A free boundary gas dynamic model as a two-body field theory problem

---

A Dissertation

presented to

the Faculty of the Graduate School

University of Missouri

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Michael Thomas Heitzman

Professor Carmen Chicone, Dissertation Supervisor

JULY 2009

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

A free boundary gas dynamic model as a two-body field theory problem

presented by Michael Thomas Heitzman,

a candidate for the degree of Doctor of Philosophy and hereby certify that in their opinion it is worthy of acceptance.

Professor Carmen Chicone

Professor Stamatis Dostoglou

Professor Adam Helfer

Professor Yuri Latushkin

Professor Frank Feng

Dedicated to my family, friends, and ex-wife, each of whom have supported me in their own way.

ACKNOWLEDGEMENTS

I would like to thank my committee members, Frank Feng, Yuri Latushkin, and in particular Stamatis Dostoglou and Adam Helfer, who have been very generous with their time and have provided valuable critique. Most of all I would like to thank my advisor, Carmen Chicone, who has spent countless Friday afternoons over these many years introducing me to a world that I can spend the rest of my life exploring.

Table of Contents

A free boundary gas dynamic model as a two-body field theory problem

Michael Thomas Heitzman

Professor Carmen Chicone, Dissertation Supervisor

ABSTRACT

Motivated by the two-body problem in the classical field theories of electrodynamics and gravitation, in which finite propagation speeds lead to radiation reaction and runaway solutions, we develop a free boundary problem in gas dynamics to explore the motion of sources in a medium whose dynamics are governed by hyperbolic, wave-like equations arising from physical conservation laws. In our linearized acoustic model, the fields can be eliminated to yield functional differential equations for the motion of the sources—delay equations with an infinite dimensional state space. Expansion and truncation gives rise to runaway solutions, just as in the classical field theories. We illustrate a scheme for eliminating runaway solutions by reducing to a finite dimensional, globally attracting, invariant manifold on which effective equations of motion for the sources can be obtained. The effective equations of motion approximate the asymptotic behavior of solutions in the full space as they approach the manifold. We also treat the full nonlinear free boundary problem and show that unique classical solutions exist locally, for initial fields close enough to their constant steady state.

0.1 Introduction

A field theory consists of partial differential equations (PDEs) governing the evolution of the field, with the field sources incorporated either as boundary data or inhomogeneous source terms. The PDEs are hyperbolic, wave-like equations in which disturbances in the fields propagate with finite speed. Additionally, the motion of the sources is determined by ordinary differential equations (ODEs) coupled to the field at the location of the source. Electrodynamics is a classical example of a field theory. The PDEs governing the electromagnetic fields are Maxwell's equations,

$$\begin{aligned}
\frac{1}{c}\frac{\partial B}{\partial t} &= -\nabla \times E \\
\frac{1}{c}\frac{\partial E}{\partial t} &= \nabla \times B - \frac{1}{c}j \\
\nabla \cdot E &= \rho \\
\nabla \cdot B &= 0,
\end{aligned}$$

where $E(x,t)$, $B(x,t)$ are the electric and magnetic fields, $\rho(x,t)$ and $j(x,t)$ are the charge and current densities (the sources of the fields), and c is the speed of light. The ODE that determines the motion of the charged particle is given by the Lorentz force law, combined with Newton's 2nd law:

$$m\ddot{q}(t) = e[E(q(t),t) + \frac{1}{c}\dot{q}(t) \times B(q(t),t)],$$

where $q(t)$ is the position of the particle and e is its charge.

Typically in electrodynamics, the PDEs and ODEs are treated separately. If the source trajectories (and hence ρ and j) are given, then the fields can be computed by solving the PDEs. Or, if the fields are given, then the motion of the sources may be determined by solving the ODEs. But this separation is unsatisfactory. Rather than prescribing either the fields or the source trajectories for all time, one might wish to give only initial data for fields and sources and then determine how the entire coupled system evolves. Unfortunately, for point charges in electrodynamics, such an initial value problem is not well posed. The difficulty occurs because the field strength due to a point source becomes infinite at the location of the point source. (Solving Maxwell's equations for the fields due to a point source gives the Lienard-Wiechert fields, expressions for $E(x,t)$ and $B(x,t)$ containing a term that diverges as $1/(x - q(t_{ret}))^2$, where the retarded time t_{ret} is delayed by the time it takes a signal to travel with speed c from the charge location $q(t_{ret})$ to the point x. But evaluation at $x = q(t)$ gives $t_{ret} = t$, so that $E(q(t),t)$ and $B(q(t),t)$ are singular (cf. [6]).)

One method of attempting to overcome this difficulty is by neglecting the particle's own contribution to the field in the PDEs, and then compensating by adding a radiation reaction term (determined indirectly by energy and momentum conservation) in the ODE to account for the effect of the force that the particle exerts on itself. The result is the Lorentz-Dirac-Abraham equation

$$m\ddot{q} = F_{ext} + \frac{2e^2}{3c^3}\dddot{q}, \tag{1}$$

where F_{ext} is the net force due to the fields of all the other charges, and the last term is the radiation reaction. This 3rd order ODE is non-Newtonian with unphysical, runaway solutions. Additional constraints must be imposed to reduce this higher order ODE to an effective 2nd order Newtonian equation of motion with no runaway solutions (cf. [1]).

An alternate approach is to replace the point charges with extended charges. For a charged spherical shell of radius r, charge e, and bare mass m_0 moving with low velocity ($v/c << 1$) on a line, A. Sommerfeld's exact representation of the electromagnetic fields do not diverge and may be used in the Lorentz force law to give the equation of motion

$$\frac{d}{dt}(m_0 v) = \frac{e^2}{3r^2 c}(v(t - \frac{2r}{c}) - v(t)). \tag{2}$$

(See [1, 5].) This retarded functional differential equation (RFDE) is again non-Newtonian. It does not have runaway solutions, but expanding in the small delay and truncating again yields the 3rd order Abraham-Dirac-Lorentz equation (incorporating an appropriate electromagnetic mass into the total mass), with runaway solutions which must be eliminated by additional constraints. Unfortunately, singularities still emerge when taking the point charge limit of the extended charge.

Similar issues arise in gravitational field theory. The difficulties encountered in the classical field theories of electrodynamics and gravity for point sources provides motivation to seek a simpler field theory model where the fields due to the point source remain finite at the location of the source, yet the model still displays issues associated with finite propagation speed and radiation reaction. Fluid dynamics provides the setting for such a model (cf. [7]).

0.2 Nonlinear gas dynamic model

Consider a tube containing an isentropic ideal gas confined between two pistons attached to springs at each end. The motion of each piston is determined by an ODE: Newton's 2nd law applied to the piston acted on by the spring force, spring damping, and gas pressure (at the location of the piston). The pressure field in the gas between the pistons is determined by the PDEs of gas dynamics, with the motion of the pistons included as boundary data. The pistons are analogous to charges in electrodynamics, with the force due to the gas pressure acting as the Lorentz force law. The PDEs of gas dynamics are analogous to Maxwell's equations for the electromagnetic fields. But, the gas dynamics model has the advantage that even though a piston behaves as a point charge, the pressure field it produces does not become singular at the location of the piston. Hence, we may directly couple the motion of the piston to its own pressure field, and thus immediately ascertain the radiation reaction, without resorting to the indirect methods required in electrodynamics.

Let us assume the tube lies along the x-axis and the gas flow is one-dimensional with velocity $u(x,t)$ and positive density $\rho(x,t)$. Assume the gas is isentropic and the pressure P is only a function of ρ, given by the equation of state

$$P(\rho) = a\rho^{\gamma}, \tag{3}$$

where $a > 0$ and $\gamma > 1$ are constants characterizing the gas. Let the ith piston (for i=1,2) have mass M_i, area A, and be attached to a spring with spring constant K_i and spring damping coefficient μ_i. Let R_1 be the displacement of the first piston from its vacuum equilibrium position of $x = 0$, and R_2 be the displacement from its vacuum equilibrium position of $x = l$.

On the state-dependent domain $R_1(t) < x < l + R_2(t)$, the field equations arise from conservation of mass and momentum of the gas, taking the form of the continuity equation and Euler's equation

$$\rho_t = -(\rho u)_x \tag{4}$$

$$\rho u_t = -\rho u u_x - P(\rho)_x. \tag{5}$$

In addition, matching the piston and gas velocity and applying Newton's 2nd law to the pistons imposes the boundary conditions (BCs)

$$
\begin{aligned}
u(R_1(t),t) &= \dot{R}_1(t) && (6)\\
u(l+R_2,t) &= \dot{R}_2(t) && (7)\\
M_1\ddot{R}_1(t) &= -K_1R_1(t)-\mu_1\dot{R}_1(t)-P(\rho(R_1(t),t))A && (8)\\
M_2\ddot{R}_2(t) &= -K_2R_2(t)-\mu_2\dot{R}_2(t)+P(\rho(l+R_2(t),t))A. && (9)
\end{aligned}
$$

Initial data consists of values for $R_1(0)$, $R_2(0)$, $\dot{R}_1(0)$, $\dot{R}_2(0)$, and the initial fields

$$\rho(x,0)=F(x),\ \ u(x,0)=G(x),\ \ x\in[R_1(0),l+R_2(0)], \tag{10}$$

for some functions F and G, where F must satisfy the condition

$$\int_{R_1(0)}^{l+R_2(0)} F(x)\,dx = M, \tag{11}$$

where M is the total mass of gas between the pistons, per cross sectional area, A. Note that $\int_{R_1(t)}^{l+R_2(t)} \rho(x,t)\,dx$ remains constant in t by the continuity equation (4) and the BCs (6)–(7).

The usual compatibility conditions imposed by matching initial and boundary data requires

$$\dot{R}_1(0)=G(R_1(0)),\ \ \dot{R}_2(0)=G(l+R_2(0)). \tag{12}$$

In addition, if Euler's equation (5) holds at the boundary, then it may be used with BCs (6)–(7) (differentiated w.r.t. t) to determine the initial piston accelerations

$$\ddot{R}_1(0)=-\left(\frac{1}{F(x)}\frac{\partial}{\partial x}P(F(x))\right)\Bigg|_{x=R_1(0)},\ \ \ddot{R}_2(0)=-\left(\frac{1}{F(x)}\frac{\partial}{\partial x}P(F(x))\right)\Bigg|_{x=l+R_2(0)}.$$

The BCs (8)–(9) then give rise to the further compatibility conditions

$$
\begin{aligned}
-\left(\frac{M_1}{F(x)}\frac{\partial}{\partial x}P(F(x))\right)\Bigg|_{x=R_1(0)} &= -K_1R_1(0) - \mu_1 G(R_1(0)) \\
&\quad -P(F(R_1(0))A \qquad (13) \\
-\left(\frac{M_2}{F(x)}\frac{\partial}{\partial x}P(F(x))\right)\Bigg|_{x=l+R_2(0)} &= -K_2R_2(0) - \mu_2 G(l+R_2(0)) \\
&\quad +P(F(l+R_2(0))A. \qquad (14)
\end{aligned}
$$

Thus the initial fields F and G determine the initial piston velocities $\dot{R}_1(0)$, $\dot{R}_2(0)$ and positions $R_1(0)$, $R_2(0)$. This implies that the state of the system (3)–(14) at any time t is completely determined by the field functions $\rho(\cdot,t)$ and $u(\cdot,t)$.

For every solution of (3)–(9), the energy

$$
\begin{aligned}
E(t) \;:=\; & A\int_{R_1(t)}^{l+R_2(t)}\left(\frac{1}{2}\rho(x,t)u^2(x,t) + \frac{P(\rho(x,t))}{\gamma-1}\right)dx \\
& +\frac{1}{2}K_1R_1^2(t) + \frac{1}{2}K_2R_2^2(t) + \frac{1}{2}M_1\dot{R}_1^2(t) + \frac{1}{2}M_2\dot{R}_2^2(t)
\end{aligned}
$$

satisfies

$$\frac{dE}{dt} = -\mu_1\dot{R}_1^2 - \mu_2\dot{R}_2^2.$$

The linearization of (3)–(14) about a steady state of constant fields, $\rho = \rho_0$ and $u = 0$, yields the acoustic problem on which we expound in chapter 1. In chapter 2, we show the full nonlinear problem is locally well posed for initial data taken close enough to that constant steady state (in C^1 norm).

0.3 Steady state solution

We now show that for a steady state solution (constant in time) of the system (3)–(14), the field functions ρ and u must be constant in space as well, and these constants, along with the steady state piston positions, are uniquely determined by the parameters a, γ, A, M, l, K_1 and K_2.

By (3) and PDEs (4)–(5), steady state solutions ρ and u must satisfy

$$\begin{aligned}\rho_x u + \rho u_x &= 0 \\ \rho u u_x + \gamma a \rho^{\gamma-1}\rho_x &= 0,\end{aligned}$$

which together imply

$$(u^2 + \gamma a \rho^{\gamma-1})\rho_x = 0.$$

Since $(u^2 + \gamma a \rho^{\gamma-1}) > 0$, we must have $\rho_x = 0$. Thus ρ is constant, say $\rho = \rho_0$. Then (4) further simplifies to become

$$\rho_0 u_x = 0,$$

which implies that u is constant, say $u = u_0$.

Let $R_1 = R_{1_0}$ and $R_2 = R_{2_0}$ be the steady state piston displacements (which are constant by the definition of steady state). Then the BCs (6)–(7) imply that in fact, $u_0 = 0$.

The condition

$$\int_{R_1(t)}^{l+R_2(t)} \rho(x,t)\, dx = M$$

implies the steady state must satisfy

$$\rho_0(l + R_{2_0} - R_{1_0}) = M.$$

Furthermore the BCs (8)–(9) along with the equation of state (3) imply

$$R_{1_0} = -\frac{aA\rho_0^\gamma}{K_1}, \quad R_{2_0} = \frac{aA\rho_0^\gamma}{K_2}. \tag{15}$$

Therefore the steady state density ρ_0 is uniquely determined by the parameters a, A, γ, M, l, K_1 and K_2 through the relation

$$l\rho_0 + \frac{K_1 K_2}{K_1 + K_2} aA\rho_0^{\gamma+1} = M. \tag{16}$$

This then uniquely determines R_{1_0} and R_{2_0} as well, by (15). Note that the values of ρ_0, R_{1_0}, and R_{2_0} are independent of the piston masses, M_1 and M_2, as well as initial data (provided that condition (11) is satisfied).

It is easy to check that $\rho = \rho_0$ and $u = 0$ (with $R_1(0)$ and $R_2(0)$ determined by the compatibility conditions (13)–(14) to be R_{1_0} and R_{2_0} as given above) are solutions to (3)–(14).

Chapter 1

Acoustic model

1.1 Linearized acoustic model

The acoustic model is the linearization about the steady state solution of (3)–(14),

$$\rho = \rho_0, \quad u = 0, \quad R_1 = R_{1_0}, \quad R_2 = R_{2_0}, \tag{1.1}$$

determined in section 0.3 by (15)–(16)

We linearize the system (3)–(14) about the steady state (1.1) by introducing the change of variables, for $i = 1, 2$,

$$\rho(x,t) = \rho_0 + \epsilon\tilde{\rho}(x,t), \quad u(x,t) = \epsilon\tilde{u}(x,t), \quad R_i(t) = R_{i_0} + \epsilon\tilde{R}_i(t),$$

$$F(x) = \rho_0 + \epsilon\tilde{F}(x), \quad G(x) = \epsilon\tilde{G}(x). \tag{1.2}$$

Keeping only terms to 1st order in ϵ (i.e., differentiating all expressions with respect to ϵ and then letting $\epsilon \to 0$), and denoting $c_0^2 = P'(\rho_0)$, we get the linearized system on the fixed domain $R_{1_0} < x < l + R_{2_0}$,

$$\tilde{\rho}_t = -\rho_0\tilde{u}_x \tag{1.3}$$

$$\rho_0\tilde{u}_t = -c_0^2\tilde{\rho}_x, \tag{1.4}$$

with boundary conditions

$$\begin{aligned}
\tilde{u}(R_{1_0}, t) &= \dot{\tilde{R}}_1(t) && (1.5)\\
\tilde{u}(l + R_{2_0}, t) &= \dot{\tilde{R}}_2(t) && (1.6)\\
M_1 \ddot{\tilde{R}}_1(t) &= -K_1 \tilde{R}_1(t) - \mu_1 \dot{\tilde{R}}_1(t) - c_0^2 \tilde{\rho}(R_{1_0}, t) A && (1.7)\\
M_2 \ddot{\tilde{R}}_2(t) &= -K_2 \tilde{R}_2(t) - \mu_2 \dot{\tilde{R}}_2(t) + c_0^2 \tilde{\rho}(l + R_{2_0}, t) A, && (1.8)
\end{aligned}$$

initial conditions, for $i = 1, 2$,

$$\tilde{R}_i(0), \quad \dot{\tilde{R}}_i(0) \tag{1.9}$$

$$\tilde{\rho}(x, 0) = \tilde{F}(x), \quad \tilde{u}(x, 0) = \tilde{G}(x), \quad x \in [R_{1_0}, l + R_{2_0}], \tag{1.10}$$

and compatibility conditions

$$\dot{\tilde{R}}_1(0) = \tilde{G}(R_{1_0})), \quad \dot{\tilde{R}}_2(0) = \tilde{G}(l + R_{2_0}). \tag{1.11}$$

$$-M_1 \frac{c_0^2}{\rho_0} \tilde{F}'(R_{1_0}) = -K_1 \tilde{R}_1(0) - \mu_1 \tilde{G}(R_{1_0}) - c_0^2 \tilde{F}(R_{1_0}) A, \tag{1.12}$$

$$-M_2 \frac{c_0^2}{\rho_0} \tilde{F}'(l + R_{2_0}) = -K_2 \tilde{R}_2(0) - \mu_2 \tilde{G}(l + R_{2_0}) + c_0^2 \tilde{F}(l + R_{2_0}) A, \tag{1.13}$$

which again determine the initial piston displacements and velocities, $\tilde{R}_i(0)$, $\dot{\tilde{R}}_i(0)$, in terms of the initial fields $\tilde{F}$ and $\tilde{G}$.

Formally, the new linearized fields given by the change of variables in (1.2) would have the same (state dependent) domain as the original functions. However, in the linearized boundary conditions (1.5)–(1.8), the linearized fields are only evaluated at the fixed positions $x = R_{1_0}$ and $x = l + R_{2_0}$, which may not even be located within the original state dependent domain. Therefore we define the new linearized field functions to have the fixed spatial domain $[R_{1_0}, l + R_{2_0}]$. We emphasize that, in the linearized system, all functions of x are on the fixed domain $[R_{1_0}, l + R_{2_0}]$;

we no longer have a free boundary. However, the functions $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are still unknown (hence the four B.C.s, rather than just the two that would be required if the $\tilde{R}$'s where prescribed).

The linearization of condition (11) leads to the condition

$$\int_{R_{1_0}}^{l+R_{2_0}} \tilde{F}(x)\, dx + \rho_0(\tilde{R}_2(0) - \tilde{R}_1(0)) = 0. \tag{1.14}$$

On the fixed domain, the linearized continuity equation (1.3) no longer conserves mass; combined with BCs (1.5)–(1.6) it yields

$$\frac{d}{dt}\int_{R_{1_0}}^{l+R_{2_0}} \tilde{\rho}(x,t)\, dx = -\rho_0(\dot{\tilde{R}}_2(t) - \dot{\tilde{R}}_1(t)),$$

so that the map $t \mapsto \int_{R_{1_0}}^{l+R_{2_0}} \tilde{\rho}(x,t)\, dx + \rho_0(\tilde{R}_2(t) - \tilde{R}_1(t))$ is a constant of the motion. The mass of the gas between the pistons would be conserved with the extra condition

$$\dot{\tilde{R}}_2(t) - \dot{\tilde{R}}_1(t) = 0, \tag{1.15}$$

but we choose not to impose this extra condition. More will be said about this later, as this issue comes back to haunt us when we seek a finite dimensional, globally attracting, invariant manifold whose dynamics capture the asymptotic behavior of solutions evolving in the full infinite dimensional state space of the linearized system (1.3)–(1.14).

It can be seen from PDEs (1.3)–(1.4) that $\tilde{\rho}$ and $\tilde{u}$ each satisfy the wave equation with wave speed c_0 (by differentiating one equation in t, the other in x, and canceling the mixed partials after multiplying by the appropriate constants). We may then use d'Alembert's solution to express $\tilde{\rho}$ and $\tilde{u}$ as a sum of functions of $x + ct$ and $x - ct$, respectively, where the functions can be determined by initial and boundary conditions (yielding a finite series solution for each t, in which the number of terms increases with t due to successive reflections at the boundaries). Instead, we use the more geometric method of characteristics, which is equivalent in this case but may also be applied to more general hyperbolic systems, including the nonlinear system of the previous section.

First, we note that if the fluid energy term of the previous section is modified by defining the total energy of the linearized system to be

$$\tilde{E} = \frac{1}{2}M_1\dot{\tilde{R}}_1^2 + \frac{1}{2}M_2\dot{\tilde{R}}_2^2 + \frac{1}{2}K_1\tilde{R}_1^2 + \frac{1}{2}K_2\tilde{R}_2^2 + A\int_{R_{1_0}}^{l+R_{2_0}} (\frac{1}{2}\rho_0\tilde{u}^2 + \frac{c_0^2}{2\rho_0}\tilde{\rho}^2)\, dx, \quad (1.16)$$

then PDEs and BCs (1.3)–(1.8) may be used to show

$$\frac{d\tilde{E}}{dt} = -\mu_1\dot{\tilde{R}}_1^2 - \mu_2\dot{\tilde{R}}_2^2. \quad (1.17)$$

This will be used to show that, for positive spring damping coefficients μ_1 and μ_2, the displacements $\tilde{R}_1$ and $\tilde{R}_2$ approach zero as t grows toward $+\infty$, and that the fields $\tilde{\rho}$ and $\tilde{u}$ approach zero as well, uniformly for almost all $x \in [R_{1_0}, l + R_{2_0}]$. Although these results may be expected, they are not obvious. In the next section, we show that if a steady state is approached, then it must be the trivial solution.

1.2 Linearized steady state solution

For a steady state solution of the linearized system (1.3)–(1.14), the PDEs (1.3)–(1.4) immediately imply the field functions $\tilde{\rho}$ and $\tilde{u}$ must be constant in space as well, say $\tilde{\rho} = \tilde{\rho}_0$ and $\tilde{u} = \tilde{u}_0$ (not to be confused with ρ_0 in (1.1)). By definition of a steady state, each $\tilde{R}_i$ is constant, say $\tilde{R}_i = \tilde{R}_{i_0}$, for $i = 1, 2$ (not to be confused with R_{i_0} in (1.1)). Then the BCs (1.5)–(1.6) imply $\tilde{u}_0 = 0$, and BCs (1.7)–(1.8) give

$$\tilde{R}_{1_0} = -\frac{c_0^2}{K_1}\tilde{\rho}_0, \quad \tilde{R}_{2_0} = \frac{c_0^2}{K_2}\tilde{\rho}_0. \quad (1.18)$$

Suppose for now that any solution of the linearized system (1.3)–(1.14) approaches a steady state (as will later be shown), i.e., assume that for all $x \in [R_{1_0}, l + R_{2_0}]$, we have

$$\lim_{t\to+\infty} \tilde{\rho}(x,t) = \tilde{\rho}_0, \quad \lim_{t\to+\infty} \tilde{u}(x,t) = 0, \quad \lim_{t\to+\infty} \tilde{R}_i(t) = \tilde{R}_{i_0}, \ \ i = 1, 2.$$

Then by BCs (1.5)–(1.6), we also have

$$\lim_{t\to+\infty} \dot{\tilde{R}}_i(t) = 0, \ \ i = 1, 2.$$

Define

$$\tilde{M}(t) := \int_{R_{1_0}}^{l+R_{2_0}} \tilde{\rho}(x,t)\, dx. \tag{1.19}$$

and denote $\tilde{M}(\infty) := \lim_{t\to+\infty} \tilde{M}(t)$. Then by our assumption, we have

$$\tilde{M}(\infty) = \tilde{\rho}_0(l + R_{2_0} - R_{1_0}). \tag{1.20}$$

Differentiating (1.19) with respect to t, and using PDE (1.3) and BCs (1.5)–(1.6), we obtain

$$\begin{aligned} \tilde{M}'(t) &= \int_{R_{1_0}}^{l+R_{2_0}} \tilde{\rho}_t(x,t)\, dx \\ &= -\rho_0 \int_{R_{1_0}}^{l+R_{2_0}} \tilde{u}_x(x,t)\, dx \\ &= -\rho_0(\tilde{u}(l + R_{2_0}, t) - \tilde{u}(R_{1_0}, t)) \\ &= -\rho_0(\dot{\tilde{R}}_2(t) - \dot{\tilde{R}}_1(t)). \end{aligned}$$

Now, perhaps surprisingly, we integrate the last expression for $\tilde{M}'(t)$ from $t = 0$ to $t = \sigma$, and let $\sigma \to +\infty$, which yields

$$\tilde{M}(\infty) - \tilde{M}(0) = -\rho_0(\tilde{R}_{2_0} - \tilde{R}_{1_0}) + \rho_0(\tilde{R}_2(0) - \tilde{R}_1(0)).$$

Using (1.18) and (1.20) in the above equation, and then solving for $\tilde{\rho}_0$, we have

$$\tilde{\rho}_0 = \frac{\tilde{M}(0) + \rho_0(\tilde{R}_2(0) - \tilde{R}_1(0))}{l + R_{2_0} - R_{1_0} + \rho_0 c_0^2 \frac{K_1+K_2}{K_1 K_2}}, \tag{1.21}$$

where

$$\tilde{M}(0) = \int_{R_{1_0}}^{l+R_{2_0}} \tilde{F}(x)\, dx.$$

Thus, combining (1.21) with condition (1.14) gives $\tilde{\rho} = 0$, so that (1.18) implies $\tilde{R}_{1_0} = 0$ and $\tilde{R}_{2_0} = 0$ as well.

Therefore any steady state that is approached by a solution of the linearized system must be the trivial solution

$$\tilde{\rho} = 0, \quad \tilde{u} = 0, \quad \tilde{R}_1 = 0, \quad \tilde{R}_2 = 0.$$

Before showing that solutions of the linearized system (1.3)–(1.14) approach a constant steady state, we must first introduce the method of characteristics.

1.3 Method of characteristics

The PDEs (1.3)–(1.4) can be written in the form

$$\begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix}_t = \begin{bmatrix} 0 & -\rho_0 \\ -\frac{c_0^2}{\rho_0} & 0 \end{bmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix}_x,$$

where the matrix on the right hand side has eigen values $\pm c_0$ with corresponding left eigen vectors $(\pm\frac{c_0}{\rho_0}, -1)$. Left multiplying the above system by each left eigen vector, respectively, and collecting derivatives, yields the characteristic form of the system

$$\begin{aligned} (\frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x})(\tilde{u} - \frac{c_0}{\rho_0}\tilde{\rho}) &= 0 \\ (\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x})(\tilde{u} + \frac{c_0}{\rho_0}\tilde{\rho}) &= 0, \end{aligned}$$

which may be written in the form

$$
\begin{aligned}
w_t - c_0 w_x &= 0 && (1.22)\\
z_t + c_0 z_x &= 0, && (1.23)
\end{aligned}
$$

where the Riemann invariant functions, defined by

$$
\begin{aligned}
w(x,t) &= \tilde{u}(x,t) - \frac{c_0}{\rho_0}\tilde{\rho}(x,t) && (1.24)\\
z(x,t) &= \tilde{u}(x,t) + \frac{c_0}{\rho_0}\tilde{\rho}(x,t), && (1.25)
\end{aligned}
$$

are constant along the integral curves of $\frac{dx}{dt} = \mp c_0$, the characteristics, which in this case are straight lines in the (x,t) plane with slopes $\mp 1/c_0$, respectively.

Of course, $\tilde{u}$ and $\tilde{\rho}$ may be determined from w and z by adding and subtracting, respectively. In the next section, we will use the Riemann invariants combined with the boundary and initial conditions to eliminate the field functions $\tilde{\rho}$ and $\tilde{u}$. Thus we obtain equations of motion for just the pistons, but these become functional differential equations (FDEs) involving a time delay rather than ODEs after the pistons begin to interact.

