathematical thinking. I also focused on Lexi's use of language such as "always" or "any" as indicators of her attending to the generality requirement. Afterwards, I made constant comparisons between each unit of analysis both within and across categories to look for similarities and differences in her understanding. For example, after documenting the criteria Lexi used when evaluating provided responses in interview 1, I compared the criteria with her constructed response on the same proof task in interview 1 to triangulate the data and then compared the criteria she used during interview 1 with the criteria she used in interviews 2 – 4 in order to capture changes in her criteria. I also noted instances where Lexi's statements could be interpreted in multiple ways and conducted multiple reviews of the data to look for disconfirming evidence and possible rival interpretations (Yin, 2014; Corbin & Strauss, 1998).

Findings

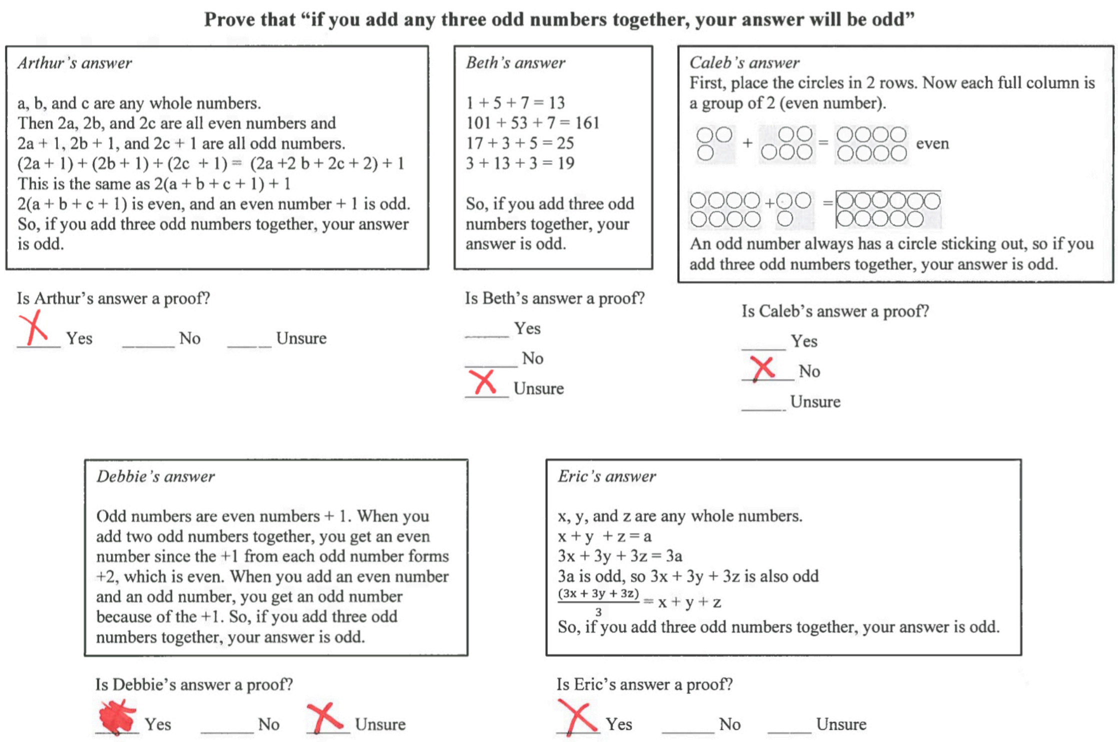
This section is organized around a description of Lexi's demonstrated understanding of the generality requirement across three different contexts: when constructing and evaluating arguments for true mathematical claims[10], when proving false mathematical claims, and when using variables to fulfill the generality requirement. Within each context, I describe Lexi's understanding chronologically and conclude with a brief summary of her learning as it developed, or remained the same, across the selected moments.

[10] I initially analyzed the constructed responses and evaluations of provided responses separately, but chose to combine the two categories for this paper since constructed an argument for all of the tasks where I asked her to evaluate provided arguments.

Lexi's Understanding of the Generality Requirement When Constructing and Evaluating Solutions to Proof Tasks

Initial interview. In order to gain baseline evidence of students' initial understanding of proof, the interviewer posed the following proof task: "Sarah said, 'If you add any three odd numbers together, your answer will be odd.' Is she right? Explain your answer." After trying one example with smaller numbers (1+3+5) and another one with a larger number (51+3+9), Lexi verbally concluded that the statement was true "because the two problems that I did that were different both added up to an odd number." Lexi's empirical-based argument was typical of the broader group and unsurprising given her lack of formal proof instruction at the time of the initial interview. When asked if her answer proved that the statement was *always* true, she replied, "No, because I only gave 2 examples. So, I could probably do more but…" In this response, Lexi appears to recognize that her provided examples were insufficient to prove the statement was *always* true even though she had endorsed the claim as true, which raises the possibility that she had not been aware of the generality requirement embedded within the original prompt. Indeed, Lexi expressed uncertainty in the interview whether the sum of three odds would *always* be odd. "I know in math we always have special cases, so probably not, but I think in the majority of the time, it'll probably end up odd."

After Lexi discussed her response to the *sum of three odd numbers* proof task, the interviewer asked her to look at five provided "student" arguments for the same task and determine whether she thought each one was a proof (see Figure 4.2). Although she was unsure whether Beth's answer was a proof "because it seemed kind of easy," she decided that Arthur's answer was a proof "because you can always plug in numbers and see if it works out or not." Similarly, she was unsure how to evaluate Debbie's answer "because it doesn't like give any examples or anything." In her evaluations for both Arthur and Debbie's responses, Lexi drew on the idea that proofs should contain or generate examples, mirroring her own approach to solving the task. In summary, Lexi consistently drew on the idea that proofs should contain examples when evaluating the provided student responses.


Prove that "if you add any three odd numbers together, your answer will be odd"

Arthur's answer

a, b, and c are any whole numbers.
Then 2a, 2b, and 2c are all even numbers and
2a + 1, 2b + 1, and 2c + 1 are all odd numbers.
(2a + 1) + (2b + 1) + (2c + 1) = (2a +2 b + 2c + 2) + 1
This is the same as 2(a + b + c + 1) + 1
2(a + b + c + 1) is even, and an even number + 1 is odd.
So, if you add three odd numbers together, your answer is odd.

Is Arthur's answer a proof?
X Yes ______ No ____ Unsure

Beth's answer

1 + 5 + 7 = 13
101 + 53 + 7 = 161
17 + 3 + 5 = 25
3 + 13 + 3 = 19

So, if you add three odd numbers together, your answer is odd.

Is Beth's answer a proof?
_____ Yes
______ No
X Unsure

Caleb's answer

First, place the circles in 2 rows. Now each full column is a group of 2 (even number).

even

An odd number always has a circle sticking out, so if you add three odd numbers together, your answer is odd.

Is Caleb's answer a proof?
_____ Yes
X No
_____ Unsure

Debbie's answer

Odd numbers are even numbers + 1. When you add two odd numbers together, you get an even number since the +1 from each odd number forms +2, which is even. When you add an even number and an odd number, you get an odd number because of the +1. So, if you add three odd numbers together, your answer is odd.

Is Debbie's answer a proof?
Yes ______ No X Unsure

Eric's answer

x, y, and z are any whole numbers.
x + y + z = a
3x + 3y + 3z = 3a
3a is odd, so 3x + 3y + 3z is also odd
$\frac{(3x + 3y + 3z)}{3} = x + y + z$
So, if you add three odd numbers together, your answer is odd.

Is Eric's answer a proof?
X Yes _____ No _____ Unsure

Figure 4.2. Lexi's evaluation of five student responses for the sum of three odd numbers proof task. The evaluations marked in the image were her answers during the initial interview.

First focus interview. The first five sessions in the design experiment aimed to motivate an intellectual need for proofs through problematizing whether it was possible to know if a statement in math was always true. This was primarily achieved through exploring the questions, "do all quadrilaterals tessellate?" (Sessions 1–2) and "do all regular polygons tessellate?" (Session 3). In Session 4, I summarized the main ideas that surfaced in the whole group discussions and provided a mathematical explanation in the form of an informal proof for why all quadrilaterals, but *not* all regular polygons, tessellated. Session 5 aimed to perturb students' belief that examples were sufficient evidence to determine whether a given statement was always true through placing them in situations where the pattern held true for some, but not all cases. This was achieved through engaging students in the Circle and Spots problem and discussing the Monster Counterexample (Stylianides, G. J., & Stylianides, A. J., 2009). By exposing students to the limitations of generalizing patterns from a few cases, I aimed to establish an intellectual need for an alternate method to prove that a statement is always true.

During the first focus interview (immediately after Session 5), Lexi was asked to prove that all triangles tessellate. Lexi's argument drew on her prior experiences with tessellating quadrilaterals as well as a "theory" she posed in Session 3 regarding the relationship between the sum of the angles in a polygon and whether the polygon would always tessellate. Her verbal argument that all triangles tessellate is shown below.

> So I said I know that quadrilaterals tessellate, because angles add up to 360º, which that's just like a starting off, I guess. And then I said this works with triangles as well, because the angles all add up to 180º. So anytime the shape adds up to a multiple of 180, it will tessellate. So like, if it adds up to 720º, that'll tessellate too. So like 6, like a hexagon.

Although the use of the word "always" in the original proof statement may have primed her to construct a general argument, she nevertheless included statements in her response that were consistent with someone who was operating under the generality requirement. Furthermore, Lexi's generalization beyond the original proof statement towards the end of her argument provides further evidence to suggest that she was considering the generality of the claim being proven. Lexi used the words "so like" to indicate a transition in her argument from statements that referenced features of all possible shapes to statements that provided a specific example to further illustrate, or explain, her mathematical claim. This verbal cue suggests a possible shift in Lexi's use of examples when working on a proof task. Namely, whereas Lexi used examples as her sole justification during the initial interview, she was now using examples to illustrate the broader mathematical relationships that she was attempting to describe.

After completing the proof task, the interviewer asked Lexi to evaluate three provided "student" solutions for the same task and state whether she thought each of them was a proof (see Figure 4.3). In her evaluations, Lexi drew on the ideas that proofs should include examples, particularly visual examples, and should explain why the statement was true. For example, both of these criteria were evoked when she justified her conclusion that Rebecca's answer was not a proof.

Lexi: Um, for this one [Rebecca's answer] I don't think it is a proof, because like… she just said "since I haven't found any example that doesn't work, then all triangles must tessellate," but she never said like, like she gave examples of one's she's tried but not like visual examples I guess.

Interviewer: Okay. So if she had given a picture, do you think that would have made it a proof?

Lexi: Yeah, it'd make it like, more of a proof I guess. And she never said why it worked and why it didn't work or something.

At the end of her explanation, Lexi voiced the idea that proofs should explain why the statement "worked." Although she still drew on the existence of examples when evaluating Rebecca's answer, Lexi now expected those examples to explain *why* the statement was true instead of just providing evidence *that* it was true. The dual criteria that proofs should contain examples and explain why was also evoked when as she justified her conclusion that Terri's answer (Figure 4.3) was a proof.

> It gave, like, a visual example but it also explained how they got that, so they said, like, if you add up the three angles it equals 180 degree line, and that the triangles add up to 180 degrees, which is all true, so.

Lexi's reference to the diagram as a "visual example" raises the possibility that she was viewing it as a specific example instead of viewing it as representative of all possible triangles. However, her justification primarily focused on the explanatory component of Terri's answer that she noted had been missing in Rebecca's answer.

Prove that all triangles tessellate

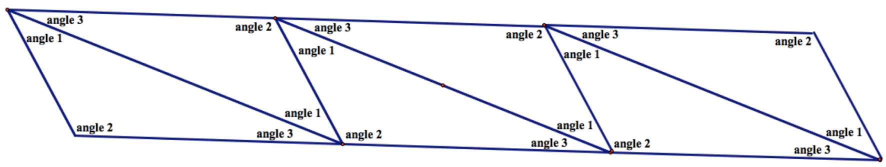
Rebecca's answer

I tried ten different triangles – including obtuse, acute, right, and equilateral – and they all tessellated. Since I haven't found an example that doesn't work, all triangles must tessellate.

Is Rebecca's answer a proof? _____ Yes _____ No _____ Unsure

Terri's answer

If you take 3 copies of a triangle and place them together so that their sides match up but the angles that are touching are different, you'll get a straight line. I know this will work because the sum of the angles in a triangle is 180 and the sum of the angles in a straight line is also 180. So, all triangles tessellate.



Is Terri's answer a proof? _____ Yes _____ No _____ Unsure

Figure 4.3. Two provided student responses for the task: "prove that all triangles tessellate". Lexi viewed Terri's answer as a proof but not Rebecca's.

Second focus interview. Between the first focus interview (after Session 5) and the second focus interview (after Session 11), Lexi was introduced to the idea of generic examples and the use of variables to represent all possible numbers (Session 6). This session was the only time in the design experiment where we talked about number-based proofs; all of the other instructional tasks before and after this session were situated within a geometric context. In the session immediately preceding the second focus interview (Session 11), Lexi worked with her small group members to prove the conjecture, "all squares are similar." This was the first time that students had the opportunity to construct arguments for proof tasks in their small groups; however, we did not discuss how to prove the conjecture as a whole group until Session 12, after the second focus interview.

During the second focus interview, Lexi was asked to prove that the sum of two consecutive numbers was an odd number. Similar to her approach in the initial interview, Lexi began working on the task by trying two smaller examples (1+2 and 3+4) and one "larger" example (20+21). This time, she also wrote the following two statements: "If you add 2 even numbers it's always even. So, therefore when you add an odd and an even you basically add two even numbers and the[n] add the number." In other words, Lexi was thinking about adding two consecutive numbers in terms of adding two even numbers, which she knew would always be even, plus adding the extra one from the second consecutive number. Her verbal explanation, shown below, further clarified this approach.

> Whenever you add an even and an odd, which is what you're going to do when you add consecutive numbers, you just add an even and an even, so like 1+1 or, 2+2 is 4, when you add like two numbers, and so like 1+2, for example, you add 1+1 that equals 2, but then since it has to be consecutive, 1+2 which equals 3.

Although Lexi stated in her verbal argument that the first consecutive number would always be even, her example of 1+1 demonstrated that the pattern also held true when beginning with an odd number. Throughout her verbal explanation, Lexi alternated between making statements about all numbers and using specific examples to help illustrate her point. Similar to her explanation in the first focus interview, she prefaced her examples with the phrases “so like” and “for example” to indicate that they should be viewed as illustrations, or generic examples, to represent the broader mathematical relationships she was trying to describe.

Lexi’s inclusion of statements that pertained to all possible sums of consecutive numbers appeared to be motivated by her conception that proofs should explain why the statement was true more so than her recognition that proofs should adhere to the generality requirement. When asked why she chose to continue writing after trying the three examples, she explained, “if you were to put just three examples like that no one knows what you’re talking about instead of like adding something to it to show what you mean.” In addition to referencing the idea that proofs should explain, Lexi’s justification suggests a recognition that proofs should be written for a broader audience beside herself (need for communication).

