CHAPMAN-ENSKOG SOLUTIONS TO ARBITRARY ORDER IN SONINE POLYNOMIALS

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by

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In memory of Dr. Igor Nikolaevich Ivchenko who lived and breathed rarefied gas dynamics and who helped introduce me to the subject.

This work is dedicated to my wife Annie and my parents (both Tipton and 許氏家庭).

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ABSTRACT

The Chapman-Enskog solutions of the Boltzmann equations provide a basis for the computation of important transport coefficients for simple gases and gas mixtures. These coefficients include the viscosity, the thermal conductivity, and also, for gas mixtures, the diffusion and the thermal diffusion coefficients. In the standard method for computing the transport coefficients, the Chapman-Enskog solutions are expressed as expansions in Sonine polynomials because of the rapid convergence of this series for the transport coefficients. Due to the complex nature of the expansions, direct, general expressions have been limited to low-order solutions. In this work, the Chapman-Enskog solutions have been been explored to arbitrary, relatively high orders of the Sonine polynomial expansions. Explicit, symbolic expressions containing the full dependence of the problem on the molecular masses, the molecular sizes, the mole fractions, and the intermolecular potential model via the omega integrals have been generated and archived for orders of expansion of 150 for simple gases, of 60 for the viscosity-related solutions for a binary gas mixture, and of 70 for the diffusion- and thermal conductivity-related solutions for a binary gas mixture. Numerical results using high-precision arithmetic are reported for the above orders of expansion using the rigid-sphere potential model, as analytical expressions are available for the omega-integrals. These benchmark results are then compared with the rigid-sphere results of other authors reported in the literature and a good agreement between the results is demonstrated.

Chapter 1

Introduction and organization of the dissertation

1.1 Introduction

The transport coefficients form the basis for predicting and modeling gas flows for many problems of interest in physical sciences and engineering. These coefficients include the viscosity, the thermal conductivity, and also, for gas mixtures, the diffusion and the thermal diffusion coefficients. For many problems, one is able to apply the Navier-Stokes equations and obtain a sufficiently accurate description for a system and thus, the associated transport coefficients. However, for systems where the effects of molecular interactions (both gas and gas-surface) are not negligible, a more detailed model must be used. The traditional method for modeling this type of system, the kinetic theory of gases, due to the work of Maxwell [3–5], Boltzmann [6] and others, describes the state of a gas (or each component thereof) by means of the Boltzmann equation and use of a molecular distribution function. The Boltzmann equation accounts for the molecular collisions through a collision operator term. A general method for the solution of this integro-differential equation did not exist until the problem was independently solved by Chapman [7, 8] and Enskog [9]. The formalized version of these two solution approaches, now well known as the Chapman-Enskog method, has formed the basis for many noteworthy texts on the subject [10–12].

The Chapman-Enskog solutions of the Boltzmann equations provide a basis for the computation of the transport coefficients. In the standard method for computing the transport coefficients, the Chapman-Enskog solutions are expressed as expansions in Sonine polynomials [13] because of the rapid convergence of this series for the transport coefficients [14]. Due to the complex nature of the expansions, direct, general expressions that can be related to the transport coefficients have been limited to low-order solutions. In this dissertation, the Chapman-Enskog solutions have been been explored to arbitrary, relatively high orders of the Sonine polynomial expansions. Explicit, symbolic expressions containing the full dependence of the problem on the molecular masses, the molecular sizes, the mole fractions, and the intermolecular potential model via the omega integrals have been generated and archived for orders of expansion of 150 for simple gases, of 60 for the viscosity-related solutions for a binary gas mixture, and of 70 for the diffusion- and thermal conductivity-related solutions for a binary gas mixture. High-precision numerical values for transport coefficients and related solutions are reported for the above orders of expansion using the rigid-sphere potential model as analytical expressions are available for the omega-integrals, thus elimating the introduction of of numerical integration errors. These results are presented as benchmark set of tables for normalized values of the tranport coefficients for simple gases and gas mixtures. The results are also compared with the rigid-sphere results of other authors reported in the literature and a good agreement between the results is demonstrated.

1.2 The organization of the dissertation

This dissertation has been organized into five separate, more-or-less self-contained chapters (Chapters 2-6) each of which contains all of the definitions, theoretical discussion, and results associated with one of a series of journal articles [15–19]. In Chapter 2, the use of high order Sonine polynomial expansions to obtain error free results for the transport coefficients and the related Chapman-Enskog solutions for a simple gas are explored. Explicit results up to order 150 are reported for the rigid-sphere potential model. In Chapter 3, the use of high order Sonine polynomial expansions to arbitrary order for the viscosity-related Chapman-Enskog solutions for binary gas mixtures is explored and explicit results up to order 60 are reported for the rigid-sphere potential model. The results are compared with those of Garcia and Siewert [1] and Takata et. al. [2] and in general, there is good agreement between the results reported. Also, extensive tables of the viscosity coefficient (and related results) for various binary mixtures of the noble gases (using the rigid-sphere potential model) up to order 60 are presented. It is notable that the precision of our order 60 results can be nearly doubled by the application of an extrapolation method to the order 1 through 60 results for the viscosity coefficient. In Chapter 4, the use of high order Sonine polynomial expansions for the diffusion- and the thermal conductivity-related Chapman-Enskog solutions for binary gas mixtures is explored and results up to order 70 are reported for the rigid-sphere potential model. The results are compared with those from the work Takata et. al. [2] and both sets of results appear to be in agreement. Extensive tables of both the diffusion and thermal conductivity coefficients (and related solutions) for various binary mixtures of the noble gases (using the rigid-sphere potential model) are presented. Again, it is notable that the precision of the order 70 results can be significantly increased by the application of an extrapolation method to the order 1 through 70 results for the both the diffusion and the thermal conductivity coefficient solutions. In Chapters 5 and 6, explicit binary gas mixture bracket integral expressions required for order 5 (in Sonine polynomial expansions) viscosity- (Chapter 5) and the diffusion- and thermal conductivity-related (Chapter 6) Chapman-Enskog solutions are presented. Chapter 7 gives a brief summary of the work and some suggestions for future work.

Chapter 2

Chapman-Enskog solutions for a simple, rigid-sphere gas

2.1 Introduction

The Chapman-Enskog solutions of the Boltzmann equations provide a basis for the computation of transport coefficients such as the viscosity, the thermal conductivity, and diffusion coefficients [10–12, 20–28]. These solutions are also useful for computations of slip and jump coefficients [28–30]. Generally, the Chapman-Enskog solutions are expressed as Sonine polynomial expansions and it has been found that relatively, low-order expansions (of order 4) provide reasonable precision (to about 1 part in 1,000) in the computation of the transport coefficients [10]. For the computation of slip and jump coefficients, the adequacy of low-order expansions needs to be investigated further [28, 31–33]. Additionally, low-order expansions do not provide good convergence in velocity space for the actual Chapman-Enskog solutions and, thus, it is of some interest in molecular gas dynamics to have better solutions available. One should note that full solutions for the special case of rigid-sphere molecules in a simple gas were obtained by Pekeris and Alterman [26]. Although these authors provided results for the Chapman–Enskog solutions to a higher accuracy than previously reported, their technique was specific to a simple, rigid-sphere gas. They converted the relevant integral equation(s) to a fourth order ordinary differential equation on an infinite domain, with boundary conditions at both extrema, and then solved the problem numerically using a shooting method. This technique is not ideal for solving integral equations, and it is not clear if it can be adopted for real potentials. Later, the relevant integral equation(s) was solved directly [34] (by Neumann iteration in conjunction with various splines and collocation) and a good agreement with Pekeris and Alterman [26] was obtained. These results have been further confirmed [35, 36]. The use of Neumann iteration together with splines and collocation has been effective for a simple, rigid-sphere gas, but again its extension to real potentials has not been explored, and is also likely to pose substantial challenges with accurate evaluations of the multifold integrals that are associated with real potentials. Thus, despite these previous works, [26, 34–36] there has remained a need for further exploration of the Sonine Polynomial expansions.

The purpose in this chapter is to report on the use of high-order standard Sonine polynomial expansions for the determination of the viscosity and thermal conductivity coefficients as well as the Chapman-Enskog solutions. The results presented here may be obtained to any desired precision for a simple, rigid-sphere gas. The current approach can also be used to obtain results of arbitrary accuracy for real potentials, without further mathematical and numerical complications, for single gases as well as gas mixtures. An additional and important advantage of the method is that for gas mixtures, where a number of parameters such as molecular mole fractions, masses, diameters, interaction potentials, etc., are involved, the associated matrix elements can all be obtained as algebraic expressions defined in terms of the standard omega integrals, and numerical results can then be obtained in a straightforward manner and to a desired accuracy by inversion of the associated matrices (with the use of the values for the omega integrals appropriate to a specified potential). There does exist considerable literature on omega integrals and their numerical computation [10–12, 27] and one can further improve on the accuracy of these computations as appropriate to higher order Sonine expansions.

2.2 The viscosity and thermal conductivity equations for a simple gas

The Chapman-Enskog approach for the determination of the transport coefficients from the Boltzmann equation is described elegantly and concisely in the text of Chapman and Cowling [10]. Many other authors have followed this text and expanded upon the theory [11,12,20–25,27]. While this approach could be described again here, doing so might tend to distract the reader from the more essential developments described below. Thus, the reader is referred to the text of Chapman and Cowling for their development of the molecular distribution function (and notations) which leads to the equations and expressions that will be considered in this work. However, it is important to note that in the Chapman-Enskog approach, for a simple, monatomic gas, the molecular distribution, f, to the order of interest here, is developed as:

$$f = f^{(0)} + f^{(0)}\Phi^{(1)} \ , \tag{2.1}$$

where $f^{(0)}$ represents the local equilibrium Maxwellian distribution:

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\mathscr{C}^2\right) \ , \tag{2.2}$$

in which, n is the local molecular number density, m is the molecular mass, k is the Boltzmann constant, T is the local temperature, and:

$$\boldsymbol{\mathscr{C}} \equiv \left(\frac{m}{2kT}\right)^{1/2} \mathbf{C} \ , \tag{2.3}$$

where, $\mathbf{C} = \mathbf{c} - \mathbf{c}_0$ is the molecular velocity, and $\mathbf{c}_0$ is the local mean molecular velocity. The function, $\Phi^{(1)}$, is expressed as:

$$\Phi^{(1)} = -\frac{1}{n}\left(\frac{2kT}{m}\right)^{1/2} \mathbf{A} \cdot \nabla \ln(T) - \frac{2}{n}\mathsf{B} : \nabla \boldsymbol{c}_0 \ , \tag{2.4}$$

where $\mathbf{A}$ is a vector and B is a tensor which have the forms:

$$\mathbf{A}(\boldsymbol{\mathscr{C}}) = \boldsymbol{\mathscr{C}} A(\mathscr{C}) \ , \tag{2.5}$$

$$\mathsf{B}(\boldsymbol{\mathscr{C}}) = \overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}\, B(\mathscr{C}) \ , \tag{2.6}$$

and are determined from the linearized Boltzmann equation. Note that using the expression for $\Phi^{(1)}$ given in Eq. (2.4), the molecular distribution, f, can be written as:

$$f = f^{(0)}\left(1 - \frac{1}{n}\left(\frac{2kT}{m}\right)^{1/2}\mathbf{A}\cdot\boldsymbol{\nabla}\ln(T) - \frac{2}{n}\mathbf{B} : \nabla\mathbf{c}_0\right) \ . \tag{2.7}$$

In the notations of Chapman and Cowling [10], the Chapman-Enskog equations for the thermal conductivity and the viscosity are:

$$nI(\mathbf{A}) = f^{(0)}\left(\mathscr{C}^2 - \tfrac{5}{2}\right)\boldsymbol{\mathscr{C}} \ , \tag{2.8}$$

$$nI(\mathsf{B}) = f^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}} \ , \tag{2.9}$$

where I is the linearized Boltzmann collision operator defined by:

$$n^2 I(\phi) = \iint f^{(0)} f_1^{(0)}\left(\phi + \phi_1 - \phi' - \phi_1'\right) g\alpha_1 \mathrm{d}\mathbf{e}'\mathrm{d}\mathbf{c}_1 \ . \tag{2.10}$$

One should note that $\mathbf{g} = \mathbf{c}_1 - \mathbf{c}$ is the dimensional pre-collision relative velocity, $\alpha(g,\chi)$ is the collision cross-section, $\mathbf{e}'$ is a unit vector (dimensionless) in the direction of the post-collision relative velocity, $\mathbf{g}' = \mathbf{c}_1' - \mathbf{c}'$, and χ is the angle between $\mathbf{g}$ and $\mathbf{g}'$. For computational purposes, the Boltzmann collision operator may also be expressed (equivalently) as:

$$n^2 I(\phi) = \iint f^{(0)} f_1^{(0)}\left(\phi + \phi_1 - \phi' - \phi_1'\right) g b \mathrm{d}b \mathrm{d}\varepsilon \mathrm{d}\mathbf{c}_1 \ , \tag{2.11}$$

where b is the impact distance (also known as the impact parameter) and ε is the azimuthal angle of the scattering plane (the plane containing $\mathbf{g}$ and $\mathbf{g}'$) relative to a fixed direction in space. Note that a non-dimensional relative velocity, $\hat{\boldsymbol{g}}$, may also be defined as:

$$\hat{\boldsymbol{g}} \equiv \left(\frac{m}{2kT}\right)^{1/2}\mathbf{g} \ . \tag{2.12}$$

For later purposes, it is useful to note that one can also express the Boltzmann operator as:

$$
\begin{aligned}
n^2 I(\phi) = & n^2 \left(\frac{m}{2kT}\right) \sigma_0^2 \iint \exp\left(-\mathscr{C} - \mathscr{C}\right) \\
& \times \left[\phi\left(\boldsymbol{\mathscr{C}}\right) + \phi\left(\boldsymbol{\mathscr{C}}_1\right) - \phi\left(\boldsymbol{\mathscr{C}}'\right) - \phi\left(\boldsymbol{\mathscr{C}}'_1\right)\right] g b^* \mathrm{d}b^* \mathrm{d}\varepsilon \mathrm{d}\boldsymbol{\mathscr{C}}_1 \ ,
\end{aligned} \tag{2.13}
$$

where $b^* = b/\sigma_0$ and σ_0 has dimensions of length. Thus, the Chapman-Enskog equations that one must solve are written as:

$$
nI\left(\boldsymbol{\mathscr{C}} A\left(\mathscr{C}\right)\right) = f^{(0)} \left(\mathscr{C}^2 - \tfrac{5}{2}\right) \boldsymbol{\mathscr{C}} \ , \tag{2.14}
$$

$$
nI\left(\overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}} B\left(\mathscr{C}\right)\right) = f^{(0)} \overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}} \ . \tag{2.15}
$$

The heat flux, $\mathbf{q}$, and the pressure tensor, p, are given by:

$$
\mathbf{q} = \tfrac{1}{2} m \int f C^2 \mathbf{C} \mathrm{d}\mathbf{c} \ , \tag{2.16}
$$

$$
\mathsf{p} = m \int f \, \overset{\circ}{\mathbf{C}\mathbf{C}} \, \mathrm{d}\mathbf{c} \ , \tag{2.17}
$$

which, using Eq. (2.7) and the condition:

$$
\int f^{(0)} A\left(\mathscr{C}\right) \mathscr{C} \mathrm{d}\mathbf{c} = \mathbf{0} \ , \tag{2.18}
$$

can be expressed as:

$$
\mathbf{q} = -\lambda \boldsymbol{\nabla} T \ , \tag{2.19}
$$

$$
\mathsf{p} = p \mathsf{U} - 2\mu \overline{\overline{\overset{\circ}{\boldsymbol{\nabla}\mathbf{c}_0}}} \ , \tag{2.20}
$$

where, p is the scalar pressure, U is the unit tensor, and $\overline{\overline{\overset{\circ}{\boldsymbol{\nabla}\mathbf{c}_0}}}$ is the traceless strain tensor. The thermal conductivity, λ, and the viscosity, μ, are given by:

$$
\lambda = \frac{2k^2 T}{3mn} \int f^{(0)} \mathscr{C}^2 A\left(\mathscr{C}\right) \left(\mathscr{C}^2 - \tfrac{5}{2}\right) \mathrm{d}\mathbf{c} \ , \tag{2.21}
$$

$$
\mu = \frac{2kT}{5n} \int f^{(0)} \overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}} : \overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}} B\left(\mathscr{C}\right) \mathrm{d}\mathbf{c} \ , \tag{2.22}
$$

which can also be written as:

$$
\lambda = \frac{2k^2 T}{3m} \left[\mathbf{A}\left(\boldsymbol{\mathscr{C}}\right) \ , \ \mathbf{A}\left(\boldsymbol{\mathscr{C}}\right)\right] \ , \tag{2.23}
$$

$$\mu = \frac{2kT}{5}\Big[\mathsf{B}\left(\boldsymbol{\mathscr{C}}\right)\ ,\ \mathsf{B}\left(\boldsymbol{\mathscr{C}}\right)\Big] . \tag{2.24}$$

In Eqs. (2.23) and (2.24), the terms in square brackets are commonly known as the bracket integrals which are defined by:

$$\Big[\varphi_1\ ,\ \varphi_2\Big] \equiv \int \varphi_1 I\left(\varphi_2\right) \mathrm{d}\mathbf{c} , \tag{2.25}$$

such that:

$$\Big[\mathbf{A}\left(\boldsymbol{\mathscr{C}}\right)\ ,\ \mathbf{A}\left(\boldsymbol{\mathscr{C}}\right)\Big] = \int \mathbf{A}\left(\boldsymbol{\mathscr{C}}\right) \cdot I\left(\mathbf{A}\left(\boldsymbol{\mathscr{C}}\right)\right) \mathrm{d}\mathbf{c} , \tag{2.26}$$

$$\Big[\mathsf{B}\left(\boldsymbol{\mathscr{C}}\right)\ ,\ \mathsf{B}\left(\boldsymbol{\mathscr{C}}\right)\Big] = \int \mathsf{B}\left(\boldsymbol{\mathscr{C}}\right) : I\left(\mathsf{B}\left(\boldsymbol{\mathscr{C}}\right)\right) \mathrm{d}\mathbf{c} . \tag{2.27}$$

For future purposes, it is also useful to note that Eqs. (2.21) and (2.22) can also be written as:

$$\lambda = -\frac{5k^2T}{2m}\left(\frac{16}{15\pi^{1/2}}\int_0^\infty \exp\left(-\mathscr{C}^2\right)\mathscr{C}^4\left(\tfrac{5}{2}-\mathscr{C}^2\right)A\left(\mathscr{C}\right)\mathrm{d}\mathscr{C}\right) , \tag{2.28}$$

$$\begin{aligned}\mu &= \frac{4kT}{15n}\int f^{(0)}\mathscr{C}^4 B\left(\mathscr{C}\right)\,\mathrm{d}\mathbf{c}\\ &= \frac{16kT}{15\pi^{1/2}}\int_0^\infty \exp\left(-\mathscr{C}^2\right)\mathscr{C}^6 B\left(\mathscr{C}\right)\mathrm{d}\mathscr{C} .\end{aligned} \tag{2.29}$$

2.3 The Chapman-Enskog solutions

In the standard development [10], the functions $A\left(\mathscr{C}\right)$ and $B\left(\mathscr{C}\right)$ are expressed as:

$$A\left(\mathscr{C}\right) = \sum_{p=1}^{N} a_p S_{3/2}^{(p)}\left(\mathscr{C}^2\right) , \tag{2.30}$$

$$B\left(\mathscr{C}\right) = \sum_{p=1}^{N} b_p S_{5/2}^{(p-1)}\left(\mathscr{C}^2\right) , \tag{2.31}$$

where $S_m^{(n)}\left(x\right)$ are the un-normalized Sonine polynomials defined by:

$$S_m^{(n)}\left(x\right) \equiv \sum_{p=0}^{n}\left(-x\right)^p \frac{\left(m+n\right)!}{\left(m+p\right)!\left(p\right)!\left(n-p\right)!} . \tag{2.32}$$

The first two Sonine polynomials are, thus, $S_m^{(0)}\left(x\right) = 1$ and $S_m^{(1)}\left(x\right) = m+1-x$. Also, the Sonine polynomials are orthogonal, such that:

$$\int_0^\infty \exp\left(-x\right) S_m^{(p)}\left(x\right) S_m^{(q)}\left(x\right) x^m dx = \left\{\Gamma\left(m+p+1\right)/p!\right\}\delta_{p,q} , \tag{2.33}$$

where $\delta_{p,q}$ is the Kronecker delta.

Note that, in terms of Sonine polynomials, the Chapman-Enskog equations, Eqs. (2.14) and (2.15), can be written as:

$$nI\left(\boldsymbol{\mathscr{C}}\, A\left(\mathscr{C}\right)\right) = -f^{(0)} S_{3/2}^{(1)}\left(\mathscr{C}^2\right)\boldsymbol{\mathscr{C}}\ , \tag{2.34}$$

$$nI\left(\overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}\, B\left(\mathscr{C}\right)\right) = f^{(0)} S_{5/2}^{(0)}\left(\mathscr{C}^2\right)\overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}\ . \tag{2.35}$$

When Eqs. (2.30) and (2.31) are substituted into Eqs. (2.34) and (2.35), respectively, multiplied by $S_{3/2}^{(q)}\left(\mathscr{C}^2\right)\boldsymbol{\mathscr{C}}$ and $S_{5/2}^{(q-1)}\left(\mathscr{C}^2\right)\overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}$, respectively, and then integrated with respect to $\mathbf{c}$, one obtains the following systems of linear, algebraic equations for the expansion coefficients, a_p and b_p:

$$\sum_{p=1}^{N} a_{pq} a_p = \alpha_q \quad (q = 1, 2, \ldots, N)\ , \tag{2.36}$$

$$\sum_{p=1}^{N} b_{pq} b_p = \beta_q \quad (q = 1, 2, \ldots, N)\ , \tag{2.37}$$

where a_{pq} and b_{pq} are defined by:

$$\begin{aligned} a_{pq} &\equiv \int I\left(\boldsymbol{\mathscr{C}}\, S_{3/2}^{(p)}\left(\mathscr{C}^2\right)\right)\cdot \boldsymbol{\mathscr{C}}\, S_{3/2}^{(q)}\left(\mathscr{C}^2\right) \mathrm{d}\mathbf{c} \\ &= \left[\boldsymbol{\mathscr{C}}\, S_{3/2}^{(p)}\left(\mathscr{C}^2\right)\ ,\ \boldsymbol{\mathscr{C}}\, S_{3/2}^{(q)}\left(\mathscr{C}^2\right)\right]\ , \end{aligned} \tag{2.38}$$

$$\begin{aligned} b_{pq} &\equiv \int I\left(\overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}\ S_{5/2}^{(p-1)}\left(\mathscr{C}^2\right)\right) : \overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}\ S_{5/2}^{(q-1)}\left(\mathscr{C}^2\right) \mathrm{d}\mathbf{c} \\ &= \left[\overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}\ S_{5/2}^{(p-1)}\left(\mathscr{C}^2\right)\ ,\ \overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}\, S_{5/2}^{(q-1)}\left(\mathscr{C}^2\right)\right]\ , \end{aligned} \tag{2.39}$$

and α_q and β_q are given by:

$$\alpha_q = -n^{-1}\int f^{(0)}\boldsymbol{\mathscr{C}}\, S_{3/2}^{(1)}\left(\mathscr{C}^2\right)\cdot \boldsymbol{\mathscr{C}}\, S_{3/2}^{(q)}\left(\mathscr{C}^2\right) \mathrm{d}\mathbf{c}\ , \tag{2.40}$$

$$\beta_q = n^{-1}\int f^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}\ S_{5/2}^{(0)}\left(\mathscr{C}^2\right) : \overset{\circ}{\boldsymbol{\mathscr{C}}\boldsymbol{\mathscr{C}}}\, S_{5/2}^{(q-1)}\left(\mathscr{C}^2\right) \mathrm{d}\mathbf{c}\ , \tag{2.41}$$

which evaluate to:

$$\alpha_q = -\tfrac{15}{4}\delta_{q,1}\ , \tag{2.42}$$

$$\beta_q = \tfrac{5}{2}\delta_{q,1}\ . \tag{2.43}$$

Based upon these notations, one may express the transport coefficients as:

$$\lambda = -\frac{5k^2T}{2m}a_1 \ , \tag{2.44}$$

$$\mu = kTb_1 \ . \tag{2.45}$$

Now, if a_{pq} and b_{pq} are evaluated to a given order, N, then a_p and b_p can be obtained by solving the systems of N linear, algebraic equations represented by Eqs. (2.36) and (2.37), respectively, as straightforward matrix inversion problems. Obviously, the matrix coefficients, a_{pq} and b_{pq}, depend upon the intermolecular collision cross-section, $\alpha\left(g, \chi\right)$, and are discussed further below. Note that the variable N is used in the current chapter to represent the order of expansion in Sonine polynomials to conform to the first of the publications in this series [15]. In the subsequent chapters, the variable m is used to represent this order of expansion.

2.4 The matrix elements a_{pq} and b_{pq}

For general intermolecular force laws, a_{pq} and b_{pq} are determined by evaluating the eight-fold integrals implicit in Eqs. (2.10), (2.38), and (2.39). These integrals may be expressed as:

$$a_{pq} = \sum_{r,\ell} \hat{A}_{pqr\ell} \mathit{\Omega}_{\,1}^{(\ell)}\left(r\right) \ , \tag{2.46}$$

$$b_{pq} = \sum_{r,\ell} \hat{B}_{pqr\ell} \mathit{\Omega}_{\,1}^{(\ell)}\left(r\right) \ , \tag{2.47}$$

where $\hat{A}_{pqrl}$ and $\hat{B}_{pqrl}$ are pure numbers and the quantities, $\mathit{\Omega}_{\,1}^{(\ell)}\left(r\right)$, are known as the omega integrals which are defined for a simple gas as:

$$\begin{aligned}\mathit{\Omega}_{\,1}^{(\ell)}\left(r\right) &\equiv \pi^{1/2} \iint \exp\left(-\hat{g}^2\right) \hat{g}^{2r+2} \left(1 - \cos^{\ell}\chi\left(b, \hat{g}\right)\right) g b \mathrm{d}b \mathrm{d}\hat{g} \\ &= \left(\frac{kT}{\pi m}\right)^{1/2} \int_0^{\infty} \exp\left(-\hat{g}^2\right) \hat{g}^{2r+3} \phi_1^{(\ell)}\left(\hat{g}\right) \mathrm{d}\hat{g} \ ,\end{aligned} \tag{2.48}$$

in which:

$$\phi_1^{(\ell)}\left(\hat{g}\right) = 2\pi \int_0^{\infty} \left(1 - \cos^{\ell}\chi\left(b, \hat{g}\right)\right) b \mathrm{d}b \ , \tag{2.49}$$

where the notation follows that of Chapman and Cowling [10] by using the non-dimensional, pre-collision, relative velocity, $\hat{g}$ (Note: The dimensional relative velocity, g, in the first part of Eq. (2.48) is not a typographical error but is retained in order to conform to the equation as given by Chapman and Cowling. In the second part of the equation it has been converted to the non-dimensional relative velocity, $\hat{g}$). Chapman and Cowling have not given explicit expressions for the expansion coefficients, $\hat{A}_{pqrl}$ and $\hat{B}_{pqrl}$, and advise that a_{pq} and b_{pq} be evaluated directly by working with expansions of a more general nature. They have presented explicit expressions for a_{pq} and b_{pq} only up to $N = 3$ which are reproduced below in Eqs. (2.50)-(2.61):

$$a_{11} = 4\Omega_1^{(2)}(2) \ , \tag{2.50}$$

$$a_{12} = 7\Omega_1^{(2)}(2) - 2\Omega_1^{(2)}(3) \ , \tag{2.51}$$

$$a_{22} = \tfrac{77}{4}\Omega_1^{(2)}(2) - 7\Omega_1^{(2)}(3) + \Omega_1^{(2)}(4) \ , \tag{2.52}$$

$$a_{13} = \tfrac{63}{8}\Omega_1^{(2)}(2) - \tfrac{9}{2}\Omega_1^{(2)}(3) + \tfrac{1}{2}\Omega_1^{(2)}(4) \ , \tag{2.53}$$

$$a_{23} = \tfrac{945}{32}\Omega_1^{(2)}(2) - \tfrac{261}{16}\Omega_1^{(2)}(3) + \tfrac{25}{8}\Omega_1^{(2)}(4) - \tfrac{1}{4}\Omega_1^{(2)}(5) \ , \tag{2.54}$$

$$\begin{aligned} a_{33} =& \tfrac{14553}{256}\Omega_1^{(2)}(2) - \tfrac{1215}{32}\Omega_1^{(2)}(3) + \tfrac{313}{32}\Omega_1^{(2)}(4) - \tfrac{9}{8}\Omega_1^{(2)}(5) \\ &+ \tfrac{1}{16}\Omega_1^{(2)}(6) + \tfrac{1}{6}\Omega_1^{(4)}(4) \ , \end{aligned} \tag{2.55}$$

$$b_{11} = 4\Omega_1^{(2)}(2) \ , \tag{2.56}$$

$$b_{12} = 7\Omega_1^{(2)}(2) - 2\Omega_1^{(2)}(3) \ , \tag{2.57}$$

$$b_{22} = \tfrac{301}{12}\Omega_1^{(2)}(2) - 7\Omega_1^{(2)}(3) + \Omega_1^{(2)}(4) \ , \tag{2.58}$$

$$b_{13} = \tfrac{63}{8}\Omega_1^{(2)}(2) - \tfrac{9}{2}\Omega_1^{(2)}(3) + \tfrac{1}{2}\Omega_1^{(2)}(4) \ , \tag{2.59}$$

$$b_{23} = \tfrac{1365}{32}\Omega_1^{(2)}(2) - \tfrac{321}{16}\Omega_1^{(2)}(3) + \tfrac{25}{8}\Omega_1^{(2)}(4) - \tfrac{1}{4}\Omega_1^{(2)}(5) \ , \tag{2.60}$$

$$\begin{aligned} b_{33} =& \tfrac{25137}{256}\Omega_1^{(2)}(2) - \tfrac{1755}{32}\Omega_1^{(2)}(3) + \tfrac{381}{32}\Omega_1^{(2)}(4) - \tfrac{9}{8}\Omega_1^{(2)}(5) \\ &+ \tfrac{1}{16}\Omega_1^{(2)}(6) + \tfrac{1}{2}\Omega_1^{(4)}(4) \ , \end{aligned} \tag{2.61}$$

with $a_{qp} = a_{pq}$ and $b_{qp} = b_{pq}$. For the current work, a program has been written which enables one to construct explicit expressions for a_{pq} and b_{pq}, from the expansions of Chapman and Cowling, to any arbitrary order. As an example of such calculations, the additional explicit expressions for a_{pq} and b_{pq} that are needed in the Sonine polynomial expansions to order 5 have been reported in Appendix A. Further details of this work will be reported in subsequent chapters and in future publications where gas mixtures and more general intermolecular potentials will be addressed.

2.5 The matrix elements, a_{pq} and b_{pq}, for rigid-sphere molecules

For a simple gas of rigid-sphere molecules, the omega integrals can be evaluated analytically such that:

$$\Omega_1^{(\ell)}(r) = \sigma^2 \left(\frac{\pi kT}{m}\right)^{1/2} W_1^{(\ell)}(r) \ , \tag{2.62}$$

in which:

$$W_1^{(\ell)}(r) = \tfrac{1}{4}\left[2 - \frac{1}{(l+1)}\left(1 + (-1)^{\ell}\right)\right](r+1)! \ , \tag{2.63}$$

and σ is the diameter of the sphere. Now, defining:

$$a_{pq} \equiv 8\sigma^2 \left(\frac{\pi kT}{m}\right)^{1/2} \hat{a}_{pq} \ , \tag{2.64}$$

$$b_{pq} \equiv 8\sigma^2 \left(\frac{\pi kT}{m}\right)^{1/2} \hat{b}_{pq} \ , \tag{2.65}$$

one has:

$$\begin{gathered} \hat{a}_{11} = 1 \ , \ \hat{a}_{12} = -\tfrac{1}{4} \ , \ \hat{a}_{22} = \tfrac{45}{16} \ , \ \hat{a}_{13} = -\tfrac{1}{32} \ , \\ \hat{a}_{23} = -\tfrac{103}{128} \ , \ \hat{a}_{33} = \tfrac{5657}{1024} \ , \ \ldots \ , \end{gathered} \tag{2.66}$$

$$\begin{gathered} \hat{b}_{11} = 1 \ , \ \hat{b}_{12} = -\tfrac{1}{4} \ , \ \hat{b}_{22} = \tfrac{205}{48} \ , \ \hat{b}_{13} = -\tfrac{1}{32} \ , \\ \hat{b}_{23} = -\tfrac{163}{128} \ , \ \hat{b}_{33} = \tfrac{11889}{1024} \ , \ \ldots \ , \end{gathered} \tag{2.67}$$

The use of Sonine polynomials in this problem was first suggested by Burnett [13] who did not use the omega integrals discussed above but, rather, evaluated the bracket integrals directly. Essentially, Burnett showed that

$\hat{a}_{pq}$ = the coefficient of $s^p t^q$ in the expansion of:

$$\tfrac{1}{8}\left\{1-\tfrac{1}{2}\left(s+t\right)\right\}^{1/2}\frac{st}{\left(1-st\right)^3}-\tfrac{1}{8}\left\{1-\tfrac{1}{2}\left(s+t\right)\right\}^{-1/2}\frac{s^2t^2}{\left(1-st\right)^2}\ , \tag{2.68}$$

$\hat{b}_{pq}$ = the coefficient of $s^p t^q$ in the expansion of:

$$\begin{aligned}&\frac{\left(1+\tfrac{2}{3}st+\tfrac{1}{3}s^2\right)}{\left(1-st\right)^4}\left\{1-\tfrac{1}{2}\left(s+t\right)\right\}^{1/2}-\tfrac{1}{3}\frac{st}{\left(1-st\right)^3}\left\{1-\tfrac{1}{2}\left(s+t\right)\right\}^{1/2}\\&-\tfrac{1}{48}\frac{s^2t^2}{\left(1-st\right)^2}\left\{1-\tfrac{1}{2}\left(s+t\right)\right\}^{-3/2}\ ,\end{aligned} \tag{2.69}$$

where a typographical error in Burnett's work has been corrected for Eq. (2.68). In spite of the typographical error, it is notable that the original numerical values reported by Burnett for both $\hat{a}_{pq}$ and $\hat{b}_{pq}$ are consistent with those obtained from Eqs. (2.68) and (2.69).

For more explicit expressions for $\hat{a}_{pq}$ and $\hat{b}_{pq}$ for rigid-sphere molecules the works of Mott-Smith [37] and Foch and Ford [38] are considered for comparison. Following them, one can express $\hat{a}_{pq}$ and $\hat{b}_{pq}$ as:

$$\hat{a}_{pq}=-\frac{1}{2\sqrt{2}}\hat{J}^1_{p,q}\ , \tag{2.70}$$

$$\hat{b}_{pq}=-\frac{1}{3\sqrt{2}}\hat{J}^2_{p-1,q-1}\ , \tag{2.71}$$

where the coefficients $\hat{J}^\ell_{p,q}$ are expressed as:

$$\begin{aligned}\hat{J}^\ell_{p,q}&=2^{-p-q-\ell+(3/2)}\ell!\ \times\\&\qquad\sum_{k=0}^{\ell}\sum_{j=0}^{\min(p,q)}\frac{4^j\Gamma\left(p+q-2j+\ell-k-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\left(p-j\right)!\left(q-j\right)!\left(\ell-k\right)!}B^j_k\left(\infty\right)\ ,\end{aligned} \tag{2.72}$$

with:

$$B^j_k\left(\infty\right)=\frac{(2j+k+1)!}{(2j+1)!k!}-2^{k-1}\frac{(j+k+1)!}{j!k!}\left(1+\delta_{j,0}\delta_{k,0}\right)\ . \tag{2.73}$$

For $N = 3$, the results of Burnett [13] and Foch and Ford [38] for a_{pq} and b_{pq} are in complete agreement with those of Chapman and Cowling [10] as noted in Eqs. (2.66) and (2.67). It is important to note that Eqs. (2.72) and (2.73) are not quite the same as the corresponding equations given by Foch and Ford. A slight reinterpretation of their results was necessary to achieve consistency with the bracket integral notation used by Chapman and Cowling. Essentially, the difference is strictly a notational issue that arises because Foch and Ford used normalized Sonine polynomials while Chapman and Cowling used un-normalized Sonine polynomials. Also, for the case of a simple gas of rigid spheres, factors of $\sqrt{\pi}$ and $\sqrt{2}$ cancel in Eqs. (2.70) and (2.71) such that all of the expressions for $\hat{a}_{pq}$ and $\hat{b}_{pq}$, for any arbitrary order of expansion, are fractions involving only integers.

For rigid-sphere molecules, it is also possible to evaluate all a_{pq} and b_{pq} analytically in a manner that is slightly different from that used by Chapman and Cowling [10] and Foch and Ford [38]. For a rigid-sphere gas, one can express the Boltzmann operator as [10–12, 20–26, 28, 34]:

$$\begin{aligned} I(\phi) = -\frac{m\sigma^2}{2\pi kT} \exp\left(-\mathscr{C}^2\right) \Big[& -\upsilon(\mathscr{C})\, \phi(\boldsymbol{\mathscr{C}}) \\ & + \int \exp\left(-\mathscr{C}'^2\right) K(\boldsymbol{\mathscr{C}}, \boldsymbol{\mathscr{C}}')\, \phi(\boldsymbol{\mathscr{C}}')\, \mathrm{d}\boldsymbol{\mathscr{C}}' \Big] , \end{aligned} \tag{2.74}$$

where:

$$\upsilon(\mathscr{C}) = \left(2\mathscr{C} + \frac{1}{\mathscr{C}}\right) \frac{\sqrt{\pi}}{2} \operatorname{erf}(\mathscr{C}) + \exp\left(-\mathscr{C}^2\right) , \tag{2.75}$$

with the error function defined as:

$$\operatorname{erf}(\mathscr{C}) \equiv \frac{2}{\sqrt{\pi}} \int_0^{\mathscr{C}} \exp\left(-x^2\right) \mathrm{d}x . \tag{2.76}$$

The kernel, $K(\boldsymbol{\mathscr{C}}, \boldsymbol{\mathscr{C}}')$, can be expanded in Legendre polynomials as [28, 34]:

$$K(\boldsymbol{\mathscr{C}}, \boldsymbol{\mathscr{C}}') = \frac{1}{2\pi} \sum_{n=0}^{\infty} \left(n + \tfrac{1}{2}\right) k_n(\mathscr{C}, \mathscr{C}')\, P_n(\cos\theta_0) , \tag{2.77}$$

where θ_0 is the angle between $\boldsymbol{\mathscr{C}}$ and $\boldsymbol{\mathscr{C}}'$. This yields:

$$k_1(\mathscr{C},\mathscr{C}') = \begin{cases} \left(\dfrac{\frac{2}{15}\mathscr{C}'^5 - \frac{2}{3}\mathscr{C}'^3\mathscr{C}^2 - 4\mathrm{g}(\mathscr{C}')}{-\frac{1}{2}\mathscr{C}'^2\mathscr{C}^2}\right) & ; \ (\mathscr{C}' \le \mathscr{C}) \ , \\ \left(\dfrac{\frac{2}{15}\mathscr{C}^5 - \frac{2}{3}\mathscr{C}^3\mathscr{C}'^2 - 4\mathrm{g}(\mathscr{C})}{-\frac{1}{2}\mathscr{C}^2\mathscr{C}'^2}\right) & ; \ (\mathscr{C}' \ge \mathscr{C}) \ , \end{cases} \tag{2.78}$$

$$k_2(\mathscr{C},\mathscr{C}') = \begin{cases} \dfrac{\left(\begin{array}{l} \frac{2}{35}\mathscr{C}'^7 - \frac{2}{15}\mathscr{C}'^5\mathscr{C}^2 + \mathscr{C}'\mathscr{C}^2 \\ -3\mathscr{C}'^3 + 18\mathscr{C}' + \big(-6\mathscr{C}'^4 \\ +2\mathscr{C}^2\mathscr{C}'^2 - 3\mathscr{C}^2 + 15\mathscr{C}'^2 \\ -18\big)P(\mathscr{C}') \end{array}\right)}{\left(-\frac{1}{2}\mathscr{C}'^3\mathscr{C}^3\right)} & ; \ (\mathscr{C}' \le \mathscr{C}) \ , \\ \dfrac{\left(\begin{array}{l} \frac{2}{35}\mathscr{C}^7 - \frac{2}{15}\mathscr{C}^5\mathscr{C}'^2 + \mathscr{C}\mathscr{C}'^2 \\ -3\mathscr{C}^3 + 18\mathscr{C} + \big(-6\mathscr{C}^4 \\ +2\mathscr{C}'^2\mathscr{C}^2 - 3\mathscr{C}'^2 + 15\mathscr{C}^2 \\ -18\big)P(\mathscr{C}) \end{array}\right)}{\left(-\frac{1}{2}\mathscr{C}^3\mathscr{C}'^3\right)} & ; \ (\mathscr{C}' \ge \mathscr{C}) \ , \end{cases} \tag{2.79}$$

where:

$$P(\mathscr{C}) = \tfrac{1}{2}\sqrt{\pi}\exp\left(\mathscr{C}^2\right)\operatorname{erf}(\mathscr{C}) \ , \tag{2.80}$$

$$g(\mathscr{C}) = \mathscr{C} + \left(\mathscr{C}^2 - 1\right)P(\mathscr{C}) \ . \tag{2.81}$$

The higher-order $k_n(\mathscr{C},\mathscr{C}')$ are known, but are not needed for current discussion.

For $\hat{a}_{pq}$ and $\hat{b}_{pq}$, we find that:

$$\hat{a}_{pq} = -\tfrac{1}{2}\left(\frac{2}{\pi}\right)^{1/2} \times \int_0^\infty \exp\left(-\mathscr{C}^2\right)\mathscr{C}^2\mathscr{L}_1\left(\mathscr{C}S_{3/2}^{(p)}\left(\mathscr{C}^2\right)\right)\mathscr{C}S_{3/2}^{(q)}\left(\mathscr{C}^2\right)\mathrm{d}\mathscr{C} \ , \tag{2.82}$$

$$\hat{b}_{pq} = -\tfrac{1}{3}\left(\frac{2}{\pi}\right)^{1/2} \times \int_0^\infty \exp\left(-\mathscr{C}^2\right)\mathscr{C}^2\mathscr{L}_2\left(\mathscr{C}^2S_{5/2}^{(p-1)}\left(\mathscr{C}^2\right)\right)\mathscr{C}^2S_{5/2}^{(q-1)}\left(\mathscr{C}^2\right)\mathrm{d}\mathscr{C} \ , \tag{2.83}$$

where:

$$\mathscr{L}_1\varphi(\mathscr{C}) = -\upsilon(\mathscr{C})\varphi(\mathscr{C}) + \int_0^\infty \exp\left(-\mathscr{C}'^2\right)\mathscr{C}'^2 k_1(\mathscr{C},\mathscr{C}')\varphi(\mathscr{C}')\,\mathrm{d}\mathscr{C}' \ , \tag{2.84}$$

$$\mathscr{L}_2\varphi(\mathscr{C}) = -\upsilon(\mathscr{C})\varphi(\mathscr{C}) + \int_0^\infty \exp\left(-\mathscr{C}'^2\right)\mathscr{C}'^2 k_2(\mathscr{C},\mathscr{C}')\,\varphi(\mathscr{C}')\,\mathrm{d}\mathscr{C}' \,. \tag{2.85}$$

For this work, $\hat{a}_{pq}$ and $\hat{b}_{pq}$ have been evaluated analytically (for low orders) from Eqs. (2.82) and (2.83) (using *Mathematica* ®) and the results are in agreement with those from Burnett [13] and from Foch and Ford [38]. The present, analytical results have also been verified by numerically evaluating the relevant two-fold integrals as well. In Tables 2.1 and 2.2, values for $\hat{a}_{pq}$ and $\hat{b}_{pq}$ up to order $N = 5$ as obtained by all of the methods discussed above have been reported explicitly.

Table 2.1: The normalized bracket integrals, $\hat{a}_{pq}$, for rigid-sphere molecules with $N = 5$ where $\hat{a}_{pq}$ is defined by Eq. (2.64).

p	q				
	1	2	3	4	5
1	$+1$	$-\frac{1}{4}$	$-\frac{1}{32}$	$-\frac{1}{128}$	$-\frac{5}{2048}$
2	$-\frac{1}{4}$	$+\frac{45}{16}$	$-\frac{103}{128}$	$-\frac{59}{512}$	$-\frac{267}{8192}$
3	$-\frac{1}{32}$	$-\frac{103}{128}$	$+\frac{5657}{1024}$	$-\frac{6783}{4096}$	$-\frac{16619}{65536}$
4	$-\frac{1}{128}$	$-\frac{59}{512}$	$-\frac{6783}{4096}$	$+\frac{149749}{16384}$	$-\frac{734875}{262144}$
5	$-\frac{5}{2048}$	$-\frac{267}{8192}$	$-\frac{16619}{65536}$	$-\frac{734875}{262144}$	$+\frac{57292281}{4194304}$

Table 2.2: The normalized bracket integrals, $\hat{b}_{pq}$, for rigid-sphere molecules with $N = 5$ where $\hat{b}_{pq}$ is defined by Eq. (2.65).

p	q				
	1	2	3	4	5
1	$+1$	$-\frac{1}{4}$	$-\frac{1}{32}$	$-\frac{1}{128}$	$-\frac{5}{2048}$
2	$-\frac{1}{4}$	$+\frac{205}{48}$	$-\frac{163}{128}$	$-\frac{287}{1536}$	$-\frac{1321}{24576}$
3	$-\frac{1}{32}$	$-\frac{163}{128}$	$+\frac{11889}{1024}$	$-\frac{15043}{4096}$	$-\frac{38203}{65536}$
4	$-\frac{1}{128}$	$-\frac{287}{1536}$	$-\frac{15043}{4096}$	$+\frac{1220437}{49152}$	$-\frac{6323017}{786432}$
5	$-\frac{5}{2048}$	$-\frac{1321}{24576}$	$-\frac{38203}{65536}$	$-\frac{6323017}{786432}$	$+\frac{575512075}{12582912}$

2.6 Results for rigid-sphere molecules

For rigid-sphere molecules, since a_{pq} and b_{pq} can be computed analytically, the coefficients, a_p and b_q, may be readily computed from Eqs. (2.36) and (2.37) by use of a linear equation solver. For computational purposes, as well as for ease in the reporting of results, it is convenient (but not necessary) to introduce non-dimensional coefficients, $\hat{a}_p$ and $\hat{b}_p$. These can be expressed as:

$$a_p = \left[\sigma^2 \left(\frac{kT}{m}\right)^{1/2}\right]^{-1} \hat{a}_p \ , \ b_p = \left[\sigma^2 \left(\frac{kT}{m}\right)^{1/2}\right]^{-1} \hat{b}_p \ . \tag{2.86}$$

Here, the non-dimensional coefficients, $\hat{a}_p$ and $\hat{b}_p$, are determined from:

$$\sum_{p=1}^{N} \hat{a}_{pq}\hat{a}_p = \frac{1}{8\pi^{1/2}}\alpha_q \quad (q = 1, 2, \ldots, N) \ , \tag{2.87}$$

$$\sum_{p=1}^{N} \hat{b}_{pq}\hat{b}_p = \frac{1}{8\pi^{1/2}}\beta_q \quad (q = 1, 2, \ldots, N) \ . \tag{2.88}$$

Also, one should note that the Chapman-Enskog solutions, $A(\mathscr{C})$ and $B(\mathscr{C})$, can be similarly expressed in non-dimensional form as:

$$A(\mathscr{C}) = \left[\sigma^2 \left(\frac{kT}{m}\right)^{1/2}\right]^{-1} \hat{A}(\mathscr{C}) \ , \tag{2.89}$$

$$B(\mathscr{C}) = \left[\sigma^2 \left(\frac{kT}{m}\right)^{1/2}\right]^{-1} \hat{B}(\mathscr{C}) \ , \tag{2.90}$$

where:

$$\hat{A}(\mathscr{C}) = \sum_{p=1}^{N} \hat{a}_p S_{3/2}^{(p)}\left(\mathscr{C}^2\right) \ , \tag{2.91}$$

$$\hat{B}(\mathscr{C}) = \sum_{p=1}^{N} \hat{b}_p S_{5/2}^{(p-1)}\left(\mathscr{C}^2\right) \ . \tag{2.92}$$

Furthermore, the transport coefficients can now be written as:

$$\lambda = \frac{1}{\sigma^2}\left(\frac{k^3T}{m}\right)^{1/2}\left(-\tfrac{5}{2}\hat{a}_1\right) \ , \tag{2.93}$$

$$\mu = \frac{1}{\sigma^2}\left(mkT\right)^{1/2}\hat{b}_1 \ . \tag{2.94}$$

In this context, it is instructive to note that in the first approximation, $N = 1$, one has:

$$\hat{a}_1 = -\frac{15}{32\pi^{1/2}} = -0.2644638672 \ , \ \hat{b}_1 = \frac{5}{16\pi^{1/2}} = 0.1763092448 \ . \qquad (2.95)$$

Since the coefficients, $\hat{a}_{pq}$ and $\hat{b}_{pq}$, are pure fractions, the necessary inversions of the associated matrices can be made exactly (infinite precision). This has permitted the evaluation of $\hat{a}_p$ and $\hat{b}_p$ as pure fractions as well (the factor $\sqrt{\pi}$ appears just as a post-inversion multiplicative factor) to any order of approximation, N. In doing this, integer arithmetic and a numerical precision of 400 for multiplicative factors such as $\sqrt{\pi}$ in the post-inversion stage have been used. In Table 3, numerical values of the coefficients, $\hat{a}_p$ and $\hat{b}_p$, for orders, $N = 1, 2, 3, 4, 5, 10$ have been reported. Several aspects of the results in Table 2.3 should be mentioned here. First, the coefficients, $\hat{a}_1$ and $\hat{b}_1$, converge very rapidly with increasing order of the expansion and, thus, the low-order expansions are actually quite good. Since these coefficients are direct measures of the thermal conductivity and viscosity, respectively, it is also clear that, for a rigid-sphere gas, one can obtain results for these transport coefficients to any desired precision without any need for numerical integrations and floating point arithmetic. Next, it can also be seen that $\hat{a}_p$ and $\hat{b}_p$, for $p > 1$, also converge very rapidly with increasing order of the expansion and, thus, again $\mathscr{C}\hat{A}(\mathscr{C})$ and $\mathscr{C}^2\hat{B}(\mathscr{C})$ can be computed to any desired precision.

Table 2.3: The coefficients, $\hat{a}_p$ and $\hat{b}_p$, in the thermal conductivity and viscosity solutions, respectively, for a simple, rigid-sphere gas for selected orders of the expansion, N.

N	p	$\hat{a}_p$	$\hat{b}_p$
1	1	−2.644638672 E−01	1.763092448 E−01
2	1	−2.704744097 E−01	1.789276989 E−01
	2	−2.404216975 E−02	1.047381652 E−02
3	1	−2.710274701 E−01	1.791088438 E−01
	2	−2.559671789 E−02	1.098749518 E−02
	3	−5.261547568 E−03	1.687204703 E−03

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Table 2.3 – Continued

N	p	$\hat{a}_p$	$\hat{b}_p$
4	1	−2.711110227 E−01	1.791312363 E−01
	2	−2.581667108 E−02	1.104727263 E−02
	3	−5.774728997 E−03	1.823251999 E−03
	4	−1.603510094 E−03	4.091736849 E−04
5	1	−2.711276380 E−01	1.791350752 E−01
	2	−2.585960743 E−02	1.105736149 E−02
	3	−5.867440808 E−03	1.844580304 E−03
	4	−1.801454829 E−03	4.531142486 E−04
	5	−5.887995382 E−04	1.257183628 E−04
10	1	−2.711331397 E−01	1.791361732 E−01
	2	−2.587374859 E−02	1.106023815 E−02
	3	−5.897114182 E−03	1.850509236 E−03
	4	−1.859175579 E−03	4.642795019 E−04
	5	−7.025420616 E−04	1.469644729 E−04
	6	−2.987432846 E−04	5.419264743 E−05
	7	−1.378800291 E−04	2.223377525 E−05
	8	−6.712664712 E−05	9.806088761 E−06
	9	−3.313900504 E−05	4.453953963 E−06
	10	−1.476957983 E−05	1.851082163 E−06

The reader should note that the values in this table have been truncated at 10 significant figures for the purpose of inclusion in this table even though the integer arithmetic used yielded infinite precision through the entire matrix inversion process. Multiplicative constants included at the post-inversion stage of the calculations were retained to a precision of 400.

It is worthwhile to note that, for the lower-order approximations with $N = 1, 2, 3, 4, 5$, the values for $\hat{a}_1$ and $\hat{b}_1$ have been found to be the following multiples of $1/8\sqrt{\pi}$, respectively:

$$\begin{aligned} \text{for } \hat{a}_1 \ : \ \Bigg\{ & -\frac{15}{4}, \ -\frac{675}{176}, \ -\frac{914835}{238048}, \ -\frac{3221214885}{837929728}, \\ & -\frac{4459785339426885}{1160046081791488} \Bigg\}, \end{aligned} \tag{2.96}$$

$$\begin{aligned} \text{for } \hat{b}_1 \ : \ \Bigg\{ & +\frac{5}{2}, \ +\frac{1025}{404}, \ +\frac{5893845}{2320688}, \ +\frac{284672445045}{112075112864}, \\ & +\frac{639165671950185757}{251633169670177536} \Bigg\}. \end{aligned} \tag{2.97}$$

From these multiples, it is straightforward to express in analytical form (using the notation of Chapman and Cowling [10]) the ratios of the transport coefficients obtained from the order N approximations to those obtained from the order 1 approximations, i.e. for $N = 1, 2, 3, 4, 5$:

$$\frac{[\lambda]_N}{[\lambda]_1} = 1 \, , \; \frac{45}{44} \, , \; \frac{60989}{59512} \, , \; \frac{214747659}{209482432} \, , \; \frac{297319022628459}{290011520447872} \, , \tag{2.98}$$

$$\frac{[\mu]_N}{[\mu]_1} = 1 \, , \; \frac{205}{202} \, , \; \frac{1178769}{1160344} \, , \; \frac{56934489009}{56037556432} \, , \; \frac{639165671950185757}{629082924175443840} \, . \tag{2.99}$$

These fractions, when converted to numerical values, are in agreement with the values reported in Chapman and Cowling for orders up to $N = 4$. In the current work, this process has been carried out much further going up to order $N = 150$ and, in principle, there was no reason that the process could not have been carried out to any order necessary to achieve a specific desired degree of convergence in the values of the transport coefficients since only integer math was involved in the calculations. In Table 2.4 results have been presented for $N = 150$ in an abbreviated form that allows one to clearly see the degree to which convergence has been obtained for the transport coefficients. Based upon these results, which have been presented to 32 (or less) significant digits in Table 2.4, the following ratios for simple, rigid-sphere gases are believed to be precise to at least 25 digits:

$$\frac{[\lambda]_\infty}{[\lambda]_1} = 1.0252181683234523152755255 \, , \tag{2.100}$$

$$\frac{[\mu]_\infty}{[\mu]_1} = 1.0160339416559622539545479 \, . \tag{2.101}$$

Finally, by way of comparison, the values reported by Chapman and Cowling [10] for the order 4 approximation were:

$$\frac{[\lambda]_4}{[\lambda]_1} = 1.025213 \, , \tag{2.102}$$

$$\frac{[\mu]_4}{[\mu]_1} = 1.01600 \, , \tag{2.103}$$

and the values reported by Pekeris and Alterman [26], after solving their differential equation, were:

$$\frac{[\lambda]}{[\lambda]_1} = 1.025218 \; , \tag{2.104}$$

$$\frac{[\mu]}{[\mu]_1} = 1.016034 \; . \tag{2.105}$$

Table 2.4: The convergence of the thermal conductivity (λ) and viscosity (μ) transport coefficients with increasing order of the approximations.