1.4 Functional differential equations of motion

Here we continue to work formally with the linearized system (1.3)–(1.14). Well-posedness will be shown in section 1.5. Let $l_0 = l + R_{2_0} - R_{1_0}$ be the steady state distance between the pistons, which is also the fixed distance between the boundaries in the linearized system. Then for $t \in [0, l_0/c_0]$, the characteristic line of slope $-1/c_0$ going through the boundary point (R_{1_0}, t) will intersect the x-axis at the point $(R_{1_0} + c_0 t, 0)$, which lies in the fixed domain. Since the Riemann invariant w is constant along this line, we have

$$\tilde{u}(R_{1_0}, t) - \frac{c_0}{\rho_0}\tilde{\rho}(R_{1_0}, t) = \tilde{u}(R_{1_0} + c_0 t, 0) - \frac{c_0}{\rho_0}\tilde{\rho}(R_{1_0} + c_0 t, 0).$$

Using the BCs (1.5) and (1.7) in the left hand side and the ICs (1.10) in the right hand side of the above equation, we get the ODE

$$M_1\ddot{\tilde{R}}_1(t) = -(\mu_1 + \rho_0 c_0 A)\dot{\tilde{R}}_1(t) - K_1\tilde{R}_1(t) - A\tilde{F}(R_{1_0} + c_0 t) + \rho_0 c A\tilde{G}(R_{1_0} + c_0 t). \quad (1.26)$$

Similarly, again for $t \in [0, l_0/c_0]$, the characteristic line of slope $1/c_0$ going through the boundary point $(l + R_{2_0}, t)$ will intersect the x-axis at the point $(l + R_{2_0} - c_0 t, 0)$, which lies in the fixed domain. Since the Riemann invariant z is constant along this line, we have

$$\tilde{u}(l + R_{2_0}, t) + \frac{c_0}{\rho_0}\tilde{\rho}(l + R_{2_0}, t) = \tilde{u}(l + R_{2_0} - c_0 t, 0) + \frac{c_0}{\rho_0}\tilde{\rho}(l + R_{2_0} - c_0 t, 0).$$

Using the BCs (1.6) and (1.8) in the left hand side and the ICs (1.10) in the right hand side of the above equation, we get the ODE

$$\begin{aligned} M_2\ddot{\tilde{R}}_2(t) &= -(\mu_2 + \rho_0 c_0 A)\dot{\tilde{R}}_2(t) - K_2\tilde{R}_2(t) + A\tilde{F}(l + R_{2_0} - c_0 t) \\ &\quad + \rho_0 c_0 A\tilde{G}(l + R_{2_0} - c_0 t). \end{aligned} \quad (1.27)$$

From the ODEs (1.26)–(1.27) we see that for $t \in [0, l_0/c_0]$, the pistons behave as decoupled damped harmonic oscillators with forcing terms given by the initial fields $\tilde{F}$ and $\tilde{G}$. They are decoupled because l_0/c_0 is the time required for a signal to travel from one piston to the other (in the linearized approximation that the distance between the pistons is their steady state separation of l_0). Thus the pistons have not had time to interact yet. Note that the damping proportional to the piston velocity has two parts: spring damping with coefficient μ_i and fluid damping with coefficient

$\rho_0 c_0 A$. The fluid damping may be associated with the radiation reaction—it arises from the coupling of the piston's motion to its own pressure field (the analogous direct coupling could not be done for point charges in classical electrodynamics). Some justification for associating the damping term $\rho_0 c_0 A\dot{\tilde{R}}_i$ with self-force can be provided by noting that it is present even for zero initial fields $\tilde{F}$ and $\tilde{G}$, and the only other field source, namely the other piston, is beyond the range of influence for $t < l_0/c_0$.

Starting at a point on the boundary for $t > l_0/c$ (after the pistons have begun to interact), tracing back in time along a characteristic line will hit the other boundary before the $x-$axis. The characteristic line of slope $-1/c_0$ going through the left boundary point (R_{1_0}, t) will intersect the right boundary at the point $(l + R_{2_0}, t - \frac{l_0}{c_0})$. The characteristic line of slope $1/c_0$ going through the right boundary point $(l+R_{2_0}, t)$ will intersect the left boundary at the point $(R_{1_0}, t - \frac{l_0}{c_0})$. Making use of the Riemann invariants w and z as before, we have

$$\begin{aligned}
\tilde{u}(R_{1_0}, t) - \frac{c_0}{\rho_0}\tilde{\rho}(R_{1_0}, t) &= \tilde{u}(l + R_{2_0}, t - \frac{l_0}{c_0}) - \frac{c_0}{\rho_0}\tilde{\rho}(l + R_{2_0}, t - \frac{l_0}{c_0}) \\
\tilde{u}(l + R_{2_0}, t) + \frac{c_0}{\rho_0}\tilde{\rho}(l + R_{2_0}, t) &= \tilde{u}(R_{1_0}, t - \frac{l_0}{c_0}) + \frac{c_0}{\rho_0}\tilde{u}(R_{1_0}, t - \frac{l_0}{c_0}).
\end{aligned}$$

Applying the BCs (1.5)–(1.8) to eliminate the field functions $\tilde{\rho}$ and $\tilde{u}$ yields the neutral functional differential equations (NFDEs)

$$
\begin{aligned}
M_1\ddot{\tilde{R}}_1(t)+M_2\ddot{\tilde{R}}_2(t-\frac{l_0}{c_0}) &= -(\mu_1+\rho_0c_0A)\dot{\tilde{R}}_1(t)-K_1\tilde{R}_1(t)\\
&\quad +(\rho_0c_0A-\mu_2)\dot{\tilde{R}}_2(t-\frac{l_0}{c_0})\\
&\quad -K_2\tilde{R}_2(t-\frac{l_0}{c_0}) \qquad (1.28)\\
M_2\ddot{\tilde{R}}_2(t)+M_1\ddot{\tilde{R}}_1(t-\frac{l_0}{c_0}) &= -(\mu_2+\rho_0c_0A)\dot{\tilde{R}}_2(t)-K_2\tilde{R}_2(t)\\
&\quad +(\rho_0c_0A-\mu_1)\dot{\tilde{R}}_1(t-\frac{l_0}{c_0})\\
&\quad -K_1\tilde{R}_1(t-\frac{l_0}{c_0}). \qquad (1.29)
\end{aligned}
$$

NFDEs are functional differential equations with a delay appearing in the highest order derivatives (see [15]). By themselves, the NFDEs (1.28) and (1.29) are not Newtonian: they require initial functions for $\tilde{R}_1$ and $\tilde{R}_2$ on the interval $[0, l_0/c_0]$. On the other hand, these initial functions are generated by the solutions of ODEs (1.26)–(1.27), which are Newtonian: they only require the initial data $\tilde{R}_1(0)$, $\tilde{R}_2(0)$, $\dot{\tilde{R}}_1(0)$, and $\dot{\tilde{R}}_2(0)$ (assuming that the initial fields $\tilde{F}$ and $\tilde{G}$ are given without the compatibility conditions (1.11)–(1.13)). In this way, the ODE/NFDE system (1.26)–(1.29) may be considered as a Newtonian system.

Unfortunately, viewed in this manner, these equations do not define a dynamical system; the time $t = 0$ is special because the initial fields are specified at that time only. To form a true dynamical system with a flow, the gas density and velocity fields, $\tilde{\rho}$ and $\tilde{u}$, must be included in the state space (making it infinite dimensional), and the equations that determine their evolution must be incorporated into the solution (and the compatibility conditions (1.11)–(1.13) must be included to ensure the fields remain smooth). The mechanism for determining the evolution of the fields from the ODE/NFDE system (1.26)–(1.29) may be seen as follows. Given the initial fields $\tilde{F}$

and $\tilde{G}$ and compatibility conditions (1.11)–(1.13), the ODEs (1.26)–(1.27) determine $\tilde{R}_1$ and $\tilde{R}_2$ on the time interval $[0, l_0/c_0]$, which are then used as initial functions for the NFDEs (1.28)–(1.29), which in turn determine $\tilde{R}_1$ and $\tilde{R}_2$ for $t > l_0/c_0$. Knowing $\tilde{R}_1$ and $\tilde{R}_2$ on any time interval of length l_0/c_0, the fields at the beginning of the time interval may be obtained by solving for the initial fields in the ODEs (1.26)–(1.27), translated in time appropriately. Of course, this translation assumes existence and uniqueness of solutions of the ODE/NFDE system (1.26)–(1.29).

Alternatively, the fields $\tilde{\rho}$ and $\tilde{u}$ may be constructed from $\tilde{R}_1$, $\tilde{R}_2$, $\tilde{F}$, and $\tilde{G}$ using Riemann invariants, as in the next section, where we show well-posedness of the linearized PDE/ODE system (1.3)–(1.14). First, we state a proposition summarizing the relationship between the two systems.

Proposition 1. *The linearized PDE/ODE system* (1.3)–(1.14) *is well posed if and only if the ODE/NFDE system* (1.26)–(1.29) *with conditions* (1.11)–(1.14) *is well posed.*

Proof. We have shown that for any solution of the system (1.3)–(1.14), the piston displacements $\tilde{R}_1$ and $\tilde{R}_2$ must also satisfy the system (1.26)–(1.29). Conversely, given functions $\tilde{R}_1$ and $\tilde{R}_2$ which satisfy the ODE/NFDE system (1.26)–(1.29) with conditions (1.11)–(1.14), the procedure that was used to eliminate the fields in deriving (1.26)–(1.29) can be reversed to construct the fields $\tilde{\rho}$ and $\tilde{u}$ from $\tilde{R}_1$, $\tilde{R}_2$ and the initial fields $\tilde{F}$ and $\tilde{G}$. This construction can be seen explicitly in the next section, where in fact the functions $\tilde{R}_1$ and $\tilde{R}_2$ will be constructed as well, rather than being assumed given. □

1.5 Existence and uniqueness for the acoustic model

In this section we use characteristics and Riemann invariants to construct (unique) solutions to the linearized system (1.3)–(1.14), rather than eliminating the fields. Let $\tilde{F}$ and $\tilde{G}$ be C^1 functions with domain $[R_{1_0}, l + R_{2_0}]$. We begin by constructing the solution on the boundary in terms of $\tilde{F}$ and $\tilde{G}$.

For $t \in [0, l_0/c_0]$, define $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ as the solution of the ODEs (1.26) and (1.27), respectively, where the initial values $\tilde{R}_1(0)$, $\dot{\tilde{R}}_1(0)$, $\tilde{R}_2(0)$, and $\dot{\tilde{R}}_2(0)$ are determined from $\tilde{F}$ and $\tilde{G}$ by conditions (1.11)–(1.13). Clearly $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are well defined by the standard existence, uniqueness, and extension theorems for ODEs. In fact they are C^3 functions of t, and depend smoothly on their initial data. We can then impose the boundary conditions by defining $\tilde{u}(R_{1_0}, t)$, $\tilde{u}(l+R_{2_0}, t)$, $\tilde{\rho}(R_{1_0}, t)$, and $\tilde{\rho}(l + R_{2_0}, t)$ by BCs (1.5)–(1.8), for $t \in [0, l_0/c_0]$.

We now divide the (x, t) plane into four regions on which we will construct the field solutions separately. The characteristic lines $x = R_{1_0} + c_0 t$ and $x = l + R_{2_0} - c_0 t$ emanating from the boundary points $(R_{1_0}, 0)$ and $(l + R_{2_0}, 0)$, respectively, form the following three regions:

$$
\begin{aligned}
\mathcal{R}_1 &= \{(x,t) : R_{1_0} + c_0 t \le x \le l + R_{2_0} - c_0 t, \quad t \ge 0\} \\
&= \{(x,t) : t \le \frac{x - R_{1_0}}{c_0}, \quad t \le \frac{l + R_{2_0} - x}{c_0}, \quad t \ge 0\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_2 &= \{(x,t) : x \le R_{1_0} + c_0 t, \quad x \le l + R_{2_0} - c_0 t, \quad x \ge 0\} \\
&= \{(x,t) : \frac{x - R_{1_0}}{c_0} \le t \le \frac{l + R_{2_0} - x}{c_0}, \quad t \ge 0\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_3 &= \{(x,t) : x \ge R_{1_0} + c_0 t, \quad x \ge l + R_{2_0} - c_0 t, \quad x \le l + R_{2_0}\} \\
&= \{(x,t) : \frac{l + R_{2_0} - x}{c_0} \le t \le \frac{x - R_{1_0}}{c_0}, \quad x \le l + R_{2_0}\}.
\end{aligned}
$$

The two additional characteristics $x = R_{1_0} - l_0 + c_0 t$ and $x = l + R_{2_0} + l_0 - c_0 t$ emanating from the boundary points $(R_{1_0}, \frac{l_0}{c_0})$ and $(l + R_{2_0}, \frac{l_0}{c_0})$, respectively, form the upper boundary to the fourth region:

$$
\begin{aligned}
\mathcal{R}_4 &= \{(x,t) : l + R_{2_0} - c_0 t \le x \le R_{1_0} + c_0 t, \\
&\quad R_{1_0} - l_0 + c_0 t \le x \le l + R_{2_0} + l_0 - c_0 t\} \\
&= \{(x,t) : \frac{l + R_{2_0} - x}{c_0} \le t \le \frac{x - R_{1_0} + l_0}{c_0}, \\
&\quad \frac{x - R_{1_0}}{c_0} \le t \le \frac{l + R_{2_0} + l_0 - x}{c_0}\}.
\end{aligned}
$$

In region $\mathcal{R}_1$, both families of characteristics with slopes $\mp 1/c_0$ (on which w and z must be constant, respectively) can be traced backward in time to the segment $[R_{1_0}, l + R_{2_0}] \times \{t = 0\}$, where initial data is given. So we impose the initial conditions by defining, for $x \in [R_{1_0}, l + R_{2_0}]$, $\tilde{\rho}(x,0) = \tilde{F}(x)$ and $\tilde{u}(x,0) = \tilde{G}(x)$. Then for a point $(x,t) \in \mathcal{R}_1$, to see how we must define $\tilde{\rho}(x,t)$, we express it in terms of w and z and trace back in time along the appropriate characteristics to $t = 0$, as follows.

$$
\begin{aligned}
\tilde{\rho}(x,t) &= \frac{\rho_0}{2c_0}\big(z(x,t) - w(x,t)\big) \\
&= \frac{\rho_0}{2c_0}\big(z(x - c_0 t, 0) - w(x + c_0 t, 0)\big) \\
&= \frac{\rho_0}{2c_0}\big(\tilde{u}(x - c_0 t, 0) + \frac{c_0}{\rho_0}\tilde{\rho}(x - c_0 t, 0) \\
&\quad - \tilde{u}(x + c_0 t, 0) + \frac{c_0}{\rho_0}\tilde{\rho}(x + c_0 t, 0)\big) \\
&= \frac{1}{2}\big(\tilde{F}(x - c_0 t) + \tilde{F}(x + c_0 t)\big) + \frac{\rho_0}{2c_0}\big(\tilde{G}(x - c_0 t) - \tilde{G}(x + c_0 t)\big).
\end{aligned}
$$

Doing the same thing for $\tilde{u}$, we have

$$\begin{aligned}
\tilde{u}(x,t) &= \frac{1}{2}\big(w(x,t)+z(x,t)\big) \\
&= \frac{1}{2}\big(w(x+c_0t,0)+z(x-c_0t,0)\big) \\
&= \frac{1}{2}\big(\tilde{u}(x+c_0t,0)-\frac{c_0}{\rho_0}\tilde{\rho}(x+c_0t,0) \\
&\quad +\tilde{u}(x-c_0t,0)+\frac{c_0}{\rho_0}\tilde{\rho}(x+c_0t,0)\big) \\
&= \frac{c_0}{2\rho_0}\big(\tilde{F}(x-c_0t)-\tilde{F}(x+c_0t)\big)+\frac{1}{2}\big(\tilde{G}(x-c_0t)+\tilde{G}(x+c_0t)\big).
\end{aligned}$$

From a point (x,t) in region $\mathcal{R}_2$, characteristics of slope $-1/c_0$ (on which w is constant) can be traced back to $t=0$ as before, but characteristics of slope $+1/c_0$ (on which z is constant) will intersect the boundary at $(R_{1_0}, t-\frac{x-R_{1_0}}{c_0})$ before reaching the x-axis. This will incorporate the solution at the boundary, which we already have defined above in terms of the function $\tilde{R}_1$ by BCs (1.5) and (1.7), and $\tilde{R}_1$ itself has already been defined in terms of the given functions $\tilde{F}$ and $\tilde{G}$. For notational convenience, let $t_1(x)=\frac{x-R_{1_0}}{c_0}$, which physically represents the time it takes for a signal with speed c_0 to travel from the boundary at R_{1_0} to the point x. Then, for $(x,t)\in\mathcal{R}_2$, we must have

$$\begin{aligned}
\tilde{\rho}(x,t) &= \frac{\rho_0}{2c_0}\big(z(x,t)-w(x,t)\big) \\
&= \frac{\rho_0}{2c_0}\big(z(R_{1_0},t-t_1(x))-w(x+c_0t,0)\big) \\
&= \frac{\rho_0}{2c_0}\Big(\tilde{u}(R_{1_0},t-t_1(x))+\frac{c_0}{\rho_0}\tilde{\rho}(R_{1_0},t-t_1(x)) \\
&\quad -\tilde{u}(x+c_0t,0)+\frac{c_0}{\rho_0}\tilde{\rho}(x+c_0t,0)\Big) \\
&= \frac{\rho_0}{2c_0}\bigg(-\frac{M_1}{\rho_0c_0A}\ddot{\tilde{R}}(t-t_1(x))-\big(\frac{\mu_1}{\rho_0c_0A}-1\big)\dot{\tilde{R}}_1(t-t_1(x)) \\
&\quad -\frac{K_1}{\rho_0c_0A}\tilde{R}_1(t-t_1(x))-\tilde{G}(x+c_0t)+\frac{c_0}{\rho_0}\tilde{F}(x+c_0t)\bigg).
\end{aligned}$$

Similarly, we see that $\tilde{u}$ must satisfy

$$\begin{aligned}
\tilde{u}(x,t) &= \frac{1}{2}\big(w(x,t)+z(x,t)\big) \\
&= \frac{1}{2}\big(w(x+c_0t,0)+z(R_{1_0},t-t_1(x))\big) \\
&= \frac{1}{2}\Big(\tilde{u}(x+c_0t,0)-\frac{c_0}{\rho_0}\tilde{\rho}(x+c_0t,0) \\
&\quad +\tilde{u}(R_{1_0},t-t_1(x))+\frac{c_0}{\rho_0}\tilde{\rho}(R_{1_0},t-t_1(x))\Big) \\
&= \frac{1}{2}\left(-\frac{M_1}{\rho_0c_0A}\ddot{\tilde{R}}(t-t_1(x))-\big(\frac{\mu_1}{\rho_0c_0A}-1\big)\dot{\tilde{R}}_1(t-t_1(x))\right. \\
&\quad \left.-\frac{K_1}{\rho_0c_0A}\tilde{R}_1(t-t_1(x))+\tilde{G}(x+c_0t)-\frac{c_0}{\rho_0}\tilde{F}(x+c_0t)\right).
\end{aligned}$$

From a point (x,t) in region $\mathcal{R}_3$, characteristics of slope $+1/c_0$ (on which z is constant) can be traced back to $t=0$, but characteristics of slope $-1/c_0$ (on which w is constant) will intersect the boundary at $(l+R_{2_0},t-\frac{l+R_{2_0}-x}{c_0})$ before reaching the x-axis. Letting $t_2(x)=\frac{l+R_{2_0}-x}{c_0}$, we find by a computation similar to the one above, that in region $\mathcal{R}_3$, $\tilde{\rho}$ must satisfy

$$\begin{aligned}
\tilde{\rho}(x,t) &= \frac{\rho_0}{2c_0}\left(\frac{M_2}{\rho_0c_0A}\ddot{\tilde{R}}_2(t-t_2(x))+\big(\frac{\mu_2}{\rho_0c_0A}-1\big)\dot{\tilde{R}}_2(t-t_2(x))\right. \\
&\quad \left.+\frac{K_2}{\rho_0c_0A}\tilde{R}_2(t-t_2(x))+\tilde{G}(x+c_0t)+\frac{c_0}{\rho_0}\tilde{F}(x+c_0t)\right),
\end{aligned}$$

and $\tilde{u}$ must satisfy

$$\begin{aligned}
\tilde{u}(x,t) &= \frac{1}{2}\left(-\frac{M_2}{\rho_0c_0A}\ddot{\tilde{R}}_2(t-t_2(x))-\big(\frac{\mu_2}{\rho_0c_0A}-1\big)\dot{\tilde{R}}_2(t-t_2(x))\right. \\
&\quad \left.-\frac{K_2}{\rho_0c_0A}\tilde{R}_2(t-t_2(x))+\tilde{G}(x+c_0t)+\frac{c_0}{\rho_0}\tilde{F}(x+c_0t)\right).
\end{aligned}$$

From a point (x,t) in region $\mathcal{R}_4$, neither characteristic can be traced back to $t=0$ before intersecting the boundary: the characteristic of slope $+1/c_0$ (on which z is constant) will intersect the boundary at $(R_{1_0},t-\frac{x-R_{1_0}}{c_0})$ and the characteristic of slope $-1/c_0$ (on which w is constant) will intersect the boundary at $(l+R_{2_0},t-\frac{l+R_{2_0}-x}{c_0})$.

Therefore treating z as in region $\mathcal{R}_2$ and w as in region $\mathcal{R}_3$, we see that in region $\mathcal{R}_4$, $\tilde{\rho}$ and $\tilde{u}$ must satisfy

$$\begin{aligned}\tilde{\rho}(x,t) \;=\;& \frac{\rho_0}{2c_0}\Bigg(-\frac{M_1}{\rho_0 c_0 A}\ddot{\tilde{R}}_1(t-t_1(x)) - \big(\frac{\mu_1}{\rho_0 c_0 A}-1\big)\dot{\tilde{R}}_1(t-t_1(x)) \\ & -\frac{K_1}{\rho_0 c_0 A}\tilde{R}_1(t-t_1(x)) + \frac{M_2}{\rho_0 c_0 A}\ddot{\tilde{R}}_2(t-t_2(x)) \\ & +\big(\frac{\mu_2}{\rho_0 c_0 A}-1\big)\dot{\tilde{R}}_2(t-t_2(x)) + \frac{K_2}{\rho_0 c_0 A}\tilde{R}_2(t-t_2(x))\Bigg),\end{aligned} \tag{1.30}$$

$$\begin{aligned}\tilde{u}(x,t) \;=\;& -\frac{1}{2}\Bigg(\frac{M_1}{\rho_0 c_0 A}\ddot{\tilde{R}}_1(t-t_1(x)) + \big(\frac{\mu_1}{\rho_0 c_0 A}-1\big)\dot{\tilde{R}}_1(t-t_1(x)) \\ & +\frac{K_1}{\rho_0 c_0 A}\tilde{R}_1(t-t_1(x)) + \frac{M_2}{\rho_0 c_0 A}\ddot{\tilde{R}}_2(t-t_2(x)) \\ & +\big(\frac{\mu_2}{\rho_0 c_0 A}-1\big)\dot{\tilde{R}}_2(t-t_2(x)) + \frac{K_2}{\rho_0 c_0 A}\tilde{R}_2(t-t_2(x))\Bigg),\end{aligned} \tag{1.31}$$

where we recall that the functions $\tilde{R}_1$ and $\tilde{R}_2$ on the interval $[0, l_0/c_0]$ have been defined in terms of $\tilde{F}$ and $\tilde{G}$.

We have shown that it is necessary to define the field functions $\tilde{\rho}$ and $\tilde{u}$ in the regions $\bigcup_{i=1}^{4}\mathcal{R}_i$ as described, in order to both satisfy conditions (1.5)–(1.14) and ensure that the functions w and z, given in terms of $\tilde{\rho}$ and $\tilde{u}$ by (1.24)–(1.25), are constant along the characteristics of slope $\mp 1/c_0$, respectively. The invariance of w and z on their appropriate characteristics is required by any solution of PDEs (1.22)–(1.23), and hence by any corresponding solution of PDEs (1.3)–(1.4). Therefore, if system (1.3)–(1.14) has a solution on the domain $\bigcup_{i=1}^{4}\mathcal{R}_i$, it must be the one we have constructed. On the other hand, our constructed solution candidate does satisfy conditions (1.5)–(1.14), by construction. Also, the corresponding w and z are constant on their appropriate characteristics, again by construction. If w and z corresponding to our constructed $\tilde{\rho}$ and $\tilde{u}$ were C^1 (which would be the case if our constructed $\tilde{\rho}$ and $\tilde{u}$ were C^1), then they would in fact be C^1 solutions of PDEs (1.22)–(1.23), and hence our constructed $\tilde{\rho}$ and $\tilde{u}$ would be C^1 solutions of PDEs (1.3)–(1.4). Thus it remains to check that our constructed $\tilde{\rho}$ and $\tilde{u}$ are C^1. Since $\tilde{F}$ and $\tilde{G}$ are C^1, and $\tilde{R}_1$ and $\tilde{R}_2$ are C^3, it is clear that our constructed $\tilde{\rho}$ and $\tilde{u}$ are C^1 in each region $\mathcal{R}_i$ separately. Furthermore, it can be checked that our constructed $\tilde{\rho}$ and $\tilde{u}$ are C^1 on the boundaries of each region by using the compatibility conditions (1.11)–(1.13). Thus our constructed $\tilde{\rho}$ and $\tilde{u}$ is the unique C^1 solution of the linearized system (1.3)–(1.14) in the domain $\cup_{i=1}^4 \mathcal{R}_i$. In particular, the solution is defined on $[R_{1_0}, l + R_{2_0}] \times [0, l_0/c_0]$. We can now consider the fields, say, at time $t = l_0/2c_0$ to be the new initial data functions and repeat the entire process from that time forward, thus yielding a unique C^1 solution on the time interval $[l_0/2c_0, 3l_0/2c_0]$. Repeating this process, and patching these unique solutions together, by induction we have a unique global C^1 solution of the linearized system (1.3)–(1.14). We state this result in the following theorem.

Theorem 2. *Given $\tilde{F}, \tilde{G} \in C^1[R_{1_0}, l + R_{2_0}]$, the linearized system* (1.3)–(1.14) *has a unique global C^1 solution.*

Remark 3. Note that for $r \in \mathbb{N}$, if the initial fields $\tilde{F}$ and $\tilde{G}$ are in C^r , then linearized system (1.3)–(1.14) has a unique global C^r solution, by a similar proof.

Remark 4. If the initial fields $\tilde{F}$ and $\tilde{G}$ merely C^0, then the solution constructed in the previous proof satisfies a weak formulation of the linearized system (1.3)–(1.11) and (1.14), where (1.3)–(1.4) are replaced with the conditions that the Riemann invariants must remain constant on the appropriate characteristics, and the compatibility conditions (1.12)–(1.13) have been dropped. We call such a solution a *weak solution.*

1.6 Globally attracting steady states

We are now finally prepared to show that, as $t \to +\infty$, solutions of the linearized system (1.3)–(1.14) approach a steady state. As discussed in section 1.2, such a steady steady state must be the trivial solution

$$\tilde{\rho} = 0, \quad \tilde{u} = 0, \quad \tilde{R}_1 = 0, \quad \tilde{R}_2 = 0.$$

We introduce some more precise notation for clarification. Denote the state space of the linearized system (1.3)–(1.14) by

$$\mathcal{B} = C^1([R_{1_0}, l + R_{2_0}]) \times C^1([R_{1_0}, l + R_{2_0}]),$$

where the state at time t is given by the pair of field functions $(\tilde{\rho}(\cdot,t),\ \tilde{u}(\cdot,t)) \in \mathcal{B}$. Note that the piston displacements and velocities, $\tilde{R}_1(t)$, $\tilde{R}_2(t)$, $\dot{\tilde{R}}_1(t)$, and $\dot{\tilde{R}}_2(t)$, are determined by the values of $\tilde{\rho}(\cdot,t)$ and $\tilde{u}(\cdot,t)$ and their spatial derivatives at the boundaries, through the compatibility conditions (1.11)–(1.13) evolved to time t, i.e., the conditions

$$\dot{\tilde{R}}_1(t) = \tilde{u}(R_{1_0}, t)), \quad \dot{\tilde{R}}_2(t) = \tilde{u}(l + R_{2_0}, t). \tag{1.32}$$

$$-M_1 \frac{c_0^2}{\rho_0} \tilde{\rho}_x(R_{1_0}, t) = -K_1 \tilde{R}_1(t) - \mu_1 \tilde{u}(R_{1_0}, t) - c_0^2 \tilde{\rho}(R_{1_0}, t) A, \tag{1.33}$$

$$-M_2 \frac{c_0^2}{\rho_0} \tilde{\rho}_x(l + R_{2_0}, t) = -K_2 \tilde{R}_2(t) - \mu_2 \tilde{u}(l + R_{2_0}, t) + c_0^2 \tilde{\rho}(l + R_{2_0}, t) A. \tag{1.34}$$

Suppose, for the moment, that $\tilde{\rho}$ and $\tilde{u}$ were C^2. Then we could differentiate the PDEs (1.3)–(1.4) in x and differentiate the BCs (1.5)–(1.8) in t, and find that if we define the new energy for the spatial derivatives,

$$\tilde{E}_1 = \frac{1}{2} M_1 \ddot{\tilde{R}}_1^2 + \frac{1}{2} M_2 \ddot{\tilde{R}}_2^2 + \frac{1}{2} K_1 \dot{\tilde{R}}_1^2 + \frac{1}{2} K_2 \dot{\tilde{R}}_2^2 + A \int_{R_{1_0}}^{l+R_{2_0}} (\frac{1}{2} \rho_0 \tilde{u}_x^2 + \frac{c_0^2}{2\rho_0} \tilde{\rho}_x^2) \, dx, \quad (1.35)$$

then the new PDEs and BCs may be used to show

$$\frac{d\tilde{E}_1}{dt} = -\mu_1 \ddot{\tilde{R}}_1^2 - \mu_2 \ddot{\tilde{R}}_2^2. \quad (1.36)$$

Thus $\tilde{E}_1$ is bounded since it is positive and decreasing. This implies the $\ddot{\tilde{R}}_i$ are bounded as well. In fact, (1.35) is well defined even if $\tilde{\rho}$ and $\tilde{u}$ are merely C^1. Since in this case, the $\tilde{R}_i$ will be C^3, our only concern is whether we can differentiate the integral in (1.35) with respect to t. Using the standard procedure of approximating the functions $\tilde{\rho}$ and $\tilde{u}$ by their mollifications (C^∞ functions which satisfy the same linear PDEs as they do), we still obtain (1.36), with $\tilde{\rho}$ and $\tilde{u}$ merely assumed to be C^1.

