THE VANISHING OF THE CHOW COHOMOLOGY RING FOR AFFINE TORIC VARIETIES AND ADDITIONAL RESULTS

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Ryan Richey

Dr. Dan Edidin, Dissertation Supervisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by Ryan Richey, a candidate for the degree of Doctor of Philosophy of

Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

Professor Dan Edidin

Professor Zhenbo Qin

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Professor Saku Aura

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Ryan Richey

Dr. Dan Edidin, Dissertation Supervisor

ABSTRACT

From the recent work of Edidin and Satriano, given a good moduli space morphism between a smooth Artin stack and its good moduli space X, they prove that the Chow cohomology ring of X embeds into the Chow ring of the stack. In the context of toric varieties, this implies that the Chow cohomology ring of any toric variety embeds into the Chow ring of its canonical toric stack. Furthermore, the authors give a conjectural description of the image of this embedding in terms of strong cycles. One consequence of their conjectural description, and an additional conjecture, is that the Chow cohomology ring of any affine toric variety ought to vanish. We prove this result without any assumption on smoothness. Afterwards, we present a series of results related to their conjectural description, and finally, we provide a conjectural toric description of the image of this embedding for complete toric varieties by utilizing Minkowski weights.

Chapter 1

Introduction

Intersection theory on singular schemes has historically been troublesome due to the difficulty in defining intersection multiplicities and an intersection product satisfying the expected geometric properties. For example, a natural condition an intersection product ought to satisfy is some form of the moving lemma [9, Theorem 1.6]. This lemma guarantees that given two Chow classes $\alpha, \beta \in A_*(X)$, there ought to exist generically transverse cycles $A, B \in Z(X)$ representing α, β respectively such that $\alpha \cdot \beta = [A \cap B]$ (the intersection product is represented by the class of $A \cap B$). In the discussion after [9, Theorem 1.6], they enumerate several examples detailing the failure of the moving lemma in a variety of ways; in particular, there are examples of varieties with no intersection product satisfying this lemma.

In the work of [10, Chapter 17], Fulton introduces the Chow cohomology ring of any scheme which does carry an intersection product. If X is smooth, then this ring is isomorphic as rings to the ordinary Chow ring with the intersection product as defined in [10, Chapter 6]. The definition of this ring and its intersection product essentially guarantees it to be the weakest intersection theory on any scheme, in the sense that any other conception of intersection theory ought to map to the Chow cohomology ring [18, Section 8]. Unfortunately, Fulton's Chow cohomology ring and intersection product still seem to lack geometric significance. However, there is an approach using intersection theory on stacks.

If G is an algebraic group acting on a scheme X, the work of [5] defined the G-equivariant Chow groups and Chow cohomology rings of X in a manner similar to Borel's equivariant cohomology. Furthermore, they show that their constructions satisfy all of the same properties as the ordinary Chow groups and cohomology rings. One of the most important applications of this theory is towards providing a means of understanding intersection theory on quotient stacks. Given a quotient stack $[X/G]$, the intersection theory of this stack is defined to be the G-equivariant intersection theory of X.

Now suppose $\mathcal{X}$ is a smooth Deligne-Mumford quotient stack, X is a scheme, and there exists a morphism $\mathcal{X} \to X$. If this morphism is a coarse moduli space morphism, then there is a pushforward isomorphism $A^*(\mathcal{X})_{\mathbb{Q}} \cong A^*(X)_{\mathbb{Q}}$. The intersection product on $\mathcal{X}$ thus descends to an intersection product on $A^*(X)_{\mathbb{Q}}$; since $\mathcal{X}$ is smooth, its intersection product does possess geometric significance. Furthermore, by [19], every variety with at worst finite quotient singularities can be realized as the coarse moduli space of some smooth Deligne-Mumford quotient stack, and hence, such a variety possesses an intersection product on its rational Chow groups.

Even if $\mathcal{X} \to X$ is not a coarse moduli space morphism and $\mathcal{X}$ is not Deligne-Mumford (but is assumed to be Artin), the work of [8] demonstrates that there can exist maps between their Chow cohomology rings. In particular, if $\mathcal{X} \to X$ is a good moduli space morphism, then the authors of the previous paper is able to construct a pullback map $A^*_{op}(X)_{\mathbb{Q}} \to A^*(\mathcal{X})_{\mathbb{Q}}$ and show that $A^*_{op}(X)_{\mathbb{Q}}$ embeds as a subring into $A^*(\mathcal{X})_{\mathbb{Q}}$. Thus, the Chow cohomology ring once again inherits a geometrically significant intersection product. However, merely knowing that the Chow cohomology ring embeds as a subring is still somewhat unsatisfactory; instead, it would be far more meaningful to be able to explicitly identify what subring the Chow cohomology ring corresponds to. In this same paper, the authors are also able to show that the subring is contained in the group of topologically strong cycles; however, there are examples of topologically strong cycles which are not the pullback of operational classes, and hence, the image is not characterized by being topologically strong. To attempt in identifying the image, the authors conjecture that the Chow cohomology ring ought to correspond to the ring of strong cycles. They are able to prove this in dimension 3 and in a variety of other cases.

1.1 Main Results

Following their conjecture on the relationship of the image to strong cycles, we first consider the consequence of this conjecture for affine toric varieties. In conjunction with another conjecture, one ought to predict that the Chow cohomology ring of any affine toric variety should vanish. We confirm this conjecture, and hence, provide indirect evidence of towards the conjecture concerning strong cycles. The proof is provided in Chapter 6.

Theorem 1.1.1. *Given any affine toric variety $X(\sigma)$, $A^k_{op}(X(\sigma)) = 0$ for $k > 0$. In particular, $A^*_{op}(X(\sigma)) = \mathbb{Z}$.*

Besides this primary result, we also give a series of minor results which are consequences of what [8] has proved. In particular, we show that the Chow cohomology ring of any three dimensional toric variety has the structure of an ideal in the Chow ring of its canonical toric stack, and furthermore, we show this structure fails in higher dimensions. Furthermore, in dimension three, we give a conjectural description of what this ideal I is in terms of the intersection of kernels. Also, we prove that any variable corresponding to a ray which does not lie in any non-simplicial cone generates an ideal contained in the Chow cohomology ring. Finally, we give a conjectural toric description of the image in terms of Minkowski weights.

Chapter 2

Intersection Theory

In this chapter, we will review the relevant material from intersection theory on schemes and quotient stacks necessary for the later chapters. We begin by reviewing the Chow groups of a scheme and the properties they enjoy following [10]. Note, these are the algebraic-geometric analogues of the homology groups of a topological space and serve as a foundation for the remaining discussion on intersection theory; in particular, the Chow groups are also referred to as the Chow homology groups. Following this, we will proceed to discuss the algebraic-geometric analogue of the cohomology ring of a topological space called the Chow cohomology ring. However, the construction of the Chow cohomology ring requires extended work in contrast to its topological analogue.

Afterwards, we discuss equivariant intersection theory and the construction of the equivariant Chow homology and cohomology groups following [5]. These satisfy the same formal properties as the ordinary Chow homology and cohomology groups, but their construction is quite different. With these in hand, the construction of the Chow homology and cohomology groups of a quotient stack are defined as the equivariant Chow homology and cohomology groups. We recall an important computation due to [3] to be used later in the paper concerning computing the equivariant Chow rings of certain open subschemes of $\mathbb{A}^n$.

Finally, we end this section with a brief survey of the main result of [8]. Given a good moduli space of a stack, the Chow cohomology ring of the good moduli space embeds into the Chow cohomology ring of the stack. This will serve as the motivating result for Chapters 5 and 6.

For this section, we adhere to the following notation and definitions. X will be always be a scheme over $\mathrm{Spec}(k)$ where k is an algebraically closed field. A subvariety of X is an integral, separated, closed subscheme of X. A point will be a closed point. The field of rational functions of a subvariety $V \subset X$ is the field of fractions of $\mathcal{O}_{V,X}$, denoted $R(V)$, and the group of non-zero rational functions is denoted $R(V)^*$.

2.1 Chow Groups

Following [10, Chapter 1], a k-dimensional **prime cycle** is a k-dimensional subvariety of X, let $Z_k(X)$ denote the free $\mathbb{Z}$-module generated by all k-dimensional prime cycles; a general element of $Z_k(X)$ is called a **k-cycle**.

Let $W \subset X$ be a $(k+1)$-dimensional subvariety, then there is a natural means of associating a k-cycle to each non-zero rational function on W. For if $r \in R(W)^*$ and $V \subset W$ is a codimension one subvariety of W, then by [10, Section 1.2 and Appendix A], there is a group homomorphism

$$\mathrm{ord}_V : R(W)^* \to \mathbb{Z},$$

wherein the value of $\mathrm{ord}_V(r)$ represents the order of vanishing of r along V. Since there are only finitely many subvarieties of W for which $\mathrm{ord}_V(r) \neq 0$, this produces a well-defined k-cycle defined by:

$$\operatorname{div}(r) = \sum_{V \subset W, \operatorname{codim}_W(V)=1} \operatorname{ord}_V(r)[V].$$

The free $\mathbb{Z}$-module generated by all k-cycles derived from rational functions on subvarieties of X is denoted $\operatorname{Rat}_k(X)$ which is a subgroup of $Z_k(X)$.

Definition 2.1.1. The k-th **Chow group** of X is the quotient group

$$A_k(X) := Z_k(X)/\operatorname{Rat}_k(X).$$

The direct sum of all Chow groups is denoted $A_*(X) = \oplus_{k \geqslant 0} A_k(X)$.

The Chow groups satisfy functorial properties with respect to proper morphisms and flat morphisms. Associated to any proper map $f : X \to Y$, [10, Section 1.4] demonstrates the existence of an induced covariant map on Chow groups called the **pushforward** of f:

$$f_* : A_k(X) \to A_k(Y).$$

Hence, if X is complete over k, there is a natural morphism of 0-th Chow groups sending a 0-cycle on X to a 0-cycle on $\operatorname{Spec}(k)$. Since $A_0(\operatorname{Spec}(k))$ is freely generated by the class $[\operatorname{Spec}(k)]$, $A_0(\operatorname{Spec}(k)) \cong \mathbb{Z} \cdot [\operatorname{Spec}(k)]$, it follows that any class in $A_0(\operatorname{Spec}(k))$ is uniquely determined by the integer for which it is a mutliple of $[\operatorname{Spec}(k)]$.

Definition 2.1.2. If X is complete over k, then the **degree** of $\alpha \in A_0(X)$, denoted as either $\deg_X(\alpha)$ or $\int_X(\alpha)$, is defined to be the unique integer such that the pushfoward of α is equal to this integer multiplied against $[\operatorname{Spec}(k)]$ in $A_0(\operatorname{Spec}(k)$.

Furthermore, associated to any flat map of relative dimension n $g : X \to Y$, [10, Section 1.7] demonstrates the existence of an induced contravariant map on Chow groups called the **pullback** of g:

$$g^* : A_k(Y) \to A_{k+n}(X)$$

. Moreover, the pullback and pushforward satisfy further compatibility properties together.

With the previous two concepts defined, the following proposition provides one of the primary means of computing Chow groups.

Proposition 2.1.3. *[10, Proposition 1.8] Let Y be a closed subscheme of X and $U = X \setminus Y$. If $i : Y \to X$ and $j : U \to X$ dnenote the canonical inclusions, and $i_* : A_*Y \to A_*X$ is the pushforward of i and $j^* : A_*X \to A_*U$ is the pullback of j, then the following sequence is exact for all k:*

$$A_k(Y) \to A_k(X) \to A_k(U) \to 0.$$

2.1.1 Chern Classes

In preparation for our discussion of the Chow cohomology ring, for each vector bundle $V \to X$ of rank r, there exists a collection of morphisms on the Chow groups called the **Chern classes** of E, denoted $c_i(E)$. Although our utility for them is limited in this thesis, the theory of Chern classes is absolutely integral to the study of intersection theory and will serve a motivation for the definition of the Chow cohomology ring. Geometrically, the Chern classes can be identified with classes in $A_*(X)$ representing the degeneracy loci of the appropriate number of global sections of E, see [9, Theorem 5.3(b)] and the previous discussion.

In particular, each Chern class defines a morphism of groups:

$$c_i(E) : A_k(X) \to A_{k-i}(X),$$

whose value on $\alpha \in A_k(X)$ is denoted $c_i(E) \cap \alpha$. These morphisms satisfy a variety of compatibility properties: dimensional vanishing, commutativity, compatible with proper pushforwards, compatible with flat and l.c.i. pullbacks, and more [10, Section 3.2 and Ch. 6].

We conclude this section with one last useful result: the Chow groups of a vector bundle is isomorphic to the Chow groups of X with a shift in degree.

Theorem 2.1.4. *[10, Theorem 3.3(a)] Let $\pi : E \to X$ be a vector bundle of rank r over X, then the flat pullback*

$$\pi^* : A_{k-r}X \to A_kE$$

is an isomorphism for all k.

2.2 Chow Cohomology

2.2.1 Chow Rings of Smooth Schemes

Assume that X is smooth, then the diagonal morphism $\delta_X : X \to X \times_k X$ is a regular embedding; hence, one can define the **intersection product** of two Chow classes $\alpha, \beta \in A_*(X)$ by utilizing the Gysin homomorphism of the diagonal [10, Section 6.2]:

$$\alpha \cdot \beta := \delta_X^*(\alpha \times \beta).$$

The Gysin homomorphism is derived from Fulton's construction of the intersection product by utilizing the deformation to the normal cone [10, Chapter 5 and Section

6.1]. If X is further assumed to be equidimensional and $\dim(X) = n$, then the Chow cohomology ring of X has the following simple definition.

Definition 2.2.1. Under the above assumptions, the k-th **Chow cohomology group** of X is denoted by $A^k(X)$ and defined to be

$$A^k(X) = A_{n-k}(X).$$

The **Chow ring** of X is the direct sum of all Chow cohomology groups, $A^*(X) = \oplus_{k \geqslant 0} A^k(X)$.

With this definition, the intersection product described above defines a product structure on $A^*(X)$:

$$A^k(X) \times A^l(X) \to A^{(k+l)}(X).$$

This makes $A^*(X)$ into a commutative, associative, graded ring with unity.

Remark 2.2.2. Given $\alpha \in \mathbb{A}^k(X)$, then intersection against α defines a morphism of groups: $A_l(X) \to A_{l-k}(X)$ where $\beta \mapsto \alpha \cdot \beta$. From this perspective, elements of the Chow ring of X define operations on the Chow groups of X analogous to the definition of Chern classes in the previous subsection; this will be the fundamental observation used to generalize the concept of Chow rings to singular schemes in the next section.

2.2.2 Chow Cohomology Rings

If X is not smooth, then the diagonal morphism is not a regular embedding, and hence, Fulton's construction of the intersection product is not applicable. To ameliorate this situation and to make sense of the Chow ring of a singular scheme, [10, Chapter

17] generalizes the concept of Chern classes and defines the Chow cohomology ring to consist of all formal operations on the Chow groups which satisfy analogous properties of Chern classes.

Definition 2.2.3. [10, Definition 17.1 and 17.3] The k-th **Chow cohomology group**, denoted $A^k_{op}(X)$, is defined as follows: an element $c \in A^k_{op}(X)$ is a collection of homomorphisms of groups of the form:

$$c_g^{(p)} : A_p X' \to A_{p-k} X',$$

for every morphism $g : X' \to X$, for every p, and which are compatible with respect to proper pushforward, and flat and l.c.i. pullbacks. An element $c \in A^k_{op}(X)$ is often called an **operation** or **operational class**.

Given two such operational classes as defined above, the composition of these classes will serve the role as the intersection product.