Lexi’s conception that proofs should explain was also the main criteria she evoked when evaluating four provided “student” responses for the sum of two consecutive numbers proof task (see Figure 4.4). She thought that Aaron’s answer was a proof “because it explains like, it doesn’t have any examples, but it does explain like how any number, or any like pair of consecutive numbers can be, or is going to be odd.” Even though Lexi’s explanation mentions the lack of examples in the paragraph proof written by Aaron, she ultimately appeared to evaluate it as a proof due to the explanatory nature of the response. She referenced similar criteria when explaining why she did not think Carter's answer was a proof. "You can have, like, different examples and it might not prove all of them once you're, well, it's not proving like the general, like I don't know how to say it, but it's not proving all of them." Lexi's justifications for Aaron and Carter were the first time that she explicitly mentioned the generality requirement when evaluating an argument. Her mention of the generality requirement is especially notable given the fact that the original proof conjecture did not explicitly prime her to be aware of the generality requirement due to the use of the word "any" instead of "always" in the task prompt.

Prove that "the sum of two consecutive numbers is an odd number"

Aaron's answer	*Carter's answer*
If I pick two consecutive numbers, I will always end up with an even number and an odd number. An odd number can be written as an even number + 1. The sum of two even numbers will always be even, so an even number + an even number + 1 is an odd number. So, the sum of two consecutive numbers is an odd number.	5 + 6 = 11 6 + 7 = 13 59 + 60 = 121 103 + 104 = 207 799 + 800 = 1599 So, the sum of two consecutive numbers is an odd number.

Is Aaron's answer a proof?

_____ Yes ______ No _____ Unsure

Is Carter's answer a proof?

___ Yes ____ No ___ Unsure

Figure 4.4. Two student responses for task: "Prove that 'the sum of two consecutive numbers is an odd number.'" Lexi thought that Aaron's answer, but not Carter's answer, was a proof.

Final interview. Lexi completed three proof tasks during the final interview, including repeating the *sum of three odd numbers* proof task that she had completed during the initial interview. After attempting to represent the three odd numbers using variables, Lexi wrote the following statements: "Sarah is correct. I know this because anytime you add two even numbers together it equals an even number. Therefore, I know that an odd number is just an even number plus one." Similar to her work during the second focus interview, Lexi's written argument began by stating a fact about the sum of any two even numbers, followed by a statement that attempted to describe how the stated fact could be used to prove the given conjecture (see Figure 4.7). Lexi's verbal explanation, shown below, provided additional details into her approach:

> Anytime you add two even numbers together, it equals even, two odds equals an odd, no two odds (*mumbles 7 + 7 = 14*) is an even so then once you add another odd number it equals an odd number because you have like, … so you have *(writing)* 7 + 7 = 14 and then you'll have, so then 14 + 7 and that equals 21, cause really 7 is just 6 + 1, so 14 + 6 equals 20, but then you add the 1, so it's always going to be odd.

Notice that her verbal explanation both began and ended with statements that acknowledged and adhered to the generality requirement ("Anytime... so it's always going to be odd"). In the middle, she utilized specific examples to illustrate her thinking and explain it in a way that presumably would make sense to others.

When evaluating five provided student responses for the *sum of three odd numbers* task[11], Lexi drew on the criteria that the student's work should adhere to the generality requirement and "explain why" in order to be considered a proof. For example, she immediately decided that Bonnie's answer (see Figure 4.2) was not a proof because it "just gives you examples of specific numbers, or yeah and so therefore it doesn't explain

[11] These were the same student solutions that she evaluated during the initial interview.

like all odd numbers, it just covers these ones." On the other hand, she thought that Caleb's answer was a proof even though it contained specific circles.

> I think it's like a really general proof, but I think it still is because, like… like looking at these diagrams, this one will make a perfect rectangle, like this and this one doesn't, it has one sticking out every time, but I think it could be a little more specific and like explaining how they did that.

Lexi's explanation of Caleb's answer only focused on the generic aspects of the example—namely, determining whether the sum was even or odd based on whether there was a circle "sticking out"—and not on the specific number of circles in the argument. Whereas Lexi evaluated Bonnie and Caleb's answers differently due to whether she thought it fulfilled the generality requirement, she thought both answers could be improved by adding more text that explained why the statement was true.

Summary. During the initial interview, Lexi both constructed an examples-based argument and used the presence of examples as her criterion for whether the provided solutions should be considered a proof. Additionally, she did not appear to be aware of the generality embedded within the proof statement until it was explicitly pointed out. Lexi continued to reference criterion that focused on the presence of examples in the first focus interview but sought examples that explained *why* the statement was true instead of just providing evidence *that* it was true. In second focus interview, Lexi transitioned to using examples not only as a way of gaining initial certainty regarding the validity of the conjecture but also to illustrate her thinking. After trying a few examples, she constructed the written portion of her argument, which fulfilled the generality requirement through referencing properties of number sums that she knew to be always true. Collectively, Lexi's work during the first and second focus interviews suggests that her transition from using examples to determine whether a statement is true to using examples to illustrate her thinking appears to be the result of a perturbation that occurred due to establishing an intellectual need for causality (understand why) rather than an intellectual need for certainty. During the second focus interview and final interview, she evaluated the provided solutions to proof tasks based on whether they fulfilled the generality requirement and explained why the statement was true. Although both criterion were referenced, her primary emphasis shifted from whether it explained why in the second focus interview to whether it pertained to all cases in the final interview.

Lexi's Understanding the Generality Requirement When Disproving Mathematical Conjectures

Students who have a strong grasp of the generality requirement for universal claims recognize that a) in order for a statement to be true in mathematics, it must hold true for all cases; and b) only one counterexample is needed to demonstrate that a statement is false. Although disproving mathematical claims was not a primary focus of the design research study, students had the opportunity to investigate three false universal claims. The idea of disproving a mathematical statement first arose in Session 3 after students explored the question, "do all regular polygons tessellate?" At this point, students had not yet been exposed to the process of writing a proof, so our conversation focused on the idea that a single counterexample was sufficient to prove that the claim was false. During Session 11, students were asked to conjecture which polygons they thought would be similar to one another (e.g., "all squares are similar"). The classes' list included squares and equilateral triangles (true conjectures) as well as right triangles and rhombuses (false conjectures). Students worked in small groups proving their conjectures for squares, right triangles, and equilateral triangles during Sessions 11 and 12 and then individually proved the conjecture for rhombuses during the final interview.

Session 11. Lexi and her small group members initially thought that the conjecture, “all right triangles are similar,” was true based on Lexi’s voiced assumption that the angles of a right triangle were 45º, 45º, and 90º. After hearing her statement, I perturbed her assumption by asking the group whether *all* right triangles had those angle measurements. Lexi replied, “yep, cause they all have a 90º angle and they all have to… well… actually they might not.” She and the other small group members debated this question back and forth until she suggested that they “do a little drawing for a minute” and then constructed the two right triangles shown below.

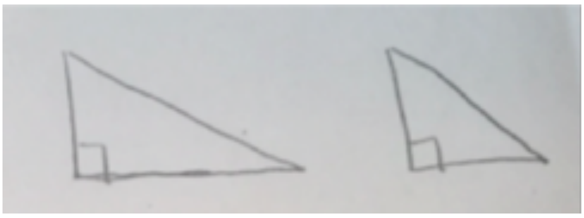


Figure 4.5. Lexi’s drawing when investigating the conjecture, “all right triangles are similar”.

It appears from this drawing that Lexi drew the right triangles so that the vertical sides of the triangles were the same length and the horizontal sides were different lengths. Although this drawing did not help her resolve the group’s question about the angle measurements, it did allow her to voice the conclusion that the two triangles were not similar to one another because the sides were not proportional. Later in the conversation, Lexi decided that the angles did not have to be 45º because, “if you keep stretching this (horizontal side) out, this (top angle) will be bigger and this (bottom angle) will get smaller.” The small group ran out of time before they could construct a written response proving that the claim was false. However, Lexi’s contributions to the group’s investigation of the conjecture demonstrated at least some awareness that a single counterexample was sufficient to disprove the claim. Notably, she concluded the conjecture was false based on the lack of proportional sides *before* deciding that the corresponding angles would not always be congruent, suggesting an awareness that one failed criteria for similar polygons was sufficient for the triangles to be considered not similar.

Final interview. Lexi was asked to explore and then prove the conjecture, "all rhombuses are similar," during the final interview. After introducing the task, the interviewer provided a ruler, protractor, and Geometer's Sketchpad (GSP) document to aid her exploration of the conjecture (see Figure 4.6). The GSP document contained two rhombuses that could be dynamically manipulated; all of the angle and side length measurements were listed on the screen. Lexi began the task by manipulating the bottom rhombus on the screen. Prior to writing down her response for the proof task, Lexi wondered out loud, "A square's a rhombus, isn't it? Yes it is, because it has two parallel sides." At this point, she began writing her written argument:

> This is not a true statement. By the definition of a rhombus, we know that there are two parallel sides and the diagonal angles are the same. Also, all side lengths are the same. The reason this is not true is because a square is a rhombus but a rhombus is not a square.

The overall structure of her argument parallels the structure she used in the *sum of three odd numbers* task: namely, the argument began by stating the validity of the conjecture, followed by statements referencing definitions or properties related to the claim being proven. In the final statement, Lexi provided justification for her assertion that the statement was false. Instead of providing a single counterexample, Lexi provided a class of counterexamples—the similarity between squares and (non-square) rhombuses.

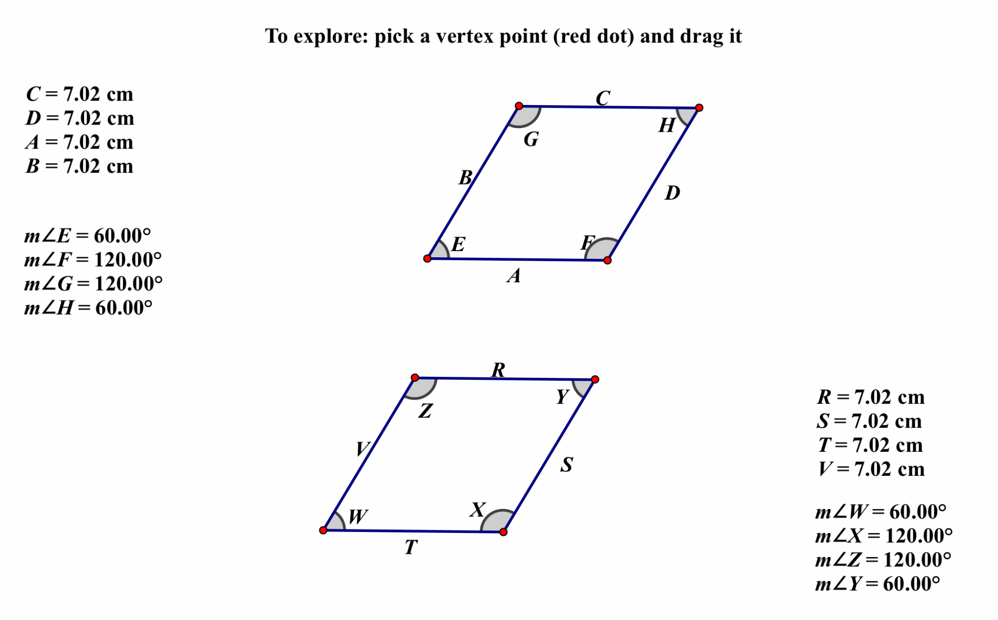


Figure 4.6. Screenshot of the Geometer's Sketchpad document students used to explore the similar rhombuses conjecture.

Lexi's verbal explanation of her response further clarified her justification for why the conjecture was false and provided additional insight into her understanding of the process for disproving a mathematical claim.

Lexi: I don't really know how to explain it but like… A rhombus is not a square because, like, squares have to have the four, all the same angles, so therefore, like a square has diagonals are the same, and they have the same side lengths, and they both have two sets of parallel sides, so therefore, not all polygon or not all rhombuses are similar.

Interviewer: Okay, so how do you prove that a statement in math is NOT true?

Lexi: Well, I didn't include like a formula for it, which I could have done, but um… and I didn't like write the diagrams on here either, but I just found like alternate example [the similarity between squares and non-square rhombuses] that didn't work so I proved that it wasn't true.

Interviewer: Okay. And how many alternate examples do you have to have to prove that it's not true?

Lexi: Just one.

In her first response, Lexi referenced the properties of rhombuses to establish that a square is a rhombus and then explained that not all rhombus were squares given the additional property that squares have “all the same angles,” or 90º angles. This verbal explanation revealed that Lexi was not only attending to the overall appearance of squares and rhombuses, but was also attending to the specific angle measurements of the two shapes when using it as a counterexample.

When asked how to disprove a mathematical statement, Lexi evaluated her own argument based on some of the group’s criteria for good proofs as well as her understanding of how to prove that a mathematical statement is false. Given that we did not construct a proof for a false conjecture as a whole group, it is possible that Lexi thought that the criteria for “good proofs” applied for both arguments that were true and false. Regardless, her follow-up responses indicated an awareness that only one “alternate example,” or counterexample, was needed to prove that a conjecture was false. Lexi’s decision to provide a class of counterexamples (the similarity between squares and non-square rhombuses) can be interpreted in a few ways. One interpretation is that it provides evidence to suggest that she was not fully convinced that a single counterexample disproved the claim. A second interpretation is that she made a conscious choice to go beyond providing a single counterexample so that her explanation also explained *why* the statement was false. Finally, it could be that this class of counterexamples was the first thing she came up with and she did not see the need to provide a single counterexample after she had provided a class of counterexamples.

Summary. Given that Lexi only investigated and proved two false counterexamples (similarity statements for right triangles and rhombuses) towards the end of the study, it is not clear whether there were any shifts in her understanding of how to disprove a mathematical claim over the course of the design experiment. Regardless, her work on the tasks involving false conjectures was consistent with someone who was attending to the generality requirement for universal claims. When disproving the claim that all rhombuses were similar, Lexi provided a class of counterexamples instead of a single counterexample. It is possible that this decision was motivated by her conception that proofs should explain *why* a statement is true or false. However, more evidence is needed to determine the extent to which Lexi drew on the explanatory nature of proofs when constructing responses to proof tasks for false universal statements.

Using Variables to Fulfill the Generality Requirement

There are a range of ways that students can demonstrate understanding of and construct proofs that adhere to the generality requirement, including using variables to represent all possible numbers, justifying statements using definitions or properties that apply to all possible cases, or including the words "always" or "any" in their proof statements. In this section, I specifically look at the way in which Lexi interpreted and used variables in her arguments to represent all possible cases.