Note that $\tilde{E}$ and $\tilde{E}_1$ provide bounds on both the L^2 norms of $\tilde{\rho}$ and $\tilde{u}$, and their derivatives $\tilde{\rho}_x$ and $\tilde{u}_x$, so that as they evolve, the solutions $\tilde{\rho}$ and $\tilde{u}$ remain bounded in the Sobolev space H^1. Thus by a standard Sobolev embedding theorem, we have that $\tilde{\rho}$ and $\tilde{u}$ remain bounded in the C^0 norm as well. This can also be seen from equations (1.30)–(1.31), since (1.16)–(1.17) and (1.35)–(1.36) imply that $\tilde{R}_i$, $\dot{\tilde{R}}_i$, and $\ddot{\tilde{R}}_i$, for $i = 1, 2$, are all bounded functions.

By (1.16)–(1.17), we have that $\tilde{E}$ is decreasing and bounded below, hence it converges to a constant value as $t \to +\infty$. It is tempting to then conclude that $\frac{d\tilde{E}}{dt} \to 0$, from which (1.17) would imply $\dot{\tilde{R}}_i \to 0$, for $i = 1, 2$. However, the fact that $\tilde{E}$ is decreasing to a constant value is not enough to conclude that $\frac{d\tilde{E}}{dt} \to 0$. It would be true, however, if $\frac{d^2\tilde{E}}{dt^2}$ were bounded. In fact, we can see that $\frac{d^2\tilde{E}}{dt^2}$ does remain bounded, by differentiating (1.17) and recalling that $\tilde{R}_i$ and $\dot{\tilde{R}}_i$, for $i = 1, 2$, are bounded. Thus we do indeed have $\frac{d\tilde{E}}{dt} \to 0$, and hence $\dot{\tilde{R}}_i \to 0$ for $i = 1, 2$, as $t \to +\infty$. Even so, this does not imply $\tilde{R}_i$ must converge to a steady state (for example, consider the function $\sin\sqrt{t}$). To show that requires more sophisticated analysis.

Let $S(t) : \mathcal{B} \to \mathcal{B}$ be the evolution operator for the linearized system (1.3)–(1.14), which satisfies the usual group properties: for $\xi \in \mathcal{B}$,

$$S(0)\xi = \xi,$$

$$S(t)S(s)\xi = S(t+s)\xi, \quad t, s \in \mathbb{R},$$

$$S(t)\xi \text{ is jointly continuous in } t \text{ and } \xi.$$

Note that the energies $\tilde{E}$ and $\tilde{E}_1$ as defined by (1.16) and (1.35) provide alternative norms on the space $\mathcal{B}$. From these we may define yet another norm: $\tilde{E}_2 := \tilde{E} + \tilde{E}_1$. By the Rellich-Kondrachov compactness theorem, $H^1([R_{1_0}, l + R_{2_0}])$ is compactly embedded in $L^2([R_{1_0}, l + R_{2_0}])$, from which it can be seen that $\mathcal{B}$ equipped with the $\tilde{E}_2$ norm is compactly embedded in $\mathcal{B}$ equipped with the $\tilde{E}$ norm (Rellich-Kondrachov takes care of the integral terms, for the rest of the terms, going from $\tilde{E}_2$ to $\tilde{E}$ just involves projection from a finite dimensional space to one of lower dimension). Thus, a sequence in $\mathcal{B}$ that is bounded in the $\tilde{E}_2$ norm will have a subsequence that converges in the $\tilde{E}$ norm. Given $\xi \in \mathcal{B}$, for any $t \geq 0$, we have that $||S(t)\xi||_{\tilde{E}_2} \leq ||\xi||_{\tilde{E}_2}$ (since $\tilde{E}_2$ is decreasing along orbits by (1.17) and (1.36)). The aforementioned compact embedding then implies that, for each $s \geq 0$, the set $\overline{\{S(t)\xi : t \geq s\}}$ is compact in the

$\tilde{E}$ norm. Thus we have that the omega limit set of ξ, defined by

$$\omega(\xi) = \bigcap_{s>0} \overline{\{S(t)\xi : t \geq s\}}, \tag{1.37}$$

is nonempty and compact in the $\tilde{E}$ norm. We would like to show that $\omega(\xi)$ consists of constant fields by applying the evolution operator $S(t)$ to an arbitrary element of $\omega(\xi)$. Unfortunately, since the closure in (1.37) is only in the $\tilde{E}$ norm, $\omega(\xi)$ may not be in the domain of $S(t)$.

Now suppose, for the moment, that ξ happens to consist of fields $\tilde{\rho}$ and $\tilde{u}$ that are C^2, i.e. let $\xi \in \mathcal{B}_2$, where

$$\mathcal{B}_2 = C^2[R_{1_0}, l + R_{2_0}] \times C^2[R_{1_0}, l + R_{2_0}].$$

Then by remark 3 following theorem 2, $S(t)\xi$ will be in $\mathcal{B}_2$ as well. Repeating the same argument that yielded (1.35)–(1.36) for initial data in $\mathcal{B}_1$ (including approximation by mollifications), we now find that the new energy defined by

$$\tilde{E}_3 = \frac{1}{2}M_1\dddot{\tilde{R}}_1^2 + \frac{1}{2}M_2\dddot{\tilde{R}}_2^2 + \frac{1}{2}K_1\ddot{\tilde{R}}_1^2 + \frac{1}{2}K_2\ddot{\tilde{R}}_2^2 + A\int_{R_{1_0}}^{l+R_{2_0}} (\frac{1}{2}\rho_0\tilde{u}_{xx}^2 + \frac{c_0^2}{2\rho_0}\tilde{\rho}_{xx}^2)\, dx, \tag{1.38}$$

satisfies

$$\frac{d\tilde{E}_3}{dt} = -\mu_1\dddot{\tilde{R}}_1^2 - \mu_2\dddot{\tilde{R}}_2^2, \tag{1.39}$$

for the solution $S(t)\xi$ in $\mathcal{B}_2$.

Let $\tilde{E}_4 := \tilde{E}_2 + \tilde{E}_3$. We again use the Rellich-Kondrachov compactness theorem, which implies $H^2[R_{1_0}, l + R_{2_0}]$ is compactly embedded in $H^1[R_{1_0}, l + R_{2_0}]$. Therefore $\mathcal{B}_2$ equipped with the $\tilde{E}_4$ norm is compactly embedded in $\mathcal{B}_2$ equipped with the $\tilde{E}_2$ norm.

Again, the aforementioned compact embedding then implies that, for each $s \geq 0$, the set $\overline{\{S(t)\xi : t \geq s\}}$ (closure in $\tilde{E}_2$ norm) is compact in the $\tilde{E}_2$ norm. Thus we have that the omega limit set of ξ, in the $\tilde{E}_2$ norm, defined by

$$\omega_2(\xi) = \bigcap_{s>0} \overline{\{S(t)\xi : t \geq s\}}, \quad \text{(closure in } \tilde{E}_2 \text{ norm)}, \tag{1.40}$$

is nonempty and compact in the $\tilde{E}_2$ norm. Due to the closure in $\tilde{E}_2$ norm, $\omega_2(\xi)$ may no longer be contained in $\mathcal{B}_2$, but rather is contained in the larger space

$$\mathcal{H} := \{\zeta \in H^1([R_{1_0}, l + R_{2_0}]) \times H^1([R_{1_0}, l + R_{2_0}]) : ||\zeta||_{\tilde{E}_2} < +\infty\}.$$

We claim that the norm $\tilde{E}_2$ is constant on the set $\omega_2(\xi)$. An alternate expression for the set $\omega_2(\xi)$ is

$$\omega_2(\xi) = \{\zeta \in \mathcal{H} : \exists \{t_k\}_{k=1}^{\infty} \subset \mathbb{R}, \text{ such that } t_k \to +\infty, S(t_k)\xi \to \zeta \text{ as } k \to +\infty\},$$

where the convergence $S(t_k)\xi \to \zeta$ is with respect to the $\tilde{E}_2$ norm.

Thus for any $\zeta \in \omega(\xi)$, there is an unbounded, increasing sequence $\{t_k\}_{k=1}^{\infty} \subset \mathbb{R}$ such that

$$\begin{aligned} ||\zeta||_{\tilde{E}_2} &= ||\lim_{k\to+\infty} S(t_k)\xi||_{\tilde{E}_2} \\ &= \lim_{k\to+\infty} ||S(t_k)\xi||_{\tilde{E}_2} \\ &= \lim_{t\to+\infty} ||S(t)\xi||_{\tilde{E}_2}, \end{aligned}$$

where we have used the fact that the map $\xi \mapsto ||S(t)\xi||_{\tilde{E}_2}$ is continuous. (Since $\tilde{E}_2$ is decreasing along orbits, we have $||S(t)\xi_1 - S(t)\xi_2||_{\tilde{E}_2} \leq ||\xi_1 - \xi_2||_{\tilde{E}_2}$ for all $\xi_1, \xi_2 \in \mathcal{B}_2$). Furthermore, the limit in the last equality above exists (and hence is unique) because the map $t \mapsto ||S(t)\xi||_{\tilde{E}_2}$ is decreasing, continuous, and bounded below. Thus the norm $\tilde{E}_2$ is indeed constant on $\omega(\xi)$.

Next we will show that every point in $\omega_2(\xi)$ is a steady state. Each element of $\omega_2(\xi)$ resides in $\mathcal{H}$, and therefore each element is an equivalence class of functions which has a unique continuous representative. This continuous representative may be used as initial data for the weak formulation of the linearized system (1.3)–(1.14), as described in remark 4. In this way, the (weak) evolution operator $S(t)$ is well defined on $\mathcal{H}$, and in particular, $\omega_2(\xi)$ is invariant under $S(t)$ by (1.40).

Let $\zeta \in \omega_2(\xi)$ be represented by

$$\zeta = (\tilde{\rho}(x,0), \tilde{u}(x,0), \tilde{R}_1(0), \tilde{R}_2(0), \dot{\tilde{R}}_1(0), \dot{\tilde{R}}_2(0)),$$

which can be considered as initial data for the (weak) solution

$$S(t)\zeta = (\tilde{\rho}(x,t), \tilde{u}(x,t), \tilde{R}_1(t), \tilde{R}_2(t), \dot{\tilde{R}}_1(t), \dot{\tilde{R}}_2(t)).$$

We will show that, in fact, $S(t)\zeta$ is independent of t, i.e. that $S(t)\zeta = \zeta$. The invariance of $\omega_2(\xi)$ implies that $||S(t)\zeta||_{\tilde{E}_2}$ remains constant for all $t \geq 0$, from which we conclude that $\dot{\tilde{R}}_1 = 0$ and $\dot{\tilde{R}}_2 = 0$ by (1.17). Thus $\tilde{R}_1$ and $\tilde{R}_2$ are constant, say, $\tilde{R}_1 = r_1$ and $\tilde{R}_2 = r_2$. For any point (x,t) for which both families of characteristics, lines of slopes $\pm 1/c_0$, can be traced back in time to the boundaries, which will always be the case for $t \geq l_0/c_0$, the fields $\tilde{\rho}(x,t)$ and $\tilde{u}(x,t)$ can be expressed in terms of the $\tilde{R}_i$ and their derivatives at retarded times (the retardation depending on x) by (1.30)–(1.31). Thus we find that, at least for $t \geq l_0/c_0$, the fields must be constant in x and t, with the values

$$\tilde{\rho} = \frac{1}{2c_0^2 A}(K_2 r_2 - K_1 r_1), \quad \tilde{u} = -\frac{1}{\rho_0 c_0 A}(K_1 r_1 + K_2 r_2).$$

Since we already have the $\dot{\tilde{R}}_i = 0$, the BCs (1.5)–(1.6) imply that the only constant

$\tilde{u}$ could be is zero. We thus obtain, for $t \geq l_0/c_0$,

$$\tilde{\rho} = -\frac{1}{c_0^2 A} K_1 r_1 = \frac{1}{c_0^2 A} K_2 r_2, \quad \tilde{u} = 0. \tag{1.41}$$

But this implies the fields had to have these same constant values for all $t \geq 0$; in fact, for all $t \in \mathbb{R}$, since the proof of theorem 2 is equally valid working backward in time, giving global backward existence and uniqueness as well (of course, going backward in time the energy norms are increasing rather than decreasing, but these were not used in proving existence and uniqueness). Thus ζ is in fact a constant steady state. Therefore, $\omega_2(\xi)$ consists only of constant steady states of the form (1.41).

Actually, we can conclude more. Different constant values of the steady states given by (1.41) would have different $\tilde{E}_2$ norms. But we have shown every point in $\omega_2(\xi)$ has the same $\tilde{E}_2$ norm, namely $\lim_{t\to+\infty} ||S(t)\xi||_{\tilde{E}_2}$. Therefore $\omega_2(\xi)$ consists of only one point, a constant steady state. And we have that $S(t)\xi$ converges in the $\tilde{E}_2$ norm to that steady state as $t \to +\infty$. This implies that the fields $\tilde{\rho}$ and $\tilde{u}$ converge to some constant and zero, respectively, in the H^1 norm, $\dot{\tilde{R}}_1$ and $\dot{\tilde{R}}_2$ converge to zero (which we already knew), and $\tilde{R}_1$ and $\tilde{R}_2$ converge to some constants r_1 and r_2, related to $\tilde{\rho}$ by (1.41).

What we haven't quite shown yet is that this constant steady state must be the trivial solution, as in section 1.2, where the fields were assumed to converge pointwise to steady states. However, in that derivation, the only place the assumption that $\lim_{t\to+\infty} \tilde{u}(x,t) = 0$ was used, was to show that $\lim_{t\to+\infty} \dot{\tilde{R}}_i(t) = 0$. But that is already known from other arguments. Furthermore, the assumption that $\lim_{t\to+\infty} \tilde{\rho}(x,t) = \tilde{\rho}_0$ was only used when letting $t \to +\infty$ in (1.19) to conclude (1.20). In fact, even if $\tilde{\rho} \to \tilde{\rho}_0$ merely in L^1 as $t \to +\infty$, we could still conclude that (1.20) holds. Therefore, knowing that $S(t)\xi$ converges in the $\tilde{E}_2$ norm to the constant steady state in $\omega_2(\xi)$ as $t \to +\infty$, and thus the corresponding field solution $\tilde{\rho}$ converges in H^1, and hence L^1 as well, to its constant value, we can run through the derivation of (1.21) again and conclude that the constant value of the steady state in $\omega_2(\xi)$ is indeed the trivial solution.

Now, we remove the assumption that ξ is C^2. Let $\xi \in \mathcal{B}$ (whose fields are merely C^1). Let $\{\xi_n\} \subset \mathcal{B}_2$ be a sequence (with C^2 fields), such that $\xi_n \to \xi$ in $\tilde{E}_2$ norm as $n \to +\infty$. Clearly such a sequence exists. As argued above, $\omega_2(\xi_n)$ will consist of a single point, the equivalence class of functions which has the trivial solution as its continuous representative. We denote this point by ζ_n (which is actually independent of n), for which $S(t)\xi_n \to \zeta_n$ in the $\tilde{E}_2$ norm as $t \to +\infty$. By the continuity of the operator $S(t)$, we have that $\lim_{n\to+\infty} S(t)\xi_n = S(t)\xi$. We would like to argue that the following limit (in $\tilde{E}_2$ norm) exists:

$$\begin{aligned}
\lim_{t\to+\infty} S(t)\xi &= \lim_{t\to+\infty}\left(\lim_{n\to+\infty} S(t)\xi_n\right) \\
&= \lim_{n\to+\infty}\left(\lim_{t\to+\infty} S(t)\xi_n\right) \\
&= \lim_{n\to+\infty} \zeta_n.
\end{aligned}$$

The above limit would exist, and the interchange of the limits in t and n would be justified, if we knew that $S(t)\xi_n \to S(t)\xi$ in the $\tilde{E}_2$ norm, as $n \to +\infty$, uniformly in t. But this follows from the fact that $S(t)$ is linear and $\tilde{E}_2$ decreases along orbits, which implies

$$\begin{aligned}
||S(t)\xi_n - S(t)\xi||_{\tilde{E}_2} &= ||S(t)(\xi_n - \xi)||_{\tilde{E}_2} \\
&\leq ||\xi_n - \xi||_{\tilde{E}_2}.
\end{aligned}$$

Thus, as $t \to +\infty$, $S(t)\xi$ converges in the $\tilde{E}_2$ norm to a point in $\omega_2(\xi)$, which therefore must consist of only one point—the equivalence class of functions which has the trivial solution as its continuous representative. In particular, as $t \to +\infty$, the piston displacements and velocities, $\tilde{R}_1(t)$, $\tilde{R}_2(t)$, $\dot{\tilde{R}}_1(t)$, and $\dot{\tilde{R}}_2(t)$, converge to zero, and the fields $\tilde{\rho}$ and $\tilde{u}$ converge to zero in H^1 norm. We summarize this result in the following theorem.

Theorem 5. *Let $\tilde{F}, \tilde{G} \in C^1[R_{1_0}, l + R_{2_0}]$. Then the unique global C^1 solution of the linearized system* (1.3)–(1.14) *with positive spring damping parameters μ_1 and μ_2, $(\tilde{\rho}(\cdot,t), \tilde{u}(\cdot,t), \tilde{R}_1(t), \tilde{R}_2(t), \dot{\tilde{R}}_1(t), \dot{\tilde{R}}_2(t))$, converges to the trivial solution* $(0,0,0,0,0,0)$ *as $t \to +\infty$, where the convergence of $\tilde{\rho}$ and $\tilde{u}$ to zero is in the H^1 norm.*

The trivial steady state, which we have just shown all solutions must approach, clearly forms a finite dimensional, invariant manifold within the infinite dimensional state space $\mathcal{B}$. Of course, the dynamics on that manifold are not very interesting. The next task that naturally arises is to explore how this steady state is approached. Is there a larger, yet still finite dimensional, invariant manifold which attracts all solutions and captures the effective dynamics of their asymptotic behavior? We explore this idea in the equivalent ODE/NFDE system (1.26)–(1.29) with conditions (1.11)–(1.13), although those conditions will be dropped along with the initial fields altogether as we focus our attention on just the NFDEs.

1.7 Identical pistons in dimensionless form

We now return to the ODE/NFDE system (1.26)–(1.29) from section 1.4, without the compatibility conditions (1.11)–(1.13). For simplicity, we consider the case of identical pistons; that is,

$$M = M_1 = M_2, \quad K = K_1 = K_2, \quad \mu = \mu_1 = \mu_2.$$

Introducing the dimensionless parameters and variables

$$\omega = \sqrt{\frac{K}{M}}, \quad \tau = \omega t, \quad \mathcal{R}_i(\tau) = \frac{1}{l_0}\tilde{R}_i(t), \quad \alpha = \frac{\rho_0 c A}{M\omega}, \quad \beta = \frac{\mu}{M\omega},$$

$$\gamma = \frac{\omega l_0}{c_0}, \quad \mathcal{F}(\tau) = \frac{c_0^2 A}{K l_0}\tilde{F}(c_0 t), \quad \mathcal{G}(\tau) = \frac{\rho_0 c_0 A}{K l_0}\tilde{G}(c_0 t), \quad \delta = \frac{\omega R_{1_0}}{c_0}$$

and decoupling the system by introducing the new state variables $X = \mathcal{R}_1 + \mathcal{R}_2$ and $Y = \mathcal{R}_1 - \mathcal{R}_2$, we obtain

$$\begin{aligned} X''(\tau) &= -(\alpha+\beta)X'(\tau) - X(\tau) - \mathcal{F}(\delta+\tau) + \mathcal{F}(\delta+\gamma-\tau) \\ &\quad + \mathcal{G}(\delta+\tau) + \mathcal{G}(\delta+\gamma-\tau), \quad \tau < \gamma \qquad (1.42) \\ X''(\tau) + X''(\tau-\gamma) &= -(\alpha+\beta)X'(\tau) - X(\tau) \\ &\quad + (\alpha-\beta)X'(\tau-\gamma) - X(\tau-\gamma), \quad \tau > \gamma \qquad (1.43) \end{aligned}$$

and

$$\begin{aligned} Y''(\tau) &= -(\alpha+\beta)Y'(\tau) - Y(\tau) - \mathcal{F}(\delta+\tau) - \mathcal{F}(\delta+\gamma-\tau) \\ &\quad + \mathcal{G}(\delta+\tau) - \mathcal{G}(\delta+\gamma-\tau), \quad \tau < \gamma \qquad (1.44) \\ Y''(\tau) - Y''(\tau-\gamma) &= -(\alpha+\beta)Y'(\tau) - Y(\tau) \\ &\quad - (\alpha-\beta)Y'(\tau-\gamma) + Y(\tau-\gamma), \quad \tau > \gamma \qquad (1.45) \end{aligned}$$

which may be considered as two separate Newtonian systems. Indeed, an initial position and velocity $(X(0), X'(0))$ (respectively, $(Y(0), Y'(0))$) given for the ODE (1.42) (respectively, (1.44)) produces a solution on the interval $[0, \gamma]$ that provides the initial function required for a unique solution of the NFDE (1.43) (respectively, (1.45)). This viewpoint assumes that specific initial fields $\mathcal{F}$ and $\mathcal{G}$ are given. On the other hand, we may consider the NFDEs (1.43) and (1.45) more abstractly with initial data from the space of all solutions of the ODEs (1.42) and (1.44) given all possible C^1 initial fields $\mathcal{F}$ and $\mathcal{G}$. This viewpoint is perhaps more aligned with the electrodynamic analogue. There, the corresponding FDEs do not usually come together with an ODE whose solution provides the required initial function. Instead, the initial electromagnetic fields are most often prescribed in the infinite past with a stipulation of fast enough spatial decay. In a similar manner, if we specify our initial acoustic fields at $t = -\infty$ instead of $t = 0$, then we will have only the NFDEs (1.43) and (1.45). In this case there are no ODEs to determine initial functions, and the initial fields don't appear (nor do the compatibility conditions (1.11)–(1.13)). These NFDEs are non-Newtonian: initial trajectory functions are required to determine their solutions just as in electrodynamics (for example, (2)). The question then becomes: Can we approximate the non-Newtonian infinite-dimensional NFDEs with finite-dimensional Newtonian ODEs? In other words, are there effective equations of motion?

1.8 Expansion in the small delay and runaway solutions

We examine the NFDE (1.43)

$$X''(\tau) + X''(\tau - \gamma) = -(\alpha + \beta)X'(\tau) - X(\tau) + (\alpha - \beta)X'(\tau - \gamma) - X(\tau - \gamma),$$

under the assumption that the delay γ is small. Expanding to zeroth order in γ gives

$$X''(\tau) = -\beta X'(\tau) - X(\tau), \tag{1.46}$$

a damped harmonic oscillator where β is the dimensionless spring damping coefficient. The characteristic polynomial,

$$\lambda^2 + \beta\lambda + 1, \tag{1.47}$$

of this linear system has, for $\beta > 0$, two imaginary roots with negative real parts and, for $\beta = 0$, two purely imaginary roots; hence, there are no runaway solutions. In case the NFDE (1.43) has a two-dimensional invariant submanifold on which the effective dynamics takes place, we claim that (1.46) gives the zeroth-order approximation of the dynamics restricted to this manifold.

Expanding (1.43) to first-order in γ gives the third-order ODE

$$\gamma X'''(\tau) = [(\alpha - \beta)\gamma + 2]X''(\tau) + (2\beta - \gamma)X'(\tau) + 2X(\tau), \tag{1.48}$$

whose characteristic polynomial is

$$\gamma\lambda^3 - [\gamma(\alpha - \beta) + 2]\lambda^2 - (2\beta - \gamma)\lambda - 2.$$

It can be shown that two of the roots of this polynomial have non-positive real parts (which continue from the two roots of polynomial (1.47)); but, the new third root (coming from $\lambda = \infty$ in the perturbation) has positive real part. This third root thus corresponds to a one-dimensional family of runaway solutions– an artifact of expansion and truncation, similar to the runaway solutions of the Lorentz-Dirac-Abraham equation (1). The removal of these unphysical solutions is the goal of the next section.

1.9 Singular perturbation theory

(1.48) is equivalent to the first-order ODE system

$$\begin{aligned} X'(\tau) &= V(\tau) \\ V'(\tau) &= Q(\tau) \\ \gamma Q'(\tau) &= 2X(\tau) + (2\beta - \gamma)V(\tau) + [\gamma(\alpha - \beta) + 2]Q(\tau). \end{aligned} \tag{1.49}$$

This system has a three-dimensional state space and a one-dimensional subspace of unphysical solutions, which we will eliminate by using geometric singular perturbation theory. We will restrict the dynamics to a two-dimensional (slow) submanifold that we claim approximates the effective dynamics of NFDE (1.43).

Introducing a fast time s by the singular change of variables $\tau = \gamma s$, and the new functions

$$\tilde{X}(s) = X(\tau), \quad \tilde{V}(s) = V(\tau), \quad \tilde{Q}(s) = Q(\tau), \tag{1.50}$$

the ODE system (1.49) becomes

$$\begin{aligned} \tilde{X}'(s) &= \gamma \tilde{V}(s) \\ \tilde{V}'(s) &= \gamma \tilde{Q}(s) \\ \tilde{Q}'(s) &= 2\tilde{X}(s) + (2\beta - \gamma)\tilde{V}(s) + [2 + \gamma(\alpha - \beta)]\tilde{Q}(s). \end{aligned} \tag{1.51}$$

The unperturbed system ($\gamma = 0$) has a two-dimensional manifold of rest points given by $\{(\tilde{X}, \tilde{V}, \tilde{Q}) : \tilde{X} + \beta\tilde{V} + \tilde{Q} = 0\}$. This manifold is normally hyperbolic. Indeed, the characteristic polynomial for system (1.51) with $\gamma = 0$ is $\lambda^2(\lambda - 2)$. The eigenspace corresponding to its two zero roots is tangent to the invariant manifold of rest points; the (one-dimensional) eigenspace corresponding to the nonzero eigenvalue is transverse to the invariant manifold. Since the nonzero eigenvalue has positive real part, solutions not on the manifold move away from it exponentially fast. According to a theorem of Fenichel (see [2, 3, 4]), this two-dimensional normally hyperbolic invariant manifold persists when γ is perturbed from zero and the perturbed invariant manifold is the graph of a function

$$\tilde{Q} = h(\tilde{X}, \tilde{V}) = -\tilde{X} - \beta\tilde{V} + \gamma h_1(\tilde{X}, \tilde{V}) + O(\gamma^2), \tag{1.52}$$

where the perturbed function h_1 may be determined by equating the tangent vector $(\tilde{X}', \tilde{V}', \tilde{Q}')$ as given by the right hand side of system (1.51) with a linear combination of the basis vectors $(1, 0, h_{\tilde{X}})$ and $(0, 1, h_{\tilde{V}})$. By carrying out this procedure, we find that

$$h_1(\tilde{X}, \tilde{V}) = \frac{\alpha}{2}\tilde{X} + \frac{\alpha\beta}{2}\tilde{V}. \tag{1.53}$$

The fast system (1.51), restricted to the perturbed manifold given by (1.52), is

$$\begin{aligned} \tilde{X}'(s) &= \gamma V(s) \\ \tilde{V}'(s) &= \gamma[(\frac{\gamma\alpha}{2} - 1)\tilde{X}(s) + \beta(\frac{\gamma\alpha}{2} - 1)\tilde{V}(s)]. \end{aligned} \tag{1.54}$$

It follows that the slow system (1.49), restricted to the corresponding slow manifold, is

$$\begin{aligned} X'(\tau) &= V(\tau) \\ V'(\tau) &= (\frac{\gamma\alpha}{2} - 1)X(\tau) + \beta(\frac{\gamma\alpha}{2} - 1)V(\tau), \end{aligned} \tag{1.55}$$

which may be rewritten as the single second-order ODE

$$X''(\tau) = (\frac{\gamma\alpha}{2} - 1)X(\tau) + \beta(\frac{\gamma\alpha}{2} - 1)X'(\tau). \tag{1.56}$$

This equation, which we will show in the next section agrees to first-order with the effective dynamics, has no runaway solutions; in fact all of its solutions decay to zero exponentially fast.

