

WEAK PHASE RETRIEVAL AND PHASELESS RECONSTRUCTION

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ABSTRACT

Phase retrieval and phaseless reconstruction for Hilbert space frames is a very active area of research. Recently, it was shown that these concepts are equivalent. In this thesis, we make a detailed study of a weakening of these concepts to weak phase retrieval and weak phaseless reconstruction. We will give several necessary and/or sufficient conditions for frames to have these weak properties. We will prove three surprising results: (1) Weak phaseless reconstruction is equivalent to phaseless reconstruction. I.e. It was never *weak*; (2) Weak phase retrieval is not equivalent to weak phaseless reconstruction; (3) Weak phase retrieval requires at least $2m - 1$ vectors in an m -dimensional Hilbert space. We also gives several examples illustrating the relationship between these concepts.

Chapter 1

Introduction

We first give the background required to understand the rest of the thesis. This includes a brief introduction to Hilbert space frame theory. Hilbert space frame theory has broad applications in pure mathematics as well as in applied mathematics, computer science, and engineering.

1.1 Preliminaries

In this section, we introduce some of the basic definitions and results from frame theory. Throughout this paper, \mathbb{H}^m denotes an m dimensional real or complex Hilbert space and we will write \mathbb{R}^m or \mathbb{C}^m when it is necessary to differentiate between the two. We start with the definition of a frame in \mathbb{H}^m .

Definition 1.1.1. *A family of vectors $\Phi = \{\phi_i\}_{i=1}^n$ in \mathbb{H}^m is a **frame** if there are constants $0 < A \leq B < \infty$ so that for all $x \in \mathbb{H}^m$*

$$A\|x\|^2 \leq \sum_{i=1}^n |\langle x, \phi_i \rangle|^2 \leq B\|x\|^2,$$

where A and B are the **lower and upper frame bounds** of the frame, respectively. The frame is called an **A-tight frame** if $A = B$ and is a **Parseval frame** if $A = B = 1$.

In addition, Φ is called an **equal norm frame** if $\|\phi_i\| = \|\phi_j\|$ for all i, j and is called a **unit norm frame** if $\|\phi_i\| = 1$ for all $i = 1, 2, \dots, n$.

Next, we give the formal definitions of phase retrieval, phaseless reconstruction, and norm retrieval.

Definition 1.1.2. Let $\Phi = \{\phi_i\}_{i=1}^n \in \mathbb{H}^m$ be such that for $x, y \in \mathbb{H}^m$

$$|\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle|, \text{ for all } i = 1, 2, \dots, n.$$

Φ yields

(i) **phase retrieval** with respect to an orthonormal basis $\{e_i\}_{i=1}^m$ if there is a $|\theta| = 1$ such that x and θy have the same phase i.e. $\text{phase}(x_i) = \theta \text{phase}(y_i)$, for all $i = 1, 2, \dots, m$, where $x_i = \langle x, e_i \rangle$.

(ii) **phaseless reconstruction** if there is a $|\theta| = 1$ such that $x = \theta y$.

(iii) **norm retrieval** if $\|x\| = \|y\|$.

We note that tight frames $\{\phi_i\}_{i=1}^m$ for \mathbb{H}^m do norm retrieval. Indeed, if

$$|\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle|, \text{ for all } i = 1, 2, \dots, m,$$

then

$$A\|x\|^2 = \sum_{i=1}^m |\langle x, \phi_i \rangle|^2 = \sum_{i=1}^m |\langle y, \phi_i \rangle|^2 = A\|y\|^2.$$

Phase retrieval in \mathbb{R}^m is classified in terms of a fundamental result called the complement property, which we define below:

Definition 1.1.3 ([2]). *A frame $\Phi = \{\phi_i\}_{i=1}^n$ in \mathbb{H}^m satisfies the **complement property** if for all subsets $\mathcal{I} \subset \{1, 2, \dots, n\}$, either $\text{span}\{\phi_i\}_{i \in \mathcal{I}} = \mathbb{H}^m$ or $\text{span}\{\phi_i\}_{i \in \mathcal{I}^c} = \mathbb{H}^m$.*

A fundamental result from [2] is:

Theorem 1.1.4 ([2]). *If Φ does phaseless reconstruction then it has complement property. In \mathbb{R}^m , if Φ does phase retrieval then it has complement property.*

It follows that if $\Phi = \{\phi_i\}_{i=1}^n$ does phase retrieval in \mathbb{R}^m then $n \geq 2m - 1$. It is also known [2] that Φ never does phase retrieval when $n < 2m - 1$.