Definition 2.2.4. Given $c \in A^k_{op}(X)$ and $d \in A^l_{op}(X)$, define $c \cdot d \in A^{k+l}_{op}(X)$ as follows: if $g : X' \to X$ is a morphism of schemes, then

$$(c\dot{d})_g^{(p)} = c_g^{(p-l} \circ d_g^{(p)} : A_p(X') \to A_{p-l}(X') \to A_{p-l-k}(X').$$

Under the above intersection product, the graded group $\oplus_{k \geqslant 0} A^k_{op}(X)$ is a commutative, associative, graded ring with unity. Note, in [10, Example 17.3.3], this graded ring does satisfy dimensional vanishing: if $k > \dim(X)$, then $A^k_{op}(X) = 0$.

Definition 2.2.5. The **Chow cohomology** or **operational Chow ring** of X is the previously defined graded ring denoted

$$A^*_{op}(X) := \oplus_{k \geqslant 0} A^k_{op}(X).$$

If X is smooth, then [10, Corollary 17.4] proves that the Poincaré duality map $A^k_{op}(X) \to A^k(X)$ is an isomorphism of rings, wherein, the intersection product defined by composition corresponds to the intersection product as defined in the previous section. Hence, this conception of the Chow cohomology ring through operations does generalize the construction in the previous section.

Remark 2.2.6. In [10] and other sources, the Chow cohomology ring is also simply denoted $A^*(X)$ without the subscript 'op', regardless if X is smooth is or not. We will deviate from this convention to emphasize the distinction between operational classes in the smooth and non-smooth cases.

2.2.3 Kimura's Exact Sequence

Due to the general definition of operational classes, the Chow cohomology ring is often difficult to compute. However, there is a method due to [16] which can allow us to compute the Chow cohomology ring of X, given that we have an envelope $X' \to X$.

Definition 2.2.7. A morphism $X' \to X$ is an **envelope** if it is proper, and for each closed subvariety $V \subset X$, there exists a closed subvariety $V' \subset X'$ such that the restriction $V' \to V$ is birational.

The following exact sequence due to Kimura provides of means of computing the Chow cohomology of X inside the Chow cohomology of an envelope.

Theorem 2.2.8. *[16, Theorem 2.3] (Kimura's Exact Sequence) Let $X' \to X$ be an envelope which is an isomorphism outside of $S \subset X$ and $E \subset X'$, then*

$$0 \to A^k_{op}(X) \to A^k_{op}(X') \oplus A^k_{op}(S) \to A^k_{op}(E)$$

is exact for each k.

Remark 2.2.9. By [10, Example 17.3.2] or [16, Lemma 2.1], given an enveleope $X' \to X$, the pullback along this morphism already induces an injection $A^*_{op}(X) \subset A^*_{op}(X')$.

Remark 2.2.10. If $X' \to X$ is merely proper and surjective, then above sequence is still exact if tensored with $\mathbb{Q}$.

Thus, if one is able to compute the Chow cohomology rings of X', S, and E along with the kernel of the above map, then the Chow cohomology ring of X is in theory computable.

2.3 Equivariant Intersection Theory

For this section, let G be an algebraic group over $\mathrm{Spec}(k)$, with $\dim(G) = g$, which acts on X. The quotient stack $[X/G]$ is the fibered category defined in the usual manner: over a scheme S, $[X/G](S)$ consists of all G-torsors over S with a G-equivariant morphism to X, and morphisms consist of isomorphisms forming Cartesian diagrams which commute with the G-equivariant morphisms to X. If $X = pt$, then the quotient stack of X by G is the classifying stack of G, denoted BG.

2.3.1 Chow Homology and Cohomology of Quotient Stacks

Following [5], the equivariant Chow groups are constructed in a somewhat analogous manner to Borel's construction of equivariant cohomology, with the role of EG replaced by certain open subsets in representations of G. To construct such groups, begin by choosing an l-dimensional representation V of G which contains an open subset $U \subset V$ on which G acts freely and $\mathrm{codim}(V - U) \geqslant (n - i)$.

Definition 2.3.1. [5, Definition-Proposition 1] Let $X_G := (X \times U)/G$, then the i-th **equivariant Chow group** of X, denoted $A_i^G(X)$, is defined as:

$$A_i^G(X) := A_{i+l-g}(X_G).$$

Remark 2.3.2. Note, the fact that this definition is independent of the choice of representation is also contained in [5, Definition-Proposition 1]. Also, due to the above definition, the equivariant Chow groups may not vanish in high degree, which is unlike the ordinary theory of Chow groups.

The equivariant Chow groups satisfy the same formal properties ([10, Chapters 1-6]) as the ordinary Chow groups (see [5, Sections 2.1-2.5]).

Definition 2.3.3. The i-th **equivariant Chow cohomology group** of X, denoted $A^i_G(X)$, is defined to consist of operations on the equivariant Chow groups satisfying the same properties as mentioned previously for all G-equivariant morphisms $X' \to X$. Furthermore, the **equivariant Chow cohomology ring** of X is $A^*_G(X) := \oplus_{k \geqslant 0} A^k_G(X)$.

With the above definitions established, we can next turn towards intersection theory on quotient stacks.

Definition 2.3.4. The i-th **Chow group** of the quotient stack $[X/G]$, denoted $A_i([X/G])$, is the $(i-g)$-th G-equivariant Chow group of X:

$$A_i([X/G]) := A^G_{i-g}(X).$$

The i-th **Chow cohomology group** of the quotient stack $[X/G]$, denoted $A^i([X/G])$, is defined to consist of operations on $A_pX' \to A_{p-i}X'$ for all morphisms $X' \to [X/G]$ where X' is a scheme and satisfy the same properties as mentioned previously. Furthermore, the **Chow cohomology ring** of $[X/G]$ is $A^*([X/G]) := \oplus_{k\geqslant} A^k([X/G])$.

Proposition 2.3.5. *[5, Section 5.3] The above definitions are independent of the presentation of $[X/G]$; i.e., if $[X/G] \cong [Y/H]$, then the above groups are all isomorphic. Furthermore, the Chow cohomology ring of $[X/G]$ is isomorphic to the G-equivariant Chow cohomology ring of X:*

$$A^*([X/G]) \cong A^*_G(X).$$

2.3.2 The Chow Ring of Certain Open Subschemes of $\mathbb{A}^n$

We end this section with a computation due to [3] computing the G-equivariant Chow rings of certain open subschemes of $\mathbb{A}^n$ where G is diagonalizable. The usefulness of this result will be seen in later sections dealing with computing Chow rings of canonical toric stacks. Let G be a diagonalizable group with a faithful action on $\mathbb{A}^n$ (G injects into $\mathbb{G}_m^n$), and $X = \mathbb{A}^n - Z$ where $Z = L_1 \cup ... \cup L_l$ with L_i being linear, G-invariant subspaces, then the goal of this subsection is to compute $A^*_G(\mathbb{A}^n - Z)$.

We begin by applying the excision exact sequence for equivariant Chow groups to identify $A^*_G(\mathbb{A}^n - Z)$ as a cokernel:

$$A^*_G(Z) \to A^*_G(\mathbb{A}^n) \to A^*_G(\mathbb{A}^n - Z) \to 0.$$

By the vector bundle property of Chow groups, $A^*_G(\mathbb{A}^n) \cong \mathbb{A}^*_G(pt)$; for brevity, $A^*_G(pt)$ will be denoted simply by A^*_G. Recall, BG denotes the classifying stack of G and is the quotient stack $[pt/G]$; hence, A^*_G is the Chow cohomology ring of BG. If G is diagonalizable, then $A^*_G \cong Sym(X(G))$ is the symmetric algebra of the character group of G, and hence, morphisms between character groups give rise to morphisims of equivariant Chow rings of points. In our context, since the action is assumed to be faithful, the injection $G \subset \mathbb{G}_m^n$ induces a surjection of character groups $X(\mathbb{G}_m^n) \to X(G)$, and further, it induces a surjection of equivariant Chow rings $A^*_{\mathbb{G}_m^n} \to A^*_G$.

Definition 2.3.6. In the above setup, define $C(G) = \ker(A^*_{\mathbb{G}_m^n} \to A^*_G$. Equivalently, it is the linear ideal generated by the relations among the image of $X(\mathbb{G}_m^n)$ in $X(G)$.

Hence, $A^*_G(\mathbb{A}^n)$ in the excision sequence has been computed as a quotient of $A^*_{\mathbb{G}_m^n}$ which has been computed as mentioned previously. For the image of the kernel, since each L_i is linear and isomorphic to $\mathbb{A}^k$ for some k, it follows that the image of $A^*_G(Z)$ is the ideal generated by the equivariant fundamental classes of each L_i in A^*_G.

Definition 2.3.7. In the above setup, let $\mathcal{Z}$ denote the ideal of A^*_G generated by the images of the equivariant fundamental classes of each L_i.

Thus, by combining the above definitions and results, we obtain:

$$A^*_G(\mathbb{A}^n - Z) \cong A^*_G/\mathcal{Z} \cong A^*_{\mathbb{G}_m^n}/(C(G), \mathcal{Z}).$$

Each of the above ideals can be explicitly computed. Let $e_1, ..., e_n$ denote the canonical basis characters of $X(\mathbb{G}_m^n) \cong \mathbb{Z}^n$, then associated to each character is a 1-dimensional representation, and we set $t_1, ..., t_n$ to be the first Chern class of the respective 1-dimensional representation. By [5], $A^*_{\mathbb{G}_m^n} = \mathbb{Z}[t_1, ..., t_n]$. For the linear equivalence ideal, let $v_1, ..., v_k$ be a basis for $\ker(X(\mathbb{G}_m^n) \to X(G))$, then $C(G)$ is the ideal generated by the corresponding first Chern classes of $v_1, ..., v_k$ expressed in terms of $t_1, ..., t_n$:

$$v_i = \sum_j a_{ij} e_j \Rightarrow \sum_j a_{ij} t_j \in C(G).$$

For the ideal $\mathcal{Z}$, let L_i be an irreducible component, then $L_i = V(x_{i_1}, ..., x_{i_l})$ where $x_1, ..., x_n$ are coordinates for $\mathbb{A}^n$, then the equivariant fundamental class of L_i is $t_{i_1} \cdot ... \cdot t_{i_l}$.

Chapter 3

Toric Varieties

The purpose of this chapter is to review the relevant material from the theory of toric varieties necessary for the later chapters. Recall, a torus is an algebraic group isomorphic as a group and isomorphic as a scheme to $\mathbb{G}_m^n$. A **toric variety** is a variety containing a torus T as a dense open subscheme, wherein the action of T on itself extends to an action on the entire variety. We begin by discussing the correspondence between normal toric varieties and cones and fans along with recalling an assortment of related notions. Following this, we discuss the geometry of the action of the torus on the variety through the Orbit-Cone Correspondence. Then we conclude the first section with a discussion of toric morphisms, the behavior of orbit closures under such morphisms, and define the additional concepts to be used later.

In the second section, we review the Cox construction of a toric variety. This expresses a toric variety as a good categorical quotient of an open subscheme of $\mathbb{A}^n$ modulo a diagonalizable subgroup of the torus of $\mathbb{A}^n$. The importance of this construction will be seen throughout Chapter 4; it will serve as the foundation for the construction of the canonical toric stack, and the various notions associated to the Cox construction will appear in computing the Chow ring of the canonical toric stack. Finally, we end this chapter with discussing the main results of [12] concerning intersection theory on toric varieties.

3.1 Basic Concepts

3.1.1 Cones

Following [11] and [4], let k be an algebraically closed field, let N and M be dual lattices over $\mathbb{Z}$ with $\langle -, - \rangle$ thier associated perfect pairing (N and M will be the lattices of 1-parameter subgroups and characters of a torus respectively), and we denote their associated vector spaces over $\mathbb{R}$ by $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Recall, a subset $\sigma \subset N_{\mathbb{R}}$ is a convex polyhedral cone if for each $v, w \in \sigma$, $\lambda v + \mu w \in \sigma$ for all $\lambda \geqslant 0$ and $\mu \geqslant 0$, and σ is generated by finitely many vectors. The polyhedral cones corresponding to affine toric varieties must satisfy the following two additional properties.

1. A cone σ is **strongly convex** if $\sigma \cap (-\sigma) = 0$ (equivalently, σ contains no positive-dimensional linear subspace).

2. A cone σ is **rational** if σ is generated by vectors in the lattice N.

Throughout, a **cone** will always refer to a strongly convex, rational, polyhedral cone unless specified otherwise. Associated to any cone $\sigma \subset N_{\mathbb{R}}$, the **dual cone** of σ, denoted $\check{\sigma}$, can be constructed in $M_{\mathbb{R}}$ as follows:

$$\check{\sigma} = \{u \in M_{\mathbb{R}} : \langle v, u \rangle \geqslant 0, \text{ for all } v \in \sigma\}.$$

We have the following additional basic definitions:

1. The dimension of σ is the the dimension of the linear space $\mathbb{R} \cdot \sigma$ spanned by σ, denoted $\dim(\sigma)$.

2. A face of σ is the intersection of σ with the vanishing locus of any fixed $u \in \check{\sigma}$.

3. The codimension of a face τ of σ is $\dim(\sigma) - \dim(\tau)$.

4. A facet of σ is a face of codimension 1.

5. A ray of σ is a face of dimension 1.

6. A primitive generator of a ray of σ is the unique minimal lattice point on the ray.

Remark 3.1.1. The strong convexity assumption is further equivalent to the statement that $\{0\}$ is a face of σ.

Let $S_\sigma := \check{\sigma} \cap M$ be the integral points of the dual cone, and note that S_σ is an additive semigroup. In order to construct an affine variety from this semigroup, we need that S_σ is finitely generated, and this has been established in the following well-known lemma.

Lemma 3.1.2. *[4, Proposition 1.2.17] (Gordan's Lemma) If σ is a rational, convex, polyhedradl cone, then S_σ is a finitely generated semigroup.*

Associated to any finitely generated semigroup S, one can define the associated finitely generated group ring $k[S]$. As a k-vector space, it is additively generated by the symbols x^s where $s \in S$. Furthermore, the multiplication on $k[S]$ is induced from the addition on S:

$$x^s \cdot x^t := x^{s+t}.$$

Definition 3.1.3. Given a cone σ, the affine toric variety associated to σ, denoted $X(\sigma)$, is defined to be:

$$X(\sigma) := \mathrm{Spec}(k[S_\sigma]).$$

Remark 3.1.4. If $\sigma = \{0\}$, then $k[M] \cong k[x_i^{\pm 1}]_{i \in I}$ is the coordinate ring of $\mathbb{G}_m^n$ if $n = \dim(M_{\mathbb{R}})$. In this case, $X(0)$ is denoted T_N and will be the torus acting on our affine toric varieties as mentioned in the introduction. The following discussion will demonstrate how T_N embeds into and acts on $X(\sigma)$.

Suppose $\tau \subset \sigma$ is a face of σ, then there is a natural induced morphism of semigroups $S_\sigma \to S_\tau$. This morphism further induces a morphism of k-algebras $k[S_\sigma] \to k[S_\tau]$, and hence, it induces a morphism of affine varieties $X(\tau) \to X(\sigma)$. Relying on [11, Proposition 1.2], the following lemma demonstrates that this morphism is an open embedding.

Lemma 3.1.5. *[11] If τ is a face of σ, then the above map $X(\tau) \to X(\sigma)$ is an open embedding, and moreover, the image of $X(\tau)$ is a principle open subscheme of $X(\sigma)$.*

Thus, we can now properly motivate the definition of strong convexity and explain the existence of the torus T_N. For if σ is strongly convex, then $\{0\}$ is a face of σ, and hence, T_N is a principle open subscheme of $X(\sigma)$. To determine the action of T_N on $X(\sigma)$, note that a morphism of semigroups $S_\sigma \to k$ (where the latter is viewed as a multiplicative semigroup) corresponds bijectively to points of $X(\sigma)$. Hence, given a point $x : S_\sigma \to k$ of $X(\sigma)$ and a point $t : M \to k^*$ of T_N, the action t on x is determined by the morphism of semigroups obtaining by merely producing x and t.