The NFDE (1.45) contains a degeneracy that complicates its analysis. We will outline the main features of its reduction to an effective equation of motion.

Expanding to zeroth-order in γ, both second derivative terms and the terms without derivatives cancel. This procedure leads to the one-dimensional equation

$$Y'(\tau) = 0 \tag{1.57}$$

with no runaways—the solutions are all constant.

After expansion to first-order in γ, the NFDE (1.45) becomes the third-order ODE

$$\gamma Y'''(\tau) = \gamma(\alpha - \beta)Y''(\tau) - (2\alpha + \gamma)Y'(\tau), \tag{1.58}$$

whose characteristic polynomial has one zero root (persisting in the perturbation from $\gamma = 0$) and two roots with positive real parts (which come from ∞ in the perturbation). Thus (1.58) has a two-dimensional subspace of runaway solutions, which we may try to eliminate as before. (1.58) is equivalent to the first-order ODE system

$$\begin{aligned} Y'(\tau) &= W(\tau) \\ \sqrt{\gamma}\, W'(\tau) &= Z(\tau) \\ \sqrt{\gamma}\, Z'(\tau) &= -(2\alpha + \gamma)W(\tau) + \sqrt{\gamma}\,(\alpha - \beta)Z(\tau). \end{aligned} \tag{1.59}$$

Introducing a new fast time s by the change of variables $\tau = \sqrt{\gamma}\, s$ and the new functions

$$\tilde{Y}(s) = Y(\tau), \quad \tilde{W}(s) = W(\tau), \quad \tilde{Z}(s) = Z(\tau), \tag{1.60}$$

we transform the the slow system (1.59) to the fast system

$$\begin{aligned}
\tilde{Y}'(s) &= \sqrt{\gamma}\,\tilde{W}(s) \\
\tilde{W}'(s) &= \tilde{Z}(s) \\
\tilde{Z}'(s) &= -(2\alpha+\gamma)\tilde{V}(s) + \sqrt{\gamma}\,(\alpha-\beta)\tilde{Z}(s).
\end{aligned} \tag{1.61}$$

Setting $\gamma = 0$ yields a system with a one-dimensional invariant manifold of rest points, given by $\{(\tilde{Y}, \tilde{W}, \tilde{Z}) : \tilde{W} = \tilde{Z} = 0\}$. This manifold is not normally hyperbolic because the nonzero roots of the characteristic polynomial $\lambda(\lambda^2 - 2\alpha)$ of the unperturbed system ($\gamma = 0$) are pure imaginary. For $\gamma > 0$, the characteristic polynomial becomes $\lambda[\lambda^2 - \sqrt{\gamma}\,(\alpha-\beta)\lambda + 2\alpha + \gamma]$. Its nonzero roots have positive real part for $\beta < \alpha$ (small spring damping). In this case, nonconstant solutions must spiral out exponentially fast. Proceeding formally to find the perturbed slow manifold as before, we find that it remains unchanged (to first-order in γ). We conjecture that the NFDE (1.45) has a one-dimensional invariant manifold attracting all solutions and equation (1.57) is (to first order) the corresponding effective equation of motion. Unfortunately, the existence of the desired manifold does not follow easily from known methods.

In view of the degeneracy of the Y-system, we will restrict our attention to the nondegenerate X-system. As a physical motivation, note that we may view our model as a single extended body problem rather than a two-body problem. Thus, we may wish to analyze the motion of the body as a whole, which X measures, and not concern ourselves with the body's internal relative vibrations, which Y measures.

Note that if we had chosen to adopt the condition (1.15), then equation (1.57) would be an immediate consequence. In that case, (1.57) would not just be an effective equation of motion, but rather the exact equation of motion (for Y in the linearized system).

1.10 Quasi-inertial manifolds

It has been shown (see [10]) that RFDEs of the form

$$\dot{x}(t) = f(x(t), x(t-\gamma)), \tag{1.62}$$

where γ is a fixed delay and x is an n-dimensional vector, has an n-dimensional inertial manifold. Similarly, we conjecture that the NFDE (1.43) extended to all τ has a two-dimensional invariant manifold, determined by a flow η for sufficiently small γ, which contains the constant steady states that all solutions must approach exponentially fast. Although, strictly speaking, it may not be an inertial manifold—there may not be a uniform lower bound on that exponential rate. We will refer to this conjectured manifold as quasi-inertial, as it does have the main desired properties of being finite dimensional and globally attracting. If the quasi-inertial manifold exists, then the formal calculations that produce the effective equation of motion are justified because the effective equation of motion is exactly the equation that produces the dynamics on the quasi-inertial manifold. In turn, the equation of motion, obtained by expansion and truncation in the retarded time and reduction to a slow manifold of a singular perturbation problem, would be justified because we will show that the slow vector field agrees with the quasi-inertial vector field. We rewrite the NFDE (1.43) extended to all τ as the first order system

$$\begin{aligned} w'(\tau) - Cw'(\tau-\gamma) &= f(w(\tau), w(\tau-\gamma)), \\ &= Aw(\tau) + Bw(\tau-\gamma), \end{aligned} \tag{1.63}$$

where

$$w = \begin{pmatrix} X \\ V \end{pmatrix}, \ V = X', \ A = \begin{bmatrix} 0 & 1 \\ -1 & -(\alpha+\beta) \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 0 \\ -1 & \alpha-\beta \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

For $\xi = (\xi_1, \xi_2) \in R^2$, we suppose that there exists a smooth flow $\eta(\tau, \xi, \gamma)$, which depends smoothly on the parameter γ and the initial state ξ, that is a global solution of the NFDE (1.63) and satisfies the usual flow properties

$$\begin{aligned} \eta(0, \xi, \gamma) &= \xi \\ \eta(\tau, \eta(s, \xi, \gamma), \gamma) &= \eta(\tau + s, \xi, \gamma). \end{aligned}$$

Although solutions of NFDEs do not generally have continuous derivatives, the special solutions given by the smooth flow η do have continuous derivatives.

The quasi-inertial vector field $\mathcal{X}$ is defined to be

$$\mathcal{X}(\xi, \gamma) = \eta_\tau(0, \xi, \gamma).$$

We will show that the slow vector field agrees with the quasi-inertial vector field up to first-order in γ. For simplicity, the continuation of this process to higher order is not considered here.

The vector field $\mathcal{X}$ expanded in powers of γ has the form

$$\mathcal{X}(\xi, \gamma) = \mathcal{X}(\xi, 0) + \gamma \mathcal{X}_\gamma(\xi, 0) + O(\gamma^2). \tag{1.64}$$

Since η is a solution of (1.63), we have

$$\eta_\tau(\tau, \xi, \gamma) - C\eta_\tau(\tau - \gamma, \xi, \gamma) = f(\eta(\tau, \xi, \gamma), \eta(\tau - \gamma, \xi, \gamma)).$$

Setting $\tau = 0$ gives

$$\mathcal{X}(\xi, \gamma) - C\mathcal{X}(\eta(-\gamma, \xi, \gamma), \gamma) = f(\xi, \eta(-\gamma, \xi, \gamma)), \tag{1.65}$$

where we have used the identity $\eta_\tau(-\gamma,\xi,\gamma) = \mathcal{X}(\eta(-\gamma,\xi,\gamma),\gamma)$, obtained from the definition of $\mathcal{X}$ and the flow property. Setting $\gamma = 0$ in (1.65) gives the zeroth order quasi-inertial vector field approximation

$$\begin{aligned}\mathcal{X}(\xi,0) &= (I-C)^{-1}f(\xi,\xi)\\ &= \begin{bmatrix} 0 & 1 \\ -1 & -\beta \end{bmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},\end{aligned} \tag{1.66}$$

which corresponds to the ODE system

$$\begin{pmatrix} X' \\ V' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\beta \end{bmatrix}\begin{pmatrix} X \\ V \end{pmatrix},$$

or equivalently the single ODE

$$X'' = -\beta X' - X, \tag{1.67}$$

in agreement with the zeroth order slow manifold result (1.46).

To determine $\mathcal{X}(\xi,\gamma)$ to first order in γ, we differentiate (1.65) with respect to γ and evaluate at $\gamma = 0$ to obtain the partial derivative

$$\begin{aligned}\mathcal{X}_\gamma(\xi,0) &= -(I-C)^{-1}[C\mathcal{X}_\xi(\xi,0) + D_2 f(\xi,\xi)]\mathcal{X}(\xi,0)\\ &= \frac{1}{2}\begin{bmatrix} 0 & 0 \\ \alpha & \alpha\beta \end{bmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.\end{aligned} \tag{1.68}$$

(1.64), (1.67), and (1.68) give the quasi-inertial vector field, to first order in γ

$$\mathcal{X}(\xi,\gamma) = \begin{bmatrix} 0 & 1 \\ a(\gamma) & b(\gamma) \end{bmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

where

$$a(\gamma) = \frac{\alpha\gamma}{2} - 1, \quad b(\gamma) = \beta(\frac{\alpha\gamma}{2} - 1).$$

It corresponds to the ODE system

$$\begin{pmatrix} X' \\ V' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ a(\gamma) & b(\gamma) \end{bmatrix}\begin{pmatrix} X \\ V \end{pmatrix},$$

or equivalently the single ODE

$$X'' = (\frac{\alpha\gamma}{2} - 1)X + \beta(\frac{\alpha\gamma}{2} - 1)X', \tag{1.69}$$

in agreement with the first-order slow manifold ODEs (1.55) and (1.56).

1.11 Effective dynamics

The ODE (1.69) gives the effective dynamics to first-order in γ of the full system (1.42) and (1.43), or equivalently the hybrid PDE/ODE system (1.3)–(1.14) (restricted to looking at the sum $R_1 + R_2$ for identical pistons). To be useful, the effective dynamics must correctly predict long-term behavior. This is indeed the case for our model: the globally attracting steady state $X = 0$ predicted by inspection of the ODE (1.69) is exactly the globally attracting steady state of the full system (a fact that is implied by (1.18) for identical pistons).

The effective dynamical equation can also be used to predict transient behavior. The zeroth-order effective dynamics given by ODE (1.46), which is the equation of a damped harmonic oscillator, is the exact equation of motion for the sum $X = \mathcal{R}_1 + \mathcal{R}_2$ in case the pistons are completely decoupled and there is no fluid between them. On the other hand, in case there is fluid between the pistons, the zeroth-order effective dynamics approximates the motions of the sum. The first-order correction, i.e. the ODE (1.69), reveals that the effect of the fluid is to decrease both the natural oscillation frequency and the damping by a factor of $1 - \alpha\gamma/2$, where $\alpha\gamma/2 = \rho_0 Al/(2M)$ is the ratio of the total fluid mass to the total mass of the pistons. This effect is not obvious from inspection of the exact system (1.42)–(1.43), but it can be confirmed there by numerical calculation.

1.12 Summary

We have derived a two-body acoustic field theory model with the aim of elucidating some of the issues that occur in electrodynamic and gravitational field theories, such as self-force, radiation damping, and runaway solutions— with a focus on discussing a scheme for elimination of these runaway solutions. Our model is a hybrid of field equations (PDEs for fluid density and velocity) coupled to mechanical equations of motion (ODEs for pistons on springs responding to fluid pressure determined by density). Even though a piston behaves as a point charge, this coupling involves the action of the field produced by that piston on itself (as well as the other piston); this is not possible in electrodynamics due to the singularity of the field at the point charge. Thus our model automatically incorporates the self-force, and the corresponding radiation reaction is manifested as fluid damping.

The full model combines the nonlinear field equations of gas dynamics (a system of hyperbolic conservation laws) with state dependent boundary conditions (ODEs determining the motion of the pistons)– a free boundary value problem. Linearizing both the PDEs and the ODE boundary conditions about the steady state essentially yields the wave equation on a fixed domain, but the boundary conditions are still ODEs for the piston motion which must be determined. This linearization is the acoustic model.

Our system is infinite-dimensional with a state space consisting of C^1 fluid density and velocity field functions (which determine piston positions and velocities by compatibility conditions). We have shown global existence and uniqueness of C^1 field solutions of the linearized system. Using the solution of the field equations to eliminate the field functions from the PDE/ODE system, we arrive at an equivalent system that starts with ODEs (involving initial fields and fluid damping) before the pistons interact, and then changes to NFDEs (with no initial fields) after the pistons begin interacting. This system is also infinite-dimensional because the initial fields in the initial ODEs must be specified. Alternatively, we may presume the pistons have always been interacting by prescribing the initial fields in the distant past (as customary in electrodynamics). In this case, the initial fields and the initial ODEs play no role, and we are left with only the NFDEs. These NFDEs are still infinite-dimensional because they require, as initial data, piston trajectories on a time interval of nonzero length.

In case spring damping is included, we have used several different energy norms (which decrease along orbits) and Sobolev embeddings to show that all field solutions of the acoustic PDE/ODE model (and hence the equivalent ODE/NFDE system) converge to zero (in the H^1 norm), and that the piston displacements and velocities converge to zero as well. This suggests the possibility of an invariant, finite dimensional manifold within the full infinite dimensional state space which contains the globally attracting steady states, and whose dynamics approximate the behavior of solutions in the full space as the steady states are approached. The dynamics on this manifold would be the finite dimensional effective equations of motion.

Taking the case of identical pistons in the linearized system, we convert to dimensionless units and decouple the NFDEs by looking at the sum, X, and difference, Y, of the piston displacements. To determine finite-dimensional effective equations of motion, we expand each NFDE in the small delay and truncate. This process results in high-order ODEs with extraneous runaway solutions whose presence is due to the expansion and truncation, similar to those found in the Lorentz-Dirac-Abraham equation from electrodynamics.

In previous work, a scheme has been developed for the systematic elimination of the corresponding runaway solutions obtained from expanding and truncating the class of RFDEs in the form of (1.62). We have applied the scheme (in part) here to our NFDE and found it to be successful, at least for the sum of the piston displacements, X. There is a degeneracy in the relative piston displacement Y-system that seems to be related to the particular way one chooses to linearize the original system. The scheme consists of three main steps: (1) using singular perturbation theory to reduce to an appropriate slow manifold of lower dimension where there are no runaways, (2) a proof of the existence of an inertial manifold in the unreduced equations of motion, and (3) a computation showing that the reduced dynamics on the slow manifold agree with the dynamics on the inertial manifold. Without a formal proof of the inertial manifold, we have carried out steps (1) and (3) of this program (to first-order in the delay) for our acoustic model. In addition, we have obtained the corresponding first-order effective dynamics, (1.69). The globally attracting steady state that is predicted by analyzing the effective equation of motion is exactly the same as that of the hybrid system, namely the sum of the piston displacements approaches zero. Moreover, the behavior of the solution of the effective equation agrees with the (numerical) solution of the exact system as the steady state is approached. In fact, the first-order effective equation produces the correct amplitude and frequency over time of the sum of the displacements of the oscillating pistons. Furthermore, the effective equation reveals phenomena not obvious in the original model; namely, the interaction with the fluid has the effect of decreasing the natural frequency and the spring damping (for the sum of the displacements, X).

The results obtained here suggest further research on the interaction of fluids coupled to vibrating sources. We also emphasize that our acoustic field theory model provides some evidence that the two-body problems of the more complicated electrodynamic and gravitational field theories might yield effective dynamics by similar methods. Furthermore, we note that the appearance of NFDEs in our model, rather than RFDEs, seems to stem from the direct coupling of the body to its own field, and that NFDEs would arise in other field theory two-body settings were such direct coupling possible.

In our linearized model, the NFDEs have fixed delays (due to the fixed boundaries). In the full nonlinear problem, the NFDEs have state dependent delays (due to the free boundaries). However, even showing existence and uniqueness of solutions to the full nonlinear PDE/ODE system is not trivial. That is the topic of the next (somewhat technical) chapter, where we show short time existence and uniqueness of C^1 solutions to the full nonlinear PDE/ODE system.

Chapter 2

Full nonlinear model

It is well known that the system of hyperbolic conservation laws (4)–(5), when imposed on the whole real line, does not in general have global classical solutions due to the development of shocks (discontinuous generalized solutions whose derivatives blow up in finite time) (cf. [16]). However, we conjecture that when equations (4)–(5) are restricted to the finite (state-dependent) spatial interval $R_1(t) < x < l + R_2(t)$ coupled with the boundary conditions (6)–(9), which incorporate spring damping, then shocks may be avoided if the initial data (satisfying appropriate compatibility conditions) are close enough to the steady state $\rho = \rho_0$, $u = 0$, $R_1 = R_{1_0}$, and $R_2 = R_{2_0}$, determined in section 0.3. (The linearization about this steady state gave the acoustic approximation that was analyzed in chapter 1.)

A first step in showing this is to convert the free boundary value problem to one with a fixed boundary. Rather than linearizing as in chapter 1, this may be done for the full nonlinear system by changing from Eulerian coordinates to a form of Lagrangian coordinates (transforming the spatial coordinate to a mass coordinate by integrating the gas density). Having fixed the boundary through this coordinate transformation, the new problem is still nontrivial due to the nonlinearity. As in chapter 1, there are again Riemann invariant functions, although the characteristics on which they are constant are no longer straight lines—they are now curves whose slopes are state dependent. That is the crux of the problem. If the characteristics were known, the solutions could be found trivially with the Riemann invariants. Or, if the solutions were known, then the characteristics could be ascertained. Obtaining both at once is the challenge.

We overcome this by first solving a linear version of the problem, not by linearization as in chapter 1, but by replacing nonlinear parts of the PDEs and BCs with a known function. This determines the characteristics, so that the solution of this new linear problem may be found. Now this solution can be used to form a new known function that is substituted into the nonlinear parts of the original PDEs and ODEs, which generates a new solution, and so on. Iteration of this procedure produces a sequence of differentiable functions which converges. Actually, we form the operator which acts on the known function and produces the solution of the linear problem, and we show that this operator has a fixed point by contraction mapping, but that amounts to the same thing. Finding a complete space that the operator maps back to itself, and on which it is a contraction, leads to requirements of small time and initial data close enough to a steady state.

If the limit of this sequence of functions (or the fixed point of the operator) were C^1, than it would be a classical solution of the original nonlinear PDEs and BCs. Otherwise, it is only a solution in some weak sense. To show this limit function is C^1, we use estimates on the modulus of continuity of the sequence of derivatives to show that sequence can be made to be equicontinuous, and other estimates show they are uniformly bounded, so that the Arzela-Ascoli theorem can be applied to obtain a subsequence of the derivatives which converges uniformly, so that what they converge to is the derivative of the limit of the original sequence. Thus, the original limit function was indeed differentiable and therefore a classical solution of the full nonlinear problem.

But first, we have to fix the free boundary.

2.1 Lagrangian coordinates

In order to fix the boundary, we change from Eulerian coordinates (x, t) to a form of Lagrangian coordinates (m, t), where the transformation is given by

$$m(x,t) = \int_{R_1(t)}^{x} \rho(y,t)\, dy.$$

(A similar transformation is used in [18].) Physically, $m(x, t)$ is the mass of gas, per cross sectional area, A, contained between piston 1 and the point x at time t. Under this change of coordinates, the state dependent domain $R_1(t) \leq x \leq l + R_2(t)$ is transformed to the fixed domain $0 \leq m \leq M$, where M is the total mass of gas per cross sectional area between the pistons. If $\rho > 0$, then $m(x, t)$ is an increasing function of x and the transformation can be inverted to give $x(m, t)$, which satisfies

$$m = \int_{R_1(t)}^{x(m,t)} \rho(y,t)\, dy. \tag{2.1}$$

The Jacobian of the transformation and its inverse are then

$$\frac{\partial m}{\partial x}(x,t) = \rho(x,t), \quad \frac{\partial x}{\partial m}(m,t) = \frac{1}{\rho(x(m,t),t)}.$$

We define the new functions

$$\tilde{\rho}(m,t) = \rho(x(m,t),t), \quad \tilde{v} = \frac{1}{\tilde{\rho}}, \quad \tilde{u}(m,t) = u(x(m,t),t), \quad \tilde{P}(\tilde{v}) = P(1/\tilde{v}),$$

$$\tilde{F}(m) = F(x(m,0)), \;\; \tilde{G}(m) = G(x(m,0)).$$

Note that the fluid velocity in Lagrangian coordinates is given by

$$\frac{\partial x}{\partial t}(m,t) = \tilde{u}(m,t),$$

which can be verified by differentiating (2.1) w.r.t. t and applying the continuity equation (4) and the BC (6).

Under this transformation, the free boundary value problem (3)–(11) is equivalent to the system with fixed domain $0 \leq m \leq M$,

$$\tilde{P}(\tilde{v}) \;=\; a\tilde{v}^{-\gamma} \tag{2.2}$$

$$\tilde{v}_t(m,t) \;=\; \tilde{u}_m(m,t) \tag{2.3}$$

$$\tilde{u}_t(m,t) \;=\; -\tilde{P}(\tilde{v}(m,t))_m, \tag{2.4}$$

with boundary conditions

$$\tilde{u}(0,t) \;=\; \dot{R}_1(t) \tag{2.5}$$

$$\tilde{u}(M,t) \;=\; \dot{R}_2(t) \tag{2.6}$$

$$M_1\ddot{R}_1(t) \;=\; -K_1R_1(t) - \mu_1\dot{R}_1(t) - \tilde{P}(\tilde{v}(0,t))A \tag{2.7}$$

$$M_2\ddot{R}_2(t) \;=\; -K_2R_2(t) - \mu_2\dot{R}_2(t) + \tilde{P}(\tilde{v}(M,t))A, \tag{2.8}$$

and initial fields

$$\tilde{v}(m,0) = \tilde{F}(m), \;\; \tilde{u}(m,0) = \tilde{G}(m), \tag{2.9}$$

where $\tilde{F}$ satisfies

$$\int_0^M \tilde{F}(m)\, dm = l + R_2(0) - R_1(0). \tag{2.10}$$

Note that $\int_0^M \tilde{v}(m,t)\, dm = l + R_2(t) - R_1(t)$, as can be verified by change of variables, from Lagrangian coordinates back to Eulerian coordinates.

The new compatibility conditions become

$$\begin{aligned}
\dot{R}_1(0) &= \tilde{G}(0) && (2.11)\\
\dot{R}_2(0) &= \tilde{G}(M) && (2.12)\\
-\left(M_1 \frac{\partial}{\partial m}\tilde{P}(\tilde{F}(m))\right)\Bigg|_{m=0} &= -K_1 R_1(0) - \mu_1 \tilde{G}(0) - \tilde{P}(\tilde{F}(0))A && (2.13)\\
-\left(M_2 \frac{\partial}{\partial m}\tilde{P}(\tilde{F}(m))\right)\Bigg|_{m=M} &= -K_2 R_2(0) - \mu_2 \tilde{G}(M) + \tilde{P}(\tilde{F}(M))A. && (2.14)
\end{aligned}$$

Again the initial fields $\tilde{F}$ and $\tilde{G}$ determine the initial piston velocities $\dot{R}_1(0)$, $\dot{R}_2(0)$ and positions $R_1(0)$, $R_2(0)$.

The energy in Lagrangian coordinates becomes

$$\begin{aligned}
E(t) &= A\int_0^M \left(\frac{1}{2}\rho(x,t)u^2(x,t) + \frac{\tilde{P}(\tilde{v}(m,t))\tilde{v}(m,t)}{\gamma-1}\right) dm \\
&\quad + \frac{1}{2}K_1 R_1^2(t) + \frac{1}{2}K_2 R_2^2(t) + \frac{1}{2}M_1 \dot{R}_1^2(t) + \frac{1}{2}M_2 \dot{R}_2^2(t),
\end{aligned}$$

which again, for a solution of (2.2)-(2.8), satisfies

$$\frac{dE}{dt} = -\mu_1 \dot{R}_1^2 - \mu_2 \dot{R}_2^2.$$

2.2 Riemann invariants

The field equations (2.3)–(2.4) may be expressed in the conservation law form

$$\begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix}_t + \begin{bmatrix} 0 & -1 \\ -c^2(\tilde{v}) & 0 \end{bmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix}_m = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.15}$$

where $c(\tilde{v}) := \sqrt{-\tilde{P}'(\tilde{v})} = \gamma a \tilde{v}^{-\gamma-1}$. Let B denote the matrix in (2.15), whose eigenvalues $\pm c(\tilde{v})$ are real and distinct for $\tilde{v} > 0$. Thus the conservation law is strictly hyperbolic (c.f. [16]). The corresponding right eigenvectors are

$$r_{\pm} = \begin{pmatrix} 1 \\ \mp c(\tilde{v}) \end{pmatrix},$$

where $Br_\pm = \pm cr_\pm$. Similarly, denote the left eigenvectors $l_\pm$, where $l_\pm B = \pm cl_\pm$. The Riemann invariants may be determined by finding functions $w(\tilde{v}, \tilde{u})$ and $z(\tilde{v}, \tilde{u})$ such that $\nabla w \cdot r_+ = 0$ and $\nabla z \cdot r_- = 0$ (where ∇ is w.r.t. $\tilde{v}$ and $\tilde{u}$). Note that the left and right eigenvectors are biorthogonal, i.e. $l_\pm \cdot r_\mp = 0$. Reaping the benefits of having only two dimensions, we have that ∇w is parallel to l_-, and ∇z is parallel to r_+, i.e. $\nabla w \cdot B = -c\nabla w$ and $\nabla z \cdot B = c\nabla z$. Then left multiplying (2.15) by ∇w and ∇z, respectively, and making use of the chain rule yields

$$
\begin{aligned}
\frac{\partial}{\partial t} w(\tilde{v}(m,t), \tilde{u}(m,t)) - c(\tilde{v}(m,t)) \frac{\partial}{\partial m} w(\tilde{v}(m,t), \tilde{u}(m,t)) &= 0 \\
\frac{\partial}{\partial t} z(\tilde{v}(m,t), \tilde{u}(m,t)) + c(\tilde{v}(m,t)) \frac{\partial}{\partial m} z(\tilde{v}(m,t), \tilde{u}(m,t)) &= 0.
\end{aligned}
$$

Abusing notation and dropping the composition of w and z with $\tilde{v}$ and $\tilde{u}$, we henceforth treat w and z as functions of m and t directly, denoted by $w(m,t)$ and $z(m,t)$. The field equations then take the characteristic form

$$
\begin{aligned}
w_t - c(\tilde{v}) w_m &= 0 \\
z_t + c(\tilde{v}) z_m &= 0,
\end{aligned}
$$

from which it can be seen that w and z are constant along the characteristics—the integral curves of $\frac{dm}{dt} = -c(\tilde{v}(m,t))$ and $\frac{dm}{dt} = c(\tilde{v}(m,t))$, respectively.