Full spark is another important notion of vectors in frame theory. A formal definition is given below:

Definition 1.1.5. *Given a family of vectors $\Phi = \{\phi_i\}_{i=1}^n$ in \mathbb{H}^m , the **spark** of Φ is defined as the cardinality of the smallest linearly dependent subset of Φ . When $\text{spark}(\Phi) = m + 1$, every subset of size m is linearly independent, and in that case, Φ is said to be **full spark**.*

We note that from the definitions it follows that full spark frames with $n \geq 2m - 1$ have the complement property and hence do phaseless reconstruction. Moreover, if $n = 2m - 1$ then the complement property implies full spark. We conclude this section by providing the definition of norm retrieval for projections. Towards the end of this thesis, we show that the weak phase retrieval does not imply norm retrieval.

Definition 1.1.6. Let $\{W_i\}_{i=1}^n$ be a collection of subspaces in \mathbb{H}^m and let $\{P_i\}_{i=1}^n$ be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i=1}^n$ (or $\{P_i\}_{i=1}^n$) yields **phaseless reconstruction** (Resp. **norm retrieval**) if for all $x, y \in \mathbb{H}^m$ satisfying $\|P_i\|x = \|P_i\|y$ for all $i = 1, 2, \dots, n$ we have $x = cy$ for some $|c| = 1$ (Resp. $\|x\| = \|y\|$).

Chapter 2

Weak Phase Retrieval

2.1 Defining weak phase retrieval

In this section, we define the notion of weak phase retrieval and obtain the minimum number of vectors required to do weak phase retrieval. First we define the notion of vectors having weakly the same phase. In this section we will make a detailed study of weak phase retrieval. We start with the definition.

Definition 2.1.1. *Two vectors in \mathbb{H}^m , $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ weakly have the same phase if there is a $|\theta| = 1$ so that*

$$\text{phase}(a_i) = \theta \text{phase}(b_i), \text{ for all } i = 1, 2, \dots, m, \text{ for which } a_i \neq 0 \neq b_i.$$

In the real case, if $\theta = 1$ we say x, y weakly have the same signs and if $\theta = -1$ they weakly have opposite signs.

In the definition above note that we are only comparing the phase of x and y for entries where both are nonzero. Hence, two vectors may *weakly* have the same phase but not have the same phase in the usual sense. We define weak phase retrieval formally as follows:

Definition 2.1.2. *A family of vectors $\{\phi_i\}_{i=1}^n$ in \mathbb{H}^m does **weak phase retrieval** if for any $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{H}^m , with*

$$|\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle|, \text{ for all } i = 1, 2, \dots, m,$$

then x, y weakly have the same phase.

Observe that the difference between phase retrieval and weak phase retrieval is that in the later it is possible for $a_i = 0$ but $b_i \neq 0$.

2.1.1 Real Case

Now we begin our study of weak phase retrieval in \mathbb{R}^m . The following proposition provides a useful criteria for determining when two vectors have weakly the same or opposite phases.

Proposition 2.1.3. *Let $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{R}^m . The following are equivalent:*

1. *We have*

$$\text{sgn}(a_i a_j) = \text{sgn}(b_i b_j), \text{ for all } a_i a_j \neq 0 \neq b_i b_j.$$

2. *Either x, y have weakly the same signs or they have weakly opposite signs.*

Proof. (1) \Rightarrow (2): Let

$$I = \{1 \leq i \leq m : a_i = 0\} \text{ and } J = \{1 \leq i \leq n : b_i = 0\}.$$

Let

$$K = [m] \setminus (I \cup J).$$

So $i \in K$ if and only if $a_i \neq 0 \neq b_i$. Let $i_0 = \min K$. We examine two cases:

Case 1: $\text{sgn } a_{i_0} = \text{sgn } b_{i_0}$.

For any $i_0 \neq k \in K$, $\text{sgn } (a_{i_0} a_k) = \text{sgn } (b_{i_0} b_k)$, implies $\text{sgn } a_k = \text{sgn } b_k$. Since all other coordinates of either x or y are zero, it follows that x, y weakly have the same signs.

Case 2: $\text{sgn } a_{i_0} = -\text{sgn } b_{i_0}$.

For any $i_0 \neq k \in K$, $a_{i_0} a_k = b_{i_0} b_k$

(2) \Rightarrow (1): This is immediate. □

The next lemma will be useful in the following proofs as gives a criteria for showing when vectors do not weakly have the same phase.