Finally, we conclude this section with passing reference to the relationship between arbitrary affine toric varieties, as defined in the introduction, and affine toric varieties as described in this section.

Theorem 3.1.6. *[4, Theorem 3.1.5] Suppose X is an affine toric variety as defined in the introduction, then there exists a cone σ such that $X \cong X(\sigma)$ if and only if X*

is normal.

3.1.2 Fans

Definition 3.1.7. A **fan** Δ is a collection of cones in $N_{\mathbb{R}}$ such that:

1. for each cone $\sigma \in \Delta$, every face of σ is contained in Δ,

2. for each pair of cones $\sigma, \sigma' \in \Delta$, $\sigma \cap \sigma'$ is a face of each.

The concept of a fan is an analogue of a simplicial complex for cones. The importance of fans is that they induce toric varieties (which are in general not affine). Based upon the previous, for each cone $\sigma \in \Delta$, we obtain an affine toric variety $X(\sigma)$. For cones $\sigma, \sigma' \in \Delta$, their affine toric varieties can be glued together since $\sigma \cap \sigma'$ is simultaneously a face of each, and hence, $X(\sigma \cap \sigma'$ is simultaneously a principle open subset of $X(\sigma)$ and $X(\sigma')$.

Definition 3.1.8. Given a fan Δ, the toric variety associated to Δ, denoted $X(\Delta)$, is defined to be the variety obtained by previously described gluing procedure.

Remark 3.1.9. The scheme obtained by this gluing procedure is indeed separated [11, Section 1.4], and furthermore, the action of the torus T_N on each $X(\sigma)$ extends to an action on $X(\Delta)$. Thus, this definition does produce a toric variety as described in the introduction.

Similar to the conclusion of the previous section, we remark on the relationship between an arbitrary toric variety and one that arises from fans.

Theorem 3.1.10. *[4, Corollary 3.1.8] Suppose X is a toric variety as defined in the introduction, then there exists a fan Δ such that $X \cong X(\Delta)$ if and only if X is normal.*

3.1.3 Orbit-Cone Correspondence

The geometry of the action of T_N on $X(\Delta)$ can be easily described in terms of the combinatorics of Δ as follows. First, note that for each cone $\sigma \in \Delta$, there exists a distinguished point, denoted γ_σ, determined by the morphism of semigroups $S_\sigma \to k$ defined by either: if m vanishes on σ, then $m \mapsto 1$, and otherwise, $m \mapsto 0$.

Definition 3.1.11. Let $\sigma \in \Delta$, then:

1. $O(\sigma) := T_N \cdot \gamma_\sigma$ is the orbit of the distinguished point of σ,

2. $V(\sigma) = \overline{O(\sigma)}$ is the orbit-closure of the previous orbit.

The Orbit-Cone correspondence, detailed in the following theorem, demonstrates that all T_N-orbits arise in the above manner.

Theorem 3.1.12. *[4, Theorem 3.2.6(a),(b)] (Orbit-Cone Correspondence) There is a bijective correspondence between cones $\sigma \in \Delta$ and T_N-orbits defined by sending $\sigma \mapsto O(\sigma)$. Furthermore,* $\dim(O(\sigma)) = \dim(N_{\mathbb{R}}) - \dim(\sigma)$.

We conclude this subsection by describing how $V(\sigma)$ has the structure of a toric variety arising from a fan related to Δ.

Definition 3.1.13. 1. Given a cone σ, define $N(\sigma) = N/N_\sigma$ where N_σ is the lattice generated by $\sigma \cap N$.

2. [4, 3.2.8] Given a fan Δ and a cone $\sigma \in \Delta$, define the fan, denoted $Star(\sigma)$, as follows:

$$Star(\sigma) = \{\overline{\tau} \subset N(\sigma)_{\mathbb{R}} : \sigma \text{ is a face of } \tau \in \Delta\}.$$

Proposition 3.1.14. *[4, Proposition 3.2.7] If $\sigma \in \Delta$, then $V(\sigma) \cong X(Star(\sigma))$.*

3.1.4 Toric Morphisms and Star Subdivisions

The purpose of this subsection is to review the correspondence between equivariant morphisms of toric varieties and certain morphisms of lattices, and then describe an important class of examples which will be used throughout the later chapters.

Definition 3.1.15. 1. [4, Definition 3.3.1] Given a pair of lattices N', N and a pair of fans Δ', Δ in N', N respectively, a morphism of lattices $\phi : N' \to N$ is **compatible** with respect to the fans if for every cone $\sigma' \in \Delta'$, there exists a cone $\sigma \in \Delta$ such that $\phi_{\mathbb{R}}(\sigma') \subset \sigma$.

2. [4, Definition 3.3.3] Suppose $X(\Delta)$ and $X(\Delta')$ are toric varieties, then a morphism of varieties $\varphi : X(\Delta') \to X(\Delta)$ is **toric** if it sends the torus of $X(\Delta')$ to the torus of $X(\Delta)$ and the restriction of φ to the tori is a group morphism.

Theorem 3.1.16. *[4, Theorem 3.3.4] Toric morphisms correspond to compatible morphisms of lattices.*

Suppose $N = N'$ and $\phi = \mathrm{id}_N$, then if ϕ is compatible with respect to fans $\Delta', \Delta \subset N_{\mathbb{R}}$ and $|\Delta| = |\Delta'|$, we say that Δ' is a **refinement** of Δ. Note, although the underlying map of lattices is trivial, the associated map of toric varieties $X(\Delta') \to X(\Delta)$ is far from trivial. Suppose we fix a particular fan Δ, then among all possible refinements, we distinguish a special collection of such refinements.

Definition 3.1.17.
- Suppose σ has primitive generators $u_1, ..., u_n$, then the **star** of σ is the vector $Star(\sigma) = \sum_{i=1}^{n} u_i$.
- The **star subdivision** of σ is the fan, denoted σ^*, whose maximal cones are all of the form $Cone(Star(\sigma), \mu)$ where μ is a facet of σ.

- The **star subdivision** of a fan Δ with respect to a vector v, denoted $\Delta^*(v)$, is the fan consisting of all cones not containing v, and all cones of the form $Cone(\tau, v)$ where $v \notin \tau$ and $\{v\} \cup \tau$ is contained in a cone of Δ.

Proposition 3.1.18. *[4, Proposition 3.3.15, Lemma 11.1.3, and Proposition 11.1.6] Given a fan Δ, then $\Delta^*(\sigma)$ is a refinement for each cone $\sigma \in \Delta$, and the induced toric morphism identifies $X(\Delta^*(\sigma))$ with the blowup of $X(\Delta)$ at the distinguished point γ_σ. In particular, the induced toric morphism is a birational, proper map which is in fact projective.*

We conclude this section with understanding the behavior of orbit closures under toric morphisms.

Theorem 3.1.19. *[4, Lemma 3.3.21 and Theorem 11.1.10] With notation as above, suppose σ is a cone in Δ with orbit closure $V(\sigma) \subset X(\Delta)$. If $\sigma_1, ..., \sigma_k$ denote the minimal cones of Δ' such that σ is minimal over $\phi(\sigma_i)$ then the irreducible decomposition of the inverse image of $V(\sigma)$ is:*

$$\varphi^{-1}(V(\sigma)) = \bigcup_{i=1}^{k} V(\sigma_i).$$

In particular, in the case of a refinement, σ_i can be characterized as the minimal cones intersecting the relative interior of σ.

3.1.5 Simplicial, Smooth, Semi-Proper, and Semi-Projective

In this final subsection, we review a series of definitions and a few results to be used in later chapters.

Definition 3.1.20. 1. Given a cone σ, we say that σ is **simplicial** if its primitive generators are linearly independent over $\mathbb{R}$. A fan Δ is said to be **simplicial** if every cone contained in it is simplicial. The toric variety $X(\Delta)$ is simplicial if Δ is simplicial.

2. Given a cone σ, we say that σ is **smooth** if its primitive generators form part of an integral basis for N. A fan Δ is said to be **smooth** if every cone contained in it is smooth.

Simplicial toric varieties correspond to toric orbifolds; these are varieties which possess at worst finite quotient singularities. Smooth toric varieties are known to correspond to toric varieties whose associated fans are smooth [4, Theorem 3.1.19(b)]. Although the existence of resolutions of singularities is famously difficult, the construction of a resolution of singularities for singular toric varieties is quite explicit.

Theorem 3.1.21. *[4, Theorem 11.1.9] Every fan Δ has a refinement Δ' such that Δ' is smooth, Δ' contains every smooth cone of Δ, Δ' is obtained by a sequence of star subdivisions of Δ, and the associated toric morphism $X(\Delta') \to X(\Delta)$ is a projective resolution of singularities.*

Finally, we introduce two additional definitions to be used later.

Definition 3.1.22. A toric variety $X(\Delta)$ is **semi-projective/semi-proper** if the natural map $X(\Delta) \to \operatorname{Spec}(\Gamma(X(\Delta), \mathcal{O}_{X(\Delta)}))$ is projective/proper and $X(\Delta)$ has a torus fixed point.

Remark 3.1.23. Given a cone σ, the toric variety associated to the star subdivision of σ has a special relationship between $X(\sigma^*)$ and $X(\sigma)$, namely, $X(\sigma^*) \to X(\sigma)$ is a projective toric morphism, or in terms of the language that follows, $X(\sigma^*)$ is a semi-proper and semi-projective toric variety.

3.2 Cox Construction of a Toric Variety

The Cox construction recognizes a toric variety as a good quotient of an open subspace of affine space modulo a diagonalizable subgroup of a torus. The importance of this construction will be seen in Chapter 4 concerning the construction of the canonical stack of an arbitrary toric variety, and moreover, the related notions appearing in the Cox construction serve an essential role in computing the Chow rings of the canonical stacks.

3.2.1 Exceptional Set

Following [4, Chapter 5], given a full-dimensional fan Δ on a lattice N, the total coordinate ring of $X(\Delta)$ is the polynomial ring $k[x_\rho : \rho \in \Delta(1)]$ freely generated by the variables x_ρ in bijective correspondence with the rays. Suppose $n = |\Delta(1)|$ such that $\mathbb{A}^n = \mathrm{Spec}(k[x_\rho : \rho \in \Delta(1)])$, then for each cone $\sigma \in \Delta$, define the monomial:

$$x^\sigma = \prod x_{\rho_i} \text{ where } \{\rho_i\} \text{ are the rays not ontained in } \sigma.$$

As σ runs over all cones of Δ, these monomials form an ideal whose vanishing locus will be the exceptional set.

Definition 3.2.1. Define $B(\Delta) = (x^\sigma)_{\sigma \in \Delta}$ to be the ideal generated by these monomials, then the **exceptional set** of Δ is the following closed subscheme of $\mathbb{A}^n$:

$$Z(\Delta) := \mathbb{V}(B(\Delta)) \subset \mathbb{A}^n.$$

The open subscheme $\mathbb{A}^n - Z(\Delta)$ modulo a certain diagonalizable group will present $X(\Delta)$ as a good quotient. We remark that $\mathbb{A}^n - Z(\Delta)$ is a toric variety described by the following fan.

Definition 3.2.2. Given a fan Δ, the **canonical fan** of Δ is the fan denoted Σ_Δ which is generated by cones of the form $\overline{\sigma}$, for $\sigma \in \Delta$, where:

$$\overline{\sigma} = Cone(e_\rho : \rho \in \sigma(1)) \subset \mathbb{R}^{|\Delta(1)|}.$$

Proposition 3.2.3. *[4, Proposition 5.1.9 (a) and (b)] The toric variety $X(\Sigma_\Delta)$ is isomorphic to $\mathbb{A}^n - Z(\Delta)$. Furthermore, the morphism of lattices $\mathbb{Z}^n \to N$ defined by sending the canonical basis vectors to the primitive generators induces a toric morphism $\mathbb{A}^n - Z(\Delta) \to X(\Delta)$.*

Before proceeding to discuss the diagonalizable group of interest, the irreducible components of $Z(\Delta)$ serve an important role in describing the Chow ring of the canonical stack, and fortunately, they possess a simple combinatorial interpretation.

Definition 3.2.4. A set of rays $C \subset \Delta(1)$ is a **primitive collection** if the following holds:

1. For each proper subset $C' \subset C$, there exists $\sigma \in \Delta$ such that $C' \subset \sigma(1)$.

2. There does not exist a cone $\sigma \in \Delta$ such that $C \subset \sigma$.

Note, given a primitive collection C, we can generate a linear subspace $\mathbb{V}((x_{\rho_i})_{i \in C})$.

Proposition 3.2.5. *[4, Proposition 5.1.6] The irreducible components of $Z(\Delta)$ are all of the form $\mathbb{V}((x_{\rho_i})_{i \in C})$ where C is a primitive collection of rays.*

3.2.2 The Subgroup of the Torus

To define the diagonalizable group G for which $\mathbb{A}^n - Z(\Delta)$ will be the quotient of, we begin as follows: let $\dim(\Delta) = k$, then define a morphism of lattices $\mathbb{Z}^n \to \mathbb{Z}^k$ by sending the canonical basis vectors to the minimal generators of the rays of Δ. Note, this morphism has finite cokernel; hence, the dual map is injective. If we set L to be the cokernel of this dual map, then the group G is defined to be $\mathrm{Hom}(L, \mathbb{G}_m)$, the group of characters of L, which injects into the torus of $\mathbb{A}^n$. The following lemma provides an additional information concerning the structure of G.

Lemma 3.2.6. *[4, Lemma 5.1.1] Let G be defined as above, then*

1. *$Cl(X(\Delta))$ is the character group of G.*
2. *G is isomorphic to a product of a torus and a finite abelian group, and in particular, G is reductive.*
3. *[4, Theorem 5.1.9c] The toric morphism $\mathbb{A}^n - Z(\Delta) \to X(\Delta)$ is constant on G-orbits.*

3.2.3 The Cox Construction

With the group G and $\mathbb{A}^n - Z(\Delta)$ defined as above, the Cox construction is summarized in the following theorem.

Theorem 3.2.7. *[4, Theorem 5.1.11(a)] Given the toric morphism $\mathbb{A}^n - Z(\Delta) \to X(\Delta)$ described above, then $X(\Delta)$ is the good quotient of $\mathbb{A}^n - Z(\Delta)$ modulo the action of G.*

3.3 Intersection Theory on Toric Varieties

3.3.1 Chow Groups of Toric Varieties

Following [12, Section 1], the generators and relations of the Chow groups of a toric variety are well-understood and are described in terms of torus-invariant subvarieties and divisors of characters. Given $\tau \in \Delta$ of codimension $k+1$, the orbit closure of each σ containing τ as a facet is a divisor in $V(\tau)$. Furthermore, N_σ/N_τ is a 1-dimensional lattice; let $n_{\sigma,\tau} \in N$ denote an integral point whose image generates this quotient lattice. As remarked above, the lattice containing the fan of $V(\tau)$ is $N(\tau)$. Its dual lattice is denoted $M(\tau) = \tau^\perp \cap M$. Given $u \in M(\tau)$, let x^u denote the associated rational function defined on $T_{N(\tau)}$, then by [12],

$$[\mathrm{div}(x^u)] = \sum_\sigma \langle u, n_{\sigma,\tau}\rangle \cdot [V(\sigma)].$$

The sum is over all σ containing τ as a facet.