With the choice of functions

$$
w = \tilde{u} - \frac{2\sqrt{\gamma a}}{\gamma - 1} \tilde{v}^{(1-\gamma)/2}, \quad z = \tilde{u} + \frac{2\sqrt{\gamma a}}{\gamma - 1} \tilde{v}^{(1-\gamma)/2} \tag{2.16}
$$

$$
\hat{P}(q) = a \left(\frac{\gamma - 1}{4\sqrt{\gamma a}} q \right)^{\frac{2\gamma}{\gamma - 1}}, \quad \hat{c}(q) = \sqrt{\gamma a} \left(\frac{\gamma - 1}{4\sqrt{\gamma a}} q \right)^{\frac{\gamma + 1}{\gamma - 1}}
$$

$$
f = \tilde{G} - \frac{2\sqrt{\gamma a}}{\gamma - 1} \tilde{F}^{(1-\gamma)/2}, \quad g = \tilde{G} + \frac{2\sqrt{\gamma a}}{\gamma - 1} \tilde{F}^{(1-\gamma)/2},
$$

the initial boundary value problem (2.2)–(2.10) is equivalent to the system of field equations with domain $0 \le m \le M$,

$$\begin{aligned} w_t - \hat{c}(z-w)w_m &= 0 && (2.17) \\ z_t + \hat{c}(z-w)z_m &= 0, && (2.18) \end{aligned}$$

with boundary conditions

$$\begin{aligned} \dot{R}_1(t) &= \frac{1}{2}(z(0,t) + w(0,t)) && (2.19) \\ \dot{R}_2(t) &= \frac{1}{2}(z(M,t) + w(M,t)) && (2.20) \\ M_1\ddot{R}_1(t) &= -K_1R_1(t) - \mu_1\dot{R}_1(t) - \hat{P}(z(0,t) - w(0,t))A && (2.21) \\ M_2\ddot{R}_2(t) &= -K_2R_2(t) - \mu_2\dot{R}_2(t) + \hat{P}(z(M,t) - w(M,t))A, && (2.22) \end{aligned}$$

and initial fields

$$w(m,0) = f(m), \quad z(m,0) = g(m), \qquad (2.23)$$

where f and g satisfy

$$\int_0^M \left(\frac{\gamma - 1}{4\sqrt{\gamma a}}(g(m) - f(m)) \right)^{\frac{2}{1-\gamma}} dm = l + R_2(0) - R_1(0). \qquad (2.24)$$

The new compatibility conditions become

$$\begin{aligned} \dot{R}_1(0) &= \frac{1}{2}(f(0) + g(0)) && (2.25) \\ \dot{R}_2(0) &= \frac{1}{2}(f(M) + g(M)) && (2.26) \\ -\frac{1}{2}M_1\hat{c}(g(0) - f(0))(g'(0) - f'(0)) &= -K_1R_1(0) - \frac{\mu_1}{2}(f(0) + g(0)) \\ &\quad -\hat{P}(g(0) - f(0))A && (2.27) \\ -\frac{1}{2}M_2\hat{c}(g(M) - f(M))(g'(M) - f'(M)) &= -K_2R_2(0) - \frac{\mu_2}{2}(f(M) + g(M)) \\ &\quad +\hat{P}(g(M) - f(M))A. && (2.28) \end{aligned}$$

Again the initial fields f and g determine the initial piston velocities $\dot{R}_1(0)$, $\dot{R}_2(0)$ and positions $R_1(0)$, $R_2(0)$.

The energy in terms of Riemann invariants becomes

$$\begin{aligned} E(t) &= A\int_0^M \left(\frac{1}{8}(z(m,t)+w(m,t))^2 + \frac{(\gamma-1)^2}{16\gamma}(z(m,t)-w(m,t))^2\right)\, dm \\ &\quad + \frac{1}{2}K_1R_1^2(t) + \frac{1}{2}K_2R_2^2(t) + \frac{1}{2}M_1\dot{R}_1^2(t) + \frac{1}{2}M_2\dot{R}_2^2(t), \end{aligned}$$

which again, for a solution of (2.17)–(2.22), satisfies

$$\frac{dE}{dt} = -\mu_1\dot{R}_1^2 - \mu_2\dot{R}_2^2.$$

The field equations (2.17)-(2.18) may be expressed as

$$\begin{aligned} \frac{d}{d\tau}w(\alpha(\tau;m,t),\tau) &= 0 \\ \frac{d}{d\tau}z(\beta(\tau;m,t),\tau) &= 0, \end{aligned}$$

where $\alpha(\tau;m,t)$ and $\beta(\tau;m,t)$ denote the characteristic curves (parameterized by τ) through the point (m,t), defined by

$$\begin{aligned} \frac{d\alpha}{d\tau} &= -\hat{c}(z(\alpha,\tau)-w(\alpha,\tau)), \quad \alpha(t;m,t)=m \\ \frac{d\beta}{d\tau} &= \hat{c}(z(\beta,\tau)-w(\beta,\tau)), \quad \beta(t;m,t)=m. \end{aligned}$$

Unfortunately, the function $z-w$ must be known to determine the characteristics α and β.

2.3 Linear problem

Let $q(m,t)$ be a positive function that is Lipshitz in m and continuous in t. Replacing $z(m,t)-w(m,t)$ with $q(m,t)$ for the argument of $\hat{c}$ and $\hat{P}$ in (2.17)–(2.22) results in the new linear system

$$w_t - \hat{c}(q) w_m = 0 \tag{2.29}$$

$$z_t + \hat{c}(q) z_m = 0, \tag{2.30}$$

with boundary conditions

$$\dot{R}_1(t) = \frac{1}{2}(z(0,t) + w(0,t)) \tag{2.31}$$

$$\dot{R}_2(t) = \frac{1}{2}(z(M,t) + w(M,t)) \tag{2.32}$$

$$M_1 \ddot{R}_1(t) = -K_1 R_1(t) - \mu_1 \dot{R}_1(t) - \hat{P}(q(0,t))A \tag{2.33}$$

$$M_2 \ddot{R}_2(t) = -K_2 R_2(t) - \mu_2 \dot{R}_2(t) + \hat{P}(q(M,t))A, \tag{2.34}$$

and the same initial fields

$$w(m,0) = f(m), \quad z(m,0) = g(m), \tag{2.35}$$

where f and g satisfy

$$\int_0^M \left(\frac{\gamma - 1}{4\sqrt{\gamma a}} (g(m) - f(m)) \right)^{\frac{2}{1-\gamma}} dm = l + R_2(0) - R_1(0). \tag{2.36}$$

If $q(m,t)$ is consistent with initial conditions on $z(m,t) - w(m,t)$, i.e. $q(m,0) = g(m) - f(m)$, then the compatibility conditions remain the same,

$$\dot{R}_1(0) = \frac{1}{2}(f(0) + g(0)) \tag{2.37}$$

$$\dot{R}_2(0) = \frac{1}{2}(f(M) + g(M)) \tag{2.38}$$

$$-\frac{1}{2} M_1 \hat{c}(g(0) - f(0))(g'(0) - f'(0)) = -K_1 R_1(0) - \frac{\mu_1}{2}(f(0) + g(0)) - \hat{P}(g(0) - f(0))A \tag{2.39}$$

$$-\frac{1}{2} M_2 \hat{c}(g(M) - f(M))(g'(M) - f'(M)) = -K_2 R_2(0) - \frac{\mu_2}{2}(f(M) + g(M)) + \hat{P}(g(M) - f(M))A. \tag{2.40}$$

Again the initial fields f and g determine the initial piston velocities $\dot{R}_1(0)$, $\dot{R}_2(0)$ and positions $R_1(0)$, $R_2(0)$. Again, the field equations may be expressed as

$$\begin{aligned}\frac{d}{d\tau}w(\alpha_q(\tau;m,t),\tau) &= 0\\ \frac{d}{d\tau}z(\beta_q(\tau;m,t),\tau) &= 0.\end{aligned}$$

where α_q and β_q are now explicitly determined by the characteristic ODEs

$$\frac{d\alpha_q}{d\tau} = -\hat{c}(q(\alpha_q,\tau)), \quad \alpha_q(t;m,t) = m \tag{2.41}$$

$$\frac{d\beta_q}{d\tau} = \hat{c}(q(\beta_q,\tau)), \quad \beta_q(t;m,t) = m, \tag{2.42}$$

and the subscript q indicates the dependence of α and β on the function q. If $q(\cdot,t) \in C^1([0,M])$ for each $t \geq 0$, and $q(m,\cdot) \in C^0([0,+\infty))$ for each $m \in [0,M]$, then $\alpha_q(\tau;m,t)$ and $\beta_q(\tau;m,t)$ are C^1 with respect to τ, m, and t.

2.4 Constructing a solution of the linear problem

We will use the method of characteristics to construct a solution to the linear system (2.29)–(2.40), for $q(\cdot,t) \in C^1([0,M])$ for each $t \geq 0$, and $q(m,\cdot) \in C^0([0,+\infty))$ for each $m \in [0,M]$, with certain hypotheses on the initial data f and g. We first assume $g - f > 0$. (Note that this assumption is consistent with the notion of taking initial data close enough to the constant steady state, since by (2.16), $w < 0$ and $z > 0$ for $|\tilde{u}|$ and $|\tilde{v} - \tilde{v}_0|$ small enough, where $\tilde{v}_0 > 0$ is constant). We further assume that f and g are differentiable and satisfy (2.24), in which $R_1(0)$ and $R_2(0)$ are eliminated by using (2.39)–(2.40). (Physically, this ensures that integrating the initial density distribution gives M, the total mass of gas between the pistons, per area).

We begin by constructing the solution on the boundary. Define $R_1^q(t)$ and $R_2^q(t)$ as solutions to the ODEs (2.33) and (2.34), respectively, where the initial values $R_1^q(0)$,

$R_2^q(0)$, $\dot{R}_1^q(0)$ and $\dot{R}_2^q(0)$ are determined from f and g through the compatibility conditions (2.37)–(2.40), and the superscript q indicates the dependence of the solutions on the function q. The solutions $R_1^q(t)$ and $R_2^q(t)$ may be written explicitly as

$$\begin{aligned} R_1^q(t) &= e^{-\frac{\nu_1}{2}t}((\cos\omega_1 t + \frac{\nu_1}{2\omega_1}\sin\omega_1 t)R_1^q(0) + \frac{1}{\omega_1}(\sin\omega_1 t)\dot{R}^q(0)) \\ &\quad -\frac{A}{M_1}\int_0^t e^{-\frac{\nu_1}{2}(t-s)}\frac{1}{\omega_1}\sin\omega_1(t-s)\hat{P}(q(0,t-s))\,ds \qquad (2.43) \\ R_2^q(t) &= e^{-\frac{\nu_2}{2}t}((\cos\omega_2 t + \frac{\nu_2}{2\omega_2}\sin\omega_1 t)R_2^q(0) + \frac{1}{\omega_2}(\sin\omega_2 t)\dot{R}^q(0)) \\ &\quad +\frac{A}{M_2}\int_0^t e^{-\frac{\nu_2}{2}(t-s)}\frac{1}{\omega_2}\sin\omega_2(t-s)\hat{P}(q(M,t-s))\,ds, \qquad (2.44) \end{aligned}$$

with their derivatives

$$\begin{aligned} \dot{R}_1^q(t) &= e^{-\frac{\nu_1}{2}t}(-(\frac{\nu_1^2}{4\omega_1}+\omega_1)(\sin\omega_1 t)R_1^q(0) + (-\frac{\nu_1}{2\omega_1}\sin\omega_1 t + \cos\omega_1 t)\dot{R}_1^q(0)) \\ &\quad +\frac{A}{M_1}\int_0^t e^{-\frac{\nu_1}{2}(t-s)}(\frac{\nu_1}{2\omega_1}\sin\omega_1(t-s) - \cos\omega_1(t-s))\hat{P}(q(0,t-s))\,ds, \end{aligned} \qquad (2.45)$$

$$\begin{aligned} \dot{R}_2^q(t) &= e^{-\frac{\nu_2}{2}t}(-(\frac{\nu_2^2}{4\omega_2}+\omega_2)(\sin\omega_2 t)R_2^q(0) + (-\frac{\nu_2}{2\omega_2}\sin\omega_2 t + \cos\omega_2 t)\dot{R}_2^q(0)) \\ &\quad -\frac{A}{M_2}\int_0^t e^{-\frac{\nu_2}{2}(t-s)}(\frac{\nu_2}{2\omega_2}\sin\omega_2(t-s) - \cos\omega_2(t-s))\hat{P}(q(M,t-s))\,ds, \end{aligned} \qquad (2.46)$$

where

$$\nu_i = \frac{\mu_i}{M_i}, \quad \omega_i = \frac{1}{2}\sqrt{\frac{K_i}{M_i} - \nu_i^2}, \quad i = 1, 2.$$

Since $\hat{c}(q) > 0$, the characteristics $\alpha_q(\tau; m, t)$ and $\beta_q(\tau; m, t)$ defined by (2.41)–(2.42) have inverses $\alpha_q^{-1}(\xi; m, t)$ and $\beta_q^{-1}(\xi; m, t)$ where

$$\begin{aligned} \alpha_q^{-1}(\xi; m, t) = \tau &\iff \alpha_q(\tau; m, t) = \xi \\ \beta_q^{-1}(\xi; m, t) = \tau &\iff \beta_q(\tau; m, t) = \xi. \end{aligned} \qquad (2.47)$$

For $q \in C^1$ with respect to m and $q \in C^0$ with respect to t, the nonautonomous vector fields for the characteristic ODEs (2.41)–(2.42) are C^1, with C^0 time dependence. Thus the characteristics depend smoothly on initial data. In particular, $\alpha_q(\tau; m, t)$ and $\beta_q(\tau; m, t)$ are C^1 with respect to m and t as well as τ. The implicit function theorem then implies that the inverses of the characteristics, $\alpha_q^{-1}(\xi; m, t)$ and $\beta_q^{-1}(\xi; m, t)$, are also C^1 with respect to ξ, m, and t.

The two characteristics $\alpha_q(\tau; M, 0)$ and $\beta_q(\tau; 0, 0)$, emanating from the points $(M, 0)$ and $(0, 0)$, respectively, partition the domain $[0, M] \times [0, \infty)$ into the following three regions:

$$\begin{aligned} \mathcal{R}_1 &= \{(m,t) : \beta_q(t;0,0) \le m \le \alpha_q(t;M,0), \;\; t \ge 0\} \\ &= \{(m,t) : t \le \beta_q^{-1}(m;0,0), \;\; t \le \alpha_q^{-1}(m;M,0), \;\; t \ge 0\}, \end{aligned} \tag{2.48}$$

$$\begin{aligned} \mathcal{R}_2 &= \{(m,t) : m \le \beta_q(t;0,0), \;\; m \le \alpha_q(t;M,0), \;\; m \ge 0\} \\ &= \{(m,t) : \beta_q^{-1}(m;0,0) \le t \le \alpha_q^{-1}(m;M,0), \;\; m \ge 0\}, \end{aligned} \tag{2.49}$$

$$\begin{aligned} \mathcal{R}_3 &= \{(m,t) : m \ge \beta_q(t;0,0), \;\; m \ge \alpha_q(t;M,0), \;\; m \le M\} \\ &= \{(m,t) : \alpha_q^{-1}(m;M,0) \le t \le \beta_q^{-1}(m;0,0), \;\; m \le M\}. \end{aligned} \tag{2.50}$$

The two additional characteristics, $\alpha_q(\tau; M, \beta_q^{-1}(M; 0, 0))$ and $\beta_q(\tau, 0, \alpha_q^{-1}(0; M, 0))$, provide an upper bound to the fourth region:

$$\begin{aligned} \mathcal{R}_4 = {} & \{(m,t) : \alpha_q(t;M,0) \le m \le \beta_q(t;0,0), \\ & \beta_q(t,0,\alpha_q^{-1}(0;M,0)) \le m \le \alpha_q(t;M,\beta_q^{-1}(M;0,0))\} \\ = {} & \{(m,t) : t \ge \beta_q^{-1}(m;0,0), \;\; t \ge \alpha_q^{-1}(m;M,0) \\ & t \le \beta_q^{-1}(m;0,\alpha_q^{-1}(0;M,0)), \;\; t \le \alpha_q^{-1}(m;M,\beta^{-1}(M;0,0))\}. \end{aligned} \tag{2.51}$$

In region $\mathcal{R}_1$, both the α_q and the β_q characteristics (on which w and z must be constant, respectively) can be traced backwards in time to the segment $[0, M] \times \{t = 0\}$, where initial data is given. So we impose the initial conditions $w^q(m, 0) = f(m)$ and $z^q(m, 0) = g(m)$, and define, for $(m, t) \in \mathcal{R}_1$,

$$w^q(m,t) := w^q(\alpha_q(0; m, t), 0) = f(\alpha_q(0; m, t))$$

$$z^q(m,t) := z^q(\beta_q(0; m, t), 0) = g(\beta_q(0; m, t),$$

where again the superscript q indicates the dependence of the solutions on the function q. By construction, w^q and z^q thus defined are classical solutions of the field equations (2.29)-(2.30) which satisfy the initial conditions (2.35), and the boundary conditions (2.31)-(2.34) (namely at the points $(0, 0)$ and $(M, 0)$, due to the compatibility conditions (2.37)-(2.40)).

In region $\mathcal{R}_2$, the α_q characteristics (on which w must be constant) can be traced backwards in time to the segment $[0, M] \times \{t = 0\}$, just as in region $\mathcal{R}_1$. So for $(m, t) \in \mathcal{R}_2$, we again define

$$w^q(m,t) := w^q(\alpha_q(0; m, t), 0) = f(\alpha_q(0; m, t)).$$

However, the β_q characteristics (on which z must be constant) will intersect the boundary $m = 0$ before reaching $t = 0$. The β_q characteristic through the point $(m, t) \in \mathcal{R}_2$ will intersect the boundary $m = 0$ at time $\beta_q^{-1}(0; m, t)$. To ensure the boundary condition (2.31) holds, we define, for $0 \leq t \leq \alpha_q^{-1}(0; M, 0)$,

$$z^q(0,t) := 2\dot{R}_1^q(t) - w^q(0,t).$$

We then define for any point $(m,t) \in \mathcal{R}_2$,

$$\begin{aligned} z^q(m,t) &:= z^q(0, \beta_q^{-1}(0;m,t)) \\ &= 2\dot{R}_1^q(\beta_q^{-1}(0;m,t)) - w^q(0,\beta_q^{-1}(0;m,t)) \\ &= 2\dot{R}_1^q(\beta_q^{-1}(0;m,t)) - f(\alpha_q(0;0,\beta_q^{-1}(0;m,t))). \end{aligned}$$

By construction, w^q and z^q are solutions of (2.29)-(2.40) in region $\mathcal{R}_2$.

Similarly, in region $\mathcal{R}_3$, the β_q characteristics (on which z must be constant) can be traced backwards in time to the segment $[0,M] \times \{t=0\}$, just as in region $\mathcal{R}_1$. So for $(m,t) \in \mathcal{R}_3$, we again define

$$z^q(m,t) := z^q(\beta_q(0;m,t),0) = g(\beta_q(0;m,t)).$$

However, the α_q characteristics (on which w must be constant) will intersect the boundary $m = M$ before reaching $t = 0$. The α_q characteristic through the point $(m,t) \in \mathcal{R}_3$ will intersect the boundary $m = M$ at time $\alpha_q^{-1}(M;m,t)$. To ensure the boundary condition (2.32) holds, we define, for $0 \le t \le \beta_q^{-1}(M;0,0)$,

$$w^q(M,t) := 2\dot{R}_2^q(t) - z^q(M,t).$$

We then define for any point $(m,t) \in \mathcal{R}_3$,

$$\begin{aligned} w^q(m,t) &:= w^q(M, \alpha_q^{-1}(M;m,t)) \\ &= 2\dot{R}_2^q(\alpha_q^{-1}(M;m,t)) - z^q(M,\alpha_q^{-1}(M;m,t)) \\ &= 2\dot{R}_2^q(\alpha_q^{-1}(M;m,t)) - g(\beta_q(0;M,\alpha_q^{-1}(M;m,t))). \end{aligned}$$

Finally, in region $\mathcal{R}_4$, neither α_q nor β_q characteristics can be traced back to $[0,M] \times \{t=0\}$ before intersecting the boundaries $m = M$ and $m = 0$, respectively. So here we define w^q as in region $\mathcal{R}_3$ and z^q as in region $\mathcal{R}_2$.

We then have that

$$w^q(m,t) = \begin{cases} f(\alpha_q(0,m,t)) & (m,t) \in \mathcal{R}_1 \cup \mathcal{R}_2 \\ 2\dot{R}_2^q(\alpha_q^{-1}(M;m,t)) - g(\beta_q(0;M,\alpha_q^{-1}(M;m,t))) & (m,t) \in \mathcal{R}_3 \cup \mathcal{R}_4 \end{cases} \tag{2.52}$$

$$z^q(m,t) = \begin{cases} g(\beta_q(0,m,t)) & (m,t) \in \mathcal{R}_1 \cup \mathcal{R}_3 \\ 2\dot{R}_1^q(\beta_q^{-1}(0;m,t)) - f(\alpha_q(0;0,\beta_q^{-1}(0;m,t))) & (m,t) \in \mathcal{R}_2 \cup \mathcal{R}_4. \end{cases} \tag{2.53}$$

are classical solutions to the linear initial boundary value problem (2.29)–(2.40) in $\cup_{i=1}^4 \mathcal{R}_i$.

The solutions w^q and z^q may be constructed beyond region $\mathcal{R}_4$ in a similar manner, yielding a solution akin to that of a boundary value problem for the wave equation, which for each t may be expressed as a finite series where the number of terms in the series increases with t due to successive reflections at the boundary.

Alternatively, the solution has been defined at least until the time that the characteristics $\alpha_q(\tau; M, 0)$ and $\beta_q(\tau; 0, 0)$ intersect, say at time $\tau = t_0 > 0$. Then we can regard $w^q(m,t_0)$ and $z^q(m,t_0)$ as new initial conditions and repeat the previous construction of the solution to extend from time t_0 to $2t_0$ (notice the time t_0 is independent of initial data and depends only on the fixed function q). By induction we have a global solution of (2.29)–(2.40).

We summarize this result in the following theorem.

Theorem 6. *Let $q : [0,M] \times [0,\infty) \to \mathbb{R}^+$, such that $q(\cdot,t) \in C^1([0,M])$ for each $t \in [0,\infty)$, $q(m,\cdot) \in C^0([0,\infty))$ for each $m \in [0,M]$, and $q(m,0) = g(m) - f(m)$, where $f, g \in C^1([0,M],\mathbb{R})$ such that $g - f > 0$ and*

$$\int_0^M \left(\frac{\gamma - 1}{4\sqrt{\gamma a}} (g(m) - f(m)) \right)^{\frac{2}{1-\gamma}} dm = l + \psi_2(f,g) - \psi_1(f,g), \tag{2.54}$$

where

$$\psi_1(f,g) = \frac{M_1}{2K_1}\hat{c}(g(0)-f(0))(g'(0)-f'(0)) - \frac{\mu_1}{2K_1}(f(0)+g(0)) - \frac{A}{K_1}\hat{P}(g(0)-f(0)) \tag{2.55}$$

and

$$\begin{aligned}\psi_2(f,g) &= \frac{M_2}{2K_2}\hat{c}(g(M)-f(M))(g'(M)-f'(M)) - \frac{\mu_2}{2K_2}(f(M)+g(M)) \\ &\quad + \frac{A}{K_2}\hat{P}(g(M)-f(M)).\end{aligned} \tag{2.56}$$

Then the linear IBVP (2.29)–(2.40) *has a unique,* C^1*, global solution.*

In particular, for $(m,t) \in \cup_{i=1}^{4}\mathcal{R}_i$*, a formula for the solution is given by* (2.52)–(2.53)*, where the regions* $\mathcal{R}_i$ *are given by* (2.48)-(2.51)*,* α_q *and* β_q *are determined by the ODEs* (2.41)–(2.42)*,* $R_1^q(t)$ *and* $R_2^q(t)$ *are given by* (2.43)–(2.44)*, in which* $R_i(0)$*,* $\dot{R}_i(0)$ *for* $i = 1, 2$ *are determined from* f *and* g *by the compatibility conditions* (2.37)–(2.40)*.*

In case $q \notin C^1$, if α_q, β_q, α_q^{-1}, or β_q^{-1} are not all C^1 with respect to m and t, then we call w and z given by (2.52)–(2.53) a *weak solution* of the IBVP (2.29)–(2.40).

2.5 Steady state in transformed coordinates

The unique steady state solution of the full nonlinear system (3)–(14) in Eulerian coordinates was found in section 0.3 to be $\rho = \rho_0$, $u = 0$, $R_1 = R_{1_0}$, and $R_2 = R_{2_0}$, where ρ_0, R_{1_0}, and R_{2_0} are determined by (15)–(16).

Similarly, the steady states $\tilde{v} = \tilde{v}_0$ and $\tilde{u} = 0$ are solutions to the full nonlinear problem in Lagrangian coordinates, (2.2)–(2.14), where $\tilde{v}_0$ is determined by the condition

$$l + \frac{K_1K_2}{K_1+K_2}aA\tilde{v}_0^{-\gamma} = M\tilde{v}_0.$$

The Riemann invariant formulation of the full nonlinear problem, (2.17)–(2.28), has the steady state solution $w = -\frac{1}{2}q_0$ and $z = \frac{1}{2}q_0$ where q_0 satisfies

$$l + \frac{K_1 K_2}{K_1 + K_2} aA \left(\frac{\gamma - 1}{4\sqrt{\gamma a}} q_0 \right)^{\frac{2\gamma}{1-\gamma}} = M \left(\frac{\gamma - 1}{4\sqrt{\gamma a}} q_0 \right)^{\frac{2}{1-\gamma}}. \tag{2.57}$$

Next we will show short time existence and uniqueness of solutions for initial data taken close enough to this steady state.

2.6 Short time existence and uniqueness of weak solutions

From now on, we will use the notation $||\cdot||$ (without subscripts) to represent sup norm. Let ϵ_1, δ_1, and δ_1' be fixed positive constants. Let T, ϵ, L, J, δ, and δ' be positive constants, to be determined, such that $\epsilon \leq \epsilon_1$, $\delta \leq \delta_1$, and $\delta' \leq \delta_1'$. Let q_0 be the positive constant determined by (2.57) (i.e., q_0 is the steady state value of $z - w$ from the previous section). Let $\alpha_{q_0+\epsilon_1}$ and $\beta_{q_0+\epsilon_1}$ be solutions of the characteristic ODEs (2.41)–(2.42) for $q = q_0 + \epsilon_1$. Suppose $T \leq t_1$, where $\alpha_{q_0+\epsilon_1}(t_1; M, 0) = \beta_{q_0+\epsilon_1}(t_1; 0, 0)$ (so that, in particular, $[0, M] \times [0, T] \subset \cup_{i=1}^{3} \mathcal{R}_i$ where the regions $\mathcal{R}_i$ are given by (2.48)–(2.50)).

Let $f, g \in C^1([0, M])$ satisfy (2.54)-(2.56). In addition, suppose

$$\sup_{m \in [0,M]} |f(m) + \frac{1}{2}q_0| \leq \frac{1}{2}\delta, \quad \sup_{m \in [0,M]} |g(m) - \frac{1}{2}q_0| \leq \frac{1}{2}\delta, \tag{2.58}$$

and

$$\sup_{m \in [0,M]} |f'(m)| \leq \frac{1}{2}\delta', \quad \sup_{m \in [0,M]} |g'(m)| \leq \frac{1}{2}\delta' \tag{2.59}$$

(i.e. g and f are close to the constant steady state values of z and w, which are $\frac{1}{2}q_0$ and $-\frac{1}{2}q_0$, respectively).

Let $X = [0, M] \times [0, T]$, and define

$$\mathcal{B} = \{q \in C^0(X, [q_0 - \epsilon, q_0 + \epsilon]) : Lip_m(q) \leq L, Lip_t(q) \leq J,\ q(m, 0) = g(m) - f(m)\},$$

a closed subset of the Banach space $C^0(X, \mathbb{R})$, with the C^0 norm.

We use the solution of the linear problem given by Theorem 6 to define the non-linear operator S on $\mathcal{B}$ as

$$S(q)(m,t) = \begin{cases} g(\beta_q(0; m, t)) - f(\alpha_q(0; m, t)), & (m,t) \in \mathcal{R}_1 \\ 2\dot{R}_1^q(\beta_q^{-1}(0; m, t)) - f(\alpha_q(0; 0, \beta_q^{-1}(0; m, t)) & \\ -f(\alpha_q(0; m, t)), & (m,t) \in \mathcal{R}_2 \\ -2\dot{R}_2^q(\alpha_q^{-1}(M; m, t)) + g(\beta_q(0; M, \alpha_q^{-1}(M; m, t))) & \\ +g(\beta_q(0; m, t)) & (m,t) \in \mathcal{R}_3. \end{cases} \tag{2.60}$$

We will show that $S : \mathcal{B} \to \mathcal{B}$ is a contraction (in the C^0 norm), for appropriate choice of T, ϵ, δ, and δ'. This will require use of the fact that the characteristics defined by (2.41)–(2.42), and their inverses, are Lipshitz with respect to m, t, and q, with bounds on the Lipshitz constants given by the following lemma.

First, we define a new norm.