Lemma 2.1.4. *Let $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ be vectors in \mathbb{R}^m . If there exists $i \in [m]$ such that $a_i b_i \neq 0$ and $\langle x, y \rangle = 0$, then x and y do not have weakly the same or opposite signs.*

Proof. We proceed by way of contradiction. If x and y weakly have the same phase then $a_j b_j \geq 0$ for all $j \in [m]$ and in particular we arrive at the following contradiction

$$\langle x, y \rangle = \sum_{j=1}^n a_j b_j \geq a_i b_i > 0.$$

If x and y weakly have opposite phases then $a_j b_j \leq 0$ for all $j \in [m]$ and by reversing the inequalities in the expression above we get the desired result. \square

The following result relates weak phase retrieval and phase retrieval. Recall that in the real case, it is known that phase retrieval, phaseless reconstruction and the complement property are equivalent [2, 7].

Corollary 2.1.5. *Suppose $\Phi = \{\phi_i\}_{i=1}^n \in \mathbb{R}^m$ does weak phase retrieval but fails complement property, then there exists two vectors $v, w \in \mathbb{R}^m$ such that $v \perp w$ and*

$$|\langle v, \phi_i \rangle| = |\langle w, \phi_i \rangle| \text{ for all } i \in [n] \quad (2.1)$$

Further, v and w are disjointly supported.

Proof. By the assumption, $\Phi = \{\phi_i\}_{i=1}^n$ fails complement property, there exists $I \subset [n]$, s.t. $A = \text{Span}\{\phi_i\}_{i \in I} \neq \mathbb{R}^m$ and $B = \text{Span}\{\phi_i\}_{i \in I^c} \neq \mathbb{R}^m$. Choose $\|x\| = \|y\| = 1$ such that $x \perp A$ and $y \perp B$. Then

$$|\langle x + y, \phi_i \rangle| = |\langle x - y, \phi_i \rangle| \text{ for all } i=1, 2, \dots, n.$$

Let $w = x + y$ and $v = x - y$. Then $w \perp v$. Observe

$$\langle w, v \rangle = \langle x + y, x - y \rangle = \|x\|^2 + \langle y, x \rangle - \langle x, y \rangle - \|y\|^2 = 0.$$

Moreover, the assumption that Φ does weak phase retrieval implies u and w have weakly the same or opposite phases. Then it follows from Lemma 2.1.4 that u and w are disjointly supported. \square

In \mathbb{R}^2 let $\phi_1 = (1, 1)$ and $\phi_2 = (1, -1)$. These vectors clearly fail complement

property. But if $x = (a_1, a_2)$, $b = (b_1, b_2)$ and we have,

$$|\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle|, \text{ for } i = 1, 2,$$

then

$$|a_1 + a_2|^2 = |b_1 + b_2|^2 \text{ and } |a_1 - a_2|^2 = |b_1 - b_2|^2.$$

By squaring these out and subtracting the result we get:

$$2a_1a_2 = 2b_1b_2.$$

Hence, either x, y have the same signs or opposite signs. I.e. These vectors do weak phase retrieval. With some particular assumptions, the following proposition gives the specific form of vectors which do weak phase retrieval but not phase retrieval.

Proposition 2.1.6. *Let $\Phi = \{\phi_i\}_{i=1}^n \in \mathbb{R}^m$ such that Φ does weak phase retrieval but fails complement property. Let $x = (a_1, a_2, \dots, a_m)$, $y = (b_1, b_2, \dots, b_m) \in \mathbb{R}^m$ such that $x + y \perp x - y$ and satisfy equation 2.1. If $a_i b_i \neq 0$, $a_j b_j \neq 0$ for some i, j and all other co-ordinates of x and y are zero, then*

$$|a_i| = |b_i|, \text{ for } i = 1, 2.$$

Proof. Without loss of generality, take $x = (a_1, a_2, 0, \dots, 0)$ and $y = (b_1, b_2, 0, \dots, 0)$. Observe that both $x + y$ and $x - y$ either weakly have the same phase or weakly have the opposite phase. Thus, by Lemma 2.1.4, $x + y$ and $x - y$ have disjoint support as these vectors are orthogonal. Since,

$$x + y = (a_1, b_1, a_2 + b_2, 0, \dots, 0) \text{ and } x - y = (a_1 - b_1, a_2 - b_2, 0, \dots, 0)$$

it reduces to the cases where either $a_1 = b_1$, $a_2 = -b_2$ or $a_1 = -b_1$, $a_2 = b_2$. In both cases, it follows from equation 2.1 that $|a_i| = |b_i|$ for all $i = 1, 2, \dots, m$. \square

The next theorem gives the main result about the minimum number of vectors required to do weak phase retrieval in \mathbb{R}^m . Recall that phase retrieval requires $n \geq 2m - 1$ vectors.