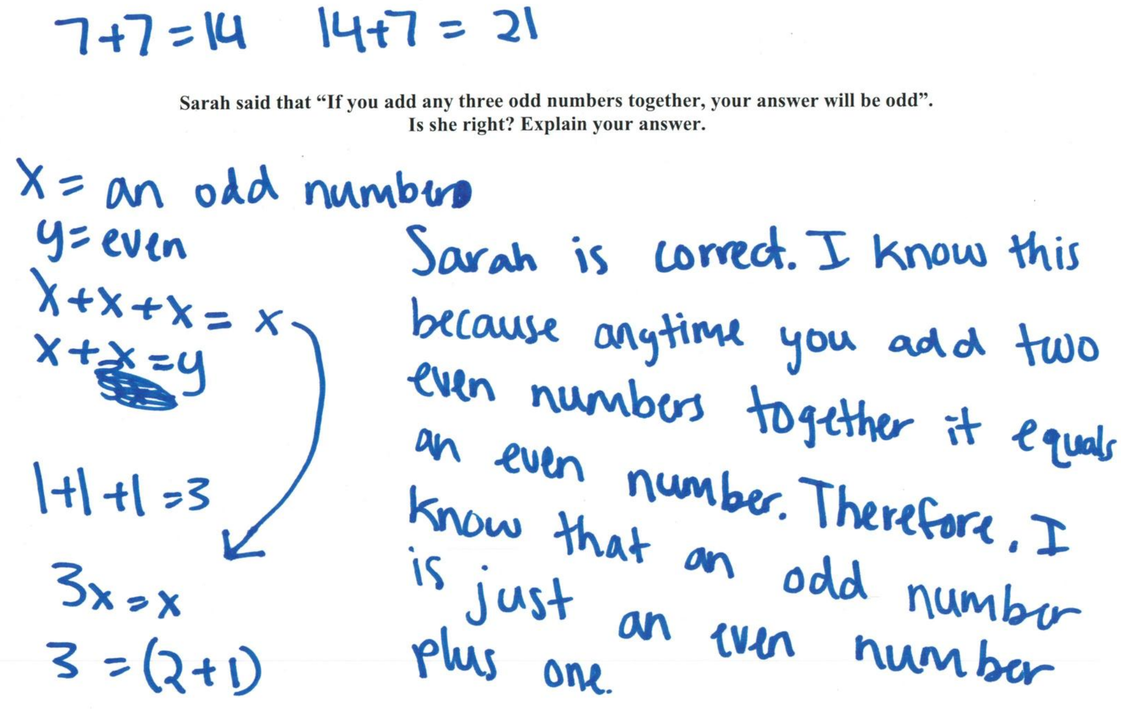
Initial interview. During the initial interview, Lexi constructed an argument for the *sum of three odd numbers* proof task and then was asked to evaluate five provided arguments for the same task. Both Arthur's and Eric's arguments contained variables—Arthur's was a correct algebraic proof and Eric's was an incorrect argument that contained circular reasoning (see Figure 4.2). Lexi thought that Arthur's answer was a proof, "since it had like an equation with it…because you can always plug in numbers and see if it works out or not." This justification suggests that Lexi was interpreting the variables in Arthur's answer as a placeholder that could generate multiple examples, but may not have been thinking about the variable as a representation for *all* possible numbers. She also thought Eric's answer was a proof because it “had an equation and it looks like it worked out.” The fact that Lexi evaluated both Arthur's and Eric’s responses as proofs suggests that she was not attending to the details within each answer, but rather was evaluating them based on their use of variables. Notably, Arthur and Eric's answers were the only two solutions that she thought were proofs; she was unsure whether the paragraph proof (Debbie) and examples-based answer (Bonnie) were proofs and thought the generic example (Caleb) was definitely not a proof (see Figure 4.2).

Session 6. Recall that after I introduced the generality requirement in Sessions 1–4 and problematized examples-based arguments in Session 5, I introduced the idea of using variables to represent all possible numbers in Session 6. First, I led students through a number trick and then showed them a generic example[12] to prove the number trick would always work. When discussing the generic example, I emphasized the idea that students should not count the specific number of circles on the page but instead should focus on the overall structure of the pictures by imagining a larger number of circles that we could not physically represent on paper. Once we talked through the different aspects of the generic example as a whole group, I introduced the idea of completing the number trick using a variable to represent our number. By using a variable, I explained, we could prove that the answer would always be even no matter what number we selected. Next, I asked students to work on the following task in their small groups: “9×11 equals 1 less than 10^2, 3×5 equals 1 less than 4^2. Is this a coincidence? If it is not a coincidence, how could you prove that this will always work?”

[12] The number trick and description of the generic example can be found at https://nrich.maths.org/2280.

Lexi's group quickly decided that the pattern was not a coincidence. During their initial conversation, Lexi conjectured that the relationship could be generalized even further. “However many they're apart is going to be that many less than the middle number squared.” She also provided the following example to help illustrate her idea: “like if it's 3×7, the middle number is 5, so 3×7 is 21, 5 squared is 25, so it's 4 less because 3 and 7 are 4 apart.” Although Lexi's conjecture is not true for all possible combinations of numbers, it shows that she was attending to the overall structure of the numbers in the examples instead of deciding it was true based on trying a few specific cases. Amanda, one of Lexi's small group members, initially proposed the equation $n(n+2) = n^2 - 1$ to represent the number relationship, based on her recognition that the two numbers on the left hand side were two apart. Lexi responded to Amanda’s proposal by suggesting they plug in a number to check the equation. During this process, Lexi also questioned whether they would need to use two different variables to represent the two numbers on the two sides of the equation. Although Amanda was later able to correct the right side of the equation to $(n+1)^2 - 1$, it was not clear from Lexi's involvement in the small group discussion whether she ever understood why only one variable was needed to represent the given numerical relationship. While plugging in specific numbers is certainly a valid approach for verifying their equation, Lexi's question about using different variables to represent the two numbers suggests that she was viewing the variables as placeholders and may or may not have been attending to the relationship between the three numbers in the statement.

Final interview. When completing the *sum of three odd numbers* proof task during the final interview, Lexi attempted to use variables to represent the three odd numbers but ultimately neglected that route in favor of proving the statement using properties of even and odd numbers (see Figure 4.7).


7+7=14 14+7 = 21

Sarah said that "If you add any three odd numbers together, your answer will be odd".
Is she right? Explain your answer.

x = an odd numbers
y = even
x+x+x = x
x+x =y
1+1+1 =3
3x = x
3 = (2+1)

Sarah is correct. I know this because anytime you add two even numbers together it equals an even number. Therefore, I know that an odd number is just an even number plus one.

Figure 4.7. Lexi's written argument for the sum of three odd numbers proof task during the final interview.

Notice that Lexi used "*x*" to represent odd numbers and "*y*" to represent even numbers. This decision mirrors the comments she made in Session 6 about using two different variables to represent the two numbers and suggests that either she did not realize that even and odd numbers could be written in terms of each other or was unable to represent this relationship on her own. Additionally, Lexi represented the three odd numbers with the same variable even though she understood that the three odd numbers in the proof statement did not have to be the same number. Although she recognized that you could use variables to represent the three odd numbers, she acknowledged during the interview that, "I don't quite know how to do that with this one."

After completing her argument for the *sum of three odd numbers* proof task, she was asked to evaluate five student solutions. Her evaluations of Arthur and Eric's responses (see Figure 4.2) showed an increased awareness of the generality requirement; however, her justifications focused on plugging in numbers for the variables to determine whether she thought the student's work was a proof.

> I think [Arthur's argument] is a proof because, I was, like, doing different examples in my head and, like, any time you, like any number, since you're multiplying by 2 that doesn't really affect it at all really, but like once you add the 1 it'll affect it, cause any number times 2 is going to be an even number, cause it's a multiple of 2, so.

This justification suggests that Lexi continued to approach variables in terms of verifying the equations by plugging in specific numbers. However, she extended beyond examples when she stated, "any number times 2 is going to be an even number." One way to interpret this statement is to consider it evidence of her drawing on the definition of variables as generalized numbers when reading Arthur's argument. Another interpretation is that this statement occurred only because it drew on her prior knowledge of even and odd numbers, and more specifically, on properties she referenced in her own written argument. This latter interpretation is further supported by her suggestion that Debbie's paragraph argument could be improved by adding variables. "I think it could have, like, had an examples with it, not like specific ones, just had, like, variables representing different numbers I think would have helped it." This statement suggests a view of variables as placeholders, since variables can represent "different numbers".

Unlike during the initial interview, Lexi was unsure whether Eric's answer should be considered a proof. “Well, I, like, for *x*, I did 1, (for) *y* I did 2, and (for) *z* I did 3, so 1 + 2 + 3 is 6 and then 3 + 6 is 9 + 9 is 18, so then *a* would have to be 6, and 3*a* is not odd, so I don’t really think it is.” Lexi recognized that Eric’s answer began by stating that *a*, *b*, and *c* were whole numbers, which meant that they did not all have to be odd numbers like the original proof statement required. On the other hand, the counterexample she stated above did not appear to provide enough evidence for her to conclude that the statement was definitely not a proof. She explained her lack of certainty by saying, “I mean it looks like it would be a proof, cause it looks like, very detailed and stuff, but I don’t, I don’t really think it is.” This response highlights one of the challenges students experience as they are beginning to learn about proofs—that is, learning how to look beyond the surface features of an argument to determine whether it proves the given statement for all possible cases.

Summary. Although Lexi demonstrated growth in her understanding of the generality requirement and recognition of the limitations of specific examples, she struggled throughout the study with appropriately using variables in her arguments. This difficulty appeared to stem, at least in part, by her understanding of variables as placeholders instead of viewing them as varying quantities. As a result, she struggled to use variables meaningfully when working on proof tasks and consistently evaluated the provided student solutions that contained variables based on their ability to generate examples rather than on their adherence to the generality requirement. Lexi's limited understanding of a variable is especially notable given that she was currently learning

Algebra 2 concepts and had experienced continued success in her prior and current math classes.

Discussion

This study sought to understand how one student, Lexi, developed understanding of the generality requirement while participating in an introduction-to-proof design study. During the initial interview, Lexi constructed an examples-based response that was consistent with the responses given by a significant percentage of middle school (Knuth et al., 2009) and honors Algebra 1 students (Healy & Hoyles, 2000) on similar number-based tasks. Additionally, her response was reasonable given that she had not yet learned about formal, deductive proofs and empirical evidence regularly serves as justification in everyday life (Harel, 2010). As Lexi began thinking about proof in terms of persuading others as well as herself regarding the validity of the statement, she adopted the conception that proofs should explain *why* the statement is true (the need for causality). The need for causality appeared to perturb her understanding of what constitutes a proof, evidenced by a shift in her approach when constructing and evaluating responses for proof tasks in later interviews, and ultimately appeared to facilitate her transition from using examples as justification for a claim's validity to using examples as illustration of her written statements that adhered to the generality requirement. Lexi's conception that proofs should explain, and the subsequent shift away from examples-based responses as evidence of the validity of a claim, aligns with other studies (e.g., Bieda & Lepak, 2009) and provides further support to de Villiers' (1990) call for secondary teachers to utilize the explanatory feature of proofs as a way of positioning them as meaningful activity in the classroom. As researchers, teachers, and curriculum developers seek to find ways to improve the way in which proofs are first introduced to students, they should seek tasks and opportunities for classroom discussions that emphasize the explanatory feature of proofs.

Lexi’s conception that proofs should explain “why” not only appeared to influence the types of arguments she constructed and the ways that she evaluated provided arguments, but also may have contributed to her use of a class of counterexamples when disproving the conjecture, “all rhombuses are similar.” When disproving the claim, Lexi referred to the similarity between squares and (non-square) rhombuses instead of providing a single specific example that disproved the conjecture. The desire to understand why a mathematical statement was false was similarly found in a study by Komatsu (2010) with a pair of fifth graders, who renewed exploration of the task after being presented with a specific counterexample in order to “overcome the problem” by revising the scope of the proof claim (p. 5). Lexi’s class of counterexamples raises a curricular question for secondary teachers: namely, to what extent should they emphasize the use of a single counterexample to disprove a universal mathematical claim? On the one hand, accepting answers such as Lexi’s potentially mask the ability to assess students’ understanding that only a single counterexample is needed to disprove a claim (Buchbinder & Zaslavsky, 2013). On the other hand, only accepting a single counterexample potentially loses the opportunity students have to make sense of the underlying reasons for why the mathematical claim is false. In this study, I intentionally chose to prioritize developing students’ understanding of the overall concept of proofs over their understanding of the “proper” form of proofs. Given this particular focus, Lexi and other students’ choice to provide a class of counterexamples was not interpreted as a lack of understanding of the generality requirement for false universal statements.

However, more research is needed to determine at what point teachers should introduce students to different aspects of the form of proofs and to what extent they should hold students accountable to writing proofs in the “proper” form.

Whereas Lexi demonstrated significant growth over the course of the study in terms of her understanding of generality when proving true and false universal mathematical claims, she struggled throughout the study to use variables when constructing or evaluating arguments for number based tasks due to her use of variables as placeholders. Lexi was not the only student in the study to struggle using variables in her arguments for number-based proof tasks, as only 3 of 10 study participants attempted to use variables when proving the *sum of three odd numbers* task during the final interview and all three students’ variable use ignored the relationship between even and odd numbers. Students’ tendency not to use variables, or to use them incorrectly, when constructing arguments for number-based proof tasks in the present study parallel the findings from Healy and Hoyles (2000) and raises larger questions as to why only a small number of successful algebra students would use variables to prove number-based claims. One possible explanation for these findings is that the students do not view variables as a way of representing a relationship for all possible cases. Subsequently, would developing students’ understanding of variables as varying quantities increase the likelihood of students successfully constructing algebraic proofs, or would students continue to use other methods of proving the statement due to the increased explanatory power of non-algebraic arguments (Healy & Hoyles, 2000)? As number-based conjectures offer an opportunity for students to engage in proving universal claims at the beginning of high school Geometry courses, and have the potential to support students’ understanding of both algebra and proof (Martinez et al., 2011), more research is needed to understand ways to support students in developing understanding of variables as varying quantities and, more broadly, to support students' view of variables as representing an infinite number of values, instead of multiple, but finite, number of values. Specifically, what kind of tasks help to perturb students' understanding of variables as placeholders and facilitate a transition to viewing variables as varying quantities?

When assessing Lexi's understanding of the generality requirement, I did so through analyzing her construction of arguments for true and false universal claims as well as her evaluation of provided student solutions. I did not, however, fully assess the boundaries of her understanding of the generality requirement. For example, it is still unknown whether she recognized that a proof demonstrating the diagonals of a rectangle are congruent also proves that the diagonals of a square are congruent, since squares are a subset of rectangles, but does *not* prove that the diagonals of a parallelogram are congruent, since not all parallelograms are rectangles. Given that Lexi only worked on proof tasks involving universal claims, it is additionally possible that she might have over-generalized when analyzing a proof of a particular statement. More research is needed to assess whether having students only or primarily engage in proving universal claims during the introduction to proof unit results in them making over-generalizations when they encounter proof tasks about particular statements, which are the more common types of proof tasks students experience as homework problems (Otten et al., 2014).