Theorem 3.3.1. *[12, Proposition 1.1] Given a toric variety $X(\Delta)$,*

1. *$A_k(X(\Delta))$ is generated by the classes of orbit closures $[V(\sigma)]$ where σ has codimension k in Δ.*

2. *The group $\mathrm{Rat}_k(X(\Delta))$ is generated by all relations of the above form as τ runs through all cones of codimension $k+1$ in Δ and u runs over a generating set of $M(\tau)$.*

3.3.2 Chow Cohomology of Toric Varieties

Although one may hope that there is a similar toric description of the Chow cohomology of an arbitrary toric variety, such a description simply isn't known except in certain cases. The goal of this subsection is to review the main case in which the Chow cohomology is known - the complete case. For the entirety of this subsection, we will assume Δ is a complete fan. As in the previous chapter, the complete toric variety has a degree map $A_0(X(\Delta)) \to \mathbb{Z}$. Hence, given an arbitrary $c \in A^k_{op}(X(\Delta))$, this gives rise to a morphism $A_k(X(\Delta)) \to A_0(X(\Delta))$ which can be composed with the degree map. The function which sends an operational class to a morphism of groups defined in this manner is denoted D_X:

$$D_X : A^k_{op}(X(\Delta)) \to \mathrm{Hom}(A_k(X(\Delta)), \mathbb{Z}).$$

In general, this map is far from an isomorphism, but in the case of complete toric varieties, Totaro has shown that this map is an isomorphism.

Theorem 3.3.2. *[12, Proposition 1.4] If X is a complete toric variety, then D_X is an isomorphism.*

This isomorphism leads to the following simple description of the Chow cohomology in terms of the Minkowski ring. The notation $\Delta^{(k)}$ denotes the collection of codimension k cones of Δ.

Definition 3.3.3. [12, Section 2] A function $c : \Delta^{(k)} \to \mathbb{Z}$ is a **Minkowski weight** of degree k if

$$\sum_{\sigma \in \Delta^{(k)} : \tau \subset \sigma} \langle u, n_{\sigma,\tau} \rangle c(\sigma) = 0,$$

for every cone τ in $\Delta^{(k+1)}$ and $u \in M(\tau)$. The collection of all such functions is denoted $MW^k(X(\Delta)$, and the **Minkowski ring** of $X(\Delta)$ is defined to be the graded ring $MW^*(X(\Delta))$.

The defining condition of a Minkowski weight corresponds exactly to the relations of the Chow cohomology ring; thus, the following theorem is established in light of the previous discussion:

Theorem 3.3.4. *[12, Theorem 2.1] The Chow cohomology of a complete toric variety is isomorphic as groups to its Minkowski ring: $A^k_{op}(X(\Delta)) \cong MW^k(X(\Delta)$. In fact,* *this isomorphism extends to an isomorphism of rings once multiplication of Minkowski weights is suitably defined [12, Section 3].*

For examples of computing the Minkowski ring of some interesting toric varieties, see the computations in Chapter 6.

Chapter 4

Toric Stacks

The purpose of this chapter is to review the recent theory of toric stacks following [13]. In this paper, they consolidate a variety of other definitions that have been in use by introducing the theory of stacky fans. We begin by recalling the basic definitions concerning toric stacks and stacky fans along with the basics of toric morphisms, then we move forward to discuss our main object of interest - the canonical toric stack. In particular, the canonical toric stack of a toric variety is the stacky generalization of the Cox construction and presents the toric variety as a good moduli space of this stack. Finally, by utilizing the results discussed at the end of Chapter 2, we present generators and relations for the Chow ring of the canonical toric stack.

4.1 Stacky Fans and Toric Stacks

Following [13], we begin with the definition of a toric stack. Let $X(\Delta)$ be a toric variety associated to a fan Δ, let T_0 be the associated torus, and let $G \subset T_0$ be a subgroup. Note, since $T_0 \subset X(\Delta)$ is a dense open subscheme of $X(\Delta)$, it follows that the quotient stack $[X(\Delta)/G]$ contains the torus T_0/G as a dense open subscheme. Furthermore, the action of T_0/G on itself necessarily extends to an action on $[X/G]$.

Definition 4.1.1. [13, Definition 1.1] Under the above notation, a **toric stack** is any Artin quotient stack of the above form $[X(\Delta)/G]$.

Remark 4.1.2. Note, since $[X(\Delta)/G]$ contains T_0/G as a dense open subscheme, it follows that these quotient stacks have trivial generic stabilizers. In [13], they also introduce non-strict toric stacks to accommodate other definitions allowing generic stackiness. These non-strict toric stacks can be shown to arise from non-strict stacky fans, just as toric stacks, in the above definition, can be shown to arise from stacky fans. Since the canonical toric stack is a toric stack, as in the previous definition, we will refrain from defining non-strict toric stacks.

The theory of stacky fans serves the same role as fans in the theory of toric varieties: the geometry of the quotient stack can generally be understood in terms of the combinatorics of the stacky fan. Furthermore, just as normal toric varieties correspond to fans, toric stacks correspond to stacky fans (under the above definition, the toric stacks are necessarily normal).

Definition 4.1.3. A pair $(\Delta, \beta : N \to L)$ is a **stacky fan** if N and L are lattices, Δ is a fan on N, and the cokernel of β is finite.

We next describe how a stacky fan gives rise to a toric stack. Since the cokernel of β is finite, it follows that the dual of β is injective: $\beta^* : L^* \hookrightarrow N^*$. Furthermore, since this latter map is injective, it follows that the induced map on tori $T_\beta : T_N \to T_L$ is surjective. Note, T_N is the torus of $X(\Delta)$.

Definition 4.1.4. Under the above notation, let $G_{beta} = \ker(T_\beta)$, then the toric stack associated to a stacky fan (Δ, β) is defined to be the quotient stack $[X(\Delta)/G_\beta]$.

Remark 4.1.5. Under our beginning definition, it's easy to see every toric stack arises as the toric stack associated to a stacky fan since $T_L = T_N/G_\beta$.

4.1.1 Toric Morphisms of Toric Stacks

Similar to toric morphisms between toric varieties, toric morphisms between toric stacks correspond to morphisms of stacky fans.

Definition 4.1.6. [13, Definition 3.1] A **toric morphism** of toric stacks is a morphism which restricts to a homomorphism of tori and is equivariant with respect to that homomorphism.

Definition 4.1.7. [13, Definition 3.2] Given two stacky fans $(\Delta', \beta' : N' \to L')$ and $(\Delta, \beta : N \to L)$, then a **morphism of stacky fans** is a pair (Φ, ϕ) where $\Phi : N' \to N'$, $\phi : L' \to L$, Φ is compatible with respect to the fans Δ' and Δ, and the following diagram commutes:

$$\begin{array}{ccc} N' & \longrightarrow & N \\ \downarrow & & \downarrow \\ L' & \longrightarrow & L \end{array}$$

The result of [13, Theorem 3.4] establishes that toric morphisms equivalently arise from morphisms of stacky fans, but the proof is not immediate from the correspondence in the theory of toric varieties.

4.2 Canonical Toric Stacks

Given a full-dimensional fan Δ on a lattice N and its associated toric variety $X(\Delta)$, the canonical toric stack, denoted $\mathcal{X}(\Delta)$, is a smooth quotient stack possessing a special relationship with $X(\Delta)$. Following [13, Section 5], there are two equivalent definitions for the canonical stack. The first is given in terms of the Cox construction, and the second constructs $\mathcal{X}(\Delta)$ as a toric stack arising from a canonical stacky fan.

Definition 4.2.1. The canonical stack of $X(\Delta)$ will be denoted by $\mathcal{X}(\Delta)$ and is equipped with a toric morphism $\mathcal{X}(\Delta) \to X(\Delta)$; it can be constructed in either of the following ways:

1. The Cox construction of a toric variety produces an open subscheme $U \subset \mathbb{A}^{|\Delta(1)|}$ and a diagonalizable group G such that $X(\Delta) \cong U/G$, [4, Theorem 5.1.11]. The canonical stack of $X(\Delta)$ is the quotient stack $[U/G]$ with toric morphism $[U/G] \to X(\Delta)$.

2. Alternatively, let $(\Sigma_\Delta, \beta : \mathbb{Z}^{|\Delta(1)|} \to N)$ be the stacky fan where β is defined by sending the canonical basis vectors to the primitive generators of Δ, and Σ_Δ is the canonical fan defined in Chapter 2. The toric stack associated to this stacky fan is the canonical stack $\mathcal{X}(\Delta)$.

   Furthermore, the toric morphism $\mathcal{X}(\Delta) \to X(\Delta)$ is easily constructed from the following morphism of stacky fans:

$$\begin{array}{ccc} \mathbb{Z}^{|\Delta(1)|} & \longrightarrow & N \\ \downarrow & & \| \\ N & = & N \end{array}$$

In what follows, the most significant consequence of the existence of $\mathcal{X}(\Delta) \to X(\Delta)$ is that this is a good moduli space morphism. From [1, Definition 4.1], a morphism from an Artin stack to an algebraic space is a **good moduli space morphism** if it is cohomologically affine and Stein; in Chapter 5, we will further review the definition of good moduli space morphisms and their importance, but for the present chapter, we simply wish to mention the following result which will be integral to the work of Chapter 5 and 6.

Theorem 4.2.2. *[13, Theorem 6.3 and Example 6.23] The canonical stack morphism $\mathcal{X}(\Delta) \to X(\Delta)$ is a good moduli space morphism.*

Example 4.2.3. Let σ denote the cone generated by $\{(1,0,1),(0,-1,1),(-1,0,1),(0,1,1)\}$ in $\mathbb{R}^3$, then $X(\sigma)$ is a 3-dimensional, affine, singular toric variety with four rays. Let $\beta : \mathbb{Z}^4 \to \mathbb{Z}^3$ be defined by sending the canonical basis vectors to the primitive generators of σ; note, the preimage of σ under β is the cone defining $\mathbb{A}^4$. Furthermore, the cokernel of the dual of β is $\mathbb{Z}^4 \to \mathbb{Z}^3$ defined as the matrix $[1,-1,1,-1]$; hence, if we apply $D(-) := \mathrm{Hom}(-, \mathbb{G}_m)$ to this cokernel, we see that $G_\beta = \mathbb{G}_m$ and defines an action of $\mathbb{A}^4$ with weights $(1,-1,1,-1)$. Thus, the canonical stack of $X(\sigma)$ is $\mathcal{X}(\sigma) = [\mathbb{A}^4/\mathbb{G}_m]$.

4.3 The Chow Ring of a Canonical Toric Stack

The Chow ring of $\mathcal{X}(\Delta)$ admits a simple presentation due to the structure of the open subscheme of $\mathbb{A}^n$ for which it is the quotient of. Indeed, if $n = |\Delta(1)|$, then as discussed in Chapter 2, the toric variety $X(\Sigma_\Delta) \cong \mathbb{A}^n - Z(\Delta)$ where $Z(\Delta)$ is a finite union of G_β-invariant, linear subspaces. Thus, by [3], we can directly compute the Chow ring of any canonical toric stack. We begin by re-defining the ideals discussed in Chapter 2 relative to the current situation.

Definition 4.3.1.

1. The **linear equivalence ideal** of Δ, denoted $L(\Delta)$, is defined to be $L(\Delta) := C(G_\beta)$ as in Chapter 2.
2. The **Stanley-Reisner ideal** of Δ, denoted $\mathcal{Z}(\Delta)$, is defined to be $\mathcal{Z}(\Delta) := \mathcal{Z}$ as in Chapter 2.

Corollary 4.3.2. *Under the above notation,*

$$A^*(\mathcal{X}(\Delta)) \cong A^*_G/\mathcal{Z}(\Delta) \cong A^*_{\mathbb{G}_m^n}/(L(\Delta), \mathcal{Z}(\Delta)).$$

As mentioned at the end of Chapter 2, the above ring can be explicitly computed. But in the context of toric stacks, the Stanley-Reisner ideal has a simpler, combinatorial interpretation.

Remark 4.3.3. The irreducible components of $Z(\Delta)$ correspond to primitive collections of rays as mentioned in Chapter 3. Furthermore, this correspondence is of the following form: $\{\rho_{i_1}, ..., \rho_{i_k}\}$ is a primitive collection if and only if $V(x_{i_1}, ..., x_{i_k})$ is an irreducible component of $Z(\Delta)$. Hence, since $\mathcal{Z}(\Delta)$ is generated by the equivariant fundamental classes of $Z(\Delta)$, if $\{t_i\}$ denote the generators of $A^*_{\mathbb{G}_m^n}$ as in Chapter 2, then $t_{i_1} \cdot ... \cdot t_{i_k}$ generates the Stanley-Reisner ideal.

Example 4.3.4. Let Δ be the fan generated by the rays

$$\{(1,0,1), (0,-1,1), (-1,0,1), (0,1,1), (0,0,1)\},$$

and whose maximal cones are described as $\{0123, 014, 124, 234, 304\}$ (see our convention on defining toric varieties in Chapter 5.5). Then define $\beta : \mathbb{Z}^5 \to \mathbb{Z}^3$ by sending the canonical basis vectors to each of the primitive generators.

Let $t_0, ..., t_4$ denote the generators of Chow ring, then the linear-equivalence ideal is generated by the dot product between the column vectors of the transpose of β and the variable vector $(t_0, ..., t_4)$. Hence,

$$L(\Delta) = (t_0 - t_2, t_1 - t_3, t_0 + ... + t_3 - t_4).$$

For the Stanley-Reisner ideal, we utilize the previous remark. There are two primitive collections of rays $\{0, 2, 4\}$ and $\{1, 3, 4\}$. Hence,

$$\mathscr{Z}(\Delta) = (t_0 t_2 t_4, t_1 t_3 t_4).$$

Thus,

$$A^*(\mathcal{X}(\Delta)) = \mathbb{Z}[t_0, ..., t_4]/(t_0 - t_2, t_1 - t_3, t_0 + ... + t_3 - t_4, t_0 t_2 t_4, t_1 t_3 t_4) \cong$$

$$\mathbb{Z}[t_0, t_1]/(2t_0^2(t_0 + t_1), 2t_1^2(t_0 + t_1)).$$

Chapter 5

The Image of the Embedding of Chow Cohomology Rings

We begin this chapter with an outline of the material in [8]. The central result of their paper, which serves as the guiding result for this chapter and the next, is that for any good moduli space morphism $\pi : \mathcal{X} \to X$, there is a pullback map which embeds $A^*_{op}(X)$ into $A^*(\mathcal{X})$. As mentioned in the introduction, the value of this result is that it provides a significant step towards giving a geometric interpretation of the intersection product of operational classes along with a further geometric interpretation of the operational classes themselves. However, the fact that this embedding exists is insufficient in obtaining such geometric interpretations, unless one can characterize the image of this embedding.

Hence, a natural question following this result is whether one can give a geometric characterization of the image, or in the context of the canonical toric stack of a toric variety, can one give a toric characterization of the image of this embedding. Working towards identifying the image, the authors of [8] are able to show that the image of any operational class is generated by topologically strong cycles; these are cycles on $\mathcal{X}$ which are almost saturated with respect to π. However, there are examples of topologically strong cycles which are not the pullback of any operational class for certain specific schemes X; hence, topologically strong cycles are insufficient to characterize the image. As a refinement of topologically strong cycles, the authors introduce the concept of a strong cycle and are able to show, in a variety of instances, that the image seems to coincide with the group generated by strong cycles. Thus, they are led to conjecturing that the image of the Chow cohomology ring of X is equivalently described as Chow cohomology classes generated by strong cycles.

Following our summary of the above results, we then focus on the toric situation. We provide a series of results related to the above discussion. Afterwards, we provide a conjectural toric characterization of the image for complete toric varieties by utilizing Minkowski weights.

5.1 The Embedding of Chow Cohomology Rings

The concept of a good moduli space of a stack is a generalization of good categorical quotients in geometric invariant theory.

Definition 5.1.1. [1, Definition 4.1] Given an algebraic stack $\mathcal{X}$ and a scheme X, a morphism $\pi : \mathcal{X} \to X$ is a **good moduli space morphism** if

1. π is cohomologically affine: the pushfoward π_* is an exact functor,

2. π is Stein: $\mathcal{O}_X \cong \pi_*\mathcal{O}_{\mathcal{X}}$.