Theorem 2.1.7. *If $\{\phi_i\}_{i=1}^n$ does weak phase retrieval on \mathbb{R}^m then $n \geq 2m - 2$.*

Proof. For a contradiction assume $n \leq 2m - 3$. If $I = [m - 2]$ then $|I| = m - 2$ and $|I^c| \leq m - 1$. For this partition of $[n]$, let $x + y$ and $x - y$ be as in Corollary 2.1.5. Then $x + y$ and $x - y$ must be disjointly supported and therefore for each $i = 1, 2, \dots, m$ $a_i = \epsilon_i b_i$, where $\epsilon_i = \pm 1$ for each i . Observe the conclusion holds for a fixed x and any $y \in (\text{span}\{\phi_i\}_{i \in I})^\perp$ and $\dim (\text{span}\{\phi_i\}_{i \in I})^\perp \geq 2$. However this poses a contradiction since there are infinitely many distinct choices of y in this space, however our argument shows that there are at most 2^m choices of y . \square

Contrary to the initial hopes, the previous result shows that the minimal number of vectors doing weak phase retrieval is only one less than the number of vectors doing phase retrieval. However it is interesting to note that a minimal set of vectors doing weak phase retrieval is necessarily full spark, as is true for the minimal number of vectors doing phase retrieval, as the next result shows.

Theorem 2.1.8. *If $\Phi = \{\phi_i\}_{i=1}^{2n-2}$ does weak phase retrieval in \mathbb{R}^n , then Φ is full spark.*

Proof. We proceed by way of contradiction. Assume Φ is not full spark. Then there exists $I \subset \{1, 2, \dots, 2n - 2\}$ with $|I| = n$ such that $\dim \text{span}\{\phi_i\}_{i \in I} \leq n - 1$. Observe that the choice of I above implies $|I^c| = n - 2$. Now we arrive at a contradiction by applying the same argument used in (the proof of) Theorem 2.1.7. \square

It is important to note that the converse of Theorem 2.1.8 does not hold. For example, the canonical basis in \mathbb{R}^2 is trivially full spark but does not do weak phase retrieval.

If Φ is as in Theorem 2.1.8, then the following corollary guarantees it is possible to add a vector to this set and obtain a collection which does phaseless reconstruction.

Corollary 2.1.9. *If Φ as in Theorem 2.1.8, then there exists a dense set of vectors F in \mathbb{R}^n such that $\{\psi\} \cup \Phi$ does phaseless reconstruction for any $\psi \in F$.*

Proof. The result follows almost instantly from the observation that the set of $\psi \in \mathbb{R}^n$ such that $\Phi \cup \{\psi\}$ is full spark is dense in \mathbb{R}^n . To see this let $G = \bigcup_{\substack{I \subset [2n-2] \\ |I|=n-1}} \text{span}\{\phi_i\}_{i \in I}$. Then G is the finite union of hyperplanes so G^c is dense and $\{\psi\} \cup \Phi$ is full spark for any $\psi \in G^c$. It suffices to verify that this collection of vectors is full spark. Either a sub-collection of m -vectors is contained in Φ , then it spans \mathbb{R}^n , or the subcollection contains the vector ψ . In this case, denote $I \subset [2n-2]$ with $|I| = n-1$ and suppose $\sum_{i \in I} a_i \phi_i + a\psi = 0$. Therefore $a\psi = -\sum_{i \in I} a_i \phi_i$ and if $a \neq 0$ then $a\psi \in \text{span}\{\phi_i\}_{i \in I}$, a contradiction. It follows $a = 0$ and since Φ is full spark (see Theorem 2.1.8), in particular $\{\phi_i\}_{i \in I}$ are linearly independent, it follows that $a_i = 0$ for all $i \in I$. \square

2.1.2 Complex Case

An extension of Proposition 2.1.3 in the complex case is given below:

Proposition 2.1.10. *Let $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{C}_m . The following are equivalent:*

1. *If there is a $|\theta| = 1$ such that $\text{phase}(a_i) = \theta \text{phase}(b_i)$, for some i , then $\text{phase}(a_i a_j) = \theta^2 \text{phase}(b_i b_j)$, $i \neq j$ and $a_i \neq 0 \neq b_i$, for any i .*
2. *x and y weakly have the same phase.*

Proof. (1) \Rightarrow (2): Let the index sets I, J and K be as in proposition 2.1.3. By (1), there is a $|\theta| = 1$ such that $\text{phase}(a_i) = \theta \text{phase}(b_i)$ for some $i \in K$.

Now, for any $j \in K, j \neq i$,

$$\text{phase}(a_i a_j) = \text{phase}(a_i) \text{phase}(a_j) = \theta \text{phase}(b_i) \text{phase}(a_j).$$

But $\text{phase}(a_i a_j) = \theta^2 \text{phase}(b_i b_j) = \theta^2 \text{phase}(b_i) \text{phase}(b_j)$.