Conclusion

The generality requirement represents one of the most important, yet challenging aspects for students to understand when developing their understanding of and ability to construct proofs. Whereas students' difficulties with the generality requirement are well-documented (e.g., Reid & Knipping, 2010; G. J. Stylianides, Stylianides, & Weber, 2017), this study provides a contrasting view of a student who experienced success, across multiple proof tasks, in demonstrating understanding of the generality requirement. Lexi's learning process lends support to the idea of leveraging the explanatory feature of proofs with beginning proof students (de Villiers, 1990); in particular, her experience provides support for the idea that the explanatory feature of proofs could be used to help motivate an intellectual need for proofs (Harel, 2008) and support students' transition away from examples-based responses as the sole justification for a claim's validity. Although Lexi stopped using examples as a way of demonstrating that a statement was true, she continued to use examples during the proving process as a way of convincing herself of the claim's validity and illustrating her mathematical ideas when explaining statements that adhered to the generality requirement. This latter use of examples could represent a possible intermediate step in students' transition from empirical arguments to deductive arguments. Given the lack of follow-up interviews with Lexi, it is not known whether she would continue to use examples to illustrate her thinking as she became more comfortable with the proving process. Regardless, the case of Lexi provides evidence to support that secondary students are capable of developing nuanced understanding of the use of examples in the proving process and can recognize both their value in explaining their thinking as well as their limitation in proving that a statement is always true.

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CHAPTER 5

Implications and Recommendations for Future Research

This study was the first iteration of a design study that aimed to introduce students to proof by developing their understanding of the generality and purpose of proof. Implications of this study are limited by the small number of participants, the participants' above-average understanding and success in Algebra, and the fact that the instructional sequence has yet to be refined over multiple iterations. Nonetheless, the findings offer multiple implications for classroom teachers and mathematics education researchers and suggest directions for future research. In the following sections, I summarize the key implications of each article presented in chapters 2–4 and then propose recommendations for future research based on the findings of that article. Next, I propose additional opportunities for future research based on aspects of the instructional sequence that fell beyond the scope of the three articles.

An Analytic Framework for Assessing Students' Understanding of Proof Components

In chapter 2, I proposed a framework designed to assess students' understanding of the different components of proof, including their understanding that proofs should attend to the generality requirement, include mathematical justifications, and make explicit connections between the diagram (if present) and the written argument. I asserted that one advantage of analyzing students' arguments on proof tasks using the proposed framework is that it does not require a judgment of whether their overall argument should be considered a proof. This is especially valuable when assessing work produced by students who are in the beginning stages of learning how to construct proofs, as it allows for classroom teachers and researchers to assess students' understanding of proof in a way that acknowledges the understanding they demonstrate, even if they are not able to construct a complete "proof", and identify ways that the students can improve their understanding and construction of proofs in the future. From a research perspective, one advantage of this framework is that it aligns with Styliandies' (2007) definition of proof, which is commonly cited by others who conduct research on proof at the K-12 level. Thus, this framework allows for researchers to assess students' work on proof tasks using a framework that aligns with their definition of proof. Finally, use of this framework has the potential to shift the conversation within the research community about what students know and can do in regards to constructing proofs. For example, coding the work students produced on the three proof tasks in the present study resulted in illuminating multiple aspects of proof that the students consistently demonstrated evidence of understanding, such as attention to the generality requirement and providing mathematically accurate justifications, even though many of their arguments would not necessarily be considered complete proofs.

In regards to implications for the classroom, the inclusion of multiple illustrations provides examples of ways in which students could demonstrate understanding (or not) of a particular aspect of proof, as well as the possible arguments that students could produce relatively early in the proof learning process. In my study, I chose not to expect that students' arguments contained all of the elements of a proof from the very beginning; this decision opened the door for students to engage in richer proof tasks (e.g., tasks involving universal claims or statements where the validity was not immediately apparent) earlier in the learning process. In contrast, teachers who require students to construct complete and valid proofs at the beginning of the learning process must necessarily choose tasks that are relatively simple and straightforward so that students are able to meet their expectations. For example, a typical proof task during the introduction to proof unit would be the following: “Given: C is the midpoint of $\overline{AD}$, Prove: $x = 6$” alongside a diagram where $\overline{AC} = 4x$ and $\overline{CD} = 2x + 12$.” This task is fairly straightforward since it requires students to apply the definition of a segment midpoint and then solve a simple algebraic equation. Straightforward tasks, such as the example above, also tend to involve proving statements that obvious to students, which can cloud the purpose of constructing a proof and result in an over-emphasis on form of the proof rather than its function. In the example shown above, a deductive argument is unnecessary since a student could easily “prove” that $x = 6$ by replacing x in both expressions with 6 and then showing that the two segments have equal lengths. By shifting expectations in regards to the quality and completeness of students’ work on proof tasks, the teacher is able to select tasks that reveal the purpose of proof and position the proving process as a meaningful activity.

Using this framework to assess students’ understanding of proof components requires that teachers also shift their approach to teaching proof in the classroom in order to maintain alignment between the way that proofs are taught and assessed. Traditionally, proof instruction in the classroom has placed a significant emphasis on the form of proofs (Martin & McCrone, 2003; Schoenfeld, 1988), with some teachers assessing students’ proofs according to whether they placed each statement and reason in the “proper” order (Schoenfeld, 1989). In contrast, my proposed framework places minimal emphasis on the form of proofs and instead prioritizes an understanding of the generality and justification requirements. One possible explanation for the difference in focus between my framework and classroom instruction from prior studies could be the timing of the study;

in other words, because the prior studies observed teaching that occurred later in the year, it makes sense that the teachers would place additional emphasis on the form of proofs than someone who was teaching an introduction-to-proof unit. Nontheless, this gap between what is prioritized in my framework and what has been prioritized in traditional classrooms offers multiple opportunities for future research. For example, what kinds of supports do teachers need in order to be able to interpret and use my proposed framework to assess students' arguments? Would teachers need to shift their approach to teaching proof before using this framework to assess students' work on proof tasks, or could the use of the framework help to shift their instructional approach? What impact, if any, would using this framework have on students' beliefs about their ability to construct proofs or their broader understanding of the purpose of proofs? Additionally, future research is needed to understand at what point teachers should begin to introduce and hold students accountable for constructing arguments that contain the logical structure and form found in mathematical proofs.

Engaging Students in Tasks Involving Universal Claims in an Introduction-to-Proof Unit

In chapter 3, I analyzed my design decision to use universal claims in the instructional sequence in terms of the opportunities it provided for reasoning-and-proving, statements *about* reasoning-and-proving, and developing an intellectual necessity for proof. I provided evidence to suggest that the use of universal claims allowed for students to engage in a variety of reasoning-and-proving activity, including some not seen in typical classrooms (e.g., constructing a diagram or revising a proof) and provided opportunities for the teacher (myself) to facilitate discussion *about* reasoning-and-proving. I also proposed possible revisions to the tasks based on my analysis of students' understanding when engaging in each task. For example, some students interpreted the claim, "all rhombuses are similar" to mean selecting an "arbitrary" rhombus, dilating it to form the second rhombus, and then determining if the two rhombuses are similar. Future research should consider the types of supports, including technological supports (e.g., dynamic geometry software), that could be developed and implemented to increase students' understanding of the scope of universal claims.

Although I intentionally chose to use universal claims (e.g., "all squares are similar") in all of my instructional tasks in order to emphasize the generality requirement, I did not provide students with any opportunities to consider particular claims (e.g., "square ABCD is similar to square EFGH"). Subsequently, it is not clear from my dissertation data whether the students would have recognized the scope of a particular claim or whether they would have over-generalized a statement to be about all possible cases instead of a specific case or finite number of cases. Future iterations of the instructional sequence should include tasks designed to assess students' understanding of the boundaries of a claim by asking students to prove both a universal claim and a particular claim and then make comparisons between the two claims. Alongside assessing students' understanding of the scope of a claim, future research should also analyze the similarities and difference between the proof approach students take in tasks involving universal and particular claims. For example, does the student produce the same proof for the two claims, or do they recognize that a particular claim could also be proven through other means, such as proof-by-exhaustion or measuring the angles and sides of the shape?

Throughout the instructional sequence, I regularly incorporated opportunities for students to reflect and talk *about* the reasoning-and-proving process. I captured some of the *about* reasoning-and-proving instances during the study in chapter 3 but did not draw any links between the use of this strategy and the extent to which students took up these ideas about proof and incorporated them into their subsequent work. Future analysis of the data collected during my dissertation study should expand upon the statements *about* reasoning-and-proving and specifically analyze the ways in which these statements were reflected (or not) in students' work. For example, after introducing the idea that students' statements in their argument should adhere to the generality requirement and include a mathematical justification for each claim, to what extent (and when) were these ideas evident in student's subsequent work on the proof tasks? Additionally, future research could continue to investigate the ways in which statements *about* reasoning-and-proving can be incorporated into proof instruction, with the goal of developing guidelines for teachers regarding the fundamental aspects of proof that should be the object of focus in these statements. In what ways do statements *about* reasoning-and-proving parallel prior research on problem solving strategies, and in what ways are the statements unique to constructing proofs?

Developing Understanding of the Generality Requirement of Proofs: The Case of Lexi

In chapter 4, I described how one student, Lexi, developed understanding of the generality requirement over the course of the design study. In this article, I described how Lexi's conception that proofs should explain why appeared to support her transition from producing examples-based arguments to deductive arguments. Lexi's experience provides further support for the suggestion that the explanatory feature of proofs should be emphasized in the classroom to help develop students' conception of proving as a meaningful and central part of mathematics (de Villiers, 1990; Hersh, 1993). Future research should investigate what features of proofs students find explanatory, including whether the form or formality of the proof influences whether it is seen as explanatory. What are some other strategies for supporting students' view of proof as a meaningful activity in the classroom? More direction is also needed regarding the balance high school teachers should strike between expecting students to construct formal deductive arguments versus less formal, deductive arguments that may be more convincing and explanatory for students.

A second implication from the case of Lexi relates to the way that her understanding of the role of variables impacted the ways in which she used them in number based proof tasks. Given that Lexi was a strong algebra student, it is likely her difficulty in using variables in a way that captures the relationship between multiple numbers in a task is shared among many high school students. In a future study, I intend to investigate my design conjecture that developing students' ability to generate and use generic examples to explain why universal mathematical claims are true will support their understanding of the generality requirement and facilitate their use of variables when proving elementary number theory conjectures. During this design study, I will pay careful attention to students' explanations of their generic examples to look for instances where they are focusing on the structure of the numbers and ability to see relationships between the numbers in the mathematical conjecture as evidence of readiness for the use of variables to algebraically prove the given conjecture. During the final interview with participants, I will also assess whether students' understanding of generic examples for number tasks translates into improved understanding of diagrams used in geometry proof tasks. In other words, when presented with a proof demonstrating that the diagonals of a rectangle are congruent, would students recognize that a) the proof demonstrates the statement is true for all rectangles, and not just the rectangle shown in the diagram, and b) that the proof also demonstrates that the diagonals of a square are congruent (since squares are also rectangles), but does *not* prove that the diagonals of a parallelogram are congruent.

Additional Opportunities for Future Research

Given the large scope of topics that I covered in the introduction-to-proof unit, it was not possible for me to investigate all of my design conjectures during retrospective analysis. Additionally, my dissertation study did not always provide enough experiences related to a design conjecture to be able to analyze whether the instructional decision resulted in the types of learning I had conjectured would occur. For example, in sessions 9–12 I investigated the conjecture that having students construct definitions for key mathematical terms would support their use of the definition in related proof tasks. Specifically, students constructed and refined a definition for similar polygons and then used their definition to prove their conjectures about classes of polygons that were similar to one another (e.g., "all squares are similar"). Data from this study provided positive, albeit very limited, evidence to suggest the potential benefits of engaging students in this sequence of tasks in terms of both their understanding of the associated definition and its use in proofs. Future research should investigate in more detail the potential relationship between constructing definitions and students' use of those definitions in related proofs. For example, does the process of considering the boundaries of a class of objects when constructing a definition help students consider the boundaries or scope of a claim to be proved? Which geometric terms are commonly used in worthwhile proof tasks and, as a result, are worthy of spending the additional class time needed to develop and refine the associated definition? To what extent does engaging students in constructing definitions support their understanding and use of other geometry definitions that they did *not* construct?

Concluding Remarks

This study demonstrated both the potential advantages and challenges for introducing students to proof using tasks involving universal claims and an instructional sequence that aimed to develop students' understanding of the generality and purpose of proof. The results showed that universal claims can provide rich reasoning-and-proving opportunities, help motivate a need for proof, and develop students' understanding of the generality requirement; however, more work is needed within the instructional sequence to support students' understanding of the scope of universal claims (e.g., what does it mean for all squares to be similar to one another?). Additionally, the results provided further evidence that emphasizing the causality purpose of proof can facilitate students' transition from empirical to deductive arguments. Future iterations of the instructional sequence, as well as smaller studies targeting particular aspects of the instructional sequence, are needed to make claims about the link between the instruction and students' learning, further flesh out the instructional sequence, and develop a learning trajectory for how students might develop understanding of the generality and purpose of proof.

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APPENDIX A: EXPANDED INSTRUCTIONAL SEQUENCE

This appendix contains a detailed list of the overall learning goals for each session, the tasks that the students worked on (closed circle bullet points), a brief summary of student approaches to the task, and notes for future iterations (open circle bullet points). I also include a brief description of the main ideas that surfaced in the class discussion and a few student strategies on select tasks. Students were given a composition notebook at the beginning of the study so all of their work and responses to the writing prompts were recorded in the provided notebooks.

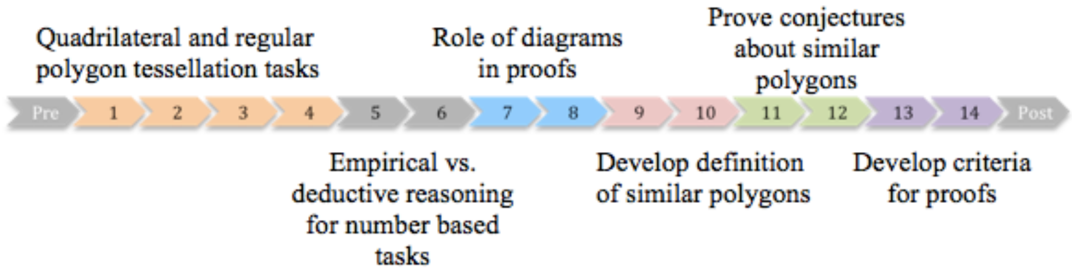

Figure 1. Overview of the instructional sequence.

Session 1:

<u>Learning goals:</u> Observe student dynamics when working in small groups, begin tessellation task.

- Intro task: give each group 15 unifix cubes, 3 perspective images (example shown below), and a page describing the front-right-top views of a 3D object. One person in each group is the "builder" and is not supposed to see the images. The other group members are supposed to describe the images and tell the builder how to put the pieces together. During debrief afterwards, we talked about what types of information was helpful for the builders and general ideas about communicating with each other in small groups.
    - o This was a great task and fairly challenging for some groups!

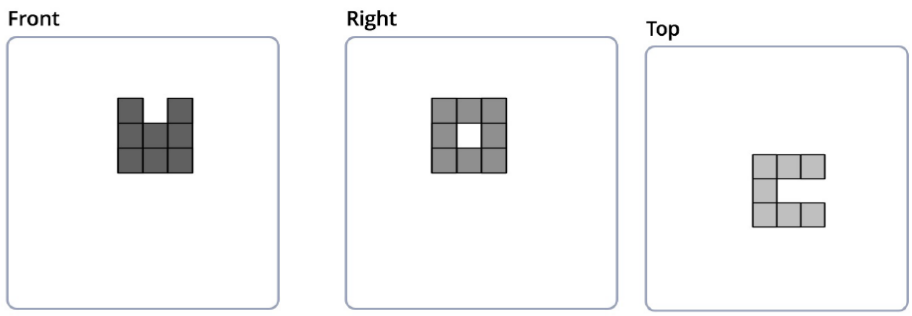

Figure 2. Sample front, right, and top perspective images given to groups as the initial team building activity.