N	$[\lambda]_N \, / \, [\lambda]_1$	$[\mu]_N \, / \, [\mu]_1$
1	1.0000000	1.00000000
2	1.0227272	1.01485148
3	1.0248185	1.01587891
4	1.0251344	1.01600591
5	1.0251972	1.01602769
6	1.0252122	1.01603232
7	1.0252163	1.01603347
8	1.0252175	1.01603379
9	1.0252179	1.01603389
10	1.0252180	1.01603392
$\vdots$	$\vdots$	$\vdots$
100	1.02521816832345231527555251	1.01603394165596225395457984
$\vdots$	$\vdots$	$\vdots$
140	1.0252181683234523152755252740	1.016033941655962253954579863833
150	1.0252181683234523152755252741	1.016033941655962253954579863836

These results are presented in the form used by Chapman and Cowling [10]. Specifically, they have been normalized with respect to the values obtained from the order 1 approximations and have been truncated at 32 (or less) significant figures.

The work presented here has shown that, for the transport coefficients associated with a simple, rigid-sphere gas, the use of Sonine polynomial expansions can yield results to any desired precision. If calculations of the transport coefficients to a very high precision were the main purpose of an investigation, it would be difficult to do better than this approach.

In Tables 2.5 and 2.6 the results for the Chapman-Enskog solutions for thermal conductivity and viscosity, $\mathscr{C}\hat{A}\,(\mathscr{C})$ and $\mathscr{C}^2\hat{B}\,(\mathscr{C})$, respectively, have been reported as functions of $\mathscr{C}$ from $\mathscr{C} = 0.0$ to $\mathscr{C} = 6.0$ for two approximations of order $N = 140, 150$. In these tables the results have been truncated at 16 significant figures for the purpose of including them in the tables even though the integer arithmetic that was used to generate the results was exact for the inversion of the associated matrices and, thus, exhibits infinite precision. Note that where the results involve multiplicative constants included at the post-inversion stage, a numerical precision of 400 has been used in the calculations. This 16 significant figure precision is adequate to permit the reader to determine how well the results have converged. In Table 2.7 results for the modified Chapman-Enskog solutions for thermal conductivity and viscosity, $(2\pi)^{1/2}\mathscr{C}\hat{A}(\mathscr{C})$ and $(2\pi)^{1/2}\mathscr{C}^2\hat{B}(\mathscr{C})$, respectively, have been reported as functions of $\mathscr{C}$ from $\mathscr{C} = 0.0$ to $\mathscr{C} = 6.0$ for $N = 150$. These results have been compared with values obtained by two other methods previously reported in the literature [26,34] (where the factor $(2\pi)^{1/2}$ has been introduced strictly to compare with the functions that have been tabulated in earlier works. The results of Siewert and co-workers [35, 36] have not been included in this table as they are also in good agreement with the previously reported values, and the values reported here). From an examination of the results for $N = 140, 150$ in Tables 2.5 and 2.6, it appears that the results as presented in Table 2.7 represent convergence to the number of figures given. It is notable, however, that approximations of even higher order than $N = 150$ will be needed if comparable levels of precision are desired for higher values of $\mathscr{C}$.

Table 2.5: Values of the Chapman-Enskog solution for thermal conductivity, for a simple, rigid-sphere gas.

$\mathscr{C}$	$\mathscr{C}\hat{A}(\mathscr{C})$ $(N = 140)$	$\mathscr{C}\hat{A}(\mathscr{C})$ $(N = 150)$
0.0	0.0	0.0
0.1	–0.08607773065808671	–0.08607773065808957
0.2	–0.1692707734107010	–0.1692707734107006
0.3	–0.2467894864225835	–0.2467894864225829
0.4	–0.3160227955545736	–0.3160227955545744
0.5	–0.3746013706876520	–0.3746013706876515
0.6	–0.4204360467288163	–0.4204360467288164

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Table 2.5 – Continued

$\mathscr{C}$	$\mathscr{C}\hat{A}(\mathscr{C})$ $(N = 140)$	$\mathscr{C}\hat{A}(\mathscr{C})$ $(N = 150)$
0.7	–0.4517317206003278	–0.4517317206003281
0.8	–0.4669805525509980	–0.4669805525509976
0.9	–0.4649402245991789	–0.4649402245991793
1.0	–0.4446033173345084	–0.4446033173345083
1.1	–0.4051630446501128	–0.4051630446501126
1.2	–0.3459792278213244	–0.3459792278213248
1.3	–0.2665469736960984	–0.2665469736960981
1.4	–0.1664693276608043	–0.1664693276608045
1.5	–0.04543430493782625	–0.04543430493782639
1.6	0.09680384657847260	0.09680384657847301
1.7	0.2604395964897753	0.2604395964897748
1.8	0.4456278610934124	0.4456278610934127
1.9	0.6524930897100649	0.6524930897100650
2.0	0.8811362388016972	0.8811362388016967
2.1	1.131640090895628	1.131640090895629
2.2	1.404073295567058	1.404073295567057
2.3	1.698493434692079	1.698493434692079
2.4	2.014949348789879	2.014949348789880
2.5	2.353482906971993	2.353482906971992
2.6	2.714130359397194	2.714130359397195
2.7	3.096923376924544	3.096923376924544
2.8	3.501889856314451	3.501889856314449
2.9	3.929054549322168	3.929054549322171
3.0	4.378439558998862	4.378439558998859
3.1	4.850064735318494	4.850064735318496
3.2	5.343947993958881	5.343947993958885
3.3	5.860105575953075	5.860105575953067
3.4	6.398552261432833	6.398552261432843
3.5	6.959301547382858	6.959301547382853
3.6	7.542365796894370	7.542365796894362
3.7	8.147756365616057	8.147756365616082
3.8	8.775483709773325	8.775483709773291
3.9	9.425557479139708	9.425557479139729
4.0	10.09798659760684	10.09798659760686
5.0	18.05338483530590	18.05338483530922
5.5	22.87154695379960	22.87154695381391
6.0	28.25065932487901	28.25065932464951

Results are truncated at 16 significant figures from the infinite precision arithmetic results and are presented for the expansion approximations of order $N = 140$ and $N = 150$ and for values of $\mathscr{C}$ from $\mathscr{C} = 0.0$ to $\mathscr{C} = 6.0$.

Table 2.6: Values of the Chapman-Enskog solution for viscosity, for a simple, rigid-sphere gas.

$\mathscr{C}$	$\mathscr{C}^2 \hat{B}(\mathscr{C})$ $(N = 140)$	$\mathscr{C}^2 \hat{B}(\mathscr{C})$ $(N = 150)$
0.0	0.0	0.0
0.1	0.002471778056461653	0.002471778056462115
0.2	0.009841135524255944	0.009841135524256284
0.3	0.02197390741041948	0.02197390741041921
0.4	0.03865835753092612	0.03865835753092620
0.5	0.05962082039231741	0.05962082039231749
0.6	0.08454392037081006	0.08454392037080992
0.7	0.1130849263351761	0.1130849263351762
0.8	0.1448923018631456	0.1448923018631455
0.9	0.1796192562408724	0.1796192562408724
1.0	0.2169338406589923	0.2169338406589925
1.1	0.2565257063394405	0.2565257063394404
1.2	0.2981099908704467	0.2981099908704467
1.3	0.3414289472973078	0.3414289472973078
1.4	0.3862519353597600	0.3862519353597599
1.5	0.4323743174502101	0.4323743174502102
1.6	0.4796156920599805	0.4796156920599804
1.7	0.5278177858218355	0.5278177858218355
1.8	0.5768422278281085	0.5768422278281086
1.9	0.6265683524628100	0.6265683524628099
2.0	0.6768911193869824	0.6768911193869825
2.1	0.7277191986906966	0.7277191986906966
2.2	0.7789732419074355	0.7789732419074353
2.3	0.8305843421021580	0.8305843421021583
2.4	0.8824926757054153	0.8824926757054151
2.5	0.9346463129504795	0.9346463129504795
2.6	0.9870001810541189	0.9870001810541192
2.7	1.039515163520589	1.039515163520588
2.8	1.092157319361425	1.092157319361425
2.9	1.144897207091162	1.144897207091161
3.0	1.197709299745165	1.197709299745164
3.1	1.250571478659984	1.250571478659985
3.2	1.303464595232603	1.303464595232602
3.3	1.356372091261850	1.356372091261851
3.4	1.409279669737781	1.409279669737783
3.5	1.462175009069577	1.462175009069573
3.6	1.515047514730157	1.515047514730162
3.7	1.567888103153649	1.567888103153648
3.8	1.620689013461474	1.620689013461469
3.9	1.673443643227991	1.673443643228004
4.0	1.726146405039169	1.726146405039151
5.0	2.249607322611573	2.249607322612014

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Table 2.6 – Continued

$\mathscr{C}$	$\mathscr{C}^2\hat{B}(\mathscr{C})$ $(N = 140)$	$\mathscr{C}^2\hat{B}(\mathscr{C})$ $(N = 150)$
5.5	2.508650364393547	2.508650364403088
6.0	2.765919684662051	2.765919684750519

Results are truncated at 16 significant figures from the infinite precision arithmetic results and are presented for the expansion approximations of order $N = 140$ and $N = 150$ and for values of $\mathscr{C}$ from $\mathscr{C} = 0.0$ to $\mathscr{C} = 6.0$.

Table 2.7: The modified Chapman-Enskog solution, $(2\pi)^{1/2}\mathscr{C}\hat{A}(\mathscr{C})$, for the thermal conductivity for a simple, rigid-sphere gas.

$\mathscr{C}$	$(2\pi)^{1/2}\mathscr{C}\hat{A}(\mathscr{C})$		
	Pekeris and Altermann [26]	Loyalka and Hickey [34]	Present Work $(N = 150)$
0.0	0.0	0.0	0.0
0.1	–0.215765	–0.21576	–0.215764873483
0.2	–0.424299	–0.42429	–0.424298906699
0.3	–0.618610	–0.61860	–0.618609504548
0.4	–0.792152	–0.79215	–0.792151674765
0.5	–0.938986	–0.93898	–0.938986387481
0.6	–1.053877	–1.0538	–1.053876882404
0.7	–1.132324	–1.1323	–1.132323503404
0.8	–1.170543	–1.1705	–1.170546656727
0.9	–1.165432	–1.1654	–1.165432312993
1.0	–1.114455	–1.1144	–1.114455246225
1.1	–1.015593	–1.0155	–1.015593143555
1.2	–0.867241	–0.86724	–0.867241314891
1.3	–0.668134	–0.66813	–0.668134180783
1.4	–0.417277	–0.41727	–0.417276723573
1.5	–0.113887	–0.11388	–0.113886913395
1.6	0.242651	0.24265	0.2426512589266
1.7	0.652825	0.65282	0.6528252563947
1.8	1.117023	1.1170	1.1170233965800
1.9	1.635558	1.6355	1.6355576276685
2.0	2.208681	2.2086	2.2086810099823
2.1	2.836601	2.8365	2.8366010485449
2.2	3.519490	3.5194	3.5194898223227
2.3	4.257492	4.2574	4.2574916676742
2.4	5.050729	5.0507	5.0507290096260
2.5	5.899307	5.8993	5.899306798476
2.6	6.803316	6.8033	6.803315899899
2.7	7.762836	7.7628	7.762835700964
2.8	8.777936	8.7779	8.777936128481
2.9	9.848679	9.8486	9.848679225898

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Table 2.7 – Continued

$\mathscr{C}$	$(2\pi)^{1/2}\,\mathscr{C}\hat{A}\,(\mathscr{C})$		
	Pekeris and Altermann [26]	Loyalka and Hickey [34]	Present Work ($N = 150$)
3.0	10.975	10.9751	10.97512039734
3.1	12.157	12.1573	12.15730939934
3.2	13.395	13.3952	13.39529113981
3.3	14.689	14.6891	14.68910632900
3.4	16.039	16.0387	16.03879201521
3.5	17.444	17.4443	17.44438203035
3.6	18.906	18.9059	18.90590736410
3.7	20.423	20.4233	20.42339648085
3.8	21.997	21.9968	21.99687559048
3.9	23.626	23.6263	23.62636888137
4.0	25.319	25.3118	25.31189872240
4.1		27.0534	27.05348583905
4.2		28.8511	28.85114946776
4.3		30.7049	30.7049074921
4.4		32.6147	32.6147765631
4.5		34.5807	34.5807722066
4.6			36.6029089186
4.7			38.681200250
4.8			40.815658887
4.9			43.006296713
5.0	45	45.2531	45.253124880
5.1			47.556153859
5.2			49.915393492
5.3			52.33085304
5.4			54.80254122
5.5		57.3304	57.33046627
5.6			59.91463594
5.7			62.5550575
5.8			65.2517380
5.9			68.0046839
6.0		70.8138	70.8139014

Note that the values given for the current work have been truncated to the number of significant figures which appear to have converged based upon a comparison of the order $N = 140$ and $N = 150$ expansion results.

Table 2.8: The modified Chapman-Enskog solution, $(2\pi)^{1/2}\mathscr{C}^2\hat{B}(\mathscr{C})$, for the viscosity for a simple, rigid-sphere gas.

$\mathscr{C}$	$(2\pi)^{1/2}\mathscr{C}^2\hat{B}(\mathscr{C})$		
	Pekeris and Altermann [26]	Loyalka and Hickey [34]	Present Work ($N = 150$)
0.0	0.0	0.0	0.0
0.1	0.006196	0.006195	0.00619582876
0.2	0.024668	0.024668	0.02466806855
0.3	0.055080	0.055080	0.055080417619
0.4	0.096902	0.096902	0.096902132037
0.5	0.149447	0.14944	0.149447234152
0.6	0.211920	0.21192	0.211920181249
0.7	0.283462	0.28346	0.283461873786
0.8	0.363191	0.36319	0.363191140626
0.9	0.450239	0.45023	0.450238706361
1.0	0.543772	0.54377	0.543772498720
1.1	0.643015	0.64301	0.643014588680
1.2	0.747251	0.74725	0.747250932065
1.3	0.855835	0.85583	0.855835453072
1.4	0.968190	0.96818	0.968190022303
1.5	1.083802	1.0838	1.083801689344
1.6	1.202218	1.2022	1.202218254674
1.7	1.323043	1.3230	1.323042985794
1.8	1.445929	1.4459	1.445929038275
1.9	1.570574	1.5705	1.570573948272
2.0	1.696714	1.6967	1.696714418702
2.1	1.824122	1.8241	1.824121519429
2.2	1.952596	1.9525	1.952596353346
2.3	2.081966	2.0819	2.081966196379
2.4	2.212081	2.2120	2.212081093077
2.5	2.342811	2.3428	2.342810874821
2.6	2.474043	2.4740	2.474042560896
2.7	2.605678	2.6056	2.605678100788
2.8	2.737632	2.7376	2.737632417056
2.9	2.869832	2.8698	2.869831710840
3.0	3.0022	3.0022	3.002211995529
3.1	3.1347	3.1347	3.134717827856
3.2	3.2673	3.2673	3.267301209390
3.3	3.3999	3.3999	3.399920634877
3.4	3.5325	3.5325	3.532540267027
3.5	3.6651	3.6651	3.665129220192
3.6	3.7977	3.7976	3.797660937832
3.7	3.9301	3.9301	3.930112650822
3.8	4.0625	4.0624	4.062464905526
3.9	4.1947	4.1947	4.194701152116
4.0	4.3268	4.3268	4.326807385023

Continued on Next Page...

Table 2.8 – Continued

$\mathscr{C}$	$(2\pi)^{1/2}\,\mathscr{C}^2\hat{B}\,(\mathscr{C})$		
	Pekeris and Altermann [26]	Loyalka and Hickey [34]	Present Work ($N = 150$)
4.1		4.4587	4.458771828523
4.2		4.5905	4.59058466148
4.3		4.7222	4.72223777611
4.4		4.8537	4.85372456633
4.5		4.9850	4.98503974186
4.6			5.11617916482
4.7			5.2471397060
4.8			5.3779191182
4.9			5.5085159249
5.0	5.6	5.6389	5.6389293216
5.1			5.769159089
5.2			5.899205518
5.3			6.029069338
5.4			6.158751663
5.5		6.2882[a]	6.288253934
5.6			6.417577874
5.7			6.546725447
5.8			6.67569882
5.9			6.80450034
6.0		6.9331	6.9331324

Note that the values given for the current work have been truncated to the number of significant figures which appear to have converged based upon a comparison of the order $N = 140$ and $N = 150$ expansion results.
[a]This value has been corrected from the value of 6.2868 reported by Loyalka and Hickey [34] which has been determined to have been a typographical error. The previous authors have recomputed the value using their original program and have verified that the actual result was 6.28825394 which was truncated to five significant figures.

In this paper, the reported results for a simple, rigid-sphere gas have been limited to expansions of up to order $N = 150$ even though it is theoretically possible to extend the expansions to any desired higher order. In practice, the matrix coefficients, $\hat{a}_{pq}$ and $\hat{b}_{pq}$, for simple, rigid-sphere gases, can be determined in the manner described using integer arithmetic and, hence, infinite precision can be obtained. The necessary matrices can be readily generated for expansions up to $N = 200$; however, the infinite precision, integer arithmetic, matrix inversion process to find

$\hat{a}_p$ and $\hat{b}_p$ is slow on existing computer systems available to the author for expansions higher than about $N = 150$. Thus, it has been more convenient to explore the higher-order matrix inversions using floating point operations. Specifically, it should be noted that when this approach was used to invert the matrix for the order $N = 200$ expansion, a substantial savings in computational time was achieved. During this process, all numbers initially represented to a precision of 400 in all of the calculations and numerical results were successfully obtained that confirm the values reported in Table 6 which were computed using integer arithmetic (in so far as was appropriate).

It is also important to take note of the quantities, ε_t and ε_p, introduced in some previous works [34–36]. These quantities are expressed in the present nomenclature as:

$$\varepsilon_t = \sqrt{2\pi}\,\tfrac{2}{5}\left[\frac{1}{\sigma^2}\left(\frac{k^3T}{m}\right)^{1/2}\right]^{-1}\lambda = -\sqrt{2\pi}\,\hat{a}_1\ , \tag{2.106}$$

$$\varepsilon_p = \sqrt{2\pi}\left[\frac{1}{\sigma^2}\,(mkT)^{1/2}\right]^{-1}\mu = \sqrt{2\pi}\,\hat{b}_1\ , \tag{2.107}$$

such that the present work yields values of (truncated to 34 significant figures):

$$\varepsilon_t = \begin{cases} 0.6796300490785916669371706138604712\ ; & N = 140\ , \\ 0.6796300490785916669371706138617483\ ; & N = 150\ , \end{cases} \tag{2.108}$$

$$\varepsilon_p = \begin{cases} 0.4490278062878924346090494895345538\ ; & N = 140\ , \\ 0.4490278062878924346090494895346545\ ; & N = 150\ . \end{cases} \tag{2.109}$$

As noted previously, the computer programs constructed in this work permit one to express the bracket integrals, a_{pq} and b_{pq}, in terms of the omega integrals to any order of approximation and, for rigid spheres, the expressions generated from these programs (using analytical values for the omega integrals for rigid spheres) have been verified against several other results of both an analytical as well as a numerical nature. This implies that one should be able to obtain the Chapman-Enskog solutions to any arbitrary precision for a given potential if an adequate program to compute the omega integrals (numerically) to the requisite order and precision is available. A program has been constructed for omega integrals that is working quite well for arbitrary potential models for reasonably high orders. It is anticipated that some results for the Chapman-Enskog solutions for selected realistic potential models (e.g. Lennard-Jones) will be reported in the near future. The present work has also been extended to gas mixtures and, for rigid-sphere molecules, the results have shown convergence similar to that reported here for all of the cases that have been computed with respect to both the mixture transport coefficients and the Chapman-Enskog solutions. It is expected that these gas-mixture computations will be extended to realistic potential models as well since all of the pieces needed for the computations are in place. The results for binary mixtures will be reported in subsequent chapters. The work for ternary and higher mixtures and for realistic potentials is ongoing and will be reported in the future.

Chapter 3

Viscosity in a binary, rigid-sphere, gas mixture

3.1 Introduction

The Chapman-Enskog solutions of the Boltzmann equations provide a basis for the computation of important transport coefficients for both simple gases and gas mixtures [10–12,20–26,28–31,39]. The use of Sonine polynomial expansions for the Chapman-Enskog solutions was first suggested by Burnett [13] and has become the general method for obtaining the transport coefficients due to the relatively rapid convergence of this series [10–13,15,20–24]. While it has been found that relatively, low-order expansions (of order 4) can provide reasonable accuracy in computations of the transport coefficients (to about 1 part in 1,000), the adequacy of the low-order expansions for computation of the slip and jump coefficients associated with gas-surface interfaces still needs to be explored. Also of importance is the fact that such low-order expansions do not provide good convergence (in velocity space) for the actual Chapman-Enskog solutions even though the transport coefficients derived from these solutions appear to be reasonable. Thus, it is of some interest to explore Sonine polynomial expansions to higher orders. In Chapter 2 [15], it is shown that for simple, rigid-sphere gases (i.e. single-component, monatomic gases), the use of higher-order Sonine polynomial expansions enables one to obtain results of arbitrary precision that are error free. The purpose in this chapter is to report the results of the current investigation of relatively high-order, standard, Sonine polynomial expansions for the viscosity-related Chapman-Enskog solutions for binary gas mixtures of rigid-sphere molecules. In the following sections the basic theory, the theory relating to viscosity, the solution technique in terms of the Sonine polynomials, the bracket integrals, details related to the specific case of rigid-spheres molecules, and the associated results are described.

A part of the motivation with respect to this work has been some of the recently reported results on direct numerical solutions of the linearized Boltzmann equations for rigid-sphere, gas mixtures. In particular, results for the transport coefficients and the Chapman-Enskog solutions have been reported both by Takata et al. [2] and by Garcia and Siewert [1]. This work provides a benchmark for assessing the precision of the numerical results reported by these authors and, indeed, some such comparisons are reported. The present work does have an important distinguishing feature in that, for rigid-sphere gas mixtures, no numerical integrations are required and thus, in principle, results of arbitrary precision can be obtained for any given order of the Sonine polynomial expansions. The computational resources available to the author at the present time have realistically permitted expansions only to order 60 given the manner in which this technique has been implemented but, even here it has been possible to obtain convergence of 9 or more significant digits in the normalized gas mixture viscosities (depending upon the specific mass ratios, size ratios, and mole fractions considered) and it is certain that further improvements in the implementation of the technique or the availability of better computational resources will allow even higher-order expansions and greater convergence of the results. Further, note that in this work the full dependence of the solutions on the molecular masses, the molecular sizes, the mole fractions, and the intermolecular potential model via the omega integrals has been retained, and explicit (symbolic) expressions for the necessary matrix elements (the bracket integrals) used in evaluating the coefficients in the Sonine polynomial expansions for the coupled Chapman-Enskog equations have been obtained. These generalized matrix elements, once determined, need not be determined again. For rigid spheres (or for any other potential model of interest that can be represented via the omega integrals), one can then determine in a straightforward manner a set of matrix elements that are specific to the potential model being used and store them. These specific matrix elements require only the input of the appropriately computed omega integrals which, for rigid spheres, are known exactly such that no numerical integrations are needed. In this fashion, this method requires only a matrix inversion at the final step. This is important, as all that is needed for finding both the viscosity and the Chapman-Enskog viscosity solutions for arbitrary, binary, rigid-sphere gas mixtures is precomputed in a general form. Thus, one is able to study parametric dependencies and convergence of the results in an economical and systematic way as, once the matrix elements up to the highest order are computed and stored, results can be processed to any order up to this highest order without any new computations of matrix elements being required. Further, since these values for the viscosity converge with increasing order, since one can use arbitrarily high numerical precision as needed in *Mathematica*® for the final matrix inversion step, and since one can easily compare results for a given order with the results for immediately preceding orders, one can be confident in the results and the degree of convergence obtained.

3.2 The basic theory

Following the work and notations of Chapman and Cowling [10], an abbreviated version of the relevant theory is given below. For an arbitrary, rarefied, gas mixture, one begins with the Boltzmann equations describing the molecular distribution functions of the constituent gases:

$$\left(\frac{\partial}{\partial t} + \mathbf{c}_i \cdot \boldsymbol{\nabla}_r + \mathbf{F}_i \cdot \boldsymbol{\nabla}_{c_i}\right) f_i\left(\mathbf{r},\, \mathbf{c}_i, t\right) \\ = \sum_j \iiint \left(f_i' f_j' - f_i f_j\right) g b \mathrm{d}b \mathrm{d}\varepsilon \mathrm{d}\mathbf{c}_j = \sum_j J\left(f_i f_j\right) \ , \tag{3.1}$$

in which the left-hand side (LHS) is known as the streaming term of the equation which contains the differential streaming operator in the brackets, the right-hand side (RHS) is a sum over what are known as the collision integrals in which $J(f_i f_j)$ is called the collision operator, $f_i\left(\mathbf{r},\, \mathbf{c}_i, t\right)$ is the molecular distribution function of the i-th constituent, g is the magnitude of the pre-collision relative velocity, $\mathbf{g} = \mathbf{c}_j - \mathbf{c}_i$, b is the 'impact parameter' associated with the binary scattering events, ε is an angle corresponding to the azimuthal orientation of the scattering plane, and $\mathbf{c}$ is the molecular velocity. A prime ($'$) indicates a function of a post-collision velocity while the corresponding lack of a prime indicates a pre-collision velocity dependence, e.g. $f_i = f_i\left(\mathbf{r},\, \mathbf{c}_i, t\right)$ while $f_i' = f_i\left(\mathbf{r},\, \mathbf{c}_i', t\right)$. In the summation over the different constituents, scattering between like constituents (i.e. when $i = j$) is treated in the same way as scattering between unlike constituents with the various pre- and post-collision velocities retained as separate variables for purposes of integration. In this circumstance, for clarity, it is common practice to drop the i subscript inside the collision integral in order to facilitate the necessary discrimination between the velocities (i.e. $\mathbf{c}_i \to \mathbf{c}$ and $f_i \to f$). Of course it follows from this that, if one is dealing with a simple gas having only one constituent (Chapter 2), one obtains from this process the single Boltzmann equation describing the gas in which $j = 1$ and no subscript is necessary on the LHS:

$$\left(\frac{\partial}{\partial t} + \mathbf{c} \cdot \boldsymbol{\nabla}_r + \mathbf{F} \cdot \boldsymbol{\nabla}_c\right) f\left(\mathbf{r},\, \mathbf{c}, t\right) = \iiint \left(f' f_1' - f f_1\right) g b \mathrm{d}b \mathrm{d}\varepsilon \mathrm{d}\mathbf{c}_1 \ . \tag{3.2}$$

Equivalent expressions for the above equations are often encountered in which $b\mathrm{d}b\mathrm{d}\varepsilon$ is expressed as $\alpha_{ij}(g, \chi)\mathrm{d}\mathbf{e}'$ or $\sigma_{ij}(g, \chi)\mathrm{d}\boldsymbol{\Omega}$ where χ is the scattering angle (the angle between $\mathbf{g}$ and $\mathbf{g}'$) and $\alpha_{ij}(g, \chi) = \sigma_{ij}(g, \chi)$ is known as the differential collision cross-section which describes the probability per unit time per unit volume that two molecules colliding with velocities, $\mathbf{c}_i$ in $\mathrm{d}\mathbf{c}_i$ and $\mathbf{c}_j$ in $\mathrm{d}\mathbf{c}_j$, will have a relative velocity after collision, $\mathbf{g}' = \mathbf{c}'_j - \mathbf{c}'_i$, that lies within the solid angle, $\mathrm{d}\mathbf{e}' = \mathrm{d}\mathbf{\Omega} = \sin(\chi)\,\mathrm{d}\chi\mathrm{d}\varepsilon$.

For the specific case of a binary gas mixture, one expresses the distribution functions f_1 and f_2 in the form:

$$f_1 = f_1^{(0)} + f_1^{(1)} + f_1^{(2)} + \cdots \ , \tag{3.3}$$

$$f_2 = f_2^{(0)} + f_2^{(1)} + f_2^{(2)} + \cdots \ , \tag{3.4}$$

where the lowest-order approximations are chosen to be:

$$f_1^{(0)} = n_1 \left(\frac{m_1}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m_1}{2kT}\left(\mathbf{c}_1 - \mathbf{c}_0\right)^2\right) \ , \tag{3.5}$$

$$f_2^{(0)} = n_2 \left(\frac{m_2}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m_2}{2kT}\left(\mathbf{c}_2 - \mathbf{c}_0\right)^2\right) \ , \tag{3.6}$$

in which m_1 and m_2 are the molecular masses of the constituent gases, k is Boltzmann's constant, and n_1, n_2, $\mathbf{c}_0$ and T are, in general, arbitrary functions of $\mathbf{r}$ and t. Note that in choosing the lowest-order approximations to be of this form (which correspond to Maxwellian distributions), one has effectively equated the functions n_1 and n_2 to the number densities of the two gases in the mixture, T to the temperature of the mixture, and $\mathbf{c}_0$ to the mass-velocity of the mixture where, $\mathbf{c}_0 = M_1 x_1 \mathbf{c}_1 + M_2 x_2 \mathbf{c}_2$ in which, $M_i = m_i/m_0$, $m_0 = m_1 + m_2$, $x_i \equiv n_i/n$ denote the proportion by number of the constituent gases in the mixture (the mole fractions), and $n = n_1 + n_2$ is the total molecular number density of the binary mixture.

If one now limits further consideration only to the second approximation (up to $f^{(1)}$), that is equivalent to assuming that the distribution functions for each of the constituents can be expressed as small linear perturbations from equilibrium states specified by the Maxwellian distributions of Eqs. (3.5) and (3.6), i.e. $f_i = f_i^{(0)}(1 + \Phi_i^{(1)})$. Thus, $f_1^{(1)}$ and $f_2^{(1)}$ are written in the form:

$$f_1^{(1)} = f_1^{(0)} \Phi_1^{(1)} \ , \tag{3.7}$$

$$f_2^{(1)} = f_2^{(0)} \Phi_2^{(1)} \ , \tag{3.8}$$

where the perturbations, $\Phi_1^{(1)}$ and $\Phi_2^{(1)}$, satisfy the time derivative expressions given by:

$$\mathcal{D}_1^{(1)} = -n_1^2 I_1 \left(\Phi_1^{(1)} \right) - n_1 n_2 I_{12} \left(\Phi_1^{(1)} + \Phi_2^{(1)} \right) \ , \tag{3.9}$$

$$\mathcal{D}_2^{(1)} = -n_2^2 I_2 \left(\Phi_2^{(1)} \right) - n_1 n_2 I_{21} \left(\Phi_1^{(1)} + \Phi_2^{(1)} \right) \ , \tag{3.10}$$

in which:

$$\begin{aligned} \mathcal{D}_i^{(r)} = & \frac{\partial_{r-1} f_i^{(0)}}{\partial t} + \frac{\partial_{r-2} f_i^{(1)}}{\partial t} + \cdots + \frac{\partial_0 f_i^{(r-1)}}{\partial t} \\ & + \left(\mathbf{c}_i \cdot \frac{\partial}{\partial \mathbf{r}} + \mathbf{F}_i \cdot \frac{\partial}{\partial \mathbf{c}_i} \right) f_i^{(r-1)} \ , \end{aligned} \tag{3.11}$$

and:

$$n_i^2 I_i (F) = \iint f_i^{(0)} f^{(0)} \left(F_i + F - F_i' - F' \right) g \alpha_i \mathrm{d}\mathbf{e}' \mathrm{d}\mathbf{c} \ , \tag{3.12}$$

$$n_i n_j I_{ij} (K) = \iint f_i^{(0)} f_j^{(0)} \left(K - K' \right) g \alpha_{ij} \mathrm{d}\mathbf{e}' \mathrm{d}\mathbf{c}_j \ , \tag{3.13}$$

are linear functions of their arguments such that $I(\phi + \psi) = I(\phi) + I(\psi)$ and $I(a\phi) = aI(\phi)$ regardless of subscripts (where a is any arbitrary constant). The LHSs of Eqs. (3.9) and (3.10) can be expressed in the form:

$$\begin{aligned} \mathcal{D}_1^{(1)} = f_1^{(0)} \Big\{ & \left(\mathscr{C}_1^2 - \tfrac{5}{2} \right) \mathbf{C}_1 \cdot \boldsymbol{\nabla} \ln (T) + x_1^{-1} \mathbf{d}_{12} \cdot \mathbf{C}_1 \\ & + 2 \boldsymbol{\mathscr{C}}_1 \overset{\circ}{\boldsymbol{\mathscr{C}}}_1 : \boldsymbol{\nabla} \mathbf{c}_0 \Big\} \ , \end{aligned} \tag{3.14}$$

$$\begin{aligned} \mathcal{D}_2^{(1)} = f_2^{(0)} \Big\{ & \left(\mathscr{C}_2^2 - \tfrac{5}{2} \right) \mathbf{C}_2 \cdot \boldsymbol{\nabla} \ln (T) + x_2^{-1} \mathbf{d}_{21} \cdot \mathbf{C}_2 \\ & + 2 \boldsymbol{\mathscr{C}}_2 \overset{\circ}{\boldsymbol{\mathscr{C}}}_2 : \boldsymbol{\nabla} \mathbf{c}_0 \Big\} \ , \end{aligned} \tag{3.15}$$

in which $\boldsymbol{\mathscr{C}}_i \equiv (m_i / 2kT)^{1/2} \mathbf{C}_i$, $\mathbf{C}_i \equiv \mathbf{c}_i - \mathbf{c}_0$, and the bold sans serif notation, $\boldsymbol{\mathsf{w}}$, denotes a dyadic tensor, $\boldsymbol{\mathsf{w}} = \mathbf{ab}$, constructed from the components of the vectors, $\mathbf{a}$ and $\mathbf{b}$. Note that $\mathbf{d}_{12}$ is given as either of the two forms noted below:

$$\mathbf{d}_{12} = \frac{\rho_1 \rho_2}{\rho p} \left\{ \mathbf{F}_2 - \frac{1}{\rho_2} \boldsymbol{\nabla} p_2 - \left(\mathbf{F}_1 - \frac{1}{\rho_1} \boldsymbol{\nabla} p_1 \right) \right\} \ , \tag{3.16}$$

$$\mathbf{d}_{12} = \boldsymbol{\nabla} x_1 + \frac{n_1 n_2 (m_2 - m_1)}{n \rho} \boldsymbol{\nabla} \ln (p) - \frac{\rho_1 \rho_2}{\rho p} \left(\mathbf{F}_1 - \mathbf{F}_2 \right) \ , \tag{3.17}$$

where ρ_1 and ρ_2 are the mass densities of the constituent gases, $\rho = \rho_1 + \rho_2$ is the total mass density of the mixture, p_1 and p_2 are the partial pressures of the constituent gases, and $p = p_1 + p_2$ is the total pressure of the mixture. Since $\boldsymbol{\nabla} x_2 = -\boldsymbol{\nabla} x_1$, either Eq. (3.16) or Eq. (3.17) can be used to show that $\mathbf{d}_{21} = -\mathbf{d}_{12}$. The functions $\mathit{\Phi}_1^{(1)}$ and $\mathit{\Phi}_2^{(1)}$ can then be expressed as:

$$\mathit{\Phi}_1^{(1)} = -\mathbf{A}_1 \cdot \frac{\partial \ln(T)}{\partial \mathbf{r}} - \mathbf{D}_1 \cdot \mathbf{d}_{12} - 2\mathsf{B}_1 : \frac{\partial}{\partial \mathbf{r}} \mathbf{c}_0 \ , \tag{3.18}$$

$$\mathit{\Phi}_2^{(1)} = -\mathbf{A}_2 \cdot \frac{\partial \ln(T)}{\partial \mathbf{r}} - \mathbf{D}_2 \cdot \mathbf{d}_{12} - 2\mathsf{B}_2 : \frac{\partial}{\partial \mathbf{r}} \mathbf{c}_0 \ , \tag{3.19}$$

where the functions $\mathbf{A}$ and $\mathbf{D}$ are vectors and the functions B are non-divergent tensors, such that:

$$\mathbf{A} = \mathbf{C} A(C) \ , \ \mathbf{D} = \mathbf{C} D(C) \ , \ \mathsf{B} = \overset{\circ}{\mathbf{C}\mathbf{C}} B(C) \ , \tag{3.20}$$

with the appropriate subscript 1 or 2 implied throughout each expression, and where $\mathbf{A}$, $\mathbf{D}$, and B, must satisfy the following pairs of equations, respectively:

$$f_1^{(0)} \left(\mathscr{C}_1^2 - \tfrac{5}{2}\right) \mathbf{C}_1 = n_1^2 I_1(\mathbf{A}_1) + n_1 n_2 I_{12}(\mathbf{A}_1 + \mathbf{A}_2) \ , \tag{3.21}$$

$$f_2^{(0)} \left(\mathscr{C}_2^2 - \tfrac{5}{2}\right) \mathbf{C}_2 = n_2^2 I_2(\mathbf{A}_2) + n_1 n_2 I_{21}(\mathbf{A}_1 + \mathbf{A}_2) \ , \tag{3.22}$$

$$x_1^{-1} f_1^{(0)} \mathbf{C}_1 = n_1^2 I_1(\mathbf{D}_1) + n_1 n_2 I_{12}(\mathbf{D}_1 + \mathbf{D}_2) \ , \tag{3.23}$$

$$-x_2^{-1} f_2^{(0)} \mathbf{C}_2 = n_2^2 I_2(\mathbf{D}_2) + n_1 n_2 I_{21}(\mathbf{D}_1 + \mathbf{D}_2) \ , \tag{3.24}$$

$$f_1^{(0)} \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1} = n_1^2 I_1(\mathsf{B}_1) + n_1 n_2 I_{12}(\mathsf{B}_1 + \mathsf{B}_2) \ , \tag{3.25}$$

$$f_2^{(0)} \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2} = n_2^2 I_2(\mathsf{B}_2) + n_1 n_2 I_{21}(\mathsf{B}_1 + \mathsf{B}_2) \ . \tag{3.26}$$

Note, that the forms for the distribution functions $\mathit{\Phi}_1^{(1)}$ and $\mathit{\Phi}_2^{(1)}$ have been chosen such that $\mathbf{A}$ and $\mathbf{D}$ must also satisfy the relationships:

$$\int f_1^{(0)} m_1 \mathbf{C}_1 \cdot \mathbf{A}_1 \mathrm{d}\mathbf{c}_1 + \int f_2^{(0)} m_2 \mathbf{C}_2 \cdot \mathbf{A}_2 \mathrm{d}\mathbf{c}_2 = 0 \ , \tag{3.27}$$

$$\int f_1^{(0)} m_1 \mathbf{C}_1 \cdot \mathbf{D}_1 \mathrm{d}\mathbf{c}_1 + \int f_2^{(0)} m_2 \mathbf{C}_2 \cdot \mathbf{D}_2 \mathrm{d}\mathbf{c}_2 = 0 \ , \tag{3.28}$$

such that the second-order Chapman-Enskog approximations yield:

$$f_1^{(1)} = f_1^{(0)}\Big\{1 - A_1(C_1)\,\mathbf{C}_1 \cdot \boldsymbol{\nabla} \ln(T) - D_1(C_1)\,\mathbf{C}_1 \cdot \mathbf{d}_{12} \\ - 2B_1(C_1)\,\mathbf{C}_1\overset{\circ}{}\mathbf{C}_1 : \boldsymbol{\nabla}\mathbf{c}_0\Big\}\ , \tag{3.29}$$

and:

$$f_2^{(1)} = f_2^{(0)}\Big\{1 - A_2(C_2)\,\mathbf{C}_2 \cdot \boldsymbol{\nabla} \ln(T) - D_2(C_2)\,\mathbf{C}_2 \cdot \mathbf{d}_{12} \\ - 2B_2(C_2)\,\mathbf{C}_2\overset{\circ}{}\mathbf{C}_2 : \boldsymbol{\nabla}\mathbf{c}_0\Big\}\ . \tag{3.30}$$

Equations (3.29) and (3.30) then allow one to verify that the mean kinetic energies of the peculiar motions of the molecules of each constituent gas are the same up to this level of approximation.

From Eqs. (3.21)-(3.26), one may then construct the following general expressions:

$$n^2\{\mathbf{A}, \mathbf{a}\} = \int f_1^{(0)}\left(\mathscr{C}_1^2 - \tfrac{5}{2}\right)\mathbf{C}_1 \cdot \mathbf{a}_1 \mathrm{d}\mathbf{c}_1 \\ + \int f_2^{(0)}\left(\mathscr{C}_2^2 - \tfrac{5}{2}\right)\mathbf{C}_2 \cdot \mathbf{a}_2 \mathrm{d}\mathbf{c}_2\ , \tag{3.31}$$

$$n^2\{\mathbf{D}, \mathbf{a}\} = x_1^{-1}\int f_1^{(0)}\mathbf{C}_1 \cdot \mathbf{a}_1 \mathrm{d}\mathbf{c}_1 - x_2^{-1}\int f_2^{(0)}\mathbf{C}_2 \cdot \mathbf{a}_2 \mathrm{d}\mathbf{c}_2\ , \tag{3.32}$$

$$n^2\{\mathsf{B}, \mathsf{b}\} = \int f_1^{(0)}\boldsymbol{\mathscr{C}}_1\overset{\circ}{}\boldsymbol{\mathscr{C}}_1 : \mathsf{b}_1 \mathrm{d}\mathbf{c}_1 + \int f_2^{(0)}\boldsymbol{\mathscr{C}}_2\overset{\circ}{}\boldsymbol{\mathscr{C}}_2 : \mathsf{b}_2 \mathrm{d}\mathbf{c}_2\ , \tag{3.33}$$

where $\mathbf{a}$ is any vector-function defined in both velocity domains, b is any tensor function defined in both velocity domains, and $\{F, G\}$ are known as the bracket integrals which are defined as:

$$n^2\{F, G\} \equiv n_1^2[F, G]_1 + n_1 n_2 [F_1 + F_2, G_1 + G_2]_{12} + n_2^2 [F, G]_2\ , \tag{3.34}$$

where:

$$[F, G]_1 \equiv \int G_1 I_1(F)\,\mathrm{d}\mathbf{c}_1\ , \tag{3.35}$$

$$[F, G]_2 \equiv \int G_2 I_2(F)\,\mathrm{d}\mathbf{c}_2\ , \tag{3.36}$$

$$[F_1 + G_2, H_1 + K_2]_{12} \equiv \int F_1 I_{12} (H_1 + K_2) \, \mathrm{d}\mathbf{c}_1 + \int G_2 I_{21} (H_1 + K_2) \, \mathrm{d}\mathbf{c}_2 \; . \tag{3.37}$$

Here, due to symmetry and linearity, one has that $[F, G]_1 = [G, F]_1$, $[F, G]_2 = [G, F]_2$, and $[F_1+G_2, H_1+K_2]_{12} = [H_1+K_2, F_1+G_2]_{12}$ such that $\{F, G\} = \{G, F\}$, $\{F, G + H\} = \{F, G\} + \{F, H\}$, and $\{F, aG\} = a\{F, G\}$ (where a is any arbitrary constant).

3.3 The theory for viscosity

Focusing solely on the viscosity solution, one notes that the lowest-order approximation to the pressure system for a binary gas mixture reduces simply to the equilibrium hydrostatic pressure of the mixture. For the second-order approximation, one then adds to this the pressure system $\mathbf{p}^{(1)}$ given by:

$$\mathbf{p}^{(1)} \equiv n_1 m_1 \left(\overline{\mathbf{C_1 C_1}}\right)^{(1)} + n_2 m_2 \left(\overline{\mathbf{C}_2 \mathbf{C}_2}\right)^{(1)} \;, \tag{3.38}$$

which represents the deviation of the pressure system from the hydrostatic value. Now, in terms of the second approximation distribution functions of Eqs. (3.29) and (3.30):

$$\begin{aligned} \mathbf{p}^{(1)} &= -2m_1 \int f_1^{(0)} \mathbf{C}_1 \mathbf{C}_1 \left(\mathbf{B}_1 : \nabla \mathbf{c}_0\right) \mathrm{d}\mathbf{c}_1 \\ &\quad - 2m_2 \int f_2^{(0)} \mathbf{C}_2 \mathbf{C}_2 \left(\mathbf{B}_2 : \nabla \mathbf{c}_0\right) \mathrm{d}\mathbf{c}_2 \\ &= -\tfrac{2}{5} \left\{ m_1 \int f_1^{(0)} \, \overset{\circ}{\mathbf{C}_1 \mathbf{C}_1} : \mathbf{B}_1 \mathrm{d}\mathbf{c}_1 + m_2 \int f_2^{(0)} \, \overset{\circ}{\mathbf{C}_2 \mathbf{C}_2} : \mathbf{B}_2 \mathrm{d}\mathbf{c}_2 \right\} \overset{\circ}{\mathbf{e}} \\ &= -\tfrac{4}{5} k n^2 T \left\{\mathbf{B}, \mathbf{B}\right\} \overset{\circ}{\mathbf{e}} \;, \end{aligned} \tag{3.39}$$

where $\mathbf{e} = \overline{\overline{\nabla \mathbf{c}_0}}$ is the rate-of-strain tensor and $\overset{\circ}{\mathbf{e}}$ is the non-divergent symmetrical rate-of-shear tensor. If one then writes the viscosity as:

$$\mu \equiv \tfrac{2}{5} k n^2 T \left\{\mathbf{B}, \mathbf{B}\right\} \;, \tag{3.40}$$

Eq. (3.39) becomes:

$$\mathbf{p}^{(1)} = -2\mu \overset{\circ}{\mathbf{e}} \equiv -2\mu \, \overset{\circ}{\overline{\overline{\frac{\partial}{\partial \mathbf{r}} \mathbf{c}_0}}} \;, \tag{3.41}$$

which is the expression for the viscous stress system in any medium.

3.4 Solution in terms of Sonine polynomials

For the Chapman-Enskog functions of viscosity of a binary mixture, $\mathbf{B}_1$ and $\mathbf{B}_2$, it is assumed that both $\mathbf{B}_1$ and $\mathbf{B}_2$ can be expressed in series form as follows:

$$\mathbf{B}_1 = \sum_{\substack{p=-\infty \\ p\neq 0}}^{+\infty} b_p \mathbf{b}_1^{(p)} , \quad \mathbf{B}_2 = \sum_{\substack{p=-\infty \\ p\neq 0}}^{+\infty} b_p \mathbf{b}_2^{(p)} , \tag{3.42}$$

where the same expansion coefficients, b_p, are used in both series, the value $p = 0$ is explicitly not included in either series (Note: In Chapman and Cowling [10], summations that explicitly omit the 0^{th} term are denoted with primes on the summation symbol), and the functions, $\mathbf{b}^{(p)}$, in the two series are defined in the two velocity domains by the equations:

$$\begin{array}{lll} \mathbf{b}_1^{(p)} = \mathbf{b}_1^{(0)} \equiv 0 , & \mathbf{b}_2^{(-p)} = \mathbf{b}_2^{(0)} \equiv 0 , & (p = 0) , \\ \mathbf{b}_1^{(p)} \equiv 0 , & \mathbf{b}_2^{(-p)} \equiv 0 , & (p < 0) , \\ \mathbf{b}_1^{(p)} \equiv S_{5/2}^{(p-1)} (\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1} , & \mathbf{b}_2^{(-p)} \equiv S_{5/2}^{(p-1)} (\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2} , & (p > 0) , \end{array} \tag{3.43}$$

where:

$$\begin{aligned} S_m^{(n)} (x) &= \sum_{p=0}^{n} \frac{(m+n)_{n-p}}{(p)!\,(n-p)!} (-x)^p \\ &= \sum_{p=0}^{n} \frac{(m+n)!}{(p)!\,(n-p)!\,(m+p)!} (-x)^p , \end{aligned} \tag{3.44}$$

(with $S_m^{(0)} (x) = 1$ and $S_m^{(1)} (x) = m+1-x$) are numerical multiples (un-normalized) of the Sonine polynomials originally used in the kinetic theory of gases by Burnett [13]. As it is used in Chapman and Cowling (and in the present work), one should note the following orthogonality property of the Sonine polynomials:

$$\int_0^{\infty} \exp(-x)\, S_m^{(p)} (x)\, S_m^{(q)} (x)\, x^m \mathrm{d}x = \{\Gamma(m+p+1)/p!\}\, \delta_{p,q} , \tag{3.45}$$

where $\delta_{p,q}$ is the Kronecker delta and $\Gamma(x)$ is the Gamma function..

From Eq. (3.33), one may write:

$$\{\mathbf{B}, \mathbf{b}^{(q)}\} = \beta_q , \tag{3.46}$$

where:

$$n^2\beta_q = \int f_1^{(0)} \boldsymbol{\mathscr{C}}_1 \overset{\circ}{\boldsymbol{\mathscr{C}}}_1 : \mathbf{b}_1^{(q)} \mathrm{d}\mathbf{c}_1 + \int f_2^{(0)} \boldsymbol{\mathscr{C}}_2 \overset{\circ}{\boldsymbol{\mathscr{C}}}_2 : \mathbf{b}_2^{(q)} \mathrm{d}\mathbf{c}_2 \ . \tag{3.47}$$

Integrating Eq. (3.47), one finds that:

$$\beta_1 = \tfrac{5}{2}\frac{n_1}{n^2} \ , \quad \beta_{-1} = \tfrac{5}{2}\frac{n_2}{n^2} \ , \quad \beta_q = 0 \ \ (q \neq \pm 1) \ . \tag{3.48}$$

Combining Eqs. (3.42)-(3.47) yields:

$$\sum_{\substack{p=-\infty \\ p\neq 0}}^{\infty} b_p b_{pq} = \beta_q \ , \tag{3.49}$$

where:

$$b_{pq} = \{\mathbf{b}^{(p)}, \mathbf{b}^{(q)}\} \ . \tag{3.50}$$

If values for b_{pq} are known, all of the b_p can be determined by solving the algebraic system of equations represented by Eq. (3.49). In the general matrix terminology of Chapman and Cowling, this is:

$$b_p = \lim_{m\to\infty} \left\{ \beta_1 \mathscr{B}_{1p}^{(m)} + \beta_{-1}\mathscr{B}_{-1p}^{(m)} \right\} \Big/ \mathscr{B}^{(m)} \ , \tag{3.51}$$

where $\mathscr{B}^{(m)}$ is the determinant of the $2m \times 2m$ symmetric coefficient matrix of the system of Eq. (3.49) and $\mathscr{B}_{qp}^{(m)}$ is the cofactor of b_{qp} in the expansion of $\mathscr{B}^{(m)}$. Since, from Eq. (3.48) $\beta_q = 0$ $(q \neq \pm 1)$, this yields:

$$\{\mathbf{B}, \mathbf{B}\} = \sum_p b_p \{\mathbf{B}, \mathbf{b}^{(p)}\} = \beta_1 b_1 + \beta_{-1} b_{-1} \ , \tag{3.52}$$

which, in turn and still using general matrix terminology, yields the following expression for the viscosity of a binary gas mixture via Eqs. (3.40) and (3.51):

$$\mu = \tfrac{5}{2} kT \lim_{m\to\infty} \left\{ x_1^2 \mathscr{B}_{11}^{(m)} + 2x_1 x_2 \mathscr{B}_{1-1}^{(m)} + x_2^2 \mathscr{B}_{-1-1}^{(m)} \right\} \Big/ \mathscr{B}^{(m)} \ . \tag{3.53}$$

It can be shown that the successive approximations to μ as m is increased form a monotonically increasing series that converges to an exact value for μ in the limit as $m \to \infty$. Using Eqs. (3.40), (3.48), and (3.52), the viscosity from Eq. (3.53) can be expressed in a more convenient form as:

$$\mu = p\left(x_1 b_1 + x_2 b_{-1}\right) \ , \tag{3.54}$$

which is the form used in the current work.

3.5 The bracket integrals

In order to complete the evaluation of the viscosity it is necessary to evaluate the bracket integrals defined in Eq. (3.34) for $\{\mathbf{b}^{(p)}, \mathbf{b}^{(q)}\}$. Hence, it is first necessary to be able to evaluate the square bracket integrals of Eqs. (3.35)-(3.37), specifically, $[\mathbf{b}_1^{(p)}, \mathbf{b}_1^{(q)}]_1$, $[\mathbf{b}_1^{(p)}, \mathbf{b}_1^{(q)}]_{12}$, and $[\mathbf{b}_1^{(p)}, \mathbf{b}_2^{(q)}]_{12}$. Completion of this task requires integration over all of the collision variables and, additionally, requires knowledge of the form of the intermolecular potential. However, the intermolecular potential affects only integrations over the variables, g and b, because they determine the scattering angle, χ. All six of the other integrals (there are eight total in the collision operator) can be evaluated in some manner without specific knowledge of the intermolecular potential.

The specific bracket integrals mentioned above that require evaluation are given explicitly in Chapman and Cowling but how they have been identified is not readily apparent in the text which focuses on very broad generalized expressions that can be difficult to interpret. In practice, the needed bracket integrals are most readily determined by considering Eqs. (3.25) and (3.26) directly. If these equations are expressed as:

$$n_1^2 I_1\left(\mathbf{B}_1\right) + n_1 n_2 I_{12}\left(\mathbf{B}_1\right) + n_1 n_2 I_{12}\left(\mathbf{B}_2\right) = f_1^{(0)} \overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1} \ , \tag{3.55}$$

$$n_2^2 I_2\left(\mathbf{B}_2\right) + n_1 n_2 I_{21}\left(\mathbf{B}_2\right) + n_1 n_2 I_{21}\left(\mathbf{B}_1\right) = f_2^{(0)} \overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2} \ , \tag{3.56}$$

then one can insert Sonine polynomial approximations for the solutions, $\mathbf{B}_1$ and $\mathbf{B}_2$, can multiply through the equations with additional Sonine polynomials, and can then integrate the equations. By using the orthogonality of the Sonine polynomials as given in Eq. (3.45), this process will yield a set of simultaneous equations involving the matrix coefficients necessary to specify the solutions. The RHSs of the equations determine the constants, β_q, and the LHSs yield combinations of bracket integrals that correspond to the desired b_{pq} coefficients of the matrix used to determine the b_p values which, in turn, determine the viscosity via Eq. (3.54).