Thus, it follows that $\text{phase}(a_j) = \theta \text{phase}(b_j)$. Since all other coordinates of either x or y are zero, it follows that x, y weakly have the same phase.

(2) \Rightarrow (1): By definition, there is a $|\theta| = 1$ such that $\text{phase}(a_i) = \theta \text{phase}(b_i)$ for all $a_i \neq 0 \neq b_i$. Now, (1) follows immediately as $\text{phase}(a_i a_j) = \text{phase}(a_i) \text{phase}(a_j)$.

□

Chapter 3

Weak Phaseless Reconstruction

3.1 Weak Phaseless Reconstruction

In this section, we define weak phaseless reconstruction and study its characterizations. A formal definition is given below:

Definition 3.1.1. *A family of vectors $\{\phi_i\}_{i=1}^n$ in \mathbb{H}^m does **weak phaseless reconstruction** if for any $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{H}^m , with*

$$|\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle|, \text{ for all } i = 1, 2, \dots, m, \quad (3.1)$$

there is a $|\theta| = 1$ so that

$$a_i = \theta b_i, \text{ for all } i = 1, 2, \dots, m, \text{ for which } a_i \neq 0 \neq b_i.$$

In particular, $\{\phi_i\}$ does phaseless reconstruction for vectors having all non-zero coordinates.

Note that if $\Phi = \{\phi_i\}_{i=1}^n \in \mathbb{R}^m$ does weak phaseless reconstruction, then it does weak phase retrieval. The converse is not true in general. Let $x = (a_1, a_1, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$. If $\Phi = \{\phi_i\}_{i=1}^n \in \mathbb{R}^m$ does weak phase retrieval and $|\{i | a_i b_i \neq 0\}| = 2$ then Φ may not do weak phaseless reconstruction. If $a_i a_j = b_i b_j$ where $a_i b_i \neq 0$ and $a_j b_j \neq 0$ then we certainly cannot conclude in general that $|a_i| = |b_i|$ (see Example 4.1.1).

Theorem 3.1.2. *If $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{R}_m . The following are equivalent:*

I. *There is a $\theta = \pm 1$ so that*

$$a_i = \theta b_i, \text{ for all } a_i \neq 0 \neq b_i.$$

II. *We have $a_i a_j = b_i b_j$ for all $1 \leq i, j \leq m$, and $|a_i| = |b_i|$ for all i such that $a_i \neq 0 \neq b_i$.*

III. *The following hold:*

A. *Either x, y have weakly the same signs or they have weakly the opposite signs.*

B. *One of the following holds:*

(i) *There is a $1 \leq i \leq m$ so that $a_i = 0$ and $b_j = 0$ for all $j \neq i$.*

(ii) *There is a $1 \leq i \leq m$ so that $b_i = 0$ and $a_j = 0$ for all $j \neq i$.*

(iii) *If (i) and (ii) fail and $I = \{1 \leq i \leq m : a_i \neq 0 \neq b_i\}$, then the following hold:*

(a) *If $i \in I^c$ then $a_i = 0$ or $b_i = 0$.*

(b) *For all $i \in I$, $|a_i| = |b_i|$.*

Proof. (I) \Rightarrow (II) : By (I) $a_i = \theta b_i$ for all i such that both are non-zero, so $a_i a_j = (\theta b_i)(\theta b_j)$ then $a_i a_j = \theta^2 b_i b_j$. Since $\theta = \pm 1$ it follows that $a_i a_j = b_i b_j$ for all i, j (that are non-zero). The second part is trivial.

(II) \Rightarrow (III) :

(A) This follows from Proposition 2.1.3.

(B) (i) Assume $a_i = 0$ but $b_i \neq 0$. Then for all $j \neq i$ we have $a_i a_j = 0 = b_i b_j$ and so $b_j = 0$.

(ii) This is symmetric to (i).

(iii) If (i) and (ii) fail, then by definition, for any i , either both a_i and b_i are zero or they are both non-zero, which proves (A). (B) is immediate.

(III) \Rightarrow (I) : The existence of θ is clear by part A. In part B, (i) and (ii) trivially imply I. Assume (iii) then for each i such that $a_i \neq 0 \neq b_i$ and $|a_i| = |b_i|$ then $a_i = \pm b_i$. □

Corollary 3.1.3. *Let Φ be a frame for \mathbb{R}^m . The following are equivalent:*

1. Φ does weak phaseless reconstruction.
2. For any $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{R}^m , if

$$|\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle| \text{ for all } i,$$

then each of the equivalent conditions in Theorem 3.1.2 holds.