Following [7], if $\mathcal{X} \to X$ is a good moduli space morphism, then a closed point $x \in \mathcal{X}$ is **stable** if $\pi^{-1}(\pi(x)) = x$ under the induced map of topological spaces; furthermore, the point is **properly stable** if it is stable and the stabilizer of x is finite. The stack $\mathcal{X}$ is stable or properly stable respectively if there exists a good moduli space morphism such that the set of stable points or properly stable points respectively is non-empty. Once again, these definitions are modeled after the related notions in geometric invariant theory. If $\mathcal{X}^s$ and $\mathcal{X}^{ps}$ denotes the open substack of stable and properly stable points respectively, then note the following remark.

Remark 5.1.2. The open substack $\mathcal{X}^{ps}$ is the maximal, Deligne-Mumford substack of $\mathcal{X}$ such that $\pi^{-1}(\pi(\mathcal{X}^{ps})) = \mathcal{X}^{ps}$.

Given a morphism from a quotient stack to a scheme, $\pi : \mathcal{X} \to X$, [8] constructs a pullback map $\pi^* : A^*_{op}(X) \to A^*(\mathcal{X})$, wherein, the pullback is defined by sending an operational class on X to its evaluation on the fundamental class of $\mathcal{X}$: $c \cap [\mathcal{X}]$. The main result of [8] establishes that this is an embedding of the Chow cohomology ring of the good moduli space into the Chow cohomology ring of the stack over $\mathbb{Q}$.

Remark 5.1.3. Before stating this theorem, in the case that $\mathcal{X}$ is Deligne-Mumford, then this embedding is in fact an isomorphism over $\mathbb{Q}$. Hence, if we restrict π to $\mathcal{X}^{ps}$ and its image $\pi(\mathcal{X}^{ps})$, the Chow cohomology rings are isomorphic over $\mathbb{Q}$.

Theorem 5.1.4. *[8, Theorem 1.1] Let $\mathcal{X}$ be a smooth, connected, properly stable, Artin stack with good moduli space $\pi : \mathcal{X} \to X$, then π^* is injective over $\mathbb{Q}$.*

5.2 Background on Strong Cycles and Related Notions

5.2.1 Strong and Topologically Strong Cycles

Following [8], let $\mathcal{X}$ be a smooth Artin stack with good moduli space X such that the good moduli space morphism $\pi : \mathcal{X} \to X$ is properly stable. If $\mathcal{Z} \subset \mathcal{X}$ is a closed substack, let $Z = \pi(\mathcal{Z}) \subset X$; note, since π is cohomologically affine and Stein, it follows that the coherent sheaf of ideals defining $\mathcal{Z}$ pushes forward to a coherent sheaf of ideals in $\mathcal{O}_X$, hence, Z has the structure of a closed subscheme of X.

Definition 5.2.1. Let $\mathcal{Z} \subset \mathcal{X}$ be an integral substack, then

1. $\mathcal{Z}$ is **saturated** with respect to π if $\pi^{-1}(\pi(\mathcal{Z})) = \mathcal{Z}$.

2. $\mathcal{Z}$ is **strong** if $\dim(\mathcal{Z}) = \dim(Z)$ and $\mathcal{Z}$ is saturated.

3. $\mathcal{Z}$ is **topologically strong** if $\dim(\mathcal{Z}) = \dim(Z)$ and $\pi^{-1}(\pi(\mathcal{Z}))_{red} = \mathcal{Z}$.

With these two notions defined, we can construct the following two graded groups:

1. Let $A^k_{st}(\mathcal{X}/X) \subset A^k(\mathcal{X})$ denote the **relative strong Chow group** of degree k; this is the subgroup generated by codimension k strong cycles on $\mathcal{X}$.

2. Let $A^k_{tst}(\mathcal{X}/X) \subset A^k(\mathcal{X})$ denote the **relative topologically strong Chow group** of degree k; this is the subgroup generated by codimension k topologically strong cycles on $\mathcal{X}$.

The following lemma allows us to reduce the study of strong and topologically strong cycles to their étale local structures.

Lemma 5.2.2. *[8, Lemma 3.3] A cycle $\mathcal{Z} \subset \mathcal{X}$ is strong or topologically strong if and only if it is étale locally strong or topologically strong respectively.*

Étale locally, these notions have the following descriptions by the local structure theorem of [2]: given a point $x \in \mathcal{X}$, there is an étale neighborhood of the form $[U/G_x] \to \mathcal{X}$ of x, where G_x is the stabilizer of x, such that $U/G_x \to X$ is an étale neighborhood of $\pi(x)$ and the relevant diagram involving $\pi : \mathcal{X} \to X$ is Cartesian; furthermore, by shrinking U if needed, we may assume $U = \mathrm{Spec}(A)$. Under this assumption, if $\mathcal{Z} \subset [U/G]$ is a cycle with defining ideal $I \subset A$ and Z denotes its image under π, then the conditions defining strong and topologically strong cycles are equivalent to the following:

- $\dim(\mathcal{Z}) = \dim(Z)$ if and only if $\dim(A/I) = \dim(A^G/I^G)$,

- $\mathcal{Z} = \pi^{-1}(Z)$ if and only if I is generated by invariant functions of A^G.

- $\mathcal{Z} = \pi^{-1}(Z)_{red}$ if and only if there exists an ideal J generated by invariant functions such that $\sqrt{J} = I$.

Remark 5.2.3. The above description provides an effective means of verifying if a cycle is strong or topologically strong. By working locally, if the substack is defined by G-invariant functions, then the substack is strong. Hence, this will be our primary method used in the last section of this chapter to verify if a cycle is strong or not.

5.2.2 Stratification by Stabilizers

In [8, Section 3.1], they demonstrate the existence of a stratification of X of the form $X_0 \subset ... \subset X_n = X$ such that:

1. $X_i \setminus X_{i-1}$ has finite quotient singularities,

2. $X - X_{n-1}$ is the image of $\mathcal{X}^{ps}$, the maximal saturated Deligne-Mumford open substack of $\mathcal{X}$ (see the remark of the previous subsection).

This stratification is constructed by iteratively considering the locus of maximal dimensional stabilizers. In order for a cycle $\mathcal{Z}$ to be saturated with respect to π and satisfy the dimension condition, $\mathcal{Z}$ must necessarily intersect the maximal saturated Deligne-Mumford substack, and hence, its image must also intersect $X - X_{n-1}$. The following proposition establishes an analogous property for the intersection with each element of the stratification. Moreover, it provides a useful restriction for a cycle to be strong and topologically strong.

Proposition 5.2.4. *[8, Proposition 3.8] If $\mathcal{Z}$ is topologically strong, then*

1. $Z \cap (X - X_{n-1}) \neq \emptyset$,

2. *For $k \leqslant n-1$ and every connected component Y of X_k,* $\dim(\pi^{-1}(Z \cap Y)) < \dim(Z)$.

Conversely, given an integral closed subscheme satisfying nontrivial intersection with $X - X_{n-1}$, define $\mathcal{Z} = \overline{\pi^{-1}(Z \cap (X - X_{n-1}))}$, then if $\mathcal{Z} = \pi^{-1}(Z)_{red}$ then $\mathcal{Z}$ is topologically strong.

As a useful corollary of the previous result, we can detect whether an integral closed subscheme of X is the image of a topologically strong cycle in small degree due to the fiber over X_{n-1}.

Corollary 5.2.5. *[8, Corollary 3.11] If* $\dim(Z) = 0$ *or* 1*, then Z is the image of a topologically strong cycle if and only if $Z \subset X - X_{n-1}$.*

5.2.3 The Image is Topologically Strong and Consequences

Beyond demonstrating the pullback map is an embedding, [8] further establish that the pullback sends $A^k_{op}(X)$ into the topologically strong subgroup.

Theorem 5.2.6. *[8, Theorem 4.1] If $c \in A^k_{op}(X)$, then $\pi^*(c) \in A^k_{tst}(\mathcal{X}/X)$.*

To outline the proof of the above argument, the authors begin by showing the pullback of the first Chern class of a line bundle is strong if X possesses an ample line bundle, topologically strong cycles push foward to topologically strong cycles under proper maps, and the pullback of any polynomial in the Chern classes of arbitrary vector bundles is topologically strong. The proof then proceeds by passing to a projective (hence, an ample line bundle exists) resolution of singularities wherein the Chow cohomology ring is known to be generated by Chern classes of vector bundles; hence, the authors then pushforward and pullback appropriately to apply the above results and prove that the image is topologically strong.

Lastly, the authors of [8] provide an additional collection of partial results towards identifying the relationship between the image of the pullback and strong or topologically strong cycles.

Theorem 5.2.7. *By [8, Theorem 1.5]:*

1. *If X is smooth or has finite quotient singularities, then $A^*(X)_{\mathbb{Q}} \cong A^*_{tst}(\mathcal{X}/X)_{\mathbb{Q}}$.*

2. *If X has an ample line bundle, then $\pi^*(A^1_{op}(X)) = A^1_{st}(\mathcal{X}/X)$.*

3. *If $\mathcal{Z}$ is regularly embedded strong substack of codimension k, then $\mathcal{Z} \in \pi^*(A^k_{op}(X))$.*

4. *If $k = \dim(X)-1$ or $k = \dim(X)$, then π^* is an isomorphism onto $A^k_{tst}(\mathcal{X}/X)_{\mathbb{Q}}$; furthermore, if the maximal saturated Deligne-Mumford open substack is representable, then $A^k_{st}(\mathcal{X}/X)_{\mathbb{Q}} = A^k_{tst}(\mathcal{X}/X)_{\mathbb{Q}}$.*

Furthermore, if $\dim(X) \leqslant 3$, X possesses an ample line bundle, and the maximal saturated Deligne-Mumford open substack is representable, then the pullback induces an isomorphism $A^*_{op}(X)_{\mathbb{Q}} \cong A^*_{st}(X)_{\mathbb{Q}}$, see [8, Corollary 1.6]

5.2.4 Conjectures

We now specialize to the toric situation. Let $\mathcal{X}(\Delta) \to X(\Delta)$ denote the canonical toric stack, our first conjecture is a specialization of the conjecture discussed in the introduction.

Conjecture 5.2.8. *Given an arbitrary fan* Δ*, its associated toric variety* $X(\Delta)$*, and its canonical stack* $\mathcal{X}(\Delta) \to X(\Delta)$*, the image of* $A^*_{op}(X(\Delta))_{\mathbb{Q}}$ *is equal to* $A^*_{st}(\mathcal{X}(\Delta)/X(\Delta))$*.*

Furthermore, partially based upon the results of Chapter 6 along with some general conjectures in intersection theory, we further refine the above conjecture in the context of toric varieties and toric stacks:

Conjecture 5.2.9. *Given an arbitrary fan* Δ *and its canonical stack* $\mathcal{X}(\Delta)$*, the Chow ring of* $\mathcal{X}(\Delta)$ *is generated by strong complete intersections.*

Remark 5.2.10. By reading the introduction to Chapter 6, the combination of these two previous conjectures leads us to conclude that $A^*_{op}(X(\sigma))$ ought to vanish for any affine toric variety.

Finally, we provide one additional question/conjecture that is yet to be formalized.

Conjecture 5.2.11. *Suppose* $Y \to X(\Delta)$ *is another good moduli space morphism, what conditions are necessary to guarantee there exists a map* $Y \to \mathcal{X}(\Delta)$ *commuting with* X *and such that* $A^*(\mathcal{X}(\Delta)) \subset A^*(\mathcal{Y})$*?*

In [13, Section 4], they introduce the theory of fantastacks; these are slight generalizations of the Cox construction which preserve the good moduli space. Hence, if Y is any fantastack over $X(\Delta)$, then $Y \to X(\Delta)$ is a good moduli space morphism and the embedding theorem of Edidin-Satriano applies. Furthermore, it's somewhat straightforward to verify that $A^*(\mathcal{X}(\Delta)) \subset A^*(Y)$ for any fantastack over $X(\Delta)$. Hence, the heart of the previous question/conjecture concerns to what extent this result can be extended. Due to the canonical nature of the canonical toric stack, it seems reasonable that its Chow ring ought to be particularly natural among all other good moduli space morphisms $Y \to X(\Delta)$; however, one has to be careful since, in general, there may not even exist a map $Y \to \mathcal{X}(\Delta)$.

5.3 Some Results Related to these Ideas and Conjectures

As a consequence of the above discussion, in dimension three, we can recognize that the Chow cohomology ring has a special algebraic structure in the Chow ring of the canonical toric stack. Note, this algebraic structure is only noticeable through the embedding of the previous section; even in the case that Δ is complete and we identify the Chow cohomology with the Minkowski ring, it is essentially undetectable that the Minkowski ring possesses this structure.

Proposition 5.3.1. *If* $\dim(\Delta) = 3$*, then there exists an ideal* $I \subset A^*(\mathcal{X}(\Delta))$ *such that:*

$$\pi^*(A^*_{op}(X(\Delta))) = 1 + I \subset A^*(\mathcal{X}).$$

Proof. Given $x \in A^*(\mathcal{X})$, it suffices to show $x \cdot z \in \pi^* A^*_{op}(X)$ where $z \in \pi^* A^k_{op}(X)$ for $k = 1, 2, 3$.

If $k = 1$, then z is represented by a strong cycle by the last theorem of the previous section. Let $\mathcal{Z}$ denote this strong cycle. Since $X(\Delta)$ is 3-dimensional, the locus where $\mathcal{X} \to X$ fails to be Deligne-Mumford consists of finitely many points (these are the subvarieties corresponding to the non-simplicial cones; however, since every cone in dimension 2 or less is simplicial, it follows that the only non-simplicial cones are in dimension 3 which necessarily correspond to points). Let Z denote the image of $\mathcal{Z}$ in $X(\Delta)$, then since Z is Cartier, the support of Z can be moved away from this finite set of points. In which case, $\mathcal{Z}$ and Z are supported over the Deligne-Mumford locus wherein the Chow rings are isomorphic, and hence, the class of $x \cdot z$ is also supported on the Deligne-Mumford locus since $x \cdot z$ is supported in $\mathcal{Z}$. Thus, the product is the pullback of an operational class.

For $k = 2, 3$, z is still represented by a strong cycle $\mathcal{Z}$ because of the low degree corollary mentioned above. Furthermore, since strong cycles are topologically strong, they must satisfy the transversality condition listed above. Hence, it follows that $\mathcal{Z}$ and Z must be contained in the Deligne-Mumford locus again, and the previous argument carries through. ∎

We also have a conjectural description of the ideal I in dimension 3. To describe this, for each $\sigma \in \Delta$, let $\mathcal{X}(\sigma) \to X(\sigma)$ denote the canonical stack of the open affine toric variety $X(\sigma)$. Set K_σ to be the kernel of the pullback $A^*(\mathcal{X}) \to A^*(\mathcal{X}(\sigma))$.

Conjecture 5.3.2. *If* $\dim(\Delta) = 3$, *then the ideal* I *from the previous proposition is equal to:*

$$I = \bigcap_{\sigma \in \Delta} K_\sigma.$$

The verification of this conjecture in dimension 3 relies on the second conjecture mentioned above concerning complete intersections.

Proof. First, by chapter 6, $A^*_{op}(X(\sigma)) = 0$ in positive degree. Hence, the ideal I is included in the right hand side of the above, and it merely suffices to verify the leftward inclusion. Since the operational Chow ring is known to coincide with the strong Chow ring, it suffices to verify that each element in the intersection of the kernels is generated by strong cycles.

Let $[\mathcal{Z}]$ be such a cycle of codimension 1, then since $\mathcal{X}$ is smooth, this cycle is locally of the form on each affine open set $[V(f)/G_\beta]$. But since this cycle is in kernel of each affine, this is equivalent to each such representative being a G_β-invariant function, and hence, it must be strong.

For higher codimension, if we assume the above conjecture that $\mathcal{Z}$ is a complete intersection locally generated by $[V(f_1, ..., f_k)/G_\beta]$, then the previous argument still applies since being in the kernel is equivalent to strong on each open affine. ∎

Remark 5.3.3. However, this result does not generalize to higher dimensions. In the next section, we give an example of a dimension 4, complete toric variety whose Chow cohomology ring cannot be of the above form due to the ranks of its Minkowski ring.