Definition 7. Let $\hat{h}$ be a continuous, real valued function on an interval containing q_0. For $\sigma > 0$, we define

$$||\hat{h}||_\sigma = \sup_{|q-q_0|\leq\sigma} |\hat{h}(q)|.$$

Lemma 8. *Let $q \in \mathcal{B}$ and suppose α_q and β_q are solutions of* (2.41)–(2.42)*. Then for $(m,t) \in X$, $\tau \in [0,T]$, $\alpha_q(\tau; m, t)$ and $\beta_q(\tau; m, t)$ are Lipshitz with respect to m,* *t, and q, with their respective Lipshitz constants satisfying the following bounds:*

$$\begin{aligned} Lip_m(\alpha_q),\ Lip_m(\beta_q) &\leq e^{Lip(\hat{c})LT} \\ Lip_t(\alpha_q),\ Lip_t(\beta_q) &\leq ||\hat{c}||_\epsilon e^{Lip(\hat{c})LT} \\ Lip_q(\alpha_q),\ Lip_q(\beta_q) &\leq T Lip(\hat{c}) e^{Lip(\hat{c})LT}. \end{aligned}$$

Furthermore, for $\xi \in [0, M]$, $\alpha_q^{-1}(\xi; m, t)$ and $\beta_q^{-1}(\xi; m, t)$ are Lipshitz with respect to m, t, and q, with respective Lipshitz constants satisfying

$$Lip_m(\alpha_q^{-1}),\ Lip_m(\beta_q^{-1}) \leq ||1/\hat{c}||_\epsilon e^{Lip(1/\hat{c})JM}$$

$$Lip_t(\alpha_q^{-1}),\ Lip_t(\beta_q^{-1}) \leq e^{Lip(1/\hat{c})JM}$$

$$Lip_q(\alpha_q^{-1}),\ Lip_q(\beta_q^{-1}) \leq M Lip(1/\hat{c}) e^{Lip(1/\hat{c})JM}.$$

Proof. We will prove the results for the α_q characteristics. The proof for the β_q characteristics is similar.

For $q \in \mathcal{B}$, $(m, t) \in X$ and $\tau \in [0, T]$, integrating (2.41) from t to τ yields

$$\alpha_q(\tau; m, t) = m - \int_t^\tau \hat{c}(q(\alpha_q(s; m, t), s))\, ds. \tag{2.61}$$

For $m_1, m_2 \in [0, M]$, $t_1, t_2 \in [0, T]$, $q_1, q_2 \in \mathcal{B}$, (2.61) yields, respectively,

$$\begin{aligned} |\alpha_q(\tau; m_1, t) - \alpha_q(\tau; m_2, t)| &\leq |m_1 - m_2| \\ &\quad + Lip(\hat{c}) L \left| \int_t^\tau |\alpha_q(s; m_1, t) - \alpha_q(s, m_2, t)|\, ds \right| \end{aligned}$$

$$\begin{aligned} |\alpha_q(\tau; m, t_1) - \alpha_q(\tau; m, t_2)| &\leq ||\hat{c}||_\epsilon |t_1 - t_2| \\ &\quad + Lip(\hat{c}) L \left| \int_{t_2}^\tau |\alpha_q(s; m, t_1) - \alpha_q(s, m, t_2)|\, ds \right| \end{aligned}$$

$$\begin{aligned}|\alpha_{q_1}(\tau;m,t)-\alpha_{q_2}(\tau;m,t)| &\leq Lip(\hat{c})|\tau-t|\cdot||q_1-q_2|| \\ &\quad +Lip(\hat{c})L\left|\int_t^\tau |\alpha_q(s;m_1,t)-\alpha_q(s,m_2,t)|\,ds\right|.\end{aligned}$$

In case $\tau \geq t$, or $\tau \geq t_2$, the absolute values outside of the integrals in the above inequalities may be removed, allowing for the application of Gronwall's inequality (see [12], for example), which gives, respectively,

$$\begin{aligned}|\alpha_q(\tau;m_1,t)-\alpha_q(\tau;m_2,t)| &\leq |m_1-m_2|e^{Lip(\hat{c})L|\tau-t|} \\ |\alpha_q(\tau;m,t_1)-\alpha_q(\tau;m,t_2)| &\leq ||\hat{c}||_\epsilon|t_1-t_2|e^{Lip(\hat{c})L|\tau-t_2|} \\ |\alpha_{q_1}(\tau;m,t)-\alpha_{q_2}(\tau;m,t)| &\leq Lip(\hat{c})|\tau-t|\cdot||q_1-q_2||e^{Lip(\hat{c})L|\tau-t|}.\end{aligned} \tag{2.62}$$

In case $\tau < t$ or $\tau < t_2$, we must reverse the flow before we can apply Gronwall's inequality. Define $\phi_q(\tau;m,t) := \alpha_q(-\tau;m,t)$ for $\tau \in [-T,0]$. Then by (2.41) we have

$$\frac{d\phi_q}{d\tau} = \hat{c}(q(\phi_q,-\tau)), \quad \phi_q(-t;m,t) = m.$$

Integration from $\tau = -t$ to $\tau = \sigma$, for $\sigma \in [-T,0]$, yields

$$\phi_q(\sigma;m,t) = m + \int_{-t}^{\sigma} \hat{c}(q(\phi_q(s;m,t),-s)\,ds. \tag{2.63}$$

This implies the corresponding inequalities,

$$\begin{aligned}|\phi_q(\sigma;m_1,t)-\phi_q(\sigma;m_2,t)| &\leq |m_1-m_2| \\ &\quad +Lip(\hat{c})L\left|\int_{-t}^{\sigma} |\phi_q(s;m_1,t)-\phi_q(s;m_2,t)|\,ds\right|\end{aligned}$$

$$\begin{aligned}|\phi_q(\sigma;m,t_1)-\phi_q(\sigma;m,t_2)| &\leq ||\hat{c}||_\epsilon|t_1-t_2| \\ &\quad +Lip(\hat{c})L\left|\int_{-t_2}^{\sigma} |\phi_q(s;m,t_1)-\phi_q(s;m,t_2)|\,ds\right|\end{aligned}$$

$$
\begin{aligned}
|\phi_{q_1}(\sigma;m,t)-\phi_{q_2}(\sigma;m,t)| \;&\le\; Lip(\hat{c})|\sigma+t|\cdot||q_1-q_2|| \\
&\qquad +Lip(\hat{c})L\left|\int_{-t}^{\sigma}|\phi_{q_1}(s;m,t)-\phi_{q_2}(s;m,t)|\,ds\right|.
\end{aligned}
$$

In case $\sigma > -t$ or $\sigma > -t_2$, the absolute values outside of the integrals may be removed and Gronwall's inequality yields

$$
\begin{aligned}
|\phi_q(\sigma;m_1,t)-\phi_q(\sigma;m_2,t)| \;&\le\; |m_1-m_2|e^{Lip(\hat{c})L|\sigma+t|} \\
|\phi_q(\sigma;m,t_1)-\phi_q(\sigma;m,t_2)| \;&\le\; ||\hat{c}||_\epsilon|t_1-t_2|e^{Lip(\hat{c})L|\sigma+t_2|} \\
|\phi_{q_1}(\sigma;m,t)-\phi_{q_2}(\sigma;m,t)| \;&\le\; Lip(\hat{c})|\sigma+t|\cdot||q_1-q_2||e^{Lip(\hat{c})L|\sigma+t|}.
\end{aligned}
$$

We can now reverse the flow back again by setting $\tau=-\sigma$, so that for $\tau<t$ or $\tau<t_2$, we again obtain the inequalities in (2.62). Taking the supremum over all $\tau\in[0,T]$ and $m\in[0,M]$, $t\in[0,T]$, and $q\in\mathcal{B}$, in the inequalities in (2.62), yields the desired bounds for $Lip_m(\alpha_q)$, $Lip_t(\alpha_q)$, and $Lip_q(\alpha_q)$. The proof for $Lip_m(\beta_q)$, $Lip_t(\beta_q)$, and $Lip_q(\beta_q)$ is similar.

The inverse characteristics α_q^{-1} and β_q^{-1} defined by (2.47) satisfy the inverse characteristic ODEs

$$
\frac{d\alpha_q^{-1}}{d\xi} = -(1/\hat{c})(q(\xi,\alpha_q^{-1})), \quad \alpha_q^{-1}(m;m,t)=t \tag{2.64}
$$
$$
\frac{d\beta_q^{-1}}{d\xi} = (1/\hat{c})(q(\xi,\beta_q^{-1})), \quad \beta_q^{-1}(m;m,t)=t. \tag{2.65}
$$

Integrating (2.64) from $\xi=m$ to $\xi=\lambda$, for $\lambda\in[0,M]$, yields

$$
\alpha_q^{-1}(\lambda;m,t)=t-\int_m^{\lambda}(1/\hat{c})(q(\xi,\alpha_q^{-1}(\xi;m,t)))\,d\xi, \tag{2.66}
$$

from which we obtain the inequalities

$$\begin{aligned}|\alpha_q^{-1}(\lambda;m_1,t)-\alpha_q^{-1}(\lambda;m_2,t)| &\le ||1/\hat{c}||_\epsilon |m_1-m_2| \\ &\quad +Lip(1/\hat{c})J\left|\int_{m_2}^{\lambda}|\alpha_q^{-1}(\xi;m_1,t)-\alpha_q^{-1}(\xi,m_2,t)|\,d\xi\right|\end{aligned}$$

$$\begin{aligned}|\alpha_q^{-1}(\lambda;m,t_1)-\alpha_q^{-1}(\lambda;m,t_2)| &\le |t_1-t_2| \\ &\quad +Lip(1/\hat{c})J\left|\int_{m}^{\lambda}|\alpha_q^{-1}(\xi;m,t_1)-\alpha_q^{-1}(\xi,m,t_2)|\,d\xi\right|\end{aligned}$$

$$\begin{aligned}|\alpha_{q_1}^{-1}(\lambda;m,t)-\alpha_{q_2}^{-1}(\lambda;m,t)| &\le Lip(1/\hat{c})|\lambda-m|\cdot||q_1-q_2|| \\ &\quad +Lip(1/\hat{c})J\left|\int_{m}^{\lambda}|\alpha_q^{-1}(\xi;m_1,t)-\alpha_q^{-1}(\xi,m_2,t)|\,d\xi\right|.\end{aligned}$$

In case $\lambda \ge m_2$ or $\lambda \ge m$, the absolute value outside the integrals may be removed and Gronwall's inequality yields

$$\begin{aligned}|\alpha_q^{-1}(\lambda;m_1,t)-\alpha_q^{-1}(\lambda;m_2,t)| &\le ||1/\hat{c}||_\epsilon |m_1-m_2| e^{Lip(1/\hat{c})J|\lambda-m_2|} \\ |\alpha_q^{-1}(\lambda;m,t_1)-\alpha_q^{-1}(\lambda;m,t_2)| &\le |t_1-t_2| e^{Lip(1/\hat{c})J|\lambda-m|} \\ |\alpha_{q_1}^{-1}(\lambda;m,t)-\alpha_{q_2}^{-1}(\lambda;m,t)| &\le Lip(1/\hat{c})|\lambda-m|\cdot||q_1-q_2|| e^{Lip(1/\hat{c})J|\lambda-m|}.\end{aligned} \tag{2.67}$$

In case $\lambda < m_2$ or $\lambda < m$, we must again reverse the flow in order to apply Gronwall's inequality. Define $\psi_q(\xi;m,t) := \alpha_q^{-1}(-\xi;m,t)$ for $\xi \in [-M,0]$. Then by (2.64), we have

$$\frac{d\psi_q}{d\xi} = (1/\hat{c})(q(-\xi,\psi_q(\xi;m,t)), \quad \psi_q(-m,m,t)=t.$$

Integration from $\xi = -m$ to $\xi = \gamma$, for $\gamma \in [-M, 0]$, yields

$$\psi_q(\gamma; m, t) = t + \int_{-m}^{\gamma} (1/\hat{c})(q(-\xi, \psi_q(\xi; m, t)))\, d\xi, \tag{2.68}$$

from which we obtain the inequalities

$$\begin{aligned} |\psi_q(\gamma; m_1, t) - \psi_q(\gamma; m_2, t)| \;&\leq\; ||1/\hat{c}||_\epsilon |m_1 - m_2| \\ &\quad + Lip(1/\hat{c}) J \left| \int_{-m_2}^{\gamma} |\psi_q(\xi; m_1, t) - \psi_q(\xi, m_2, t)|\, d\xi \right| \end{aligned}$$

$$\begin{aligned} |\psi_q(\gamma; m, t_1) - \psi_q^{-1}(\gamma; m, t_2)| \;&\leq\; |t_1 - t_2| \\ &\quad + Lip(1/\hat{c}) J \left| \int_{-m}^{\gamma} |\psi_q(\xi; m, t_1) - \psi_q(\xi, m, t_2)|\, d\xi \right| \end{aligned}$$

$$\begin{aligned} |\psi_{q_1}(\gamma; m, t) - \psi_{q_2}(\gamma; m, t)| \;&\leq\; Lip(1/\hat{c}) |\gamma + m| \cdot ||q_1 - q_2|| \\ &\quad + Lip(1/\hat{c}) J \left| \int_{-m}^{\gamma} |\psi_{q_1}(\xi; m, t) - \alpha_{q_2}^{-1}(\xi, m, t)|\, d\xi \right|. \end{aligned}$$

In case $\gamma > -m_2$ or $\gamma > -m$, the absolute values outside of the integrals may be removed and Gronwall's inequality gives

$$\begin{aligned} |\psi_q(\gamma; m_1, t) - \psi_q(\gamma; m_2, t)| \;&\leq\; ||1/\hat{c}||_\epsilon |m_1 - m_2| e^{Lip(1/\hat{c}) J |\gamma + m_2|} \\ |\psi_q(\gamma; m, t_1) - \psi_q(\gamma; m, t_2)| \;&\leq\; |t_1 - t_2| e^{Lip(1/\hat{c}) J |\gamma + m|} \\ |\psi_{q_1}(\gamma; m, t) - \psi_{q_2}(\gamma; m, t)| \;&\leq\; Lip(1/\hat{c}) |\gamma + m| \cdot ||q_1 - q_2|| e^{Lip(1/\hat{c}) J |\gamma + m|}. \end{aligned}$$

Reversing the flow back again by letting $\gamma = -\lambda$, for $\lambda < m_2$ or $\lambda < m$, respectively, we again obtain the inequalities in (2.67). Taking the supremum over all $\lambda \in [0, M]$

and $m \in [0, M]$, $t \in [0, T]$, and $q \in \mathcal{B}$, in the inequalities in (2.67), yields the desired bounds for $Lip_m(\alpha_q^{-1})$, $Lip_t(\alpha_q^{-1})$, and $Lip_q(\alpha_q^{-1})$. The proof for $Lip_m(\beta_q^{-1})$, $Lip_t(\beta_q^{-1})$, and $Lip_q(\beta_q^{-1})$ is similar. $\square$

We will also need bounds for the Lipshitz constants of the piston velocities $\dot{R}_i^q(t)$, both with respect to q and t, where the displacements $R_i^q(t)$ are as in Theorem 6. The former is straightforward; the latter requires more work. To this end, we first give a bound for $\dot{R}_i^q(t)$.

Lemma 9. *Let $q \in \mathcal{B}$ and $R_i^q(t)$ for $i = 1, 2$ be as in Theorem 6. Then for $t \in [0, T]$,*

$$|\dot{R}_i^q(t)| \leq C_{1_i}\delta + C_{2_i}T, \quad i = 1, 2,$$

where $C_{1_i} = \frac{1}{2}\left(\frac{\nu_1}{2\omega_1} + 1\right)$ and

$$C_{2_i} = \left(\frac{\nu_1^2}{4} + \omega_1^2\right)\left(\frac{M_i}{2K_i}||\hat{c}||_{\delta_1}\delta_1' + \frac{\mu_i}{2K_i}\delta_1 + \frac{A}{K_i}||\hat{P}||_{\delta_1}\right) + \frac{A}{M_i}\left(\frac{\nu_i}{2\omega_i} + 1\right)||\hat{P}||_{\epsilon_1}.$$

Proof. The result follows directly from (2.45)-(2.46), in which $R_i(0)$ and $\dot{R}_i(0)$ are expressed in terms of f, g, f' and g' through the compatibility conditions (2.37)–(2.40).

We show the case $i = 1$. The case $i = 2$ is similar. From (2.45), we have

$$|\dot{R}_1^q(t)| \leq \left(\frac{\nu_1^2}{4} + \omega_1^2\right)|R_1^q(0)|t + \left(\frac{\nu_1}{2\omega_1} + 1\right)|\dot{R}_1^q(0)| + \frac{A}{M_1}\left(\frac{\nu_1}{2\omega_1} + 1\right)||\hat{P}||_{\epsilon_1}t. \quad (2.69)$$

From the compatibility conditions (2.37) and (2.39) and the bounds on f, g, f', and g' given by (2.58)–(2.59), we have

$$|R_1^q(0)| \leq \frac{M_1}{2K_1}||\hat{c}||_\delta\delta' + \frac{\mu_1}{2K_1}\delta + \frac{A}{K_1}||\hat{P}||_\delta, \qquad |\dot{R}_1^q(0)| \leq \frac{1}{2}\delta. \quad (2.70)$$

Combining the inequalities (2.69)–(2.70) yields the desired bound on $|\dot{R}_1^q(t)|$. $\square$

We can use this estimate to give a bound for the piston accelerations.

Lemma 10. *Let $q \in \mathcal{B}$ and $R_i^q(t)$ for $i = 1, 2$ be as in Theorem 6. Then for $t \in [0, T]$,*

$$|\ddot{R}_i^q(t)| \leq C_{4_i}\delta + C_{5_i}\delta' + C_{6_i}T + C_{7_i}\epsilon, \quad i = 1, 2,$$

where

$$C_{4_i} = \frac{\mu_i}{M_i}\left(C_{1_i} + \frac{1}{2}\right), \quad C_{5_i} = \frac{1}{2}||\hat{c}||_{\delta_1}, \quad C_{7_i} = \frac{2A}{M_i}Lip(\hat{P}),$$

$$\begin{aligned} C_{6_i} &= \left(\nu_i + \frac{\omega_i^2}{2}t_1\right)\left(\frac{1}{2}||\hat{c}||_{\delta_1}\delta_1' + \frac{\mu_i}{2M_i}\delta_1 + \frac{A}{M_i}||\hat{P}||_{\delta_1}\right) \\ &\quad + \frac{K_i}{2M_i}\delta_1 + \frac{AK_i}{2M_i^2}t_1||\hat{P}||_{\epsilon_1} + \frac{\mu_i}{M_i}C_{2_i} \end{aligned}$$

Proof. We show the case $i = 1$. The case $i = 2$ is similar. From the BC (2.33), we have

$$\begin{aligned} |\ddot{R}_1^q(t)| &= \left|-\frac{\mu_i}{M_1}\dot{R}_1^q(t) - \frac{K_1}{M_1}R_1^q(t) - \frac{A}{M_1}\hat{P}(q(0,t))\right| \\ &\leq \frac{\mu_1}{M_1}|\dot{R}_1^q(t)| + \frac{K_1}{M_1}|R_1^q(t) - R_1^q(0)| + \left|\frac{K_1}{M_1}R_1^q(0) + \frac{A}{M_1}\hat{P}(q(0,0))\right| \\ &\quad + \frac{A}{M_1}|\hat{P}(q(0,t)) - \hat{P}(q(0,0))|. \end{aligned} \tag{2.71}$$

We already have a bound for $|\dot{R}_1^q(t)|$ from Lemma 9. Using the fact that $\sin x \leq x$, we note that

$$\begin{aligned} \left|\int_0^t e^{\frac{\nu_1}{2}(t-s)}\frac{1}{\omega_1}\sin\omega_1(t-s)\hat{P}(q(0,t-s))\,ds\right| &\leq ||\hat{P}||_{\epsilon_1}\int_0^t (t-s)\,ds \\ &= \frac{1}{2}||\hat{P}||_{\epsilon_1}t^2 \\ &\leq \frac{1}{2}||\hat{P}||_{\epsilon_1}t_1 t. \end{aligned}$$

Thus by (2.43), we have

$$\begin{aligned} |R_1^q(t) - R_1^q(0)| &\leq \left|e^{-\frac{\nu_1}{2}t}\cos\omega_1 t - 1\right||R_1^q(0)| \\ &\quad + \left(\frac{\nu_1}{2}|R_1^q(0)| + |\dot{R}_1^q(0)| + \frac{A}{2M_1}||\hat{P}||_{\epsilon_1}t_1\right)t. \end{aligned} \tag{2.72}$$

Note that

$$\begin{aligned}\left|e^{-\frac{\nu_1}{2}t}\cos\omega_1 t - 1\right| &\leq \left|e^{-\frac{\nu_1}{2}t} - 1\right| + |\cos\omega_1 t - 1| \\ &\leq \frac{\nu_1}{2}t + \frac{\omega_1^2}{2}t^2, \end{aligned} \tag{2.73}$$

where the last inequality comes from estimating the error in approximating $e^{-\frac{\nu_1}{2}t}$ and $\cos\omega_1 t$ with the first term in their Maclaurin series expansion (using the alternating series error estimate). Combining the inequalities (2.70) and (2.72), we have

$$|R_1^q(t) - R_1^q(0)| \leq C_{3_1}t, \tag{2.74}$$

where

$$C_{3_i} = \left(\nu_i + \frac{\omega_i^2}{2}t_1\right)\left(\frac{M_i}{2K_i}||\hat{c}||_{\delta_1}\delta_1' + \frac{\mu_i}{2K_i}\delta_1 + \frac{A}{K_i}||\hat{P}||_{\delta_1}\right) + \frac{1}{2}\delta_1 + \frac{A}{2M_i}||\hat{P}||_{\epsilon_1}t_1. \tag{2.75}$$

Recalling that $q(0,0) = g(0) - f(0)$, the compatibility condition (2.39) combined with the estimates (2.58)–(2.59) gives

$$\begin{aligned}\left|\frac{K_1}{M_1}R_1^q(0) + \frac{A}{M_1}\hat{P}(q(0,0))\right| &\leq \left|\frac{1}{2}\hat{c}(q(0,0))(g'(0) - f'(0)) - \frac{\mu_1}{2M_1}(f(0) + g(0))\right| \\ &\leq \frac{1}{2}||\hat{c}||_{\delta_1}\delta' + \frac{\mu_1}{2M_1}\delta. \end{aligned} \tag{2.76}$$

We also have

$$|\hat{P}(q(0,t)) - \hat{P}(q(0,0))| \leq 2Lip(\hat{P})\epsilon. \tag{2.77}$$

Finally, combining the result of Lemma 9 and the estimates (2.74), (2.76), and (2.77) with the estimate (2.71) yields the required estimate of $|\ddot{R}_1^q(t)|$. □

Now we can give bounds for the Lipshitz constants of the piston velocities.

Lemma 11. *Let $q \in \mathcal{B}$ and $R_i^q(t)$ for $i = 1, 2$ be as in Theorem 6. Then for $t \in [0, T]$, $\dot{R}_i^q(t)$ is Lipshitz both with respect to q and t, and the respective Lipshitz constants* *satisfy the following bounds:*

$$Lip_q(\dot{R}_i^q) \leq \left(\frac{\nu_i}{2\omega_i} + 1\right) \frac{A}{M_i} Lip(\hat{P})T, \quad i = 1, 2$$

$$Lip_t(\dot{R}_i^q) \leq C_{4_i}\delta + C_{5_i}\delta' + C_{6_i}T + C_{7_i}\epsilon, \quad i = 1, 2.$$

Proof. The first estimate uses (2.45)–(2.46). The second estimate follows directly from Lemma 10. $\square$

We are now prepared to show $S : \mathcal{B} \to \mathcal{B}$ is a contraction for δ, δ', ϵ and T small enough. We first provide (in the next three propositions) sufficient conditions for S to map $\mathcal{B}$ to itself.

Proposition 12. *Let $q \in \mathcal{B}$ and S the operator on $\mathcal{B}$ defined by* (2.60). *Suppose $\delta \leq \frac{1}{2}(2C_{1_i} + 1)^{-1}\epsilon$ and $T \leq \frac{1}{4C_{2_i}}\epsilon$. Then*

$$\sup_{(m,t)\in X} |S(q)(m,t) - q_0| \leq \epsilon.$$

Proof. For $(m, t) \in \mathcal{R}_1$, (2.60), with the condition (2.58) and the hypothesis on δ, yields

$$\begin{aligned} |S(q)(m,t) - q_0| &= |g(\beta_q(0; m,t)) - f(\alpha_q(0; m,t)) - q_0| \\ &\leq \sup_{\xi\in[0,M]} \left| g(\xi) - \frac{1}{2}q_0 \right| + \sup_{\xi\in[0,M]} \left| f(\xi) + \frac{1}{2}q_0 \right| \\ &\leq \delta \\ &\leq \epsilon. \end{aligned}$$

For $(m, t) \in \mathcal{R}_2$, (2.60), with the condition (2.58), Lemma 9 (noting that $\beta_q^{-1}(0; m, t) \in$

$[0,T]$), and the hypotheses on δ and T, yields

$$\begin{aligned}
|S(q)(m,t)-q_0| &= |2\dot{R}_1^q(\beta_q^{-1}(0;m,t)) - f(\alpha_q(0;0,\beta^{-1}(0;m,t))) \\
&\quad -f(\alpha_q(0;m,t)) - q_0| \\
&\leq 2|\dot{R}_1^q(\beta_q^{-1}(0;m,t))| + \delta \\
&\leq 2(C_{1_1}\delta + C_{2_1}T) + \delta \\
&= (2C_{1_1}+1)\delta + 2C_{2_1}T \\
&\leq \epsilon.
\end{aligned}$$

The case $(m,t) \in \mathcal{R}_3$ is similar to the case $(m,t) \in \mathcal{R}_2$. Since $X \subset \cup_{i=1}^3 \mathcal{R}_i$, we thus have

$$\sup_{(m,t)\in X} |S(q)(m,t) - q_0| \leq \epsilon.$$

□

Proposition 13. *Let $q \in \mathcal{B}$ and S be the operator on $\mathcal{B}$ defined by* (2.60)*. Suppose $\epsilon \leq \frac{1}{12C_{8_i}}L$, $\delta \leq \frac{1}{2}\epsilon$, $\delta' \leq \frac{1}{12C_{9_i}}L$ and $T \leq \frac{1}{12C_{10_i}}L$, for $i = 1,2$, where*

$$C_{8_i} = (C_{4_i} + 2C_{7_i})||1/\hat{c}||_{\epsilon_1} e^{Lip(1/\hat{c})JM}$$

$$C_{9_i} = 2C_{5_i}||1/\hat{c}||_{\epsilon_1} e^{Lip(1/\hat{c})JM} + \frac{1}{2}||\hat{c}||_{\epsilon_1}||1/\hat{c}||_{\epsilon_1} e^{Lip(\hat{c})Lt_1 + Lip(1/\hat{c})JM} + \frac{1}{2}e^{Lip(\hat{c})Lt_1}$$

$$C_{10_i} = 2C_{6_i}||1/\hat{c}||_{\epsilon_1} e^{Lip(1/\hat{c})JM}.$$

Then $Lip_m(S(q)) \leq L$.

Proof. For $(m_1,t),(m_2,t) \in \mathcal{R}_1$, (2.60) with the condition (2.59), Lemma 8, and the hypothesis on δ' yields

$$\begin{aligned}
|S(q)(m_1,t) - S(q)(m_2,t)| &= |g(\beta_q(0;m_1,t)) - f(\alpha_q(0;m_1,t)) \\
&\quad -g(\beta_q(0;m_2,t)) + f(\alpha_q(0;m_2,t))| \\
&\leq (||g'||Lip_m(\beta_q) + ||f'||Lip_m(\alpha_q))|m_1 - m_2| \\
&\leq \delta' e^{Lip(\hat{c})LT}|m_1 - m_2| \\
&\leq \frac{1}{4}L|m_1 - m_2|.
\end{aligned}$$

For $(m_1,t),(m_2,t) \in \mathcal{R}_2$, (2.60) with the condition (2.59), Lemma 8, Lemma 9, and the hypotheses on ϵ, δ, δ', and T give the following estimate on $\Delta^t_{m_1,m_2} := S(q)(m_1,t) - S(q)(m_2,t)$,

$$\begin{aligned}
|\Delta^t_{m_1,m_2}| &\leq |2\dot{R}^q_1(\beta_q^{-1}(0;m_1,t)) - 2\dot{R}^q_1(\beta_q^{-1}(0;m_2,t))| \\
&\quad +|f(\alpha_q(0;0,\beta_q^{-1}(0;m_1,t))) - f(\alpha_q(0;0,\beta_q^{-1}(0;m_2,t)))| \\
&\quad +|f(\alpha_q(0;m_1,t)) - f(\alpha_q(0;m_2,t))| \\
&\leq 2Lip_t(\dot{R}^q_1)Lip_m(\beta_q^{-1})|m_1 - m_2| \\
&\quad +(||f'||Lip_t(\alpha_q)Lip_m(\beta_q^{-1}) + ||f'||Lip_m(\alpha_q))|m_1 - m_2| \\
&\leq 2(C_{4_1}\delta + C_{5_1}\delta' + C_{6_1}T + C_{7_1}\epsilon)||1/\hat{c}||_{\epsilon_1} e^{Lip(1/\hat{c})JM}|m_1 - m_2| \\
&\quad +|\frac{1}{2}\delta'(||\hat{c}||_{\epsilon_1} e^{Lip(\hat{c})LT}||1/\hat{c}||_{\epsilon_1} e^{Lip(1/\hat{c})JM} + e^{Lip(\hat{c})LT})|m_1 - m_2| \\
&\leq (C_{8_1}\epsilon + C_{9_1}\delta' + C_{10_1}T)|m_1 - m_2| \\
&\leq \frac{1}{4}L|m_1 - m_2|.
\end{aligned}$$

Similarly, we have $|\Delta^t_{m_1,m_2}| \leq \frac{1}{4}L|m_1 - m_2|$ for $(m_1,t),(m_2,t) \in \mathcal{R}_3$.