The easiest way to follow this process is via a low-order example. If one considers the Sonine polynomials used in the definition of the $\boldsymbol{\mathsf{b}}_1^{(p)}$ and $\boldsymbol{\mathsf{b}}_2^{(p)}$ Chapman-Enskog expansion tensors, $p = 1$ is the lowest order with $S_m^{(0)}(x) = 1$, such that $\boldsymbol{\mathsf{b}}_1^{(1)} \equiv S_{5/2}^{(0)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1} = \overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}$ and $\boldsymbol{\mathsf{b}}_2^{(-1)} \equiv S_{5/2}^{(0)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2} = \overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}$. Using only this order of approximation for $\boldsymbol{\mathsf{B}}_1$ and $\boldsymbol{\mathsf{B}}_2$, i.e. assuming that $\boldsymbol{\mathsf{B}}_1 = b_1\boldsymbol{\mathsf{b}}_1^{(1)} = b_1\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}$ and $\boldsymbol{\mathsf{B}}_2 = b_{-1}\boldsymbol{\mathsf{b}}_2^{(-1)} = b_{-1}\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}$, one may express Eqs. (3.55) and (3.56) as:

$$
\begin{aligned}
n_1^2 I_1\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right)b_1 + n_1n_2I_{12}\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right)b_1 + n_1n_2I_{12}\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right)b_{-1}& \\
= f_1^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\ ,&
\end{aligned}
\tag{3.57}
$$

$$
\begin{aligned}
n_2^2 I_2\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right)b_{-1} + n_1n_2I_{21}\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right)b_{-1} + n_1n_2I_{21}\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right)b_1& \\
= f_2^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\ .&
\end{aligned}
\tag{3.58}
$$

Now, multiplying through Eq. (3.57) by $\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}$ and through Eq. (3.58) by $\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}$, one has:

$$
\begin{aligned}
n_1^2\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1} : I_1\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right)b_1 + n_1n_2\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1} : I_{12}\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right)b_1& \\
+ n_1n_2\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1} : I_{12}\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right)b_{-1} = \overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1} : f_1^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\ ,&
\end{aligned}
\tag{3.59}
$$

and:

$$
\begin{aligned}
n_2^2\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2} : I_2\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right)b_{-1} + n_1n_2\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2} : I_{21}\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right)b_{-1}& \\
+ n_1n_2\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2} : I_{21}\left(\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right)b_1 = \overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2} : f_2^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\ .&
\end{aligned}
\tag{3.60}
$$

From these, after integrating throughout Eqs. (3.59) and (3.60) in the manner of Eqs. (3.12) and (3.13) for I_i and I_{ij}, one can express Eqs. (3.59) and (3.60) in bracket integral notation as:

$$\begin{aligned}\left(n_1^2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1},\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\big]_1+n_1n_2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1},\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\big]_{12}\right)b_1&\\ +n_1n_2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1},\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\big]_{12}b_{-1}=\iint\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}:f_1^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\,g\alpha_1\mathrm{d}\mathbf{e}'\mathrm{d}\mathbf{c}\ ,&\quad(3.61)\end{aligned}$$

and:

$$\begin{aligned}&n_1n_2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2},\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\big]_{21}b_1\\ &+\left(n_2^2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2},\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\big]_2+n_1n_2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2},\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\big]_{21}\right)b_{-1}\\ &\qquad\qquad=\iint\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}:f_2^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\,g\alpha_2\mathrm{d}\mathbf{e}'\mathrm{d}\mathbf{c}\ .\quad(3.62)\end{aligned}$$

Given the way that the solution has been approached and the low order of this example, it is readily apparent that these equations (when divided through on both sides by n^2) must be equivalent to:

$$b_{11}b_1+b_{1-1}b_{-1}=\beta_1\ , \quad (3.63)$$

$$b_{-11}b_1+b_{-1-1}b_{-1}=\beta_{-1}\ , \quad (3.64)$$

where:

$$b_{11}=x_1^2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1},\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\big]_1+x_1x_2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1},\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\big]_{12}\ , \quad (3.65)$$

$$b_{1-1}=x_1x_2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1},\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\big]_{12}\ , \quad (3.66)$$

$$b_{-11}=x_1x_2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2},\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\big]_{21}\ , \quad (3.67)$$

$$b_{-1-1}=x_2^2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2},\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\big]_2+x_1x_2\big[\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2},\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\big]_{21}\ , \quad (3.68)$$

and the integrals on the RHSs correspond to the β_q constants which, due to the orthogonality of the Sonine polynomials, are defined only in the current example where $q=\pm1$ and are otherwise zero. Equations (3.63) and (3.64) can, of course, be rearranged to cast them into the ordered form that corresponds directly to the form that the more general, higher-order problem has been expressed in, i.e.:

$$\begin{bmatrix} b_{-1-1} & b_{-11} \\ b_{1-1} & b_{11} \end{bmatrix} \begin{bmatrix} b_{-1} \\ b_1 \end{bmatrix} = \begin{bmatrix} \beta_{-1} \\ \beta_1 \end{bmatrix} . \tag{3.69}$$

Thus, with the bracket integrals expressed in terms of the appropriate Sonine polynomials, the matrix elements of Eqs. (3.65)-(3.68) are simply:

$$\begin{aligned} b_{11} = x_1^2 \big[S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1} \big]_1 \\ + x_1 x_2 \big[S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1} \big]_{12} , \end{aligned} \tag{3.70}$$

$$b_{1-1} = x_1 x_2 \big[S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2} \big]_{12} , \tag{3.71}$$

$$b_{-11} = x_1 x_2 \big[S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2}, S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1} \big]_{21} , \tag{3.72}$$

$$\begin{aligned} b_{-1-1} = x_2^2 \big[S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2}, S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2} \big]_2 \\ + x_1 x_2 \big[S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2}, S_{5/2}^{(0)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2} \big]_{21} , \end{aligned} \tag{3.73}$$

which, for higher-order expansions, become simply:

$$\begin{aligned} b_{pq} = x_1^2 \big[S_{5/2}^{(p-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(q-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1} \big]_1 \\ + x_1 x_2 \big[S_{5/2}^{(p-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(q-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1} \big]_{12} , \end{aligned} \tag{3.74}$$

$$b_{p-q} = x_1 x_2 \big[S_{5/2}^{(p-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(q-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2} \big]_{12} , \tag{3.75}$$

$$b_{-pq} = x_1 x_2 \big[S_{5/2}^{(p-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2}, S_{5/2}^{(q-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_1 \boldsymbol{\mathscr{C}}_1} \big]_{21} , \tag{3.76}$$

$$\begin{aligned} b_{-p-q} = x_2^2 \big[S_{5/2}^{(p-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2}, S_{5/2}^{(q-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2} \big]_2 \\ + x_1 x_2 \big[S_{5/2}^{(p-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2}, S_{5/2}^{(q-1)}(\mathscr{C}) \overset{\circ}{\boldsymbol{\mathscr{C}}_2 \boldsymbol{\mathscr{C}}_2} \big]_{21} . \end{aligned} \tag{3.77}$$

From the definitions of I_i and I_{ij} in Eqs. (3.12) and (3.13), it follows that Eqs. (3.76) and (3.77) are essentially identical to Eqs. (3.75) and (3.74), respectively, with the only difference being the interchange of the subscripts 1 and 2 representing the different components of the mixture. Thus, in general, the complete Chapman-Enskog solution for viscosity requires only the bracket integrals:

$$\begin{aligned}
&\left[S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1\right]_1 ,\\
&\left[S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1\right]_{12} ,\\
&\left[S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_2\overset{\circ}{\boldsymbol{\mathscr{C}}}_2\right]_{12} ,
\end{aligned} \tag{3.78}$$

which are exactly those identified by Chapman and Cowling. Some further simplification of the problem can be obtained by noting from Chapman and Cowling that, in the limit of a simple (single) gas where $m_1 = m_2$, $n_1 = n_2 = n$, and $\alpha_{21} = \alpha_{12} = \alpha_1$, one has that $[F, G]_1 = [F_1, G_1 + G_2]_{12} = ([F_1, G_1]_{12} + [F_1, G_2]_{12})$ which, in the current problem, equates to:

$$\begin{aligned}
&\left[S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1\right]_1\\
&= \Bigg(\left[S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1\right]_{12}\\
&\qquad + \left[S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_2\overset{\circ}{\boldsymbol{\mathscr{C}}}_2\right]_{12}\Bigg)\Bigg|_{\substack{m_1=m_2\\ n_1=n_2=n\\ \alpha_{12}=\alpha_{21}=\alpha_1}} .
\end{aligned} \tag{3.79}$$

At this point, it still remains to perform the six integrations unrelated to the intermolecular potential model that is employed in order to complete the evaluation of the two necessary bracket integrals on the RHS of Eq. (3.79). For the relevant details of this integral evaluation process, one should refer to the text of Chapman and Cowling [10]. Note that Chapman and Cowling make use of the following definition of the Sonine polynomials:

$$\begin{aligned}
(\mathrm{S}/s)^{m+1}\exp\left(-x\mathrm{S}\right) &\equiv (1-s)^{-m-1}\exp\left(-xs/(1-s)\right)\\
&= \sum_{n=0}^{\infty} s^n S_m^{(n)}\left(x\right) ,
\end{aligned} \tag{3.80}$$

where $\mathrm{S} = s/(1-s)$ and, likewise, $\mathrm{T} = t/(1-t)$, are used to express the needed bracket integrals in terms of the coefficients of expansions in the arbitrarily introduced variables, s and t. Thus, it is possible after following Chapman and Cowling to determine that:

$$\begin{aligned}&\left[S_{5/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_2\overset{\circ}{\boldsymbol{\mathscr{C}}}_2\right]_{12}\\&\quad= \text{coeff}\left[s^p t^q\right]\left(\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3}\iiint\Big\{L_{12}(0)-L_{12}(\chi)\Big\}gb\mathrm{d}b\mathrm{d}\varepsilon\mathrm{d}\boldsymbol{g}\right),\end{aligned}\qquad(3.81)$$

and:

$$\begin{aligned}&\left[S_{5/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1\right]_{12}\\&\quad= \text{coeff}\left[s^p t^q\right]\left(\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3}\iiint\left\{L_1(0)-L_1(\chi)\right\}gb\mathrm{d}b\mathrm{d}\varepsilon\mathrm{d}\boldsymbol{g}\right),\end{aligned}\qquad(3.82)$$

where $\boldsymbol{g} \equiv (m_0 M_1 M_2/2kT)^{1/2}\mathbf{g}_{21}$. Also, note that the retention of a single g in the integrands of Eqs. (3.81) and (3.82) (as opposed to g) is not a typographical error but, rather, is the exact notation used by Chapman and Cowling. After integration over ε and the directions of $\boldsymbol{g}$ (which changes the constants somewhat), one can express the χ-dependent portions of the RHS bracketed integrals of Eqs. (3.81) and (3.82) as:

$$\begin{aligned}&\tfrac{3}{2}\left(\mathrm{ST}/st\right)^{7/2}\left(M_1M_2\right)^{-1}\pi^{-3/2}L_{12}(\chi)\\&= e^{-g^2}\sum_r\sum_n\left\{2M_1M_2st\left(1-\cos(\chi)\right)\right\}^r\left(g^{2r}/r!\right)\left(M_2s+M_1t\right)^n\\&\times\Big[(n+1)(n+2)S_{r+1/2}^{(n+2)}\left(g^2\right)+2(n+1)g^2\left(1-\cos(\chi)\right)S_{r+3/2}^{(n+1)}\left(g^2\right)\\&+g^4\left\{\left(1-\cos(\chi)\right)^2-\tfrac{1}{2}\sin^2(\chi)\right\}S_{r+5/2}^{(n)}\left(g^2\right)\Big],\end{aligned}\qquad(3.83)$$

and:

$$\begin{aligned}&\tfrac{3}{2}\left(\mathrm{ST}/st\right)^{7/2}\pi^{-3/2}L_1(\chi)\\&= e^{-g^2}\sum_r\sum_n\left[st\left(M_1^2+M_2^2+2M_1M_2\cos(\chi)\right)\right]^r\left(g^{2r}/r!\right)\\&\times\left[M_2(s+t)-(M_2-M_1)st\right]^n\Big[M_1^2(n+1)(n+2)S_{r+1/2}^{(n+2)}\left(g^2\right)\\&+2(n+1)g^2M_1\left(M_1+M_2\cos(\chi)\right)S_{r+3/2}^{(n+1)}\left(g^2\right)\\&+g^4\left\{\left(M_1+M_2\cos(\chi)\right)^2-\tfrac{1}{2}M_2^2\sin^2(\chi)\right\}S_{r+5/2}^{(n)}\left(g^2\right)\Big].\end{aligned}\qquad(3.84)$$

In both of these cases, the coefficient of $[s^p t^q]$ yields a polynomial in powers of g^2 and $\cos(\chi)$ that is multiplied by $\exp(-g^2)$ and in which each term is some function of the molecular masses via M_1 and M_2. The χ-independent portions of the RHS bracketed integrals of Eqs. (3.81) and (3.82) are obtained by the simple expedient of setting $\chi = 0$ in Eqs. (3.83) and (3.84) which yields overall terms in the combined polynomial involving $(1 - \cos^{\ell}(\chi))$. Thus, it is possible to express Eqs. (3.81) and (3.82) as:

$$\begin{aligned}&\left[S_{5/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1^{\circ}\boldsymbol{\mathscr{C}}_1, S_{5/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_2^{\circ}\boldsymbol{\mathscr{C}}_2\right]_{12}\\ &= \tfrac{16}{3}\pi^{1/2}M_2^{p+1}M_1^{q+1}\iint \exp\left(-g^2\right)\sum_{r,\ell}\left(B_{pqr\ell}\right)g^{2r+2}\\ &\quad\times\left(1-\cos^{\ell}(\chi)\right)\mathrm{g}b\mathrm{d}b\mathrm{d}g\\ &= \tfrac{16}{3}M_2^{p+1}M_1^{q+1}\sum_{r,\ell}\left(B_{pqr\ell}\right)\Omega_{12}^{(\ell)}(r)\ ,\end{aligned} \tag{3.85}$$

and:

$$\begin{aligned}&\left[S_{5/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1^{\circ}\boldsymbol{\mathscr{C}}_1, S_{5/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1^{\circ}\boldsymbol{\mathscr{C}}_1\right]_{12}\\ &= \tfrac{16}{3}\pi^{1/2}\iint \exp\left(-g^2\right)\sum_{r,\ell}\left(B'_{pqr\ell}\right)g^{2r+2}\\ &\quad\times\left(1-\cos^{\ell}(\chi)\right)\mathrm{g}b\mathrm{d}b\mathrm{d}g\\ &= \tfrac{16}{3}\sum_{r,\ell}\left(B'_{pqr\ell}\right)\Omega_{12}^{(\ell)}(r)\ ,\end{aligned} \tag{3.86}$$

where the omega integrals are defined as:

$$\Omega_{12}^{(\ell)}(r) \equiv \left(\frac{kT}{2\pi m_0 M_1 M_2}\right)^{1/2}\int_0^{\infty}\exp\left(-g^2\right)g^{2r+3}\phi_{12}^{(\ell)}\mathrm{d}g\ , \tag{3.87}$$

with:

$$\phi_{12}^{(\ell)} \equiv 2\pi\int_0^{\pi}\left(1-\cos^{\ell}(\chi)\right)b\mathrm{d}b\ . \tag{3.88}$$

As stated by Chapman and Cowling [10]:

> "Explicit expressions for [$B_{pqr\ell}$ and $B'_{pqr\ell}$] can be obtained from [Eqs. (3.83) and (3.84)] using [Eq. (3.44)] for $S_m^{(n)}(x)$. In view of the complication of these expressions it is, however, better in practice to calculate any desired values of [$B_{pqr\ell}$ and $B'_{pqr\ell}$] directly from [Eqs. (3.83) and (3.84)]."

This approach as suggested by Chapman and Cowling has been explored using *Mathematica*® up to order 60 and the order 60 viscosity results are reported here. Note that in all of the expressions generated for this work, the fully general dependence of the expressions on the mole fractions, the molecular masses, and the models of the intermolecular potential (that can be employed) have been retained. For example, from Eq. (3.49), since there exists symmetry in the off-diagonal elements such that $b_{pq} = b_{qp}$, one can write the following matrix equation for order 1 (as shown previously in a low-order example):

$$\begin{bmatrix} b_{-1-1} & b_{-11} \\ b_{1-1} & b_{11} \end{bmatrix} \begin{bmatrix} b_{-1} \\ b_1 \end{bmatrix} = \begin{bmatrix} \beta_{-1} \\ \beta_1 \end{bmatrix} . \tag{3.89}$$

Here, the general expressions for the b_{pq} coefficients have been determined to be:

$$\begin{aligned} b_{-1-1} = x_2^2\Big\{4\Omega_2^{(2)}(2)\Big\} + x_1x_2\Big\{\tfrac{80}{3}\big(M_1M_2\big)\Omega_{12}^{(1)}(1) \\ + 8\big(M_1^2\big)\Omega_{12}^{(2)}(2)\Big\} , \end{aligned} \tag{3.90}$$

$$b_{-11} = x_1x_2\left\{-\tfrac{80}{3}\left(M_1M_2\right)\Omega_{12}^{(1)}(1) + 8\left(M_1M_2\right)\Omega_{12}^{(2)}(2)\right\} , \tag{3.91}$$

$$b_{1-1} = x_1x_2\left\{-\tfrac{80}{3}\left(M_1M_2\right)\Omega_{12}^{(1)}(1) + 8\left(M_1M_2\right)\Omega_{12}^{(2)}(2)\right\} , \tag{3.92}$$

$$\begin{aligned} b_{11} = x_1^2\Big\{4\Omega_1^{(2)}(2)\Big\} + x_1x_2\Big\{\tfrac{80}{3}\big(M_1M_2\big)\Omega_{12}^{(1)}(1) \\ + 8\big(M_2^2\big)\Omega_{12}^{(2)}(2)\Big\} . \end{aligned} \tag{3.93}$$

The symmetry in the off-diagonal elements is obvious, with $b_{-11} = b_{1-1}$. Likewise, for order 2, Eq. (3.49) plus symmetry gives the following matrix equation:

$$\begin{bmatrix} b_{-2-2} & b_{-2-1} & b_{-21} & b_{-22} \\ b_{-1-2} & b_{-1-1} & b_{-11} & b_{-12} \\ b_{1-2} & b_{1-1} & b_{11} & b_{12} \\ b_{2-2} & b_{2-1} & b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{-2} \\ b_{-1} \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \beta_{-1} \\ \beta_1 \\ 0 \end{bmatrix} , \tag{3.94}$$

and the general expressions for the b_{pq} coefficients have been determined to be:

$$\begin{aligned}
b_{-2-2} =& x_2^2\Big\{ \tfrac{301}{12}\Omega_2^{(2)}(2) - 7\Omega_2^{(2)}(3) + \Omega_2^{(2)}(4)\Big\} \\
&+ x_1x_2\Big\{ \tfrac{1540}{3}\Big(\tfrac{4}{11}M_1M_2^3 + \tfrac{7}{11}M_1^3M_2\Big)\Omega_{12}^{(1)}(1) \\
&- \tfrac{784}{3}\left(M_1^3M_2\right)\Omega_{12}^{(1)}(2) + \tfrac{128}{3}\left(M_1^3M_2\right)\Omega_{12}^{(1)}(3) \\
&+ \tfrac{602}{3}\left(\tfrac{22}{43}M_1^2M_2^2 + \tfrac{21}{43}M_1^4\right)\Omega_{12}^{(2)}(2) \\
&- 56\left(M_1^4\right)\Omega_{12}^{(2)}(3) + 8\left(M_1^4\right)\Omega_{12}^{(2)}(4) \\
&+ 16\left(M_1^3M_2\right)\Omega_{12}^{(3)}(3)\Big\}\ ,
\end{aligned} \tag{3.95}$$

$$\begin{aligned}
b_{-2-1} = b_{-1-2} =& x_2^2\Big\{7\Omega_2^{(2)}(2) - 2\Omega_2^{(2)}(3)\Big\} \\
&+ x_1x_2\Big\{\tfrac{280}{3}\left(M_1^2M_2\right)\Omega_{12}^{(1)}(1) \\
&- \tfrac{112}{3}\left(M_1^2M_2\right)\Omega_{12}^{(1)}(2) + 28\left(M_1^3\right)\Omega_{12}^{(2)}(2) \\
&- 8\left(M_1^3\right)\Omega_{12}^{(2)}(3)\Big\}\ ,
\end{aligned} \tag{3.96}$$

$$\begin{aligned}
b_{-21} = b_{1-2} =& x_1x_2\Big\{-\tfrac{280}{3}\left(M_1^2M_2\right)\Omega_{12}^{(1)}(1) + \tfrac{112}{3}\left(M_1^2M_2\right)\Omega_{12}^{(1)}(2) \\
&+ 28\left(M_1^2M_2\right)\Omega_{12}^{(2)}(2) - 8\left(M_1^2M_2\right)\Omega_{12}^{(2)}(3)\Big\}\ ,
\end{aligned} \tag{3.97}$$

$$\begin{aligned}
b_{-22} = b_{2-2} =& x_1x_2\Big\{-\tfrac{1540}{3}\left(M_1^2M_2^2\right)\Omega_{12}^{(1)}(1) + \tfrac{784}{3}\left(M_1^2M_2^2\right)\Omega_{12}^{(1)}(2) \\
&- \tfrac{128}{3}\left(M_1^2M_2^2\right)\Omega_{12}^{(1)}(3) + \tfrac{602}{3}\left(M_1^2M_2^2\right)\Omega_{12}^{(2)}(2) \\
&- 56\left(M_1^2M_2^2\right)\Omega_{12}^{(2)}(3) + 8\left(M_1^2M_2^2\right)\Omega_{12}^{(2)}(4) \\
&- 16\left(M_1^2M_2^2\right)\Omega_{12}^{(3)}(3)\Big\}\ ,
\end{aligned} \tag{3.98}$$

$$\begin{aligned}
b_{-1-1} =& x_2^2\Big\{4\Omega_2^{(2)}(2)\Big\} + x_1x_2\Big\{\tfrac{80}{3}\big(M_1M_2\big)\Omega_{12}^{(1)}(1) \\
&+ 8\big(M_1^2\big)\Omega_{12}^{(2)}(2)\Big\}\ ,
\end{aligned} \tag{3.99}$$

$$b_{-11} = x_1x_2\left\{-\tfrac{80}{3}\left(M_1M_2\right)\Omega_{12}^{(1)}(1) + 8\left(M_1M_2\right)\Omega_{12}^{(2)}(2)\right\}\ , \tag{3.100}$$

$$\begin{aligned}
b_{-12} = b_{2-1} =& x_1x_2\Big\{ - \tfrac{280}{3}\left(M_1M_2^2\right)\Omega_{12}^{(1)}(1) + \tfrac{112}{3}\left(M_1M_2^2\right)\Omega_{12}^{(1)}(2) \\
&+ 28\left(M_1M_2^2\right)\Omega_{12}^{(2)}(2) - 8\left(M_1M_2^2\right)\Omega_{12}^{(2)}(3)\ ,
\end{aligned} \tag{3.101}$$

$$\begin{aligned}
b_{11} =& x_1^2\Big\{4\Omega_1^{(2)}(2)\Big\} + x_1x_2\Big\{\tfrac{80}{3}\big(M_1M_2\big)\Omega_{12}^{(1)}(1) \\
&+ 8\big(M_2^2\big)\Omega_{12}^{(2)}(2)\Big\}\ ,
\end{aligned} \tag{3.102}$$

$$
\begin{aligned}
b_{12} = b_{21} = & x_1^2 \Big\{ 7\Omega_1^{(2)}(2) - 2\Omega_1^{(2)}(3) \Big\} + x_1 x_2 \Big\{ \tfrac{280}{3} \left(M_1 M_2^2\right) \Omega_{12}^{(1)}(1) \\
& - \tfrac{112}{3} \left(M_1 M_2^2\right) \Omega_{12}^{(1)}(2) + 28 \left(M_2^3\right) \Omega_{12}^{(2)}(2) \\
& - 8 \left(M_2^3\right) \Omega_{12}^{(2)}(3) \Big\} ,
\end{aligned}
\tag{3.103}
$$

$$
\begin{aligned}
b_{22} = & x_1^2 \Big\{ \tfrac{301}{12} \Omega_1^{(2)}(2) - 7\Omega_1^{(2)}(3) + \Omega_1^{(2)}(4) \Big\} \\
& + x_1 x_2 \Big\{ \tfrac{1540}{3} \Big(\tfrac{4}{11} M_1 M_2^3 + \tfrac{7}{11} M_1 M_2^3 \Big) \Omega_{12}^{(1)}(1) \\
& - \tfrac{784}{3} \left(M_1 M_2^3\right) \Omega_{12}^{(1)}(2) + \tfrac{128}{3} \left(M_1 M_2^3\right) \Omega_{12}^{(1)}(3) \\
& + \tfrac{602}{3} \left(\tfrac{22}{43} M_1^2 M_2^2 + \tfrac{21}{43} M_2^4 \right) \Omega_{12}^{(2)}(2) - 56 \left(M_2^4\right) \Omega_{12}^{(2)}(3) \\
& + 8 \left(M_2^4\right) \Omega_{12}^{(2)}(4) + 16 \left(M_1 M_2^3\right) \Omega_{12}^{(3)}(3) \Big\} .
\end{aligned}
\tag{3.104}
$$

In Eqs. (3.95)-(3.104), the off-diagonal symmetry has been expressed explicitly. While one could, certainly, go even higher in order detailing explicit expressions for the matrix coefficients, the expressions rapidly become unwieldy in analytical form. If one wishes to verify the order 2 expressions presented here or if additional explicit expressions for the order 3 matrix coefficients are desired, they can be constructed directly from the relevant bracket integral expressions which have been reported previously in the literature [40,41]. In producing analytical order 60 results computationally this work has gone far beyond the hand calculated orders 2 and 3 reported in the previous literature and the expressions from the present work are in essential agreement with those bracket integral results from the literature except for an apparent typographical error in Ref. [41] where one coefficient in the bracket integral corresponding to (in the current notation) $\left[S_{5/2}^{(2)}(\mathscr{C})\,\mathring{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(1)}(\mathscr{C})\,\mathring{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}$ for the omega-integral $\Omega_{ij}^{(2)}(2)$ should be changed from $\frac{4095}{16}$ to $\frac{4095}{32}$. The bracket integrals appropriate to higher orders have not been previously reported. In going computationally to higher orders, all of the previously generated expressions are retained and need not be recomputed. Most importantly, regardless of the order of the expansion used, the dependencies of the matrix coefficients on mole fractions and molecular masses are carried through in general form. Molecular diameters and the specific intermolecular potential model used are also carried through in general form via the omega integrals which may also be defined in terms of the quantity, σ_{12}. This quantity is, in purely general terms, only a convenient, arbitrarily chosen length in the range of b. Thus, it is often convenient to express the omega-integrals as [10]:

$$\Omega_{12}^{(\ell)}(r) = \tfrac{1}{2}\sigma_{12}^2 \left(2\pi kT/m_0 M_1 M_2\right)^{1/2} W_{12}^{(\ell)}(r) \ , \tag{3.105}$$

where:

$$\begin{aligned} W_{12}^{(\ell)}(r) \equiv & \int_0^\infty \exp\left(-g^2\right) g^{2r+2} \int_0^\pi \left(1-\cos^\ell(\chi)\right) \\ & \times (b/\sigma_{12})\, \mathrm{d}(b/\sigma_{12})\, \mathrm{d}\left(g^2\right) \\ = & \, 2\int_0^\infty \exp\left(-g^2\right) g^{2r+3} \int_0^\pi \left(1-\cos^\ell(\chi)\right) \\ & \times (b/\sigma_{12})\, \mathrm{d}(b/\sigma_{12})\, \mathrm{d}g \ , \end{aligned} \tag{3.106}$$

and where the corresponding simple gas expressions are:

$$\Omega_1^{(\ell)}(r) = \sigma_1^2 \left(\pi kT/m_1\right)^{1/2} W_1^{(\ell)}(r) \ , \tag{3.107}$$

with:

$$\begin{aligned} W_1^{(\ell)}(r) = & 2\int_0^\infty \exp\left(-g^2\right) g^{2r+3} \int_0^\pi \left(1-\cos^\ell(\chi)\right) \\ & \times (b/\sigma_1)\, \mathrm{d}(b/\sigma_1)\, \mathrm{d}g \ , \end{aligned} \tag{3.108}$$

and:

$$\Omega_2^{(\ell)}(r) = \sigma_2^2 \left(\pi kT/m_2\right)^{1/2} W_2^{(\ell)}(r) \ , \tag{3.109}$$

with:

$$\begin{aligned} W_2^{(\ell)}(r) = & 2\int_0^\infty \exp\left(-g^2\right) g^{2r+3} \int_0^\pi \left(1-\cos^\ell(\chi)\right) \\ & \times (b/\sigma_2)\, \mathrm{d}(b/\sigma_2)\, \mathrm{d}g \ . \end{aligned} \tag{3.110}$$

In Eqs. (3.105)-(3.110), σ_1 and σ_2 are arbitrary scale lengths associated with collisions between like molecules of type 1 and type 2, respectively, while σ_{12} is associated with collisions between unlike molecules of types 1 and 2. These scale lengths are commonly associated with some concept of the molecular diameters depending upon the specific details of the intermolecular potential model that is employed. In the current work, results are reported for the case of rigid-sphere molecules because the form of the potential model allows the omega integrals to be evaluated analytically eliminating the need for any numerical integrations in the current work. Some of the details of this potential model are described in the following section.

3.6 The case of rigid-sphere molecules

For the specific case of a binary, rigid-sphere, gas mixture, one has an intermolecular potential model of the form [27]:

$$U(r) = \begin{cases} \infty, & r < \sigma_{12} \, , \\ 0, & r \geq \sigma_{12} \, , \end{cases} \tag{3.111}$$

where $\sigma_{12} = \frac{1}{2}(\sigma_1 + \sigma_2)$, and σ_1 and σ_2 are the actual diameters of the colliding spherical molecules. Under the rigid-sphere assumption, one then has the collisional relationships $b = \sigma_{12} \cos\left(\frac{1}{2}\chi\right)$ and $b\mathrm{d}b = -\frac{1}{4}\sigma_{12}^2 \sin(\chi)\,\mathrm{d}\chi$. Using these in Eqs. (3.105) and (3.106) yields:

$$W_{12}^{(\ell)}(r) = \tfrac{1}{4}\left[2 - \frac{1}{\ell+1}\left(1 + (-1)^{\ell}\right)\right](r+1)! \, , \tag{3.112}$$

such that:

$$\begin{aligned} \mathit{\Omega}_{12}^{(\ell)}(r) =& \tfrac{1}{2}\sigma_{12}^2 \left(2\pi kT/m_0 M_1 M_2\right)^{1/2} \\ & \times \tfrac{1}{4}\left[2 - \frac{1}{\ell+1}\left(1 + (-1)^{\ell}\right)\right](r+1)! \, , \end{aligned} \tag{3.113}$$

and, since $W_1^{(\ell)}(r) = W_2^{(\ell)}(r) = W_{12}^{(\ell)}(r)$, the corresponding simple-gas expressions:

$$\begin{aligned} \mathit{\Omega}_{1}^{(\ell)}(r) =& \sigma_1^2 \left(\pi kT/m_1\right)^{1/2} \\ & \times \tfrac{1}{4}\left[2 - \frac{1}{\ell+1}\left(1 + (-1)^{\ell}\right)\right](r+1)! \, , \end{aligned} \tag{3.114}$$

and:

$$\begin{aligned} \mathit{\Omega}_{2}^{(\ell)}(r) =& \sigma_2^2 \left(\pi kT/m_2\right)^{1/2} \\ & \times \tfrac{1}{4}\left[2 - \frac{1}{\ell+1}\left(1 + (-1)^{\ell}\right)\right](r+1)! \, . \end{aligned} \tag{3.115}$$

All of these omega integrals are readily evaluated for the purpose of the current work and, thus, one has the following simplified, rigid-sphere, matrix elements for the order 2 approximation:

$$
\begin{aligned}
b_{-2-2} =& x_2^2\Big\{\sqrt{\frac{\pi kT}{m_2}}\tfrac{205}{6}\sigma_2^2\Big\} + x_1x_2\Big\{\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\Big(34M_1^4 + \tfrac{370}{3}M_1^3M_2 \\
&+ \tfrac{308}{3}M_1^2M_2^2 + \tfrac{280}{3}M_1M_2^3\Big)\sigma_{12}^2\Big\}\ ,
\end{aligned}
\tag{3.116}
$$

$$
\begin{aligned}
b_{-2-1} = b_{-1-2} =& x_2^2\Big\{-\sqrt{\frac{\pi kT}{m_2}}2\sigma_2^2\Big\} + x_1x_2\Big\{-\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\Big(4M_1^3 \\
&+ \tfrac{28}{3}M_1^2M_2\Big)\sigma_{12}^2\Big\}\ ,
\end{aligned}
\tag{3.117}
$$

$$
b_{-21} = b_{1-2} = x_1x_2\Big\{\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\tfrac{16}{3}M_1^2M_2\sigma_{12}^2\Big\}\ ,
\tag{3.118}
$$

$$
b_{-22} = b_{2-2} = x_1x_2\Big\{-\sqrt{\frac{2\pi kT}{m_0M_1M_2}}80M_1^2M_2^2\sigma_{12}^2\Big\}\ ,
\tag{3.119}
$$

$$
\begin{aligned}
b_{-1-1} =& x_2^2\Big\{\sqrt{\frac{\pi kT}{m_2}}\tfrac{8}{3}\sigma_2^2\Big\} + x_1x_2\Big\{\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\Big(8M_1^2 \\
&+ \tfrac{40}{3}M_1M_2\Big)\sigma_{12}^2\Big\}\ ,
\end{aligned}
\tag{3.120}
$$

$$
b_{-11} = b_{1-1} = x_1x_2\Big\{-\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\tfrac{16}{3}M_1M_2\sigma_{12}^2\Big\}\ ,
\tag{3.121}
$$

$$
b_{-12} = b_{2-1} = x_1x_2\Big\{\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\tfrac{16}{3}M_1M_2^2\sigma_{12}^2\Big\}\ ,
\tag{3.122}
$$

$$
\begin{aligned}
b_{11} =& x_1^2\Big\{\sqrt{\frac{\pi kT}{m_1}}\tfrac{8}{3}\sigma_1^2\Big\} + x_1x_2\Big\{\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\Big(8M_2^2 \\
&+ \tfrac{40}{3}M_1M_2\Big)\sigma_{12}^2\Big\}\ ,
\end{aligned}
\tag{3.123}
$$

$$
\begin{aligned}
b_{12} = b_{21} =& x_1^2\Big\{-\sqrt{\frac{\pi kT}{m_1}}2\sigma_1^2\Big\} + x_1x_2\Big\{-\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\Big(4M_2^3 \\
&+ \tfrac{28}{3}M_1M_2^2\Big)\sigma_{12}^2\Big\}\ ,
\end{aligned}
\tag{3.124}
$$

$$
\begin{aligned}
b_{22} =& x_1^2\Big\{\sqrt{\frac{\pi kT}{m_1}}\tfrac{205}{6}\sigma_1^2\Big\} + x_1x_2\Big\{\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\Big(34M_2^4 + \tfrac{370}{3}M_1M_2^3 \\
&+ \tfrac{308}{3}M_1^2M_2^2 + \tfrac{280}{3}M_1^3M_2\Big)\sigma_{12}^2\Big\}\ .
\end{aligned}
\tag{3.125}
$$

In these results, this methodology is continued to arbitrary order with the full dependencies of the matrix elements on the molecular masses, mole fractions, and molecular diameters being retained explicitly up to the final point of actual evaluation via matrix inversion. As indicated previously, adaptation of this work to more realistic potential models is straightforward since the potential model is present in the bracket integral expressions via the omega integrals.

3.7 Results

The quantities of major interest in the present work are the viscosity:

$$\mu = p\left(x_1 b_1 + x_2 b_{-1}\right) \ , \tag{3.126}$$

and the Chapman-Enskog viscosity solutions:

$$\mathbb{B}_1\left(\mathscr{C}\right) = \sum_{p=1}^{+\infty} b_p S_{5/2}^{(p-1)}\left(\mathscr{C}\right)\mathscr{C} \ , \tag{3.127}$$

$$\mathbb{B}_2\left(\mathscr{C}\right) = \sum_{p=-1}^{-\infty} b_p S_{5/2}^{(-p-1)}\left(\mathscr{C}\right)\mathscr{C} \ . \tag{3.128}$$

Here, note that b_1 and b_{-1} are actually integrals on $\mathbb{B}_1$ and $\mathbb{B}_2$ and, thus, may be expressed as:

$$b_1 = \frac{16}{15\sqrt{\pi}} \int_0^\infty \exp\left(-\mathscr{C}\right)\mathscr{C}\mathbb{B}\left(\mathscr{C}\right) \mathrm{d}\mathscr{C} \ , \tag{3.129}$$

and:

$$b_{-1} = \frac{16}{15\sqrt{\pi}} \int_0^\infty \exp\left(-\mathscr{C}\right)\mathscr{C}\mathbb{B}\left(\mathscr{C}\right) \mathrm{d}\mathscr{C} \ . \tag{3.130}$$

Note that $\mathbb{B}_1$, $\mathbb{B}_2$, b_1, b_{-1}, and μ are all dependent upon x_1, x_2, m_1, m_2, σ_1, σ_2, and temperature, T (in addition to the Boltzmann constant), although these dependencies are not displayed explicitly in the above equations. In the m-th order of the expansion, these are written as:

$$\left[\mu\right]_m = p\left(x_1 b_1^{(m)} + x_2 b_{-1}^{(m)}\right) \ , \tag{3.131}$$

$$\mathbb{B}_1^{(m)}\left(\mathscr{C}\right) = \sum_{p=1}^{m} b_p^{(m)} S_{5/2}^{(p-1)}\left(\mathscr{C}\right)\mathscr{C} \ , \tag{3.132}$$

$$\mathbb{B}_2^{(m)}\left(\mathscr{C}\right) = \sum_{p=-1}^{-m} b_p^{(m)} S_{5/2}^{(-p-1)}\left(\mathscr{C}\right)\mathscr{C} \ , \tag{3.133}$$

$$b_1^{(m)} = \frac{16}{15\sqrt{\pi}} \int_0^{\infty} \exp\left(-\mathscr{C}_1^2\right) \mathscr{C}_1^4 \mathbb{B}_1^{(m)}\left(\mathscr{C}_1\right) \mathrm{d}\mathscr{C}_1 \, , \tag{3.134}$$

and:

$$b_{-1}^{(m)} = \frac{16}{15\sqrt{\pi}} \int_0^{\infty} \exp\left(-\mathscr{C}_2^2\right) \mathscr{C}_2^4 \mathbb{B}_2^{(m)}\left(\mathscr{C}_2\right) \mathrm{d}\mathscr{C}_2 \, . \tag{3.135}$$

As a part of this work, the following sets of calculations have been carried out:

(i) A comparison of the current results with the results previously reported by Takata et al. [2] has been conducted. For this comparison, the current results have been adapted so as to present them using the same non-dimensionalization scheme employed by Takata et al. and have then been plotted in graphical form with the same scaling and for the same set of virtual gas mixtures reported by Takata et al.

(ii) A comparison of the current results with the results previously reported by Garcia and Siewert [1] has been conducted. In this comparison, these results have again been adapted so as to present them using the same non-dimensionalization scheme employed by Garcia and Siewert which, in part, is the same as that used by Takata et al. [2]. However, Garcia and Siewert presented their results numerically rather than graphically for selected real gas mixtures and the current results include (among others) similarly adapted numerical results corresponding to the same real gas mixtures. Additionally, Garcia and Siewert presented numerical results for the non-dimensionalized quantities, $\varepsilon_{p,1}$, $\varepsilon_{p,2}$, $\widehat{B}_1$, and $\widehat{B}_2$ (defined below) which have also been computed in the same non-dimensionalized forms for comparison purposes. Here, some actual numerical values from orders 58-60 have been included which allows the degree of convergence of the current results to be explicitly assessed in each case.

(iii) Using the non-dimensionalization/normalization scheme of Chapman and Cowling [10] which normalizes viscosity relative to the first-order approximate results, a comprehensive set of order 60 results for all binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe has been obtained. The two cases considered by Garcia and Siewert [1] are a subset of these collected results which are presented numerically for orders 1 (un-normalized) and 60 (normalized to order 1). Additionally, extrapolated limiting values of the viscosities (corresponding to $m = \infty$) are presented that have been obtained by applying the *Mathematica*® function `SequenceLimit` to the sequence of normalized viscosity results corresponding to orders 1 through 60.

In the work of Takata et al. [2], the authors have considered cases involving a selection of 'virtual' gas mixtures for which the size and mass ratios of the constituents are general values only and do not reflect the sizes and masses of specific gas constituents. The size ratios that have been considered in this work include $\sigma_2/\sigma_1 = \frac{1}{2}, 1, 2$ and the mass ratios that have been considered include $m_2/m_1 = 1, 2, 3, 4, 5, 8, 10$. Mole fractions are specified by $x_1 \in (0, 1)$ with $x_1 + x_2 = 1$. The authors define a non-dimensional viscosity as:

$$\hat{\mu} = f_1 \mu \; , \tag{3.136}$$

where:

$$f_1 = \left(\frac{\sqrt{m_1 k T}}{\sigma_1^2} \frac{1}{4\sqrt{\pi}} \right)^{-1} \; , \tag{3.137}$$

and, thus, $\hat{\mu}$ depends only upon the masses, m_1 and m_2, and the molecular diameters, σ_1 and σ_2. Dependence upon the temperature, T, has been totally eliminated for rigid-sphere molecules which is quite convenient because the rigid-sphere model is known to exhibit an unrealistic temperature dependence. The authors have presented their results graphically and have not given their precise numerical results explicitly although the current indistinguishable comparison results indicate that they have obtained precise results for the cases studied. The present results are reported in the same format and for exactly the same cases considered by Takata

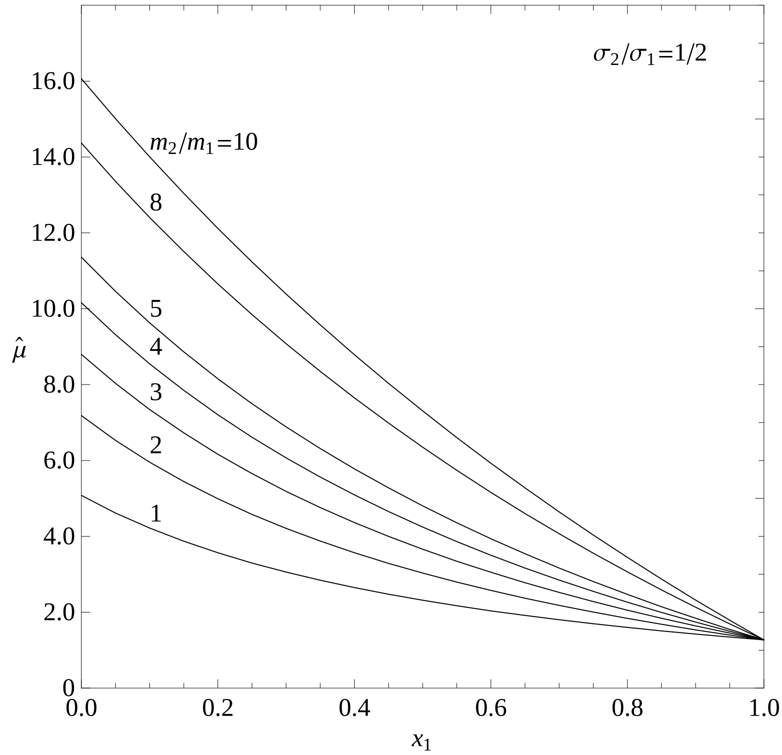

Figure 3.1: The $\hat{\mu}$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 1/2$.

et al., in Figs. 3.1-3.3 where they conform to the same non-dimensionalization scheme described in Eqs. (3.136)-(3.137).

In the work of Garcia and Siewert [1], the authors have considered Ne:Ar and He:Xe gas mixtures. They have reported values of the non-dimensionalized quantities $\hat{\mu}$, $\varepsilon_{p,1}$, $\varepsilon_{p,2}$, $\widehat{B}_1$, and $\widehat{B}_2$ which are related to the current results in the following manner:

$$\hat{\mu} = f_1\mu = 2^{3/2}\left(\frac{\sigma_1}{x_1\sigma_1 + x_2\sigma_2}\right)^2 (x_1\varepsilon_{p,1} + x_2\varepsilon_{p,2}) \ , \tag{3.138}$$

$$\varepsilon_{p,1} = f_2 b_1 \ , \tag{3.139}$$

$$\varepsilon_{p,2} = f_2 b_{-1} \ , \tag{3.140}$$

$$\widehat{B}_1 = f_2 \mathbb{B}_1 \ , \tag{3.141}$$

$$\widehat{B}_2 = f_2 \mathbb{B}_2 \ , \tag{3.142}$$

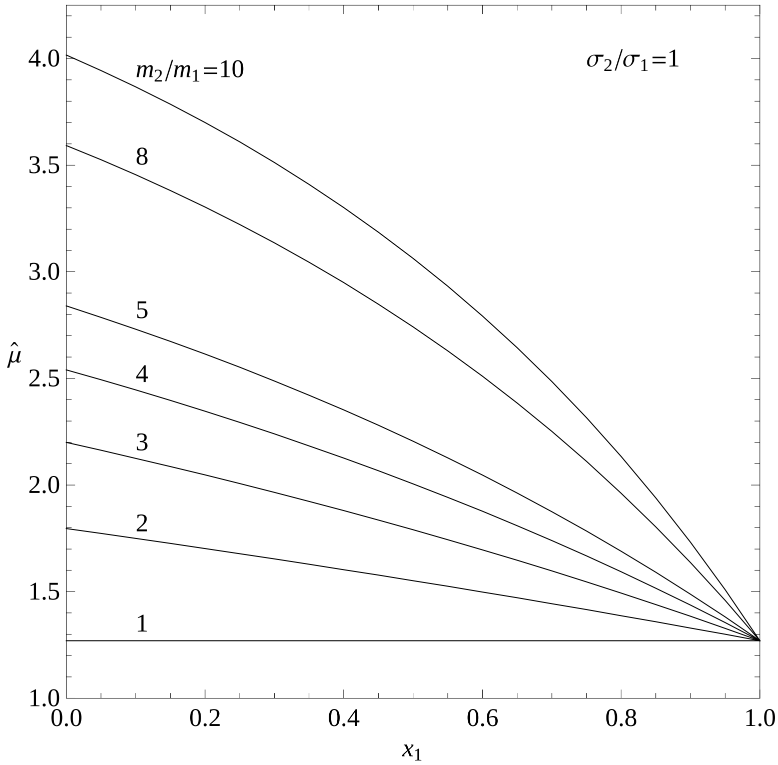

Figure 3.2: The $\hat{\mu}$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 1$.

where f_1 is the same non-dimensionalization used by Takata et al. [2] in Eq. (3.136) and:

$$f_2 = f_1 2^{-3/2} p \left(\frac{\sigma_1}{x_1\sigma_1 + x_2\sigma_2} \right)^{-2} . \tag{3.143}$$

Here, again, the current results have been adapted in a corresponding manner in order to conduct a comparison with Garcia and Siewert in their notation and this comparison is given in numerical form in Tables 3.1-3.7 for selected cases. Note that for the Ne:Ar mixture, Garcia and Siewert used molecular weights of 20.183 and 39.948 for Ne and Ar, respectively, with a molecular diameter ratio of $\sigma_{\text{Ar}}/\sigma_{\text{Ne}} = 1.406$. Additionally, for the He:Xe mixture, Garcia and Siewert have used molecular weights of 4.0026 and 131.30 for He and Xe, respectively, with a molecular diameter ratio of $\sigma_{\text{Xe}}/\sigma_{\text{He}} = 2.226$. For the present comparisons with their results in Tables 3.1-3.7, the same molecular parameters used by Garcia and Siewert (as listed above) have been used. The values reported later as the primary

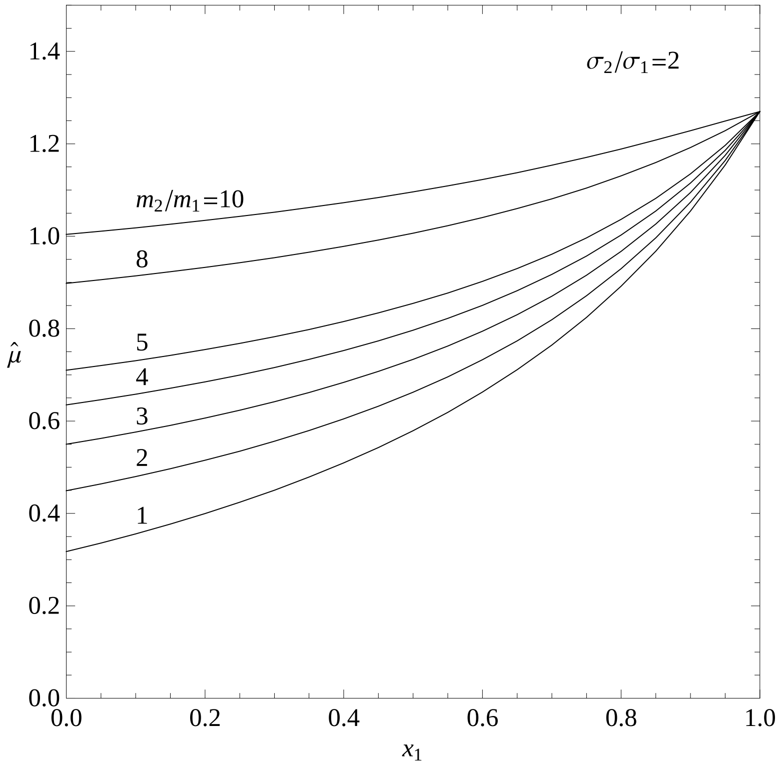

Figure 3.3: The $\hat{\mu}$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 2$.

results of the current work were obtained with slightly different parameter values that are specified later. Note from the results presented in Tables 3.1-3.5 the high number of digits of precision in the current results for $\varepsilon_{p,1}$, $\varepsilon_{p,2}$, and $\hat{\mu}$, and that these results support those of Garcia and Siewert to the number of digits that they had reported. The current results for $\widehat{B}_1$ and $\widehat{B}_2$, however, have not converged fully (to the digits reported) for all values of $\mathscr{C}$ and that, for the lower and higher values of $\mathscr{C}$, expansions beyond order 60 will be needed for a more complete comparison. Also, however, note that it is the middle range of $\mathscr{C}$ ($\approx$ 1-3) where the current work converges best that contributes the most to the macroscopic quantities of interest (such as viscosity and the slip coefficients) and, thus, the current results are the most appropriate ones to use in calculations of such quantities.

Table 3.1: Values of $\varepsilon_{p,1}$, $\varepsilon_{p,1}$, and $\hat{\mu}$ presented in Garcia and Siewert [1] for a binary, rigid-sphere, gas mixture of Ne:Ar.

	Garcia and Siewert [1]		
x_1	$\varepsilon_{p,1}$	$\varepsilon_{p,2}$	$\hat{\mu}^\dagger$
0.0	0.5595420	0.6317248	0.903862
0.1	0.5476674	0.6176388	0.926428
0.2	0.5359138	0.6036851	0.951027
0.3	0.5242967	0.5898802	0.977937
0.4	0.5128351	0.5762439	1.00749
0.5	0.5015515	0.5627999	1.04008
0.6	0.4904734	0.5495769	1.0762
0.7	0.4796344	0.5366103	1.11643
0.8	0.4690763	0.5239439	1.1615
0.9	0.4588517	0.5116331	1.21232
1.0	0.4490278	0.4997484	1.27004

†Calculated here (see Eq. (3.138)) from $\varepsilon_{p,1}$ and $\varepsilon_{p,2}$ as reported by Garcia and Siewert.

Table 3.2: Values of $\varepsilon_{p,1}$, $\varepsilon_{p,1}$, and $\hat{\mu}$ presented in the dimensionless form used by Garcia and Siewert [1] and computed using the same molecular masses and sizes that they used for a binary, rigid-sphere, gas mixture of Ne:Ar with the current order 58, 59, and 60 results arbitrarily truncated at 20 significant figures.

x_1	m	$\varepsilon_{p,1}$	$\varepsilon_{p,2}$	$\hat{\mu}$
10^{-100}	58	0.5595419676692080135	0.63172475587517940212	0.90386225002530434319
	59	0.5595419676692087027	0.63172475587517940213	0.90386225002530434320
	60	0.5595419676692092190	0.63172475587517940213	0.90386225002530434321
0.1	58	0.5476674230030515696	0.61763878392026992253	0.92642788323526232109
	59	0.5476674230030519489	0.61763878392026989958	0.92642788323526234729
	60	0.5476674230030522315	0.61763878392026988248	0.92642788323526236680
0.2	58	0.5359137600871278461	0.60368509577823719786	0.95102658536050854817
	59	0.5359137600871280432	0.60368509577823717168	0.95102658536050857795
	60	0.5359137600871281891	0.60368509577823715230	0.95102658536050859997
0.3	58	0.5242967335652768020	0.58988023823505222751	0.97793688289859194166
	59	0.5242967335652768976	0.58988023823505220635	0.97793688289859196545
	60	0.5242967335652769678	0.58988023823505219079	0.97793688289859198292
0.4	58	0.5128351292637218386	0.57624393073746343774	1.0074903509531462084
	59	0.5128351292637218813	0.57624393073746342354	1.0074903509531462240
	60	0.5128351292637219124	0.57624393073746341318	1.0074903509531462354
0.5	58	0.5015515418963773753	0.56279988306568113001	1.0400848946575599796
	59	0.5015515418963773925	0.56279988306568112178	1.0400848946575599884
	60	0.5015515418963774049	0.56279988306568111583	1.0400848946575599947
0.6	58	0.4904734073043349838	0.54957688246219324868	1.0762022481161614336
	59	0.4904734073043349898	0.54957688246219324456	1.0762022481161614378
	60	0.4904734073043349942	0.54957688246219324160	1.0762022481161614407
0.7	58	0.4796343927427824630	0.53661026065287905164	1.1164313496741517441
	59	0.4796343927427824648	0.53661026065287904989	1.1164313496741517458

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Table 3.2 – Continued

x_1	m	$\varepsilon_{p,1}$	$\varepsilon_{p,2}$	$\hat{\mu}$
	60	0.4796343927427824660	0.53661026065287904865	1.1164313496741517469
0.8	58	0.4690763001600726542	0.52394390652487269288	1.1615000475714855626
	59	0.4690763001600726546	0.52394390652487269228	1.1615000475714855631
	60	0.4690763001600726549	0.52394390652487269186	1.1615000475714855635
0.9	58	0.4588517197238235193	0.51163307906334812957	1.2123188450384572282
	59	0.4588517197238235193	0.51163307906334812942	1.2123188450384572283
	60	0.4588517197238235194	0.51163307906334812931	1.2123188450384572284
1-10^{-100}	58	0.4490278062878924345	0.49974842186381900536	1.2700424270699528174
	59	0.4490278062878924346	0.49974842186381900534	1.2700424270699528174
	60	0.4490278062878924346	0.49974842186381900532	1.2700424270699528174

Table 3.3: Values of $\varepsilon_{p,1}$, $\varepsilon_{p,1}$, and $\hat{\mu}$ presented in Garcia and Siewert [1] for a binary, rigid-sphere, gas mixture of He:Xe.

	Garcia and Siewert [1]		
x_1	$\varepsilon_{p,1}$	$\varepsilon_{p,2}$	$\hat{\mu}^{\dagger}$
0.0	0.6872939	2.571784	1.46801
0.1	0.6591318	2.496050	1.47828
0.2	0.6313460	2.424505	1.48925
0.3	0.6040327	2.358467	1.50078
0.4	0.5773241	2.299872	1.51252
0.5	0.5514060	2.251684	1.52364
0.6	0.5265471	2.218677	1.53232
0.7	0.5031460	2.209062	1.53438
0.8	0.4818064	2.238244	1.51971
0.9	0.4634256	2.338849	1.46101
1.0	0.4490278	2.593733	1.27004

†Calculated here (see Eq. (3.138)) from $\varepsilon_{p,1}$ and $\varepsilon_{p,2}$ as reported by Garcia and Siewert.