Finally, we end this section with a further structural result.

Proposition 5.3.4. *Suppose ρ is a ray not contained in any non-simplicial cone of Δ, and let t_ρ denote the corresponding variable in $A^*(\mathcal{X}(\Delta))$, then the ideal $(t_\rho) \subset \pi^* A^*_{op}(X(\Delta))$.*

Proof. The proof proceeds along similar lines to the previous ones. Since ρ is not contained in any non-simplicial cone, it follows that the closed subscheme $\mathbb{V}(x_\rho) \subset \mathbb{A}^{|\Delta(1)|}$ in the canonical toric stack and its image are apart of the Deligne-Mumford locus. Let $x \in A^*(\mathcal{X}(\Delta)$, then $x \cdot t_\rho$ is supported on $\mathbb{V}(x_\rho)$ which is contained in the Deligne-Mumford locus. Hence, once again, it follows that the product is the pullback of an operational class, and altogether, the ideal of t_ρ must be contained in the pullback of the Chow cohomology ring. ∎

5.4 Conjectural Algorithm for the Image for Complete Toric Varieties

Recall from the previous chapters, the Chow cohomology ring of any complete toric variety is isomorphic to the Minkowski ring of the toric variety. By representing operational classes as Minkowski weights, we can provide a conjectural, effective algorithm for computing the image of $A^*_{op}(X(\Delta))$ modulo certain kernels living in the Chow ring of the canonical stack of a simplicialization. An outline of the method is given below:

1. Given a complete fan Δ and an operational class $c \in A^k_{op}(X(\Delta)$, begin by identifying c with a Minkowski weight $w_c \in MW^k(X(\Delta))$.

2. Construct a simplicialization Δ' by star subdivisions of Δ, and let $\phi : X(\Delta') \to X(\Delta)$ be the associated toric morphism.

3. Construct $\phi^*(w_c) \in MW^k(X(\Delta'))$; by [12], this Minkowski weight is defined by extension by zero. That is, given a codimension k cone $\sigma \in \Delta'$, if $\sigma \in \Delta$, then $\phi^*(w_c)(\sigma) = w_c(\sigma)$; and if σ is a new cone constructed from the star subdivisions, then $\phi^*(w_c)(\sigma) = 0$.

4. By [12, Section 5], the correspondence between $A^*_{op}(X(\Delta'))_{\mathbb{Q}}$ and $MW^*(X(\Delta')_{\mathbb{Q}}$ is known when Δ' is non-singular. In particular, the map sending a homogeneous polynomial to the following defines the isomorphism:

$$p \mapsto (\sigma \mapsto \deg(p \cdot x^\sigma)).$$

   In the case that Δ' is simplicial, one can generalize the degree map used in [12] to construct a correspondence between $MW^*(X(\Delta'))$ and $A^*(\mathcal{X}(\Delta')_{\mathbb{Q}}$ following [6].

Hence, let p_c denote the homogeneous polynomial corresponding to the pulled back Minkowski weight.

5. Conjecturally, we claim that there is a means of identifying this homogeneous polynomial with a class in $A^k(\mathcal{X}(\Delta))$ unique up to a subgroup of $A^*(\mathcal{X}(\Delta')$. In particular, this is done by either annihilating the additional variables corresponding to the new rays of the simplicialization in degree 1, or by expressing the additional variables in terms of a combination of the original variables modulo a subgroup in degree $\geqslant 2$.

5.5 Computations

For each of the following examples, we compute generators and relations of their strong Chow rings and their Minkowski rings, apply the previously described algorithm and output its results, and emphasize some additional details specialized to each. Throughout this section, we adhere to the following notational conventions unless specified otherwise:

1. All sequences will be zero-indexed.

2. Given the sequence of primitive generators of the fan, the ray ρ_i corresponding to the i-th primitive generator will be written as $'i'$.

3. Given a cone determined by the rays $\rho_{i_1}, ..., \rho_{i_k}$, the cone will be written as $'i_1 i_2 ... i_k'$.

4. Once an ordering of $\Delta^{(k)}$ has been established, then a Minkowski weight of degree k, $c : \Delta^{(k)} \to \mathbb{Z}$, will be written as a sequence $(a_1, .., a_n)$ if $n = |\Delta^{(k)}|$.

Wherein implicitly, if σ_i denotes the i-th cone of $\Delta^{(k)}$ in the above ordering, then $c(\sigma_i) = a_i$.

5.5.1 Dimension 3 - Cone over the Square with an additional Ray

- Primitive Generators = $((1,0,1),(0,-1,1),(-1,0,1),(0,1,1),(0,0,-1))$.
- Maximal Cones = $(0123, 014, 124, 234, 304)$.
- Weight Matrix =
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix}$$
- Chow Ring of Canonical Stack:

$A^*(\mathcal{X}(\Delta)) = \mathbb{Z}[x_0, x_1, x_2, x_3, x_4]/(x_0 - x_2, x_1 - x_3, 2x_0 + 2x_1 - x_4, x_0x_2x_4, x_1, x_3, x_4)$.

The canonical toric stack is:

$$\mathcal{X}(\Delta) = [\mathbb{A}^5 \setminus (V(x_0, x_2, x_5) \cup V(x_1, x_3, x_5))/\mathbb{G}_m^2].$$

Note, $X(\Delta) \cong Proj(k[x_0x_1, x_1x_2, x_2x_3, x_3x_0, x_5])$.

Minkowski Ring

For each collection of cones $\Delta^{(k)}$, we impose the following orderings:

- Degree 0: $(0123, 014, 124, 234, 034)$
- Degree 1: $(01, 12, 23, 03, 34, 14, 04, 24)$,
- Degree 2: $(0, 1, 2, 3, 4)$

Then in accordance with Chapter 3.3.1 and the above orderings, the additive generators of the Minkowski ring in each degree are as follows:

- Degree 0: $(1, 2, 2, 2, 2)$,
- Degree 1: $(1, 1, 1, 1, 2, 2, 2, 2)$,
- Degree 2: $(1, 0, 1, 0, 2)$, and $(0, 1, 0, 1, 2)$.

Strong Generators

The following are strong generators and are computed in [8]. To verify strong-ness, compute their local descriptions under the above Proj isomorphism. Note, these are actually strong complete intersections.

1. Degree 1: $[V(x_4)/\mathbb{G}_m^2]$ with Chow class $2x_0 + 2x_1$.
2. Degree 2: $[V(x_4, x_0)/\mathbb{G}_m^2]$, $[V(x_4, x_1)/\mathbb{G}_m^2]$ with Chow class $2x_0(x_0 + x_1)$ and $2x_1(x_0 + x_1)$.

Output of the Algorithm

Instead of relying on our Sage implementation, we will explicitly work out this example by hand. In degree 1, let c be the Minkowski weight defined by $(1, 1, 1, 1, 2, 2, 2, 2)$. We can simplicialize Δ by star subdividing the cone 0123 (in fact, this is a resolution of singularities) by introducing the ray with primitive generator $(0, 0, 1)$; let Δ' denote this cone. By [12], the pullback of c will assume the same values on cones in Δ and 0 otherwise. Hence, if $05, 15, 25, 35$ is the ordering of the new cones appended to the above ordering, then the pullback is equal to $(1, 1, 1, 1, 2, 2, 2, 2, 0, 0, 0, 0)$.

Let p_c denote the homogeneous degree 1 polynomial associated to this Minkowski weight. The Chow ring of the canonical stack admits the following presentation:

$$A^*(\mathcal{X}(\Delta')) =$$

$$\mathbb{Z}[x_0, x_1, x_2, x_3, x_4, x_5]/(x_0 - x_2, x_1 - x_3, 2x_0 + 2x_1 - x_4 + x_5, x_4x_5, x_0x_2, x_1x_3).$$

In particular, we fix x_0, x_1, x_5 as generators, then $p_c = a_0x_1 + a_1x_2 + a_5x_5$ is a unique representation of p_c. We use the known correspondence between Minkowski weights and homogeneous polynomials as mentioned in [12, Section 5] to compute the coefficients.

- 01: $\int p \cdot x_0x_1 = a_5x_0x_1x_5 = 1$ implies $a_5 = 1$.
- 12, 23, 30 are the same computation.
- 04: $\int a_1x_0x_1x_4 = 2$ implies $a_1 = 2$.
- 14: $\int a_0x_0x_1x_4 = 2$ implies $a_0 = 2$.
- 23 and 34 are the same computation.
- 05: $\int a_1x_0x_1x_5 + a_5x_0x_5^2 = 0$. However, by the above presentation and the fact that $a_1 = 2$ and $a_5 = 1$, this polynomial vanishes; hence, its degree is indeed 0.
- 15, 25, 35 are the same computations.

Thus, $p_c = 2x_0 + 2x_1 + x_5$, and if we either delete x_5 or re-write x_5 and view this polynomial as living in $A^*(\mathcal{X}(\Delta))$, then $p_c = x_4$ and is strong.

For degree 2, let c be the Minkowski weight $(1, 0, 1, 0, 2)$ and d the Minkowski weight $(0, 1, 0, 1, 2)$, then once again, they pullback to $(1, 0, 1, 0, 2, 0)$ and $(0, 1, 0, 1, 2, 0)$. Let p_c and p_d denote their respective polynomials which are of the form:

$$p_c = a_0x_0x_1 + bx_0x_5 + cx_1x_5, \qquad p_d = ex_0x_1 + fx_0x_5 + gx_1x_5.$$

If we utilize the same correspondence to compute the coefficients, for p_c we have that:

- 0: $\int cx_0x_1x_5 = 1$ implies $c = 1$.
- 1: $\int bx_0x_1x_5 = 0$ implies $b = 0$.
- 2, 3 are the same computations.
- 4: $\int ax_0x_1x_4 = 2$ implies $a = 2$.
- 5: $\int ax_0x_1x_5 + cx_1x_5^2 = 0$. However, with $a = 2$ and $c = 1$, the defining relations of the Minkowski ring imply this polynomial is indeed 0, and hence, its degree does vanish.

Thus, $p_c = 2x_0x_1 + x_1x_5$, and similarly, $p_d = 2x_0x_1 + x_0x_5$. If we use the relation $x_5 = -2x_0 + -2x_1 + x_4$, $p_c = x_1x_4$ and $p_d = x_0x_4$. Both of these are strong. Thus, our algorithm is successful in this example.

Remark 5.5.1. It is possible that non-singularity of the resolution is crucial to this algorithm, and may be an essential reason the algorithm works perfectly on this example. The work of [12] only establishes the identification of homogeneous polynomials in the Chow ring and Minkowski weights for smooth complete toric varieties. For the remaining examples, their resolutions are merely simplicial, and hence, this may affect their results even though we are utilizing the method of [6] to generalize the degree map.

5.5.2 Dimension 3 - Fan over the Cube

- Primitive Generators = $(1, 0, 1), (0, -1, 1), (-1, 0, 1), (0, 1, 1), (1, 0, -1)$, $(0, -1, -1)$, $(-1, 0, -1)$, $(0, 1, -1)$.

- Maximal Cones = $(0123, 4567, 0145, 1256, 2367, 0347)$.

- Weight Matrix =

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix}$$

Minkowski Ring

For each collection of cones $\Delta^{(k)}$, we impose the following orderings:

- Degree 0: $(0123, 4567, 0145, 1256, 2367, 0347)$,

- Degree 1: $(01, 12, 23, 03, 04, 45, 15, 26, 56, 67, 47, 37)$,

- Degree 2: $(0, 1, 2, 3, 4, 5, 6, 7)$.

Then in accordance with Section 3.3.1 and the above orderings, the additive generators of the Minkowski ring in each degree are as follows:

- Degree 0: $(1, 1, 1, 1, 1, 1)$,

- Degree 1: $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$

- Degree 2: $(1, 0, 0, 0, 0, 0, 1, 0)$, $(0, 1, 0, 0, 0, 0, 0, 1)$, $(0, 0, 1, 0, 0, 1, -1, 1)$, $(0, 0, 0, 1, 0, 1, 0, 0)$, $(0, 0, 0, 0, 1, -1, 1, -1)$.

Strong Generators

We can identify $X(\Delta)$ as $Proj(k[x0x1x2x3, x4x5x6x7, ...]$ (the remaining generators are the other maximal dimensional cones), and $\mathcal{X}(\Delta)$ is the quotient of $\mathbb{A}^8 - V(x0x1x2x3, ...)$ modulo $\mathbb{G}_m^5$. Let p be a suitably generic homogeneous polynomial of degree 1 in the variables $x0x1x2x3, ...$ (e.g. the sum of all of the monomials). The following are strong generators:

1. Degree 1: $[V(p)/\mathbb{G}_m^5]$ with Chow class $x_0 + ... + x_3$.

2. Degree 2: $V(x_i, p)/\mathbb{G}_m^5]$ with Chow class $x_i(x_0 + ... + x_3)$ for $i = 0, 1, ..., 4$.

The verification that the generator in degree 1 is strong is immediate, since a generic combination has weights $(1, 1, 1, 1, 0)$ which is the precisely the weight associated to each open set, we can simply divide through by the generator of that open set. For degree 2, we consider the case $i = 5$ (note that the other cases are simpler), and we let p be the sum of all the monomials. In this case, $V(x_4, p) = V(4, 0123{+}1256{+}2367)$. Consider the open set $D(0123)$, then interestingly, this subvariety cannot be written in terms of invariant functions valid on the entirety of $D(0123)$ (since the weights of 4 cannot be expressed integrally in terms of 0123). Instead, note that if $0123 \neq 0$, then either 1256 or 2367 are not 0; hence,

$$D(0123) = (D(0123) \cap D(1256)) \cup (D(0123) \cap D(2367)).$$

On the first open, rescale 4 to 0145 and then divide by 0123 throughout all terms to obtain invariant functions. Similarly, on the second open, rescale 4 to 0347 and then divide by 0123.

Output of the Algorithm

- Degree 1: $2 * x0 + 4 * x1 + 2 * x2 + 2 * x5 - 2 * x7 + 8 * x8 + 8 * x10 + 8 * x11$

- Degree 2:

    1. $2 * x1 * x2 + 2 * x2 * x3 - 2 * x2 * x6 + 4 * x0 * x8 + 4 * x1 * x8$

2. $2*x0*x3+2*x2*x3-2*x3*x7+4*x0*x8+4*x1*x8$

3. $2*x0*x1+2*x0*x3-2*x1*x5+2*x2*x6-2*x3*x7+8*x1*x8$

4. $2*x0*x1+2*x1*x2-2*x1*x5+4*x0*x8+4*x1*x8$

5. $-2*x0*x4+2*x1*x5-2*x2*x6+2*x3*x7+4*x0*x8-4*x1*x8$

In degree 1, we cannot re-write the additional variables in terms of the original ones to remove these additional variables. However, if instead we annihilate these variables, then indeed we obtain a strong generator after using the linear relations.

In degree 2, if re-write the additional variables in terms of the original ones and then simplify, once again, we do not directly obtain the original variables as in the previous example (there appears to still exist $x8$'s). However, it appears that modulo the kernel of the $A^k(\mathcal{X}(\Delta')) \to A^k(\mathcal{X}(\Delta))$, these possibly are the strong generators. The uncertainty is due to complications with Sage.

5.5.3 Dimension 3 - Fulton's Fan over the Cube

- Primitive Generators = $((1,2,3),(-1,1,1),(-1,1,-1),(1,1,-1),$ $(1,-1,-1)$,$(-1,-1,-1)$, $(1,-1,1)$, $(-1,-1,1))$.

- Maximal Cones = $(0167, 2345, 4567, 1257, 0123, 0346)$.

- Weight Matrix =

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 5 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

Fulton's fan over the cube modifies the $(1,1,1)$ ray to produce a complete non-projective toric variety with vanishing Picard group. Since the Picard group vanishes, we would expect to observe no strong divisors and instead see the strong cycles concentrated in codimension at least 2.