- Main task: Review quadrilaterals definition, introduce tessellations, then pose question: Do all quadrilaterals tessellate? Hand out ~6 different sets of quadrilaterals to each group (familiar quadrilaterals as well as one irregular, convex quadrilateral).

- Writing prompt: "How confident are you that all quadrilaterals tessellate? How would you explain to a friend your answer?" After all students recorded their answer, I had a few share their thoughts. Then I held up a concave polygon (Figure 3) and asked whether they thought it would tessellate. All students thought it would, so I asked them how they knew without trying it?

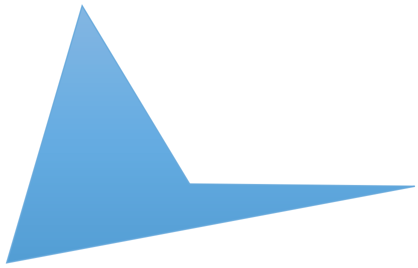

Figure 3. Convex quadrilateral I held up at the end of Session 1.

Notes:

- When tessellating the quadrilaterals, students paid a lot of attention to the sides of the shapes but not the angles.
- Students initially did a lot of guessing and checking, but became a little more systematic in their approach towards the end of the session.
- They were all pretty confident that all quadrilaterals tessellated after figuring out how to tessellate the irregular convex quadrilateral.
- ***Next time:*** Don't ask students to confirm that the shape at the end is a quadrilateral before asking them if it tessellates.

Session 2:

Learning goals: Help students become more systematic in their approach to tessellating quadrilaterals; hopefully shift their attention to the angles as well as the sides.

- Task: In each group, write a set of "step-by-step" directions for how to tessellate any quadrilateral. Each group was given some of the quadrilaterals from day 1 as well as a concave quadrilateral and another irregular convex quadrilateral. Students were allowed to label parts of the quadrilaterals if they wanted (e.g., top side, bottom side) to help in writing their directions.
    - I had initially planned for additional activities in this session, but writing the directions took the entire time.
    - Some of the groups did start to think about the angles ("mismatch the angles"; "same slant/opposite angles"), but there was still more attention placed on the sides than the angles. All of the groups had some variation

of a "flip the shape and match the same sides together" rule for tessellating quadrilaterals.
  - This was a great task to help them become more intentional in their tessellations. However, I imagine that you wouldn't want to do session 1 and 2 in the same day, as it might be a bit long for them to work on the same task.
  - There were 3 students absent: Wilson, Lauren, and Lexi. As a result, the small groups were mixed up somewhat from their original configuration.

Session 3:

<u>Learning goals:</u> Create some uncertainty regarding students' confidence that all quadrilaterals tessellate and motivate a need for understanding *why* the statement is true.

- Task 1: Put one student who was absent from session 2 into each group. Give the groups a new irregular quadrilateral and have the remaining students tell the absent student how to tessellate the quadrilateral using their directions from session 2. Students can revise or add to their directions if they want. Debrief as a whole group afterwards.

- Task 2: Ask class if they know of other polygons that they think would always tessellate. After eliciting their initial ideas, I introduced the term regular polygons. After stating the definition and examples of shapes that are regular polygons (e.g., squares, equilateral triangles), I asked students if they thought all regular polygons tessellate. All students said yes, so then I passed out regular hexagons to each group. Again, ask students if they think all regular polygons tessellate. Then I passed out regular pentagons, septagons, and octagons (1 for each group) to "speed up" the process. Then we discussed it as a whole group – do we still think that all regular polygons tessellate? I introduced the idea of counterexamples and the fact that we only needed one counterexample to disprove the statement during this discussion.
  - We had a great discussion at the end of the regular polygons tessellation task, with students posing some conjectures about how to revise our conjecture about regular polygons – the key ideas that came out are listed in session 4.

- Writing prompt: Do you still think that <u>all</u> quadrilaterals tessellate? If no, explain why. If yes, is there something special about quadrilaterals that make it so that they will always tessellate?
  - Students all still thought that quadrilaterals always tessellated, but were generally unsure why this was the case. They definitely seemed at this point ready to understand *why* the statement was true instead of just being satisfied in their belief that it was true.

Session 4:

<u>Learning goals:</u> Recap sessions 1-3 and explain what's special about quadrilaterals that make it so that they will always tessellate. Then assess students' current understanding of proofs through analysis of "student" work.

- Introduction: Reviewed what we had done in the first three sessions, including how their strategy for tessellating the quadrilaterals had shifted from "matching the sides" to more detailed strategies:
    - "Flipped and mirrored it"; "rotating 180 degrees"
    - Labeling the sides of the shapes
    - Megan: Match up each side but put opposite angles together
    - Lauren: "the angles meet in the middle"

- Next, I recapped some of the claims and counterexamples that came up in session 3. During this recap, I made sure to highlight the important characteristics of the arguments/counter-arguments, followed by talking about the general pattern of making a claim – providing support for/against claim, then revising claim and repeating.
    - Clay: (directions for tessellating) "Make a square…Try to make another quadrilateral with two matched sides"
    - Heather: "That's just for this shape, it's not for all quadrilaterals…different quadrilaterals make different shapes, not just a square"

    - Amanda: "After a shape has 5 or more sides, then maybe the angles become too wide because they have too many sides"
        - Wilson: "The hexagons worked"
    - Lexi: "Maybe like after 4 sides, the sides have to be even, because 5 and 7 didn't worked"
        - Arin: "The octagon didn't work"

- Then I revealed why quadrilaterals always tessellated using a picture with capital letters in the four angle measurements to show that the sum of the 4 angles always added up to 360 degrees, which is the number of degrees in a circle. I also showed them a picture with the regular polygons, including their angle measurements, to highlight why regular hexagons, but not regular pentagons or octagons, tessellated.

- Writing prompt: Why does talking about the properties of angles in quadrilaterals explain, or prove, why ALL quadrilaterals tessellate?

- Task: Towards the end of the session, I gave the groups four sample "student" solutions for the task: "When you add the measures of the angles in any triangle, your answer is always 180 degrees." The groups were then supposed to discuss the solutions and determine whether they thought each one was a proof.

- The student solution below (Frank) brought up a great discussion afterwards, as the groups were split on whether it should be considered a proof.
- *Note:* None of the provided solutions were proofs since the students had not yet learned about alternate interior angles. However, Frank's solution (below) came the closest, especially if you remove the word "obtuse" from the work.

Frank:

(a) I tore up the angles of the obtuse triangle and put them together (as shown below).

Since the angles formed a straight line, I know the sum of the angles is 180 degrees. So the claim is right.

Figure 4. Sample solution I gave students to evaluate during session 4.

- ***Next time:*** change the picture used in Frank's solution so that it's clearer that the three angles were "torn" off of the original triangle.

Session 5:

Learning goal: Cast doubt on the idea of using examples to determine whether a statement is always true.

- Task: Students completed the circle and spots task (Figure 5) and then were introduced to a "fun fact", also known as the Monstrous counterexample (Figure 6).

Place different numbers of spots around a circle and join each pair of spots by straight lines. Explore the relationship between the number of spots and the number of different regions in the circle.

When there are 15 spots around the circle, is there an easy way to tell for sure what the greatest number of non-overlapping regions into which the circle can be divided?

Figure 5. The Circle and Spots problem (Stylianides & Stylianides, 2009, p. 329)

The expression 1 + 1,141n2 never gives a perfect square.

People used computers to check this expression and found out that it does not give a square number for any natural number from 1 to 30,693,385,322,765,657,197,397,207.

BUT

It gives a perfect square for the next number!

Figure 6. The Monstrous Counterexample (Stylianides & Stylianides, 2009, p. 330)

Notes:

Some of the groups struggled with the circle and spots task because they wanted to jump to some of the larger examples instead of starting small and generating a pattern. They were fairly engaged in the task though and wanted to keep trying to work on it at the end of the session.

- Although I think the problems were surprising for the students, I'm not confident that many of them needed additional convincing that examples weren't sufficient to determine that a statement was always true.

- Writing prompt: Were you surprised by any of the problems we talked about today? Explain. How do these problems relate to our conversations about proving mathematical statements?

Session 6:

Learning goals: Introduce the idea of generic examples and variables to represent all possible numbers.

- Task 1: Began by having students work through the following "number trick":

  Choose any number. This is going to be your particular number for this proof. Square your chosen number.
  Subtract your starting number.
  Is the number you're left with odd or even?

  Will your result always be the answer? (https://nrich.maths.org/8065)

- Show poster with the generic example of the number trick (https://nrich.maths.org/2280) and talk through why this proves the trick will always work. As I talked through the generic example, I emphasized the idea that they should pretend there was a lot more circles on the page (77, the number one of the students had originally selected for the number trick) and how the overall shape for the different parts of the trick would still be the same.

- Next, I introduced the idea of using a variable to “prove” the trick worked for every possible number. With students’ help, we then constructed a “proof” of the trick using variables.

- Task 2: In small groups, students were asked to work on the following task: “9*11 equals 1 less than 10^2, 3*5 equals 1 less than 4^2. Will this pattern always be the case?” After all of the groups worked on the task for a while, we debriefed the solution as a whole group.
    - I’m not sure all of the students understood why you could use only 1 variable to represent the two numbers (e.g., 9 and 11). In the future, have an explicit discussion with students about how many variables you need to use to prove the statement and generally why it might be worthwhile to use as few variables as possible when writing the proof.

    - ***Next time*:** spend a lot longer on the idea of generic examples and specifically having students create and explain generic examples. The idea of variables was introduced way too quickly! Also, think more about ways to help students see the covariational relationship between variables (e.g., representing odd numbers in terms of even numbers).

Session 7:

<u>Learning goals:</u> Introduce the idea of thinking about geometric diagrams as generic examples and discuss what aspects of diagrams you can (and can’t) use when writing a proof.

- Introduction: Introduce vocabulary: conditional statements are written in the form “if (hypothesis), then (conclusion)”. Contrast the use of hypothesis in mathematics with how it is used in science. Introduce the idea that when you’re proving a statement, you begin by assuming the hypothesis is true and then write an argument showing that the conclusion must also be true. During this discussion, I showed students an informal proof that all quadrilaterals tessellate (shown below). While talking through the proof, I asked students questions such as “when I was writing the proof, could I say that angle D is 90°? Why not?”

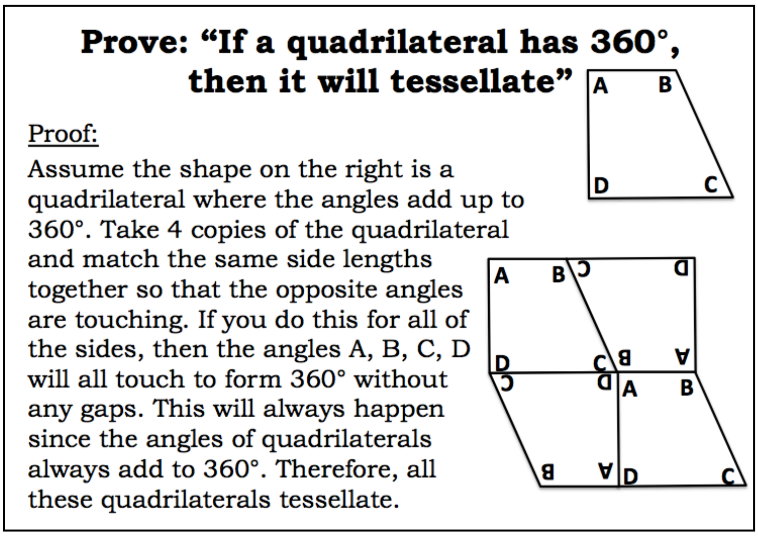

Figure 7. "Proof" for the conjecture, "If a quadrilateral has 360°, then it will tessellate."

- Main task: I gave each group 3 theorems about quadrilaterals, written as conditional statements (see below). For each statement, students were supposed to draw and label the diagram. At the end of the session, I put the three posters (1 from each group) up at the front of the room and then we discussed the different ways that students drew the diagram. During this discussion, we also talked about how to label the diagrams and whether we could put specific numbers on the sides of a quadrilateral (as one of the groups did).

<u>Statements used for Diagram Task (part 1):</u>

1. If the polygon is a rectangle, then the diagonals have the same length.
2. If a quadrilateral is a parallelogram, then the measures of the angles on the same side of the shape add to 180 degrees.
3. If a quadrilateral is an isosceles trapezoid, then the diagonals have the same length.

<u>Notes:</u>

- This task took longer and was way more challenging than expected. However, it resulted in a great discussion about notation and different aspects of their diagrams.
- Students were encouraged to come up with their own notation or use notation they had learned in previous math classes. So, there was a lot of non-standard notation (such as using colors to represent different parts that were congruent). However, by the end they all agreed that the use of letters or colors was better because it was "more generic"
- Students were unfamiliar with diagonals and isosceles trapezoids, so these terms were explained during the task.
- By the end of the session, it wasn't clear that students fully understood conditional statements and the idea that you don't know for sure the conclusion is true when you set out to write a proof. I definitely needed to revisit this in the next session!

Session 8:

Learning goals: Continue talking about diagrams as generic examples; understanding conditional statements (vocabulary, what you can and cannot take to be true when proving a conditional statement).

- Introduction: Review conditional statement vocabulary, the diagrams students produced in session 7 as well as other possible diagrams that could be used for the given theorem.

- Main task: Give groups 3 more theorems about quadrilaterals that are a little bit trickier since it is not as clear what shape they should draw for the statement (see below). For each statement, students were supposed to draw and label the diagram. After groups finished, I put the three posters (1 from each group) up at the front of the room and then we discussed the different ways that students drew the diagram.

Statements used for Diagram Task (part 2):

4. If two sides of a parallelogram that intersect have the same length, then the parallelogram is a rhombus.
5. If the diagonals of a parallelogram form a 90-degree angle, then the parallelogram is a rhombus.
6. If one angle of a parallelogram is a right angle, then the parallelogram is a rectangle.

Notes:

- Statement 6 brought out an interesting conversation, because one of the groups had initially drawn a figure that was half a parallelogram and half a rectangle (resulting in a trapezoid).
- Also, idea arose in most of the small groups when working on statement 5 that just because the hypothesis says 1 right angle doesn't mean that there can't be more than 1, just that there's definitely 1.
- Overall, the session went a lot better than session 7 and they definitely seemed to improve their understanding of diagrams and conditional statements.
- When reflecting on their work at the end of the study, students seemed to know that they should include a diagram (especially for geometry tasks), but they didn't seem to understand (as a group) how to use the diagram with writing proofs. One piece of evidence for this was that they would draw the diagram *after* they wrote part, or all, of their argument for the proof task.
- Also, this would make a really nice practitioner article – showing some of the diagrams that groups drew and then describing different ideas that came up in the small/whole group discussions.