The following theorems provide conditions under which weak phase retrieval is equivalent to weak phaseless reconstruction.

Proposition 3.1.4. *Let $\Phi = \{\phi_i\}_{i=1}^n$ do weak phase retrieval on vectors $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{H}^m . If $|I| = |\{i : a_i b_i \neq 0\}| \geq 3$ and $a_i a_j = b_i b_j$ for all $i, j \in I$, then Φ does weak phaseless reconstruction.*

Proof. If i, j, k are three members of I with $a_i a_j = b_i b_j$, $a_i a_k = b_i b_k$ and $a_k a_j = b_k b_j$, then a short calculation gives $a_i^2 a_j a_k = b_i^2 b_j b_k$ and hence $|a_i| = |b_i|$. This computation holds for each $i \in I$ and since Φ does phase retrieval, there is a $|\theta| = 1$ so that $\text{phase } a_i = \theta \text{ phase } b_i$ for all i . It follows that $a_i = \theta b_i$ for all $i = 1, 2, \dots, m$. \square

It turns out that whenever a frame contains the unit vector basis, then weak phase retrieval and phaseless reconstruction are the same.

Proposition 3.1.5. *Let the frame $\Phi = \{\phi_i\}_{i=1}^n \in \mathbb{R}^m$ does weak phase retrieval. If Φ contains the standard basis vectors, then Φ does phaseless reconstruction.*

Proof. Let $x = (a_1, a_2, \dots, a_m)$, $y = (b_1, b_2, \dots, b_m) \in \mathbb{R}^m$. By definition of weak phase retrieval, Φ satisfies the equation 3.1. In particular, for $\phi_i = e_i$, the equation 3.1 implies that $|a_i| = |b_i|$, $\forall i = 1, 2, \dots, m$. Hence the theorem. \square

We conclude this section by showing the surprising result that weak phaseless reconstruction is same as phaseless reconstruction in \mathbb{R}^m . I.e. It is not really **weak**.

Theorem 3.1.6. *Frames which do weak phaseless reconstruction in \mathbb{R}^m do phaseless reconstruction.*

Proof. For a contradiction assume $\Phi = \{\phi_i\}_{i=1}^n \subset \mathbb{R}^m$ does weak phaseless reconstruction but fail the complement property. Then there exists $I \subset [n]$ such that $\text{Span}_{i \in I} \phi_i \neq \mathbb{R}^m$ and $\text{span}_{i \in I^c} \phi_i \neq \mathbb{R}^m$. Pick non-zero vectors $x, y \in \mathbb{R}^m$ such that $x \perp \text{span}_{i \in I} \phi_i \neq \mathbb{R}^m$ and $y \perp \text{span}_{i \in I^c} \phi_i \neq \mathbb{R}^m$. Then for any $c \neq 0$ we have

$$|\langle x + cy, \phi_i \rangle| = |\langle x - cy, \phi_i \rangle| \quad \text{for all } i \in [n].$$

Now we consider the following cases where x_i and y_i denotes the i^{th} coordinate of the vectors x and y .

Case 1: $\{i : x_i \neq 0\} \cap \{i : y_i \neq 0\} = \emptyset$

Set $c = 1$ and observe since $x \neq 0$ there exists some $i \in [n]$ such that $x_i \neq 0$ and $y_i = 0$ and similarly there exists $j \in [n]$ such that $y_j \neq 0$ but $x_j = 0$. Then $x + y$ and $x - y$ have the same sign in the i^{th} but opposite signs in the j^{th} coordinate, this contradicts the assumption that Φ does weak phaseless reconstruction.

Case 2: There exists $i, j \in [n]$ such that $x_i y_i \neq 0$ and $x_j = 0, y_j \neq 0$.

Without loss of generality, we may assume $x_i y_i > 0$ otherwise consider $-x$ or $-y$. If $0 < c \leq \frac{x_i}{y_i}$, then the i^{th} coordinate of $x + cy$ and $x - cy$ have the same sign whereas the j^{th} coordinates have opposite signs which contradicts the assumption. By considering $y + cx$ and $y - cx$ this argument holds in the case that $y_j = 0$ and $x_j \neq 0$.

Case 3: $x_i = 0$ if and only if $y_i = 0$.

By choosing c small enough, we have that $x_i + cy_i \neq 0$ if and only if $x_i - cy_i \neq 0$. By weak phase retrieval, there is a $|d| = 1$ so that $x_i + cy_i = d(x_i - cy_i)$. But this forces either $x_i \neq 0$ or $y_i \neq 0$ but not both which contradicts the assumption for case 3.