Table 3.4: Values of $\varepsilon_{p,1}$, $\varepsilon_{p,1}$, and $\hat{\mu}$ presented in the dimensionless form used by Garcia and Siewert [1] and computed using the same molecular masses and sizes that they used for a binary, rigid-sphere, gas mixture of Ne:Ar with the current order 58, 59, and 60 results arbitrarily truncated at 20 significant figures.

x_1	m	$\varepsilon_{p,1}$	$\varepsilon_{p,2}$	$\hat{\mu}$
10^{-100}	58	0.6872937322013061374	2.571784078122286320	1.468010550302649127
	59	0.6872937447070151995	2.571784078122286320	1.468010550302649127
	60	0.6872937560689973469	2.571784078122286320	1.468010550302649127
0.1	58	0.6591317010899825315	2.496050032005066600	1.478278852837708127
	59	0.6591317080786303823	2.496050031666878712	1.478278853089907215
	60	0.6591317143929958823	2.496050031359873829	1.478278853316941398
0.2	58	0.6313459115366166018	2.424504832197169971	1.489249176338942153
	59	0.6313459151919377899	2.424504831768158172	1.489249176618539411
	60	0.6313459184749826248	2.424504831380797925	1.489249176868483918
0.3	58	0.6040326461214347863	2.358466807576086552	1.500783073486171713
	59	0.6040326478783327377	2.358466807190135265	1.500783073696612578
	60	0.6040326494459975592	2.358466806843711164	1.500783073883215617
0.4	58	0.5773240572503967573	2.299872149675410386	1.512521641308333831
	59	0.5773240580043236595	2.299872149389605219	1.512521641430480553
	60	0.5773240586719789073	2.299872149134824764	1.512521641537703647
0.5	58	0.5514060290073501917	2.251684192827048473	1.523643416565273163
	59	0.5514060292816985444	2.251684192650492136	1.523643416618428841
	60	0.5514060295223141155	2.251684192494498734	1.523643416664425937
0.6	58	0.5265471195182058579	2.218676851307744181	1.532318201715292502
	59	0.5265471195936374130	2.218676851223210552	1.532318201729866321
	60	0.5265471196587474937	2.218676851149646338	1.532318201742141638
0.7	58	0.5031459951586207703	2.209061619755904188	1.534376355824601918
	59	0.5031459951685818242	2.209061619735066384	1.534376355825692539

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Table 3.4 – Continued

x_1	m	$\varepsilon_{p,1}$	$\varepsilon_{p,2}$	$\hat{\mu}$
	60	0.50314599517667 15873	2.2090616197179 91686	1.5343763558265 09563
0.8	58	0.4818063818842309327	2.2382444203393 14744	1.5197101731410 90702
	59	0.4818063818813508385	2.2382444203522 76109	1.5197101731416 16425
	60	0.4818063818786078540	2.2382444203647 39004	1.5197101731421 60379
0.9	58	0.4634255855727828224	2.3388486048783 15598	1.4610143545085 04662
	59	0.4634255855713369549	2.3388486048979 30042	1.4610143545099 86315
	60	0.4634255855700366116	2.3388486049157 47730	1.4610143545113 58659
$1\text{-}10^{-100}$	58	0.4490278062878924345	2.5937331869365 66237	1.2700424270699 52817
	59	0.4490278062878924346	2.5937331869365 64630	1.2700424270699 52817
	60	0.4490278062878924346	2.5937331869365 63393	1.2700424270699 52817

Table 3.5: Values of $\varepsilon_{p,1}$, $\varepsilon_{p,1}$, and $\hat{\mu}$ computed in the current work using using the same molecular masses and sizes as Garcia and Siewert [1] for a binary, rigid-sphere, gas mixture of Ne:Ar with $x_1 = 0.5$ and demonstrating the degree of convergence of the current results as a function of the order of the approximation up to order $m = 60$.

m	$\varepsilon_{p,1}$	$\varepsilon_{p,2}$	$\hat{\mu}$
1	0.489014271942809503	0.557640318263336575	1.02279153639148582
2	0.500148966484225233	0.562686482783745180	1.03860348223764872
3	0.501287238352212758	0.562824661265745622	1.03985083033778413
4	0.501485684809746423	0.562812826116856584	1.04003318702819818
5	0.501531933091506243	0.562805024536433164	1.04007075716938611
6	0.501544924542130930	0.562801913569808127	1.04008041238468336
7	0.501549089164067012	0.562800713769189732	1.04008330960997527
8	0.501550563518324078	0.562800237499302848	1.04008428493866153
9	0.501551127737287012	0.562800040557032337	1.04008464384168076
10	0.501551357715680362	0.562799955697608877	1.04008478565203115
⋮	⋮	⋮	⋮
15	0.501551536471126114	0.562799885419491511	1.04008489165614598
⋮	⋮	⋮	⋮
20	0.501551541612327209	0.562799883193865289	1.04008489450524761
⋮	⋮	⋮	⋮
25	0.501551541875005921	0.562799883075527972	1.04008489464629812
⋮	⋮	⋮	⋮
30	0.501551541894301922	0.562799883066649675	1.04008489465647830
⋮	⋮	⋮	⋮
35	0.501551541896133184	0.562799883065796074	1.04008489465743368

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Table 3.5 – Continued

m	$\varepsilon_{p,1}$	$\varepsilon_{p,2}$	$\hat{\mu}$
⋮	⋮	⋮	⋮
40	0.501551541896343999	0.562799883065696938	1.04008489465754281
⋮	⋮	⋮	⋮
45	0.501551541896372260	0.562799883065683563	1.04008489465755736
⋮	⋮	⋮	⋮
50	0.501551541896376550	0.562799883065681524	1.04008489465755955
51	0.501551541896376807	0.562799883065681401	1.04008489465755968
52	0.501551541896376988	0.562799883065681314	1.04008489465755978
53	0.501551541896377116	0.562799883065681253	1.04008489465755984
54	0.501551541896377207	0.562799883065681210	1.04008489465755989
55	0.501551541896377272	0.562799883065681179	1.04008489465755992
56	0.501551541896377318	0.562799883065681157	1.04008489465755995
57	0.501551541896377351	0.562799883065681141	1.04008489465755996
58	0.501551541896377375	0.562799883065681130	1.04008489465755997
59	0.501551541896377392	0.562799883065681121	1.04008489465755998
60	0.501551541896377404	0.562799883065681115	1.04008489465755999

Table 3.6: Values of the Chapman-Enskog solution for viscosity presented in the dimensionless form used by Garcia and Siewert [1] and computed using the same molecular masses and sizes that they used for a binary, rigid-sphere, gas mixture of Ne:Ar with the current order 60 results truncated at 8 significant figures.

		Garcia and Siewert [1]		Present work (order 60)	
	$\mathscr{C}$	$\widehat{B}_1(\mathscr{C})$	$\widehat{B}_2(\mathscr{C})$	$\widehat{B}_1(\mathscr{C})$	$\widehat{B}_2(\mathscr{C})$
$x_1 = 0.1$	0.0	0.0	0.0	0.0	0.0
	0.1	0.008948039	0.008420111	0.0089479790	0.0084201111
	0.2	0.03545276	0.03353171	0.035452636	0.033531712
	0.3	0.07855316	0.07490027	0.078553089	0.074900268
	0.4	0.1368178	0.1318384	0.13681786	0.13183841
	0.5	0.2085412	0.2034548	0.20854121	0.20345477
	1.0	0.7100345	0.7433387	0.71003446	0.74333869
	1.5	1.339594	1.488148	1.3395936	1.4881476
	2.0	2.019132	2.338906	2.0191322	2.3389055
	2.5	2.717517	3.240245	2.7175174	3.2402453
	3.0	3.421611	4.163716	3.4216110	4.1637163
	3.5	4.125616	5.094889	4.1256150	5.0948888
	4.0	4.826999	6.026471	4.8269989	6.0264714
	4.5	5.524760	6.954864	5.5247734	6.9548639
	5.0	6.218626	7.878411	6.2187147	7.8784110
$x_1 = 0.5$	0.0	0.0	0.0	0.0	0.0
	0.1	0.007690682	0.007288256	0.0076906715	0.0072882559
	0.2	0.03053225	0.02905193	0.030532226	0.029051931
	0.3	0.06786436	0.06499422	0.067864348	0.064994222
	0.4	0.1186811	0.1146411	0.11868112	0.11464110
	0.5	0.1817459	0.1773693	0.18174593	0.17736928
	1.0	0.6346182	0.6592924	0.63461815	0.65929239

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Table 3.6 – Continued

		Garcia and Siewert [1]		Present work (order 60)	
	$\mathscr{C}$	$\widehat{B}_1(\mathscr{C})$	$\widehat{B}_2(\mathscr{C})$	$\widehat{B}_1(\mathscr{C})$	$\widehat{B}_2(\mathscr{C})$
	1.5	1.221164	1.345306	1.2211640	1.3453057
	2.0	1.865655	2.151078	1.8656546	2.1510783
	2.5	2.534725	3.023796	2.5347254	3.0237958
	3.0	3.213507	3.933287	3.2135065	3.9332871
	3.5	3.895101	4.862534	3.8951010	4.8625341
	4.0	4.576255	5.801829	4.5762547	5.8018289
	4.5	5.255462	6.745558	5.2554652	6.7455579
	5.0	5.932088	7.690450	5.9321066	7.6904498
$x_1 = 0.9$	0.0	0.0	0.0	0.0	0.0
	0.1	0.006486759	0.006248353	0.0064867580	0.0062483533
	0.2	0.02581069	0.02493178	0.025810685	0.024931775
	0.3	0.05757558	0.05586826	0.057575575	0.055868264
	0.4	0.1011612	0.09876387	0.10116121	0.098763868
	0.5	0.1557752	0.1532257	0.15577522	0.15322573
	1.0	0.5615379	0.5805261	0.56153792	0.58052610
	1.5	1.109914	1.211018	1.1099144	1.2110177
	2.0	1.727389	1.976274	1.7273894	1.9762740
	2.5	2.375998	2.826901	2.3759977	2.8269010
	3.0	3.037333	3.731066	3.0373327	3.7310662
	3.5	3.702490	4.668925	3.7024900	4.6689249
	4.0	4.367176	5.628092	4.3671764	5.6280918
	4.5	5.029391	6.600728	5.0293911	6.6007275
	5.0	5.688295	7.581783	5.6882965	7.5817829

Table 3.7: Values of the Chapman-Enskog solution for viscosity presented in the dimensionless form used by Garcia and Siewert [1] and computed using the same molecular masses and sizes that they used for a binary, rigid-sphere, gas mixture of He:Xe with the current order 60 results truncated at 8 significant figures.

		Garcia and Siewert [1]		Present work (order 60)	
	$\mathscr{C}$	$\widehat{B}_1(\mathscr{C})$	$\widehat{B}_2(\mathscr{C})$	$\widehat{B}_1(\mathscr{C})$	$\widehat{B}_2(\mathscr{C})$
$x_1 = 0.1$	0.0	0.0	0.0	0.0	0.0
	0.1	0.03050749	0.03378962	0.027923925	0.033789624
	0.2	0.1068797	0.1345907	0.10266667	0.13459069
	0.3	0.2053841	0.3007407	0.20394473	0.30074069
	0.4	0.3129096	0.5295988	0.31389900	0.52959878
	0.5	0.4244460	0.8177201	0.42507796	0.81772012
	1.0	1.009060	2.996638	1.0088808	2.9966377
	1.5	1.627307	6.012572	1.6271621	6.0125720
	2.0	2.276993	9.460218	2.2771273	9.4602184
	2.5	2.956657	13.11015	2.9569973	13.110150
	3.0	3.664323	16.84482	3.6639151	16.844819
	3.5	4.397634	20.60532	4.3942343	20.605319
	4.0	5.154014	24.36247	5.1539701	24.362467
	4.5	5.930801	28.10232	6.0289644	28.102317
	5.0	6.725367	31.81886	7.2682967	31.818855
$x_1 = 0.5$	0.0	0.0	0.0	0.0	0.0
	0.1	0.01527023	0.02786601	0.015108419	0.027866010
	0.2	0.05814503	0.1111796	0.057835996	0.11117959
	0.3	0.1219944	0.2490995	0.12186368	0.24909951
	0.4	0.2002270	0.4402630	0.20031454	0.44026295
	0.5	0.2880717	0.6828407	0.28813920	0.68284071
	1.0	0.7973243	2.579754	0.79730316	2.5797542

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Table 3.7 – Continued

		Garcia and Siewert [1]		Present work (order 60)	
	$\mathscr{C}$	$\widehat{B}_1(\mathscr{C})$	$\widehat{B}_2(\mathscr{C})$	$\widehat{B}_1(\mathscr{C})$	$\widehat{B}_2(\mathscr{C})$
	1.5	1.359373	5.352794	1.3593535	5.3527938
	2.0	1.950200	8.671471	1.9502172	8.6714712
	2.5	2.564073	12.30070	2.5641182	12.300701
	3.0	3.198823	16.09584	3.1987692	16.095839
	3.5	3.853043	19.97289	3.8525839	19.972889
	4.0	4.525390	23.88402	4.5253686	23.884024
	4.5	5.214438	27.80230	5.2276812	27.802303
	5.0	5.918679	31.71282	5.9926429	31.712822
$x_1 = 0.9$	0.0	0.0	0.0	0.0	0.0
	0.1	0.007397869	0.02506036	0.0074148273	0.025060360
	0.2	0.02936495	0.1001712	0.029393296	0.10017122
	0.3	0.06515651	0.2251231	0.065163889	0.22512312
	0.4	0.1135534	0.3995709	0.11354393	0.39957090
	0.5	0.1731147	0.6230394	0.17311118	0.62303935
	1.0	0.5922252	2.451043	0.59222745	2.4510430
	1.5	1.128489	5.375200	1.1284900	5.3752000
	2.0	1.717866	9.250880	1.7178641	9.2508803
	2.5	2.333036	13.92669	2.3330328	13.926686
	3.0	2.962572	19.26161	2.9625778	19.261605
	3.5	3.601714	25.13209	3.6017485	25.132086
	4.0	4.248524	31.43345	4.2485073	31.433451
	4.5	4.902269	38.07875	4.9012010	38.078753
	5.0	5.562708	44.99667	5.5575101	44.996673

Finally, the primary results of this work for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe are presented. Note that a more transparent way to report viscosity results may be achieved by normalizing the viscosity in the following manner:

$$[\mu]_m = [\mu]_1 \, [\mu]_m^\star \ , \tag{3.144}$$

where, $[\mu]_m^\star \equiv [\mu]_m \,/\, [\mu]_1$ and $[\mu]_1$ is the viscosity of the mixture computed with a first-order approximation ($m = 1$). In the general case, $[\mu]_1$ can be explicitly expressed as:

$$\begin{aligned}[\mu]_1 =& 5kT\Big(3x_1^2\Omega_1^{(2)}(2) + 3x_2^2\Omega_2^{(2)}(2) + 20M_1M_2\Omega_{12}^{(1)}(1) \\ &+ 6\Big(M_1x_1 - M_2x_2\Big)^2\Omega_{12}^{(2)}(2)\Big)\Big/\Delta \ ,\end{aligned} \tag{3.145}$$

where:

$$\begin{aligned}\Delta =& 8\Big\{3M_1^2x_1\Omega_1^{(2)}(2)\left(x_2\Omega_2^{(2)}(2) + 2x_1\Omega_{12}^{(2)}(2)\right) \\ &+ 3M_2^2x_2\Omega_2^{(2)}(2)\left(x_1\Omega_1^{(2)}(2) + 2x_2\Omega_{12}^{(2)}(2)\right) \\ &+ 2M_1M_2\Big[10x_1^2\Omega_1^{(2)}(2)\,\Omega_{12}^{(1)}(1) + 10x_2^2\Omega_2^{(2)}(2)\,\Omega_{12}^{(1)}(1) \\ &+ x_1x_2\Big(3\Omega_1^{(2)}(2)\,\Omega_2^{(2)}(2) + 20\Omega_{12}^{(1)}(1)\,\Omega_{12}^{(2)}(2)\Big)\Big]\Big\} \ .\end{aligned} \tag{3.146}$$

and, for the specific case of rigid-sphere molecules, one would use Eqs. (3.113)-(3.115) for the omega integrals needed. Based upon this definition, one may express Eq. (3.131) as:

$$[\mu]_m = [\mu]_1 \, \frac{p}{[\mu]_1} \left(x_1 \frac{b_1^{(m)}}{b_1^{(1)}} b_1^{(1)} + x_2 \frac{b_{-1}^{(m)}}{b_{-1}^{(1)}} b_{-1}^{(1)} \right) \ , \tag{3.147}$$

such that:

$$[\mu]_m = [\mu]_m^\star \, [\mu]_1 = p\left(x_1 b_1^{(m)\star} b_1^{(1)} + x_2 b_{-1}^{(m)\star} b_{-1}^{(1)}\right) \tag{3.148}$$

where:

$$b_1^{(m)\star} \equiv \frac{b_1^{(m)}}{b_1^{(1)}} \ , \tag{3.149}$$

$$b_{-1}^{(m)\star} \equiv \frac{b_{-1}^{(m)}}{b_{-1}^{(1)}} \ , \tag{3.150}$$

are also normalized to their order 1 values, $b_1^{(1)}$ and $b_{-1}^{(1)}$, which are, explicitly:

$$\begin{aligned} b_1^{(1)} = & n^{-1}\Big(5\Big(3x_1\varOmega_1^{(2)}(2) + 20M_1M_2\varOmega_{12}^{(1)}(1) \\ & + 6\Big(M_2^2x_2 - M_1M_2x_1\Big)\varOmega_{12}^{(2)}(2)\Big)\Big)\Big/\Delta \ , \end{aligned} \tag{3.151}$$

$$\begin{aligned} b_{-1}^{(1)} = & n^{-1}\Big(5\Big(3x_2\varOmega_2^{(2)}(2) + 20M_1M_2\varOmega_{12}^{(1)}(1) \\ & + 6\Big(M_1^2x_1 - M_1M_2x_2\Big)\varOmega_{12}^{(2)}(2)\Big)\Big)\Big/\Delta \ . \end{aligned} \tag{3.152}$$

Again, for the specific case of rigid-sphere molecules, one uses Eqs. (3.113)-(3.115) for the omega integrals needed. Here, some representative order 60 results for $[\mu]_1$, $b_1^{(1)}$, $b_{-1}^{(1)}$, and $[\mu]_{60}^{\star}$, $b_1^{(60)\star}$, $b_{-1}^{(60)\star}$ for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe are reported in Tables 3.8 and 3.9, respectively. The molecular mass and diameter values used in computing the results given in Tables 3.8 and 3.9 are summarized in Table 3.10.

Table 3.8: Values of $[\mu]_1$, $b_1^{(1)}$, and $b_{-1}^{(1)}$ for binary, rigid-sphere gas mixtures of He, Ne, Ar, Kr, and Xe (arbitrarily truncated at 21 significant figures).

	x_1	$[\mu]_1 \times 10^4$(gm cm^{-1} s^{-1})	$b_1^{(1)} \times 10^{10}$(s^{-1})	$b_{-1}^{(1)} \times 10^{10}$(s^{-1})
He:Ne	10^{-100}	2.92741878308280513703	2.88913770844589700176	1.34368924951280000719
	0.1	2.87457091754218394474	2.99860390470612425053	1.38237406112066703749
	0.2	2.81409265010785212121	3.11589653970997464552	1.42288139658991696790
	0.3	2.74472292566742726819	3.24178717070666655686	1.46526632178454263040
	0.4	2.66495469170310718033	3.37713346770320234228	1.50956427340968496508
	0.5	2.57297981381332066176	3.52288635057964958548	1.55578093550635221419
	0.6	2.46662044687919315972	3.68009563162441021352	1.60387809422126770323
	0.7	2.34324347097656398304	3.84991258242714015516	1.65375389479859582585
	0.8	2.19965389952399640659	4.03358687577877131889	1.70521529625885566101
	0.9	2.03196249535873792164	4.23245384538498686289	1.75793964358374222415
	1-10^{-100}	1.83542239893324326724	4.44790567000913691538	1.81142106975893734739
He:Ar	10^{-100}	2.08288258264550698739	2.05564528265038932879	0.829153625748764596205
	0.1	2.08912595200792372568	2.19336527037006398775	0.877782658038027373833
	0.2	2.09429607819828807954	2.35054849330352297306	0.932353661615851752165
	0.3	2.09772294109462499412	2.53156135931888861716	0.993995420556691406144
	0.4	2.09838608792901871679	2.74215695429361885582	1.06412970038542593978
	0.5	2.09470554052516077882	2.99006118786990180082	1.14456608185654434537
	0.6	2.08418333820625434606	3.28587620906805406215	1.23763090825063251035
	0.7	2.06276663965863243481	3.64451499737534967737	1.34633984467443589106
	0.8	2.02366815070522618899	4.08755640237272916364	1.47461737696076070069
	0.9	1.95507315698396541303	4.64725512274322074671	1.62753519387231249376
	1-10^{-100}	1.83542239893324326724	5.37364030429427803665	1.81142106975893734739
He:Kr	10^{-100}	2.29069445315765001595	2.26073965275859858470	0.665778798598662287728
	0.1	2.29996588414412040408	2.44303615068828806806	0.711573080954387442542

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Table 3.8 – Continued

	x_1	$[\mu]_1 \times 10^4$(gm cm^{-1} s^{-1})	$b_1^{(1)} \times 10^{10}$(s^{-1})	$b_{-1}^{(1)} \times 10^{10}$(s^{-1})
	0.2	2.30864526678849505724	2.65705292525240597026	0.764066940927188639269
	0.3	2.31598174702150130554	2.91178412864209581937	0.824824604585504706833
	0.4	2.32068645519376118207	3.21994761558125931037	0.895927220086687846857
	0.5	2.32050109284246933754	3.60011551554361606980	0.980197522388719163445
	0.6	2.31134391844667025898	4.08048097387730268622	1.08154450157776047415
	0.7	2.28551296355833244016	4.70592443991018336117	1.20549798306247147150
	0.8	2.22760711632037560795	5.55227656735689917586	1.36002747340920304707
	0.9	2.10432159509426816375	6.75790474954513806567	1.55668163099663574224
	1-10^{-100}	1.83542239893324326724	8.60270648697165432120	1.81142106975893734739
He:Xe	10^{-100}	2.07245362110288412252	2.04535269785628830251	5.21793887055332997073
	0.1	2.08699110979597286944	2.22608453023373598385	5.62240064767561632089
	0.2	2.10275645301134304187	2.44171000558585276358	6.09456316227242988984
	0.3	2.11966104887099464023	2.70336656086767784013	6.65287380250258870534
	0.4	2.13735634836514919031	3.02747173775668941418	7.32309167026819296708
	0.5	2.15491994307889401534	3.43925548062224684473	8.14225778847566163596
	0.6	2.17012784001680663964	3.97954826746530917399	9.16550583457713350475
	0.7	2.17763803894805052416	4.71893608547599563561	10.4782974497266187420
	0.8	2.16397106269447652236	5.79057815385998744863	12.2194719937669635565
	0.9	2.09150692441684148171	7.47807843578246574375	14.6261000291820395342
	1-10^{-100}	1.83542239893324326724	10.5055599727433085104	18.1142106975893734739
Ne:Ar	10^{-100}	2.08288258264550698739	2.05564528265038932879	1.79079008360150007741
	0.1	2.13444789659795181421	2.13381497011060281924	1.86102817646030435012
	0.2	2.19070714612184236145	2.21830122856319785949	1.93709435109062016556
	0.3	2.25232215799943131862	2.30991769133478242145	2.01975585978436883506
	0.4	2.32008396966820725237	2.40962647579006482559	2.10992241231590757870

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Table 3.8 – Continued

	x_1	$[\mu]_1 \times 10^4$(gm cm^{-1} s^{-1})	$b_1^{(1)} \times 10^{10}$(s^{-1})	$b_{-1}^{(1)} \times 10^{10}$(s^{-1})
	0.5	2.39494573365483538915	2.51857417157399298047	2.20868116255842328234
	0.6	2.47806615998287277431	2.63813895189007635444	2.31734261498778654141
	0.7	2.57086763987752163732	2.76999307293261056454	2.43750166794660934188
	0.8	2.67511516124266597363	2.91618701681492024662	2.57111998801520156137
	0.9	2.79302520834962376628	3.07926463832057530512	2.72063899868058390053
	1-10^{-100}	2.92741878308280513703	3.26242363787507244099	2.88913770844589700176
Ne:Kr	10^{-100}	2.29069445315765001595	2.26073965275859858470	1.43875513127484465816
	0.1	2.32626768312174477996	2.38260190450511323634	1.51505987027801028291
	0.2	2.36553152923020777062	2.51827892104716194002	1.59987483754936245766
	0.3	2.40908648566156674660	2.67024752790335195916	1.69470078232722028380
	0.4	2.45766924144659698770	2.84161064022732404186	1.80141143034886586838
	0.5	2.51219299398853531394	3.03630711693035922887	1.92237631556613287858
	0.6	2.57380265744499891992	3.25941146431176393440	2.06063523473877950046
	0.7	2.64395158679786193943	3.51757134460689872021	2.22015133385637432725
	0.8	2.72450983718089370522	3.81966222856419355921	2.40618712874705141925
	0.9	2.81791925712920618360	4.17779481300207183895	2.62587844143078789869
	1-10^{-100}	2.92741878308280513703	4.60891576019373779941	2.88913770844589700176
Ne:Xe	10^{-100}	2.07245362110288412252	2.04535269785628830251	1.12659967856692117209
	0.1	2.11029336364559657894	2.18077584471551245090	1.19999359934273759839 6
	0.2	2.15315130597330988644	2.33534557640444017216	1.28359329750738062123 0
	0.3	2.20209227720914312297	2.51341698597652313610	1.37968071216595008015 2
	0.4	2.25850364545503902581	2.72076918202165363878	1.49127071612379643748 1
	0.5	2.32422656317606976570	2.96523656275628933770	1.62242998187942695285 0
	0.6	2.40175634604303764907	3.25770687687340477821	1.77877744690557890035 1
	0.7	2.49455796191427376289	3.61375987934964924373	1.96829904139997058126 9

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Table 3.8 – Continued

	x_1	$[\mu]_1 \times 10^4$(gm cm^{-1} s^{-1})	$b_1^{(1)} \times 10^{10}$(s^{-1})	$b_{-1}^{(1)} \times 10^{10}$(s^{-1})
	0.8	2.60758047145323730071	4.05648018816788498053	2.20273224934741599761
	0.9	2.74812998659531872419	4.62154986037212065587	2.500042708548524932133
	1-10^{-100}	2.92741878308280513703	5.36708604328363454937	2.889137708445897001762
Ar:Kr	10^{-100}	2.29069445315765001595	2.26073965275859858470	1.661013423695964636143
	0.1	2.27792580456086075762	2.30967533841138706717	1.694301718036581360884
	0.2	2.26383490878451817545	2.36058777623194628792	1.728805612489032886593
	0.3	2.24828394295070345281	2.41358491132955562281	1.764580985171257137729
	0.4	2.23111866690908974380	2.46878107410759718559	1.801685696781090636133
	0.5	2.21216619817109373792	2.52629719333286078498	1.840179388302656092244
	0.6	2.19123245210886234367	2.58626095040217083954	1.880123182680915375293
	0.7	2.16809919349290761120	2.64880684858500104682	1.921579262901246932426
	0.8	2.14252063667854862465	2.71407616272158469824	1.964610291516184906921
	0.9	2.11421952175011211605	2.78221672408010974717	2.009278627280417793045
	1-10^{-100}	2.08288258264550698739	2.85338248109014473994	2.055645282650389328791
Ar:Xe	10^{-100}	2.07245362110288412252	2.04535269785628830251	1.31599130047056829750
	0.1	2.07885303892681738921	2.12786532131841598307	1.36589643013590815535
	0.2	2.08487334166880826240	2.21711018318303104219	1.41960931251242684238
	0.3	2.09034736388545465690	2.31391855913599781331	1.47756485847280140756
	0.4	2.09505717958736839229	2.41925908328860085569	1.54026306415510363204
	0.5	2.09871763017990481586	2.53426592941643817744	1.60828058956684297897
	0.6	2.10095380481519096206	2.66027374032161325255	1.68228449356499894861
	0.7	2.10126994229715787569	2.79886105163642599614	1.76304840030450147477
	0.8	2.09900602272995144993	2.95190436678978995862	1.85147125931408896474
	0.9	2.09327647662780293721	3.12164546382420226068	1.94859856896764058793
	1-10^{-100}	2.08288258264550698739	3.31077483494224526969	2.05564528265038932879

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Table 3.8 – Continued

	x_1	$[\mu]_1 \times 10^4$(gm cm^{-1} s^{-1})	$b_1^{(1)} \times 10^{10}$(s^{-1})	$b_{-1}^{(1)} \times 10^{10}$(s^{-1})
Kr:Xe	10^{-100}	2.07245362110288412252	2.04535269785628830251	1.79353101667798172505
	0.1	2.09014777588804023440	2.08853033226441748996	1.83138171860657601645
	0.2	2.10860200168842971318	2.13356950439597467224	1.87086386096744457862
	0.3	2.12786634714630409637	2.18059320630540174178	1.91208520682306201486
	0.4	2.14799535353944957717	2.22973550492645557374	1.95516322062524281055
	0.5	2.16904857045027546492	2.28114281691041605543	2.00022618471855106012
	0.6	2.19109114412954451343	2.33497536365499571956	2.04741447362476313561
	0.7	2.21419449079810592301	2.39140883694560743056	2.09688201274197582646
	0.8	2.23843706954468899106	2.45063631164389853508	2.14879795335560443243
	0.9	2.26390527244181719856	2.51287044924241225284	2.20334860232182446697
	1-10^{-100}	2.29069445315765001595	2.57834604521498857140	2.26073965275859858470

Table 3.9: Values of $[\mu]^\star_{60}$, $b_1^{(60)\star}$, and $b_{-1}^{(60)\star}$ for binary, rigid-sphere gas mixtures of He, Ne, Ar, Kr, and Xe (arbitrarily truncated at 21 significant figures).

	x_1	$[\mu]^\star_{60}$	$b_1^{(60)\star}$	$b_{-1}^{(60)\star}$
He:Ne	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.04485118491390073388
	0.1	1.01507981005595704974	1.01371253815614662426	1.04177241284408011604
	0.2	1.01425878361375016253	1.01146554561865102285	1.03872587025744242162
	0.3	1.01358127106503753285	1.00929376018953282452	1.03571476375207361499
	0.4	1.01306346604592649143	1.00719893038545592346	1.03274330346693874360
	0.5	1.01272954522889912551	1.00518354872922256718	1.02981658329674914769
	0.6	1.01261532435082903314	1.00325053005258540094	1.02694032769208383891
	0.7	1.01277457235078867075	1.00140267898088736701	1.02412039477035941321
	0.8	1.01329052801883149671	0.999641789518708382692	1.02136185083153100733
	0.9	1.01429844951434984249	0.997967117200199526458	1.01866730196333090790
	1-10^{-100}	1.01603394165596225394	0.996372787771616987210	1.01603394165596225394
He:Ar	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.05792170948470323283
	0.1	1.01587330369910437037	1.01414040770254626373	1.05484407254589890505
	0.2	1.01572351651591546094	1.01216930500376273738	1.05156548017718878262
	0.3	1.01558510378211889495	1.01011928485306677439	1.04806660553474875775
	0.4	1.01545990027647383972	1.00799171156045055591	1.04432707541903814538
	0.5	1.01535234856768505558	1.00579255958604055537	1.04032631420527959139
	0.6	1.01527198434076274506	1.00353557181296176032	1.03604522050124807820
	0.7	1.01523823494731039261	1.00124756995304609840	1.03146928492961485769
	0.8	1.01528982741619837232	0.998977447114818741648	1.02659407751873486611
	0.9	1.01550365889632625106	0.996811310231086621115	1.02143410673055207608
	1-10^{-100}	1.01603394165596225394	0.994896796246408654503	1.01603394165596225394
He:Kr	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.06672812265169489641
	0.1	1.01613267018074011140	1.01459883217027165082	1.06352765501439215433

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Table 3.9 – Continued

	x_1	$[\mu]^\star_{60}$	$b_1^{(60)\star}$	$b_{-1}^{(60)\star}$
	0.2	1.01619235268418533591	1.01304035552809960009	1.06003680041700754320
	0.3	1.01620154994363010615	1.01134395742205251696	1.05621402657755079311
	0.4	1.01614732639360126813	1.00949475081562635451	1.05201119634227042454
	0.5	1.01601694909473345520	1.00747955862365593122	1.04737347795258903733
	0.6	1.01580289424595185290	1.00529171115686511912	1.04224081553684045984
	0.7	1.01551637720177399937	1.00294218768787700613	1.03655327407293334150
	0.8	1.01522390859716212394	1.00048608938610277721	1.03026559890478261044
	0.9	1.01515202001727141556	0.998088657774696332632	1.02338267442456988958
	1-10^{-100}	1.01603394165596225394	0.996201924433832422135	1.01603394165596225394
He:Xe	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.07158326797078869468
	0.1	1.01639357708144408583	1.01492473740289806586	1.06873394450867289307
	0.2	1.01672843374133128104	1.01368207158062152843	1.06554789968428277036
	0.3	1.01702098365247956234	1.01228122347753187156	1.06196054250931617906
	0.4	1.01724632590091958839	1.01069209533799732781	1.05789052786936544082
	0.5	1.01736974570885297972	1.00887886633944532755	1.05323486342579607405
	0.6	1.01734493704325970319	1.00680164772579403206	1.04786336813622424981
	0.7	1.01711728601114110855	1.00442491574089949417	1.04161464803795254707
	0.8	1.01664931897814380161	1.00174876710053285946	1.03430204780042093843
	0.9	1.01603761248639342751	0.998923328835579890353	1.02576010669253960408
	1-10^{-100}	1.01603394165596225394	0.996718146433463758077	1.01603394165596225394
Ne:Ar	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.03305997324087952532
	0.1	1.01625590361214093518	1.01475812317239841618	1.03171180980538410381
	0.2	1.01646451044346802436	1.0134438066026345691	1.03030138092865734756
	0.3	1.01665067349896640643	1.01208909970146827722	1.02882343516367989725
	0.4	1.01680332585790530321	1.01069215442363076501	1.02727217561287563806

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Table 3.9 – Continued

	x_1	$[\mu]^\star_{60}$	$b_1^{(60)\star}$	$b_{-1}^{(60)\star}$
	0.5	1.01690903414510012054	1.00925127326839768276	1.02564123102489417856
	0.6	1.01695154099883760832	1.00776508072750981716	1.02392365308246182339
	0.7	1.01691123305354701216	1.00623280108999558652	1.02211196489147566422
	0.8	1.01676453670408282857	1.00465470888707492995	1.02019830514161537200
	0.9	1.01648326079756137771	1.00303286546716734436	1.01817474768446145741
	1-10^{-100}	1.01603394165596225394	1.00137233943072385176	1.01603394165596225394
Ne:Kr	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.04675335943346634153
	0.1	1.01612540646618870920	1.01411947329532110301	1.04451640446092753344
	0.2	1.01626026630626637042	1.01214949805450020490	1.04214244357355706995
	0.3	1.01642925411275517273	1.01012233828276131435	1.03961666789409590470
	0.4	1.01661819150005820727	1.00803714143863930129	1.03692227313408755420
	0.5	1.01680600049140600516	1.00589446989086513147	1.03404027338103467439
	0.6	1.01696189406361729486	1.00369728275624862132	1.03094943390441066323
	0.7	1.01704138561416541021	1.00145260915367030770	1.02762650060263779045
	0.8	1.01698060209827332320	0.999174486510464381038	1.02404710851579557615
	0.9	1.01668819072730193264	0.996889317300041866186	1.02018821018333592048
	1-10^{-100}	1.01603394165596225394	0.994646100832690172302	1.01603394165596225394
Ne:Xe	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.05449688035856223423
	0.1	1.01635395674827701356	1.01416482405356853467	1.05215920549044405983
	0.2	1.01671720527082156164	1.01219752714433310872	1.04960927580426410750
	0.3	1.01711281297026403361	1.01012523218887672084	1.04681507874419394814
	0.4	1.01752148286086606687	1.00794185078225281187	1.04373801831561978000
	0.5	1.01791071672093858455	1.00564324201284225906	1.04033138506620501858
	0.6	1.01822722536348454433	1.00322957901003245895	1.03653865446611572123
	0.7	1.01838472081151503329	1.00071006367515523901	1.03229200801676070796

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Table 3.9 – Continued

	x_1	$[\mu]^\star_{60}$	$b_1^{(60)\star}$	$b_{-1}^{(60)\star}$
	0.8	1.01824406630492819328	0.99811271417617950 8972	1.02751237740986319173
	0.9	1.01758089599427684252	0.99550593367877239 0245	1.02211506471325713734
	1-10^{-100}	1.01603394165596225394	0.99305016234122060 4997	1.01603394165596225394
Ar:Kr	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.03194763456257257356
	0.1	1.01575993966166860749	1.01457149563101548533	1.03034074221598809712
	0.2	1.01554335978031314600	1.01312738157502405840	1.02873889449595346821
	0.3	1.01538475013551864171	1.01170099549278364452	1.02714153222377502377
	0.4	1.01528498982394893176	1.01029172830238013721	1.02554810305552652977
	0.5	1.01524535096051309078	1.00889894121112361566	1.02395804345100521118
	0.6	1.01526758058658559367	1.00752193392693871623	1.02237075470339823785
	0.7	1.01535400909566456429	1.00615990305337883670	1.02078557071084214731
	0.8	1.01550769463157585123	1.00481188706908076713	1.01920171431473931166
	0.9	1.01573261708649119216	1.00347669296599229713	1.01761823782586966815
	1-10^{-100}	1.01603394165596225394	1.00215279775046502261	1.01603394165596225394
Ar:Xe	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.04116338481901273030
	0.1	1.01577516863219155504	1.01412351474229087974	1.03893246856972455155
	0.2	1.01558136721294471638	1.01220942237811891995	1.03664624235719889835
	0.3	1.01545117827261505159	1.01029242064257873278	1.03430170219974377261
	0.4	1.01538268746018511181	1.00837374654928482157	1.03189588350851394077
	0.5	1.01537325187006465361	1.00645522811557972980	1.02942592645019163479
	0.6	1.01541927984660133819	1.00453941823365632690	1.02688916311464699540
	0.7	1.01551594556238502883	1.00262975108483850103	1.02428322804411744723
	0.8	1.01565680932678688980	1.00073071149497989684	1.02160618784767811549
	0.9	1.01583329525527487708	0.99884798825529033 0590	1.01885667075489835608
	1-10^{-100}	1.01603394165596225394	0.99698853889687131 7706	1.01603394165596225394

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Table 3.9 – Continued

	x_1	$[\mu]^{\star}_{60}$	$b_1^{(60)\star}$	$b_{-1}^{(60)\star}$
Kr:Xe	10^{-100}	1.01603394165596225394	1.01603394165596225394	1.02674835022770721873
	0.1	1.01604055057380493445	1.01509458068476239591	1.02574971228455756949
	0.2	1.01605093185160637375	1.01414694340762548643	1.02473631354196476978
	0.3	1.01606350866511655414	1.01319085701439224554	1.02370762280798898498
	0.4	1.01607650988788149051	1.01222616907110427076	1.02266310189309286303
	0.5	1.01608794888524984262	1.01125275444132994504	1.02160220980152967364
	0.6	1.01609560012224982802	1.01027052402155875100	1.02052440837830332776
	0.7	1.01609697341268705910	1.00927943577848343835	1.01942916983825879589
	0.8	1.01608928564915532847	1.00827950871904866433	1.01831598673346045032
	0.9	1.01606942988088685058	1.00727084061375568190	1.01718438508745525258
	$1\text{-}10^{-100}$	1.01603394165596225394	1.00625363054678644209	1.01603394165596225394

Table 3.10: Parameters at STP† of the gases considered in this work.

Gas	Molecular weight [42] (amu)	Molecular diameter $\times 10^8$ (cm) [10]
He	4.002602	2.193
Ne	20.1797	2.602
Ar	39.948	3.659
Kr	83.798	4.199
Xe	131.293	4.939

†STP signifies a temperature of 0 °C and a pressure of 1 atm.

As a result of obtaining normalized viscosity values for each order of approximation up to 60, there exists for each combination of parameters considered a sequence of values that is known to be increasing monotonically to a fixed limiting value of the normalized viscosity. As a part of this work, the *Mathematica*® function `SequenceLimit` has been applied to each such sequence of values in order to determine the limiting values associated with the order 60 results. The `SequenceLimit` function extrapolates to the limit using the Wynn-epsilon algorithm. The extrapolated results, which are included in Table 3.11, are truncated at the number of digits that appear to be common to all of the extrapolations done for each point using Wynn orders from 12 to 22 which is the range of Wynn orders that appears to give the most consistent extrapolated values. It appears that the extrapolated values presented in Table 3.11 are likely correct to the precision shown excepting, perhaps, only the last digit and should be considered the best rigid-sphere gas mixture benchmark values available. Even with the last digit considered to be unreliable, the extrapolated values of Table 3.11 are more than double the precision of order 60 results and are, effectively, equivalent to about order 150 results. This approximate equivalency has been determined by examination of the results for simple gases where error free results to order 200 have been previously generated [15].

Table 3.11: Limiting values of the normalized viscosity coefficients, $[\mu]^{\star}_{\infty}$, for binary mixtures of the noble gases He, Ne, Ar, Kr, Xe as functions of the mixture fractions of the lighter constituents obtained by applying the *Mathematica*® function `SequenceLimit` to the sequences of values from each of the first 60 orders of the Sonine polynomial expansion employed.

Gas Mixture	x_1	$[\mu]^{\star}_{\infty}$
He:Ne	10^{-100}	1.01603394165596225395457986383362934 8
	0.1	1.0150798100560748321347 17
	0.2	1.01425878361382491320602 8
	0.3	1.01358127106507026178191 23
	0.4	1.01306346604593788423444 147
	0.5	1.01272954522890231158274 7161
	0.6	1.01261532435082971592663 58546
	0.7	1.01277457235078877023811 18664
	0.8	1.01329052801883150411659 927535
	0.9	1.01429844951434984261706 022
	1-10^{-100}	1.01603394165596225395457986383362934 8
He:Ar	10^{-100}	1.01603394165596225395457986383362934 8
	0.1	1.01587330372317453702
	0.2	1.01572351653535980306
	0.3	1.015585103792673782204
	0.4	1.015459900280845499878 4
	0.5	1.01535234856903286746262 3
	0.6	1.01527198434102866989461 1
	0.7	1.015238234947328656501 52
	0.8	1.01528982741619967315 8
	0.9	1.01550365889632978464227
	1-10^{-100}	1.01603394165596225395457986383362934 8
He:Kr	10^{-100}	1.01603394165596225395457986383362934 8
	0.1	1.016132670599328973
	0.2	1.0161923530441204303
	0.3	1.0162015501473002976
	0.4	1.0161473264770330173 8
	0.5	1.0160169491166040727 29
	0.6	1.015802894247880386
	0.7	1.0155163772025665
	0.8	1.01522390860002452
	0.9	1.01515202001928212
	1-10^{-100}	1.01603394165596225395457986383362934 8
He:Xe	10^{-100}	1.01603394165596225395457986383362934 8
	0.1	1.01639357873148544
	0.2	1.01672843543249824
	0.3	1.017020984822171505
	0.4	1.01724632651722329 5
	0.5	1.01736974594566489 10
	0.6	1.017344937096175

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Table 3.11 – Continued

Gas Mixture	x_1	$[\mu]^\star_\infty$
	0.7	1.017117286013916
	0.8	1.016649318985848
	0.9	1.0160376124971495
	1-10^{-100}	1.016033941655962253954579863833629348
Ne:Ar	10^{-100}	1.016033941655962253954579863833629348
	0.1	1.016255903612141000200899434832
	0.2	1.01646451044346809410691073383
	0.3	1.0166506734989664588174651800740
	0.4	1.0168033258579053353927089480029
	0.5	1.0169090341451001372788428611822
	0.6	1.01695154099883761565841905350941
	0.7	1.016911233053547014797343725516189
	0.8	1.016764536704082829300794095403725
	0.9	1.0164832607975613778534729187741107
	1-10^{-100}	1.016033941655962253954579863833629348
Ne:Kr	10^{-100}	1.016033941655962253954579863833629348
	0.1	1.01612540646633012657713000
	0.2	1.01626026630639998764083589
	0.3	1.01642925411284162301126604
	0.4	1.01661819150010258481306746
	0.5	1.016806000491424450961145714
	0.6	1.016961894063623336928447243
	0.7	1.0170413856141668592268535558
	0.8	1.016980602098273539624977052011
	0.9	1.016688190727301945675343058029
	1-10^{-100}	1.016033941655962253954579863833629348
Ne:Xe	10^{-100}	1.016033941655962253954579863833629348
	0.1	1.01635395675300562285571
	0.2	1.016717205275602430599
	0.3	1.017112812973569150954283
	0.4	1.0175214828626720532952501
	0.5	1.017910716721731470814721
	0.6	1.01822722536375476204914565
	0.7	1.0183847208115804505447794
	0.8	1.018244066304937407785749851
	0.9	1.01758089599427726964048704629
	1-10^{-100}	1.016033941655962253954579863833629348
Ar:Kr	10^{-100}	1.016033941655962253954579863833629348
	0.1	1.0157599396616686610769442568237
	0.2	1.015543359780313196546727130606
	0.3	1.0153847501355186755488035433106
	0.4	1.01528498982394895063080996630058
	0.5	1.01524535096051309989459744693215

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Table 3.11 – Continued

Gas Mixture	x_1	$[\mu]^{\star}_{\infty}$
	0.6	1.0152675805865855974971517613673 99
	0.7	1.0153540090956645656740681358028 69
	0.8	1.0155076946315758516375118924852254
	0.9	1.0157326170864911922593341412913891
	1-10^{-100}	1.0160339416559622539545798638336293 48
Ar:Xe	10^{-100}	1.0160339416559622539545798638336293 48
	0.1	1.0157751686322018215488230 47
	0.2	1.0155813672129535907181893 82
	0.3	1.0154511782726203583166875551
	0.4	1.0153826874601876660550510798
	0.5	1.0153732518700656700223623825 1
	0.6	1.0154192798466016671026641045 4
	0.7	1.0155159455623851108306230444 95
	0.8	1.0156568093267869037924575293 7143
	0.9	1.0158332952552748782990149427 97563
	1-10^{-100}	1.0160339416559622539545798638336293 48
Kr:Xe	10^{-100}	1.0160339416559622539545798638336293 48
	0.1	1.0160405505738049361336061161115 38
	0.2	1.0160509318516063756903228591461 0
	0.3	1.0160635086651165557567789582855 38
	0.4	1.0160765098878814916526207997219 14
	0.5	1.0160879488852498433390948212038049
	0.6	1.0160956001222498284307998029109126
	0.7	1.0160969734126870593169696051510 87
	0.8	1.0160892856491553285677404386327 4236
	0.9	1.0160694298808868506246411745787 26821
	1-10^{-100}	1.0160339416559622539545798638336293 48

3.8 Discussion and conclusions

The purpose in this series of papers has been to explore the use of Sonine polynomial expansions to obtain error free results for the transport coefficients and related Chapman-Enskog functions for simple gases and gas mixtures. In Chapter 2 [15], the case of simple gases was explored, and here this initial work has been extended to include an exploration of viscosity and the related Chapman-Enskog viscosity solutions for selected real gas mixtures. For specific results, the focus in this work has been on rigid-sphere molecules, as the omega integrals are readily available in a simple analytical form. However, it should be emphasized that the expressions obtained are completely general and results for any potential model can now readily be obtained subject only to omega integral values for the given potential model being available to sufficient precision.

Sonine polynomial expansions up to order 60 have been considered in this chapter but the computational tools and programs that have been constructed apply to expansions of arbitrary order and the only limiting factors have been the speed and memory capacity of the computational resources available. Nevertheless, because of the high precision of the current computations, excellent convergence has been obtained in all of the results and, via extrapolation, viscosity values to a precision of more than 15 digits have been obtained for all of the rigid-sphere gas mixtures considered.

Since even in a binary-gas mixture the parameter space is very large (involving mole fractions, mass ratios, size ratios, temperature, etc.), it is always useful to have techniques or expressions that make rapid and precise computations possible. It is because the development of a very general set of such computational tools and techniques has been emphasized in this work, comparisons with the results of Takata et al. [2] and Garcia and Siewert [1] have been relatively straightforward and quick to obtain. It is this same generality that has allowed the extension of this work in a straightforward manner to additional real gas mixtures and which will allow for a much more substantial parametric study in the near future as well as for additional study of realistic potential models. The attractiveness of this work is obvious with respect to ternary and multiple gas mixtures as no new basic expressions need to be computed and the computations would be straightforward. Finally, one should note that the process of extending this work to studies of thermal conductivity and diffusion is analogous to the content of the present chapter and such results are presented in the next chapter.

Chapter 4

Diffusion, thermal diffusion, and thermal conductivity in a binary, rigid-sphere, gas mixture

4.1 Introduction

The Chapman-Enskog solutions of the Boltzmann equations provide a basis for the computation of important transport coefficients for both simple gases and gas mixtures [10–12, 20–26, 28–31, 39]. The use of Sonine polynomial expansions for the Chapman-Enskog solutions was first suggested by Burnett [13] and has become the general method for obtaining the transport coefficients due to the relatively rapid convergence of this series [10–13, 20–24]. While it has been found that relatively, low-order expansions (of order 4) can provide reasonable accuracy in computations of the transport coefficients (to about 1 part in 1,000), the adequacy of the low-order expansions for computation of the slip and jump coefficients associated with gas-surface interfaces still needs to be explored. Also of importance is the fact that such low-order expansions do not provide good convergence (in velocity space) for the actual Chapman-Enskog solutions even though the transport coefficients derived from these solutions appear to be reasonable. Thus, it is of some interest to explore Sonine polynomial expansions to higher orders. In the preceding chapters it has been shown for simple, rigid-sphere gases (i.e. single-component, monatomic gases) that, indeed, the use of higher-order Sonine polynomial expansions enables one to obtain results of arbitrary precision that are error free (Chapter 2 [15]) and that this work can be readily extended to modeling the viscosity in a binary, rigid-sphere, gas mixture (Chapter 3 [16]). For viscosity an extensive set of order 60 results has been reported which are believed to constitute the best currently available benchmark viscosity values for binary, rigid-sphere, gas mixtures. It is the purpose in this chapter to similarly report the results of an investigation of relatively high-order, standard, Sonine polynomial expansions for the diffusion- and thermal conductivity-related Chapman-Enskog solutions for binary gas mixtures of rigid-sphere molecules. In the following sections, the basic theory, the theoretical elements specific to diffusion, thermal diffusion, and thermal conductivity, the solution technique in terms of the Sonine polynomials, the bracket integrals, details related to the specific case of rigid-sphere molecules, and the results are described.

A part of the motivation with respect to this work has been some of the recently reported results on direct numerical solutions of the linearized Boltzmann equations for rigid-sphere, gas mixtures. In particular, results for the transport coefficients and the Chapman-Enskog solutions have been reported by Takata et al. [2]. The current work provides a benchmark for assessing the precision of some of the numerical results reported by these authors and, indeed, some such comparisons are reported. This work does have an important distinguishing feature in that, for rigid-sphere gas mixtures, no numerical integrations are required and thus, in principle, results of arbitrary precision can be obtained for any given order of the Sonine polynomial expansions. The computational resources available for this work at the present time have permitted expansions only to order 70 given the manner in which this technique has been implemented, but even here it has been possible to obtain results believed to be precise to 14 or more significant digits for each of the normalized gas mixture transport coefficients (depending upon the specific mass ratios, size ratios, and mole fractions considered) and it is certain that further improvements in the implementation of the technique or the availability of better computational resources will allow even higher-order expansions and greater convergence of the results. Further, note that in this work the full dependence of the solutions on the molecular masses, the molecular sizes, the mole fractions, and the intermolecular potential model via the omega integrals has been retained, and that explicit (symbolic) expressions for the necessary matrix elements (the bracket integrals) used in evaluating the coefficients in the Sonine polynomial expansions for the coupled Chapman-Enskog equations have been obtained. These generalized matrix elements, once determined, need not be determined again. For rigid spheres (or for any other potential model of interest that can be represented via the omega integrals), one can then determine in a straightforward manner a set of matrix elements that are specific to the potential model being used and store them. These specific matrix elements require only the input of the appropriately computed omega integrals which, for rigid spheres, are known exactly such that no numerical integrations are needed. In this fashion, this method requires only a matrix inversion at the final step. This is important, as all that is needed for finding both the transport coefficients and the related Chapman-Enskog solutions for arbitrary, binary, rigid-sphere gas mixtures is precomputed in a general form. Thus, one is able to study parametric dependencies and convergence of results in an economical and systematic way as, once the matrix elements up to the highest order are computed and stored, one can process results to any order up to this highest order without any new computations of matrix elements being required. Further, since values for the transport coefficients converge with increasing order, since one can use arbitrarily high numerical precision as needed in *Mathematica*® for the final matrix inversion step, and since one can easily compare results for a given order with the results for immediately preceding orders, one can be confident in the results and the degree of convergence obtained.

4.2 The basic theory

Following the work and notations of Chapman and Cowling [10], an abbreviated version of the relevant theory is presented below. For an arbitrary, rarefied, gas mixture, one begins with the Boltzmann equations describing the molecular distribution functions of the constituent gases:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathbf{c}_i \cdot \boldsymbol{\nabla}_r + \mathbf{F}_i \cdot \boldsymbol{\nabla}_{c_i}\right) f_i\left(\mathbf{r},\, \mathbf{c}_i, t\right) \\ &\quad = \sum_j \iiint \left(f_i' f_j' - f_i f_j\right) g b \mathrm{d}b \mathrm{d}\varepsilon \mathrm{d}\mathbf{c}_j = \sum_j J\left(f_i f_j\right)\ ,\end{aligned} \tag{4.1}$$

in which the left-hand side (LHS) is known as the streaming term of the equation which contains the differential streaming operator in the brackets, the right-hand side (RHS) is a sum over what are known as the collision integrals in which $J(f_i f_j)$ is called the collision operator, $f_i\left(\mathbf{r},\, \mathbf{c}_i, t\right)$ is the molecular distribution function of the i-th constituent, g is the magnitude of the pre-collision relative velocity, $\mathbf{g} = \mathbf{c}_j - \mathbf{c}_i$, b is the 'impact parameter' associated with the binary scattering events, ε is an angle corresponding to the azimuthal orientation of the scattering plane, and $\mathbf{c}$ is the molecular velocity. A prime ($'$) indicates a function of a post-collision velocity while the corresponding lack of a prime indicates a pre-collision velocity dependence, e.g. $f_i = f_i\left(\mathbf{r},\, \mathbf{c}_i, t\right)$ while $f_i' = f_i\left(\mathbf{r},\, \mathbf{c}_i', t\right)$. In the summation over the different constituents, scattering between like constituents (i.e. when $i = j$) is treated in the same way as scattering between unlike constituents with the various pre- and post-collision velocities retained as separate variables for purposes of integration. In this circumstance, for clarity, it is common practice to drop the i subscript inside the collision integral in order to facilitate the necessary discrimination between the velocities (i.e. $\mathbf{c}_i \to \mathbf{c}$ and $f_i \to f$). Of course it follows from this that, if one is dealing with a simple gas having only one constituent, one obtains from this process the single Boltzmann equation describing the gas in which $j = 1$ and no subscript is necessary on the LHS:

$$\left(\frac{\partial}{\partial t} + \mathbf{c} \cdot \boldsymbol{\nabla}_r + \mathbf{F} \cdot \boldsymbol{\nabla}_c \right) f(\mathbf{r}, \mathbf{c}, t) = \iiint (f' f'_1 - f f_1)\, g b \mathrm{d} b \mathrm{d} \varepsilon \mathrm{d} \mathbf{c}_1 \, . \tag{4.2}$$

Equivalent expressions for the above equations are often encountered in which $b\mathrm{d}b\mathrm{d}\varepsilon$ is expressed as $\alpha_{ij}(g,\chi)\mathrm{d}\mathbf{e}'$ or $\sigma_{ij}(g,\chi)\mathrm{d}\boldsymbol{\Omega}$ where χ is the scattering angle (the angle between $\mathbf{g}$ and $\mathbf{g}'$) and $\alpha_{ij}(g,\chi) = \sigma_{ij}(g,\chi)$ is known as the differential collision cross-section which describes the probability per unit time per unit volume that two molecules colliding with velocities, $\mathbf{c}_i$ in $\mathrm{d}\mathbf{c}_i$ and $\mathbf{c}_j$ in $\mathrm{d}\mathbf{c}_j$, will have a relative velocity after collision, $\mathbf{g}' = \mathbf{c}'_j - \mathbf{c}'_i$, that lies within the solid angle, $\mathrm{d}\mathbf{e}' = \mathrm{d}\boldsymbol{\Omega} = \sin(\chi)\,\mathrm{d}\chi\mathrm{d}\varepsilon$.