- Writing prompts: Why is it okay that our diagrams weren't exactly the same? Suppose a student was proving a statement about rectangles. Would it be okay for them to draw a square as their diagram? Why or why not?

Session 9:

<u>Learning goal:</u> Students will begin developing a definition for similar polygons.

- Task: Begin by eliciting students' ideas for the definition of the term "similar polygons". Then, give groups 4 examples of similar polygons and ask them to use the examples to develop a definition based on what they notice is true across all of the examples. Half way through the lesson, I gave students 4 non-examples to help them continue to refine their current ideas about the term.
    - Note: the sequence of tasks in sessions 9 and 10 is based off of the study conducted by Kobiela & Lehrer (2015).

Session 10:

<u>Learning goal:</u> Refine groups' definitions from session 9 until the entire class has the same definition for similar polygons.

- Task: I handed each group 12 "cards" and asked them to sort them into similar polygons and non-similar polygons. The cards included examples and non-examples from session 9 as well as new examples that pressed on areas of their original definitions that were imprecise (e.g., two shapes where one was rotated; congruent polygons; shapes where the side lengths were added instead of multiplied by a number).
- I also handed each group with a paper that contained all of the features of similar polygons that groups had come up with in session 9 and asked them to come up with a single definition that only included the necessary features of similar polygons (no extra information), using the cards to help make their decisions.

    - The students had a *great, lengthy* discussion in this session about whether congruent polygons should be included in the definition of similar polygons. They were initially evenly split in whether they should be included, but eventually agreed as a group that congruent polygons were also similar.
    - At the end of the session, all three groups ended up with the same definition for similar polygons (the standard textbook definition), so there was no need to further discuss/refine the definition.

- Reflection prompt: How did this activity help you think about mathematical definitions?

Session 11:

<u>Learning goal:</u> Have students conjecture and then prove claims about similar polygons.

- Introduction: Remind students of the definition of similar polygons they constructed in session 10. Introduce the idea of a conjecture and then invite students to pose conjectures about types of polygons that would all be similar to one another (e.g., "all squares are similar").
    - Students collectively conjectured the following shapes:
        - Squares
        - Rhombuses
        - Right triangles
        - Equilateral triangles
        - (Isosceles triangles were proposed and then taken off the list; Clay proposed regular hexagons in his small group, but this conjecture did not make the class list)

- Students then worked in their small groups to prove the conjectures, beginning with squares. Once groups finished writing their proof, they exchanged papers with another group and provided feedback on anything the original group could do to make their argument more convincing.

Session 12:

<u>Learning goals:</u> Construct a proof of the conjecture "all squares are similar" as a whole class and then have them continue proving the other conjectures from session 11.

- Walk through a proof of the conjecture "all squares are similar". During discussion, connect what we're writing in the class proof to the different strategies students did in their groups during session 11. Include in this discussion the idea that you begin by stating the hypothesis ("start with two squares") and finish the proof by stating the conclusion. Talk about the different ways of labeling the diagram; representing the proportional sides.
- Afterwards, students worked on constructing proofs for the other conjectures in session 11. All of the groups determined that the conjecture, "all right triangles are similar" was false, although not all wrote down their counterexample to disprove the claim. None of the groups got to explore the conjecture about rhombuses, so this was left for the final interview.

    - ***Important note****:* It is VERY easy to mess up how to represent the proportional sides (I ended up having to correct this in session 13) so that you're not assuming the sides are proportional. This ended up being way more complicated than I had anticipated!

    - **Next time:** Have students write their proof for both equilateral triangles and squares prior to going over one of them as a class. What ended up happening in session 12 is that the students used the proof we constructed for squares as a template for writing the proof for equilateral triangles. So, while they were able to "complete" the proof rather quickly, it became all about copying down what was said in the previous proof. It would also be good if students had the opportunity to go back and revise their work, if you could figure out a way of doing it so that they're not just copying what's on the board.

Session 13:

<u>Learning goals:</u> Address notation issue from session 12, have students work on another proof task.

- Introduction: Address the issue of how we talked about the scale factor/proportional sides from session 12. In this discussion, work through a "proof" for the false conjecture that all right triangles are similar to demonstrate why our original notation allows you to think you've proven something that isn't actually true.

- Task: Give students a paper with the following question:

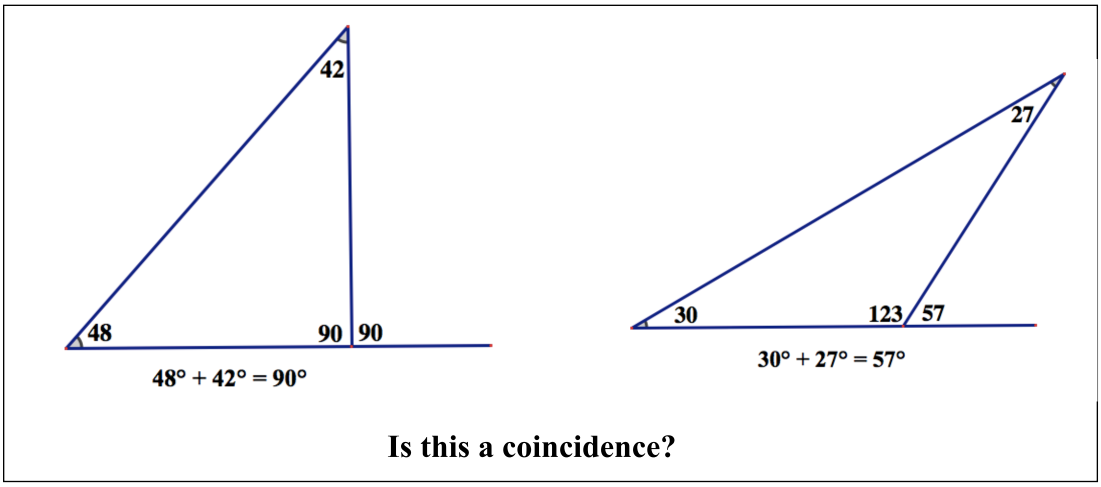

Figure 8. Task prompt given to students during session 13. Note that this is a modified version of the exterior angle theorem.

- Small groups informally discussed their answer to the task. Once they decided that it was not a coincidence, I asked the students to <u>individually</u> write an argument proving their answer. Afterwards, I had them swap papers and provide each other written feedback on their argument.

    - Note: I chose to pose the exterior angle theorem this way in order to increase uncertainty about the validity of the relationship (by phrasing it as a question) and to eliminate the need for vocabulary to describe the different angles. The two main ideas needed for the proof (sum of angles in a triangle and sum of angles in a straight line) had both surfaced in earlier sessions, so I was reasonably confident that most, if not all, students had the mathematical background needed for the task.
    - About half of the students did not draw a diagram until at least half way through writing their proof. This could be because they were using the

examples on the original sheet of paper as their diagram. Alternatively, it could be because they diagram was seen as a secondary thing you have to include when writing proofs and they were not yet viewing the diagram as a tool to help you write the proof.

- I noticed afterwards that some of the students added to their original proof after getting feedback from one of their peers. This made analyzing their work a little more challenging, but demonstrates the potential value of having students critique each other's work and then use that feedback to revise their proofs.
- **Next time:** make the original proof task either contains 1 or 3 examples. Some of the students began their proof with "start with two triangles" and it's unclear if they did this because they were copying the way that the similar squares proof started or because there were 2 examples in the original task.
- **Idea to consider:** It may be worthwhile to have this task come before the similar polygon proof tasks. Switching the order of the two tasks would mean students would have less of a "proof template" to use for the second proof task they work on. The disadvantage is that they would have less of an understanding of mathematical definitions when they go to work on the task, unless I placed this task in between constructing a definition for similar polygons and writing proofs using that definition.

Session 14:

<u>Learning goals:</u> Wrap up the overall big ideas from the sessions and assess students' conceptions of proofs.

- Task 1: Have students work in their small groups to write down 3-5 things to keep in mind when writing a proof. In other words, what are some criteria for "good proofs"? Once they finished, we compiled the lists from the three groups into a single list. Students largely came up with the same ideas, but if needed, we would have condensed it to a single "class list".

  - Class list:
    - Think about definitions
    - Diagrams with variable labels
    - Make it general (i.e., adhere to the generality requirement)
    - Formulas (e.g., "the sum of the angles in a triangle is 180°")
    - Explanations why / directions

- Task 2: Using the classes' criteria, students worked in groups to evaluate four provided "student" solutions for the exterior angle theorem proof task.
  - Note: the first two tasks were based off of the tasks used in the study conducted by Boyle, Bleiler, Yee, & Ko (2015).

- We finished the study with a written reflection and whole group discussion on their overall thoughts about the sessions and what they had learned/liked/etc.
    - There were two students absent in the final session: Heather and Brian.

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APPENDIX B: EXPANDED METHODOLOGY

The purpose of this section is to provide a more in-depth description of the study methods. I begin by describing the setting and participants, followed by a description of the data sources and interviews. A complete description of the instructional sequence is located in Appendix A.

Setting and Participants

School Setting

The design research study took place at a rural public school in the Midwest United States. The district contained only one high school, where approximately 600 predominately white (94%) students attended 9th through 12th grades. This location was selected out of convenience, as it was the first school that agreed to participate in the study. After expressing interest in being involved in the study, the mathematics department chair, principal, and their mathematics teacher offered to let me recruit students from the accelerated 9th grade mathematics class and to hold the sessions during the students' 36-minute study hall instead of after school as I had initially proposed. All students who were enrolled in the accelerated 9th grade mathematics course at the beginning of the study were recruited and all ten students agreed to be a part of the study (a few students were initially reluctant, but agreed to participate after additional conversations with their math teacher).

Instructional sessions. The school operated on an A-B block schedule, with the students' mathematics class meeting immediately after study hall on B-days. I had planned to hold all of the sessions on B days where the students had mathematics so that the sessions could run a few minutes over into the following class period if needed; however, some of the sessions had to be held on A days in order to ensure that we met twice every week and to accommodate for half days and school events where a large number of the students would be absent. Thus, some of the sessions were held on consecutive days while there were other instances where nearly a week passed between sessions (see Table 1). Each session lasted between 28 - 38 minutes. The first three sessions were held on the stage in the cafeteria and the remaining sessions were held in one of two mathematics classrooms. The cafeteria stage was set up with three long tables and three to four chairs placed at each table. The two mathematics classrooms were set up so that the desks were placed in rows, with two to three desks pushed together.

Table 1

List of when the sessions occurred each week

Week #	**First session of week**	**Second session of week**
1	Monday	Wednesday
2	Wednesday	Friday
3	Tuesday	Wednesday
4	Thursday	Friday
5	Tuesday	Wednesday
6	Tuesday	Thursday
7	Monday	Thursday

Participants

For this study, I specifically wanted to work with students who had not yet been formally introduced to proof in their mathematics classes but who would be taking high school Geometry (the course where formal proof is traditionally introduced) the following year. Since proof is traditionally introduced in chapter 2 (Otten, Males, et al., 2014) and this study was taking place in the spring, students who were enrolled in the course immediately prior to high school Geometry would most closely approximate

Geometry students in terms of their prior mathematical knowledge. Although the accelerated 9th grade mathematics students were unique in that they were currently learning Algebra 2 concepts in their regular math class during the study, they met my above criteria for participation in the study as they had not yet been taught formal deductive proofs. Since most of the tasks in the design experiment were geometric, I felt that their additional algebraic knowledge would not significantly skew the data and would be comparable to the work that other "honors" students might produce.

The ten participants included seven females (Amanda, Arin, Heather, Lauren, Lexi, Megan, and Sadie) and three males (Brian, Clay, and Wilson; all names are pseudonyms). I recruited the students by briefly describing the objective of the study and telling them that participating in the study would give them a head start on material they would be learning the following year. All students also received a graphing calculator for participating in the study. I did not collect any demographic information on the students (e.g., exact age, race, grade in their math class); I would say that they were all white and about 15 years old.

During the initial interview, I asked students to talk about their current math class and explain why they chose to be in the accelerated track. Most students used adjectives such as "challenge" and "fast paced" or "accelerated" when describing the class, but generally stated that it was "good" or "fine". Students' reasons for enrolling in the accelerated class included wanting to get ahead in math and doing so for scheduling reasons (e.g., not have to double up on math or because it allowed them to take another elective class). Two students specifically cited the value of the course in terms of being prepared for math-related careers and looking good on college transcripts. Based on my interactions with the students, I would describe them as studious and fairly comfortable with being challenged mathematically. They were very friendly with each other socially and generally worked well with each other in small groups, although some students were more outgoing and open to contributing to small group conversation than others.

I did not observe any of their regular mathematics classes; based on my interactions with the students at the beginning of the study, I would say that the students were accustomed to being taught using a traditional “show-and-tell” approach. Two norms in particular that I had to renegotiate with the students during the beginning of the study were the expectation that I would not immediately respond to their ideas as being either “correct” or “incorrect” and that students were expected to share their thinking with the entire class and respond to each other’s thinking. The group collectively became more comfortable with these new norms as the study progressed to the point where their mathematics teacher commented to me at the end of the study that she had noticed the students were more willing to share their ideas in her class.

Focus students. At the end of Session 3, I selected four students that I wanted to focus on through conducting two additional one-on-one interviews with them after Sessions 5 and 11. My goal when selecting the focus students was to select students who seemed they might have different perspectives throughout the study and might be in different places in terms of their mathematical content knowledge. When making these selections, I drew on students’ answers during the initial interview as well as their engagement during the first three sessions. Once I selected the four focus students, I placed them into a single small group in order to make it easier to collect information on their thinking. As a result, the four students worked in the same small group from Session 5 to the end of the study.

The four students I selected were Amanda, Lexi, Clay, and Heather. I selected Amanda because she seemed to be one of the strongest in the group in terms of her initial perspectives about proof. Specifically, she was one of two students who constructed a deductive argument on the *sum of three odd numbers* task, even though follow up questions revealed that she was not aware that the proof conjecture applied to all possible sums of three odd numbers. Additionally, she evaluated the algebraic proof, generic example, and paragraph proof as proofs, but thought the examples-based argument and the algebraic argument containing circular reasoning as not proofs. Finally, Amanda was one of the few students who was willing to vocalize their thinking to the entire group during the first three sessions.

I selected Lexi and Clay as students who seemed to be in the middle of the group in terms of their proof understanding and willingness to share their ideas with the class. Both students produced examples-based arguments on the *sum of three odd numbers* task; Clay also wrote a general statement to accompany his examples, but it did reach the level of a deductive argument. When evaluating the provided responses, both students were unsure whether the examples-based response should be considered a proof. Lexi thought the two algebraic arguments were both proofs and preferred the incorrect algebraic argument. Clay preferred the deductive, paragraph argument but also thought the correct algebraic proof was a proof as well. Finally, both students questioned during the initial interview whether it was possible to know if a statement in math was always true. Lexi's contributions during the initial sessions demonstrated thoughtfulness in her work and an ability to explain her thinking. Although Clay tended to let his small group members do most of the work during the first few sessions, he began forming conjectures about why quadrilaterals tessellate fairly early in the exploration process. He also seemed to be slightly more willing to explain his thinking than the other two male students.