□

It is known [1] that if $\Phi = \{\phi_i\}_{i=1}^n$ does phase retrieval or phaseless reconstruction in \mathbb{H}^m and T is an invertible operator on \mathbb{H}^m then $\{T\phi_i\}_{i=1}^n$ does phase retrieval. It now follows that the same result holds for weak phaseless reconstruction. However, this result does not hold for weak phase retrieval. Indeed, if $\phi_1 = (1, 1)$ and $\phi_2 = (1, -1)$, then we have seen that this frame does weak phase retrieval in \mathbb{R}^2 .

But the invertible operator $T(\phi_1) = (1, 0)$, $T(\phi_2) = (0, 1)$ maps this frame to a frame which fails weak phase retrieval.

Chapter 4

Examples

4.1 Illustrative Examples

In this section, we provide examples of frames that do weak phase retrieval in \mathbb{R}^3 and \mathbb{R}^4 . As seen earlier, the vectors $(1, 1)$ and $(1, -1)$ do weak phase retrieval in \mathbb{R}^2 but fail phase retrieval.

Our first example is a frame which does weak phase retrieval but fails weak phaseless reconstruction.

Example 4.1.1. *We work with the row vectors of*

$$\Phi = \left[\begin{array}{c|ccc} \phi_1 & 1 & 1 & 1 \\ \phi_2 & -1 & 1 & 1 \\ \phi_3 & 1 & -1 & 1 \\ \phi_4 & 1 & 1 & -1 \end{array} \right]$$

Observe that the rows of this matrix form an equal norm tight frame Φ (and hence do norm retrieval). Then if $x = (a_1, a_2, a_3)$ the following is the coefficient matrix

where the row E_i represents the coefficients obtained from the expansion $|\langle x, \phi_i \rangle|^2$

$$1/2 \left[\begin{array}{c|cccc} & a_1a_2 & a_1a_3 & a_2a_3 & \sum_{i=1}^3 a_i^2 \\ E_1 & 1 & 1 & 1 & 1/2 \\ E_2 & -1 & -1 & 1 & 1/2 \\ E_3 & -1 & 1 & -1 & 1/2 \\ E_4 & 1 & -1 & -1 & 1/2 \end{array} \right]$$

Then the following row operations give

$$1/2 \left[\begin{array}{c|cccc} & a_1a_2 & a_1a_3 & a_2a_3 & \sum_{i=1}^3 a_i^2 \\ F_1 = E_1 - E_2 & 1 & 1 & 0 & 0 \\ F_2 = E_3 - E_4 & -1 & 1 & 0 & 0 \\ F_3 = E_1 - E_3 & 1 & 0 & 1 & 0 \\ F_4 = E_2 - E_4 & -1 & 0 & 1 & 0 \\ F_4 = E_1 - E_4 & 0 & 1 & 1 & 0 \\ F_5 = E_2 - E_3 & 0 & -1 & 1 & 0 \end{array} \right]$$

$$1/2 \left[\begin{array}{c|cccc} & a_1a_2 & a_1a_3 & a_2a_3 & \sum_{i=1}^3 a_i^2 \\ F_1 - F_2 & 1 & 0 & 0 & 0 \\ F_3 + F_4 & 0 & 0 & 1 & 0 \\ F_5 - F_6 & 0 & 1 & 0 & 0 \end{array} \right]$$

Therefore we have demonstrated a procedure to identify $a_i a_j$ for all $1 \leq i \neq j \leq 3$.

This shows that given $y = (b_1, b_2, b_3)$ satisfying $|\langle x, \phi_i \rangle|^2 = |\langle y, \phi_i \rangle|^2$ then by the procedure outlined above we obtain

$$a_i a_j = b_i b_j, \text{ for all } 1 \leq i \neq j \leq 3.$$

By Proposition 2.1.3, these four vectors do weak sign retrieval in \mathbb{R}^3 . However this family fails to do weak phaseless reconstruction. Observe the vectors $x = (1, 2, 0)$ and $y = (2, 1, 0)$ satisfy $|\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle|$ however do not have the same absolute value in each coordinate.

Our next example is a frame which does weak phaseless reconstruction but fails phaseless reconstruction.