For the specific case of a binary gas mixture, one expresses the distribution functions f_1 and f_2 in the form:

$$f_1 = f_1^{(0)} + f_1^{(1)} + f_1^{(2)} + \cdots \, , \tag{4.3}$$

$$f_2 = f_2^{(0)} + f_2^{(1)} + f_2^{(2)} + \cdots \, , \tag{4.4}$$

where the lowest-order approximations are chosen to be:

$$f_1^{(0)} = n_1 \left(\frac{m_1}{2\pi kT} \right)^{3/2} \exp\left(-\frac{m_1}{2kT} (\mathbf{c}_1 - \mathbf{c}_0)^2 \right) \, , \tag{4.5}$$

$$f_2^{(0)} = n_2 \left(\frac{m_2}{2\pi kT} \right)^{3/2} \exp\left(-\frac{m_2}{2kT} (\mathbf{c}_2 - \mathbf{c}_0)^2 \right) \, , \tag{4.6}$$

in which m_1 and m_2 are the molecular masses of the constituent gases, k is Boltzmann's constant, and n_1, n_2, $\mathbf{c}_0$ and T are, in general, arbitrary functions of $\mathbf{r}$ and t. Note that, in choosing the lowest-order approximations to be of this form (which correspond to Maxwellian distributions), one has effectively equated the functions n_1 and n_2 to the number densities of the two gases in the mixture, T to the temperature of the mixture, and $\mathbf{c}_0$ to the mass-velocity of the mixture where, $\mathbf{c}_0 = M_1 x_1 \mathbf{c}_1 + M_2 x_2 \mathbf{c}_2$ in which, $M_i = m_i/m_0$, $m_0 = m_1 + m_2$, $x_i \equiv n_i/n$ denote the proportion by number of the constituent gases in the mixture (the mole fractions), and $n = n_1 + n_2$ is the total molecular number density of the binary mixture.

If one now limits further consideration only to the second approximation (up to $f^{(1)}$), that is equivalent to assuming that the distribution functions for each of the constituents can be expressed as small linear perturbations from equilibrium states specified by the Maxwellian distributions of Eqs. (4.5) and (4.6), i.e. $f_i = f_i^{(0)}(1+\varPhi_i^{(1)})$. Thus, $f_1^{(1)}$ and $f_2^{(1)}$ are written in the form:

$$f_1^{(1)} = f_1^{(0)}\varPhi_1^{(1)} \ , \tag{4.7}$$

$$f_2^{(1)} = f_2^{(0)}\varPhi_2^{(1)} \ , \tag{4.8}$$

where the perturbations, $\varPhi_1^{(1)}$ and $\varPhi_2^{(1)}$, satisfy the time derivative expressions given by:

$$\mathcal{D}_1^{(1)} = -n_1^2 I_1\left(\varPhi_1^{(1)}\right) - n_1 n_2 I_{12}\left(\varPhi_1^{(1)} + \varPhi_2^{(1)}\right) \ , \tag{4.9}$$

$$\mathcal{D}_2^{(1)} = -n_2^2 I_2\left(\varPhi_2^{(1)}\right) - n_1 n_2 I_{21}\left(\varPhi_1^{(1)} + \varPhi_2^{(1)}\right) \ , \tag{4.10}$$

in which:

$$\begin{aligned}\mathcal{D}_i^{(r)} = \frac{\partial_{r-1} f_i^{(0)}}{\partial t} + \frac{\partial_{r-2} f_i^{(1)}}{\partial t} + \cdots + \frac{\partial_0 f_i^{(r-1)}}{\partial t} \\ + \left(\mathbf{c}_i \cdot \frac{\partial}{\partial \mathbf{r}} + \mathbf{F_i} \cdot \frac{\partial}{\partial \mathbf{c_i}}\right) f_i^{(r-1)} \ ,\end{aligned} \tag{4.11}$$

and:

$$n_i^2 I_i\left(F\right) = \iint f_i^{(0)} f^{(0)}\left(F_i + F - F_i' - F'\right) g\alpha_i \mathrm{d}\mathbf{e}'\mathrm{d}\mathbf{c} \ , \tag{4.12}$$

$$n_i n_j I_{ij}\left(K\right) = \iint f_i^{(0)} f_j^{(0)}\left(K - K'\right) g\alpha_{ij} \mathrm{d}\mathbf{e}'\mathrm{d}\mathbf{c_j} \ , \tag{4.13}$$

are linear functions of their arguments such that $I\left(\phi+\psi\right) = I\left(\phi\right) + I\left(\psi\right)$ and $I\left(a\phi\right) = aI\left(\phi\right)$ regardless of subscripts (where a is any arbitrary constant). The LHSs of Eqs. (4.9) and (4.10) can be expressed in the form:

$$\begin{aligned}\mathcal{D}_1^{(1)} = f_1^{(0)}\Big\{ \left(\mathscr{C}_1^2 - \tfrac{5}{2}\right)\mathbf{C}_1 \cdot \boldsymbol{\nabla}\ln\left(T\right) + x_1^{-1}\mathbf{d}_{12}\cdot\mathbf{C}_1 \\ + 2\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1} : \boldsymbol{\nabla}\mathbf{c}_0\Big\} \ ,\end{aligned} \tag{4.14}$$

$$\begin{aligned}\mathcal{D}_2^{(1)} = f_2^{(0)}\Big\{ \left(\mathscr{C}_2^2 - \tfrac{5}{2}\right)\mathbf{C}_2 \cdot \boldsymbol{\nabla}\ln\left(T\right) + x_2^{-1}\mathbf{d}_{21}\cdot\mathbf{C}_2 \\ + 2\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2} : \boldsymbol{\nabla}\mathbf{c}_0\Big\} \ ,\end{aligned} \tag{4.15}$$

in which $\boldsymbol{\mathscr{C}}_i \equiv (m_i/2kT)^{1/2}\,\mathbf{C}_i$, $\mathbf{C}_i \equiv \mathbf{c}_i - \mathbf{c}_0$, and the bold sans serif notation, w, denotes a dyadic tensor, $\mathsf{w} = \mathbf{ab}$, constructed from the components of the vectors, $\mathbf{a}$ and $\mathbf{b}$. Note that $\mathbf{d}_{12}$ is given as either of the two forms noted below:

$$\mathbf{d}_{12} = \frac{\rho_1\rho_2}{\rho p}\left\{\mathbf{F}_2 - \frac{1}{\rho_2}\boldsymbol{\nabla}p_2 - \left(\mathbf{F}_1 - \frac{1}{\rho_1}\boldsymbol{\nabla}p_1\right)\right\} \ , \tag{4.16}$$

$$\mathbf{d}_{12} = \boldsymbol{\nabla}x_1 + \frac{n_1n_2\left(m_2 - m_1\right)}{n\rho}\boldsymbol{\nabla}\ln\left(p\right) - \frac{\rho_1\rho_2}{\rho p}\left(\mathbf{F}_1 - \mathbf{F}_2\right) \ , \tag{4.17}$$

where ρ_1 and ρ_2 are the mass densities of the constituent gases, $\rho = \rho_1 + \rho_2$ is the total mass density of the mixture, p_1 and p_2 are the partial pressures of the constituent gases, and $p = p_1 + p_2$ is the total pressure of the mixture. Since $\boldsymbol{\nabla}x_2 = -\boldsymbol{\nabla}x_1$, either Eq. (4.16) or Eq. (4.17) can be used to show that $\mathbf{d}_{21} = -\mathbf{d}_{12}$. The functions $\mathit{\Phi}_1^{(1)}$ and $\mathit{\Phi}_2^{(1)}$ can then be expressed as:

$$\mathit{\Phi}_1^{(1)} = -\mathbf{A}_1\cdot\frac{\partial\ln\left(T\right)}{\partial\mathbf{r}} - \mathbf{D}_1\cdot\mathbf{d}_{12} - 2\mathsf{B}_1 : \frac{\partial}{\partial\mathbf{r}}\mathbf{c}_0 \ , \tag{4.18}$$

$$\mathit{\Phi}_2^{(1)} = -\mathbf{A}_2\cdot\frac{\partial\ln\left(T\right)}{\partial\mathbf{r}} - \mathbf{D}_2\cdot\mathbf{d}_{12} - 2\mathsf{B}_2 : \frac{\partial}{\partial\mathbf{r}}\mathbf{c}_0 \ , \tag{4.19}$$

where the functions $\mathbf{A}$ and $\mathbf{D}$ are vectors and the functions B are non-divergent tensors, such that:

$$\mathbf{A} = \mathbf{C}A\left(C\right) \ , \quad \mathbf{D} = \mathbf{C}D\left(C\right) \ , \quad \mathsf{B} = \overset{\circ}{\mathbf{C}\mathbf{C}}\,B\left(C\right) \ , \tag{4.20}$$

with the appropriate subscript 1 or 2 implied throughout each expression, and where $\mathbf{A}$, $\mathbf{D}$, and B, must satisfy the following pairs of equations, respectively:

$$f_1^{(0)}\left(\mathscr{C}_1^2 - \tfrac{5}{2}\right)\mathbf{C}_1 = n_1^2 I_1\left(\mathbf{A}_1\right) + n_1n_2I_{12}\left(\mathbf{A}_1 + \mathbf{A}_2\right) \ , \tag{4.21}$$

$$f_2^{(0)}\left(\mathscr{C}_2^2 - \tfrac{5}{2}\right)\mathbf{C}_2 = n_2^2 I_2\left(\mathbf{A}_2\right) + n_1n_2I_{21}\left(\mathbf{A}_1 + \mathbf{A}_2\right) \ , \tag{4.22}$$

$$x_1^{-1}f_1^{(0)}\mathbf{C}_1 = n_1^2 I_1\left(\mathbf{D}_1\right) + n_1n_2I_{12}\left(\mathbf{D}_1 + \mathbf{D}_2\right) \ , \tag{4.23}$$

$$-x_2^{-1}f_2^{(0)}\mathbf{C}_2 = n_2^2 I_2\left(\mathbf{D}_2\right) + n_1n_2I_{21}\left(\mathbf{D}_1 + \mathbf{D}_2\right) \ , \tag{4.24}$$

$$f_1^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1} = n_1^2 I_1\left(\mathsf{B}_1\right) + n_1n_2I_{12}\left(\mathsf{B}_1 + \mathsf{B}_2\right) \ , \tag{4.25}$$

$$f_2^{(0)}\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2} = n_2^2 I_2\left(\mathsf{B}_2\right) + n_1n_2I_{21}\left(\mathsf{B}_1 + \mathsf{B}_2\right) \ . \tag{4.26}$$

Note that the forms for the distribution functions $\Phi_1^{(1)}$ and $\Phi_2^{(1)}$ have been chosen such that $\mathbf{A}$ and $\mathbf{D}$ must also satisfy the relationships:

$$\int f_1^{(0)} m_1 \mathbf{C}_1 \cdot \mathbf{A}_1 \mathrm{d}\mathbf{c}_1 + \int f_2^{(0)} m_2 \mathbf{C}_2 \cdot \mathbf{A}_2 \mathrm{d}\mathbf{c}_2 = 0 \,, \tag{4.27}$$

$$\int f_1^{(0)} m_1 \mathbf{C}_1 \cdot \mathbf{D}_1 \mathrm{d}\mathbf{c}_1 + \int f_2^{(0)} m_2 \mathbf{C}_2 \cdot \mathbf{D}_2 \mathrm{d}\mathbf{c}_2 = 0 \,, \tag{4.28}$$

and the second-order Chapman-Enskog approximations yield:

$$\begin{aligned} f_1^{(1)} = f_1^{(0)} \Big\{ 1 - A_1 \left(C_1 \right) \mathbf{C}_1 \cdot \boldsymbol{\nabla} \ln \left(T \right) - D_1 \left(C_1 \right) \mathbf{C}_1 \cdot \mathbf{d}_{12} \\ - 2 B_1 \left(C_1 \right) \mathbf{C}_1 \overset{\circ}{} \mathbf{C}_1 : \boldsymbol{\nabla} \mathbf{c}_0 \Big\} \,, \end{aligned} \tag{4.29}$$

and:

$$\begin{aligned} f_2^{(1)} = f_2^{(0)} \Big\{ 1 - A_2 \left(C_2 \right) \mathbf{C}_2 \cdot \boldsymbol{\nabla} \ln \left(T \right) - D_2 \left(C_2 \right) \mathbf{C}_2 \cdot \mathbf{d}_{12} \\ - 2 B_2 \left(C_2 \right) \mathbf{C}_2 \overset{\circ}{} \mathbf{C}_2 : \boldsymbol{\nabla} \mathbf{c}_0 \Big\} \,. \end{aligned} \tag{4.30}$$

Equations (4.29) and (4.30) then allow one to verify that the mean kinetic energies of the peculiar motions of the molecules of each constituent gas are the same up to this level of approximation.

From Eqs. (4.21)-(4.26), one may then construct the following general expressions:

$$\begin{aligned} n^2 \left\{ \mathbf{A}, \mathbf{a} \right\} = \int f_1^{(0)} \left(\mathscr{C}_1^2 - \tfrac{5}{2} \right) \mathbf{C}_1 \cdot \mathbf{a}_1 \mathrm{d}\mathbf{c}_1 \\ + \int f_2^{(0)} \left(\mathscr{C}_2^2 - \tfrac{5}{2} \right) \mathbf{C}_2 \cdot \mathbf{a}_2 \mathrm{d}\mathbf{c}_2 \,, \end{aligned} \tag{4.31}$$

$$n^2 \left\{ \mathbf{D}, \mathbf{a} \right\} = x_1^{-1} \int f_1^{(0)} \mathbf{C}_1 \cdot \mathbf{a}_1 \mathrm{d}\mathbf{c}_1 - x_2^{-1} \int f_2^{(0)} \mathbf{C}_2 \cdot \mathbf{a}_2 \mathrm{d}\mathbf{c}_2 \,, \tag{4.32}$$

$$n^2 \left\{ \mathsf{B}, \mathsf{b} \right\} = \int f_1^{(0)} \boldsymbol{\mathscr{C}}_1 \overset{\circ}{} \boldsymbol{\mathscr{C}}_1 : \mathsf{b}_1 \mathrm{d}\mathbf{c}_1 + \int f_2^{(0)} \boldsymbol{\mathscr{C}}_2 \overset{\circ}{} \boldsymbol{\mathscr{C}}_2 : \mathsf{b}_2 \mathrm{d}\mathbf{c}_2 \,, \tag{4.33}$$

where $\mathbf{a}$ is any vector-function defined in both velocity domains, b is any tensor function defined in both velocity domains, and $\{F, G\}$ are known as the bracket integrals which are defined as:

$$n^2 \left\{ F, G \right\} \equiv n_1^2 \left[F, G \right]_1 + n_1 n_2 \left[F_1 + F_2, G_1 + G_2 \right]_{12} + n_2^2 \left[F, G \right]_2 \,, \tag{4.34}$$

where:

$$[F,G]_1 \equiv \int G_1 I_1(F)\, d\mathbf{c}_1 \ , \tag{4.35}$$

$$[F,G]_2 \equiv \int G_2 I_2(F)\, d\mathbf{c}_2 \ , \tag{4.36}$$

and:

$$[F_1+G_2, H_1+K_2]_{12} \equiv \int F_1 I_{12}(H_1+K_2)\, d\mathbf{c}_1 + \int G_2 I_{21}(H_1+K_2)\, d\mathbf{c}_2 \ . \tag{4.37}$$

Here, due to symmetry and linearity, one has that $[F,G]_1 = [G,F]_1$, $[F,G]_2 = [G,F]_2$, and $[F_1+G_2, H_1+K_2]_{12} = [H_1+K_2, F_1+G_2]_{12}$ such that $\{F,G\} = \{G,F\}$, $\{F, G+H\} = \{F,G\} + \{F,H\}$, and $\{F, aG\} = a\{F,G\}$ (where a is any arbitrary constant).

4.3 The theory for diffusion and thermal diffusion

In a gas mixture, diffusion occurs when the constituents of the mixture have different mean velocities such that $\overline{\mathbf{c}}_1 - \overline{\mathbf{c}}_2 = \overline{\mathbf{C}}_1 - \overline{\mathbf{C}}_2 \neq 0$. This implies that:

$$\overline{\mathbf{C}}_1 - \overline{\mathbf{C}}_2 = n_1^{-1} \int f_1 \mathbf{C}_1 d\mathbf{c}_1 - n_2^{-1} \int f_2 \mathbf{C}_2 d\mathbf{c}_2 \neq 0 \ . \tag{4.38}$$

Now, in terms of the second approximation distribution functions of Eqs. (4.29) and (4.30):

$$\begin{aligned}
&\overline{\mathbf{C}}_1 - \overline{\mathbf{C}}_2 \\
&= -\tfrac{1}{3}\Bigg[\left\{n_1^{-1}\int f_1^{(0)} C_1^2 D_1(C_1) d\mathbf{c}_1 - n_2^{-1}\int f_2^{(0)} C_2^2 D_2(C_2) d\mathbf{c}_2\right\}\mathbf{d}_{12} \\
&+ \left\{n_1^{-1}\int f_1^{(0)} C_1^2 A_1(C_1) d\mathbf{c}_1 - n_2^{-1}\int f_2^{(0)} C_2^2 A_2(C_2) d\mathbf{c}_2\right\}\boldsymbol{\nabla}\ln(T)\Bigg] \\
&= -\tfrac{1}{3}\Bigg[\left\{n_1^{-1}\int f_1^{(0)} \mathbf{C}_1\cdot\mathbf{D}_1 d\mathbf{c}_1 - n_2^{-1}\int f_2^{(0)} \mathbf{C}_2\cdot\mathbf{D}_2 d\mathbf{c}_2\right\}\mathbf{d}_{12} \\
&+ \left\{n_1^{-1}\int f_1^{(0)} \mathbf{C}_1\cdot\mathbf{A}_1 d\mathbf{c}_1 - n_2^{-1}\int f_2^{(0)} \mathbf{C}_2\cdot\mathbf{A}_2 d\mathbf{c}_2\right\}\boldsymbol{\nabla}\ln(T)\Bigg] \\
&= \tfrac{1}{3} n \left[\{\mathbf{D},\mathbf{D}\}\,\mathbf{d}_{12} + \{\mathbf{D},\mathbf{A}\}\,\boldsymbol{\nabla}\ln(T)\right] \ .
\end{aligned} \tag{4.39}$$

The definition of the diffusion coefficient is obtained from Eq. (4.39) based on the case when the gas mixture is uniform in temperature and pressure, and when no external forces are acting on the molecules. Under these conditions, $\mathbf{d}_{12} = \boldsymbol{\nabla} x_1 = n^{-1}\boldsymbol{\nabla} n_1$ and $\boldsymbol{\nabla}\ln(T) = 0$ such that:

$$\overline{\mathbf{c}}_1 - \overline{\mathbf{c}}_2 = \overline{\mathbf{C}}_1 - \overline{\mathbf{C}}_2 = \tfrac{1}{3}\left\{\mathbf{D},\mathbf{D}\right\}\boldsymbol{\nabla} n_1 \ . \tag{4.40}$$

This may then be expressed as:

$$\overline{\mathbf{c}}_1 - \overline{\mathbf{c}}_2 = -D_{12}\left(n_1^{-1}\boldsymbol{\nabla} n_1 - n_2^{-1}\boldsymbol{\nabla} n_2\right) = -D_{12}\frac{n}{n_1 n_2}\boldsymbol{\nabla} n_1 \ , \tag{4.41}$$

in which the constant of proportionality, D_{12}, is the diffusion coefficient. In this manner, it follows that the diffusion coefficient must be defined as:

$$D_{12} \equiv \frac{n_1 n_2}{3n}\left\{\mathbf{D},\mathbf{D}\right\} \ . \tag{4.42}$$

If there is a temperature gradient present, then the second term in Eq. (4.39) also contributes in a similar manner allowing the definition of a thermal diffusion coefficient:

$$D_T \equiv \frac{n_1 n_2}{3n}\left\{\mathbf{D},\mathbf{A}\right\} \ , \tag{4.43}$$

and one then has:

$$\mathrm{k}_T \equiv D_T \big/ D_{12} = \left\{\mathbf{D},\mathbf{A}\right\} \big/ \left\{\mathbf{D},\mathbf{D}\right\} \ , \tag{4.44}$$

which is known as the thermal diffusion ratio and which allows Eq. (4.39) to be expressed as:

$$\begin{aligned}\overline{\mathbf{C}}_1 - \overline{\mathbf{C}}_2 &= -\frac{n^2}{n_1 n_2}\left\{D_{12}\mathbf{d}_{12} + D_T\boldsymbol{\nabla}\ln(T)\right\} \\ &= -\frac{n^2}{n_1 n_2}D_{12}\left\{\mathbf{d}_{12} + \mathrm{k}_T\boldsymbol{\nabla}\ln(T)\right\} \ .\end{aligned} \tag{4.45}$$

4.4 The theory for for thermal conductivity

Assuming that the molecules in the mixture have only kinetic energy of translation associated with them, i.e. that there are no internal degrees of freedom which can participate in exchanges of energy during collisions, the thermal flux may be expressed as:

$$\mathbf{q} = \int f_1 \tfrac{1}{2} m_1 C_1^2 \mathbf{C}_1 \mathrm{d}\mathbf{c}_1 + \int f_2 \tfrac{1}{2} m_2 C_2^2 \mathbf{C}_2 \mathrm{d}\mathbf{c}_2 \ , \tag{4.46}$$

such that:

$$\begin{aligned}
\mathbf{q}/kT - &\tfrac{5}{2}\left(n_1\overline{\mathbf{C}}_1 + n_2\overline{\mathbf{C}}_2\right) \\
&= \int f_1 \left(\mathscr{C}_1^2 - \tfrac{5}{2}\right) \mathbf{C}_1 \mathrm{d}\mathbf{c}_1 + \int f_2 \left(\mathscr{C}_2^2 - \tfrac{5}{2}\right) \mathbf{C}_2 \mathrm{d}\mathbf{c}_2 \\
&= -\tfrac{1}{3} \int f_1^{(0)} \left(\mathscr{C}_1^2 - \tfrac{5}{2}\right) \left\{\left(\mathbf{C}_1 \cdot \mathbf{D}_1\right) \mathbf{d}_{12} + (\mathbf{C}_1 \cdot \mathbf{A}_1) \boldsymbol{\nabla} \ln(T)\right\} \mathrm{d}\mathbf{c}_1 \\
&\quad -\tfrac{1}{3} \int f_2^{(0)} \left(\mathscr{C}_2^2 - \tfrac{5}{2}\right) \left\{\left(\mathbf{C}_2 \cdot \mathbf{D}_2\right) \mathbf{d}_{12} + (\mathbf{C}_2 \cdot \mathbf{A}_2) \boldsymbol{\nabla} \ln(T)\right\} \mathrm{d}\mathbf{c}_2 \\
&= -\tfrac{1}{3} n^2 \left[\{\mathbf{A}, \mathbf{D}\}\, \mathbf{d}_{12} + \{\mathbf{A}, \mathbf{A}\}\, \boldsymbol{\nabla} \ln(T)\right] \ .
\end{aligned} \tag{4.47}$$

Rearranging Eq. (4.47), one has:

$$\begin{aligned}
\mathbf{q} =& \tfrac{5}{2} kT \left(n_1\overline{\mathbf{C}}_1 + n_2\overline{\mathbf{C}}_2\right) - \tfrac{1}{3} k n^2 T \left[\{\mathbf{A}, \mathbf{D}\}\, \mathbf{d}_{12}\right. \\
&\left. + \{\mathbf{A}, \mathbf{A}\}\, \boldsymbol{\nabla} \ln(T)\right] \ ,
\end{aligned} \tag{4.48}$$

which can be expressed as:

$$\begin{aligned}
\mathbf{q} =& \tfrac{5}{2} kT \left(n_1\overline{\mathbf{C}}_1 + n_2\overline{\mathbf{C}}_2\right) + knT \left(\overline{\mathbf{C}}_1 - \overline{\mathbf{C}}_2\right) \left(\{\mathbf{A}, \mathbf{D}\} \,/\, \{\mathbf{D}, \mathbf{D}\}\right) \\
& - \lambda \boldsymbol{\nabla} T \\
=& - \lambda \boldsymbol{\nabla} T + \tfrac{5}{2} kT \left(n_1\overline{\mathbf{C}}_1 + n_2\overline{\mathbf{C}}_2\right) + knT \mathrm{k}_T \left(\overline{\mathbf{C}}_1 - \overline{\mathbf{C}}_2\right) \ ,
\end{aligned} \tag{4.49}$$

in which:

$$\begin{aligned}
\lambda &\equiv \tfrac{1}{3} k n^2 \left[\{\mathbf{A}, \mathbf{A}\} - \{\mathbf{A}, \mathbf{D}\}^2 / \{\mathbf{D}, \mathbf{D}\}\right] \\
&= \tfrac{1}{3} k n^2 \{\tilde{\mathbf{A}}, \tilde{\mathbf{A}}\} \ ,
\end{aligned} \tag{4.50}$$

where:

$$\begin{aligned}
\{\tilde{\mathbf{A}}, \tilde{\mathbf{A}}\} &= \{\mathbf{A}, \mathbf{A}\} - \{\mathbf{A}, \mathbf{D}\}^2 / \{\mathbf{D}, \mathbf{D}\} \\
&= \{\mathbf{A}, \mathbf{A}\} - 2\mathrm{k}_T \{\mathbf{A}, \mathbf{D}\} + \mathrm{k}_T^2 \{\mathbf{D}, \mathbf{D}\} \ ,
\end{aligned} \tag{4.51}$$

such that:

$$\tilde{\mathbf{A}}_1 \equiv \mathbf{A}_1 - \left(\{\mathbf{A}, \mathbf{D}\} / \{\mathbf{D}, \mathbf{D}\}\right) \mathbf{D}_1 = \mathbf{A}_1 - \mathrm{k}_T \mathbf{D}_1 \ , \tag{4.52}$$

and:

$$\tilde{\mathbf{A}}_2 \equiv \mathbf{A}_2 - \left(\{\mathbf{A}, \mathbf{D}\} / \{\mathbf{D}, \mathbf{D}\}\right) \mathbf{D}_2 = \mathbf{A}_2 - \mathrm{k}_T \mathbf{D}_2 \ . \tag{4.53}$$

Now, assuming the absence of any mutual diffusion such that $\overline{\mathbf{C}}_1 = \overline{\mathbf{C}}_2 = 0$, $\mathbf{q} = -\lambda \boldsymbol{\nabla} T$ and λ (which is sometimes used to represent the molecular mean free path) is clearly identical to the thermal conductivity coefficient (which is sometimes also denoted by they symbol κ). The first term in the RHS of Eq. (4.49) is the heat flow due to inequalities of temperature in the mixture, i.e. the steady-state heat flow when temperature gradients are maintained. The second term in the RHS of Eq. (4.49) only occurs because the thermal energy and flow are measured relative to $\mathbf{c}_0$ and represents heat flow carried along by the molecular flux in the presence of diffusion. The third term in the RHS of Eq. (4.49) represents the diffusion thermo-effect which is the analogous inverse process to thermal diffusion.

4.5 Solution in terms of Sonine polynomials

For the Chapman-Enskog coefficients of diffusion, thermal diffusion, and thermal conductivity, it is necessary to evaluate the bracket integrals, $\{\tilde{\mathbf{A}}, \tilde{\mathbf{A}}\}$, $\{\mathbf{A}, \mathbf{D}\}$, and $\{\mathbf{D}, \mathbf{D}\}$. It is assumed that the Chapman-Enskog functions, $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{A}}_2$, $\mathbf{D}_1$, and $\mathbf{D}_2$, may be expanded as:

$$\tilde{\mathbf{A}}_1 = \sum_{\substack{p=-\infty \\ p\neq 0}}^{+\infty} a_p \mathbf{a}_1^{(p)} \ , \quad \tilde{\mathbf{A}}_2 = \sum_{\substack{p=-\infty \\ p\neq 0}}^{+\infty} a_p \mathbf{a}_2^{(p)} \ , \tag{4.54}$$

and:

$$\mathbf{D}_1 = \sum_{p=-\infty}^{+\infty} d_p \mathbf{a}_1^{(p)} \ , \quad \mathbf{D}_2 = \sum_{p=-\infty}^{+\infty} d_p \mathbf{a}_2^{(p)} \ , \tag{4.55}$$

where, following the notations of Chapman and Cowling [10] (Note: In Chapman and Cowling, summations that explicitly omit the 0^{th} term are denoted with primes on the summation symbol), one has:

$$\begin{array}{lll}
\mathbf{a}_1^{(0)} \equiv M_1^{1/2} \rho_2 \boldsymbol{\mathscr{C}}_1 / \rho \ , & \mathbf{a}_2^{(0)} \equiv -M_2^{1/2} \rho_1 \boldsymbol{\mathscr{C}}_2 / \rho \ , & (p = 0) \ , \\
\mathbf{a}_1^{(p)} \equiv S_{3/2}^{(p)}(\mathscr{C}_1^2) \boldsymbol{\mathscr{C}}_1 \ , & \mathbf{a}_1^{(-p)} \equiv 0 \ , & (p > 0) \ , \\
\mathbf{a}_2^{(p)} \equiv 0 \ , & \mathbf{a}_2^{(-p)} \equiv S_{3/2}^{(p)}(\mathscr{C}_2^2) \boldsymbol{\mathscr{C}}_2 \ , & (p > 0) \ ,
\end{array} \tag{4.56}$$

in which $\rho_i = n_i m_i$ are the partial mass densities of the constituent gases, $\rho = \rho_1 + \rho_2$ is the total mass density of the mixture, and where:

$$\begin{aligned} S_m^{(n)}(x) &= \sum_{p=0}^{n} \frac{(m+n)_{n-p}}{(p)!\,(n-p)!}(-x)^p \\ &= \sum_{p=0}^{n} \frac{(m+n)!}{(p)!\,(n-p)!\,(m+p)!}(-x)^p \ , \end{aligned} \tag{4.57}$$

(with $S_m^{(0)}(x) = 1$ and $S_m^{(1)}(x) = m+1-x$) are numerical multiples (un-normalized) of the Sonine polynomials originally used in the kinetic theory of gases by Burnett [13]. Here, as a consequence of Eq. (4.27) which must be satisfied, it follows directly that $\mathbf{a}_1^{(0)} = \mathbf{a}_2^{(0)}$. Also, as it is used in Chapman and Cowling (and in the present work), one notes the following orthogonality property of the Sonine polynomials:

$$\int_0^\infty \exp(-x) S_m^{(p)}(x) S_m^{(q)}(x) x^m \mathrm{d}x = \left\{\Gamma(m+p+1)/p!\right\}\delta_{p,q} \ , \tag{4.58}$$

where $\delta_{p,q}$ is the Kronecker delta and $\Gamma(x)$ is the Gamma function. In Eq. (4.56), note that Chapman and Cowling express the functions $\mathbf{a}_1^{(p)}$ and $\mathbf{a}_2^{(p)}$ only in terms of $p > 0$. This varies in pattern from the manner in which they define the analogous viscosity functions, $\mathsf{b}_1^{(p)}$ and $\mathsf{b}_2^{(p)}$, where both $p > 0$ and $p < 0$ are employed [10,16], and thus the definitions may be somewhat confusing if considered at the same time. Expressed in the same pattern as the corresponding viscosity functions, Eq. (4.56) may also be written as:

$$\begin{array}{lll} \mathbf{a}_1^{(p)} = \mathbf{a}_1^{(0)} \equiv M_1^{1/2}\rho_2\boldsymbol{\mathscr{C}}_1/\rho \ , & \mathbf{a}_2^{(-p)} = \mathbf{a}_2^{(0)} \equiv -M_2^{1/2}\rho_1\boldsymbol{\mathscr{C}}_2/\rho \ , & (p=0) \ , \\ \mathbf{a}_1^{(p)} \equiv 0 \ , & \mathbf{a}_2^{(-p)} \equiv 0 \ , & (p<0) \ , \\ \mathbf{a}_1^{(p)} \equiv S_{3/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1 \ , & \mathbf{a}_2^{(-p)} \equiv S_{3/2}^{(p)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2 \ , & (p>0) \ . \end{array} \tag{4.59}$$

Now, in order to determine the expansion coefficients, a_p, note that from Eq. (4.31) one may write:

$$\{\mathbf{A}, \mathbf{a}^{(q)}\} = \alpha_q \ , \tag{4.60}$$

where:

$$n^2\alpha_q = \int f_1^{(0)}\left(\mathscr{C}_1^2 - \tfrac{5}{2}\right)\mathbf{C}_1 \cdot \mathbf{a}_1^{(q)}\mathrm{d}\mathbf{c}_1 + \int f_2^{(0)}\left(\mathscr{C}_2^2 - \tfrac{5}{2}\right)\mathbf{C}_2 \cdot \mathbf{a}_2^{(q)}\mathrm{d}\mathbf{c}_2 \ . \tag{4.61}$$

Integrating Eq. (4.61), one finds that:

$$\begin{aligned}
\alpha_1 &= -\tfrac{15}{4}\frac{n_1}{n^2}\left(\frac{2kT}{m_1}\right)^{1/2} \ , \\
\alpha_{-1} &= -\tfrac{15}{4}\frac{n_2}{n^2}\left(\frac{2kT}{m_2}\right)^{1/2} \ , \\
\alpha_q &= 0 \ (q \neq \pm 1) \ .
\end{aligned} \tag{4.62}$$

Similarly, in order to determine the expansion coefficients, d_p, note that from Eq. (4.32) one may write:

$$\{\mathbf{D}, \mathbf{a}^{(q)}\} = \delta_q \ , \tag{4.63}$$

where:

$$n^2\delta_q = x_1^{-1}\int f_1^{(0)}\mathbf{C}_1 \cdot \mathbf{a}_1^{(q)} \mathrm{d}\mathbf{c}_1 - x_2^{-1}\int f_2^{(0)}\mathbf{C}_2 \cdot \mathbf{a}_2^{(q)} \mathrm{d}\mathbf{c}_2 \ . \tag{4.64}$$

Integrating Eq. (4.64), one finds that:

$$\delta_0 = \frac{3}{2n}\left(\frac{2kT}{m_0}\right)^{1/2} \ , \quad \delta_q = 0 \ (q \neq 0) \ . \tag{4.65}$$

Combining Eqs. (4.55), (4.56), and (4.63)-(4.65) yields the system of equations:

$$\sum_{p=-\infty}^{+\infty} d_p a_{pq} = \delta_q \quad (q = 0, \pm 1, \pm 2, \cdots, \pm\infty) \ , \tag{4.66}$$

where:

$$a_{pq} = \{\mathbf{a}^{(p)}, \mathbf{a}^{(q)}\} = a_{qp} \ . \tag{4.67}$$

If values for a_{pq} are known, all of the d_p can be determined by solving the algebraic system of equations represented by Eq. (4.66). In the general matrix terminology of Chapman and Cowling, this is represented as:

$$d_p = \delta_0 \lim_{m\to\infty} \left\{\mathscr{A}_{0p}^{(m)}/\mathscr{A}^{(m)}\right\} \ , \tag{4.68}$$

where $\mathscr{A}^{(m)}$ is the determinant of the $(2m+1) \times (2m+1)$ symmetric coefficient matrix of the system of Eq. (4.66) and $\mathscr{A}_{0p}^{(m)}$ is the cofactor of a_{0p} in the expansion of $\mathscr{A}^{(m)}$.

Now, to determine the expansion coefficients, a_p, one substitutes Eq. (4.56) into Eq. (4.54). Since $\tilde{\mathbf{A}} = \mathbf{A} - \mathrm{k}_T\mathbf{D}$, one has that:

$$\{\tilde{\mathbf{A}}, \mathbf{a}^{(q)}\} = \{\mathbf{A}, \mathbf{a}^{(q)}\} - \mathrm{k}_T\{\mathbf{D}, \mathbf{a}^{(q)}\} = \alpha_q - \mathrm{k}_T\delta_q \ , \tag{4.69}$$

which, since $\delta_q = 0$ $(q \neq 0)$, yields:

$$\sum_{\substack{p=-\infty \\ p\neq 0}}^{+\infty} a_p a_{pq} = \alpha_q \quad (q = \pm 1, \pm 2, \cdots, \pm\infty) \ . \tag{4.70}$$

The system of equations represented by Eq. (4.70), together with Eq. (4.67), determine the expansion coefficients, a_p, which can be expressed in general matrix terminology as:

$$a_p = \lim_{m\to\infty} \left(\alpha_1 \mathscr{A}'^{(m)}_{1p} + \alpha_{-1}\mathscr{A}'^{(m)}_{-1p}\right) \Big/ \mathscr{A}'^{(m)} \quad (p \neq 0) \ , \tag{4.71}$$

where $\mathscr{A}'^{(m)}$ is the same as $\mathscr{A}^{(m)}_{00}$ which is the determinant of the $2m\times 2m$ coefficient matrix approximating Eq. (4.70) and $\mathscr{A}'^{(m)}_{qp}$ is the cofactor of a_{qp} in the expansion of $\mathscr{A}'^{(m)}$.

From the preceding development, it can be shown that:

$$\begin{aligned}
&\{\mathbf{D}, \mathbf{D}\} = d_0\delta_0 \ , \\
&\{\mathbf{D}, \mathbf{A}\} = d_1\alpha_1 + d_{-1}\alpha_{-1} \ , \\
&\{\tilde{\mathbf{A}}, \tilde{\mathbf{A}}\} = a_1\alpha_1 + a_{-1}\alpha_{-1} \ .
\end{aligned} \tag{4.72}$$

Thus, using Eq. (4.65) in Eq. (4.68) and Eq. (4.62) in Eq. (4.71), and retaining the general matrix notation of Chapman and Cowling, one has for the transport coefficients:

$$D_{12} = \tfrac{3}{2}\frac{x_1x_2kT}{nm_0} \lim_{m\to\infty} \mathscr{A}'^{(m)} \big/ \mathscr{A}^{(m)} \ , \tag{4.73}$$

$$D_T = -\tfrac{15}{4}\frac{x_1x_2kT}{nm_0} \lim_{m\to\infty} \left\{x_1M_1^{-1/2}\mathscr{A}^{(m)}_{01} + x_2M_2^{-1/2}\mathscr{A}^{(m)}_{0-1}\right\} \Big/ \mathscr{A}^{(m)} \ , \tag{4.74}$$

$$\mathrm{k}_T = -\tfrac{5}{2} \lim_{m\to\infty} \left\{x_1M_1^{-1/2}\mathscr{A}^{(m)}_{01} + x_2M_2^{-1/2}\mathscr{A}^{(m)}_{0-1}\right\} \Big/ \mathscr{A}'^{(m)} \ , \tag{4.75}$$

$$\begin{aligned}
\lambda = \tfrac{75}{8}k^2T \lim_{m\to\infty} \Big\{ & x_1^2 m_1^{-1}\mathscr{A}'^{(m)}_{11} \\
& + 2x_1x_2(m_1m_2)^{-1/2}\mathscr{A}'^{(m)}_{1-1} + x_2^2m_2^{-1}\mathscr{A}'^{(m)}_{-1-1}\Big\} \Big/ \mathscr{A}'^{(m)} \ ,
\end{aligned} \tag{4.76}$$

where D_{12} and λ (although not necessarily D_T) can be shown to form monotonically increasing sequences as m increases which converge to the exact values of the respective coefficients in the limit as $m \to \infty$. For any given order of the approximation, m, Eqs. (4.73)-(4.76) may be expressed in more convenient form as:

$$D_{12} = \tfrac{1}{2} x_1 x_2 \left(2kT/m_0\right)^{1/2} d_0 \ , \tag{4.77}$$

$$D_T = -\tfrac{5}{4} x_1 x_2 \left(2kT/m_0\right)^{1/2} \left(x_1 M_1^{-1/2} d_1 + x_2 M_2^{-1/2} d_{-1}\right) \ , \tag{4.78}$$

$$\mathrm{k}_T = D_T / D_{12} = -\tfrac{5}{2} \left(x_1 M_1^{-1/2} d_1 + x_2 M_2^{-1/2} d_{-1}\right) \Big/ d_0 \ , \tag{4.79}$$

and:

$$\lambda = -\tfrac{5}{4} kn \left(2kT/m_0\right)^{1/2} \left(x_1 M_1^{-1/2} a_1 + x_2 M_2^{-1/2} a_{-1}\right) \ , \tag{4.80}$$

which are the forms that have been used in the current work.

4.6 The bracket integrals

In order to complete the evaluation of the diffusion, thermal diffusion, and thermal conductivity coefficients it is necessary to determine the expansion coefficients, d_{-1}, d_0, d_1, a_{-1}, and a_1. To determine these quantities it is necessary to evaluate the bracket integrals defined in Eq. (4.34) for $\{\mathbf{a}^{(p)}, \mathbf{a}^{(q)}\}$. Hence, it is first necessary to be able to evaluate the square bracket integrals of Eqs. (4.35)-(4.37), specifically, $[\mathbf{a}_1^{(p)}, \mathbf{a}_1^{(q)}]_1$, $[\mathbf{a}_1^{(p)}, \mathbf{a}_1^{(q)}]_{12}$, and $[\mathbf{a}_1^{(p)}, \mathbf{a}_2^{(q)}]_{12}$. Completion of this task requires integration over all of the collision variables and, additionally, requires knowledge of the form of the intermolecular potential. However, the intermolecular potential affects only integrations over the variables, g and b, because they determine the scattering angle, χ. All six of the other integrals (there are eight total in the collision operator) can be evaluated in some manner without specific knowledge of the intermolecular potential.

The specific bracket integrals mentioned above that require evaluation are given explicitly in Chapman and Cowling [10] but how they have been identified is not readily apparent in the text which focuses on very broad generalized expressions that can be difficult to interpret. In practice, the needed bracket integrals are most readily determined by considering Eqs. (4.21) and (4.22) directly. If these equations are expressed as:

$$n_1^2 I_1 (\mathbf{A}_1) + n_1 n_2 I_{12} (\mathbf{A}_1) + n_1 n_2 I_{12} (\mathbf{A}_2) = -f_1^{(0)} \left(\tfrac{5}{2} - \mathscr{C}_1^2\right) \mathbf{C}_1 \ , \tag{4.81}$$

$$n_2^2 I_2 (\mathbf{A}_2) + n_1 n_2 I_{21} (\mathbf{A}_2) + n_1 n_2 I_{21} (\mathbf{A}_1) = -f_2^{(0)} \left(\tfrac{5}{2} - \mathscr{C}_2^2\right) \mathbf{C}_2 \ , \tag{4.82}$$

then one can insert Sonine polynomial approximations for the solutions, $\mathbf{A}_1$ and $\mathbf{A}_2$, can multiply through the equations with additional Sonine polynomials, and can then integrate the equations. By using the orthogonality of the Sonine polynomials as given in Eq. (4.58), this process will yield a set of simultaneous equations involving the matrix coefficients necessary to specify the solutions. The RHSs of the equations determine the constants, α_q, and the LHSs yield combinations of bracket integrals that correspond to the desired a_{pq} elements of the matrix used to determine the a_p expansion coefficients which, in turn, determine the thermal conductivity via Eq. (4.80).

The easiest way to follow this process is via a low-order example. If one considers the Sonine polynomials used in the definition of the $\mathbf{a}_1^{(p)}$ and $\mathbf{a}_2^{(p)}$ Chapman-Enskog expansion vectors, $p = 1$ is the lowest order associated with thermal conductivity with $S_m^{(1)}(x) = m + 1 - x$, such that $\mathbf{a}_1^{(1)} \equiv S_{3/2}^{(1)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1 = \left(\frac{5}{2} - \mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1$ and $\mathbf{a}_2^{(-1)} \equiv S_{3/2}^{(1)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2 = \left(\frac{5}{2} - \mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2$. Using only this order of approximation for $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$, i.e. assuming that $\tilde{\mathbf{A}}_1 = a_1 \mathbf{a}_1^{(1)} = a_1 \left(\frac{5}{2} - \mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1$ and $\tilde{\mathbf{A}}_2 = a_{-1}\mathbf{a}_2^{(-1)} = a_{-1}\left(\frac{5}{2} - \mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2$, and using the fact that $\mathbf{A}_1 = \tilde{\mathbf{A}}_1 + \mathrm{k}_T \mathbf{D}_1$

and $\mathbf{A}_2 = \tilde{\mathbf{A}}_2 + \mathrm{k}_T \mathbf{D}_2$, one may express Eqs. (4.81) and (4.82) as:

$$\begin{aligned}
&n_1^2 I_1 \left(\tilde{\mathbf{A}}_1 + \mathrm{k}_T \mathbf{D}_1\right) + n_1 n_2 I_{12} \left(\tilde{\mathbf{A}}_1 + \mathrm{k}_T \mathbf{D}_1\right) + n_1 n_2 I_{12} \left(\tilde{\mathbf{A}}_2 + \mathrm{k}_T \mathbf{D}_2\right) \\
&= n_1^2 I_1 \left(\tilde{\mathbf{A}}_1\right) + n_1 n_2 I_{12} \left(\tilde{\mathbf{A}}_1\right) + n_1 n_2 I_{12} \left(\tilde{\mathbf{A}}_2\right) \\
&\quad + \mathrm{k}_T \left[n_1^2 I_1 \left(\mathbf{D}_1\right) + n_1 n_2 I_{12} \left(\mathbf{D}_1 + \mathbf{D}_2\right)\right] \\
&= -f_1^{(0)} \left(\tfrac{5}{2} - \mathscr{C}_1^2\right) \mathbf{C}_1 \ ,
\end{aligned} \tag{4.83}$$

and:

$$\begin{aligned}
&n_2^2 I_2 \left(\tilde{\mathbf{A}}_2 + \mathrm{k}_T \mathbf{D}_2\right) + n_1 n_2 I_{21} \left(\tilde{\mathbf{A}}_2 + \mathrm{k}_T \mathbf{D}_2\right) + n_1 n_2 I_{21} \left(\tilde{\mathbf{A}}_1 + \mathrm{k}_T \mathbf{D}_1\right) \\
&= n_2^2 I_2 \left(\tilde{\mathbf{A}}_2\right) + n_1 n_2 I_{21} \left(\tilde{\mathbf{A}}_2\right) + n_1 n_2 I_{21} \left(\tilde{\mathbf{A}}_1\right) \\
&\quad + \mathrm{k}_T \left[n_2^2 I_2 \left(\mathbf{D}_2\right) + n_1 n_2 I_{21} \left(\mathbf{D}_1 + \mathbf{D}_2\right)\right] \\
&= -f_2^{(0)} \left(\tfrac{5}{2} - \mathscr{C}_2^2\right) \mathbf{C}_2 \ ,
\end{aligned} \tag{4.84}$$

which, given Eqs. (4.23) and (4.24), may be written as:

$$\begin{aligned}
&n_1^2 I_1 \left(\tilde{\mathbf{A}}_1\right) + n_1 n_2 I_{12} \left(\tilde{\mathbf{A}}_1\right) + n_1 n_2 I_{12} \left(\tilde{\mathbf{A}}_2\right) \\
&\quad = -f_1^{(0)} \left(\tfrac{5}{2} - \mathscr{C}_1^2\right) \mathbf{C}_1 - \mathrm{k}_T x_1^{-1} f_1^{(0)} \mathbf{C}_1 \\
&\quad = -f_1^{(0)} S_{3/2}^{(1)} \left(\mathscr{C}_1^2\right) \mathbf{C}_1 - \mathrm{k}_T x_1^{-1} f_1^{(0)} S_{3/2}^{(0)} \left(\mathscr{C}_1^2\right) \mathbf{C}_1 \ ,
\end{aligned} \tag{4.85}$$

and:

$$\begin{aligned}
&n_2^2 I_2 \left(\tilde{\mathbf{A}}_2\right) + n_1 n_2 I_{21} \left(\tilde{\mathbf{A}}_2\right) + n_1 n_2 I_{21} \left(\tilde{\mathbf{A}}_1\right) \\
&\quad = -f_2^{(0)} \left(\tfrac{5}{2} - \mathscr{C}_2^2\right) \mathbf{C}_2 + \mathrm{k}_T x_2^{-1} f_2^{(0)} \mathbf{C}_2 \\
&\quad = -f_2^{(0)} S_{3/2}^{(1)} \left(\mathscr{C}_2^2\right) \mathbf{C}_2 + \mathrm{k}_T x_2^{-1} f_2^{(0)} S_{3/2}^{(0)} \left(\mathscr{C}_2^2\right) \mathbf{C}_2 \ .
\end{aligned} \tag{4.86}$$

Since in the current example the discussion is limited to the first-order expansion, Eqs. (4.85) and (4.86) may be written as:

$$\begin{aligned}
&n_1^2 I_1 \left(a_1 S_{3/2}^{(1)} \left(\mathscr{C}_1^2\right) \boldsymbol{\mathscr{C}}_1\right) + n_1 n_2 I_{12} \left(a_1 S_{3/2}^{(1)} \left(\mathscr{C}_1^2\right) \boldsymbol{\mathscr{C}}_1\right) \\
&\qquad + n_1 n_2 I_{12} \left(a_{-1} S_{3/2}^{(1)} \left(\mathscr{C}_2^2\right) \boldsymbol{\mathscr{C}}_2\right) \\
&\quad = -f_1^{(0)} S_{3/2}^{(1)} \left(\mathscr{C}_1^2\right) \mathbf{C}_1 - \mathrm{k}_T x_1^{-1} f_1^{(0)} S_{3/2}^{(0)} \left(\mathscr{C}_1^2\right) \mathbf{C}_1 \ ,
\end{aligned} \tag{4.87}$$

and:

$$\begin{aligned}
&n_2^2 I_2 \left(a_{-1} S_{3/2}^{(1)} \left(\mathscr{C}_2^2\right) \boldsymbol{\mathscr{C}}_2\right) + n_1 n_2 I_{21} \left(a_{-1} S_{3/2}^{(1)} \left(\mathscr{C}_2^2\right) \boldsymbol{\mathscr{C}}_2\right) \\
&\qquad + n_1 n_2 I_{21} \left(a_1 S_{3/2}^{(1)} \left(\mathscr{C}_1^2\right) \boldsymbol{\mathscr{C}}_1\right) \\
&\quad = -f_2^{(0)} S_{3/2}^{(1)} \left(\mathscr{C}_2^2\right) \mathbf{C}_2 + \mathrm{k}_T x_2^{-1} f_2^{(0)} S_{3/2}^{(0)} \left(\mathscr{C}_2^2\right) \mathbf{C}_2 \ .
\end{aligned} \tag{4.88}$$

Now, multiplying through Eq. (4.87) by $S_{3/2}^{(1)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1$ and through Eq. (4.88) by $S_{3/2}^{(1)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2$ and integrating throughout them in the manner of Eqs. (4.12) and (4.13) for I_i and I_{ij} (with division by n^2), one has:

$$\begin{aligned}
&x_1^2 \int S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1 \cdot I_1\left(S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right)\mathrm{d}\mathbf{c}_1 a_1 \\
&\qquad + x_1x_2 \int S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1 \cdot I_{12}\left(S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right)\mathrm{d}\mathbf{c}_1 a_1 \\
&\qquad + x_1x_2 \int S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1 \cdot I_{12}\left(S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right)\mathrm{d}\mathbf{c}_1 a_{-1} \\
&\quad = -n^{-2} \int f_1^{(0)} S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1 \cdot S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\mathbf{C}_1\mathrm{d}\mathbf{c}_1 \\
&\qquad - \mathrm{k}_T x_1^{-1} n^{-2} \int f_1^{(0)} S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1 \cdot S_{3/2}^{(0)}\left(\mathscr{C}_1^2\right)\mathbf{C}_1\mathrm{d}\mathbf{c}_1 \ ,
\end{aligned} \tag{4.89}$$

and:

$$\begin{aligned}
&x_2^2 \int S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2 \cdot I_2\left(S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right)\mathrm{d}\mathbf{c}_2 a_{-1} \\
&\qquad + x_1x_2 \int S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2 \cdot I_{21}\left(S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right)\mathrm{d}\mathbf{c}_2 a_{-1} \\
&\qquad + x_1x_2 \int S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2 \cdot I_{21}\left(S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right)\mathrm{d}\mathbf{c}_2 a_1 \\
&\quad = -n^{-2} \int f_2^{(0)} S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2 \cdot S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\mathbf{C}_2\mathrm{d}\mathbf{c}_2 \\
&\qquad + \mathrm{k}_T x_2^{-1} n^{-2} \int f_2^{(0)} S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2 \cdot S_{3/2}^{(0)}\left(\mathscr{C}_2^2\right)\mathbf{C}_2\mathrm{d}\mathbf{c}_2 \ ,
\end{aligned} \tag{4.90}$$

which, after employing the orthogonality property of Eq. (4.58) to eliminate the terms involving k_T, are expressible in bracket integral notation as:

$$\begin{aligned}
&x_1^2\left[S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right]_1 a_1 \\
&\qquad + x_1x_2\left[S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right]_{12} a_1 \\
&\qquad + x_1x_2\left[S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right]_{12} a_{-1} = \alpha_1 \ ,
\end{aligned} \tag{4.91}$$

and:

$$\begin{aligned}
&x_2^2\left[S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right]_2 a_{-1} \\
&\qquad + x_1x_2\left[S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right]_{21} a_{-1} \\
&\qquad + x_1x_2\left[S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right]_{21} a_1 = \alpha_{-1} \ .
\end{aligned} \tag{4.92}$$

Given the way that the solution has been approached and the low order of this example, it is readily apparent that these equations must be equivalent to:

$$a_{11}a_1 + a_{1-1}a_{-1} = \alpha_1 \ , \tag{4.93}$$

$$a_{-11}a_1 + a_{-1-1}a_{-1} = \alpha_{-1} \ , \tag{4.94}$$

where:

$$\begin{aligned} a_{11} =& x_1^2\big[S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\big]_1 \\ &+ x_1x_2\big[S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\big]_{12} \ , \end{aligned} \tag{4.95}$$

$$a_{1-1} = x_1x_2\big[S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\big]_{12} \ , \tag{4.96}$$

$$a_{-11} = x_1x_2\big[S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(1)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\big]_{21} \ , \tag{4.97}$$

$$\begin{aligned} a_{-1-1} =& x_2^2\big[S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\big]_2 \\ &+ x_1x_2\big[S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(1)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\big]_{21} \ , \end{aligned} \tag{4.98}$$

and the integrals on the RHSs have been replaced with the appropriate α_q constants which, again due to the orthogonality of the Sonine polynomials, are defined only in the current example where $q = \pm 1$ and are otherwise zero. Equations (4.93) and (4.94) can, of course, be rearranged to cast them into the ordered form that corresponds directly to the form that the more general, higher-order problem has been expressed in, i.e.:

$$\begin{bmatrix} a_{-1-1} & a_{-11} \\ a_{1-1} & a_{11} \end{bmatrix} \begin{bmatrix} a_{-1} \\ a_1 \end{bmatrix} = \begin{bmatrix} \alpha_{-1} \\ \alpha_1 \end{bmatrix} \ . \tag{4.99}$$

From this simple example, generalization to higher-order expansions then is straightforward and one has for the relevant matrix elements:

$$\begin{aligned} a_{pq} =& x_1^2\big[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\big]_1 \\ &+ x_1x_2\big[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\big]_{12} \ , \end{aligned} \tag{4.100}$$

$$a_{p-q} = x_1x_2\big[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\big]_{12} \ , \tag{4.101}$$

$$a_{-pq} = x_1x_2\big[S_{3/2}^{(p)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(q)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\big]_{21} \ , \tag{4.102}$$

$$\begin{aligned} a_{-p-q} =& x_2^2\big[S_{3/2}^{(p)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(q)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\big]_2 \\ &+ x_1x_2\big[S_{3/2}^{(p)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(q)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\big]_{21} \ , \end{aligned} \tag{4.103}$$

where, for thermal conductivity, $p, q = 1, 2, 3, \ldots, m$.