Finally, I selected Heather as one of the students who seemed to be on the lower end of the group in terms of her mathematical confidence. Similar to most of the students during the initial interview, Heather provided two examples on the *sum of three odd numbers* task and then concluded that "if those are odd, then the rest will most likely be odd." She was also one of two students who evaluated the examples-based argument as a proof and selected the examples-based argument as the one she preferred the most out of the five possible solutions. During the middle of the study, I informally found out that Heather was considering switching out of the accelerated mathematics track because she was struggling to keep up with the pace of the class.

Outside Observers

There were two people who served as outside observers. The primary observer was another graduate student and former high school mathematics teacher; she attended sessions 2, 4, 6, 7, 8, 10, 12, and 13. The secondary observer was a mathematics education professor and attended Sessions 1 and 3. After each session, the outside observer and I would debrief how we thought the lesson went, including where the students appeared to be in their understanding of proofs and what types of tasks might be most appropriate for the following session. Whenever the primary observer did not attend a session, I debriefed with her afterwards prior to making plans for the following session. When this occurred, the outside observer and I would watch portions of the video and/or audio recordings and discuss any work that the students produced. The two outside observers served a few roles during the study: first, they listened and interacted with small groups during the sessions, giving them additional insights into students' approaches to the tasks and their current conceptions of proof. Second, they assisted me in finalizing the plans for each session, including helping me think through how to sequence tasks or brainstorm possible strategies students may use on a given task. Third, they served as my reliability and validity checks throughout the retrospective analysis process, particularly in relation to the possible ways of interpreting students' work that was unclear.

Data Sources

As is typical in design research (Cobb et al., 2003), I collected a wide range of data during the study, including audio and video recordings of every session and interview, all student written work, notes from the outside observers, and my own lesson plans and reflections that I wrote in between sessions (see Table 2). During retrospective analysis, I analyzed both the audio/video data as well as students' written work and journal reflections (if any was produced). I also reviewed my lesson plan and reflection notes in order to describe the instructional sequence. In the following sections, I provide additional detail about the data sources and methods of collection.

Table 2
List of data collected during each portion of the design study.

Name	Researchers	Key Questions or Tasks	Data Type	Length
Initial interview	• Kimberly (5 interviews) • Primary observer (2 int.) • Secondary observer (3 int.)	• Brief comments about current math class • Definition of proof? Is it possible to know if a statement in math is always true? • *Sum of three odds* task + analyze 5 solutions	1 video per student student written work	10:21 - 22:05 min. avg: 13.5
Session 1	Kimberly Secondary outside observer	Do all quadrilaterals tessellate?	2 videos 3 audio recordings observer notes student journal reflections	36 min.
Session 2	Kimberly Primary outside observer	Write how-to directions for tessellating quadrilaterals	3 videos 3 audio recordings observer notes student journals (1 per group)	33 min.
Session 3	Kimberly Secondary outside observer	Do all regular polygons tessellate? What's special about quadrilaterals?	3 videos 1.5 audio recordings observer notes student journal reflections	30 min.
Session 4	Kimberly Primary outside observer	Recap first 3 days, explain why quadrilaterals tessellate Analyze student work for sum of angles in triangle is 180	2 videos 3 audio recordings	33.5 min.
Session 5	Kimberly	Circle and Spots Problem Monster Counterexample	3 videos 3 audio recordings student journal reflections	32 min.
Focus interview #1	Kimberly (2 interviews) Primary outside observer (2 interviews)	• Analyze sum of angles in triangle solution • Prove all triangles tessellate, analyze 3 solutions • Current definition of proof • Is it possible to know statement is always true?	1 video per student student written work	Lexi: 10:34 min. Clay: 12:05 Heather: 22:51 Amanda: 15:19
Session 6	Kimberly Primary outside observer	Prove 2 number tricks Intro idea of variable to talk about "generic case"	2 videos 3 audio recordings student work in journals	28 min.
Session 7	Kimberly Primary outside observer	Construct diagram for the given statements (day 1)	3 videos 3 audio recordings 1 poster for each group	31.5 min.

Name	Researchers	Key Questions or Tasks	Data Type	Length
Session 8	Kimberly Primary outside observer	Construct diagram for the given statements (day 2)	2.5 videos 3 audio recordings 1 poster per group student journal reflections	38 min.
Session 9	Kimberly	Define similar polygons (day 1)	3 videos 2 audio recordings 1 poster (sm. group definitions) written work (1 per group)	31 min.
Session 10	Kimberly Primary outside observer	Define similar polygons (day 2)	3 videos 3 audio recordings written work; journal reflections	34 min.
Session 11	Kimberly	Prove statements about similar polygons (day 1)	3 videos 3 audio recordings pdf of work (1 per group)	32.5 min.
Focus interview #2	Kimberly (2 interviews) Secondary outside observer (2 interviews)	• Draw diagram for proof statement • Clarify journal entries from session 8 • Number proof task + analyze 4 solutions	1 video per student pdf of their work	Lexi: 18:10 min. Clay: 22:56 Heather: 19:34 Amanda: 26:38
Session 12	Kimberly Primary outside observer	Prove statements about similar polygons (day 2)	2 videos 3 audio recordings	33 min.
Session 13	Kimberly Primary outside observer	Prove exterior angle theorem	3 videos 3 audio recordings student proofs in notebooks (10) observer's notes	32 min.
Session 14	Kimberly	Construct criteria for "good proofs"	2 videos 3 audio recordings 1 poster with class criteria	32.5 min.
Final interview	Kimberly (10 interviews)	• Definition of proof? Things to think about when writing a proof? Is it possible to know if a statement in math is always true? • Should proofs be removed from HS curriculum? • Number proof task + analyze 5 solutions • Reflect on exterior angle proof from session 13 • Prove "all rhombuses are similar" conjecture	1 video per student student written work interviewer notes	32 – 44 min. avg: 37 min.

Audio and Video Data

During each session, I placed one video camera by each of the three small groups to capture their work and an audio recorded in the center of each small group to capture their small group discussions. The primary purpose of the video was to record students' hand motions and manipulation of physical objects (e.g., quadrilaterals during the tessellation task) and the primary purpose of the audio recorders was to make sure that all students' voices could be heard during the small group discussions. Collecting both audio and video data for each small group also helped to accommodate for technological issues that occurred during the sessions and ensured that I had at least one data source for each small group. When setting up the cameras during the sessions, I tried to set one camera up in such a way that it captured a slightly wider view of the classroom. All of the audio and video equipment was set up before the start of each session. The cameras were largely stationary during the sessions except in instances where the focus of the instruction significantly changed (e.g., I invited all students to the board to compare work across the three groups). When this occurred, the secondary observers or I would move one of the cameras so that it captured the instruction. As a result of this choice, there was not a single video recording of my movements during the lesson; however, since there were only 3 groups and I rotated between the groups during small group work, my movements and actions were largely captured by one of the three video cameras. Additionally, none of the cameras were regularly set up to record the white board; instead, one of the outside observers or I would reposition a camera so that it captured the front of the room on the infrequent occasions where I utilized the white boards to record anything beyond the task directions. I chose not to wear an additional mike during the sessions as I have a loud "teacher" voice and the video/audio recorders were always able to capture my comments during small group and whole class discussions. During each interview, I positioned one video camera so that the student's paper and hands, but not their face, was shown on screen. I chose not to audio record the interviews since they were all conducted one-on-one and in a small room.

Written Data

I gave each of the students a small composition notebook so that they had a single place to record their written work and journal reflections during the sessions. In order to ensure that the notebooks did not get lost, I collected the composition books after each session. Additionally, I collected all additional loose-leaf papers that students used during the sessions in instances where I gave them a paper to write on instead of using their notebooks. Finally, I made a digital copy (picture) of the posters produced during Sessions 7, 8, 9, and 14 and retained the original posters for reference if needed. All of my lesson plans and post-session reflections were recorded in a single, spiral-bound notebook. During the interviews, students were given a piece of paper for each of the proof tasks as well as a colored sharpie to record their work. Having students use sharpies made their work more visible on the video recordings and prohibited them from erasing any of their work. In the following section, I describe the overall purpose of each interview. A detailed description of the sessions is located in Appendix A.

Description of Interviews

All students were individually interviewed at the beginning and end of the study in order to track any changes in their understanding of proofs. The initial interviews lasted between 10 and 22 minutes, with an average length of 13.5 minutes. The final interviews lasted between 32 and 44 minutes, with an average length of 37 minutes. Four focus students were interviewed an additional two times, after sessions 5 and 11, to gain further understanding of their current conceptions of proof. The focus interviews lasted between 10 and 26 minutes, with the first focus interview lasting an average of 15 minutes and the second focus interview lasting an average of 22 minutes. I conducted half of the initial and focus interviews and all of the final interviews; the remaining initial and focus interviews were conducted by the two outside observers. All interviews were semi-structured (Roulston, 2010) in that the interviewers were provided an protocol for each interview but were encouraged to ask additional follow up questions to gain further insight and clarification into the student's thinking as needed. The interviews were all video recorded and I transcribed all interviews prior to analysis. I provide a brief overview of each interview in the sections below.

Initial Interview

The initial interviews were conducted between 0 and 6 days prior to the start of the sessions; specifically, 3 interviews were conducted 6 days prior to the start of the sessions, 3 interviews were conducted 5 days prior, 2 interviews were conducted 3 days prior, and 2 interviews were conducted at the end of students' math class on the day of the first session. The initial interviews were spread out in this way so that they could take place during the students' study hall and to accommodate the interviewers' schedules. The two students who conducted their interview after the first session, Heather and Wilson, did not appear to have an advantage over the other students since the first session focused on a geometric task and we did not discuss any of the topics that were covered in the initial interview. The only possible advantage they could have had as a result of the first session was an increased awareness of universal claims; however, their answers during the interview were comparable to the ones given by others in the class.

The primary purpose of the initial interview was to collect background information on the student and their understanding, if any, of proof. After asking students some initial background questions, I assessed their familiarity with the term, “prove”, both in everyday and mathematical contexts. I also asked students whether they thought it was possible to know if a statement in math is always true in order to assess students’ awareness of universal claims in mathematics. Next, I gave students the following proof task: “Sarah said, ‘If you add any three odd numbers together, your answer will be odd’. Is she right? Explain your answer.” I chose not to use the word “always” in the prompt to see whether students produced general arguments without being explicitly prompted to do so in the original task. Instead, all students were asked as a follow up question whether they thought their response proved the statement was *always* true. After giving students the original task, the interviewer provided students with marbles, a calculator and paper/sharpie that they could use during the task. Once students wrote down their answer, the interviewer asked them to verbally explain their response and then asked follow up questions to gain additional information about their approach. For example, if students gave a few examples as their answer, the interviewer would ask them why they chose to use those numbers and whether they selected those numbers for a particular reason (e.g., “random” odd numbers, numbers that were large or small). Finally, students were given a paper containing five “student” solutions for the same task. For each provided solution, students were asked to determine whether they thought the answer was or was not a proof, or they were unsure, and then to explain their answer. After evaluating all five solutions the interviewer asked them to indicate which of the following choices that you said were proofs do you prefer the most? Collectively, these tasks helped me to understand students' initial understanding of proof, including the criteria they used when determining whether an argument should be considered a proof and their initial approach on a proof task.

Final Interview

The finals interviews were conducted between 8 and 13 days after the end of the sessions; specifically, 1 interview was conducted 8 days after the end of the sessions, 3 interviews were conducted 11 days after, 4 interviews were conducted 12 days after, and 4 interviews were conducted 13 days after the end of the sessions. I conducted all of the interviews; this decision allowed me to make sure that I asked similar follow up questions to each student and make decisions based on what to cut when time ran short. The interviews lasted between 32 – 44 min, with average length of 37 minutes.

During the final interview, I asked students to repeat the sum of three odds proof task and evaluation of the five provided solutions in the final interview in order to assess any shifts in their approach since the initial interview. In addition, I asked students to revisit the exterior angle theorem argument they wrote during Session 13 so that I could ask them follow up questions on their thinking. The final proof task involved students exploring, using a Geometer's Sketchpad (GSP) app, and then proving the false conjecture, "all rhombuses are similar." A student initially posed this conjecture in Session 11, but none of the small groups had time to work on it during the sessions. One of my goals for the similar rhombus conjecture was to assess students' understanding of how to prove that a mathematical statement is false; thus, in instances where the student wrote an argument proving the statement was true (4 students), I had them explain their initial answer and then asked them more direct questions until they realized the statement was false. For example, many of the students who thought the statement was true and not explored the GSP app, so I directed them to see what would happen if they moved one of the points on a rhombus. I also asked questions such as, “do rhombuses have set angle measurements?”, in order to focus their attention to the angles of a rhombus. Once students concluded the conjecture was false, I would then ask them to briefly vocalize how they would prove the conjecture false. I also asked all students a follow up question of how many counterexamples were needed in order to prove that a mathematical statement is false.

In addition to the three proof tasks, the final interview protocol also contained questions about different elements of the design experiment and their overall thoughts on the sessions. In order to assess students’ understanding of the purpose of proofs, I posed the following hypothetical question: “Suppose some math teachers and policy makers were thinking about removing proofs from the high school curriculum. Would you think this is a good idea? Why or why not?” I also repeated my questions from the initial interview asking them to define proof and whether they thought it was possible to know that a statement in math was always true. Given that I never formally defined proof for students in the sessions, it was not surprising that many of the students tended to talk about the components of a proof rather than give a specific definition.

First Focus Interview

The purpose of the first focus interview, conducted after Session 5, was to ask follow up questions related to students’ work during Session 4 and to have them engage in a proof task on their own. The proof task was phrased as the following statement: “Write an argument that proves all triangles tessellate.” Students were given copies of different triangles to use if they wanted to explore the question before constructing their argument. Next, I asked them to evaluate three provided solutions in terms of whether they thought the argument was a proof, was not a proof, or was unsure. The first argument was empirical; the second argument did not have specific characteristics labeled on the diagram but only applied to right triangles; and the third argument was similar to the argument I presented students in Session 4. I used students' justifications for their evaluations of the provided solutions as a way to gain insight into students' understanding of proof and the features of each argument they were attending to in their evaluation. During this interview, I also re-asked students what it means to prove something in mathematics and whether they thought was possible to know if a statement in mathematics is always true in order to assess whether their answers had changed since the initial interview.

Second Focus Interview

During the second focus interview, I wanted to get a little more information about the students' understanding of diagrams as generic examples as well as assess their understanding of the boundaries of the claim. I did this through asking students to construct diagrams for the following two statements: 1) If a quadrilateral is a kite, then the diagonals form a 90 degree angle; 2) If the opposite sides of a quadrilateral have the same length, then the quadrilateral is a parallelogram. Next, I handed students other possible diagrams for the each statement and asked students to explain whether my diagrams could also be used to represent each statement. The second portion of the interview consisted of students constructing an argument for the statement, "Prove that "the sum of two consecutive numbers is an odd number." Once students explained their solution, I provided them with four possible student solutions for the same task and asked them to evaluate each one to determine whether they thought it was a proof, was not a proof, or was unsure. After evaluating each provided solution, students were asked to justify their answer.