Example 4.1.2. We provide a set of six vectors in \mathbb{R}^4 which does weak phase retrieval in \mathbb{R}^4 . In this case our vectors are the rows of the matrix:

$$\Phi = \begin{bmatrix} \phi_1 & 1 & 1 & 1 & -1 \\ \phi_2 & -1 & 1 & 1 & 1 \\ \phi_3 & 1 & -1 & 1 & 1 \\ \phi_4 & 1 & 1 & -1 & -1 \\ \phi_5 & 1 & -1 & 1 & -1 \\ \phi_6 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Note that Φ fails to do phase retrieval as it requires seven vectors in \mathbb{R}^4 to do phase retrieval in \mathbb{R}^4 . Given $x = (a_1, a_2, a_3, a_4)$, $y = (b_1, b_2, b_3, b_4)$ we assume

$$|\langle x, \phi_i \rangle|^2 = |\langle y, \phi_i \rangle|^2, \text{ for all } i = 1, 2, 3, 4, 5, 6. \quad (4.1)$$

Step 1: The following is the coefficient matrix obtained after expanding $|\langle x, \phi_i \rangle|^2$ for $i = 1, 2, \dots, 6$.

$$\frac{1}{2} \left[\begin{array}{c|ccccccc} & a_1a_2 & a_1a_3 & a_1a_4 & a_2a_3 & a_2a_4 & a_3a_4 & \sum_{i=1}^4 a_i^2 \\ E_1 & 1 & 1 & -1 & 1 & -1 & -1 & \frac{1}{2} \\ E_2 & -1 & -1 & -1 & 1 & 1 & 1 & \frac{1}{2} \\ E_3 & -1 & 1 & 1 & -1 & -1 & 1 & \frac{1}{2} \\ E_4 & 1 & -1 & -1 & -1 & -1 & 1 & \frac{1}{2} \\ E_5 & -1 & 1 & -1 & -1 & 1 & -1 & \frac{1}{2} \\ E_6 & -1 & -1 & 1 & 1 & -1 & -1 & \frac{1}{2} \end{array} \right]$$

Step 2: Consider the following row operations, the last column becomes all zeroes so we drop it and we get:

$$\left[\begin{array}{l|cccccc} F_1 = \frac{1}{2}(E_1 - E_4) & 0 & 1 & 0 & 1 & 0 & -1 \\ F_2 = \frac{1}{2}(E_2 - E_5) & 0 & -1 & 0 & 1 & 0 & 1 \\ F_3 = \frac{1}{2}(E_3 - E_6) & 0 & 1 & 0 & -1 & 0 & 1 \\ A_1 = \frac{1}{2}(F_1 + F_2) & 0 & 0 & 0 & 1 & 0 & 0 \\ A_2 = \frac{1}{2}(F_1 + F_3) & 0 & 1 & 0 & 0 & 0 & 0 \\ A_3 = \frac{1}{2}(F_2 + F_3) & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

Step 3: Subtracting out A_1, A_2 and A_3 from E_1, E_2, E_3 and E_4 , we get:

$$\left[\begin{array}{l|cccccc} E'_1 = & 1 & 0 & -1 & 0 & -1 & 0 \\ E'_2 = & -1 & 0 & -1 & 0 & 1 & 0 \\ E'_3 = & -1 & 0 & 1 & 0 & -1 & 0 \\ E'_4 = & 1 & 0 & -1 & 0 & -1 & 0 \end{array} \right]$$

Step 4: We will show that $a_i a_j = b_i b_j$ for all $i \neq j$.

Performing the given operations we get:

$$\left[\begin{array}{l|cccccc} D_1 = \frac{-1}{2}(E'_2 + E'_3) & 1 & 0 & 0 & 0 & 0 & 0 \\ A_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ D_2 = \frac{-1}{2}(E'_1 + E'_2) & 0 & 0 & 1 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 & 1 & 0 & 0 \\ D_3 = \frac{-1}{2}(E'_3 + E'_4) & 0 & 0 & 0 & 0 & 1 & 0 \\ A_3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

Doing the same operations with $y = (b_1, b_2, b_3, b_4)$ we get:

$$a_i a_j = b_i b_j, \text{ for all } 1 \leq i \neq j \leq 4.$$

Remark 4.1.3. *It should be noted that weak phase retrieval does not imply norm retrieval. We may use the previous example to illustrate this. Let $\Phi = \{\phi_i\}_{i=1}^6$ be as in Example 4.1.2. Suppose Φ does norm retrieval. Since there are only 6 vectors Φ fails the complement property. Now, take $x = (1, 1, -1, 1) \perp \{\phi_1, \phi_2, \phi_3\}$ and $y = (1, 1, 1, 1) \perp \{\phi_4, \phi_5, \phi_6\}$. Then, we have $|\langle x + y, \phi_i \rangle| = |\langle x - y, \phi_i \rangle|$ for all $i = 1, 2, \dots, 6$. From the definition 1.1.6, this implies $\|x + y\| = \|x - y\|$ which is a contradiction.*

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