One could also analyze a simple case of diffusion in a similar manner. For diffusion, one would obtain the following simple, first-order, matrix equation analogous to Eq. (4.99):

$$\begin{bmatrix} a_{-1-1} & a_{-10} & a_{-11} \\ a_{0-1} & a_{00} & a_{01} \\ a_{1-1} & a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} d_{-1} \\ d_0 \\ d_1 \end{bmatrix} = \begin{bmatrix} \delta_{-1} \\ \delta_0 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \delta_0 \\ 0 \end{bmatrix} . \tag{4.104}$$

where the matrix elements a_{pq} are defined in exactly the same way as in Eqs. (4.100)-(4.103) with the only difference being that, for diffusion, $p, q = 0, 1, 2, 3, \ldots, m$. From the definitions of I_i and I_{ij} in Eqs. (4.12) and (4.13), it follows that Eqs. (4.103) and (4.102) are essentially identical to Eqs. (4.100) and (4.101), respectively, with the only difference being the interchange of the subscripts 1 and 2 representing the different components of the mixture. Thus, in general, the complete Chapman-Enskog solutions for diffusion and thermal conductivity require only the bracket integrals:

$$\left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right]_1 , \tag{4.105}$$

$$\left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right]_{12} , \tag{4.106}$$

$$\left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right]_{12} , \tag{4.107}$$

which are exactly those identified by Chapman and Cowling. Some further simplification of the problem can be obtained by noting from Chapman and Cowling that, in the limit of a simple (single) gas where $m_1 = m_2$, $n_1 = n_2 = n$, and $\alpha_{21} = \alpha_{12} = \alpha_1$, one has that $[F, G]_1 = [F_1, G_1 + G_2]_{12} = ([F_1, G_1]_{12} + [F_1, G_2]_{12})$ which, in the current problem, equates to:

$$\begin{aligned} &\left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right]_1 \\ &= \Bigg(\left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right]_{12} \\ &\qquad + \left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right]_{12} \Bigg)\Bigg|_{\substack{m_1=m_2 \\ n_1=n_2=n \\ \alpha_{12}=\alpha_{21}=\alpha_1}} . \end{aligned} \tag{4.108}$$

At this point, it still remains to perform the six integrations unrelated to the intermolecular potential model that is employed in order to complete the evaluation of the two necessary bracket integrals on the RHS of Eq. (4.108). For the relevant details of this integral evaluation process, one should refer to the text of Chapman and Cowling [10]. Here, note that Chapman and Cowling make use of the following definition of the Sonine polynomials:

$$\begin{aligned}(\mathrm{S}/s)^{m+1} \exp(-x\mathrm{S}) &\equiv (1-s)^{-m-1} \exp(-xs/(1-s)) \\ &= \sum_{n=0}^{\infty} s^n S_m^{(n)}(x) \ ,\end{aligned} \tag{4.109}$$

where $\mathrm{S} = s/(1-s)$ and, likewise, $\mathrm{T} = t/(1-t)$, to express the needed bracket integrals in terms of the coefficients of expansions in the arbitrarily introduced variables, s and t. Thus, it is possible after following Chapman and Cowling to determine that:

$$\begin{aligned}&\left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right]_{12} \\ &= \mathrm{coeff}\left[s^p t^q\right] \times \\ &\qquad \left(\left(\frac{\mathrm{ST}}{st}\right)^{5/2} \pi^{-3} \iiint \left\{H_{12}(0) - H_{12}(\chi)\right\} g b \mathrm{d}b \mathrm{d}\varepsilon \mathrm{d}\boldsymbol{g}\right) \ ,\end{aligned} \tag{4.110}$$

and:

$$\begin{aligned}&\left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right]_{12} \\ &= \mathrm{coeff}\left[s^p t^q\right] \times \\ &\qquad \left(\left(\frac{\mathrm{ST}}{st}\right)^{5/2} \pi^{-3} \iiint \left\{H_1(0) - H_1(\chi)\right\} g b \mathrm{d}b \mathrm{d}\varepsilon \mathrm{d}\boldsymbol{g}\right) \ ,\end{aligned} \tag{4.111}$$

where $\boldsymbol{g} \equiv (m_0 M_1 M_2/2kT)^{1/2}\mathbf{g}_{21}$. Note that the retention of a single g in the integrands of Eqs. (4.110) and (4.111) (as opposed to $\mathscr{g}$) is not a typographical error but, rather, is the exact notation used by Chapman and Cowling. After integration over ε and the directions of $\boldsymbol{g}$ (which changes the constants somewhat), one can express the χ-dependent portions of the RHS bracketed integrals of Eqs.

(4.110) and (4.111) as:

$$
\begin{aligned}
&\left(\mathrm{ST}/st\right)^{5/2}\left(M_1M_2\right)^{-1/2}\pi^{-3/2}H_{12}\left(\chi\right)\\
&\quad=\exp\left(-g^2\right)\sum_r\sum_n\left\{2M_1M_2st\left(1-\cos\left(\chi\right)\right)\right\}^r\\
&\qquad\times\left(g^{2r}/r!\right)\left(M_2s+M_1t\right)^n\\
&\qquad\times\left[\left(n+1\right)S_{r+1/2}^{(n+1)}\left(g^2\right)+\left(1-\cos\left(\chi\right)\right)g^2S_{r+3/2}^{(n)}\left(g^2\right)\right]\ ,
\end{aligned}
\tag{4.112}
$$

and:

$$
\begin{aligned}
&\left(\mathrm{ST}/st\right)^{5/2}\left(M_1M_2\right)^{-1/2}\pi^{-3/2}H_{1}\left(\chi\right)\\
&=\exp\left(-g^2\right)\sum_r\sum_n\left\{st\left(M_1^2+M_2^2+2M_1M_2\cos\left(\chi\right)\right)\right\}^r\left(g^{2r}/r!\right)\\
&\qquad\times\left(M_2(s+t)-(M_2-M_1)st\right)^n\\
&\qquad\times\left[M_1(n+1)S_{r+1/2}^{(n+1)}\left(g^2\right)+\left(M_1+M_2\cos(\chi)\right)g^2S_{r+3/2}^{(n)}\left(g^2\right)\right]\ .
\end{aligned}
\tag{4.113}
$$

In both of these cases, the coefficient of $[s^pt^q]$ yields a polynomial in powers of g^2 and $\cos(\chi)$ that is multiplied by $\exp(-g^2)$ and in which each term is some function of the molecular masses via M_1 and M_2. The χ-independent portions of the RHS bracketed integrals of Eqs. (4.110) and (4.111) are obtained by the simple expedient of setting $\chi=0$ in Eqs. (4.112) and (4.113) which yields overall terms in the combined polynomials involving $(1-\cos^\ell(\chi))$. Thus, it is possible to express Eqs. (4.110) and (4.111) as:

$$
\begin{aligned}
&\left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1,S_{3/2}^{(q)}\left(\mathscr{C}_2^2\right)\boldsymbol{\mathscr{C}}_2\right]_{12}\\
&=8M_2^{p+1/2}M_1^{q+1/2}\sum_{r,\ell}A_{pqr\ell}\Omega_{12}^{(\ell)}\left(r\right)\ ,
\end{aligned}
\tag{4.114}
$$

and:

$$
\begin{aligned}
&\left[S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1,S_{3/2}^{(q)}\left(\mathscr{C}_1^2\right)\boldsymbol{\mathscr{C}}_1\right]_{12}\\
&=8\sum_{r,\ell}A'_{pqr\ell}\Omega_{12}^{(\ell)}\left(r\right)\ ,
\end{aligned}
\tag{4.115}
$$

where the omega integrals are defined as:

$$
\Omega_{12}^{(\ell)}\left(r\right)\equiv\left(\frac{kT}{2\pi m_0M_1M_2}\right)^{1/2}\int_0^\infty\exp\left(-g^2\right)g^{2r+3}\phi_{12}^{(\ell)}\mathrm{d}g\ ,
\tag{4.116}
$$

with:

$$\phi_{12}^{(\ell)} \equiv 2\pi \int_0^{\pi} \left(1 - \cos^{\ell}(\chi)\right) b db \, . \tag{4.117}$$

As stated by Chapman and Cowling [10]:

> "Explicit expressions for [$A_{pqr\ell}$ and $A'_{pqr\ell}$] can be obtained from [Eqs. (4.112) and (4.113)] using [Eq. (4.57)] for $S_m^{(n)}(x)$. In view of the complication of these expressions it is, however, better in practice to calculate any desired values of [$A_{pqr\ell}$ and $A'_{pqr\ell}$] directly from [Eqs. (4.112) and (4.113)]."

This approach has been explored as suggested by Chapman and Cowling using *Mathematica*® up to order 70 and results to this order for diffusion and thermal conductivity are reported here. Note that in all of the current expressions, the fully general dependence of the expressions on the mole fractions, the molecular masses, and the models of the intermolecular potential that can be employed has been retained. For example, from Eqs. (4.66) and (4.70), since there exists the Eq. (4.67) symmetry in the off-diagonal elements such that $a_{pq} = a_{qp}$, one can write the following matrix equations for the order 1 diffusion and thermal conductivity solutions that were arrived at previously in the low-order example:

$$\begin{bmatrix} a_{-1-1} & a_{-10} & a_{-11} \\ a_{0-1} & a_{00} & a_{01} \\ a_{1-1} & a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} d_{-1} \\ d_0 \\ d_1 \end{bmatrix} = \begin{bmatrix} \delta_{-1} \\ \delta_0 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \delta_0 \\ 0 \end{bmatrix} , \tag{4.118}$$

$$\begin{bmatrix} a_{-1-1} & a_{-11} \\ a_{1-1} & a_{11} \end{bmatrix} \begin{bmatrix} a_{-1} \\ a_1 \end{bmatrix} = \begin{bmatrix} \alpha_{-1} \\ \alpha_1 \end{bmatrix} . \tag{4.119}$$

For these first-order solutions, the general expressions for the a_{pq} matrix elements are:

$$\begin{aligned} a_{-1-1} =& x_2^2 \left(4\Omega_2^{(2)}(2)\right) + x_1 x_2 \left(10\left(5M_1^3 + 6M_1M_2^2\right)\Omega_{12}^{(1)}(1)\right. \\ & \left. -40M_1^3\Omega_{12}^{(1)}(2) + 8M_1^3\Omega_{12}^{(1)}(3) + 16M_1^2M_2\Omega_{12}^{(2)}(2)\right) , \end{aligned} \tag{4.120}$$

$$a_{-10} = x_1 x_2 \left(-20M_1^2 M_2^{1/2} \Omega_{12}^{(1)}(1) + 8M_1^2 M_2^{1/2} \Omega_{12}^{(1)}(2)\right) \ , \tag{4.121}$$

$$\begin{aligned} a_{-11} =& x_1 x_2 \left(-110M_1^{3/2} M_2^{3/2} \Omega_{12}^{(1)}(1) + 40M_1^{3/2} M_2^{3/2} \Omega_{12}^{(1)}(2) \right. \\ & \left. -8M_1^{3/2} M_2^{3/2} \Omega_{12}^{(1)}(3) + 16M_1^{3/2} M_2^{3/2} \Omega_{12}^{(2)}(2)\right) \ , \end{aligned} \tag{4.122}$$

$$a_{0-1} = x_1 x_2 \left(-20M_1^2 M_2^{1/2} \Omega_{12}^{(1)}(1) + 8M_1^2 M_2^{1/2} \Omega_{12}^{(1)}(2)\right) \ , \tag{4.123}$$

$$a_{00} = x_1 x_2 \left(8M_1 M_2 \Omega_{12}^{(1)}(1)\right) \ , \tag{4.124}$$

$$a_{01} = x_1 x_2 \left(20M_1^{1/2} M_2^2 \Omega_{12}^{(1)}(1) - 8M_1^{1/2} M_2^2 \Omega_{12}^{(1)}(2)\right) \ , \tag{4.125}$$

$$\begin{aligned} a_{1-1} =& x_1 x_2 \left(-110M_1^{3/2} M_2^{3/2} \Omega_{12}^{(1)}(1) + 40M_1^{3/2} M_2^{3/2} \Omega_{12}^{(1)}(2) \right. \\ & \left. -8M_1^{3/2} M_2^{3/2} \Omega_{12}^{(1)}(3) + 16M_1^{3/2} M_2^{3/2} \Omega_{12}^{(2)}(2)\right) \ , \end{aligned} \tag{4.126}$$

$$a_{10} = x_1 x_2 \left(20M_1^{1/2} M_2^2 \Omega_{12}^{(1)}(1) - 8M_1^{1/2} M_2^2 \Omega_{12}^{(1)}(2)\right) \ , \tag{4.127}$$

$$\begin{aligned} a_{11} =& x_1^2 \left(4\Omega_1^{(2)}(2)\right) + x_1 x_2 \left(10\left(6M_1^2 M_2 + 5M_2^3\right) \Omega_{12}^{(1)}(1) \right. \\ & \left. -40M_2^3 \Omega_{12}^{(1)}(2) + 8M_2^3 \Omega_{12}^{(1)}(3) + 16M_1 M_2^2 \Omega_{12}^{(2)}(2)\right) \ . \end{aligned} \tag{4.128}$$

The symmetry in the off-diagonal elements is obvious, with $a_{-10} = a_{0-1}$, $a_{-11} = a_{1-1}$, and $a_{01} = a_{10}$. Likewise, for order 2, Eqs. (4.66) and (4.70) plus symmetry give the following matrix equations for diffusion and thermal conductivity, respectively:

$$\begin{bmatrix} a_{-2-2} & a_{-2-1} & a_{-20} & a_{-21} & a_{-22} \\ a_{-1-2} & a_{-1-1} & a_{-10} & a_{-11} & a_{-12} \\ a_{0-2} & a_{0-1} & a_{00} & a_{01} & a_{02} \\ a_{1-2} & a_{1-1} & a_{10} & a_{11} & a_{12} \\ a_{2-2} & a_{2-1} & a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} d_{-2} \\ d_{-1} \\ d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \delta_0 \\ 0 \\ 0 \end{bmatrix} \ , \tag{4.129}$$

$$\begin{bmatrix} a_{-2-2} & a_{-2-1} & a_{-21} & a_{-22} \\ a_{-1-2} & a_{-1-1} & a_{-11} & a_{-12} \\ a_{1-2} & a_{1-1} & a_{11} & a_{12} \\ a_{2-2} & a_{2-1} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{-2} \\ a_{-1} \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_{-1} \\ \alpha_1 \\ 0 \end{bmatrix} \ , \tag{4.130}$$

and the general expressions for the a_{pq} matrix elements are:

$$\begin{aligned}
a_{-2-2} =& x_2^2\Big(\tfrac{77}{4}\Omega_2^{(2)}(2) - 7\Omega_2^{(2)}(3) + \Omega_2^{(2)}(4)\Big) + x_1 x_2\Big(\tfrac{35}{8}\left(35M_1^5\right.\\
&\left.+168M_1^3M_2^2 + 40M_1M_2^4\right)\Omega_{12}^{(1)}(1) - 49\left(5M_1^5\right.\\
&\left.+12M_1^3M_2^2\right)\Omega_{12}^{(1)}(2) + \left(133M_1^5 + 108M_1^3M_2^2\right)\Omega_{12}^{(1)}(3)\\
&- 28M_1^5\Omega_{12}^{(1)}(4) + 2M_1^5\Omega_{12}^{(1)}(5) + 28\left(7M_1^4M_2\right.\\
&\left.+4M_1^2M_2^3\right)\Omega_{12}^{(2)}(2) - 112M_1^4M_2\Omega_{12}^{(2)}(3) + 16M_1^4M_2\Omega_{12}^{(2)}(4)\\
&+ 16M_1^3M_2^2\Omega_{12}^{(3)}(3)\Big)\ ,
\end{aligned} \tag{4.131}$$

$$\begin{aligned}
a_{-2-1} =& a_{-1-2} = x_2^2\Big(7\Omega_2^{(2)}(2) - 2\Omega_2^{(2)}(3)\Big) + x_1x_2\Big(\tfrac{35}{2}\left(5M_1^4\right.\\
&\left.+12M_1^2M_2^2\right)\Omega_{12}^{(1)}(1) - 21\left(5M_1^4 + 4M_1^2M_2^2\right)\Omega_{12}^{(1)}(2)\\
&+ 38M_1^4\Omega_{12}^{(1)}(3) - 4M_1^4\Omega_{12}^{(1)}(4) + 56M_1^3M_2\Omega_{12}^{(2)}(2)\\
&- 16M_1^3M_2\Omega_{12}^{(2)}(3)\Big)\ ,
\end{aligned} \tag{4.132}$$

$$\begin{aligned}
a_{-20} =& a_{0-2} = x_1x_2\Big(-35M_1^3M_2^{1/2}\Omega_{12}^{(1)}(1)\\
&+28M_1^3M_2^{1/2}\Omega_{12}^{(1)}(2) - 4M_1^3M_2^{1/2}\Omega_{12}^{(1)}(3)\Big)\ ,
\end{aligned} \tag{4.133}$$

$$\begin{aligned}
a_{-21} =& a_{1-2} = x_1x_2\Big(-\tfrac{595}{2}M_1^{5/2}M_2^{3/2}\Omega_{12}^{(1)}(1) + 189M_1^{5/2}M_2^{3/2}\Omega_{12}^{(1)}(2)\\
&-38M_1^{5/2}M_2^{3/2}\Omega_{12}^{(1)}(3) + 4M_1^{5/2}M_2^{3/2}\Omega_{12}^{(1)}(4)\\
&+56M_1^{5/2}M_2^{3/2}\Omega_{12}^{(2)}(2) - 16M_1^{5/2}M_2^{3/2}\Omega_{12}^{(2)}(3)\Big)\ ,
\end{aligned} \tag{4.134}$$

$$\begin{aligned}
a_{-22} =& a_{2-2} = x_1x_2\Big(-\tfrac{8505}{8}M_1^{5/2}M_2^{5/2}\Omega_{12}^{(1)}(1) + 833M_1^{5/2}M_2^{5/2}\Omega_{12}^{(1)}(2)\\
&-241M_1^{5/2}M_2^{5/2}\Omega_{12}^{(1)}(3) + 28M_1^{5/2}M_2^{5/2}\Omega_{12}^{(1)}(4)\\
&-2M_1^{5/2}M_2^{5/2}\Omega_{12}^{(1)}(5) + 308M_1^{5/2}M_2^{5/2}\Omega_{12}^{(2)}(2)\\
&-112M_1^{5/2}M_2^{5/2}\Omega_{12}^{(2)}(3) + 16M_1^{5/2}M_2^{5/2}\Omega_{12}^{(2)}(4)\\
&-16M_1^{5/2}M_2^{5/2}\Omega_{12}^{(3)}(3)\Big)\ ,
\end{aligned} \tag{4.135}$$

$$\begin{aligned}
a_{-1-1} =& x_2^2\Big(4\Omega_2^{(2)}(2)\Big) + x_1x_2\Big(10\left(5M_1^3 + 6M_1M_2^2\right)\Omega_{12}^{(1)}(1)\\
&-40M_1^3\Omega_{12}^{(1)}(2) + 8M_1^3\Omega_{12}^{(1)}(3) + 16M_1^2M_2\Omega_{12}^{(2)}(2)\Big)\ ,
\end{aligned} \tag{4.136}$$

$$a_{-10} = a_{0-1} = x_1x_2\Big(-20M_1^2M_2^{1/2}\Omega_{12}^{(1)}(1) + 8M_1^2M_2^{1/2}\Omega_{12}^{(1)}(2)\Big)\ , \tag{4.137}$$

$$
\begin{aligned}
a_{-11} = & a_{1-1} = x_1 x_2 \Big(-110 M_1^{3/2} M_2^{3/2} \varOmega_{12}^{(1)}(1) + 40 M_1^{3/2} M_2^{3/2} \varOmega_{12}^{(1)}(2) \\
& -8 M_1^{3/2} M_2^{3/2} \varOmega_{12}^{(1)}(3) + 16 M_1^{3/2} M_2^{3/2} \varOmega_{12}^{(2)}(2) \Big) \ ,
\end{aligned} \tag{4.138}
$$

$$
\begin{aligned}
a_{-12} = & a_{2-1} = x_1 x_2 \Big(-\tfrac{595}{2} M_1^{3/2} M_2^{5/2} \varOmega_{12}^{(1)}(1) + 189 M_1^{3/2} M_2^{5/2} \varOmega_{12}^{(1)}(2) \\
& -38 M_1^{3/2} M_2^{5/2} \varOmega_{12}^{(1)}(3) + 4 M_1^{3/2} M_2^{5/2} \varOmega_{12}^{(1)}(4) \\
& +56 M_1^{3/2} M_2^{5/2} \varOmega_{12}^{(2)}(2) - 16 M_1^{3/2} M_2^{5/2} \varOmega_{12}^{(2)}(3) \Big) \ ,
\end{aligned} \tag{4.139}
$$

$$
a_{00} = x_1 x_2 \Big(8 M_1 M_2 \varOmega_{12}^{(1)}(1) \Big) \ , \tag{4.140}
$$

$$
a_{01} = a_{10} = x_1 x_2 \Big(20 M_1^{1/2} M_2^2 \varOmega_{12}^{(1)}(1) - 8 M_1^{1/2} M_2^2 \varOmega_{12}^{(1)}(2) \Big) \ , \tag{4.141}
$$

$$
\begin{aligned}
a_{02} = & a_{20} = x_1 x_2 \Big(35 M_1^{1/2} M_2^3 \varOmega_{12}^{(1)}(1) \\
& -28 M_1^{1/2} M_2^3 \varOmega_{12}^{(1)}(2) + 4 M_1^{1/2} M_2^3 \varOmega_{12}^{(1)}(3) \Big) \ ,
\end{aligned} \tag{4.142}
$$

$$
\begin{aligned}
a_{11} = & x_1^2 \Big(4 \varOmega_1^{(2)}(2) \Big) + x_1 x_2 \Big(10 \left(6 M_1^2 M_2 + 5 M_2^3 \right) \varOmega_{12}^{(1)}(1) \\
& -40 M_2^3 \varOmega_{12}^{(1)}(2) + 8 M_2^3 \varOmega_{12}^{(1)}(3) + 16 M_1 M_2^2 \varOmega_{12}^{(2)}(2) \Big) \ ,
\end{aligned} \tag{4.143}
$$

$$
\begin{aligned}
a_{12} = & a_{21} = x_1^2 \Big(7 \varOmega_1^{(2)}(2) - 2 \varOmega_1^{(2)}(3) \Big) + x_1 x_2 \Big(\tfrac{35}{2} \left(12 M_1^2 M_2^2 \right. \\
& \left. +5 M_2^4 \right) \varOmega_{12}^{(1)}(1) - 21 \left(4 M_1^2 M_2^2 + 5 M_2^4 \right) \varOmega_{12}^{(1)}(2) \\
& + 38 M_2^4 \varOmega_{12}^{(1)}(3) - 4 M_2^4 \varOmega_{12}^{(1)}(4) + 56 M_1 M_2^3 \varOmega_{12}^{(2)}(2) \\
& - 16 M_1 M_2^3 \varOmega_{12}^{(2)}(3) \Big) \ ,
\end{aligned} \tag{4.144}
$$

$$
\begin{aligned}
a_{22} = & x_1^2 \Big(\tfrac{77}{4} \varOmega_1^{(2)}(2) - 7 \varOmega_1^{(2)}(3) + \varOmega_1^{(2)}(4) \Big) + x_1 x_2 \Big(\tfrac{35}{8} \left(40 M_1^4 M_2 \right. \\
& \left. +168 M_1^2 M_2^3 + 35 M_2^5 \right) \varOmega_{12}^{(1)}(1) - 49 \left(12 M_1^2 M_2^3 \right. \\
& \left. +5 M_2^5 \right) \varOmega_{12}^{(1)}(2) + \left(108 M_1^2 M_2^3 + 133 M_2^5 \right) \varOmega_{12}^{(1)}(3) \\
& - 28 M_2^5 \varOmega_{12}^{(1)}(4) + 2 M_2^5 \varOmega_{12}^{(1)}(5) + 28 \left(4 M_1^3 M_2^2 \right. \\
& \left. +7 M_1 M_2^4 \right) \varOmega_{12}^{(2)}(2) - 112 M_1 M_2^4 \varOmega_{12}^{(2)}(3) + 16 M_1 M_2^4 \varOmega_{12}^{(2)}(4) \\
& + 16 M_1^2 M_2^3 \varOmega_{12}^{(3)}(3) \Big) \ .
\end{aligned} \tag{4.145}
$$

In Eqs. (4.131)-(4.145), as opposed to Eqs. (4.120)-(4.128), the symmetry has been expressed explicitly. While one could, certainly, go even higher in order detailing explicit expressions for the matrix elements, the expressions rapidly become unwieldy in analytical form. If one wishes to verify the current order 2 expressions or if additional explicit expressions for the order 3 results are desired, they can be determined from the relevant expressions which have been reported previously in the literature [11, 12, 43, 44]. In going to higher orders than those available in the current literature, as has been done in this work, it is clear from a comparison of the order 1 and order 2 expressions above that all previously generated expressions are retained at each higher order and need not be recomputed. Most importantly in this work, regardless of the order of the expansion used, the dependencies of the matrix elements on mole fractions and molecular masses are carried through in general form. Molecular diameters and the specific intermolecular potential model used are also carried through in general form via the omega integrals which may also be defined in terms of the quantity, σ_{12}. This quantity is, in purely general terms, only a convenient, arbitrarily chosen length in the range of b. Thus, it is often convenient to express the omega integrals as [10]:

$$\Omega_{12}^{(\ell)}(r) = \tfrac{1}{2}\sigma_{12}^2 \left(2\pi kT/m_0 M_1 M_2\right)^{1/2} W_{12}^{(\ell)}(r) \ , \tag{4.146}$$

where:

$$\begin{aligned} W_{12}^{(\ell)}(r) \equiv & \int_0^\infty \exp\left(-g^2\right) g^{2r+2} \\ & \times \int_0^\pi \left(1 - \cos^\ell(\chi)\right)(b/\sigma_{12})\, \mathrm{d}(b/\sigma_{12})\, \mathrm{d}\left(g^2\right) \\ = & \, 2 \int_0^\infty \exp\left(-g^2\right) g^{2r+3} \\ & \times \int_0^\pi \left(1 - \cos^\ell(\chi)\right)(b/\sigma_{12})\, \mathrm{d}(b/\sigma_{12})\, \mathrm{d}g \ , \end{aligned} \tag{4.147}$$

and where the corresponding simple gas expressions are:

$$\Omega_{1}^{(\ell)}(r) = \sigma_1^2 \left(\pi kT/m_1\right)^{1/2} W_1^{(\ell)}(r) \ , \tag{4.148}$$

with:

$$\begin{aligned} W_1^{(\ell)}(r) = & 2 \int_0^\infty \exp\left(-g^2\right) g^{2r+3} \\ & \times \int_0^\pi \left(1 - \cos^\ell(\chi)\right)(b/\sigma_1)\, \mathrm{d}(b/\sigma_1)\, \mathrm{d}g \ , \end{aligned} \tag{4.149}$$

and:

$$\Omega_2^{(\ell)}(r) = \sigma_2^2 (\pi kT/m_2)^{1/2} W_2^{(\ell)}(r) \ , \tag{4.150}$$

with:

$$\begin{aligned} W_2^{(\ell)}(r) =& 2\int_0^\infty \exp\left(-g^2\right) g^{2r+3} \\ & \times \int_0^\pi \left(1-\cos^\ell(\chi)\right)(b/\sigma_2)\,\mathrm{d}(b/\sigma_2)\,\mathrm{d}g \ . \end{aligned} \tag{4.151}$$

In Eqs. (4.146)-(4.151), σ_1 and σ_2 are arbitrary scale lengths associated with collisions between like molecules of type 1 or type 2, respectively, while σ_{12} is associated with collisions between unlike molecules of types 1 and 2. These scale lengths are commonly associated with some concept of the molecular diameters depending upon the specific details of the intermolecular potential model that is employed. In the current work, results for the case of rigid-sphere molecules are reported because the form of the rigid-sphere potential model allows all of the omega integrals to be evaluated analytically eliminating the need for any numerical integrations in the current work. Some of the details of this model are described in the following section.

4.7 The case of rigid-sphere molecules

For the specific case of a binary, rigid-sphere, gas mixture, one has an intermolecular potential model of the form [27]:

$$U(r) = \begin{cases} \infty, & r < \sigma_{12} \ , \\ 0, & r \geq \sigma_{12} \ , \end{cases} \tag{4.152}$$

where $\sigma_{12} = \frac{1}{2}(\sigma_1 + \sigma_2)$, and σ_1 and σ_2 are the actual diameters of the colliding spherical molecules. Under the rigid-sphere assumption, one then has the collisional relationships $b = \sigma_{12}\cos\left(\frac{1}{2}\chi\right)$ and $b\mathrm{d}b = -\frac{1}{4}\sigma_{12}^2 \sin(\chi)\,\mathrm{d}\chi$. Using these in Eq. (4.147) yields:

$$W_{12}^{(\ell)}(r) = \tfrac{1}{4}\left[2 - \frac{1}{\ell+1}\left(1 + (-1)^\ell\right)\right](r+1)! \ , \tag{4.153}$$

such that:

$$\begin{aligned}\Omega_{12}^{(\ell)}(r) =& \tfrac{1}{2}\sigma_{12}^2 \left(2\pi kT/m_0 M_1 M_2\right)^{1/2} \\ & \times \tfrac{1}{4}\left[2 - \frac{1}{\ell+1}\left(1+(-1)^{\ell}\right)\right](r+1)!\ ,\end{aligned} \tag{4.154}$$

and, since $W_1^{(\ell)}(r) = W_2^{(\ell)}(r) = W_{12}^{(\ell)}(r)$, the corresponding simple-gas expressions are:

$$\Omega_1^{(\ell)}(r) = \sigma_1^2 \left(\pi kT/m_1\right)^{1/2} \tfrac{1}{4}\left[2 - \frac{1}{\ell+1}\left(1+(-1)^{\ell}\right)\right](r+1)!\ , \tag{4.155}$$

and:

$$\Omega_2^{(\ell)}(r) = \sigma_2^2 \left(\pi kT/m_2\right)^{1/2} \tfrac{1}{4}\left[2 - \frac{1}{\ell+1}\left(1+(-1)^{\ell}\right)\right](r+1)!\ . \tag{4.156}$$

All of these omega integrals are readily evaluated for the purpose of the current work and, thus, one has the following simplified, rigid-sphere, matrix elements for the order 2 diffusion and thermal conductivity approximations:

$$\begin{aligned}a_{-2-2} =& x_2^2\left\{\sqrt{\frac{\pi kT}{m_2}}\left(\tfrac{45}{2}\right)\sigma_2^2\right\} + x_1x_2\left\{\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\right. \\ & \times\left(\tfrac{433}{16}M_1^5 + 68M_1^4M_2 + \tfrac{459}{2}M_1^3M_2^2 + 112M_1^2M_2^3\right. \\ & \left.\left.+\tfrac{175}{2}M_1M_2^4\right)\sigma_{12}^2\right\}\ ,\end{aligned} \tag{4.157}$$

$$\begin{aligned}a_{-2-1} = a_{-1-2} =& x_2^2\left\{-\sqrt{\frac{\pi kT}{m_2}}\,(2)\,\sigma_2^2\right\} \\ & + x_1x_2\left\{-\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\left(\tfrac{23}{4}M_1^4 + 8M_1^3M_2\right.\right. \\ & \left.\left.+21M_1^2M_2^2\right)\sigma_{12}^2\right\}\ ,\end{aligned} \tag{4.158}$$

$$a_{-20} = a_{0-2} = x_1x_2\left\{\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\left(\tfrac{1}{2}M_1^3M_2^{1/2}\right)\sigma_{12}^2\right\}\ , \tag{4.159}$$

$$a_{-21} = a_{1-2} = x_1x_2\left\{\sqrt{\frac{2\pi kT}{m_0M_1M_2}}\left(\tfrac{75}{4}M_1^{5/2}M_2^{3/2}\right)\sigma_{12}^2\right\}\ , \tag{4.160}$$

$$a_{-22} = a_{2-2} = x_1 x_2 \left\{ -\sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \left(\tfrac{2625}{16} M_1^{5/2} M_2^{5/2} \right) \sigma_{12}^2 \right\} , \tag{4.161}$$

$$\begin{aligned} a_{-1-1} = & x_2^2 \left\{ \sqrt{\frac{\pi kT}{m_2}} \, (8) \, \sigma_2^2 \right\} \\ & + x_1 x_2 \left\{ \sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \left(13 M_1^3 + 16 M_1^2 M_2 \right. \right. \\ & \left. \left. + 30 M_1 M_2^2 \right) \sigma_{12}^2 \right\} , \end{aligned} \tag{4.162}$$

$$a_{-10} = a_{0-1} = x_1 x_2 \left\{ \sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \left(2 M_1^2 M_2^{1/2} \right) \sigma_{12}^2 \right\} , \tag{4.163}$$

$$a_{-11} = a_{1-1} = x_1 x_2 \left\{ -\sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \left(27 M_1^{3/2} M_2^{3/2} \right) \sigma_{12}^2 \right\} , \tag{4.164}$$

$$a_{-12} = a_{2-1} = x_1 x_2 \left\{ \sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \left(\tfrac{75}{4} M_1^{3/2} M_2^{5/2} \right) \sigma_{12}^2 \right\} , \tag{4.165}$$

$$a_{00} = x_1 x_2 \left\{ \sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \, (4 M_1 M_2) \, \sigma_{12}^2 \right\} , \tag{4.166}$$

$$a_{01} = a_{10} = x_1 x_2 \left\{ \sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \left(2 M_1^{1/2} M_2^2 \right) \sigma_{12}^2 \right\} , \tag{4.167}$$

$$a_{02} = a_{20} = x_1 x_2 \left\{ \sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \left(\tfrac{1}{2} M_1^{1/2} M_2^3 \right) \sigma_{12}^2 \right\} , \tag{4.168}$$

$$\begin{aligned} a_{11} = & x_1^2 \left\{ \sqrt{\frac{\pi kT}{m_1}} \, (8) \, \sigma_1^2 \right\} \\ & + x_1 x_2 \left\{ \sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \left(13 M_2^3 + 16 M_1 M_2^2 + 30 M_1^2 M_2 \right) \sigma_{12}^2 \right\} , \end{aligned} \tag{4.169}$$

$$\begin{aligned} a_{12} = a_{21} = & x_1^2 \left\{ -\sqrt{\frac{\pi kT}{m_1}} \, (2) \, \sigma_1^2 \right\} \\ & + x_1 x_2 \left\{ -\sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \left(\tfrac{23}{4} M_2^4 + 8 M_1 M_2^3 \right. \right. \\ & \left. \left. + 21 M_1^2 M_2^2 \right) \sigma_{12}^2 \right\} , \end{aligned} \tag{4.170}$$

$$
\begin{aligned}
a_{22} =& x_1^2 \left\{ \sqrt{\frac{\pi kT}{m_1}} \left(\tfrac{45}{2}\right) \sigma_1^2 \right\} + x_1 x_2 \left\{ \sqrt{\frac{2\pi kT}{m_0 M_1 M_2}} \right. \\
& \times \left(\tfrac{433}{16} M_2^5 + 68 M_1 M_2^4 + \tfrac{459}{2} M_1^2 M_2^3 + 112 M_1^3 M_2^2 \right. \\
& \left. \left. + \tfrac{175}{2} M_1^4 M_2 \right) \sigma_{12}^2 \right\} .
\end{aligned} \tag{4.171}
$$

In the current results, this methodology is continued to arbitrary order for rigid-sphere molecules with the full dependencies of the matrix elements on the molecular masses, mole fractions, and molecular diameters being retained explicitly up to the final point of actual evaluation via matrix inversion. As indicated previously, adaptation of this work to more realistic potential models is straightforward since the potential model is present in a fully general form in the bracket integral expressions via the omega integrals although such results have not yet been generated.

4.8 Results

The quantities of major interest in the present work are the diffusion coefficient:

$$
D_{12} = \tfrac{1}{2} x_1 x_2 \left(2kT/m_0\right)^{1/2} d_0 \ , \tag{4.172}
$$

the thermal diffusion coefficient:

$$
D_T = -\tfrac{5}{4} x_1 x_2 \left(2kT/m_0\right)^{1/2} \left(x_1 M_1^{-1/2} d_1 + x_2 M_2^{-1/2} d_{-1} \right) \ , \tag{4.173}
$$

the thermal diffusion ratio:

$$
\mathrm{k}_T = D_T / D_{12} = -\tfrac{5}{2} \left(x_1 M_1^{-1/2} d_1 + x_2 M_2^{-1/2} d_{-1} \right) \Big/ d_0 \ , \tag{4.174}
$$

the Chapman-Enskog diffusion solutions:

$$
\mathbb{D}_1 \left(\mathscr{C}_1\right) = \sum_{p=0}^{+\infty} d_p S_{3/2}^{(p)} \left(\mathscr{C}_1^2\right) \mathscr{C}_1 \ , \tag{4.175}
$$

$$
\mathbb{D}_2 \left(\mathscr{C}_2\right) = \sum_{p=0}^{+\infty} d_{-p} S_{3/2}^{(p)} \left(\mathscr{C}_2^2\right) \mathscr{C}_2 \ , \tag{4.176}
$$

the thermal conductivity coefficient:

$$\lambda = -\tfrac{5}{4}kn\left(2kT/m_0\right)^{1/2}\left(x_1 M_1^{-1/2} a_1 + x_2 M_2^{-1/2} a_{-1}\right) , \tag{4.177}$$

and the Chapman-Enskog thermal conductivity solutions:

$$\tilde{\mathbb{A}}_1\left(\mathscr{C}_1\right) = \sum_{p=1}^{+\infty} a_p S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right)\mathscr{C}_1 , \tag{4.178}$$

$$\tilde{\mathbb{A}}_2\left(\mathscr{C}_2\right) = \sum_{p=1}^{+\infty} a_{-p} S_{3/2}^{(p)}\left(\mathscr{C}_2^2\right)\mathscr{C}_2 . \tag{4.179}$$

From Eqs. (4.172) and (4.173) it is straightforward to determine that the dimensionality of the expansion coefficients, d_0, d_1, and d_{-1}, is in units of length (cm). Likewise, from Eq. (4.177) the dimensionality of the expansion coefficients a_1 and a_{-1} is also found to be in units of length (cm).

Of additional interest, note that d_0 is actually an integral on either $\mathbb{D}_1$ or $\mathbb{D}_2$ and may be expressed as:

$$\begin{aligned} d_0 =& \frac{\rho\rho_2}{M_1^{1/2}}\frac{8}{3\sqrt{\pi}}\int_0^\infty \exp\left(-\mathscr{C}_1^2\right)\mathscr{C}_1^3\mathbb{D}_1\left(\mathscr{C}_1\right)\mathrm{d}\mathscr{C}_1 \\ =& \frac{\rho\rho_1}{M_2^{1/2}}\frac{8}{3\sqrt{\pi}}\int_0^\infty \exp\left(-\mathscr{C}_2^2\right)\mathscr{C}_2^3\mathbb{D}_2\left(\mathscr{C}_2\right)\mathrm{d}\mathscr{C}_2 , \end{aligned} \tag{4.180}$$

while d_1 and d_{-1} are also integrals on $\mathbb{D}_1$ and $\mathbb{D}_2$ which may be expressed as:

$$d_1 = \frac{16}{15\sqrt{\pi}}\int_0^\infty \exp\left(-\mathscr{C}_1^2\right)\mathscr{C}_1^3\left(\tfrac{5}{2}-\mathscr{C}_1^2\right)\mathbb{D}_1\left(\mathscr{C}_1\right)\mathrm{d}\mathscr{C}_1 , \tag{4.181}$$

and:

$$d_{-1} = \frac{16}{15\sqrt{\pi}}\int_0^\infty \exp\left(-\mathscr{C}_2^2\right)\mathscr{C}_2^3\left(\tfrac{5}{2}-\mathscr{C}_2^2\right)\mathbb{D}_2\left(\mathscr{C}_2\right)\mathrm{d}\mathscr{C}_2 . \tag{4.182}$$

Further, a_1 and a_{-1} are also expressible as integrals on $\tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{A}}_2$ and may be written as:

$$a_1 = \frac{16}{15\sqrt{\pi}}\int_0^\infty \exp\left(-\mathscr{C}_1^2\right)\mathscr{C}_1^3\left(\tfrac{5}{2}-\mathscr{C}_1^2\right)\tilde{\mathbb{A}}_1\left(\mathscr{C}_1\right)\mathrm{d}\mathscr{C}_1 , \tag{4.183}$$

and:

$$a_{-1} = \frac{16}{15\sqrt{\pi}}\int_0^\infty \exp\left(-\mathscr{C}_2^2\right)\mathscr{C}_2^3\left(\tfrac{5}{2}-\mathscr{C}_2^2\right)\tilde{\mathbb{A}}_2\left(\mathscr{C}_2\right)\mathrm{d}\mathscr{C}_2 . \tag{4.184}$$

Note that D_{12}, D_T, $\mathrm{k_T}$, λ, $\mathbb{D}_1$, $\mathbb{D}_2$, $\tilde{\mathbb{A}}_1$, $\tilde{\mathbb{A}}_2$, d_0, d_1, d_{-1}, a_1, and a_{-1} are all dependent upon the variable quantities x_1, x_2, m_1, m_2, σ_1, σ_2, and temperature, T, although these dependencies are not displayed explicitly in the above equations. In the m-th order of the expansion, one writes these as:

$$[D_{12}]_m = \tfrac{1}{2} x_1 x_2 \left(2kT/m_0\right)^{1/2} d_0^{(m)} \ , \tag{4.185}$$

$$[D_T]_m = -\tfrac{5}{4} x_1 x_2 \left(2kT/m_0\right)^{1/2} \left(x_1 M_1^{-1/2} d_1^{(m)} + x_2 M_2^{-1/2} d_{-1}^{(m)}\right) \ , \tag{4.186}$$

$$\begin{aligned} [\mathrm{k}_T]_m &= [D_T]_m / [D_{12}]_m \\ &= -\tfrac{5}{2} \left(x_1 M_1^{-1/2} d_1^{(m)} + x_2 M_2^{-1/2} d_{-1}^{(m)}\right) \Big/ d_0^{(m)} \ , \end{aligned} \tag{4.187}$$

$$[\lambda]_m = -\tfrac{5}{4} kn \left(2kT/m_0\right)^{1/2} \left(x_1 M_1^{-1/2} a_1^{(m)} + x_2 M_2^{-1/2} a_{-1}^{(m)}\right) \ , \tag{4.188}$$

$$\mathbb{D}_1^{(m)}\left(\mathscr{C}_1\right) = \sum_{p=0}^{m} d_p^{(m)} S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right) \mathscr{C}_1 \ , \tag{4.189}$$

$$\mathbb{D}_2^{(m)}\left(\mathscr{C}_2\right) = \sum_{p=0}^{m} d_{-p}^{(m)} S_{3/2}^{(p)}\left(\mathscr{C}_2^2\right) \mathscr{C}_2 \ , \tag{4.190}$$

$$\tilde{\mathbb{A}}_1^{(m)}\left(\mathscr{C}_1\right) = \sum_{p=1}^{m} a_p^{(m)} S_{3/2}^{(p)}\left(\mathscr{C}_1^2\right) \mathscr{C}_1 \ , \tag{4.191}$$

$$\tilde{\mathbb{A}}_2^{(m)}\left(\mathscr{C}_2\right) = \sum_{p=1}^{m} a_{-p}^{(m)} S_{3/2}^{(p)}\left(\mathscr{C}_2^2\right) \mathscr{C}_2 \ , \tag{4.192}$$

$$\begin{aligned} d_0^{(m)} &= \frac{\rho\rho_2}{M_1^{1/2}} \frac{8}{3\sqrt{\pi}} \int_0^\infty \exp\left(-\mathscr{C}_1^2\right) \mathscr{C}_1^3 \mathbb{D}_1^{(m)}\left(\mathscr{C}_1\right) \mathrm{d}\mathscr{C}_1 \\ &= \frac{\rho\rho_1}{M_2^{1/2}} \frac{8}{3\sqrt{\pi}} \int_0^\infty \exp\left(-\mathscr{C}_2^2\right) \mathscr{C}_2^3 \mathbb{D}_2^{(m)}\left(\mathscr{C}_2\right) \mathrm{d}\mathscr{C}_2 \ , \end{aligned} \tag{4.193}$$

$$d_1^{(m)} = \frac{16}{15\sqrt{\pi}} \int_0^\infty \exp\left(-\mathscr{C}_1^2\right) \mathscr{C}_1^3 \left(\tfrac{5}{2} - \mathscr{C}_1^2\right) \mathbb{D}_1^{(m)}\left(\mathscr{C}_1\right) \mathrm{d}\mathscr{C}_1 \ , \tag{4.194}$$

$$d_{-1}^{(m)} = \frac{16}{15\sqrt{\pi}} \int_0^\infty \exp\left(-\mathscr{C}_2^2\right) \mathscr{C}_2^3 \left(\tfrac{5}{2} - \mathscr{C}_2^2\right) \mathbb{D}_2^{(m)}\left(\mathscr{C}_2\right) \mathrm{d}\mathscr{C}_2 \ , \tag{4.195}$$

$$a_1^{(m)} = \frac{16}{15\sqrt{\pi}} \int_0^\infty \exp\left(-\mathscr{C}_1^2\right) \mathscr{C}_1^3 \left(\tfrac{5}{2} - \mathscr{C}_1^2\right) \tilde{\mathbb{A}}_1^{(m)}\left(\mathscr{C}_1\right) \mathrm{d}\mathscr{C}_1 \ , \tag{4.196}$$

and:

$$a_{-1}^{(m)} = \frac{16}{15\sqrt{\pi}} \int_0^\infty \exp\left(-\mathscr{C}_2^2\right) \mathscr{C}_2^3 \left(\tfrac{5}{2} - \mathscr{C}_2^2\right) \tilde{\mathbb{A}}_2^{(m)}\left(\mathscr{C}_2\right) \mathrm{d}\mathscr{C}_2 \ . \tag{4.197}$$

As a part of this work, two basic sets of calculations have been carried out:

(i) First, a comparison of results from the present work with the results previously reported by Takata et al. [2] is made. To conduct this comparison, the current results have been adapted so as to present them using the same non-dimensionalization scheme employed by Takata et al. and have then been plotted in graphical form with the same scaling and for the same set of virtual gas mixtures reported by Takata et al.

(ii) Second, using a non-dimensionalization/normalization scheme similar to that of Chapman and Cowling [10] which normalizes the transport coefficients relative to their first-order approximate results, a comprehensive set of order 70 results for all binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe has been obtained. Additionally, extrapolated limiting values of the transport coefficients (corresponding to $m \to \infty$) that have been obtained by applying the *Mathematica*® function `SequenceLimit` to the sequence of normalized transport coefficient results corresponding to orders 1 through 70 are presented.

In the work of Takata et al. [2], the authors have considered cases involving a selection of 'virtual' gas mixtures for which the size and mass ratios of the constituents are general values only and do not reflect the sizes and masses of specific gas constituents. The size ratios that have been considered in this work include $\sigma_2/\sigma_1 = \frac{1}{2}, 1, 2$ and the mass ratios that have been considered include $m_2/m_1 = 1, 2, 3, 4, 5, 8, 10$. Mole fractions are specified by $x_1 \in (0, 1)$ with $x_1 + x_2 = 1$. The authors define non-dimensional diffusion and thermal diffusion coefficients in a manner that may be expressed in the following way:

$$\hat{D}_{12} = f_3 D_{12} \ , \tag{4.198}$$

$$\hat{D}_T = f_3 D_T \ , \tag{4.199}$$

where:

$$f_3 = 2\sqrt{2\pi}\sqrt{\frac{m_1}{2kT}}n\sigma_1^2 \ . \tag{4.200}$$

and a non-dimensional thermal conductivity coefficient as:

$$\hat{\lambda} = \frac{f_3}{\sqrt{2}kn}\lambda \ . \tag{4.201}$$

Thus, $\hat{D}_{12}$, $\hat{D}_T$, and $\hat{\lambda}$ depend only upon the masses, m_1 and m_2, the molecular diameters, σ_1 and σ_2, and the number densities, n_1 and n_2. Dependence upon temperature, T, has been totally eliminated for rigid-sphere molecules which is quite convenient because the rigid-sphere model is known to exhibit an unrealistic temperature dependence. The authors have presented their results graphically and have not given their precise numerical results explicitly although the indistinguishable comparison results from the present work indicate that they have obtained precise results for the cases studied. These results have been reported here in the same format and for exactly the same cases considered by Takata et al., in Figs. 4.1-4.9 where the reported results conform to the same non-dimensionalization scheme described in Eqs. (4.198)-(4.201). Note that instead of presenting $\hat{D}_T$ directly, Takata et al. present results for k_T which yields $\hat{D}_T$ from the $\hat{D}_{12}$ results via $\hat{D}_T = \mathrm{k}_T\hat{D}_{12}$ and that the comparison results have been presented accordingly.

Lastly, the primary results of this work for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe are presented. Note that a more concise way to report transport coefficient results may be achieved by normalizing them in the following manner:

$$[D_{12}]_m = [D_{12}]_1[D_{12}]_m^\star \ , \tag{4.202}$$

$$[D_T]_m = [D_T]_1[D_T]_m^\star \ , \tag{4.203}$$

$$[\lambda]_m = [\lambda]_1[\lambda]_m^\star \ , \tag{4.204}$$

where:

$$[D_{12}]_m^\star \equiv \frac{[D_{12}]_m}{[D_{12}]_1} \ , \tag{4.205}$$

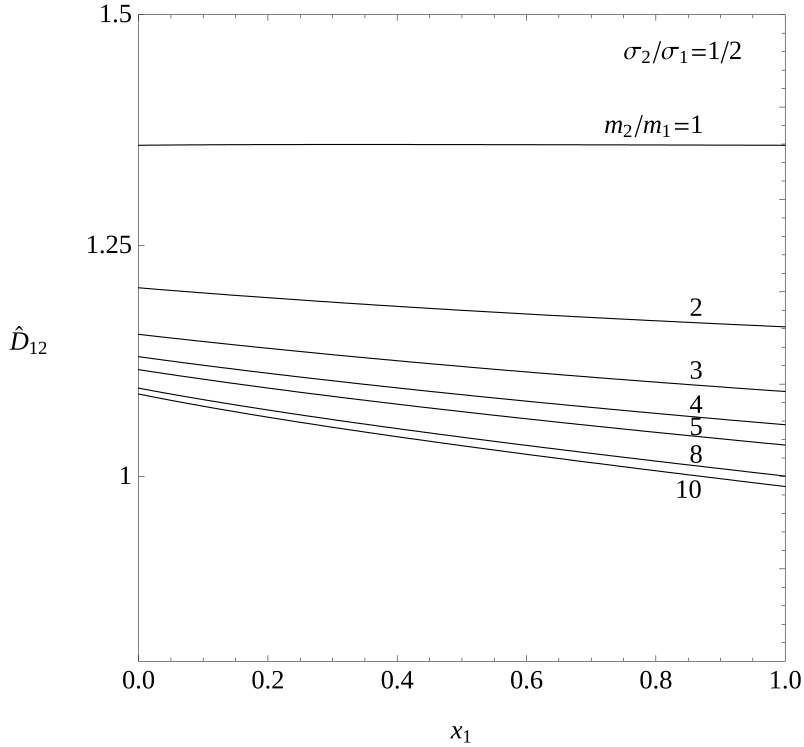

Figure 4.1: The $\hat{D}_{12}$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 1/2$.

$$[D_T]^\star_m \equiv \frac{[D_T]_m}{[D_T]_1} , \tag{4.206}$$

$$[\lambda]^\star_m \equiv \frac{[\lambda]_m}{[\lambda]_1} , \tag{4.207}$$

and where $[D_{12}]_1$, $[D_T]_1$, $[\lambda]_1$ are the transport coefficients of the mixture computed with first-order approximations ($m = 1$). In the general case, the first-order transport coefficients can be explicitly expressed as:

$$[D_{12}]_1 = \tfrac{1}{2} x_1 x_2 \left(2kT/m_0\right)^{1/2} d_0^{(1)} , \tag{4.208}$$

$$[D_T]_1 = -\tfrac{5}{4} x_1 x_2 \left(2kT/m_0\right)^{1/2} \left(x_1 M_1^{-1/2} d_1^{(1)} + x_2 M_2^{-1/2} d_{-1}^{(1)}\right) , \tag{4.209}$$

$$[\lambda]_1 = -\tfrac{5}{4} kn \left(2kT/m_0\right)^{1/2} \left(x_1 M_1^{-1/2} a_1^{(1)} + x_2 M_2^{-1/2} a_{-1}^{(1)}\right) , \tag{4.210}$$

where:

$$d_0^{(1)} = \delta_0 \left(a_{-1-1} a_{11} - a_{-11} a_{1-1}\right) / \Delta_D , \tag{4.211}$$

$$d_1^{(1)} = -\delta_0 \left(a_{-1-1} a_{10} - a_{-10} a_{1-1}\right) / \Delta_D , \tag{4.212}$$

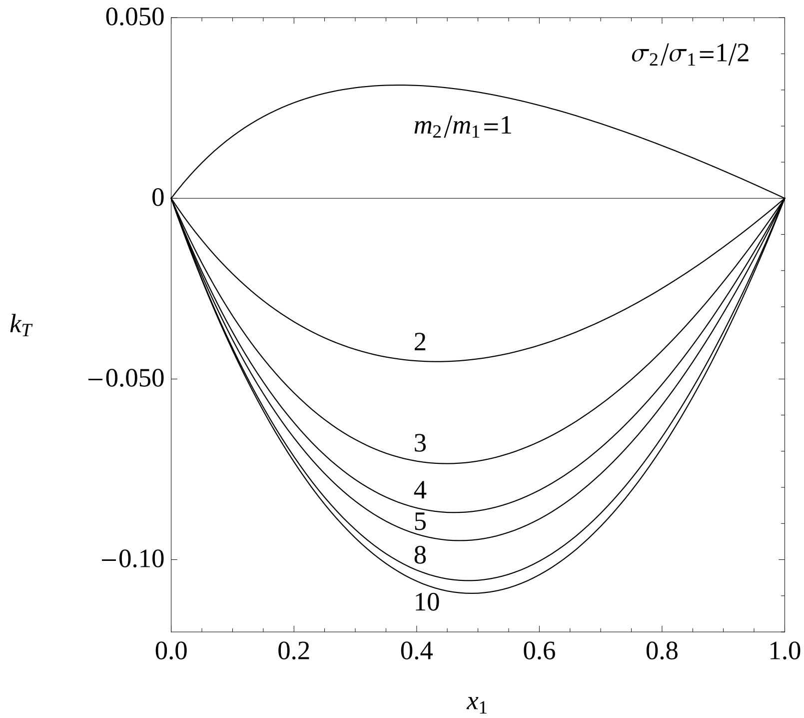

Figure 4.2: The k_T functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 1/2$.

$$d_{-1}^{(1)} = -\delta_0 \left(a_{-10}a_{11} - a_{-11}a_{10}\right) / \Delta_D \ , \tag{4.213}$$

$$a_1^{(1)} = \left(a_{-1-1}\alpha_1 - \alpha_{-1}a_{1-1}\right) / \Delta_\lambda \ , \tag{4.214}$$

and:

$$a_{-1}^{(1)} = \left(\alpha_{-1}a_{11} - a_{-11}\alpha_1\right) / \Delta_\lambda \ , \tag{4.215}$$

in which the determinants, Δ_D and Δ_λ, can be expressed as:

$$\Delta_D = a_{-1-1}a_{00}a_{11} + 2a_{-10}a_{01}a_{-11} - a_{-11}^2 a_{00} - a_{-10}^2 a_{11} - a_{01}^2 a_{-1-1} \ , \tag{4.216}$$

and:

$$\Delta_\lambda = a_{-1-1}a_{11} - a_{-11}^2 \ , \tag{4.217}$$

after the appropriate symmetries have been employed, where completely general explicit expressions for the necessary a_{pq} values have already been given in terms of the omega integrals in Eqs. (4.120)-(4.128), and where the quantities δ_0, α_1, α_{-1}

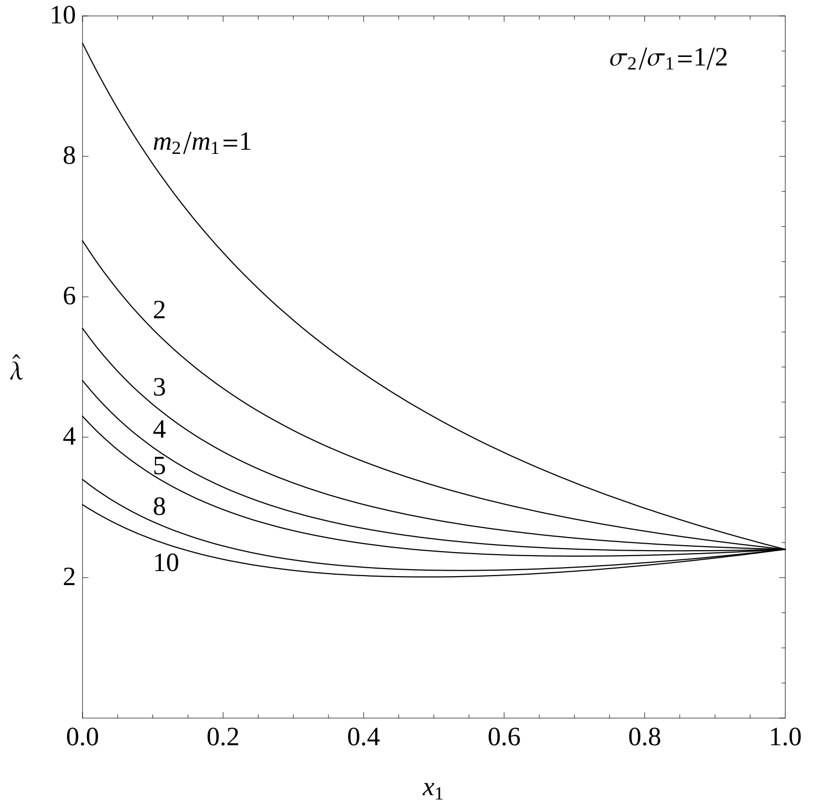

Figure 4.3: The $\hat{\lambda}$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 1/2$.