Recalling that $X \subset \cup_{i=1}^3 \mathcal{R}_i$, we have that any two points $(m_1,t),(m_2,t) \in X$ are separated at most by one region, so by the triangle inequality it is clear that

$|\Delta^t_{m_1,m_2}| \leq L|m_1 - m_2|.$ $\square$

Proposition 14. *Let $q \in \mathcal{B}$ and S be the operator on $\mathcal{B}$ defined by* (2.60)*. Suppose $\epsilon \leq \frac{1}{6C_{11_i}}J$, $\delta \leq \frac{1}{2}\epsilon$, $\delta' \leq \frac{1}{6C_{12_i}}J$, and $T \leq \frac{1}{6C_{13_i}}J$, for $i = 1, 2$, where*

$$C_{11_i} = (2C_{7_i} + C_{4_i})e^{Lip(1/\hat{c})JM}$$

$$C_{12_i} = 2C_{5_1}e^{Lip(1/\hat{c})JM} + \frac{1}{2}||\hat{c}||_{\epsilon_1}e^{Lip(\hat{c})LT}(e^{Lip(1/\hat{c})JM} + 1)$$

$$C_{13_i} = 2C_{6_i}e^{Lip(1/\hat{c})JM}.$$

Then $Lip_t(S(q)) \leq J$.

Proof. For $(m, t_1), (m, t_2) \in \mathcal{R}_1$, (2.60) with the condition (2.59), Lemma 8, and the hypothesis on δ' yields

$$\begin{aligned}
|S(q)(m,t_1) - S(q)(m,t_2)| &= |g(\beta_q(0;m,t_1)) - f(\alpha_q(0;m,t_1)) \\
&\quad -g(\beta_q(0;m,t_2)) + f(\alpha_q(0;m_2,t))| \\
&\leq (||g'||Lip_t(\beta_q) + ||f'||Lip_t(\alpha_q))|t_1 - t_2| \\
&\leq \delta'||\hat{c}||_{\epsilon_1}e^{Lip(\hat{c})LT}|t_1 - t_2| \\
&\leq \frac{1}{2}J|t_1 - t_2|.
\end{aligned}$$

For $(m, t_1), (m, t_2) \in \mathcal{R}_2$, (2.60) with the condition (2.59), Lemma 8, Lemma 11, and the hypotheses on ϵ, δ, δ', and T give the following estimate on $\Delta^m_{t_1,t_2} :=$

$S(q)(m,t_1) - S(q)(m,t_2)$,

$$
\begin{aligned}
|\Delta^m_{t_1,t_2}| &\le |2\dot{R}^q_1(\beta_q^{-1}(0;m,t_1)) - 2\dot{R}^q_1(\beta_q^{-1}(0;m,t_2))| \\
&\quad + |f(\alpha_q(0;0,\beta_q^{-1}(0;m,t_1))) - f(\alpha_q(0;0,\beta_q^{-1}(0;m,t_2)))| \\
&\quad + |f(\alpha_q(0;m,t_1)) - f(\alpha_q(0;m,t_2))| \\
&\le 2Lip_t(\dot{R}^q_1)Lip_t(\beta_q^{-1})|t_1-t_2| \\
&\quad + (||f'||Lip_t(\alpha_q)Lip_t(\beta_q^{-1}) + ||f'||Lip_t(\alpha_q))|t_1-t_2| \\
&\le 2(C_{4_1}\delta + C_{5_1}\delta' + C_{6_1}T + C_{7_1}\epsilon)e^{Lip(1/\hat{c})JM}|t_1-t_2| \\
&\quad + \frac{1}{2}\delta'||\hat{c}||_{\epsilon_1}e^{Lip(\hat{c})LT}(e^{Lip(1/\hat{c})JM}+1)|t_1-t_2| \\
&\le (C_{11_1}\epsilon + C_{12_1}\delta' + C_{13_1}T)|t_1-t_2| \\
&\le \frac{1}{2}J|t_1-t_2|.
\end{aligned}
$$

Recalling that $X \subset \cup_{i=1}^3 \mathcal{R}_i$, we have that any two points $(m,t_1),(m,t_2) \in X$ must either be in the same region or adjacent regions. So by the triangle inequality it is clear that $|\Delta^m_{t_1,t_2}| \le J|m_1 - m_2|$. $\square$

Propositions 12–14 show that the operator S maps $\mathcal{B}$ to itself (for the appropriate choice of ϵ, δ, δ', and T). We next provide sufficient conditions for S to be a contraction mapping.

Proposition 15. *Suppose* $\epsilon \le \frac{1}{4C_{14_i}}$, $\delta \le \frac{1}{2}\epsilon$, $\delta' \le \frac{1}{4C_{15_i}}$, *and* $T \le \frac{1}{4C_{16_i}}$, *for* $i = 1,2$, *where*

$$C_{14_i} = (C_{4_i} + 2C_{7_i})MLip(1/\hat{c})e^{Lip(1/\hat{c})JM}$$

$$C_{15_i} = (2C_{5_i} + \frac{1}{2}||\hat{c}||_{\epsilon_1}e^{Lip(\hat{c})Lt_1})MLip(1/\hat{c})e^{Lip(1/\hat{c})JM}$$

$$C_{16_i} = 2C_{6_i} + 2(\frac{\nu_i}{2\omega_i}+1)\frac{A}{M_i}Lip(\hat{P}) + \delta_1'Lip(\hat{c})e^{Lip(\hat{c})Lt_1}.$$

Then for every $q_1, q_2 \in \mathcal{B}$,

$$||S(q_1) - S(q_2)|| \leq \frac{3}{4}||q_1 - q_2||,$$

where S *is the operator on* $\mathcal{B}$ *defined by* (2.60).

Proof. Let $q_1, q_2 \in \mathcal{B}$.

For $(m,t) \in \mathcal{R}_1$, (2.60) with the condition (2.59), Lemma 8, and the hypotheses on δ' and T yields

$$\begin{aligned}
|S(q_1)(m,t) - S(q_2)(m,t)| &= |g(\beta_{q_1}(0;m,t)) - f(\alpha_{q_1}(0;m,t)) \\
&\quad -g(\beta_{q_2}(0;m,t)) + f(\alpha_{q_2}(0;m,t))| \\
&\leq (||g'||Lip_q(\beta) + ||f'||Lip_q(\alpha))|q_1(m,t) - q_2(m,t)| \\
&\leq \frac{1}{2}\delta' T Lip(\hat{c}) e^{Lip(\hat{c})LT}|q_1(m,t) - q_2(m,t)| \\
&\leq 2C_{16_1}T|q_1(m,t) - q_2(m,t)| \\
&\leq \frac{1}{2}|q_1(m,t) - q_2(m,t)| \\
&\leq \frac{1}{2}||q_1 - q_2||.
\end{aligned}$$

For $(m,t) \in \mathcal{R}_2$, (2.60) with the condition (2.59), Lemma 8, Lemma 11, and the hypotheses on ϵ, δ, δ', and T give the following estimate on $\Delta^{m,t}_{q_1,q_2} := S(q_1)(m,t) -$

$S(q_2)(m,t)$,

$$\begin{aligned}
|\Delta^{m,t}_{q_1,q_2}| &\leq |2\dot{R}_1^{q_1}(\beta_{q_1}^{-1}(0;m,t)) - 2\dot{R}_1^{q_2}(\beta_{q_2}^{-1}(0;m,t))| \\
&\quad +|f(\alpha_{q_1}(0;0,\beta_{q_1}^{-1}(0;m,t))) - f(\alpha_{q_2}(0;0,\beta_{q_2}^{-1}(0;m,t)))| \\
&\quad +|f(\alpha_{q_1}(0;m,t)) - f(\alpha_{q_2}(0;m,t))| \\
&\leq 2|\dot{R}_1^{q_1}(\beta_{q_1}^{-1}(0;m,t)) - \dot{R}_1^{q_1}(\beta_{q_2}^{-1}(0;m,t))| \\
&\quad +2|\dot{R}_1^{q_1}(\beta_{q_2}^{-1}(0;m,t)) - \dot{R}_1^{q_2}(\beta_{q_2}^{-1}(0;m,t))| \\
&\quad +|f(\alpha_{q_1}(0;0,\beta_{q_1}^{-1}(0;m,t))) - f(\alpha_{q_1}(0;0,\beta_{q_2}^{-1}(0;m,t)))| \\
&\quad +|f(\alpha_{q_1}(0;0,\beta_{q_2}^{-1}(0;m,t))) - f(\alpha_{q_2}(0;0,\beta_{q_2}^{-1}(0;m,t)))| \\
&\quad +|f(\alpha_{q_1}(0;m,t)) - f(\alpha_{q_2}(0;m,t)| \\
&\leq 2(Lip_t(\dot{R}_1^{q_1})Lip_q(\beta_q^{-1}) + Lip_q(\dot{R}_1^{q}))||q_1 - q_2|| \\
&\quad +||f'||(Lip_t(\alpha_{q_1})Lip_q(\beta_q^{-1}) + 2Lip_q(\alpha_q))||q_1 - q_2|| \\
&\leq 2(C_{4_1}\delta + C_{5_1}\delta' + C_{6_1}T + C_{7_1}\epsilon)MLip(1/\hat{c})e^{Lip(1/\hat{c})JM}||q_1 - q_2|| \\
&\quad +2(\frac{\nu_1}{2\omega_1} + 1)\frac{A}{M_1}Lip(\hat{P})T||q_1 - q_2|| \\
&\quad +\frac{\delta'}{2}e^{Lip(\hat{c})LT}(MLip(1/\hat{c})||\hat{c}||_{\epsilon_1}e^{Lip(1/\hat{c})JM} + 2TLip(\hat{c}))||q_1 - q_2|| \\
&\leq (C_{14_1}\epsilon + C_{15_1}\delta' + C_{16_1}T)||q_1 - q_2|| \\
&\leq \frac{3}{4}||q_1 - q_2||.
\end{aligned}$$

Similarly, for $(m,t) \in \mathcal{R}_3$, we have

$$|S(q_1)(m,t) - S(q_2)(m,t)| \leq \frac{3}{4}||q_1 - q_2||.$$

Since $X \subset \cup_{i=1}^3 \mathcal{R}_i$, we therefore have

$$||S(q_1) - S(q_2)|| \leq \frac{3}{4}||q_1 - q_2||.$$

$\square$

Propositions 12–15 show that S is a contraction which maps $\mathcal{B}$ to itself (for small enough time and initial data f and g close enough to a constant steady state). Thus S has a unique fixed point in $\mathcal{B}$, say q_f. By construction, the solution to the linear IBVP (2.29)–(2.40) given by theorem 6, for $q = q_f$, is in fact a solution of the full nonlinear IBVP (2.17)–(2.28). If $q_f \in C^1$, then the solution, w and z given by (2.52)–(2.53), is classical. In case $q_f \notin C^1$, if w or z is not C^1, then we call the solution a *weak solution*. We summarize this result in the following theorem.

Theorem 16. *Let ϵ_1, δ_1, δ_1', L, and J be fixed positive constants. Let q_0 be the positive constant determined by* (2.57)*, and let $\alpha_{q_0+\epsilon_1}$ and $\beta_{q_0+\epsilon_1}$ be solutions of the characteristic ODEs* (2.41)–(2.42) *for $q = q_0+\epsilon_1$. Let t_1 satisfy $\alpha_{q_0+\epsilon_1}(t_1; M, 0) = \beta_{q_0+\epsilon_1}(t_1; 0, 0)$. Suppose $\epsilon \leq \min\{\frac{L}{12C_{8_i}}, \frac{J}{6C_{13_i}}, \frac{1}{4C_{14_i}}\}$, $\delta \leq \frac{\epsilon}{2(2C_{1_i}+1)}$, $\delta' \leq \min\{\frac{L}{12C_{9_i}}, \frac{J}{6C_{12_i}}, \frac{1}{4C_{15_i}}\}$, and $T \leq \frac{1}{4C_{16_i}}$. Let $f, g \in C^1([0, M], \mathbb{R})$ such that $g - f > 0$, and suppose f and g satisfy* (2.54)–(2.56) *and* (2.58)–(2.59)*. Then the nonlinear IBVP* (2.17)–(2.28) *has a unique solution, w and z, defined on $[0, M] \times [0, T]$ by* (2.52)–(2.53)*, where q is taken to be q_f, the unique fixed point of the operator S defined by* (2.60)*.*

If $q_f \in C^1$, then the solution w and z is classical. In case $q_f \notin C^1$, if w or z is not C^1, then we call the solution a *weak solution* of the IBVP (2.17)–(2.28) .

2.7 Short time existence and uniqueness of classical solutions

We next show that, for initial conditions f and g close enough to the constant steady state and satisfying appropriate compatibility conditions (as in the hypotheses for theorem 16), the nonlinear IBVP (2.17)–(2.28) has a unique classical solution for small enough time. Through iteration of the operator S defined by (2.60), we construct a sequence of C^1 functions $\{q_n\}_{n=1}^{\infty} \subset \mathcal{B}$ which converges uniformly to the function q_f, the fixed point of S. To ensure that q_f itself is C^1, it is enough to show that the sequence of derivatives of q_n (or at least a subsequence) converges uniformly. In fact, it is sufficient to do this for the derivatives with respect to m only, for each fixed t, since it can be seen that, as a fixed point of S, if q_f is C^1 with respect to m, then it must also be C^1 with respect to t. We will show that for each fixed t, the sequence $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$ has a uniformly (in m) convergent subsequence by showing that it is equicontinuous (in m) and uniformly bounded, and invoking the Arzela-Ascoli theorem. As in [17], we employ the notion of modulus of continuity as a device to show equicontinuity.

Definition 17. For a bounded subset $D \in \mathbb{R}$ and a map $\phi : D \to \mathbb{R}$, define the modulus of continuity of ϕ to be the map $\omega(\cdot|\phi) : [0, \infty) \to [0, \infty)$ defined by

$$\omega(\eta|\phi) = \sup\{|\phi(x_1) - \phi(x_2)| : x_1, x_2 \in D, |x_1 - x_2| \leq \eta\}.$$

We will make use of some of the properties of the modulus of continuity given in the following lemmas (see [17]).

Lemma 18. *Suppose f is a real valued function on a bounded domain. Then*

1. *f is continuous if and only if $\lim_{\eta \to 0^+} \omega(\eta|f) = 0$,*

2. *if f is Lipshitz with Lipshitz constant L, then $\omega(\eta|f) \leq L\eta$,*

3. *if f is differentiable, then $\omega(\eta|f) \leq ||f'||\eta$,*

4. *$\omega(\eta_1|f) \leq \omega(\eta_2|f)$ for $0 \leq \eta_1 \leq \eta_2$,*

5. *$\omega(\eta_1 + \eta_2|f) \leq \omega(\eta_1|f) + \omega(\eta_2|f)$,*

6. *for any natural number* n, $\omega(n\eta|f) \leq n\omega(\eta|f)$,

7. *for any positive constant* C, $\omega(C\eta|f) \leq \lceil C \rceil \omega(\eta|f)$, *where* $\lceil C \rceil$ *is the smallest integer greater than or equal to* C.

Lemma 19. *Suppose* f *and* g *are real valued functions on a bounded domain. Then*

1. $\omega(\eta|f \pm g) \leq \omega(\eta|f) + \omega(\eta|g)$,

2. $\omega(\eta|fg) \leq ||f||\omega(\eta|g) + ||g||\omega(\eta|f)$,

3. *if* $|g| \geq a > 0$, *then* $\omega(\eta|f/g) \leq \frac{||f||}{a^2}\omega(\eta|g) + \frac{1}{a}\omega(\eta|f)$,

4. $\omega(\eta|f \circ g) \leq \omega(\omega(\eta|g)|f)$.

The following lemma conveys the utility of the notion of modulus of continuity in showing equicontinuity of a family of functions.

Lemma 20. *Suppose* $\mathcal{F}$ *is a family of real valued functions on a bounded domain. If* $\omega(\eta|f) \leq \Omega(\eta)$ *for each* $f \in \mathcal{F}$, *where* Ω *is a nonnegative function (independent of* f*) such that* $\lim_{\eta \to 0+} \Omega(\eta) = 0$, *then the family of functions* $\mathcal{F}$ *is equicontinuous.*

We construct the sequence $\{q_n\}_{n=1}^{\infty}$ inductively by defining

$$\begin{aligned} q_1(m,t) &= g(m) - f(m) \\ q_{n+1}(m,t) &= S(q_n)(m,t), \quad n \in \mathbb{N}, \end{aligned} \tag{2.78}$$

where the initial data functions f and g, and the operator S, are as in Theorem 16. Clearly $q_1 \in \mathcal{B} \cap C^1([0,M] \times [0,T])$. Now suppose $q_n \in \mathcal{B} \cap C^1([0,M] \times [0,T])$ for some fixed $n \in \mathbb{N}$. Then $q_{n+1} \in \mathcal{B}$, since S maps $\mathcal{B}$ onto itself. Furthermore, the characteristics α_{q_n}, β_{q_n}, and their inverses are C^1 with respect to m and t, as a consequence of the smoothness of q_n with respect to m (and hence that of the vector fields in the characteristic ODEs (2.41)–(2.42)). Therefore by the definition of S, q_{n+1} is C^1 with respect to m and t. Thus, by induction, $\{q_n\}_{n=1}^{\infty} \subset \mathcal{B} \cap C^1([0,M] \times [0,T])$. Because S is a contraction, the sequence $\{q_n\}_{n=1}^{\infty}$ converges uniformly to $q_f \in \mathcal{B}$, the unique fixed point of S. As an element of $\mathcal{B}$, we already know q_f is Lipshitz with respect to both m and t. The task at hand is to show that q_f is C^1 with respect to m.

We will show that for each fixed t, the sequence $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$ has a subsequence which converges uniformly in m. We already know that $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$ is uniformly bounded, since $q_n \in \mathcal{B}$ implies that $Lip_m(q_n) \leq L$, and hence $||\frac{\partial q_n}{\partial m}|| \leq L$. In order to extract a uniformly convergent subsequence by applying the theorem of Arzela-Ascoli, it remains to show that, for fixed t, $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$ is equicontinuous in m.

Applying the notion of modulus of continuity to functions of two independent variables by treating each variable separately, and restricting the domain to the various regions $\mathcal{R}_i$ defined in (2.48)–(2.50) in order to facilitate the estimates that are required to make use of lemma 20, we make the following definition.

Definition 21. Let $X = [0,M] \times [0,T]$. For a function $\psi : X \to \mathbb{R}$, let the modulus of continuity of ψ with respect to m, restricted to region $\mathcal{R}_i$ (for $i = 1,2,3$), be given by

$$\omega_m^{\mathcal{R}_i}(\eta|\psi) := \sup\{|\psi(m_1,t) - \psi(m_2,t)| : (m_1,t),(m_2,t) \in \mathcal{R}_i \cap X, |m_1 - m_2| \leq \eta\}.$$

Similarly, let the modulus of continuity of ψ with respect to t, restricted to region $\mathcal{R}_i$, $i = 1,2,3$, be given by

$$\omega_t^{\mathcal{R}_i}(\eta|\psi) := \sup\{|\psi(m,t_1) - \psi(m,t_2)| : (m,t_1),(m,t_2) \in \mathcal{R}_i \cap X, |t_1 - t_2| \leq \eta\}.$$

We will show equicontinuity (in m) of the sequence $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$ in each region separately by estimating $\omega_m^{\mathcal{R}_i}(\eta|\frac{\partial q_n}{\partial m})$ and applying Lemma 20. But first, it will be useful to have estimates of $\omega_m^X(\eta|\frac{\partial \alpha_q}{\partial m})$ and $\omega_m^X(\eta|\frac{\partial \beta_q}{\partial m})$ in terms of $\omega_m^X(\eta|\frac{\partial q}{\partial m})$, provided by the next lemma.

Lemma 22. *Assume the hypotheses of Theorem 16, and let $q \in \mathcal{B}$ be C^1 with respect to m. Then*

$$\omega_m^X(\eta|\frac{\partial \alpha_q}{\partial m}),\ \omega_m^X(\eta|\frac{\partial \beta_q}{\partial m}) \leq B_2 T \omega_m^X(\eta|\frac{\partial q}{\partial m}) + B_3\eta,$$

where $B_1 = e^{||\hat{c}'||_{\epsilon_1} L t_1}$, $B_2 = ||\hat{c}'||_{\epsilon_1} B_1(B_1+1)$, and $B_3 = L^2 B_1^2 t_1 ||\hat{c}''||_{\epsilon_1}$.

Proof. We will show the result for $\frac{\partial \alpha_q}{\partial m}$. The proof for $\frac{\partial \beta_q}{\partial m}$ is similar. For $q \in \mathcal{B}$ and C^1 in m, the characteristic ODE (2.41) has a nonautonomous C^1 vector field which is continuous in time. Hence the characteristic $\alpha_q(\tau; m, t)$ is C^1 in τ, m, and t. Differentiating (2.41) with respect to m and interchanging the order of the m and τ derivatives yields the following ODE initial value problem for $\frac{\partial \alpha_q}{\partial m}$,

$$\frac{d}{d\tau}\frac{\partial \alpha_q}{\partial m} = -\hat{c}'(q(\alpha_q, \tau))\frac{\partial q}{\partial m}\frac{\partial \alpha_q}{\partial m}, \quad \frac{\partial \alpha_q}{\partial m}(t; m, t) = 1, \tag{2.79}$$

the solution of which is

$$\frac{\partial \alpha_q}{\partial m}(\tau; m, t) = \exp\left(\int_\tau^t \hat{c}'(q(\alpha_q(s; m, t), s))\frac{\partial q}{\partial m}(\alpha_q(s; m, t), s)\, ds\right). \tag{2.80}$$

Noting that Lemma 8 implies $Lip_m(\alpha_q)$, and hence $\frac{\partial \alpha_q}{\partial m}$, is bounded by B_1, we use

Lemmas 18–19 with (2.80) to obtain the following estimate.

$$\begin{aligned}
\omega_m^X(\eta|\frac{\partial \alpha_q}{\partial m}) &\leq e^{||\hat{c}'||_{\epsilon_1} L t_1} T \omega_m^X(\eta|(\hat{c}' \circ q \circ \alpha_q)(\frac{\partial q}{\partial m} \circ \alpha_q)) \\
&\leq B_1 \left(||\hat{c}'||_{\epsilon_1} T \omega_m(\eta|\frac{\partial q}{\partial m} \circ \alpha_q) + TL\omega_m^X(\eta|\hat{c}' \circ q \circ \alpha_q) \right) \\
&\leq B_1 \left(||\hat{c}'||_{\epsilon_1} T \omega_m(B_1\eta|\frac{\partial q}{\partial m}) + t_1 L^2 ||\hat{c}''||_{\epsilon_1} Lip_m(\alpha_q)\eta \right) \\
&\leq B_1 \left(||\hat{c}'||_{\epsilon_1} T(B_1+1)\omega_m(\eta|\frac{\partial q}{\partial m}) + t_1 L^2 ||\hat{c}''||_{\epsilon_1} B_1 \eta \right) \\
&\leq B_2 T \omega_m^X(\eta|\frac{\partial q}{\partial m}) + B_3\eta. \quad \square
\end{aligned}$$

We can now show equicontinuity of $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$ with respect to m in region $\mathcal{R}_1$.

Proposition 23. *Assume the hypotheses of Theorem 16, and let the sequence* $\{q_n\}_{n=1}^{\infty}$ *be as defined by* (2.78)*. Suppose* $T \leq \frac{1}{2\delta' B_2}$*, where* $B_1 = e^{Lip(\hat{c})Lt_1}$ *and* $B_2 = ||\hat{c}'||_{\epsilon_1} B_1(B_1+1)$*. Then for each fixed* $t \in [0,T]$*, the sequence* $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$*, restricted to the domain* $\mathcal{R}_1 \cap X$*, is equicontinuous in* m*.*

Proof. For $(m,t) \in \mathcal{R}_1 \cap X$, by (2.60) and (2.78), we have

$$q_n(m,t) = g(\beta_{q_{n-1}}(0;m,t)) - f(\alpha_{q_{n-1}}(0;m,t)).$$

Differentiating with respect to m gives

$$\frac{\partial q_n}{\partial m}(m,t) = g'(\beta_{q_{n-1}}(0;m,t))\frac{\partial \beta_{q_{n-1}}}{\partial m}(0;m,t) - f'(\alpha_{q_{n-1}}(0;m,t))\frac{\partial \alpha_{q_{n-1}}}{\partial m}(0;m,t).$$

By Lemma 8, whenever they exist, $\frac{\partial \alpha_q}{\partial m}$ and $\frac{\partial \beta_q}{\partial m}$ are uniformly bounded by B_1 for all $q \in \mathcal{B}$. Using Lemmas 8, 18–19, the bounds on f' and g' assumed in Theorem 16, and finally, Lemma 22 (combined with the obvious fact that $\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial q}{\partial m}) \leq \omega_m^X(\eta|\frac{\partial q}{\partial m})$), we have the following estimate.

$$\begin{aligned}
\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial q_n}{\partial m}) &\leq \omega_m^{\mathcal{R}_1}(\eta|(g'\circ\beta_{q_{n-1}})\frac{\partial \beta_{q_{n-1}}}{\partial m}) + \omega_m^{\mathcal{R}_1}(\eta|(f'\circ\alpha_{q_{n-1}})\frac{\partial \alpha_{q_{n-1}}}{\partial m}) \\
&\leq ||g'||\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial \beta_{q_{n-1}}}{\partial m}) + \left|\left|\frac{\partial \beta_{q_{n-1}}}{\partial m}\right|\right|\omega_m^{\mathcal{R}_1}(\eta|g'\circ\beta_{q_{n-1}}) \\
&\quad +||f'||\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial \alpha_{q_{n-1}}}{\partial m}) + \left|\left|\frac{\partial \alpha_{q_{n-1}}}{\partial m}\right|\right|\omega_m^{\mathcal{R}_1}(\eta|f'\circ\alpha_{q_{n-1}}) \\
&\leq \frac{1}{2}\delta'\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial \beta_{q_{n-1}}}{\partial m}) + B_1\omega(\omega_m^{\mathcal{R}_1}(\eta|\beta_{q_{n-1}})|g') \\
&\quad +\frac{1}{2}\delta'\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial \alpha_{q_{n-1}}}{\partial m}) + B_1\omega(\omega_m^{\mathcal{R}_1}(\eta|\alpha_{q_{n-1}})|f') \\
&\leq \frac{1}{2}\delta'\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial \beta_{q_{n-1}}}{\partial m}) + B_1\omega(B_1\eta|g') \\
&\quad +\frac{1}{2}\delta'\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial \alpha_{q_{n-1}}}{\partial m}) + B_1\omega(B_1\eta|f') \\
&\leq \frac{1}{2}\delta'\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial \beta_{q_{n-1}}}{\partial m}) + B_1(B_1+1)\omega(\eta|g') \\
&\quad +\frac{1}{2}\delta'\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial \alpha_{q_{n-1}}}{\partial m}) + B_1(B_1+1)\omega(\eta|f') \\
&\leq \delta' B_2 T\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial q_{n-1}}{\partial m}) + \delta' B_3\eta \\
&\quad +B_1(B_1+1)(\omega(\eta|f')+\omega(\eta|g')). \qquad (2.81)
\end{aligned}$$

We will next use induction on n. Let

$$\Omega_1(\eta) := 2\delta' B_3\eta + 2B_1(B_1+1)(\omega(\eta|f')+\omega(\eta|g')).$$

Since $B_1 \geq 1$, we have, using Lemma 19,

$$\begin{aligned}
\omega_m(\eta|\frac{\partial q_1}{\partial m}) &= \omega_m(\eta|g'-f') \\
&\leq \omega_m(\eta|f')+\omega_m(\eta|g') \\
&\leq \Omega_1(\eta).
\end{aligned}$$

Suppose $\omega_m(\eta|\frac{\partial q_k}{\partial m}) \leq \Omega_1(\eta)$ for some $k \in \mathbb{N}$. Then by the estimate in (2.81), we have

$$\begin{aligned}
\omega_m^{\mathcal{R}_1}(\eta|\frac{\partial q_{k+1}}{\partial m}) &\leq \delta' B_2 T \omega_m^{\mathcal{R}_1}(\eta|\frac{\partial q_k}{\partial m}) + \delta' B_3 \eta + B_1(B_1+1)(\omega(\eta|f') + \omega(\eta|g')) \\
&\leq \delta' B_2 T \Omega_1(\eta) + \frac{1}{2}\Omega_1(\eta) \\
&\leq \Omega_1(\eta),
\end{aligned}$$

for $T \leq \frac{1}{2\delta' B_2}$. Thus, by induction, we have $\omega_m(\eta|\frac{\partial q_n}{\partial m}) \leq \Omega_1(\eta)$ for all $n \in \mathbb{N}$.