Some Strong Generators

The computation of strong generators in this case will utilize the properties of strong cycles discussed in the previous sections. In this case, since $\dim(X(\Delta)) = 3$, a strong cycle must miss the non-Deligne-Mumford locus, or concretely, it must miss the 6 singular points corresponding to the maximal dimensional cones (note, this is an equivalence; if it misses the 6 singular points, then it's contained in the Deligne-Mumford locus wherein the Chow rings are known to be isomorphic, hence, it then follows that the cycle must be strong). Note, the weights corresponding to the maximal dimensional cones are:

1. 0167: 11110

2. 2345: 51110

3. 4567: 41110

4. 0123: 21110

5. 0346: 31110

6. 1257: 31110

Let $w = (a, b, c, d, e)$ be an arbitrary weight, then all semi-invariants of this weight are of the form:

$$x_1^{(a-5a_6-a_7+2a_8)/2} x_2^{b-a_6-a_7+a_8} x_3^{c-a_7} x_4^{d-a_8} x_5^{e+a_6+a_7-a_8} x_6^{a_6} x_7^{a_7} x_8^{a_8}.$$

Where a_6, a_7, a_8 are arbitrary. We claim that if we fix any weight, then any combination of semi-invariants in that weight must necessarily pass through one of the singular points (and hence, cannot be strong). For example, for the weight $(1, 1, 1, 1, 0)$, all of the semi-invariants pass through 0123; similarly, for $(2, 1, 1, 1, 0)$, all of the semi-invariants pass through 0123 as well. For $(3, 1, 1, 1, 0)$, the semi-invariants miss 4 of the 6 singular points, but they still pass through 0123. However, for $(4, 1, 1, 1, 0)$ and $(5, 1, 1, 1, 0)$, although all semi-invariants miss 0123, they do pass through 0346. Furthermore, it is easy to see that if $e \neq 0$, then we can never move away from 4567. Thus, it seems reasonable to conclude that no matter the weight, there will always be a singular point passing through each of the semi-invariants.

For degree 2, to obtain strong cycles, our strategy will be to choose semi-invariants of different weights such that their intersection misses all singular points. For example, consider the weights $(5, 1, 0, 0, 1)$ and $(3, 1, 1, 1, 0)$, the former misses 0123 and 0167 only, while the latter misses the remainder. Hence, by considering a generic sum of semi-invariants in $(5, 1, 0, 0, 1)$, denoted p, and a generic sum in $(3, 1, 1, 1, 0)$, denoted q, it follows that $[V(p, q)/G]$ is strong since it misses the singular points. However, note that we can utilize Sage in this problem: to compute all semi-invariants of a given weight, to determine which singular points the semi-invariants miss, and then to determine if the semi-invariants of two different weights miss all of the singular points.

Remark 5.5.2. To use Sage for this problem, we use the Polyhedron package to generate a closed polyhedron satisfying the desired inequalities from the above generic semi-invariant, that is, requiring the exponents to be non-negative. To obtain all semi-invariants of a given weight, we use the integral points command.

Unfortunately, if one generates a large range of weights to choose pairs from, the computational time can be immense, and as the weights grow in size, it seems the prospect of missing all singular points becomes rarer. In addition to the above semi-invariants, Sage has also found the following:

1. $(5,1,0,0,1)$ and $(3,1,1,1,0)$ with semi-invariants:
   (a) $(5,1,0,0,1)$: $x0^5 * x1 * x4, x4^2 * x5$,
   (b) $(3,1,1,1,0)$: $x0^3{*}x1{*}x2{*}x3, x0^2{*}x3{*}x4{*}x6, x0^4{*}x1{*}x6{*}x7, x1{*}x2{*}x5{*}x7$.
2. $(0,0,0,0,1)$ and $(0,1,0,0,0)$ with semi-invariants:
   - $(0,0,0,0,1)$: $x4$,
   - $(0,1,0,0,0)$: $x1$.
3. $(0,0,0,0,1)$ and $(3,1,1,1,0)$ with the above semi-invariants.

Minkowski Ring

For each collection of cones $\Delta^{(k)}$, we impose the following orderings:

- Degree 0: $(0167, 2345, 4567, 1257, 0123, 0346)$,
- Degree 1: $(01, 12, 23, 03, 34, 45, 25, 46, 06, 57, 17, 67)$,
- Degree 2: $(0, 1, 2, 3, 4, 5, 6, 7)$.

Then in accordance with Section 3.3.1 and the above orderings, the additive generators of the Minkowski ring in each degree are as follows:

- Degree 1: $(0,0,0,0,0,0,0,0,0,0,0,0,0)$,
- Degree 2: $(2,0,0,0,0,5,1,-2)$,$(0,1,0,0,0,1,1,-1)$, $(0,0,1,0,0,0,1,0)$, $(0,0,0,1,0,0,0,1)$, $(0,0,0,0,1,-1,-1,1)$.

For Minkowski weights, we see once again that there are 5 generators in degree 2 and no generators in degree 0. By utilizing the algorithm mentioned above, one could conjecturally construct the Chow classes of the strong generators.

5.5.4 Dimension 4 - Cone over the 3-dimensional Cube with an additional Ray

- Primitive Generators $= ((1,0,1,1),(0,-1,1,1),(0,1,1,1),(-1,0,1,1),(1,0,-1,1),(0,-1,-1,1),(0,1,-1,1),(-1,0,-1,1),(0,0,0,-1))$.
- Maximal Cones $= (01234567, 01238, 45678, 02468, 13578, 23678, 01458)$.
- Weight Matrix =

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

The purpose of this example is to note the failure of Proposition 5.3.1. The ranks of the Chow groups of $X(\Delta)$ are $1,1,1,5,1$ for degrees equal to $0,1,2,3,4$. If $\pi^*A^*(X(\Delta)) = 1 + I$ and if s is the degree 1 generator, then multiplying s against the variable corresponding to any ray would be contained in $1+I$. However, it's easy to verify through Sage (or any computer algebra system by implementing the Chow rings) that multiplying s against the other rays are independent of one another. This would imply the 2nd degree group should have more than 1 generator; however, this contradicts the ranks of the Chow groups.

Chapter 6

The Chow Cohomology Ring of an Affine Toric Variety Vanishes

Our main goal in this section is to the prove the following theorem:

Theorem 6.0.1. *Let $X(\sigma)$ be any affine toric variety, then*

$$A^k_{op}(X(\sigma)) = 0 \text{ for all } k > 0.$$

In particular, $A^*_{op}(X(\sigma)) = \mathbb{Z}$.

Recall from the previous chapter, we conjecture that the pullback of $A^*_{op}(X(\Delta))$ in $A^*(\mathcal{X}(\Delta))$ ought to be generated by strong complete intersections. Since we are not imposing any assumptions on Δ, a simple test case for this conjecture should be when $\Delta = \sigma$ is the fan of a single cone, $X(\sigma)$ is an affine toric variety, $\mathcal{X}(\Delta) = [\mathbb{A}^n/G]$, and $A^*(\mathcal{X}(\Delta)) = A^*_G$.

In this context, a strong complete intersection of $[\mathbb{A}^n/G]$ is of the form $[\mathbb{V}(p_1, ..., p_k)/G]$ where $p_1, ..., p_k$ are G-invariant polynomials; however, if $\mathbb{V}(p_1, ..., p_k)$ is a complete intersection, then the Chow class of this cycle is the product of the Chow classes of $[\mathbb{V}(p_i)/G]$. But since p_i is G-invariant, its Chow class is 0, and hence, the Chow class of the cycle $[\mathbb{V}(p_1, ..., p_k)/G]$ is necessarily 0. Thus, if our conjectures are true, then we would expect $A^k_{op}(X(\sigma) = 0$ for all $k > 0$.

For smooth affine toric varieties, this can be shown directly since every smooth affine toric variety is of the form $\mathbb{A}^n \times \mathbb{G}_m^k$, and a short argument involving the vector bundle property of Chow groups along with the excision sequence will demonstrate the Chow ring of such a variety necessarily vanishes. However, for singular affine toric varieties, this result is far from obvious. Interestingly, we claim this result holds even if spite of the fact that it's easy to produce examples of singular affine toric varieties with non-trivial Chow homology groups; in other words, we can produce prime cycles which are not rationally equivalent to zero.

Nonetheless, there are other reasons for expecting this result to be true. [14] notes that singular affine toric varieties 'constitute a class of intuitively contractible varieties, generalizing in a natural way affine spaces', and in particular, all vector bundles on singular affine toric varieties are free so the Grothendieck ring is isomorphic to $\mathbb{Z}$. Furthermore, the following arguments can be modified to also show that the operational K-theory of any affine toric variety vanishes as well.

For an outline of the argument, we begin by star subdividing σ to produce $\phi : X(\sigma^*) \to X(\sigma)$. By the results in Chapter 3, this map is an isomorphism outside of $S = V(\sigma)$ and $E = \phi^{-1}(S)$. Hence, if we apply Kimura's exact sequence to this subdivision, to prove $A^k_{op}(X(\sigma))$ vanishes it suffices to prove $A^k_{op}(X(\sigma^*)) \to A^k_{op}(E)$ is injective.

If $\mathcal{X}(\sigma^*)$ and $\mathcal{E}$ denote the canonical stacks of $X(\sigma^*)$ and E respectively, then by utilizing the main theorem of [8], we will construct a diagram over $\mathbb{Q}$ involving the Chow rings of these canonical toric stacks and Kimura's exact sequence. Hence, to show the previous map is injective over $\mathbb{Q}$, it will suffice to show the map between

Chow rings of canonical stacks is an injection over $\mathbb{Q}$. But in fact, this map will be an isomorphism over $\mathbb{Z}$. To conclude the argument and remove the rationality assumption, we prove that the Chow cohomology ring of any affine toric variety must be torsion-free.

6.1 Proof of the Theorem

To fix notation, let σ^* denote the star subdivision of σ with respect to its star $v := Star(\sigma)$, let ρ_v be the corresponding ray in σ^*, let $\phi : X(\sigma^*) \to X(\sigma)$ be the associated toric morphism, and let $S = V(\sigma) \subset X(\sigma)$ and $E = \phi^{-1}(S)$. Note, the morphism ϕ is an isomorphism outside of S and E. Furthermore, let $\mathcal{X}(\sigma^*) \to X(\sigma^*)$ and $\mathcal{E} \to E$ denote the canonical the stacks of $X(\sigma^*)$ and E respectively.

6.1.1 Kimura's Exact Sequence

We begin by explicitly identifying the divisor E.

Lemma 6.1.1. *Under the above notation, $E = V(\rho_v)$.*

Proof. By construction, ρ_v is contained in the interior of σ, and if τ is a cone of σ^* non-trivially intersecting the interior of σ, then τ necessarily contains ρ_v. Hence, ρ_v is the unique minimal cone of σ^* intersecting the interior of σ. The lemma then follows from Theorem 3.1.19. ∎

Remark 6.1.2. Note, Payne proved in [17, Lemma 1] that any proper birational, toric morphism is an envelope. Since star subdivisions, and more generally refinements, produce such morphisms, it follows that $\phi : X(\sigma^*) \to X(\sigma)$ is an envelope. Thus, we can apply Kimura's exact sequence to ϕ.

Since the closed subscheme S is 0-dimensional and irreducible, $A^0(S) = \mathbb{Z}$ and $A^k(S) = 0$ for $k > 0$. Therefore, if we apply Kimura's exact sequence to ϕ, we obtain the following exact sequence:

$$0 \to A^k_{op}(X(\sigma)) \to A^k_{op}(X(\sigma^*)) \to A^*_{op}(V(\rho_v)) \text{ for } k > 0.$$

Thus, to show $A^k_{op}(X(\sigma)) = 0$ for $k > 0$, it suffices to show $A^k_{op}(X(\sigma^*)) \to A^k_{op}(V(\rho_v))$ is injective. To accomplish this, the following two steps will further demonstrate that it suffices to consider the Chow rings of the associated canonical stacks, and in subsection 6.1.3, we will show that these Chow rings are in fact isomorphic.

6.1.2 A Commutative Diagram of Canonical Stacks

The goal of this subsection is to show that we can construct an injectve morphism $\mathcal{E} \to \mathcal{X}(\sigma^*)$ such that the following diagram of stacks and good moduli spaces is commutative:

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{X}(\sigma^*) \\ \downarrow & & \downarrow \\ E & \longrightarrow & X(\sigma^*) \end{array} \tag{6.1}$$

Let Σ_{σ^*} denote the canonical fan as constructed in Chapter 3.2.1 for the variety $\mathbb{A}^{|\sigma^*(1)|} \setminus Z(\sigma^*)$, then as remarked in Chapter 3, this toric variety is equipped with a toric quotient morphism $X(\Sigma_{\sigma^*}) \to X(\sigma^*)$. If $n = |\sigma(1)|$, let $x_1, ..., x_{n+1}$ be coordinates on $\mathbb{A}^{|\sigma^*(1)|}$ such that x_{n+1} corresponds to the ray ρ_v. Note, since orbit closures in $X(\Delta)$ correspond to vanishing sets of rays in $X(\Sigma_\Delta)$, we have a commutative diagram of G-quotients:

$$\begin{array}{ccc} (\mathbb{V}(x_{n+1}) \setminus Z) & \longrightarrow & X(\Sigma_\sigma) \\ \downarrow & & \downarrow \\ E & \longrightarrow & X(\sigma^*) \end{array} \tag{6.2}$$

where G is the group acting on $X(\Sigma_{\sigma^*})$ such that $\mathcal{X}(\sigma^*) = [X(\Sigma_{\sigma^*})/G]$. We claim that if we quotient Diagram 6.2 by G and then consider the corresponding diagram of quotient stacks and good moduli spaces that $[(\mathbb{V}(x_{n+1}) \setminus Z/G)] \cong \mathcal{E}$.

We begin with the following basic result of toric varieties:

Lemma 6.1.3. *A collection of rays of σ^* is primitive if and only if the corresponding collection in $Star(\rho_v)$ is primitive.*

Proof. Let C be a primitive collection of σ^* and and let C' be the associated collection in $Star(\rho_v)$. If $D' \subset C'$ is a proper subset and $D \subset C$ denotes the corresponding subset, then since D is contained in some cone, it follows that D' will be contained in the image of that cone, and similarly, if $C' \subset \overline{\tau}(1)$, then $C \subset \tau$ would contradict C being primitive. Thus, if C is primitive, then C' is primitive; the converse holds by a similar argument. ∎

The existence of the commutative diagram (6.1) now follows from the following lemma.

Lemma 6.1.4. *Let $\mathcal{E}$ denote the canonical stack over E, then $\mathcal{E} \cong [(\mathbb{V}(x_{n+1}) \setminus Z)/G]$.*

Proof. By definition, $\mathcal{E}$ is the quotient stack of the form $[(\mathbb{A}^n \setminus Z')/G']$ where Z' and G' are defined as previously. By the previous lemma, there is an obvious isomorphism between $\mathbb{A}^n \setminus Z'$ and $\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*)$. We claim that $G = G'$ and the action of G' on

$\mathbb{A}^n \setminus Z'$ coincides with the action of G on $\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*)$ under this isomorphism. Indeed, this follows from the commutativity of the following diagram of character groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{k-1} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & X(G') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \mathbb{Z}^{n+1} & \longrightarrow & X(G) & \longrightarrow & 0 \end{array}$$

The vertical maps are the duals of the canonical quotient maps for the primitive generators of ρ_v and e_{n+1} respectively. ∎

Remark 6.1.5. Note that unless σ^* is a resolution of singularities, diagram (6.2) is not Cartesian. The following examples shows that even if σ^* is a simplicialization the diagram fails to be Cartesian.