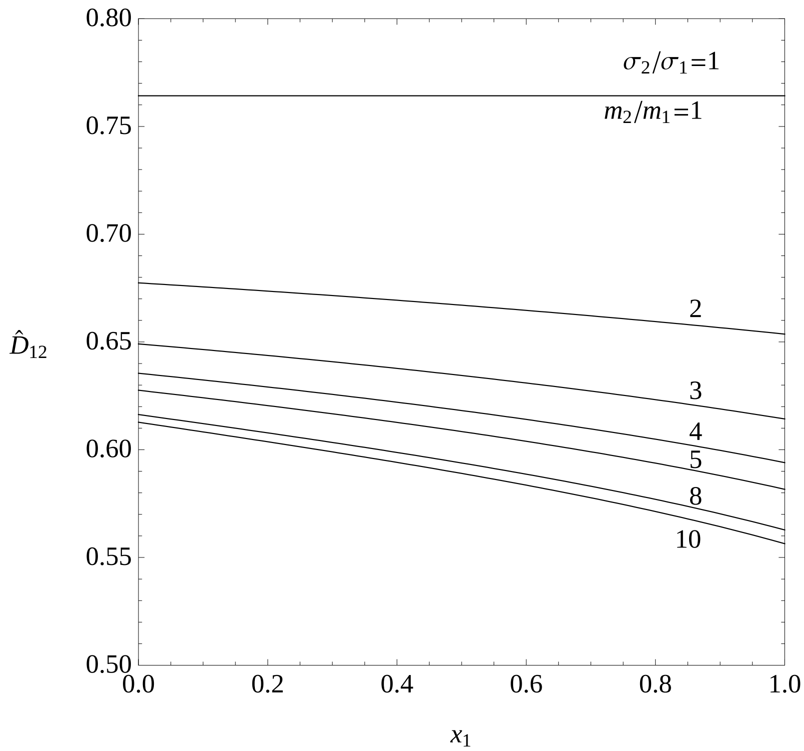

Figure 4.4: The $\hat{D}_{12}$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 1$.

have been given in Eqs. (4.62) and (4.65). Additionally, one may also normalize the quantities d_0^m, d_{-1}^m, d_1^m, a_{-1}^m, and a_1^m in the following manner:

$$\begin{aligned} d_0^{(m)\star} &= \frac{d_0^{(m)}}{d_0^{(1)}},\ d_{-1}^{(m)\star} = \frac{d_{-1}^{(m)}}{d_0^{(1)}},\ d_1^{(m)\star} = \frac{d_1^{(m)}}{d_0^{(1)}}, \\ a_{-1}^{(m)\star} &= \frac{a_{-1}^{(m)}}{a_{-1}^{(1)}},\ a_1^{(m)\star} = \frac{a_1^{(m)}}{a_1^{(1)}}\ , \end{aligned} \tag{4.218}$$

such that the transport coefficients may be expressed as:

$$[D_{12}]_m = [D_{12}]_m^\star [D_{12}]_1 = \tfrac{1}{2} x_1 x_2 \left(2kT/m_0\right)^{1/2} d_0^{(m)\star} d_0^{(1)}\ , \tag{4.219}$$

$$[D_{12}]_m^\star \equiv \frac{[D_{12}]_m}{[D_{12}]_1} = \frac{d_0^{(m)\star} d_0^{(1)}}{d_0^{(1)\star} d_0^{(1)}} = d_0^{(m)\star}\ , \tag{4.220}$$

$$\begin{aligned} [D_T]_m =& [D_T]_m^\star [D_T]_1 \\ =& -\tfrac{5}{4} x_1 x_2 \left(2kT/m_0\right)^{1/2} \\ &\times \left(x_1 M_1^{-1/2} d_1^{(m)\star} d_0^{(1)} + x_2 M_2^{-1/2} d_{-1}^{(m)\star} d_0^{(1)}\right)\ , \end{aligned} \tag{4.221}$$

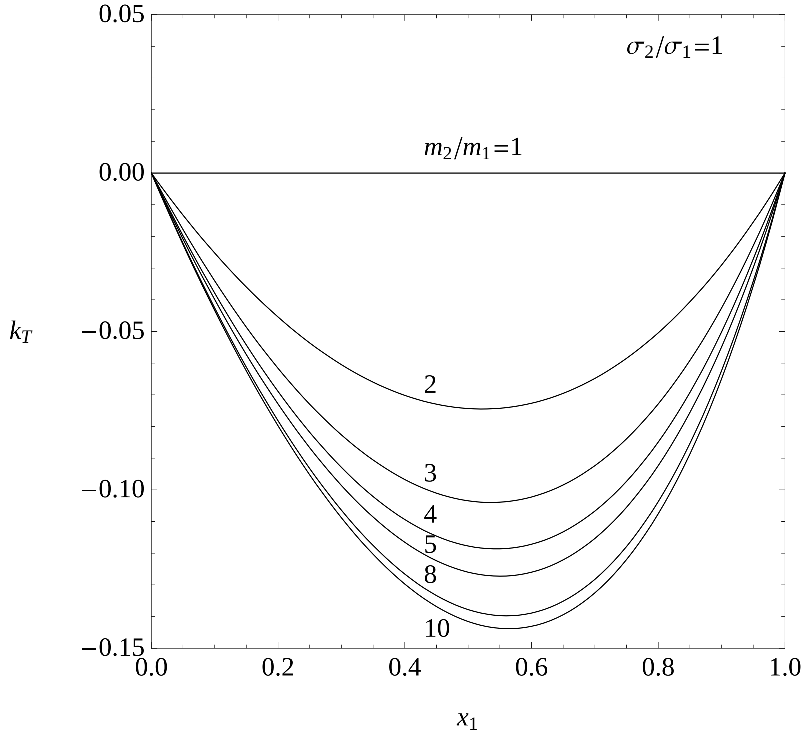

Figure 4.5: The k$_T$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 1$.

$$
\begin{aligned}
[D_T]^\star_m \equiv \frac{[D_T]_m}{[D_T]_1} &= \frac{x_1 M_1^{-1/2} d_1^{(m)\star} + x_2 M_2^{-1/2} d_{-1}^{(m)\star}}{x_1 M_1^{-1/2} d_1^{(1)\star} + x_2 M_2^{-1/2} d_{-1}^{(1)\star}} \\
&= \frac{x_1 M_2^{1/2} d_1^{(m)\star} + x_2 M_1^{1/2} d_{-1}^{(m)\star}}{x_1 M_2^{1/2} d_1^{(1)\star} + x_2 M_1^{1/2} d_{-1}^{(1)\star}} \ ,
\end{aligned}
\tag{4.222}
$$

$$
\begin{aligned}
[\lambda]_m =& [\lambda]^\star_m [\lambda]_1 \\
=& -\tfrac{5}{4} kn \left(2kT/m_0\right)^{1/2} \\
& \times \left(x_1 M_1^{-1/2} a_1^{(m)\star} a_1^{(1)} + x_2 M_2^{-1/2} a_{-1}^{(m)\star} a_{-1}^{(1)} \right) \ ,
\end{aligned}
\tag{4.223}
$$

$$
\begin{aligned}
[\lambda]^\star_m \equiv \frac{[\lambda]_m}{[\lambda]_1} =& \frac{\left(x_1 M_1^{-1/2} a_1^{(m)\star} a_1^{(1)} + x_2 M_2^{-1/2} a_{-1}^{(m)\star} a_{-1}^{(1)} \right)}{\left(x_1 M_1^{-1/2} a_1^{(1)\star} a_1^{(1)} + x_2 M_2^{-1/2} a_{-1}^{(1)\star} a_{-1}^{(1)} \right)} \\
=& \frac{\left(x_1 M_2^{1/2} a_1^{(m)\star} a_1^{(1)} + x_2 M_1^{1/2} a_{-1}^{(m)\star} a_{-1}^{(1)} \right)}{\left(x_1 M_2^{1/2} a_1^{(1)} + x_2 M_1^{1/2} a_{-1}^{(1)} \right)} \ .
\end{aligned}
\tag{4.224}
$$

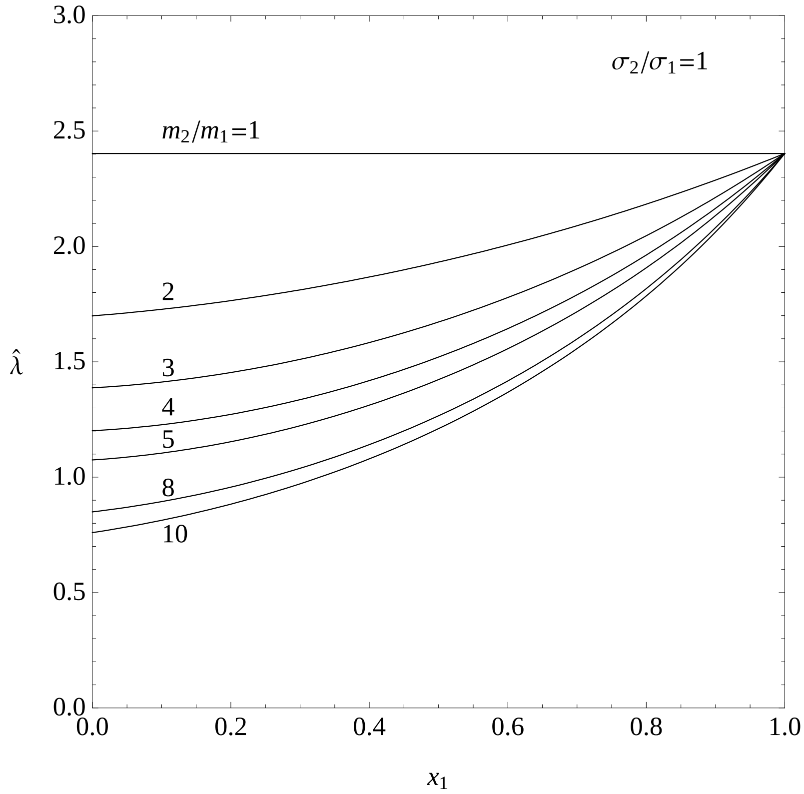

Figure 4.6: The $\hat{\lambda}$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 1$.

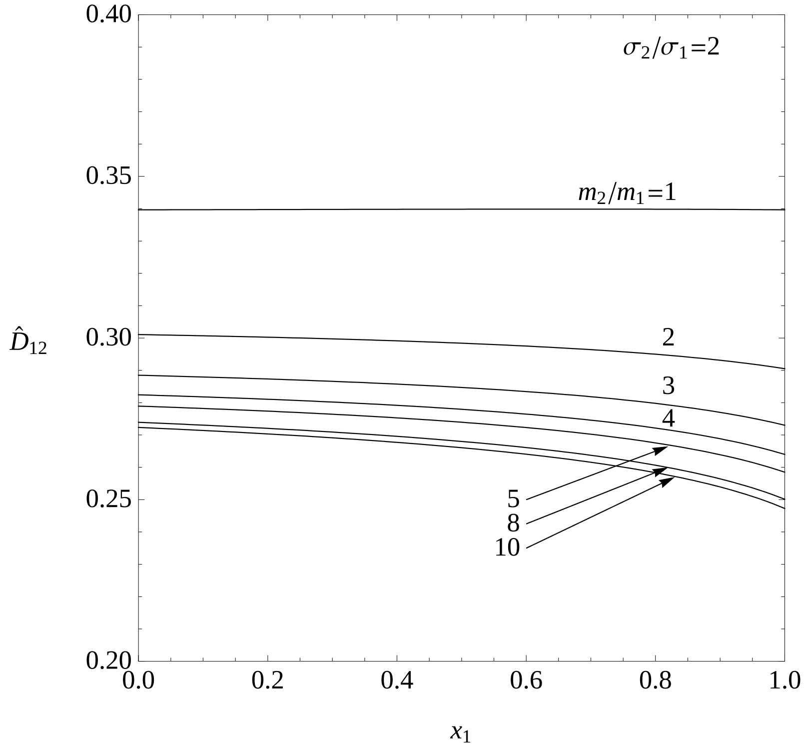

Figure 4.7: The $\hat{D}_{12}$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 2$.

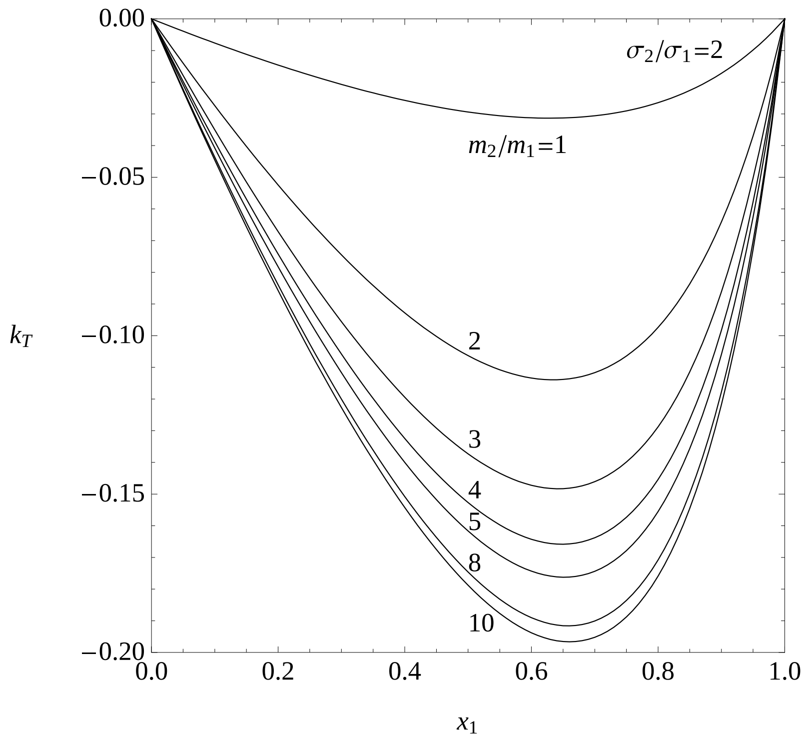

Figure 4.8: The k_T functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 2$.

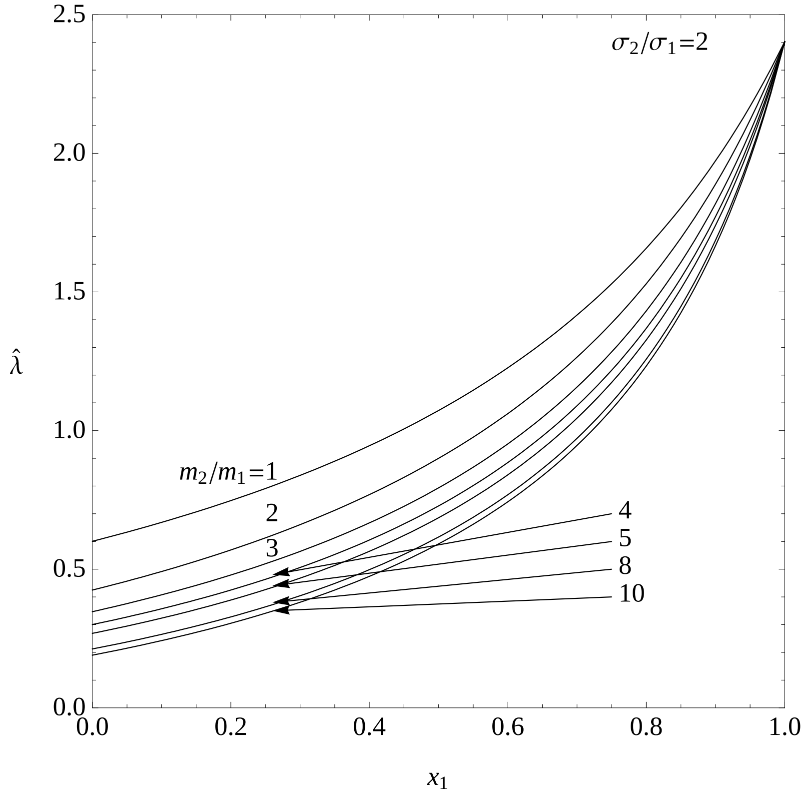

Figure 4.9: The $\hat{\lambda}$ functions calculated in this work for comparison with the values reported by Takata et al. [2] for $\sigma_2/\sigma_1 = 2$.

For the specific case of rigid-sphere molecules, one then uses Eqs. (4.154)-(4.156) for the omega integrals needed. Here, some representative order 70 results are reported for $[\lambda]_1$, $[\lambda]^{\star}_{70}$ (in Table 1), $a_1^{(1)}$, $a_{-1}^{(1)}$ (in Table 2), , $a_1^{(70)\star}$, $a_{-1}^{(70)\star}$ (in Table 3), $[D_{12}]_1$, $[D_{12}]^{\star}_{70}$ (in Table 4), $d_0^{(1)}$ (in Table 5), $[D_T]_1$,$[D_T]^{\star}_{70}$ (in Table 6), and $d_1^{(70)\star}$, $d_{-1}^{(70)\star}$ (in Table 7) for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe. The molecular mass and diameter values used in computing the results given in Tables 1-7 are summarized in Table 8.

Table 4.1: Order 1 values of the thermal conductivity coefficients, $[\lambda]_1$ and order 70, normalized values of the thermal conductivity coefficients, $[\lambda]^{\star}_{70}$, for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe as functions of the mixture fractions of the lighter constituents.

Mixture	x_1	$[\lambda]_1 \times 10^{-2}$ (erg cm^{-1} s^{-1} K^{-1})	$[\lambda]^{\star}_{70}$
He:Ne	10^{-100}	45.230993622931949432092	1.0252181683234523152732
	10^{-12}	45.230993622959561499966	1.0252181683234992998780
	10^{-9}	45.230993650544017344413	1.0252181683704369201471
	10^{-6}	45.231021235038011481513	1.0252182153080706373937
	10^{-3}	45.258643891588442209066	1.0252651662653715155855
	0.1	48.387938101542520763805	1.0299336122181635631348
	0.2	52.400524509330325834942	1.0342711214903130897172
	0.3	57.387988310715000729415	1.0377441997211695641340
	0.4	63.507872139255285497994	1.0400309786138353014717
	0.5	70.969261682002938249290	1.0409508018112383446086
	0.6	80.052478276483582685685	1.0404352117350897703442
	0.7	91.139095345248514789382	1.0385000964002596553389
	0.8	104.75907270125660531323	1.0352239452661135695676
	0.9	121.66747324170925742034	1.0307359485698434698333
	1-10^{-3}	142.73506214590358812824	1.0252776790844339602297
	1-10^{-6}	142.97459300150129227556	1.0252182278733048580938
	1-10^{-9}	142.97483283255399563591	1.0252181683830022068565
	1-10^{-12}	142.97483307238534888032	1.0252181683235118651648
	1-10^{-100}	142.97483307262542030529	1.0252181683234523152732
He:Ar	10^{-100}	16.256824822751970282052	1.0252181683234523152732
	10^{-12}	16.256824822782258518204	1.0252181683236561715647
	10^{-9}	16.256824853040206460143	1.0252181685273086064873
	10^{-6}	16.256855111014219845242	1.0252183721794181535275
	10^{-3}	16.287139175867276683665	1.0254216991572284250253
	0.1	19.567541902225391688039	1.0425705712840335307652
	0.2	23.539928793551250511756	1.0546112190247962792290
	0.3	28.361648838239252336184	1.0622186934333221356379
	0.4	34.296989480121495689575	1.0660150643963094597939

Continued on Next Page...

Table 4.1 – Continued

Mixture	x_1	$[\lambda]_1 \times 10^{-2}$ (erg cm^{-1} s^{-1} K^{-1})	$[\lambda]^{\star}_{70}$
	0.5	41.7302736989767121O3014	1.0664353047249363579152
	0.6	51.2425420642054177565578	1.0637717732775435996482
	0.7	63.7556083380921350017721	1.0582057049773514561943
	0.8	80.824394494466303843452	1.0498390730900356424888
	0.9	105.29247888187096812075	1.0387565130004953058065
	1-10^{-3}	142.49863276013777918019	1.0253633916545607905637
	1-10^{-6}	142.97435560731035884862	1.0252183136233177377633
	1-10^{-9}	142.97483259515883693443	1.0252181684687522569083
	1-10^{-12}	142.97483307214795372065	1.0252181683235976152149
	1-10^{-100}	142.97483307262542030529	1.0252181683234523152732
He:Kr	10^{-100}	8.5231373968991121279109	1.0252181683234523152732
	10^{-12}	8.5231373969264292591221	1.0252181683238784179016
	10^{-9}	8.5231374242162433616969	1.0252181687495549424323
	10^{-6}	8.5231647140528833641500	1.0252185944248187223073
	10^{-3}	8.5504771062300904651042	1.0256430118362083088582
	0.1	11.500121513991510257611	1.0575546356723539162786
	0.2	15.061120659414146582782	1.0756772661006479631344
	0.3	19.389887129167812189482	1.0850658945914598437344
	0.4	24.754821876384359333707	1.0884310310918071922230
	0.5	31.563620466317447480319	1.0871886867398702683191
	0.6	40.466352048246442605190	1.0820591608580884631148
	0.7	52.565641753550265656206	1.0733355640905965218240
	0.8	69.890789514293486821422	1.0610194419890890932426
	0.9	96.624397468245058785710	1.0449476081320856577175
	1-10^{-3}	142.34477685738630638930	1.0254296331624704143181
	1-10^{-6}	142.97420073031560194734	1.0252183798772315195570
	1-10^{-9}	142.97483244028081616119	1.0252181685350061825380
	1-10^{-12}	142.97483307199307569885	1.0252181683236638691405
	1-10^{-100}	142.97483307262542030529	1.0252181683234523152732
He:Xe	10^{-100}	4.9216330167821820114007	1.0252181683234523152732
	10^{-12}	4.9216330168049404976975	1.0252181683240983261697
	10^{-9}	4.9216330395406683274725	1.0252181689694632091047
	10^{-6}	4.9216557752877341925181	1.0252188143315938446937
	10^{-3}	4.9444107746153191591834	1.0258614343863045752151
	0.1	7.4076847095035792085458	1.0697183185162260545670
	0.2	10.398903341303874843029	1.0909460489534795397190
	0.3	14.064907578058101517237	1.1007028446396204705150
	0.4	18.660212299585055691285	1.1035361475640274222335
	0.5	24.584645450650940978795	1.1014261439483445862625
	0.6	32.503729602725701164060	1.0951928411297615309132
	0.7	43.612056052181092253450	1.0849874817193250325606
	0.8	60.286255627633844887813	1.0704532915363226700230
	0.9	88.018408944825273916226	1.0507810146504920943091
	1-10^{-3}	142.16078786736296942069	1.0254993341393084111803

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Table 4.1 – Continued

Mixture	x_1	$[\lambda]_1 \times 10^{-2}$ (erg cm^{-1} s^{-1} K^{-1})	$[\lambda]^\star_{70}$
	1-10^{-6}	142.97401506286978174515	1.0252184496491771301925
	1-10^{-9}	142.97483225461168087819	1.0252181686047781980601
	1-10^{-12}	142.97483307180740656188	1.0252181683237336411561
	1-10^{-100}	142.97483307262542030529	1.0252181683234523152732
Ne:Ar	10^{-100}	16.256824822751970282052	1.0252181683234523152732
	10^{-12}	16.256824822765709502436	1.0252181683234700489208
	10^{-9}	16.256824836491190673602	1.0252181683411859628887
	10^{-6}	16.256838561980024210522	1.0252181860570864391973
	10^{-3}	16.270571716978149590602	1.0252358884620685619511
	0.1	17.711441480628205101615	1.0268525698676256123799
	0.2	19.345192957126212032129	1.0281934714488914593858
	0.3	21.189377090608154056439	1.0292173802079323026769
	0.4	23.283083878352463674041	1.0299000812293699222141
	0.5	25.675780525893720612200	1.0302159491561190718612
	0.6	28.430998757976950359101	1.0301372619639991566582
	0.7	31.631714676888278022561	1.0296335518883510762993
	0.8	35.388402651809414235266	1.0286710816170667750579
	0.9	39.851453929045159259045	1.0272126482680727259898
	1-10^{-3}	45.171767024784271845005	1.0252409015804517672357
	1-10^{-6}	45.230934336090988009304	1.0252181910855254369741
	1-10^{-9}	45.230993563645048168086	1.0252181683462144172176
	1-10^{-12}	45.230993622872662530768	1.0252181683234750773751
	1-10^{-100}	45.230993622931949432092	1.0252181683234523152732
Ne:Kr	10^{-100}	8.5231373968991121279109	1.0252181683234523152732
	10^{-12}	8.5231373969097498650592	1.0252181683235235473900
	10^{-9}	8.5231374075368492850514	1.0252181683946844320102
	10^{-6}	8.5231480346451129255224	1.0252182395555126929495
	10^{-3}	8.5337839923515389852404	1.0252893439991010053224
	0.1	9.6816446629329232159696	1.0317538034457581865657
	0.2	11.058253391100742656625	1.0370345719623078520598
	0.3	12.706589402535584522006	1.0409660311232228314890
	0.4	14.699649191321615002990	1.0434768553555725049437
	0.5	17.139409031809005775083	1.0445071694319858267100
	0.6	20.172784347253153234791	1.0439985955546405953179
	0.7	24.019441412358268379554	1.0418865685412835500502
	0.8	29.023181278572522108607	1.0380969495951092042679
	0.9	35.754038985083526487585	1.0325535579969326603521
	1-10^{-3}	45.117319267409175033020	1.0253002118644787144220
	1-10^{-6}	45.230879720269855532863	1.0252182504515713424560
	1-10^{-9}	45.230993509029058582308	1.0252181684055805188295
	1-10^{-12}	45.230993622818046541014	1.0252181683235344434768
	1-10^{-100}	45.230993622931949432092	1.0252181683234523152732
Ne:Xe	10^{-100}	4.9216330167821820114007	1.0252181683234523152732

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Table 4.1 – Continued

Mixture	x_1	$[\lambda]_1 \times 10^{-2}$ (erg cm^{-1} s^{-1} K^{-1})	$[\lambda]^{\star}_{70}$
	10^{-12}	4.9216330167910753344736	1.0252181683236047194143
	10^{-9}	4.9216330256755050920458	1.0252181684758564561564
	10^{-6}	4.9216419101129537108420	1.0252183207273865272012
	10^{-3}	4.9305340445041817731054	1.0253703657116489256021
	0.1	5.8943304123945284371890	1.0385045278591932943301
	0.2	7.0637884852976528622716	1.0482664015249918505376
	0.3	8.4881803600790372362734	1.0549389819172681048420
	0.4	10.250696016066015082279	1.0588074265852574894028
	0.5	12.474578210118997342807	1.0600358675779038833495
	0.6	15.350481503466251519404	1.0586815011033986612090
	0.7	19.189801214506980256999	1.0546984476142186419322
	0.8	24.538136700576852048404	1.0479379936695128561528
	0.9	32.446305757994103126766	1.0381679767576370151470
	1-10^{-3}	45.064572534188571392425	1.0253627623476950184478
	1-10^{-6}	45.230826695312209880577	1.0252183130551717891806
	1-10^{-9}	45.230993456003821635549	1.0252181684681841721866
	1-10^{-12}	45.230993622765021303788	1.0252181683235970471302
	1-10^{-100}	45.230993622931949432092	1.0252181683234523152732
Ar:Kr	10^{-100}	8.5231373968991121279109	1.0252181683234523152732
	10^{-12}	8.5231373969029341875270	1.0252181683234604898855
	10^{-9}	8.5231374007211717466770	1.0252181683316269276020
	10^{-6}	8.5231412189614106637511	1.0252181764980666843098
	10^{-3}	8.5269621396743140484873	1.0252263449635295230630
	0.1	8.9329102909859355939048	1.0260419721827594171549
	0.2	9.4010877599567242765786	1.0268257874276644760644
	0.3	9.9333299875374637601988	1.0274987585311430258008
	0.4	10.536464898710989270822	1.0280006360151590403224
	0.5	11.218740263410843013411	1.0282812225666775709259
	0.6	11.990151844759242187381	1.0282996460522037112554
	0.7	12.862874418724216478170	1.0280235657720448760889
	0.8	13.851834117799969218978	1.0274284177249907781396
	0.9	14.975478121543663001491	1.0264968070980921392752
	1-10^{-3}	16.243141248290820560549	1.0252326904595083265659
	1-10^{-6}	16.256811129887982475329	1.0252181828631363017787
	1-10^{-9}	16.256824809059096999314	1.0252181683379920168076
	1-10^{-12}	16.256824822738277408760	1.0252181683234668549747
	1-10^{-100}	16.256824822751970282052	1.0252181683234523152732
Ar:Xe	10^{-100}	4.9216330167821820114007	1.0252181683234523152732
	10^{-12}	4.9216330167862284280765	1.0252181683234869262026
	10^{-9}	4.9216330208285986906293	1.0252181683580632446783
	10^{-6}	4.9216370632022921194114	1.0252182029343728047575
	10^{-3}	4.9256828694524455140116	1.0252527702850594798073
	0.1	5.3624023234483396520012	1.0285488780211025920802
	0.2	5.8834912127967103951688	1.0314708146480408198614

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Table 4.1 – Continued

Mixture	x_1	$[\lambda]_1 \times 10^{-2}$ (erg cm^{-1} s^{-1} K^{-1})	$[\lambda]^\star_{70}$
	0.3	6.4994915589469157390005	1.0337974098516583661098
	0.4	7.2292629515447565008896	1.0353850209689786282648
	0.5	8.0975603471613252400570	1.0361264682412749310272
	0.6	9.1374745762242457607419	1.0359443248669038204784
	0.7	10.394201411324449634717	1.0347850603709655614027
	0.8	11.931066234818637846176	1.0326152375828588130177
	0.9	13.839552617694560252194	1.0294215485323289035758
	1-10^{-3}	16.229515491701221314498	1.0252650502799158537620
	1-10^{-6}	16.256797477853908590549	1.0252182152530562002857
	1-10^{-9}	16.256824795407036609312	1.0252181683703819667894
	1-10^{-12}	16.256824822724625348344	1.0252181683234992449247
	1-10^{-100}	16.256824822751970282052	1.0252181683234523152732
Kr:Xe	10^{-100}	4.9216330167821820114007	1.0252181683234523152732
	10^{-12}	4.9216330167844884116317	1.0252181683234577755507
	10^{-9}	4.9216330190885822433317	1.0252181683289125928245
	10^{-6}	4.9216353231833094225720	1.0252181737837262608345
	10^{-3}	4.9239403136603149537849	1.0252236249894290960719
	0.1	5.1615166149119851359587	1.0257259902623583814072
	0.2	5.4210836674008305545319	1.0261492215722242203065
	0.3	5.7023202211719088194515	1.0264761637754997148820
	0.4	6.0075048950809476333028	1.0266958232092931551911
	0.5	6.3392640549323421272936	1.0267978427749659568629
	0.6	6.7006400046359049002837	1.0267724375701266740052
	0.7	7.0951759199421548309279	1.0266103384997581480921
	0.8	7.5270225226304726293178	1.0263027490834861141157
	0.9	8.0010732742195190087037	1.0258413224542803786291
	1-10^{-3}	8.5176573495455740352488	1.0252252251219746809347
	1-10^{-6}	8.5231319141135727158808	1.0252181753886992495033
	1-10^{-9}	8.5231373914163238490512	1.0252181683305175706570
	1-10^{-12}	8.5231373968936293396293	1.0252181683234593805286
	1-10^{-100}	8.5231373968991121279109	1.0252181683234523152732

Table 4.2: Order 1 values of the thermal conductivity related quantities, a_1 and a_{-1}, for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe as functions of the mixture fractions of the lighter constituents.

Mixture	x_1	$a_1^{(1)} \times 10^6$ (cm)	$a_{-1}^{(1)} \times 10^6$ (cm)
He:Ne	10^{-100}	$-15.235762082277183907967$	$-20.560567836453608501773$
	10^{-12}	$-15.235762082282910757147$	$-20.560567836452510851540$
	10^{-9}	$-15.235762088004033091935$	$-20.560567835355958272230$

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Table 4.2 – Continued

Mixture	x_1	$a_1^{(1)} \times 10^6$ (cm)	$a_{-1}^{(1)} \times 10^6$ (cm)
	10^{-6}	−15.235767809130253987916	−20.560566738806778066970
	10^{-3}	−15.241492823283853661098	−20.559473589249784694647
	0.1	−15.849285472703085686244	−20.485419586314615491397
	0.2	−16.553245389870214843796	−20.482507692957853305955
	0.3	−17.363463981769427446356	−20.557967623931769897757
	0.4	−18.300295079950876996768	−20.720579390524086569672
	0.5	−19.390261245686244642328	−20.982594289396504111620
	0.6	−20.668475917802252031174	−21.360981402079155359071
	0.7	−22.182323557218511221395	−21.879341994421661891460
	0.8	−23.997227989269715198034	−22.570928518511133383181
	0.9	−26.206031530731653628869	−23.483570041227100741296
	$1\text{-}10^{-3}$	−28.914309063342466166423	−24.674248929098008473390
	$1\text{-}10^{-6}$	−28.944894163038547987782	−24.688039283814368562201
	$1\text{-}10^{-9}$	−28.944924784469089457678	−24.688053093894623570332
	$1\text{-}10^{-12}$	−28.944924815090556372077	−24.688053107704723572918
	$1\text{-}10^{-100}$	−28.944924815121208491147	−24.688053107718547496864
He:Ar	10^{-100}	−8.4323761933253591554156	−10.397402827308215058740
	10^{-12}	−8.4323761933310914217505	−10.397402827311344442496
	10^{-9}	−8.4323761990576254944640	−10.397402830437598816961
	10^{-6}	−8.4323819255957932514409	−10.397405956693682862583
	10^{-3}	−8.4381125617645973432374	−10.400533923526317980475
	0.1	−9.0497831348698777801409	−10.728712359301581664450
	0.2	−9.7704620780345899163411	−11.102639591873315825266
	0.3	−10.622974627493155043775	−11.530417466566811180428
	0.4	−11.647502773190528988051	−12.027816742107354833584
	0.5	−12.902429339929236049241	−12.617713277298762831249
	0.6	−14.475961424479338553857	−13.334622552473305155498
	0.7	−16.507965933497788504917	−14.233217844375882064982
	0.8	−19.234280433993090239969	−15.405615796807564779150
	0.9	−23.086077638689591412750	−17.020137823383066122898
	$1\text{-}10^{-3}$	−28.871252488140866362500	−19.390899252221085359806
	$1\text{-}10^{-6}$	−28.944850950838639729143	−19.420776386431113933909
	$1\text{-}10^{-9}$	−28.944924741256733465538	−19.420806338612713724796
	$1\text{-}10^{-12}$	−28.944924815047344015929	−19.420806368564970567536
	$1\text{-}10^{-100}$	−28.944924815121208491147	−19.420806368594952806693
He:Kr	10^{-100}	−6.4236103707880117230623	−7.8951055393209531462766
	10^{-12}	−6.4236103707929621870928	−7.8951055393247607986081
	10^{-9}	−6.4236103757384757574486	−7.8951055431286054798546
	10^{-6}	−6.4236153212558858827084	−7.8951093469753387544248
	10^{-3}	−6.4285646814303818757128	−7.8989152469822635416820
	0.1	−6.9603421585728587656743	−8.2977474097886881544248
	0.2	−7.5958476688587996238944	−8.7503260274415129260526
	0.3	−8.3603033300904314097599	−9.2645372955727007805849

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Table 4.2 – Continued

Mixture	x_1	$a_1^{(1)} \times 10^6$ (cm)	$a_{-1}^{(1)} \times 10^6$ (cm)
	0.4	−9.2976057880465450121802	−9.8566582674279931013899
	0.5	−10.474180161884007673980	−10.550286267971477423592
	0.6	−11.995577994852039035705	−11.381429471936427427775
	0.7	−14.040293586592805681931	−12.408763866052933776804
	0.8	−16.935998577652846538887	−13.736562819065077602709
	0.9	−21.356690336198821618550	−15.573647048581970959196
	$1\text{-}10^{-3}$	−28.842196175955174114828	−18.381955737403055609076
	$1\text{-}10^{-6}$	−28.944821718258313120105	−18.418719118174360335215
	$1\text{-}10^{-9}$	−28.944924712023976049107	−18.418755994957361669721
	$1\text{-}10^{-12}$	−28.944924815018111258336	−18.418756031834258468844
	$1\text{-}10^{-100}$	−28.944924815121208491147	−18.418756031871172279568
He:Xe	10^{-100}	−4.9995786142542142848018	−5.7065238518632256560958
	10^{-12}	−4.9995786142583368968532	−5.7065238518666860743622
	10^{-9}	−4.9995786183768263396286	−5.7065238553236439246953
	10^{-6}	−4.9995827368696712091289	−5.7065273122836672308411
	10^{-3}	−5.0037046346245612308914	−5.7099864467041746748616
	0.1	−5.4489654105839980495054	−6.0758466337510474738419
	0.2	−5.9873226115769091923634	−6.4988297001281620310023
	0.3	−6.6440247940744701845761	−6.9889418372961812327820
	0.4	−7.4629944596769687719136	−7.5649850437556362081617
	0.5	−8.5129895533409595520534	−8.2542918083046264051475
	0.6	−9.9080644914865578026027	−9.0987581436560020617886
	0.7	−11.852263720875267835240	−10.167389509661436914233
	0.8	−14.750278683598265835360	−11.586040751847823852759
	0.9	−19.534749737203017957673	−13.622308244103636492321
	$1\text{-}10^{-3}$	−28.805967508064669674105	−16.962544785711679645108
	$1\text{-}10^{-6}$	−28.944785185954389988089	−17.008726091330570396601
	$1\text{-}10^{-9}$	−28.944924675491366558494	−17.008772453199577640871
	$1\text{-}10^{-12}$	−28.944924814981578648540	−17.008772499561628034430
	$1\text{-}10^{-100}$	−28.944924815121208491147	−17.008772499608036493464
Ne:Ar	10^{-100}	−11.278569674291800245673	−10.397402827308215058740
	10^{-12}	−11.278569674296427070979	−10.397402827311530872247
	10^{-9}	−11.278569678918625553815	−10.397402830624028568129
	10^{-6}	−11.278574301119442891044	−10.397406143123434209442
	10^{-3}	−11.283198837600027865677	−10.400720353139180159597
	0.1	−11.765844845579760833382	−10.746994314255090697843
	0.2	−12.307744311033637556583	−11.136584180121064814218
	0.3	−12.913860377980460475746	−11.573188337447615657630
	0.4	−13.596172877788376751166	−12.065569065724198878882
	0.5	−14.369841238085014142912	−12.624814374960686809407
	0.6	−15.254334293300828485683	−13.265164408995026769650
	0.7	−16.275078699973845245203	−14.005217183845738207578
	0.8	−17.465926688192154281350	−14.869733631996836354835

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Table 4.2 – Continued

Mixture	x_1	$a_1^{(1)} \times 10^6$ (cm)	$a_{-1}^{(1)} \times 10^6$ (cm)
	0.9	−18.872961184789427137348	−15.892420871354822097155
	1-10^{-3}	−20.542028825568351516667	−17.106877514578339270820
	1-10^{-6}	−20.560549278996124803739	−17.120360095516103671634
	1-10^{-9}	−20.560567817896132553163	−17.120373591591834993868
	1-10^{-12}	−20.560567836435051025806	−17.120373605087924233380
	1-10^{-100}	−20.560567836453608501773	−17.120373605101433832232
Ne:Kr	10^{-100}	−7.6363531059823579248179	−7.8951055393209531462766
	10^{-12}	−7.6363531059866818450029	−7.8951055393231408560234
	10^{-9}	−7.6363531103062781126590	−7.8951055415086628947279
	10^{-6}	−7.6363574299054409249450	−7.8951077270322766918259
	10^{-3}	−7.6406799261444439733855	−7.8972948267788117457903
	0.1	−8.0997827393270943273310	−8.1307575358357003282027
	0.2	−8.6347741283055621065517	−8.4053106528001039118828
	0.3	−9.2590913255519105909786	−8.7283885085970150107962
	0.4	−9.9969120798985402418962	−9.1130899049884847827577
	0.5	−10.882009511840432735664	−9.5777127663331974534090
	0.6	−11.963036424830443512359	−10.148617358255373686608
	0.7	−13.312734044410849797995	−10.865215894489722195435
	0.8	−15.044945759179923940236	−11.789191158569017289659
	0.9	−17.348423206875718301086	−13.022813787899885084459
	1-10^{-3}	−20.522189730117384920954	−14.728246477554520712732
	1-10^{-6}	−20.560529382781939616877	−14.748877261394815254149
	1-10^{-9}	−20.560567797999861118782	−14.748897933128143494481
	1-10^{-12}	−20.560567836415154754315	−14.748897953799917852820
	1-10^{-100}	−20.560567836453608501773	−14.748897953820610319686
Ne:Xe	10^{-100}	−5.4720824083065644222746	−5.7065238518632256560958
	10^{-12}	−5.4720824083103660634244	−5.7065238518652860191550
	10^{-9}	−5.4720824121082055749368	−5.7065238539235887165712
	10^{-6}	−5.4720862099505327745296	−5.7065259122275747948506
	10^{-3}	−5.4758868700761456446901	−5.7085855058141620135230
	0.1	−5.8827146795569607295900	−5.9264742056499951673442
	0.2	−6.3650285191824847675999	−6.1790243293336718962709
	0.3	−6.9398256012346274158675	−6.4734036672564679278856
	0.4	−7.6368325548298040000293	−6.8227891562343736252111
	0.5	−8.5000637322958383497389	−7.2466763175902791967620
	0.6	−9.5975790619347496365833	−7.7751935180136651718561
	0.7	−11.040504167154340729277	−8.4575112858294888041802
	0.8	−13.023497591753890053528	−9.3797338636491877774057
	0.9	−15.921409127954569084559	−10.707634067865405585515
	1-10^{-3}	−20.500408371444762677280	−12.779512828355054881396
	1-10^{-6}	−20.560507496227806083556	−12.806568545877326680384
	1-10^{-9}	−20.560567776113201394297	−12.806595681576273308540
	1-10^{-12}	−20.560567836393268094485	−12.806595708712052477232

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Table 4.2 – Continued

Mixture	x_1	$a_1^{(1)} \times 10^6$ (cm)	$a_{-1}^{(1)} \times 10^6$ (cm)
	1-10^{-100}	−20.5605678364536085 01773	−12.8065957087392154 19423
Ar:Kr	10^{-100}	−7.4818474741237029603714	−7.8951055393209531462766
	10^{-12}	−7.4818474741252781462408	−7.8951055393215524524053
	10^{-9}	−7.4818474756988888306684	−7.8951055399202592757971
	10^{-6}	−7.4818490493104672297140	−7.8951061386278845657237
	10^{-3}	−7.4834235551235611362844	−7.8957056483130435568287
	0.1	−7.6485902786934903311249	−7.9632347636237217268826
	0.2	−7.8349878353158002011363	−8.0485309780164206407939
	0.3	−8.0431114258673535595901	−8.1523559374172696018765
	0.4	−8.2754394056072791193732	−8.2763890654171941217355
	0.5	−8.5349470601747909635337	−8.4226924554273676363702
	0.6	−8.8252232839475435883082	−8.5937959922640362978707
	0.7	−9.1506235836326133702135	−8.7928096162737745099922
	0.8	−9.5164730222820670046879	−9.0235728084330363076675
	0.9	−9.9293389338326755124480	−9.2908559914623257257164
	1-10^{-3}	−10.392417184214444186475	−9.5973042214073287074964
	1-10^{-6}	−10.397397838406575321548	−9.6006323280668454416518
	1-10^{-9}	−10.397402822319310158539	−9.6006356586639083546186
	1-10^{-12}	−10.397402827303226153836	−9.6006356619945079093442
	1-10^{-100}	−10.397402827308215058740	−9.6006356619978418428349
Ar:Xe	10^{-100}	−5.2619169547344416197986	−5.7065238518632256560958
	10^{-12}	−5.2619169547366104652538	−5.7065238518640845935197
	10^{-9}	−5.2619169569032870763598	−5.7065238527221630809237
	10^{-6}	−5.2619191235812403454713	−5.7065247108015586265068
	10^{-3}	−5.2640871444160887618459	−5.7073836987615050540377
	0.1	−5.4929980379252181377905	−5.8019575206202995458849
	0.2	−5.7559022797220877600666	−5.9185059504163186516992
	0.3	−6.0568113199472182397758	−6.0598601974930554365307
	0.4	−6.4036818694714971740300	−6.2308027562305588306977
	0.5	−6.8069256597050759336150	−6.4376227476122423480447
	0.6	−7.2804286351167788473440	−6.6887385726415273079721
	0.7	−7.8431246584790138290339	−6.9956596392619818189366
	0.8	−8.5215106938464042986770	−7.3745237457600012775752
	0.9	−9.3538332436019900995030	−7.8486562973754484222293
	1-10^{-3}	−10.385663270953160019587	−8.4461957301908199751629
	1-10^{-6}	−10.397391072959357326807	−8.4530351690218901168399
	1-10^{-9}	−10.397402815553851390114	−8.4530420175570447495306
	1-10^{-12}	−10.397402827296460695056	−8.4530420244055890117226
	1-10^{-100}	−10.397402827308215058740	−8.4530420244124444113935
Kr:Xe	10^{-100}	−5.8742656972102459407328	−5.7065238518632256560958
	10^{-12}	−5.8742656972115813664334	−5.7065238518642535185051
	10^{-9}	−5.8742656985456716417796	−5.7065238528910880657978
	10^{-6}	−5.8742670326364177188001	−5.7065248797260305866633

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Table 4.2 – Continued

Mixture	x_1	$a_1^{(1)} \times 10^6$ (cm)	$a_{-1}^{(1)} \times 10^6$ (cm)
	10^{-3}	−5.8756015942589678065124	−5.7075521100164137746613
	0.1	−6.0126708821911448642501	−5.8133901134098492922604
	0.2	−6.1614448705610871210206	−5.9289456012444536681961
	0.3	−6.3216554985825581568541	−6.0540687951402705482832
	0.4	−6.4945271177771954731887	−6.1897675231965742449094
	0.5	−6.6814701455741088264609	−6.3372033578847108401428
	0.6	−6.8841175853785941048957	−6.4977217671356540152647
	0.7	−7.1043705114678023232609	−6.6728896681092584688022
	0.8	−7.3444551949324545797838	−6.8645425934489220393628
	0.9	−7.6069955006558297700437	−7.0748444673779140409638
	1-10^{-3}	−7.8920862166319807651401	−7.3039342695132394396857
	1-10^{-6}	−7.8951025185388736861725	−7.3063616759756412738790
	1-10^{-9}	−7.8951055363001696067498	−7.3063641045922375344366
	1-10^{-12}	−7.8951055393179323627356	−7.3063641070208553413890
	1-10^{-100}	−7.8951055393209531462766	−7.3063641070232863902460

Table 4.3: Normalized order 70 values of the thermal conductivity coefficient related quantities, $a_1^{(70)\star}$ and $a_{-1}^{(70)\star}$, for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe as functions of the mixture fractions of the lighter constituents.

Mixture	x_1	$a_1^{(70)\star}$	$a_{-1}^{(70)\star}$
He:Ne	10^{-100}	1.0872730381181951998864	1.0252181683234523152732
	10^{-12}	1.0872730381181435134042	1.0252181683233960496321
	10^{-9}	1.0872730380665087176882	1.0252181682671866741921
	10^{-6}	1.0872729864317084099620	1.0252181120578412679165
	10^{-3}	1.0872213470300225895264	1.0251619327268161039715
	0.1	1.0820491736037352522999	1.0198741240580900360365
	0.2	1.0766814477738833850159	1.0150314435617134428974
	0.3	1.0711266832558147250976	1.0106120503297540613266
	0.4	1.0653496646035295659507	1.0065581626514015571345
	0.5	1.0593224379890634405206	1.0028303128966243620649
	0.6	1.0530248668697698873245	0.99940738688503999148060
	0.7	1.0464468636241766023130	0.99628904158546090092218
	0.8	1.0395932460769784528097	0.99350150152838712071923
	0.9	1.0324931689235703671378	0.99110887309106257989724
	1-10^{-3}	1.0252913852507034406367	0.98924998139293332800182
	1-10^{-6}	1.0252182415422762545843	0.98923443690143894544410
	1-10^{-9}	1.0252181683966711410760	0.98923442139322123322178
	1-10^{-12}	1.0252181683235255340990	0.98923442137771305183096
	1-10^{-100}	1.0252181683234523152732	0.98923442137769752812590

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Table 4.3 – Continued

Mixture	x_1	$a_1^{(70)\star}$	$a_{-1}^{(70)\star}$
He:Ar	10^{-100}	1.1246786406007335143153	1.0252181683234523152732
	10^{-12}	1.1246786406006701074103	1.0252181683234013407534
	10^{-9}	1.1246786405373266092694	1.0252181682724777954824
	10^{-6}	1.1246785771938095349215	1.0252181173489402139158
	10^{-3}	1.1246152147323287631518	1.0251672014989919891018
	0.1	1.1181370582753568669298	1.0201960573132133428433
	0.2	1.1111443026133668172420	1.0153188819417827259622
	0.3	1.1036160984818831341805	1.0105801762743270389436
	0.4	1.0954518501239807139867	1.0059777213665793032351
	0.5	1.0865308760199950868443	1.0015169493799491926096
	0.6	1.0767088593453272501498	0.9972182980488748739083
	0.7	1.0658168822635821228506	0.9931335173272011791621 2
	0.8	1.0536708506370629141863	0.9893838988516614499424 4
	0.9	1.0401177709862649927974	0.9862580793555724560418 2
	1-10^{-3}	1.0253720873581989609769	0.9845019006538542528146 0
	1-10^{-6}	1.0252183222720458698199	0.9844959612479851970720 6
	1-10^{-9}	1.0252181684774009380899	0.9844959554675217849705 6
	1-10^{-12}	1.0252181683236062638960	0.9844959554617414810404 2
	1-10^{-100}	1.0252181683234523152732	0.9844959554617356949505 6
He:Kr	10^{-100}	1.1503483128349767016658	1.0252181683234523152732
	10^{-12}	1.1503483128349034898519	1.0252181683234125864371
	10^{-9}	1.1503483127617648877133	1.0252181682837234791772
	10^{-6}	1.1503482396231392264108	1.0252181285946128701169
	10^{-3}	1.1502750774596285027042	1.0251784361346283416015
	0.1	1.1427758039023226716517	1.0212116188191119600021
	0.2	1.1346308375906564585308	1.0171379339144574419631
	0.3	1.1257912716051758577229	1.0129995901852534725038
	0.4	1.1161048123478322696479	1.0088033082426836811597
	0.5	1.1053778661295186316289	1.0045631060246112958264
	0.6	1.0933607214406165760273	1.0003069198342868772880
	0.7	1.0797298075131560120893	0.9960925445996921064403 7
	0.8	1.0640758462852161639507	0.9920512013381457431237 0
	0.9	1.0459466997980809258315	0.9885270872701063920527 2
	1-10^{-3}	1.0254350423202215142279	0.9866344891235895541956 6
	1-10^{-6}	1.0252183852434649055998	0.9866326962385262207567 0
	1-10^{-9}	1.0252181685403723730335	0.9866326947077678262258 4
	1-10^{-12}	1.0252181683236692353310	0.9866326947062373312481 5
	1-10^{-100}	1.0252181683234523152732	0.9866326947062357992214 1
He:Xe	10^{-100}	1.1600604943154697563118	1.0252181683234523152732
	10^{-12}	1.1600604943154041605928	1.0252181683234217174629
	10^{-9}	1.1600604942498740373000	1.0252181682928545049959
	10^{-6}	1.1600604287197234051675	1.0252181377256334210342
	10^{-3}	1.1599948712122167388828	1.0251875618851580944861

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Table 4.3 – Continued

Mixture	x_1	$a_1^{(70)\star}$	$a_{-1}^{(70)\star}$
	0.1	1.1532086699382970544808	1.0220696675084938584613
	0.2	1.1456884397981849756138	1.0187338026571338247504
	0.3	1.1373473217269767208485	1.0151959481886851318287
	0.4	1.1279852846872190693664	1.0114430873481564263156
	0.5	1.1173331283160122966834	1.0074668556459334244195
	0.6	1.1050187845418018023967	1.0032703461971770278517
	0.7	1.0905145920161403317009	0.9988848663812942020469 8
	0.8	1.0730602838006018835117	0.9944181270567175648331 0
	0.9	1.0516001920064291917283	0.9902291977806969097864 9
	1-10^{-3}	1.0255032106502240916017	0.9878335042602747379444 2
	1-10^{-6}	1.0252184534849037797964	0.9878336255971384606073 6
	1-10^{-9}	1.0252181686086138839574	0.9878336261373274986014 5
	1-10^{-12}	1.0252181683237374768420	0.9878336261378681093283 4
	1-10^{-100}	1.0252181683234523152732	0.9878336261378686504806 4
Ne:Ar	10^{-100}	1.0511849005743001404009	1.0252181683234523152732
	10^{-12}	1.0511849005742811495837	1.0252181683234304177754
	10^{-9}	1.0511849005553093231415	1.0252181683015548175033
	10^{-6}	1.0511848815834785473563	1.0252181464259538404935
	10^{-3}	1.0511659054171156347499	1.0251962701195499193437
	0.1	1.0492404315562688547083	1.0230208268773704010553
	0.2	1.0471966892507730380465	1.0208062046614268618256
	0.3	1.0450397829003468895067	1.0185713208849180743994
	0.4	1.0427541353805882151620	1.0163135715653145370794
	0.5	1.0403222810501080413347	1.0140310240938126932874
	0.6	1.0377246925279724722234	1.0117228956134842380588
	0.7	1.0349396984806199935670	1.0093903388093634527008
	0.8	1.0319436197075000280933	1.0070377546951421027383
	0.9	1.0287113846178620475448	1.0046750358043694053391
	1-10^{-3}	1.0252544669421325526147	1.0023449012875361237734
	1-10^{-6}	1.0252182046361879581283	1.0023215339660112140325
	1-10^{-9}	1.0252181683597650650359	1.0023215106008177036756
	1-10^{-12}	1.0252181683234886280229	1.0023215105774525123010
	1-10^{-100}	1.0252181683234523152732	1.0023215105774291237211
Ne:Kr	10^{-100}	1.0834970858539689321932	1.0252181683234523152732
	10^{-12}	1.0834970858539338422908	1.0252181683234086794023
	10^{-9}	1.0834970858188790297823	1.0252181682798164444505
	10^{-6}	1.0834970507640544322024	1.0252181246875865529654
	10^{-3}	1.0834619838428602411258	1.0251745375159390569475
	0.1	1.0798593318451112648560	1.0209031894616197622739
	0.2	1.0759307850795801279269	1.0166780294697338293328
	0.3	1.0716557746400348147475	1.0125338454585848687219
	0.4	1.0669695701641027957819	1.0084660872191902251257
	0.5	1.0617965303722640538631	1.0044771586048062098147

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Table 4.3 – Continued

Mixture	x_1	$a_1^{(70)\star}$	$a_{-1}^{(70)\star}$
	0.6	1.0560483353527281298357	1.0005812819281029839658
	0.7	1.0496231645165848694475	0.99681371307870693217135
	0.8	1.0424083202537382342370	0.99324911861411916536843
	0.9	1.0342936077679115819449	0.99004084396319170086428
	$1\text{-}10^{-3}$	1.0253135261395093456095	0.98753298118065695024068
	$1\text{-}10^{-6}$	1.0252182637245393961487	0.98751294878502560889172
	$1\text{-}10^{-9}$	1.0252181684188534455741	0.98751292881927830419322
	$1\text{-}10^{-12}$	1.0252181683235477164035	0.98751292879931262370803
	$1\text{-}10^{-100}$	1.0252181683234523152732	0.98751292879929263804194
Ne:Xe	10^{-100}	1.1069442666657445059470	1.0252181683234523152732
	10^{-12}	1.1069442666657026283857	1.0252181683234048229137
	10^{-9}	1.1069442666238669446346	1.0252181682759599557717
	10^{-6}	1.1069442247881645174084	1.0252181208310945356915
	10^{-3}	1.1069023703986860004890	1.0251706776862948248048
	0.1	1.1025580717125662125460	1.0204849143062418182369
	0.2	1.0977238100945130293158	1.0157789797086240241436
	0.3	1.0923508247729991721718	1.0110948357866456113386
	0.4	1.0863253317536433674199	1.0064306386127324797770
	0.5	1.0795029498732825055427	1.0017923472389666478727
	0.6	1.0716988468471293416856	0.99720238378238453685954
	0.7	1.0626758782500665472169	0.99271873257619682214229
	0.8	1.0521350805908460982369	0.98848023882267819845147
	0.9	1.0397306735775055347933	0.98482605566575273007810
	$1\text{-}10^{-3}$	1.0253732060895206825256	0.98267206364278991840615
	$1\text{-}10^{-6}$	1.0252183234466689239379	0.98266413828441624981952
	$1\text{-}10^{-9}$	1.0252181684785756170737	0.98266413055168090787292
	$1\text{-}10^{-12}$	1.0252181683236074385750	0.98266413054394836588636
	$1\text{-}10^{-100}$	1.0252181683234523152732	0.98266413054394062560428
Ar:Kr	10^{-100}	1.0499416928603626880016	1.0252181683234523152732
	10^{-12}	1.0499416928603408062417	1.0252181683234265562337
	10^{-9}	1.0499416928384809281629	1.0252181682976932757685
	10^{-6}	1.0499416709786017725410	1.0252181425644175992581
	10^{-3}	1.0499198100203548164954	1.0251924140751367712450
	0.1	1.0477404842606796749271	1.0226880836183203723558
	0.2	1.0455046722798258147883	1.0202419065248053380918
	0.3	1.0432224483751893324191	1.0178696087605757580455
	0.4	1.0408832360879244358038	1.0155631748808700746587
	0.5	1.0384776279886286895840	1.0133165820087948663588
	0.6	1.0359974042848560268605	1.0111258818999988218598
	0.7	1.0334356506346579820971	1.0089894060119765025120
	0.8	1.0307870053788668738136	1.0069081284583720468261
	0.9	1.0280480852908589567710	1.0048862426782185168334
	$1\text{-}10^{-3}$	1.0252469118477304239615	1.0029512072391133774967

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Table 4.3 – Continued

Mixture	x_1	$a_1^{(70)\star}$	$a_{-1}^{(70)\star}$
	1-10^{-6}	1.0252181970713945653581	1.0029320590815687625233
	1-10^{-9}	1.0252181683522002619404	1.0029320399374436795476
	1-10^{-12}	1.0252181683234810632198	1.0029320399182995584999
	1-10^{-100}	1.0252181683234523152732	1.0029320399182803952155
Ar:Xe	10^{-100}	1.0700201810234063523819	1.0252181683234523152732
	10^{-12}	1.0700201810233736915572	1.0252181683234120328723
	10^{-9}	1.0700201809907455276399	1.0252181682831699143855
	10^{-6}	1.0700201483625750635481	1.0252181280410604589834
	10^{-3}	1.0699875136396837808084	1.0251778949589419211223
	0.1	1.0666829633245630399912	1.0212764574138503188608
	0.2	1.0631810615562466889714	1.0174938064079498908608
	0.3	1.0594806562510814286273	1.0138526946955889653890
	0.4	1.0555474343519695013500	1.0103406420510440749626
	0.5	1.0513465204139416131429	1.0069511977512331724299
	0.6	1.0468428429681176301363	1.0036857853293353794902
	0.7	1.0420023195974687457325	1.0005569596542727333786
	0.8	1.0367946382409654692024	0.99759412988816666863544
	0.9	1.0311992575538279809806	0.99485382632914106647990
	1-10^{-3}	1.0252797807776680327943	0.99246086381236463515927
	1-10^{-6}	1.0252182299530063074141	0.99243886089915602512765
	1-10^{-9}	1.0252181683850818863507	0.99243883892114730149299
	1-10^{-12}	1.0252181683235139448443	0.99243883889916931771390
	1-10^{-100}	1.0252181683234523152732	0.99243883889914731773016
Kr:Xe	10^{-100}	1.0409279615964258769814	1.0252181683234523152732
	10^{-12}	1.0409279615964119936008	1.0252181683234375334128
	10^{-9}	1.0409279615825424962921	1.0252181683086704549553
	10^{-6}	1.0409279477130438960172	1.0252181535415921143916
	10^{-3}	1.0409140769222691362274	1.0252033865799581220083
	0.1	1.0395261683042128121407	1.0237408824255047782942
	0.2	1.0380953482158022889977	1.0222643543166986275567
	0.3	1.0366322984109017011181	1.0207871223420528618654
	0.4	1.0351337834347892040098	1.0193078830244211399245
	0.5	1.0335965476291205115021	1.0178255272925145258106
	0.6	1.0320173365526416378808	1.0163391899441820818645
	0.7	1.0303929311197089008589	1.0148483168561982709987
	0.8	1.0287201996203858479440	1.0133527562223649574479
	0.9	1.0269961750289966695083	1.0118528826549812676516
	1-10^{-3}	1.0252362234477554823463	1.0103648082707662297271
	1-10^{-6}	1.0252181863813876666944	1.0103497817565262524075
	1-10^{-9}	1.0252181683415102534360	1.0103497667299443711960
	1-10^{-12}	1.0252181683234703732113	1.0103497667149177892476
	1-10^{-100}	1.0252181683234523152732	1.0103497667149027476240

Table 4.4: Order 1 and normalized order 70 values of the diffusion coefficients, $[D_{12}]_1$ and $[D_{12}]^\star_{70}$, for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe as functions of the mixture fractions of the lighter constituents.