Analysis

In this section, I provide a more detailed explanation of my analysis process for each of the three articles included in chapters 2 – 4 of my dissertation. Given that I spent nearly one year in the transcribing, analysis, and writing phases, I am going to primarily focus on the analysis process that resulted in the findings presented in each article and will not necessarily describe earlier stages of the analysis process that were not included in my final dissertation.

An Analytic Framework for Assessing Students' Understanding of Proof Components

After completing data collection, I began transcribing the initial and final interviews with students. During this process, I realized that the proof framework I had intended to use to assess students' understanding of proof (Waring, 2000) would not be able to fully capture the different aspects of proof that students understood. As a result, I chose to develop my own framework that would attend to the nuanced details of students' work on proof tasks. The framework itself was initially developed using grounded theory techniques; however, after multiple iterations of refining the framework, I realized that it aligned well with Stylianides' (2007) definition of proof and decided to use his definition to situate the framework. When coding students' work on the three proof tasks (*sum of three odd numbers* task, exterior angle theorem, and similar rhombuses task), I drew on both students' written work and their verbal explanation during the interview.

After developing the initial framework, the primary observer double coded all of students' work for the *sum of three odd numbers* task (final interview) and the secondary observer double coded all of students' work for the exterior angle theorem. I then met with each of the outside observers to compare our codes and resolve any descrepencies. Overall reliability was at least 85% for both tasks. Additionally, I discussed any questions I had about coding the remaining two sets of student work (*sum of three odd numbers* task from the initial interview and the similar rhombuses task) with the secondary observer for additional validity and reliability. I chose not to report this reliability check in the original article (chapter 2) due to the fact that I did not have the outside observers re-code the data after I made changes to the framework. Although there were a lot of similarities between my initial and final framework, I felt that the differences were large enough that the early reliability check could not be applied to the final framework. Additionally, I did not feel that the reliability scores were as important to report in the final version of this article since I used the coded data as illustrations of the framework and did not provide summary statements describing how many students' work was scored with a particular code for each category.

Engaging Students in Tasks Involving Universal Claims

The motivation for using universal claims in the instructional sequence and for investigating its impact in the types of reasoning-and-proving opportunities students had during the study was based in the textbook analysis study by Otten, Males, and Gilbertson (2014). Many of the key features of my instructional sequence and the analysis in this article originated from Otten and colleages' article; specifically, the distinction between universal and particular claims (and why that distinction might be important), the types of reasoning-and-proving opportunities found in traditional textbooks, the idea of the intellectual necessity principle (Harel, 2008), and the idea of statements *about* the reasoning-and-proving process. Many of the decisions I made about the instructional sequence were a direct attempt to build an introduction-to-proof unit that contained the various components that were largely lacking in U.S. Geometry textbooks. Thus, when it came time to decide how to analyze my instructional decision to use universal claims, it made sense to continue to draw on Otten and colleagues' (2014) article.

Data Reduction. I decided to parse the data by task based on the fact that the universal claims were embedded in the instructional sequence via the mathematical tasks. There were six main tasks involving universal claims: the tessellation tasks (Sessions 1 – 4), the Circle and Spots task (Session 5), two number based claims (Session 6), constructing diagrams task (Sessions 7 – 8), proving conjectures about similar polygons task (Session 11 – 12), and the exterior angle theorem (half of Sessions 13 and 14). Students developed a definition for similar polygons during Sessions 9 and 10; although students did have to consider features of a class of shapes in the task, it did not directly have them working with a universal claim. Thus, these sessions were removed from the possible data sources for this article. Next, I chose to elimintate the tasks in sessions 5 and 6, since they were shorter in duration (less than 30 min); additionally, removing these tasks allowed me to focus only on geometry-based tasks. This left me with four tasks involving universal claims: the tessellation tasks, constructing diagrams task, proving conjectures about similar polygons task, and the exterior angle theorem task.

Coding. In order to code for reasoning-and-proving activity, I modified the expected student activity codes by Otten and colleagues (2014) so that they were used to code the types of reasoning-and-proving activity that students actually engaged in during the sessions instead of just the expected student activity (see Table 3). I also separated out of few of the codes that had been previously collapsed into a single category (e.g., make a conjecture and refine a statement or conjecture). When analyzing the data, I looked for additional reasoning-and-proving activity that fell outside of the categories in Otten and colleagues' (2014) framework. This process yielded two emergent codes – construct a diagram and revise an argument (also included in Table 3).

Table 3

Reasoning-and-Proving activity codes

Related to Mathematical Claims	Related to Mathematical Arguments	Emergent Codes
Make a conjecture	Construct a proof	Revise an argument
Refine a statement or conjecture	Develop a rationale or non-proof argument	Construct a diagram
Draw a conclusion	Evaluate an argument	
Investigate a conjecture or statement	Find a counterexample	

I coded all instances of statements or questions *about* reasoning-and-proving using a single code ("talk ABOUT reasoning-and-proving") and then inserted a brief description of the theme of the statement/question. Examples of the code descriptions included "conditional statements", "how to format your proof", "number of counterexamples needed to prove statement false", and "assumptions we can make from the diagram." By coding all of the statements *about* reasoning-and-proving using a single code, I was able to pull the coded data across multiple video clips to look for themes. Finally, I coded for statements or questions that motivated an intellectual necessity for proof using the code describing the intellectual necessity that it aimed to motivate – certainty, causality, or communication.

My unit of analysis for reasoning-and-proving was the proof action, or the length of time that a student was engaging in a particular reasoning-and-proving action. For example, if a student was actively working on constructing a proof, then I coded the entire time as "constructing a proof". Additionally, if a whole class conversation involved a student posing a conjecture, another student providing a counterexample to the first conjecture, and then a third student posing a refined conjecture, then each student's comments would be coded with a separate reasoning-and-proving code. The remaining two categories, conversation about reasoning-and-proving and establishing an intellectual necessity for proof, were both coded by theme. In other words, all consecutive statements or questions that pertained to the same theme (e.g., how to demonstrate that a statement in math is false or establishing a need for certainty) were assigned a single code. I initially coded directly onto the video timeline using MAXQDA (see Figure 9). During the analysis process, there were some clips that I decided to transcribe since they seemed particularly memorable and possibly useful for inclusion in the article. In these instances, I would re-code the transcript with the relevant codes (see Figure 10). In Figure 1, notice that I also coded the segments where I was introducing a task or recapping a task; I used this code to help me write the descriptions of the sessions and did not directly analyze this data. I also coded multiple videos from the same session in order to capture the conversations that occurred while students worked in their small groups. Finally, I used the audio recordings in instances where students were not audible on the video recordings.

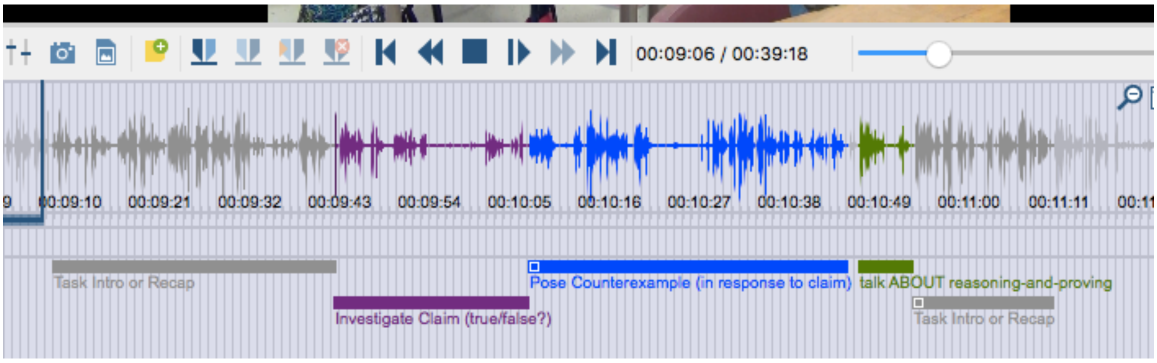

Figure 9. Excerpt of coded video data from Session 11 (constructing diagrams task). Codes are listed below the audio wavelength.

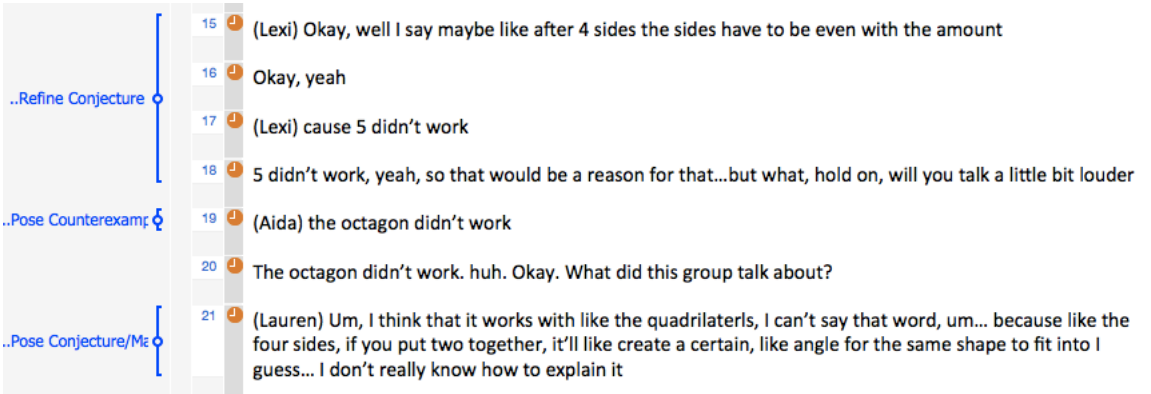

Figure 10. Example of transcribed clip from Session 3 (tessellation tasks); codes are listed on the left side.

Once I coded all of the data for a particular task, I then looked at all of the data clips that were coded for each design principle – reasoning-and-proving activity, statements *about* reasoning-and-proving, and intellectual necessity for proof – in order to identify themes and instances in the data that stuck out as being significantly different than what you might expect to see in a typical proof classroom. I conduced this portion of the analysis process using a feature in MAXQDA where that the researcher can retrieve multiple segments of data (transcribed or video/audio clips) by activating particular videos and codes. During this process, I also referenced additional data sources, such as students' written work on the task and their responses to journal prompts, to triangulate the data and gain further insight into students' current thinking. Finally, I began to outline the key ideas from looking across the coded clips within a given task and the flesh out the outline into the findings sections.

During this coding process, I looked for instances where at least one student seemed to be interpreting or engaging in a task in a way that was surprising or unexpected. During the construct a diagram task, I noticed that some of the students who used specific numbers to label sides of their diagram also acknowledged that the given quadrilateral could have other side lengths. I then took a closer look at the instances in the data where the students had produced work that seemed unexpected or non-normative in order to assess whether this behavior seemed to stem from a feature of the task, their content knowledge, or their understanding of proof. When transcribing the interview data from the final interview, I became curious about understanding why some of the students initially thought the conjecture, all rhombuses are similar, was true. In order to look more closely at this data, I transcribed all relevant data clips and consulted the related written work. Next, I wrote memos describing my interpretation of the situations. I then employed investigator triangulation by sharing the video clips with my advisor, discussing my interpretation of the data, and discussing other possible ways that the students' work could be interpreted (Stake, 1995).

Developing Understanding of the Generality Requirement of Proof: The Case of Lexi

One of my main goals for the design study was to help students develop understanding of the generality requirement. Thus, when looking closely at the learning process of a single study, it made sense to focus on this topic since it was central throughout the study. After reviewing the interview data for the four focus students –

Amanda, Lexi, Heather, and Clay – I decided to focus on Lexi for my case study analysis based on the fact that her learning trajectory seemed to mirror many of the non-case study participants, while also having some particularly interesting aspects of her trajectory (in particular, her use of variables). Lexi was also a strong candidate for the case study analysis because she was neither the strongest student (Amanda) or the weakest student (Heather) of the focus students, she regularly spoke up during the sessions, and was generally able to explain her thinking more clearly during the interviews than Clay.

Data Reduction. After deciding to focus on the case of Lexi, I began my analysis process by systematically reviewing all of the data collected in the study featuring Lexi in chronological order. As I reviewed each session or interview, I documented all instances where Lexi made a comment or question that indicated her understanding of the generality requirement at that time. During this process, I also reviewed her written work and responses to journal prompts and documented anything relevant to the generality requirement. This process allowed me to eliminate session 2 (Lexi was absent), sessions 9 – 10 (defining similar polygons), and session 13 (mostly involved students working silently on a task).

Next, I reviewed all of my summaries to look for themes in the ways that the generality requirement was evident in the tasks and then began writing short memos describing Lexi's current understanding of the generality requirement at different points in the study. When looking across the data in the study, I noticed five themes in regards to shifts in her understanding of the generality requirement. First, I noticed that Lexi had transitioned from thinking that it was not possible to know whether a statement was always true in the initial interview to thinking that it was possible during the final interview. Second, I noticed that she tended to evaluate the provided solutions to proof tasks in the four interviews according to whether then explain why during focus interview 1, focus interview 2, and the final interview; additionally, the criteria that a proof should explain why surfaced *prior* to the criteria that proofs should be general (adhere to the generality requirement). Third, Lexi began using language such as “always” while constructing arguments beginning in focus interview 2. Fourth, she seemed to view variables as a place to insert numbers, rather than something that represents all possible numbers (initial interview, session 6, final interview). Fifth, she demonstrated slightly different strategies for proving a mathematical statement was false. Specifically, in session 12, she appeared to conclude a statement was false after constructing a specific diagram, whereas in the final interview, she demonstrated that the given statement was false by referencing a class of counterexamples. Next, I re-categorized my five themes based on key features of the tasks that provided the context for revealing a particular aspect of her understanding of the generality requirement. This yielded instances where Lexi was constructing an argument for a true mathematical claim, instances where she was evaluating provided arguments for a true mathematical claim, instances where she was constructing an argument for a false mathematical claim, and instances where she used variables to represent all possible numbers.

When analyzing Lexi’s arguments for proof tasks, I attended to the language she used in order to gain insights into her understanding of proofs. For example, I analyzed Lexi’s use of the phrase “so like” when describing her response to a proof task to indicate that she was about to provide an example to help illustrate her mathematical thinking. I also focused on her use of language such as “always” or “any” as indicators of her attending to the generality requirement in her statement. Afterwards, I made constant comparisons between each unit of analysis both within and across categories to look for similarities and differences in her understanding. For example, after documenting the criteria Lexi used when evaluated provided responses in interview 1, I compared the criteria when her constructed response on the same proof task in interview 1 to triangulate the data as well as the criteria she used when evaluating provided responses in interviews 2 – 4 in order to capture changes in the criteria she evoked. I also noted instances where Lexi's statements could be interpreted multiple ways and conducted multiple reviews of the data to look for disconfirming evidence and possible rival interpretations (Strauss & Corbin, 1998; Yin, 2014). In order to ensure trustworthiness of findings, I shared data clips as well as my preliminary findings related to the selected data with my advisor, someone who was familiar with the study and literature related to proof at the secondary level, and another graduate student, someone who was not as familiar with my study or the relevant literature, to discuss my interpretation of the data as well as other possible alternate interpretations. Whenever alternate interpretations arose from our conversations, I incorporated them into my findings or adjusted my initial interpretation if our discussion revealed a more plausible explanation.

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VITA

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