Example 6.1.6. Consider the cone σ with ray generators given by $\{(1, 1, 1), (-1, 0, 1), (1, 0, 1), (0, -1, 1)\}$, and its associated star subdivision σ^* obtained by adding the ray generator $(1, 0, 4)$ and subdividing appropriately. We claim that in this example, $\mathcal{E}$ is not saturated with respect to π. Following [8, Remark 3.4], since $\dim(\mathcal{E}) = \dim(E)$ and $\pi(\mathcal{E}) = E$, it suffices to show $\mathcal{E}$ is not strong.

With the notation as in this section, the action of $\mathbb{G}_m^2$ on $\mathbb{A}^5 \setminus V(x_1, x_3) \cup V(x_2, x_4)$ is given by the matrix:

$$\begin{bmatrix} 3 & -2 & 1 & -2 & 0 \\ 2 & -3 & 0 & -3 & 1 \end{bmatrix}$$

If $\mathcal{E}$ is strong, then the ideal (x_5) is generated by an invariant function on each open set $D(x_1x_2)$, $D(x_2x_3)$, $D(x_3x_4)$, and $D(x_1x_4)$. However, on $D(x_1x_2)$ where x_1, x_2 are invertible, the weight $(0, 1)$ on x_5 cannot be expressed integrally in terms of $(3, 2)$ and $(-2, -3)$ for x_1, x_2 respectively (however, $(0, 1)$ can be expressed rationally in terms of these weights, hence, (x_5) is generated up to radical by an invariant function on $D(x_1x_2))$; thus, $\mathcal{E}$ is not strong, but the above argument does show $\mathcal{E}$ is topologically strong.

6.1.3 The Embedding of Edidin-Satriano

Since the canonical stack morphisms are good moduli space morphisms, we have that $A^*_{op}(E)_{\mathbb{Q}} \subset A^*(\mathcal{E})_{\mathbb{Q}}$ and $A^*_{\text{op}}(X(\sigma^*))_{\mathbb{Q}} \subset A^*(\mathcal{X}(\sigma^*)_{\mathbb{Q}}$. Hence, by combining these injections with the main result of the previous section and Kimura's exact sequence, the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^k_{\text{op}}(X(\sigma))_{\mathbb{Q}} & \longrightarrow & A^k_{\text{op}}(X(\sigma^*))_{\mathbb{Q}} & \longrightarrow & A^k_{\text{op}}(E)_{\mathbb{Q}} \\ & & & & \downarrow & & \downarrow \\ & & & & A^k(\mathcal{X}(\sigma^*))_{\mathbb{Q}} & \longrightarrow & A^k(\mathcal{E})_{\mathbb{Q}} \end{array}$$

Since the vertical maps are injections, to show $A^k_{\text{op}}(X(\sigma^*)) \to A^k_{\text{op}}(E)$ is injective at least over $\mathbb{Q}$, it suffices to show $A^*(\mathcal{X}(\sigma^*))_{\mathbb{Q}} \to A^*(\mathcal{E})_{\mathbb{Q}}$ is an injection. In fact, we claim this map is an isomorphism over $\mathbb{Z}$.

6.1.4 The Pullback of Chow Rings of Canonical Stacks is an Isomorphism

Theorem 6.1.7. *Let $\mathcal{E} \to \mathcal{X}(\sigma^*)$ be the injective morphism constructed in Step 2, then the pullback induces an isomorphism $A^*(\mathcal{E}) \cong A^*(\mathcal{X}(\sigma^*))$.*

Proof. By the previous lemma, $\mathcal{E} \cong [(\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*))/G] \subset \mathcal{X}(\sigma^*) = [(\mathbb{A}^{n+1} \setminus Z(\sigma^*))/G]$. Hence, the Chow rings of the canonical stacks are the following equivariant Chow rings:

$$A^*(\mathcal{X}(\sigma^*)) = A^*_G(\mathbb{A}^{n+1} \setminus Z(\sigma^*)), \text{ and } A^*(\mathcal{E}) = A^*_G(V(x_{n+1}) \setminus Z(\sigma^*)).$$

Since $Z(\sigma^*)$ does not contain x_{n+1}, we can write $\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*) = \mathbb{A}^n \setminus Z'$ and $\mathbb{A}^{n+1} \setminus Z(\sigma^*) = (\mathbb{A}^n \setminus Z') \times \mathbb{A}^1$ for the obvious exceptional set $Z' \subset \mathbb{A}^n$. In particular, the inclusion of canonical stacks is the quotient by G of the zero section $\mathbb{A}^n \setminus Z' \subset (\mathbb{A}^n \setminus Z') \times \mathbb{A}^1$ of a G-equivariant vector bundle over $\mathbb{A}^n \setminus Z'$. Hence, the pullback in G-equivariant Chow groups along this inclusion is an isomorphism. ∎

Before proceeding, we also give an alternative, direct proof of the above fact wherein we explicitly show these two rings are isomorphic; however, the alternative proof is only valid over $\mathbb{Q}$.

Proof. (Alternative Direct Proof) By the previous section, the Chow rings of the above stacks admit the following presentations:

1. $A^*(\mathcal{X}(\sigma^*))_{\mathbb{Q}} = \mathbb{Q}[s_1, ..., s_n, s]/(L, Z)$,
2. $A^*(\mathcal{E})_{\mathbb{Q}} = \mathbb{Q}[t_1, ..., t_n]/(L', Z')$.

The variables s_i correspond to the rays of $\sigma \subset \sigma^*$ and s corresponds to the ray ρ_v, while the variables t_i corresponds to the images of the rays of σ^* in $N(\rho_v)$. The restriction morphism sends $s_i \mapsto t_i$, and if $s = p(s_1, ..., s_n)$ for some polynomial p, then $s|_E = p(t_1, ..., t_n)$.

By the above lemma, it follows that if we replace s_i with t_i in Z, then $Z' = Z$. We claim a similar result holds for the linear equivalence ideals. In particular, we claim that L and L' are equivalent after expressing s as a linear polynomial in $s_1, ..., s_n$ and sending $s_i \mapsto t_i$. Suppose $\dim(N_{\mathbb{Q}}) = N$ and $\{e_i\}$ are the canonical basis vectors for N, then we can express the primitive generators of σ^* along with v as follows:

$$u_j = \sum_{i=1}^{N} a_{ji} e_i, v = \sum_{k=1}^{n} v_k = \sum_{i=1}^{N} A_i e_i \text{ where } A_i = \textstyle\sum_{j=1}^{n} a_{ij}.$$

The linear equations of L are generated by the following system of equations:

$$s_1 a_{11} + ... + s_n a_{n1} + sA_1 = 0$$
$$...$$
$$s_1 a_{1N} + ... + s_n a_{nN} + sA_N = 0$$

Without loss of generality, suppose $A_1 \neq 0$, then

$$s = \frac{1}{-A_1}(s_1 a_{11} + ... + s_n a_{n1}).$$

Let $\{\overline{e_i}\}$ denote the images of the basis vectors in $N(\rho_v)$, then since $\overline{v} = 0$,

$$\overline{v} = 0 \Rightarrow \overline{e_1} = \frac{-1}{A_1} \sum_{i=2}^{N} A_i \overline{e_i}.$$

Furthermore, note that the images of the primitive generators in this quotient lattice can be expressed as:

$$\overline{u_l} = \sum_{i=2}^{N} \left(a_{l1} \left(\frac{-A_i}{A_1} \right) + a_{li} \right) \overline{e_i}.$$

Thus, the linear equations of L' are generated by:

$$t_1 \left(a_{11}(-A_2/A_1) + a_{12}\right) + ... + t_n(a_{n1}(-A_2/A_1) + a_{n2}) = 0$$
$$...$$
$$t_1 \left(a_{n1}(-A_N/A_1) + a_{nN}\right) + ... + t_n \left(a_{nN}(-A_N/A_1) + a_{nN}\right) = 0$$

Finally, if we substitute the above description of s as a linear polynomial in $s_1, ..., s_n$ into each generator of L, we obtain exactly the previous set of equations after sending s_i to t_i. ∎

Thus, $A^k_{op}(X(\sigma))_{\mathbb{Q}} = 0$ for $k > 0$. The final step will be to show $A^k_{op}(X(\sigma))$ is torsion-free.

6.1.5 Chow Cohomology of Semi-Proper Toric Varieties

We recall from Chapter 2 the definitions of semi-proper and semi-projective toric varieties. A toric variety $X(\Delta)$ is semi-projective if it has at least one torus fixed point and the natural map $X(\Delta) \to \mathrm{Spec}(\Gamma(\mathcal{O}_{X(\Delta)}))$ is projective; equivalently, it is semi-projective if Δ has full-dimensional convex support and $X(\Delta)$ is quasi-projective. Similarly, a toric variety is semi-proper if it has at least one torus fixed point and the previous natural map is proper. Hence, for an affine toric variety $X(\sigma)$, any refinement Δ of σ produces a semi-proper toric variety $X(\Delta)$ which is proper over $X(\sigma)$. By [16, Lemma 2.1], $A^*_{op}(X(\sigma))$ injects into $A^*_{op}(X(\Delta))$; hence, if we can show the latter ring is torsion-free, it follows that the former is necessarily torsion-free.

Theorem 6.1.8. *Let $X(\Delta)$ be any semi-proper toric variety, then $A^*_{op}(X(\Delta))$ is torsion-free.*

Remark 6.1.9. Since we can always construct a resolution of singularities of σ through star subdivisions, the resulting toric variety will necessarily be projective. This is because a star subdivision is an example of a regular subdivision. Hence, although considering the semi-proper case isn't strictly necessary, since we also obtain a non-trivial result about the Chow cohomology rings of semi-proper toric varieties, we feel it is worthwhile to work through perspective.

The first step towards this result will be to reduce to smooth, semi-projective toric varieties. Indeed, by toric Chow's lemma [4, Theorem 6.1.18], there exists a projective toric variety $X(\Delta')$, and toric morphism $X(\Delta') \to X(\Delta)$, where Δ' is a smooth refinement of Δ, such that the map $X(\Delta') \to X(\sigma)$ factors through $X(\Delta) \to X(\sigma)$. Hence, $A^*_{op}(X(\Delta)) \subset A^*X(\Delta')$, and once again, if we can show the latter is torsion-free, it follows that the former is necessarily torsion-free. From this point forward, let σ' be a smooth refinement of σ such that $X(\sigma')$ is a smooth semi-projective toric variety.

Remark 6.1.10. For a smooth projective toric variety, [11, Section 5.2] demonstrates that its Chow ring is torsion-free by producing the Bialynicki-Birula decomposition of such a variety from a particular ordering of the maximal dimensional cones.

In [15], Hausel and Sturmfels generalize the construction of [11, Section 5.2] to smooth semi-projective toric varieties by utilizing the moment map associated to a 1-parameter subgroup, and in particular, they construct the Bialynicki-Birula decomposition of a smooth semi-projective toric variety by producing a collection of locally closed subsets $\{U_j\}$ satisfying the following properties:

1. Each U_i is a union of orbits, and hence, the closure of U_i is a union of U_j's.

2. $X(\sigma')$ is the disjoint union of the U_i's.

3. Each $U_i \cong \mathbb{A}^{n-k_i}$.

In particular, by [9, Definition 1.16], this decomposition of $X(\sigma')$ is an affine stratification. By [18], the classes of U_i form a basis for $A^*(X(\sigma'))$, and hence, $A^*(X(\sigma'))$ is necessarily torsion-free. This concludes the proof of the previous theorem. Thus, since $A^*_{op}(X(\sigma)) \subset A^*(X(\sigma'))$, it follows that $A^*_{op}(X(\sigma))$ is torsion-free, and by combining all of the previous subsections, this concludes the proof of the main theorem.

∎

6.2 Main Example

Arguably the simplest example of an affine singular toric variety is the cone over the square. In this following example, we explicitly verify the isomorphism of Section 6.1.4 and hence establish its Chow cohomology ring vanishes by hand.

Example 6.2.1. Let σ denote the cone generated by $\{(1,0,1),(0,-1,1),(-1,0,1),(0,1,1)\}$ in $\mathbb{R}^3$, then σ^* is actually a resolution of singularities of σ. In particular, the vertical maps are isomorphisms in Kimura's sequence. Furthermore, $E = V(\rho_v) = \mathbb{P}^1 \times \mathbb{P}^1$, and we can explicitly compute $A^*(\mathcal{X}(\sigma^*))$ as:

$$A^*(\mathcal{X}(\sigma^*)) = \mathbb{Z}[s_1, ..., s_4, s]/(s_1 - s_3, s_2 - s_4, s_1 + ... + s_4 + s, s_1 s_3, s_2 s_4).$$

Hence, $s_1 = s_3$, $s_2 = s_4$, and $s = -2s_1 - 2s_2$; and $A^*(\mathcal{X}(\sigma^*)) = \mathbb{Z}[s_1, s_2]/(s_1^2, s_2^2)$ is clearly isomorphic to the Chow ring of $\mathbb{P}^1 \times \mathbb{P}^1$ as predicted by Theorem 6.1.7. This calculation also directly verifies that $A^*_{op}(X(\sigma))$ is torsion free since it injects into the ring $\mathbb{Z}[s_1, s_2]/(s_1^2, s_2^2)$ which has no $\mathbb{Z}$-torsion.

Bibliography

[1] Jarod Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402. MR 3237451

[2] Jarod Alper, Jack Hall, and David Rydh, *A Luna étale slice theorem for algebraic stacks*, Preprint, Apr 2015.

[3] Thomas Coleman and Dan Edidin, *Inertial Chow rings of toric stacks*, Manuscripta Math. **156** (2018), no. 3-4, 341–369. MR 3811793

[4] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322

[5] Dan Edidin and William Graham, *Equivariant intersection theory*, Invent. Math. **131** (1998), no. 3, 595–634. MR 99j:14003a

[6] Dan Edidin and Yogesh More, *Integration on Artin toric stacks and Euler characteristics*, Proc. Amer. Math. Soc. **141** (2013), no. 11, 3689–3699. MR 3091761

[7] Dan Edidin and David Rydh, *Canonical reduction of stabilizers for artin stacks with good moduli spaces*, arXiv:1710.03220.

[8] Dan Edidin and Matthew Satriano, *Towards and intersection chow cohomology on git quotients*, arxiv:1707.05890, to appear in Transform. Groups (2017).

[9] David Eisenbud and Joe Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016. MR 3617981

[10] William Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984.

[11] ______, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037

[12] William Fulton and Bernd Sturmfels, *Intersection theory on toric varieties*, Topology **36** (1997), no. 2, 335–353. MR 1415592

[13] Anton Geraschenko and Matthew Satriano, *Toric stacks I: The theory of stacky fans*, Trans. Amer. Math. Soc. **367** (2015), no. 2, 1033–1071. MR 3280036

[14] Joseph Gubeladze, *K-theory of affine toric varieties*, Homology Homotopy Appl. **1** (1999), 135–145. MR 1693095

[15] Tamás Hausel and Bernd Sturmfels, *Toric hyperKähler varieties*, Doc. Math. **7** (2002), 495–534 (electronic). MR 2015052 (2004i:53054)

[16] Shun-ichi Kimura, *Fractional intersection and bivariant theory*, Comm. Algebra **20** (1992), no. 1, 285–302. MR 1145334

[17] Sam Payne, *Equivariant Chow cohomology of toric varieties*, Math. Res. Lett. **13** (2006), no. 1, 29–41. MR 2199564

[18] Burt Totaro, *Chow groups, Chow cohomology, and linear varieties*, Forum Math. Sigma **2** (2014), e17, 25pages. MR 3264256

[19] Angelo Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), no. 3, 613–670. MR MR1005008 (90k:14004)

VITA

Ryan Richey was born on November 10th, 1989 in Conroe, Texas. He graduated in 2008 from Oak Ridge High School located in Conroe, Texas. Afterwards, he attended Lone Star College - Montgomery from 2008-2009 and then transferred in 2010 to the University of Houston in Houston, Texas. In 2012, he was awarded a Bachelor's of Science in Mathematics.