Since f' and g' are continuous, Lemma 18 implies $\lim_{\eta\to 0+} \Omega_1(\eta) = 0$. Therefore by Lemma 20, the sequence of functions $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$ restricted to the domain $\mathcal{R}_1 \cap X$ is equicontinuous in m, for each fixed $t \in [0, T]$. □

Before extending this result to regions $\mathcal{R}_2$ and $\mathcal{R}_3$, we present some preliminary estimates in the following lemmas.

Lemma 24. *Assume the hypotheses of Theorem 16. Then for all $q \in \mathcal{B}$,*

$$Lip(\ddot{R}_i^q) \leq B_{4_i}, \quad i = 1, 2,$$

where R_i^q are given by (2.43)–(2.44) *and*

$$B_{4_i} = \frac{K_i}{M_i}(C_{1_i}\delta_1 + C_{2_i}t_1) + \frac{\mu_i}{M_i}(C_{4_i}\delta_1 + C_{5_i}\delta_1' + C_{6_i}t_1 + C_{7_i}\epsilon_1) + \frac{A}{M_i}||\hat{P}'||_{\epsilon_1}.$$

Proof. Since R_i^q satisfy the BCs (2.33)–(2.34), we have

$$|\ddot{R}_i(t)| \leq \frac{K_i}{M_i}|R_i(t)| + \frac{\mu_i}{M_i}|\dot{R}_i(t)| + \frac{A}{M_i}|\hat{P}(q(0,t))|.$$

Therefore

$$Lip(\ddot{R}_i^q) \leq \frac{K_i}{M_i}||\dot{R}_i|| + \frac{\mu_i}{M_i}||\ddot{R}_i|| + \frac{A}{M_i}J||\hat{P}'||_{\epsilon_1}.$$

The result then follows from Lemmas 9–10. □

Lemma 25. *Assume the hypotheses of Theorem 16. Then for all $q \in \mathcal{B}$,*

$$\omega_m^X(\eta|\frac{\partial \alpha_q^{-1}}{\partial m}),\ \omega_m^X(\eta|\frac{\partial \beta_q^{-1}}{\partial m}) \leq B_6 T \omega_m^X(\eta|\frac{\partial q}{\partial m}) + B_7\eta,$$

where

$$B_5 = ||1/\hat{c}||_{\epsilon_1} e^{||(1/\hat{c})'||_{\epsilon_1} JM},\ \ B_6 = ||1/\hat{c}||_{\epsilon_1} B_2,$$

$$B_7 = ||1/\hat{c}||_{\epsilon_1}(B_3 + ||\hat{c}'||_{\epsilon_1} L B_1 B_5) + B_1 Lip(1/\hat{c}) J B_5.$$

Proof. We prove the result for $\frac{\partial \beta_q^{-1}}{\partial m}$. The proof for $\frac{\partial \alpha_q^{-1}}{\partial m}$ is similar. Let $(m,t) \in X$. Then for any $\xi \in [0, M]$, (2.47) implies

$$\beta_q(\beta_q^{-1}(\xi; m, t), m, t) = \xi. \tag{2.82}$$

Differentiating both sides of (2.82) with respect to m gives

$$\frac{d\beta_q}{d\tau}(\beta_q^{-1}(\xi; m, t); m, t)\frac{\partial \beta_q^{-1}}{\partial m}(\xi; m, t) + \frac{\partial \beta_q}{\partial m}(\beta_q^{-1}(\xi; m, t), m, t) = 0.$$

Recalling from (2.42) that $\frac{d}{d\tau}\beta(\tau; m, t) = \hat{c}(q(\beta(\tau; m, t), \tau))$, solving for $\frac{\partial \beta_q^{-1}}{\partial m}$, and making use of (2.82) yields

$$\frac{\partial \beta_q^{-1}}{\partial m}(\xi; m, t) = -\frac{\frac{\partial \beta_q}{\partial m}(\beta_q^{-1}(\xi; m, t), m, t)}{\hat{c}(q(\xi, \beta_q^{-1}(\xi; m, t)))}. \tag{2.83}$$

Let $\psi(m,t) = \frac{\partial \beta_q}{\partial m}(\beta_q^{-1}(\xi; m, t), m, t)$ and for $\eta \geq 0$ and $|m_1 - m_2| \leq \eta$, denote

$$\Delta\psi := \frac{\partial \beta_q}{\partial m}(\beta_q^{-1}(\xi; m_1, t); m_1, t) - \frac{\partial \beta_q}{\partial m}(\beta_q^{-1}(\xi; m_2, t), m_2, t).$$

Then using Lemmas 18–19 and 8 yields

$$
\begin{aligned}
|\Delta\psi| &\leq \left|\frac{\partial\beta_q}{\partial m}(\beta_q^{-1}(\xi;m_1,t);m_1,t) - \frac{\partial\beta_q}{\partial m}(\beta_q^{-1}(\xi;m_1,t),m_2,t)\right| \\
&\quad + \left|\frac{\partial\beta_q}{\partial m}(\beta_q^{-1}(\xi;m_1,t);m_2,t) - \frac{\partial\beta_q}{\partial m}(\beta_q^{-1}(\xi;m_2,t),m_2,t)\right| \\
&\leq \omega_m^X(\eta|\frac{\partial\beta_q}{\partial m}) + \omega_m^X(\eta|\frac{\partial\beta_q}{\partial m}(\beta_q^{-1};m_2,t)) \\
&\leq \omega_m^X(\eta|\frac{\partial\beta_q}{\partial m}) + \omega_\tau(\omega_m^X(\eta|(\beta_q^{-1};m_2,t)|\frac{\partial\beta_q}{\partial m}) \\
&\leq \omega_m^X(\eta|\frac{\partial\beta_q}{\partial m}) + \omega_\tau(B_5\eta|\frac{\partial\beta_q}{\partial m}) \\
&\leq \omega_m^X(\eta|\frac{\partial\beta_q}{\partial m}) + \left|\left|\frac{d}{d\tau}\frac{\partial\beta_q}{\partial m}\right|\right| B_5\eta \\
&\leq \omega_m^X(\eta|\frac{\partial\beta_q}{\partial m}) + ||\hat{c}'||_{\epsilon_1} LB_1B_5\eta \\
&\leq B_2T\omega_m^X(\eta|\frac{\partial q}{\partial m}) + (B_3 + ||\hat{c}'||_{\epsilon_1} LB_1B_5)\eta,
\end{aligned}
$$

where we have used the equation

$$\frac{d}{d\tau}\frac{\partial\beta_q}{\partial m} = \hat{c}'(q(\beta_q,\tau))\frac{\partial q}{\partial m}\frac{\partial\beta_q}{\partial m},$$

whose derivation is similar to that of (2.79). Taking the supremum over all $|m_1 - m_2| \leq \eta$, $\xi \in [0,M]$, and $t \in [0,T]$ in the expression $\Delta\psi$ then yields

$$\omega_m^X(\eta|\psi) \leq B_2T\omega_m^X(\eta|\frac{\partial q}{\partial m}) + (B_3 + ||\hat{c}'||_{\epsilon_1} LB_1B_5)\eta.$$

We can now use Lemmas 18–19 with (2.83) and the above estimate to obtain

$$
\begin{aligned}
\omega_m^X(\eta|\frac{\partial\beta_q^{-1}}{\partial m}) &\leq ||1/\hat{c}||_{\epsilon_1}\omega_m^X(\eta|\psi) + \left|\left|\frac{\partial\beta_q}{\partial m}\right|\right|_{\epsilon_1}\omega_m^X(\eta|\frac{1}{\hat{c}}\circ q\circ\beta_q^{-1}) \\
&\leq ||1/\hat{c}||_{\epsilon_1}\left(B_2T\omega_m^X(\eta|\frac{\partial q}{\partial m}) + (B_3 + ||\hat{c}'||_{\epsilon_1} LB_1B_5)\eta\right) \\
&\quad + B_1 Lip(1/\hat{c})Lip_t(q)Lip_m(\beta_q^{-1})\eta \\
&\leq ||1/\hat{c}||_{\epsilon_1}B_2T\omega_m^X(\eta|\frac{\partial q}{\partial m}) \\
&\quad + (||1/\hat{c}||_{\epsilon_1}(B_3 + ||\hat{c}'||_{\epsilon_1} LB_1B_5) + B_1 Lip(1/\hat{c})JB_5)\eta \\
&\leq B_6T\omega_m^X(\eta|\frac{\partial q}{\partial m}) + B_7\eta. \quad \square
\end{aligned}
$$

Lemma 26. *Assume the hypotheses of Theorem 16 and let $q \in \mathcal{B}$ be C^1 in m. Then*

$$\omega_t^X(\eta|\frac{\partial \alpha_q}{\partial t}),\ \omega_t^X(\eta|\frac{\partial \beta_q}{\partial t}) \leq B_9 T \omega_m^X(\eta|\frac{\partial q}{\partial m}) + B_{11}\eta,$$

where

$$B_8 = ||\hat{c}||_{\epsilon_1} e^{Lip(\hat{c})Lt_1},\ \ B_9 = ||\hat{c}'||_{\epsilon_1} \lceil B_8 \rceil B_8,$$

$$B_{10} = ||\hat{c}'||_{\epsilon_1} + t_1 L^2 Lip(\hat{c}')B_8,\ \ B_{11} = B_8 B_{10} + Lip(\hat{c}) J e^{||\hat{c}'||_{\epsilon_1} Lt_1}.$$

Proof. We prove the result for $\frac{\partial \alpha_q}{\partial t}$. The proof for $\frac{\partial \beta_q}{\partial t}$ is similar. For $q \in \mathcal{B}$ and C^1 in m, the characteristic ODE (2.41) has a nonautonomous C^1 vector field which is continuous in time. Hence the characteristic $\alpha_q(\tau; m, t)$ is C^1 in τ, m, and t. Differentiating (2.41) with respect to t and interchanging the order of the t and τ derivatives yields the following ODE initial value problem for $\frac{\partial \alpha_q}{\partial t}$,

$$\frac{d}{d\tau}\frac{\partial \alpha_q}{\partial t} = -\hat{c}'(q(\alpha_q, \tau))\frac{\partial q}{\partial m}\frac{\partial \alpha_q}{\partial t},\ \ \frac{\partial \alpha_q}{\partial t}(t; m, t) = \hat{c}(q(m, t)), \tag{2.84}$$

the solution of which is

$$\frac{\partial \alpha_q}{\partial t}(\tau; m, t) = \hat{c}(q(m, t))e^{\int_\tau^t \hat{c}'(q(\alpha_q(s;m,t),s))\frac{\partial q}{\partial m}(\alpha_q(s;m,t),s)\, ds}. \tag{2.85}$$

For $(m, t) \in X$ and $\tau \in [0, T]$, define

$$\phi(m, t) = \int_\tau^t \hat{c}'(q(\alpha_q(s; m, t), s))\frac{\partial q}{\partial m}(\alpha_q(s; m, t, s)\, ds,$$

and for $\eta \geq 0$, $|t_1 - t_2| \leq \eta$, denote $\Delta\phi := \phi(m,t_1) - \phi(m,t_2)$. Then

$$
\begin{aligned}
|\Delta\phi| &\leq \left| \int_{t_2}^{t_1} \hat{c}'(q(\alpha_q(s;m,t_1),s)) \frac{\partial q}{\partial m}(\alpha_q(s;m,t_1,s)\; ds \right| \\
&\quad + \left| \int_{\tau}^{t_2} (\hat{c}' \circ q)\frac{\partial q}{\partial m}(\alpha_q(s;m,t_1),s) - (\hat{c}' \circ q)\frac{\partial q}{\partial m}(\alpha_q(s;m,t_2),s)\; ds \right| \\
&\leq ||\hat{c}'||_{\epsilon_1} L|t_1 - t_2| \\
&\quad + L\left| \int_{\tau}^{t_2} (\hat{c}' \circ q)(\alpha_q(s;m,t_1),s) - (\hat{c}' \circ q)(\alpha_q(s;m,t_2),s)\; ds \right| \\
&\quad + ||\hat{c}'||_{\epsilon_1} \left| \int_{\tau}^{t_2} \frac{\partial q}{\partial m}(\alpha_q(s;m,t_1),s) - \frac{\partial q}{\partial m}(\alpha_q(s;m,t_2),s)\; ds \right| \\
&\leq ||\hat{c}'||_{\epsilon_1} L|t_1 - t_2| + |t_2 - \tau| L^2 Lip(\hat{c}') Lip_t(\alpha_q)|t_1 - t_2| \\
&\quad + ||\hat{c}'||_{\epsilon_1} |t_2 - \tau| \sup_{s\in[0,T]} \left| \frac{\partial q}{\partial m}(\alpha_q(s;m,t_1),s) - \frac{\partial q}{\partial m}(\alpha_q(s;m,t_2),s) \right| \\
&\leq (||\hat{c}'||_{\epsilon_1} L + t_1 L^2 Lip(\hat{c}') B_8)|t_1 - t_2| \\
&\quad + ||\hat{c}'||_{\epsilon_1} T \sup_{s\in[0,T]} \left| \frac{\partial q}{\partial m}(\alpha_q(s;m,t_1),s) - \frac{\partial q}{\partial m}(\alpha_q(s;m,t_2),s) \right|.
\end{aligned}
$$

Taking the supremum over $|t_1 - t_2| \leq \eta$ and using Lemmas 18–19 and 8 then gives

$$
\begin{aligned}
\omega_t^X(\eta|\phi) &\leq B_{10}\eta + ||\hat{c}'||_{\epsilon_1} T \omega_t^X(\eta|\frac{\partial q}{\partial m} \circ \alpha_q) \\
&\leq B_{10}\eta + ||\hat{c}'||_{\epsilon_1} T \omega_m^X(Lip_t(\alpha_q)\eta|\frac{\partial q}{\partial m}) \\
&\leq B_{10}\eta + ||\hat{c}'||_{\epsilon_1} T \lceil B_8 \rceil \omega_m^X(\eta|\frac{\partial q}{\partial m}).
\end{aligned}
$$

Using (2.85), Lemmas 18–19, and the above inequality, we obtain

$$
\begin{aligned}
\omega_t^X(\eta|\frac{\partial \alpha_q}{\partial t}) &\leq ||\hat{c}||_{\epsilon_1} \omega_t^X(\eta|e^{\phi}) + e^{||\hat{c}'||_{\epsilon_1} L t_1} \omega_t^X(\eta|\hat{c} \circ q) \\
&\leq ||\hat{c}||_{\epsilon_1} e^{||\hat{c}'||_{\epsilon_1} L t_1} \omega_t^X(\eta|\phi) + Lip(\hat{c}) J e^{||\hat{c}'||_{\epsilon_1} L t_1} \eta \\
&\leq B_9 T \omega_m^X(\eta|\frac{\partial q}{\partial m}) + B_{11}\eta.
\end{aligned}
$$

The proof for $\omega_t^X(\eta|\frac{\partial \beta_q}{\partial t})$ is similar. $\square$

In the next proposition we use the preceding lemmas to show that, for T small enough, $\{\frac{\partial q}{\partial m}\}_{n=1}^{\infty}$ is equicontinuous in m, in regions $\mathcal{R}_2$ and $\mathcal{R}_3$.

Proposition 27. *Assume the hypotheses of Theorem 16, and let the sequence* $\{q_n\}_{n=1}^{\infty}$ *be as defined by* (2.78)*. Suppose* $T \leq \min_{i=1,2}\{\frac{1}{2B_{13_i}}\}$*, where*

$$B_{12_i} = C_{4_i}\delta_1 + C_{5_i}\delta_1' + C_{6_i}t_1 + C_{7_i}\epsilon_1,$$

$$B_{13_i} = 2B_{12_i}B_6 + \frac{1}{2}\delta_1'(B_6B_8 + B_5\lceil B_5\rceil B_9 + B_2),$$

Then for each fixed $t \in [0,T]$*, the sequence* $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$*, restricted to the domains* $\mathcal{R}_2 \cap X$*, and* $\mathcal{R}_3 \cap X$*, respectively, is equicontinuous in* m*.*

Proof. We prove the result for region $\mathcal{R}_2$. The proof for region $\mathcal{R}_3$ is similar. For $(m,t) \in \mathcal{R}_2 \cap X$, by (2.60) and (2.78), we have

$$\begin{aligned} q_n(m,t) \quad &= \quad 2\dot{R}_1^{q_{n-1}}(\beta_{q_{n-1}}^{-1}(0;m,t)) - f(\alpha_{q_{n-1}}(0;0,\beta_{q_{n-1}}^{-1}(0;m,t)) \\ &\quad -f(\alpha_{q_{n-1}}(0;m,t)). \end{aligned}$$

Differentiating with respect to m and denoting $\theta := (0;m,t)$, gives

$$\begin{aligned} \frac{\partial q_n}{\partial m}(m,t) \quad &= \quad 2\ddot{R}_1^{q_{n-1}}(\beta_{q_{n-1}}^{-1}(\theta))\frac{\partial \beta_{q_{n-1}}^{-1}}{\partial m}(\theta) \\ &\quad -f'(\alpha_{q_{n-1}}(0;0,\beta_{q_{n-1}}^{-1}(\theta)))\frac{\partial \alpha_{q_{n-1}}}{\partial t}(0;0,\beta_{q_{n-1}}^{-1}(\theta))\frac{\partial \beta_{q_{n-1}}^{-1}}{\partial m}(\theta) \\ &\quad -f'(\alpha_{q_{n-1}}(\theta))\frac{\partial \alpha_{q_{n-1}}}{\partial m}(\theta). \end{aligned} \tag{2.86}$$

Recall that for all $q \in \mathcal{B}$, Lemma 10 implies $||\ddot{R}_1^q|| \leq B_{12_1}$, and Lemma 8 implies $\left|\left|\frac{\partial \beta_q^{-1}}{\partial m}\right|\right| \leq B_5$, $\left|\left|\frac{\partial \alpha_q}{\partial t}\right|\right| \leq B_8$, and $\left|\left|\frac{\partial \alpha_q}{\partial m}\right|\right| \leq B_1$ (whenever they exist). Using these bounds and Lemmas 18–19 with (2.86) , we then have

$$\begin{aligned}
\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial q_n}{\partial m}) &\le 2B_{12}\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial \beta_{q_{n-1}}^{-1}}{\partial m}) + 2B_5\omega_m^{\mathcal{R}_2}(\eta|\ddot{R}_1^{q_{n-1}} \circ \beta_{q_{n-1}}^{-1}) \\
&+||f'||B_5\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial \alpha_{q_{n-1}}}{\partial t} \circ \beta_{q_{n-1}}^{-1}) \\
&+B_5B_8\omega_m^{\mathcal{R}_2}(\eta|f' \circ \alpha_{q_{n-1}} \circ \beta_{q_{n-1}}^{-1}) \\
&+||f'||B_8\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial \beta_{q_{n-1}}^{-1}}{\partial m}) + ||f'||\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial \alpha_{q_{n-1}}}{\partial m}) \\
&+B_1\omega_m^{\mathcal{R}_2}(\eta|f' \circ \alpha_{q_{n-1}}) \\
&\le (2B_{12} + ||f'||B_8)\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial \beta_{q_{n-1}}^{-1}}{\partial m}) + 2B_5\lceil B_5\rceil\omega(\eta|\ddot{R}_1^{q_{n-1}}) \\
&+||f'||B_5\lceil B_5\rceil\omega_t^{\mathcal{R}_2}(\eta|\frac{\partial \alpha_{q_{n-1}}}{\partial t}) + B_5B_8\lceil B_5B_8\rceil\omega(\eta|f') \\
&+||f'||\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial \alpha_{q_{n-1}}}{\partial m}) + B_1\lceil B_1\rceil\omega(\eta|f').
\end{aligned}$$

Then Lemmas 22 and 24–26 yield

$$\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial q_n}{\partial m}) \le B_{13_1}T\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial q_{n-1}}{\partial m}) + B_{14_1}\eta + B_{15}\omega(\eta|f'), \tag{2.87}$$

where

$$B_{14_i} = 2B_{12_i}B_7 + 2B_5\lceil B_5\rceil B_{4_i} + \frac{1}{2}\delta_1'(B_7B_8 + B_5\lceil B_5\rceil B_{11} + B_3), \;\; i = 1, 2,$$

and

$$B_{15} = B_5B_8\lceil B_5B_8\rceil + B_1\lceil B_1\rceil.$$

We will again use induction on n. Let

$$\Omega_2(\eta) := 2B_{14_1}\eta + 2B_{15}\omega(\eta|f') + \omega(\eta|g').$$

Note that since $B_{15} \ge 1$, we clearly have $\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial q_1}{\partial m}) \le \Omega_2(\eta)$. Suppose $\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial q_k}{\partial m}) \le$

$\Omega_2(\eta)$ for some $k \in \mathbb{N}$. Then by the inequality in (2.87), we have

$$\begin{aligned} \omega_m^{\mathcal{R}_2}(\eta|\frac{\partial q_{k+1}}{\partial m}) &\leq B_{13_1} T \Omega_2(\eta) + \frac{1}{2}\Omega_2(\eta) \\ &\leq \Omega_2(\eta), \end{aligned}$$

for $T \leq \frac{1}{2B_{13_i}}$. Thus by induction, $\omega_m^{\mathcal{R}_2}(\eta|\frac{\partial q_n}{\partial m}) \leq \Omega_2(\eta)$ for all $n \in \mathbb{N}$. Since f' and g' are continuous, we have $\lim_{\eta \to 0+} \Omega(\eta) = 0$. Therefore by Lemma 20, the sequence of functions $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$ restricted to the domain $\mathcal{R}_2 \cap X$ is equicontinuous in m, for each fixed $t \in [0, T]$. The proof for the domain $\mathcal{R}_3 \cap X$ is similar. □

We combine Propositions 23 and 27 in the following corollary.

Corollary 28. *Assume the hypotheses of Theorem 16. Then for*

$$T \leq \min\{\frac{1}{2\delta_1' B_2}, \frac{1}{2B_{13_i}}, i = 1, 2\},$$

the sequence of functions $\{\frac{\partial q_n}{\partial m}\}_{n=1}^{\infty}$ *on the entire domain* $X = [0, M] \times [0, T]$ *is equicontinuous in* m*, for each fixed* $t \in [0, T]$.

Proof. This follows from Propositions 23 and 27 combined with a simple application of the triangle inequality. □

This provides sufficient conditions for the solution in Theorem 16 to be C^1, which we state in the next theorem.

Theorem 29. *Assume the hypotheses of Theorem 16 and suppose* $T \leq \min\{\frac{1}{2\delta_1' B_2}, \frac{1}{2B_{13_i}}\}$. *Then the solution in Theorem 16 is* C^1.

Proof. Let the sequence of functions $\{q_n\}_{n=1}^{\infty}$ be as defined by (2.78), and let $t \in [0, T]$. Then $\{\frac{\partial q_n}{\partial m}(\cdot, t)\}_{n=1}^{\infty}$ is equicontinuous in m (by Corollary 28) and uniformly bounded, since $||\frac{\partial q_n}{\partial m}|| \leq L$ by the definition of $\mathcal{B}$. Thus, by Arzela-Ascoli's Theorem, for each $t \in [0, T]$, there exists a subsequence, say $\{\frac{\partial q_{n_j}}{\partial m}(\cdot, t)\}_{j=1}^{\infty}$, which converges uniformly (in m) to some function $Q(\cdot, t)$ as $j \to \infty$. Note that the original sequence $\{q_n\}_{n=1}^{\infty}$ converges uniformly (in both m and t) to q_f, the unique fixed point of the operator S. In particular, for fixed t, the subsequence $\{q_{n_j}(\cdot, t)\}_{j=1}^{\infty}$ converges to $q_f(\cdot, t)$ as $j \to \infty$. Therefore, $q_f(\cdot, t)$ is continuously differentiable (in m) and $\frac{\partial q_f}{\partial m}(\cdot, t) = Q(\cdot, t)$. Thus q_f is C^1 in m. We also have that q_f is continuous in t, since $q_f \in \mathcal{B}$. Thus, as we have argued before, the nonautonamous vector fields in the characteristic ODEs (2.41)–(2.42) for $q = q_f$ are C^1 and continuous in time, which implies that the characteristics α_{q_f}, β_{q_f}, and their inverses are C^1 in both m and t (the "initial data" for the characteristics). But then, by the definition of the operator S in (2.60), $S(q_f)$ must be C^1 in both m and t. Since $S(q_f) = q_f$, this implies that q_f is C^1 in both m and t as well. As stated in Theorem 16, this is sufficient for the solution given there to be C^1. $\square$

Bibliography

[1] T. Erber, The classical theories of radiation reaction, *Fortschritte der Physik* **9**, 343–392, 1961.

[2] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, *Indiana Univ. Math. J.* **21**, 193–226, 1971.

[3] N. Fenichel, Geometric singular perturbation theory for ordinary differential equation, *J. Diff. Eqs.*, **31**, 53–98, 1979.

[4] C. Jones, Geometric singular perturbation theory, In *Dynamical Systems, Proceedings, montecatini Terme*, R. Johnson Ed, Lecture Notes in Mathematics 1609, New York: Springer-Verlag, 44–118, 1994.

[5] P. Pearle, Classical electron models, Chapter 7 in *Electromagnitism:Paths to Research*, D. Tepliz Ed., Plenum: New York, 1982.

[6] H. Spohn, *Dynamics of Charged Particles and their Radiation Field*, Cambridge University Press, 2004.

[7] J. Templin, Radiation reaction and runaway solutions in acoustics, *Am. J. Phys.* **67** (5), 407–413, 1999.

[8] C. Chicone and M. Heitzman, The field theory two-body problem in acoustics, *Journal of General Relativity and Gravitation*, **40**, 1087–1107, May 2008.

[9] C. Chicone, Inertial and slow manifolds for delay equations with small delays, *J. Differential Equations,* **190**, 364–406, 2003.

[10] C. Chicone, Inertial flows, slow flows, and combinatorial identities for delay equations, *J. Dynamics and Differential Equations*, **16** (3), 805–831, 2004.

[11] C. Chicone, What are the equations of motion in classical mechanics? *Can. Appl. Math. Q.*, **10** (1), 15-32, 2002.

[12] C. Chicone, *Ordinary Differential Equations with Applications*, Springer, 2nd ed., 2006.

[13] C. Chicone, F. Feng and M. Heitzman, Transient response of tapered elastic bars, *Journal of Applied Mechanics*, **74**, 677–685, 2007.

[14] Alexandre J. Chorin and Jerrold E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer; 3rd ed. 1993, chapter 3.

[15] J. H. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[16] Peter Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Wave*, Society for Industrial and Applied Mathematics, 1973.

[17] Li Ta-tsien and Yu Wen-ci, *Boundary Value Problems for Quasilinear Hyperbolic Systems*, Duke University Mathematics Series V, 1985

[18] Takaaki Nishida and Joel Smoller, Mixed Problems for Nonlinear Conservation Laws, *J. Differential Equations*, **23**, 244–269, 1977

VITA

Michael Heitzman grew up in Kansas City, Missouri and received his undergraduate education at Truman State University, in Kirksville, where he studied physics and math (among other things). He then came to the University of Missouri in Columbia, where he completed a M.S. in physics. Taking a break from school, he moved to Seattle for a while, and then, after experiencing a sufficient amount of rain and lack of sunlight, to Northern Arizona. He returned to the University of Missouri in Columbia to enter graduate school in mathematics, where he has spent the last eight years, in which time he got married, got divorced, and finally finished a Ph.D.

When not doing mathematics, he enjoys biking, hiking, running, reading, and listening to and occasionally attempting to play music.