Mixture	x_1	$[D_{12}]_1 \times 10^1$ (cm^2 s^{-1})	$[D_{12}]^\star_{70}$
He:Ne	10^{-100}	8.4763042862904107387921	1.0183168264410251895622
	10^{-12}	8.4763042862901931669741	1.0183168264410047015490
	10^{-9}	8.4763042860728389206356	1.0183168264205371764332
	10^{-6}	8.4763040687184585154558	1.0183168059530133867482
	10^{-3}	8.4760865802107368829116	1.0182963397550554038402
	0.1	8.4531409327259520842838	1.0162810210604553212885
	0.2	8.4268774935701213553319	1.0142707324666846504787
	0.3	8.3970002169441953738709	1.0122868360732286968016
	0.4	8.3628585556154733300392	1.0103326982685724016705
	0.5	8.3236193007673914461865	1.0084148710654886633535
	0.6	8.2782004116703452536404	1.0065441532326442371686
	0.7	8.2251733349379864053953	1.0047371827722611333787
	0.8	8.1626149055646097075148	1.0030188170227351023179
	0.9	8.0878757140117152567131	1.0014256873072235915472
	1-10^{-3}	7.9982060829641790033697	1.0000244973007121061659
	1-10^{-6}	7.9972053694656941375862	1.0000114859182961044521
	1-10^{-9}	7.9972043676947105927527	1.0000114729196038577973
	1-10^{-12}	7.9972043666929385506321	1.0000114729066051782563
	1-10^{-100}	7.9972043666919357758141	1.0000114729065921665650
He:Ar	10^{-100}	5.5081421910969419213005	1.0276991471770545066307
	10^{-12}	5.5081421910968357900080	1.0276991471770320333917
	10^{-9}	5.5081421909908106287100	1.0276991471545812676404
	10^{-6}	5.5081420849655720494718	1.0276991247038110332178

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Table 4.4 – Continued

Mixture	x_1	$[D_{12}]_1 \times 10^1$ (cm^2 s^{-1})	$[D_{12}]^\star_{70}$
	10^{-3}	5.508035982390998733822 4	1.0276766694485398540667
	0.1	5.4966966008444349281257	1.0254049904728501454123
	0.2	5.4833120202466383740404	1.0230092014092613142592
	0.3	5.4674673638193076080134	1.0204996876698305021743
	0.4	5.4484375250491467956195	1.0178647620804553660568
	0.5	5.4251827795727345402285	1.0150946599494570970895
	0.6	5.3961575055509437075705	1.0121848343357205178638
	0.7	5.3589600474783232417197	1.0091430858738213422810
	0.8	5.3096450802455310227565	1.0060053800336177258716
	0.9	5.2412462929357231707512	1.0028725699954269374287
	$1\text{-}10^{-3}$	5.1414489800560342565034	1.0000267165334709567652
	$1\text{-}10^{-6}$	5.1402021985328269393394	1.0000000911918490880367 3
	$1\text{-}10^{-9}$	5.1402009487862549630523	1.0000000886155339534988 8
	$1\text{-}10^{-12}$	5.1402009475365054190069	1.0000000886129576425260 5
	$1\text{-}10^{-100}$	5.1402009475352544184593	1.0000000886129550636361 9
He:Kr	10^{-100}	4.5306839945775141716672	1.0352324218241692533825
	10^{-12}	4.5306839945774379824594	1.0352324218241437907153
	10^{-9}	4.5306839945013249638514	1.0352324217987065861326
	10^{-6}	4.5306839183882477336115	1.0352323963614957403356
	10^{-3}	4.5306077466437591665714	1.0352069528838151493088
	0.1	4.5224294706644430828242	1.0326195960855772365625
	0.2	4.5126732316230936372122	1.0298573546739326738725
	0.3	4.5009663047801235719219	1.0269191328237492534774
	0.4	4.4866613488719992222545	1.0237753270927140609870
	0.5	4.4687896214464131817006	1.0203937309226589043885
	0.6	4.4458334607219451188444	1.0167420028902002829000

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Table 4.4 – Continued

Mixture	x_1	$[D_{12}]_1 \times 10^1$ (cm^2 s^{-1})	$[D_{12}]^{\star}_{70}$
	0.7	4.4152737153708521473472	1.0127956318669021361727
	0.8	4.3726011924052385896215	1.0085614270339883835165
	0.9	4.3088695620171097665509	1.0041483867252117467250
	1-10^{-3}	4.2048545830459509390522	1.0000367489069942793940
	1-10^{-6}	4.2034620930733688049443	1.0000000864789661374503
	1-10^{-9}	4.2034606960361018900969	1.0000000498959049427857
	1-10^{-12}	4.2034606946390600610566	1.0000000498593219613757
	1-10^{-100}	4.2034606946376616207827	1.0000000498592853417748
He:Xe	10^{-100}	3.6157493449606025456659	1.0382919333040952019429
	10^{-12}	3.6157493449605533046527	1.0382919333040723355411
	10^{-9}	3.6157493449113615324083	1.0382919332812288000784
	10^{-6}	3.6157492957195491076837	1.0382919104376852094024
	10^{-3}	3.6157000636939913897324	1.0382690587612933982572
	0.1	3.6103872901727786145158	1.0359188682522738769257
	0.2	3.6039784536533104914039	1.0333506360869783803579
	0.3	3.5961834647925582536784	1.0305482693651365020972
	0.4	3.5864985888342557930929	1.0274637424145147932546
	0.5	3.5741425664176804106484	1.0240375819521488243020
	0.6	3.5578350699673838735157	1.0201969832418349789004
	0.7	3.5353226939598305423751	1.0158577081444922393307
	0.8	3.5022382087726656527899	1.0109426497327035504026
	0.9	3.4488641724603856983048	1.0054718673523624506821
	1-10^{-3}	3.3497713350520273034966	1.0000486313851520143937
	1-10^{-6}	3.3483298348705944357267	1.0000000569977423060979
	1-10^{-9}	3.3483283870060813690063	1.0000000085509241046406
	1-10^{-12}	3.3483283855582104635234	1.0000000085024774149925

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Table 4.4 – Continued

Mixture	x_1	$[D_{12}]_1 \times 10^1$ (cm^2 s^{-1})	$[D_{12}]^\star_{70}$
	1-10^{-100}	3.3483283855567611432913	1.0000000085024289198078
Ne:Ar	10^{-100}	2.4227813806350481562314	1.0062522721826468862490
	10^{-12}	2.4227813806350152192704	1.0062522721826422164967
	10^{-9}	2.4227813806021111952472	1.0062522721779771338901
	10^{-6}	2.4227813476980680012377	1.0062522675128933582411
	10^{-3}	2.4227484244749197300241	1.0062476012598592175778
	0.1	2.4192861674295634262605	1.0057733721101207241714
	0.2	2.4153453450163359123453	1.0052697314261031008192
	0.3	2.4108842485460844608998	1.0047400410478414128059
	0.4	2.4058103696003697360989	1.0041830960307227267830
	0.5	2.4000076900221865959484	1.0035979230487959601402
	0.6	2.3933287101359966110835	1.0029839853027986983600
	0.7	2.3855829971953904334319	1.0023415123293584777337
	0.8	2.3765203428992054154501	1.0016720343864484973468
	0.9	2.3658053170877858057442	1.0009792579461158030719
	1-10^{-3}	2.3531182791363715129719	1.0002776505333489962426
	1-10^{-6}	2.3529777535143478428812	1.0002705286485104167135
	1-10^{-9}	2.3529776128520878869209	1.0002705215264551998259
	1-10^{-12}	2.3529776127114254902000	1.0002705215193331444415
	1-10^{-100}	2.3529776127112846869999	1.0002705215193260152569
Ne:Kr	10^{-100}	1.9017257940137846498686	1.0155056865684927475768
	10^{-12}	1.9017257940137524155837	1.0155056865684809368274
	10^{-9}	1.9017257939815503648879	1.0155056865566819981913
	10^{-6}	1.9017257617794756953895	1.0155056747577401142809
	10^{-3}	1.9016935357157592616782	1.0154938725671789368343

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Table 4.4 – Continued

Mixture	x_1	$[D_{12}]_1 \times 10^1$ (cm^2 s^{-1})	$[D_{12}]^\star_{70}$
	0.1	1.8982460099217917474904	1.0142911781353565677076
	0.2	1.8941783153068666104721	1.0130060135002422162116
	0.3	1.8893829001436765890578	1.0116444068074660293919
	0.4	1.8836714489914590681657	1.0102007770729167771179
	0.5	1.8767841159918679468663	1.0086705244075222340047
	0.6	1.8683520081831108293757	1.0070515599994285398204
	0.7	1.8578332890404280886930	1.0053473318970966428332
	0.8	1.8443983891744594938648	1.0035729243629649068061
	0.9	1.8267097938606840552114	1.0017677494902959664937
	$1\text{-}10^{-3}$	1.8027496130448371347839	1.0000396945720376302292
	$1\text{-}10^{-6}$	1.8024633212551240126212	1.0000230929570504719276
	$1\text{-}10^{-9}$	1.8024630344377952628299	1.0000230763676910509640
	$1\text{-}10^{-12}$	1.8024630341509774075915	1.0000230763511017038458
	$1\text{-}10^{-100}$	1.8024630341506903026308	1.0000230763510850978927
Ne:Xe	10^{-100}	1.5043432550153760957201	1.0219922700092599257805
	10^{-12}	1.5043432550153529500433	1.0219922700092454441096
	10^{-9}	1.5043432549922304189428	1.0219922699947782549137
	10^{-6}	1.5043432318696811744195	1.0219922555275837799641
	10^{-3}	1.5043200911629506039033	1.0219777830519380255104
	0.1	1.5018325274673740506511	1.0204890428958482691599
	0.2	1.4988613058995203293079	1.0188661366760817418686
	0.3	1.4952984088022948117984	1.0171075734120072255844
	0.4	1.4909580625651764475268	1.0151951678385277499543
	0.5	1.4855682405468183320718	1.0131089292308866969869
	0.6	1.4787140801513032450678	1.0108286726184741561896
	0.7	1.4697300353273813878580	1.0083387403572518114294

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Table 4.4 – Continued

Mixture	x_1	$[D_{12}]_1 \times 10^1$ (cm^2 s^{-1})	$[D_{12}]^{\star}_{70}$
	0.8	1.4574769403500588257929	1.0056409566784675743361
	0.9	1.4398297029205057016961	1.0027907468052454654187
	1-10^{-3}	1.4126671980318517895297	1.0000304071965164664379
	1-10^{-6}	1.4123168104127010562574	1.0000045033143773098855
	1-10^{-9}	1.4123164590514489666378	1.0000044774439356508956
	1-10^{-12}	1.4123164587000867382266	1.0000044774180652428604
	1-10^{-100}	1.4123164586997350242833	1.0000044774180393465561
Ar:Kr	10^{-100}	1.0846367457840479979186	1.0068324667203306894611
	10^{-12}	1.0846367457840285717450	1.0068324667203241335750
	10^{-9}	1.0846367457646218243458	1.0068324667137748032859
	10^{-6}	1.0846367263578647124300	1.0068324601644441657352
	10^{-3}	1.0846173098853313001100	1.0068259104855013374309
	0.1	1.0825940208931575130212	1.0061735674236652710163
	0.2	1.0803385364888014556582	1.0055088149612418032894
	0.3	1.0778489540844345305070	1.0048395242421551082013
	0.4	1.0751000520238225736752	1.0041672916690566385397
	0.5	1.0720619296385905476128	1.0034940900571199587042
	0.6	1.0686989913198462764226	1.0028223857488848750847
	0.7	1.0649686387138605981237	1.0021552856317673839536
	0.8	1.0608195707249873474782	1.0014967240765948297276
	0.9	1.0561895491536178869290	1.0008517028912299836583
	1-10^{-3}	1.0510573457890158894272	1.0002327286743526543410
	1-10^{-6}	1.0510024802482199386110	1.0002266074649812343749
	1-10^{-9}	1.0510024253502667216906	1.0002266013451633539675
	1-10^{-12}	1.0510024252953687360439	1.0002266013390435374801

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Table 4.4 – Continued

Mixture	x_1	$[D_{12}]_1 \times 10^1$ (cm^2 s^{-1})	$[D_{12}]^\star_{70}$
	1-10^{-100}	1.0510024252953137831053	1.0002266013390374115377
Ar:Xe	10^{-100}	0.86216530185927002686794	1.0122348497012336614750
	10^{-12}	0.86216530185925340599140	1.0122348497012226122126
	10^{-9}	0.86216530184264915031579	1.0122348496901843991290
	10^{-6}	0.86216528523838234404493	1.0122348386519697467223
	10^{-3}	0.86214866983541734224668	1.0122237988684615845381
	0.1	0.86038597879013685047367	1.0111141344787978646606
	0.2	0.85834608039038700736633	1.0099616503843314104316
	0.3	0.85599886834377205401811	1.0087775859371034899076
	0.4	0.85328477938267472366200	1.0075630752624000603203
	0.5	0.85012679883705026500085	1.0063207555412913215961
	0.6	0.84642366476907316289043	1.0050556258810590312079
	0.7	0.84203961673801017023150	1.0037763936225805413609
	0.8	0.83678845125077317925033	1.0024976231156012613284
	0.9	0.83040782030494506050946	1.0012432298262261105002
	1-10^{-3}	0.82260409524721934836894	1.0000636769289761947249
	1-10^{-6}	0.82251610627936191206390	1.0000522751552412447238
	1-10^{-9}	0.82251601818752258107034	1.0000522637596478875179
	1-10^{-12}	0.82251601809943063875153	1.0000522637482523003543
	1-10^{-100}	0.82251601809934245862898	1.0000522637482408933601
Kr:Xe	10^{-100}	0.57835153610812004790809	1.0042215316500265988517
	10^{-12}	0.57835153610811275096383	1.0042215316500232384496
	10^{-9}	0.57835153610082310364868	1.0042215316466661967437
	10^{-6}	0.57835152881117299212724	1.0042215282896241159536
	10^{-3}	0.57834423636389162940606	1.0042181708726392850554

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Table 4.4 – Continued

Mixture	x_1	$[D_{12}]_1 \times 10^1$ (cm^2 s^{-1})	$[D_{12}]^\star_{70}$
	0.1	0.57759302691647121420036	1.0038817543497835388810
	0.2	0.57677338030395935831337	1.0035345845711090259847
	0.3	0.57588680061755142589998	1.0031801931507350181548
	0.4	0.57492668270212240465007	1.0028188299248630876719
	0.5	0.57388546790165061403053	1.0024508458219528004037
	0.6	0.57275446806227897752640	1.0020767206319061451381
	0.7	0.57152364893044244099073	1.0016970980640842120146
	0.8	0.57018136158724823304764	1.0013128302065296455202
	0.9	0.56871400677716158784997	1.0009250341669299405469
	1-10^{-3}	0.56712245310724545866441	1.0005390677522271054163
	1-10^{-6}	0.56710562857622933553051	1.0005351684743968558690
	1-10^{-9}	0.56710561174372837500463	1.0005351645751069970858
	1-10^{-12}	0.56710561172689586607075	1.0005351645712077072153
	1-10^{-100}	0.56710561172687901671245	1.0005351645712038040222

Table 4.5: Order 1 values of the diffusion coefficient related quantities, $d_0^{(1)}$, for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe as functions of the mixture fractions of the lighter constituents.

Mixture	x_1	$d_0^{(1)} \times 10^4$ (cm)
He:Ne	10^{-100}	3.9115816397097755878377 $\times 10^{99}$
	10^{-12}	3.9115816397135867660586 $\times 10^{11}$
	10^{-9}	3.9115816435209538125484 $\times 10^{8}$
	10^{-6}	3.9115854508917458012051 $\times 10^{5}$
	10^{-3}	3.9153965709038662611980 $\times 10^{2}$
	0.1	4.3343248757610913389940
	0.2	2.4304828326867268473977
	0.3	1.8452309474611796647776
	0.4	1.6080123127552058405632
	0.5	1.5364486848578973149314
	0.6	1.5917342259104439664689
	0.7	1.8074721917041147115100
	0.8	2.3542641284325308638735
	0.9	4.1470361346501712518512
	1-10^{-3}	3.6946470961887923658574 $\times 10^{2}$
	1-10^{-6}	3.6904943377904185266778 $\times 10^{5}$
	1-10^{-9}	3.6904901886967740891136 $\times 10^{8}$
	1-10^{-12}	3.6904901845476841052619 $\times 10^{11}$
	1-10^{-100}	3.6904901845435308620385 $\times 10^{99}$
He:Ar	10^{-100}	3.4267671645201063191823 $\times 10^{99}$
	10^{-12}	3.4267671645234670591428 $\times 10^{11}$
	10^{-9}	3.4267671678808462829788 $\times 10^{8}$
	10^{-6}	3.4267705252633794193613 $\times 10^{5}$
	10^{-3}	3.4301312203754157133374 $\times 10^{2}$
	0.1	3.7996072746327150899216
	0.2	2.1320747690313311977752
	0.3	1.6197439203388347247043
	0.4	1.4123430188309547098572
	0.5	1.3500623306790428807933
	0.6	1.3987910013538033525692
	0.7	1.5875984946306474422090
	0.8	2.0645478984786347835705
	0.9	3.6230265173669789336349
	1-10^{-3}	3.2018390440510426892361 $\times 10^{2}$
	1-10^{-6}	3.1978647457015279591075 $\times 10^{5}$
	1-10^{-9}	3.1978607735328342103365 $\times 10^{8}$
	1-10^{-12}	3.1978607695606676397463 $\times 10^{11}$
	1-10^{-100}	3.1978607695566914970351 $\times 10^{99}$
He:Kr	10^{-100}	3.9839103673238034196874 $\times 10^{99}$
	10^{-12}	3.9839103673277203355402 $\times 10^{11}$

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Table 4.5 – Continued

Mixture	x_1	$d_0^{(1)} \times 10^4$ (cm)
	10^{-9}	3.9839103712407192763562 $\times 10^8$
	10^{-6}	3.9839142842435215436909 $\times 10^5$
	10^{-3}	3.9878311523227998917670 $\times 10^2$
	0.1	4.4185022431330002073029
	0.2	2.4800457410686601925796
	0.3	1.8846567040008428874896
	0.4	1.6438335326258087657426
	0.5	1.5717942212325520910355
	0.6	1.6288749149837110029094
	0.7	1.8487752727308826249823
	0.8	2.4030658565357625614472
	0.9	4.2098500261061897607213
	1-10^{-3}	3.7011039213623684187090 $\times 10^2$
	1-10^{-6}	3.6961820726237055292757 $\times 10^5$
	1-10^{-9}	3.6961771516982945534355 $\times 10^8$
	1-10^{-12}	3.6961771467773700518228 $\times 10^{11}$
	1-10^{-100}	3.6961771467724442014717 $\times 10^{99}$
He:Xe	10^{-100}	3.9467279906441958789397 $\times 10^{99}$
	10^{-12}	3.9467279906480888584908 $\times 10^{11}$
	10^{-9}	3.9467279945371754338671 $\times 10^8$
	10^{-6}	3.9467318836275960384128 $\times 10^5$
	10^{-3}	3.9506248230896087193485 $\times 10^2$
	0.1	4.3787501153500986743217
	0.2	2.4586747593096589468336
	0.3	1.8692243275960435012124
	0.4	1.6311665322208495486021
	0.5	1.5605250434844225082703
	0.6	1.6181301482621653621329
	0.7	1.8375901146726645742585
	0.8	2.3892664164710787154947
	0.9	4.1828516386852860799964
	1-10^{-3}	3.6600629352010788230593 $\times 10^2$
	1-10^{-6}	3.6548330748012956193090 $\times 10^5$
	1-10^{-9}	3.6548278432238408607535 $\times 10^8$
	1-10^{-12}	3.6548278379922616600016 $\times 10^{11}$
	1-10^{-100}	3.6548278379870248439831 $\times 10^{99}$
Ne:Ar	10^{-100}	1.7629847801317523289545 $\times 10^{99}$
	10^{-12}	1.7629847801334913465037 $\times 10^{11}$
	10^{-9}	1.7629847818707698798638 $\times 10^8$
	10^{-6}	1.7629865191510265684592 $\times 10^5$
	10^{-3}	1.7647255244546084791023 $\times 10^2$
	0.1	1.9560460212531220589388
	0.2	1.0984836261076856935115

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Table 4.5 – Continued

Mixture	x_1	$d_0^{(1)} \times 10^4$ (cm)
	0.3	0.83539409367199932084619
	0.4	0.72943145348523284961258
	0.5	0.69856522153050772476972
	0.6	0.72564706751697477918656
	0.7	0.82662697183542539972465
	0.8	1.0808262632807663681958
	0.9	1.9128055787074538866555
	$1\text{-}10^{-3}$	1.7140070538494521131677 $\times 10^{2}$
	$1\text{-}10^{-6}$	1.7121925027404610762541 $\times 10^{5}$
	$1\text{-}10^{-9}$	1.7121906899044740880384 $\times 10^{8}$
	$1\text{-}10^{-12}$	1.7121906880916398143646 $\times 10^{11}$
	$1\text{-}10^{-100}$	1.7121906880898251654437 $\times 10^{99}$
Ne:Kr	10^{-100}	1.8197638784499995131832 $\times 10^{99}$
	10^{-12}	1.8197638784517884320325 $\times 10^{11}$
	10^{-9}	1.8197638802389183643226 $\times 10^{8}$
	10^{-6}	1.8197656673706148438017 $\times 10^{5}$
	10^{-3}	1.8215545650077942492129 $\times 10^{2}$
	0.1	2.0182600761553190330141
	0.2	1.1328385539298615394765
	0.3	0.86092997490448906927494
	0.4	0.75103652178552485290356
	0.5	0.71835886176258269892850
	0.6	0.74492852479562448812354
	0.7	0.84655384929580455729692
	0.8	1.1030670065106654173376
	0.9	1.9422010784697118819874
	$1\text{-}10^{-3}$	1.7267802227512430255923 $\times 10^{2}$
	$1\text{-}10^{-6}$	1.7247812143248663979317 $\times 10^{5}$
	$1\text{-}10^{-9}$	1.7247792168125574644605 $\times 10^{8}$
	$1\text{-}10^{-12}$	1.7247792148150466492398 $\times 10^{11}$
	$1\text{-}10^{-100}$	1.7247792148130471389157 $\times 10^{99}$
Ne:Xe	10^{-100}	1.7374454032581225472695 $\times 10^{99}$
	10^{-12}	1.7374454032598332605094 $\times 10^{11}$
	10^{-9}	1.7374454049688357888256 $\times 10^{8}$
	10^{-6}	1.7374471139730521520556 $\times 10^{5}$
	10^{-3}	1.7391578079106051986861 $\times 10^{2}$
	0.1	1.9272729239245678338560
	0.2	1.0819462568526167668132
	0.3	0.82238049195575874273218
	0.4	0.71749422425646927788733
	0.5	0.68630445934706428901478
	0.6	0.71160204869202896386651
	0.7	0.80831846163924818019139

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Table 4.5 – Continued

Mixture	x_1	$d_0^{(1)} \times 10^4$ (cm)
	0.8	1.0520731396921275350828
	0.9	1.8477058864749671954089
	1-10^{-3}	1.6331970847620366706341 $\times 10^{2}$
	1-10^{-6}	1.6311608376882335488071 $\times 10^{5}$
	1-10^{-9}	1.6311588023529047284051 $\times 10^{8}$
	1-10^{-12}	1.6311588003175703073128 $\times 10^{11}$
	1-10^{-100}	1.6311588003155329355208 $\times 10^{99}$
Ar:Kr	10^{-100}	1.1322620158308577512248 $\times 10^{99}$
	10^{-12}	1.1322620158319697340841 $\times 10^{11}$
	10^{-9}	1.1322620169428406115423 $\times 10^{8}$
	10^{-6}	1.1322631278148188014812 $\times 10^{5}$
	10^{-3}	1.1333751016236912033052 $\times 10^{2}$
	0.1	1.2556995522412789312494
	0.2	0.70485942290080907292843
	0.3	0.53579818060286619568782
	0.4	0.46762773990767878506349
	0.5	0.44765402104117507906836
	0.6	0.46484352131855535718715
	0.7	0.52939538221911044360158
	0.8	0.69212440838525006427672
	0.9	1.2250730360213268675642
	1-10^{-3}	1.0983064858349848813676 $\times 10^{2}$
	1-10^{-6}	1.0971520018715923874620 $\times 10^{5}$
	1-10^{-9}	1.0971508485082761883748 $\times 10^{8}$
	1-10^{-12}	1.0971508473549139916842 $\times 10^{11}$
	1-10^{-100}	1.0971508473537594749719 $\times 10^{99}$
Ar:Xe	10^{-100}	1.0587452089497119229562 $\times 10^{99}$
	10^{-12}	1.0587452089507502576089 $\times 10^{11}$
	10^{-9}	1.0587452099880465766621 $\times 10^{8}$
	10^{-6}	1.0587462472853892576279 $\times 10^{5}$
	10^{-3}	1.0597845692737380350603 $\times 10^{2}$
	0.1	1.1739557637640628807398
	0.2	0.65878448590014678870491
	0.3	0.50055846862761856940407
	0.4	0.43659994304113411689923
	0.5	0.41758473616485170012421
	0.6	0.43308931877842978102526
	0.7	0.49239558212697705011560
	0.8	0.64223890835931693951976
	0.9	1.1330519917265275682919
	1-10^{-3}	1.0111749368363685961300 $\times 10^{2}$
	1-10^{-6}	1.0100567208691140123687 $\times 10^{5}$
	1-10^{-9}	1.0100556036450346963386 $\times 10^{8}$

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Table 4.5 – Continued

Mixture	x_1	$d_0^{(1)} \times 10^4$ (cm)
	1-10^{-12}	1.0100556025278116077758 $\times 10^{11}$
	1-10^{-100}	1.0100556025266932663468 $\times 10^{99}$
Kr:Xe	10^{-100}	0.79597614549854162706557 $\times 10^{99}$
	10^{-12}	0.79597614549932756054117 $\times 10^{11}$
	10^{-9}	0.79597614628447510344619 $\times 10^{8}$
	10^{-6}	0.79597693143279930740586 $\times 10^{5}$
	10^{-3}	0.79676286183692127242534 $\times 10^{2}$
	0.1	0.88325802355353503982945
	0.2	0.49612759651541313461403
	0.3	0.37742093748173217582776
	0.4	0.32969273895322485972138
	0.5	0.31593182635730838920522
	0.6	0.32844707856599366437297
	0.7	0.37456144357015477916089
	0.8	0.49045728905361077886518
	0.9	0.86968018342411672809940
	1-10^{-3}	0.7813030377315495035907 $\times 10^{2}$
	1-10^{-6}	0.78049935989666106354738 $\times 10^{5}$
	1-10^{-9}	0.78049855701149273382672 $\times 10^{8}$
	1-10^{-12}	0.78049855620860835740776 $\times 10^{11}$
	1-10^{-100}	0.78049855620780466934407 $\times 10^{99}$

Table 4.6: Order 1 and normalized order 70 values of the thermal diffusion coefficients, $[D_T]_1$ and $[D_T]^\star_{70}$, for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe as functions of the mixture fractions of the lighter constituents.

Mixture	x_1	$[D_T]_1 \times 10^3$ (cm^2 s^{-1})	$[D_T]^\star_{70}$
He:Ne	10^{-100}	$-31.6469076801374604729750 \times 10^{-99}$	1.1727641522135545606618
	10^{-12}	$-31.646907680118735403060 \times 10^{-11}$	1.1727641522134524088207
	10^{-9}	$-31.646907661412390552449 \times 10^{-8}$	1.1727641521114027195723
	10^{-6}	$-31.646888955062234709054 \times 10^{-5}$	1.1727640500616998308119
	10^{-3}	$-31.628177295602045582489 \times 10^{-2}$	1.1726619867110624152583
	0.1	$-29.717047551859776833746$	1.1624053508495331483336
	0.2	$-55.307853254800154538740$	1.1517334502359457509954
	0.3	$-76.276452422536648304947$	1.1407145035408151851090
	0.4	$-91.969804904830071696131$	1.1293204418630852362592
	0.5	$-101.52871760112091088754$	1.1175292487057323868719
	0.6	$-103.81640112778701336482$	1.1053264335986878282488
	0.7	$-97.312443611781126866976$	1.0927083471780871580618
	0.8	$-79.951559889475667142950$	1.0796884317964479207634
	0.9	$-48.870939351579418155564$	1.0663085295817498046438
	$1\text{-}10^{-3}$	$-59.432087782463847559130 \times 10^{-2}$	1.0527967781237656594892
	$1\text{-}10^{-6}$	$-59.549417204218899741047 \times 10^{-5}$	1.0526595763539228474802
	$1\text{-}10^{-9}$	$-59.549534652441843391280 \times 10^{-8}$	1.0526594391469324378142
	$1\text{-}10^{-12}$	$-59.549534769890185255684 \times 10^{-11}$	1.0526594390097254422230
	$1\text{-}10^{-100}$	$-59.549534770007751163575 \times 10^{-99}$	1.0526594390095880978830
He:Ar	10^{-100}	$-21.024214723838819088234 \times 10^{-99}$	1.2436199063947996916504
	10^{-12}	$-21.024214723832006010265 \times 10^{-11}$	1.2436199063946698748315
	10^{-9}	$-21.024214717025741114923 \times 10^{-8}$	1.2436199062649828727582
	10^{-6}	$-21.024207910756557429181 \times 10^{-5}$	1.2436197765779464721956

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Table 4.6 – Continued

Mixture	x_1	$[D_T]_1 \times 10^3$ (cm^2 s^{-1})	$[D_T]^\star_{70}$
	10^{-3}	$-21.017397350275016965106 \times 10^{-2}$	1.2434900551961121412657
	0.1	$-20.296803112963154068559$	1.2302757015106409539546
	0.2	$-38.924804936835662152211$	1.2161249381753846915178
	0.3	$-55.478803140513369449032$	1.2010300503998733430707
	0.4	$-69.397464365025487017262$	1.1848290035165239530941
	0.5	$-79.878876195090998587899$	1.1673297086784892496578
	0.6	$-85.734509791188947246065$	1.1483046571310242587841
	0.7	$-85.121921043933103320033$	1.1274887704550815656798
	0.8	$-75.019827346313033753815$	1.1045900958343137867095
	0.9	$-50.100399379168863940708$	1.0793448926658645400110
	1-10^{-3}	$-68.368335046149426732652 \times 10^{-2}$	1.0520163604399200156974
	1-10^{-6}	$-68.596025644170986586974 \times 10^{-5}$	1.0517304388376960322879
	1-10^{-9}	$-68.596253870850547996913 \times 10^{-8}$	1.0517301528355986483420
	1-10^{-12}	$-68.596254099077764908375 \times 10^{-11}$	1.0517301525495964707777
	1-10^{-100}	$-68.596254099306220581497 \times 10^{-99}$	1.0517301525493101823116
He:Kr	10^{-100}	$-17.373920128273337679838 \times 10^{-99}$	1.2968000340186695863639
	10^{-12}	$-17.373920128269126222055 \times 10^{-11}$	1.2968000340185151612967
	10^{-9}	$-17.373920124061879893882 \times 10^{-8}$	1.2968000338642445191216
	10^{-6}	$-17.373915916812413297083 \times 10^{-5}$	1.2967998795935610511527
	10^{-3}	$-17.369705526550224988125 \times 10^{-2}$	1.2966455675893725297011
	0.1	$-16.918791080573363020679$	1.2809152871129083431000
	0.2	$-32.767164503776283324354$	1.2640206119339348769637
	0.3	$-47.234238566642085418879$	1.2459004765445057912311
	0.4	$-59.873649233985846320998$	1.2262918337576654091944
	0.5	$-70.020031335205505282305$	1.2048666354747263554431
	0.6	$-76.634749531598722967630$	1.1812088901466069346571

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Table 4.6 – Continued

Mixture	x_1	$[D_T]_1 \times 10^3$ (cm^2 s^{-1})	$[D_T]^\star_{70}$
	0.7	−77.999317989894667953860	1.1547869365756260555096
	0.8	−71.046212886181635796939	1.1249315960961032779228
	0.9	−49.703189596737326174080	1.0908788522466058119317
	1-10^{-3}	−72.779459697752492692542 $\times 10^{-2}$	1.0525993069655127511565
	1-10^{-6}	−73.088008365294018846840 $\times 10^{-5}$	1.0521925154993775819198
	1-10^{-9}	−73.088317917484597428414 $\times 10^{-8}$	1.0521921085533173031530
	1-10^{-12}	−73.088318227037794800448 $\times 10^{-11}$	1.0521921081463710891510
	1-10^{-100}	−73.088318227347657861888 $\times 10^{-99}$	1.0521921081459637355833
He:Xe	10^{-100}	−13.868513408059166448030 $\times 10^{-99}$	1.3178604054407885105683
	10^{-12}	−13.868513408056563427831 $\times 10^{-11}$	1.3178604054406483056379
	10^{-9}	−13.868513405456146246482 $\times 10^{-8}$	1.3178604053005835800959
	10^{-6}	−13.868510805036863320427 $\times 10^{-5}$	1.3178602652358067635031
	10^{-3}	−13.865908282472393554189 $\times 10^{-2}$	1.3177201491517486058756
	0.1	−13.585318936706665988944	1.3032921381179493997273
	0.2	−26.494954485415725812075	1.2874738980629910572471
	0.3	−38.512229043697521940790	1.2701286271878396279516
	0.4	−49.316268543619856830080	1.2508987137962499940943
	0.5	−58.410583843902573472529	1.2293105810522523945894
	0.6	−64.983428142682521999497	1.2047202474504596798627
	0.7	−67.609366583029714982799	1.1762288998121947205052
	0.8	−63.531793565245638787476	1.1425582057315242545406
	0.9	−46.629015437677828719495	1.1019258784644679636786
	1-10^{-3}	−74.172779147317950845888 $\times 10^{-2}$	1.0530131859856985183114
	1-10^{-6}	−74.573137142738955603498 $\times 10^{-5}$	1.0524756587751760505820
	1-10^{-9}	−74.573539264758720726561 $\times 10^{-8}$	1.0524751208933209519528
	1-10^{-12}	−74.573539666882512296013 $\times 10^{-11}$	1.0524751203554387440760

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Table 4.6 – Continued

Mixture	x_1	$[D_T]_1 \times 10^3$ (cm^2 s^{-1})	$[D_T]^\star_{70}$
	1-10^{-100}	$-74.57353966728503861567 4 \times10^{-99}$	1.05247512035490032344 71
Ne:Ar	10^{-100}	$-6.1604362857612546937808 \times10^{-99}$	1.1044242646588764172807
	10^{-12}	$-6.1604362857578160408266 \times10^{-11}$	1.1044242646588456076239
	10^{-9}	$-6.1604362823226017382092 \times10^{-8}$	1.1044242646280667605213
	10^{-6}	$-6.1604328471068979964363 \times10^{-5}$	1.1044242338492092542613
	10^{-3}	$-6.1569962295956009469345 \times10^{-2}$	1.1043934445843824048919
	0.1	$-5.8018275791320023419759$	1.1012353408148853718646
	0.2	$-10.821106443554966046832$	1.0978142571966454698468
	0.3	$-14.943117938804120319228$	1.0941342546529624547375
	0.4	$-18.025524536802534912697$	1.0901649809186930353596
	0.5	$-19.889602667258650702759$	1.0858719380587339830258
	0.6	$-20.307908059110321448339$	1.0812158840267070325286
	0.7	$-18.986556803145780952516$	1.0761522347761726413915
	0.8	$-15.539164985669461660031$	1.0706305773621532571568
	0.9	$-9.4474755683296669929697$	1.0645945394961254298434
	1-10^{-3}	$-11.407227567462140488841 \times10^{-2}$	1.0580517320233088746814
	1-10^{-6}	$-11.428799672666927294047 \times10^{-5}$	1.0579826198431211967369
	1-10^{-9}	$-11.428821264409635135185 \times10^{-8}$	1.0579825506989729518133
	1-10^{-12}	$-11.428821286001397498817 \times10^{-11}$	1.0579825506298287715896
	1-10^{-100}	$-11.428821286023010874576 \times10^{-99}$	1.0579825506297595581960
Ne:Kr	10^{-100}	$-6.7719126038873711499948 \times10^{-99}$	1.1618007435374006822356
	10^{-12}	$-6.7719126038845914198987 \times10^{-11}$	1.1618007435373355880057
	10^{-9}	$-6.7719126011076410525134 \times10^{-8}$	1.1618007434723064523022
	10^{-6}	$-6.7719098241558672923531 \times10^{-5}$	1.1618006784431444010917
	10^{-3}	$-6.7691314650864106831293 \times10^{-2}$	1.1617356229217313257867

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Table 4.6 – Continued

Mixture	x_1	$[D_T]_1 \times 10^3$ (cm^2 s^{-1})	$[D_T]^\star_{70}$
	0.1	−6.4788772927058113602607	1.1550154735750721302063
	0.2	−12.302396113057476069631	1.1476262724750767233438
	0.3	−17.341423514187456303009	1.1395450966939509555545
	0.4	−21.421578159689609398266	1.1306687705977410628483
	0.5	−24.301760726999090209598	1.1208758858944046120362
	0.6	−25.639171347371011332760	1.1100235818013765798020
	0.7	−24.929728881667926480077	1.0979451863147456201240
	0.8	−21.401034509024902592198	1.0844516171047693675273
	0.9	−13.807032130274600262743	1.0693449707586649821620
	1-10^{-3}	−17.968289909741465715057 $\times 10^{-2}$	1.0526475489940921617208
	1-10^{-6}	−18.017775070917905520 34 $\times 10^{-5}$	1.0524700717950415806372
	1-10^{-9}	−18.017827083209587445123 $\times 10^{-8}$	1.0524698942297820013612
	1-10^{-12}	−18.017827132785793922482 $\times 10^{-11}$	1.0524698940522166537597
	1-10^{-100}	−18.017827132835419754880 $\times 10^{-99}$	1.0524698940520389106689
Ne:Xe	10^{-100}	−5.6197609305460851494936 $\times 10^{-99}$	1.2065913658590160409677
	10^{-12}	−5.6197609305444088986999 $\times 10^{-11}$	1.2065913658589339368023
	10^{-9}	−5.6197609288698343547653 $\times 10^{-8}$	1.2065913657769118755530
	10^{-6}	−5.6197592542942216069185 $\times 10^{-5}$	1.2065912837548134228850
	10^{-3}	−5.6180836091018391918352 $\times 10^{-2}$	1.2065092244342720173665
	0.1	−5.4406098857934376688820	1.1979887995719276879975
	0.2	−10.469062357741253546711	1.1885143303976847522322
	0.3	−14.982224290450673858566	1.1780137720913712183998
	0.4	−18.834808162857630214371	1.1662951941043735776819
	0.5	−21.814977284377278807580	1.1531170394730369685168
	0.6	−23.600753104059358711677	1.1381723588378715854792
	0.7	−23.676932837952786952728	1.1210692633784051686495

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Table 4.6 – Continued

Mixture	x_1	$[D_T]_1 \times 10^3$ (cm^2 s^{-1})	$[D_T]^\star_{70}$
	0.8	$-21.163353558091290059621$	1.1013124150195424370659
	0.9	$-14.420253091597100610090$	1.0783113899880455156058
	1-10^{-3}	$-20.283558492747714365027 \times 10^{-2}$	1.0518275585004209956234
	1-10^{-6}	$-20.358959416892674487486 \times 10^{-5}$	1.0515411132567119723475
	1-10^{-9}	$-20.359035024970655189094 \times 10^{-8}$	1.0515408266304189888067
	1-10^{-12}	$-20.359035100578940895351 \times 10^{-11}$	1.0515408263437925150052
	1-10^{-100}	$-20.359035100654624865235 \times 10^{-99}$	1.0515408263435056016179
Ar:Kr	10^{-100}	$-2.8930139181149375335592 \times 10^{-99}$	1.1040628811513094072040
	10^{-12}	$-2.8930139181128090485410 \times 10^{-11}$	1.1040628811512728573301
	10^{-9}	$-2.8930139159864525149748 \times 10^{-8}$	1.1040628811147595333749
	10^{-6}	$-2.8930117896294879284803 \times 10^{-5}$	1.1040628446014284068086
	10^{-3}	$-2.8908850014148500368303 \times 10^{-2}$	1.1040263240968564290830
	0.1	$-2.6756143705243915345620$	1.1003340598632075619873
	0.2	$-4.8962185254238945660335$	1.0964496869794227464435
	0.3	$-6.6265399907344295433155$	1.0923985781039871202049
	0.4	$-7.8237711772878343641257$	1.0881703394320981103879
	0.5	$-8.4361378298315538775933$	1.0837553476619426452054
	0.6	$-8.4010560699362150277478$	1.0791448129716540287958
	0.7	$-7.6427461088314107142417$	1.0743309456609715822303
	0.8	$-6.0691178974129306830453$	1.0693072600999595279786
	0.9	$-3.5676665695744182038968$	1.0640690688828864201922
	1-10^{-3}	$-4.1486055147748695750741 \times 10^{-2}$	1.0586698686117168269653
	1-10^{-6}	$-4.1547677370503334044135 \times 10^{-5}$	1.0586143051789703141522
	1-10^{-9}	$-4.1547739024416099142778 \times 10^{-8}$	1.0586142496047743957235
	1-10^{-12}	$-4.1547739086070043615683 \times 10^{-11}$	1.0586142495492001890425

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Table 4.6 – Continued

Mixture	x_1	$[D_T]_1 \times 10^3$ (cm^2 s^{-1})	$[D_T]^\star_{70}$
	1-10^{-100}	$-4.1547739086131759275847 \times 10^{-99}$	1.0586142495491445592060
Ar:Xe	10^{-100}	$-2.9235509152360107495408 \times 10^{-99}$	1.1380838897696135519887
	10^{-12}	$-2.9235509152344060806533 \times 10^{-11}$	1.1380838897695546336864
	10^{-9}	$-2.9235509136313418614504 \times 10^{-8}$	1.1380838897106952497644
	10^{-6}	$-2.9235493105665489868304 \times 10^{-5}$	1.1380838308512934936705
	10^{-3}	$-2.9219456717454862907689 \times 10^{-2}$	1.1380249536088664838076
	0.1	$-2.7569658387422710906811$	1.1320068654822395637967
	0.2	$-5.1529412930326866279899$	1.1255321292052965700150
	0.3	$-7.1372891832312021267627$	1.1186170876381376962719
	0.4	$-8.6443846631183837503990$	1.1112167411098189304275
	0.5	$-9.5883058105232508309726$	1.1032835885802823967408
	0.6	$-9.8550240927414346846155$	1.0947680242313181797283
	0.7	$-9.2906140106814243782108$	1.0856196775951882064190
	0.8	$-7.6829048893311175901515$	1.0757905940661632145697
	0.9	$-4.7318953441856082073082$	1.0652420837366393819172
	1-10^{-3}	$-5.8061454856897876107603 \times 10^{-2}$	1.0540755211786314075701
	1-10^{-6}	$-5.8181857428830523826394 \times 10^{-5}$	1.0539592320636190764725
	1-10^{-9}	$-5.8181977965087995549245 \times 10^{-8}$	1.0539591157393014302539
	1-10^{-12}	$-5.8181978085624386858775 \times 10^{-11}$	1.0539591156229770774178
	1-10^{-100}	$-5.8181978085745043907267 \times 10^{-99}$	1.0539591156228606366241
Kr:Xe	10^{-100}	$-1.0838227995845962484440 \times 10^{-99}$	1.0903902205754061427737
	10^{-12}	$-1.0838227995837953696045 \times 10^{-11}$	1.0903902205753835349901
	10^{-9}	$-1.0838227987837174087314 \times 10^{-8}$	1.0903902205527983591728
	10^{-6}	$-1.0838219987055633167420 \times 10^{-5}$	1.0903901979676181922027
	10^{-3}	$-1.0830217272699306353823 \times 10^{-2}$	1.0903676084368747014188

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Table 4.6 – Continued

Mixture	x_1	$[D_T]_1 \times 10^3$ (cm^2 s^{-1})	$[D_T]^\star_{70}$
	0.1	$-1.0017361246000295077597$	1.0880849795232065695066
	0.2	$-1.8307536307976741340446$	1.0856870428441861495175
	0.3	$-2.4728751913312800023488$	1.0831905688195361120981
	0.4	$-2.9118646357275483221852$	1.0805894594508077531837
	0.5	$-3.1290639840670673370871$	1.0778773551066907014621
	0.6	$-3.1029496606419995148543$	1.0750476366842587490431
	0.7	$-2.8085861582626784271324$	1.0720934388721887733048
	0.8	$-2.2169484754364344826915$	1.0690076796720994475855
	0.9	$-1.2940750966885695874782$	1.0657831136199245401243
	$1\text{-}10^{-3}$	$-1.4926710809341307824182 \times 10^{-2}$	1.0624468737640365743670
	$1\text{-}10^{-6}$	$-1.4947570254374386527447 \times 10^{-5}$	1.0624124539748200894711
	$1\text{-}10^{-9}$	$-1.4947591122372455857985 \times 10^{-8}$	1.0624124195473693182167
	$1\text{-}10^{-12}$	$-1.4947591143240462484107 \times 10^{-11}$	1.0624124195129418597827
	$1\text{-}10^{-100}$	$-1.4947591143261351379637 \times 10^{-99}$	1.0624124195129073978623

Table 4.7: Normalized order 70 values of the thermal diffusion related quantities $d_1^{(70)\star}$ and $d_{-1}^{(70)\star}$ for binary mixtures of the noble gases He, Ne, Ar, Kr, and Xe as functions of the mixture fractions of the lighter constituents.

Mixture	x_1	$d_1^{(70)\star} \times 10^2$	$d_{-1}^{(70)\star} \times 10^2$
He:Ne	10^{-100}	6.7731889521219033258649	0.079115585974760126379754 $\times 10^{-99}$
	10^{-12}	6.7731889521182510376156	0.079115585974614355166450 $\times 10^{-11}$
	10^{-9}	6.7731889484696150750267	0.079115585828988913067970 $\times 10^{-8}$
	10^{-6}	6.7731852998321243859635	0.079115440203538368399637 $\times 10^{-5}$
	10^{-3}	6.7695351334987884233048	0.078969806273070044473211 $\times 10^{-2}$
	0.1	6.3918794667852847580958	0.064419544318686234501397
	0.2	5.9748995702774248217507	0.098683346138541817412910
	0.3	5.5159904699947163705690	0.10093607633095847446505
	0.4	5.0072057864970334265045	0.068215103919135092887566
	0.5	4.4382924888213206829316	-0.0038533127133321689529103
	0.6	3.7957716192049345547600	-0.12153380080478184842159
	0.7	3.0615327679603599249122	-0.29374015580203356257005
	0.8	2.2106112845988171179616	-0.53325739622067728763422
	0.9	1.2075312681655995592318	-0.85877543786190509243734
	$1\text{-}10^{-3}$	1.3320111715876871332661 $\times 10^{-2}$	-1.2933337835576328518242
	$1\text{-}10^{-6}$	1.3334135131141428625369 $\times 10^{-5}$	-1.2984167404658051308355
	$1\text{-}10^{-9}$	1.3334149170929273937529 $\times 10^{-8}$	-1.2984218312022470878363
	$1\text{-}10^{-12}$	1.3334149184969078175031 $\times 10^{-11}$	-1.2984218362929913183199
	$1\text{-}10^{-100}$	1.3334149184983132033143 $\times 10^{-99}$	-1.2984218362980871583983
He:Ar	10^{-100}	5.6202118882073375576658	3.4669461409060272940011 $\times 10^{-99}$
	10^{-12}	5.6202118882051655991678	3.4669461409040412989134 $\times 10^{-11}$
	10^{-9}	5.6202118860353790586203	3.4669461389200322041945 $\times 10^{-8}$
	10^{-6}	5.6202097162477443141838	3.4669441549088329537595 $\times 10^{-5}$

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Table 4.7 – Continued

Mixture	x_1	$d_1^{(70)\star} \times 10^2$	$d_{-1}^{(70)\star} \times 10^2$
	10^{-3}	5.6180388336869041526718	3.4649580375883119496560 $\times 10^{-2}$
	0.1	5.3912826475288084022528	0.032455612607802195425136
	0.2	5.1352904666355312524385	0.059410472090211074516578
	0.3	4.8454908178896878969324	0.078787170138968444099200
	0.4	4.5126379668050286840879	0.087623253701636691454855
	0.5	4.1236280252374611370588	0.081557052294983371467083
	0.6	3.6591272142184377618741	0.053912487209072618231847
	0.7	3.0891502312734682439763	-0.0060377404993051326650322
	0.8	2.3641199688434479275807	-0.11664513285848406329278
	0.9	1.3947744159988341475267	-0.31205036436599103526011
	$1\text{-}10^{-3}$	1.7112253513407576115791 $\times 10^{-2}$	-0.65900263353917745274836
	$1\text{-}10^{-6}$	1.7152391460212866431869 $\times 10^{-5}$	-0.66371383272569092516758
	$1\text{-}10^{-9}$	1.7152431702020220859250 $\times 10^{-8}$	-0.66371855936401488229710
	$1\text{-}10^{-12}$	1.7152431742262132351023 $\times 10^{-11}$	-0.66371856409066868787861
	$1\text{-}10^{-100}$	1.7152431742302414544813 $\times 10^{-99}$	-0.66371856409540007308488
He:Kr	10^{-100}	4.2200063148834996058684	1.2383598580963337931989 $\times 10^{-99}$
	10^{-12}	4.2200063148820641913475	1.2383598580962723014680 $\times 10^{-11}$
	10^{-9}	4.2200063134480850842319	1.2383598580348420616961 $\times 10^{-8}$
	10^{-6}	4.2200048794682403971605	1.2383597966039907608708 $\times 10^{-5}$
	10^{-3}	4.2185701615167994620797	1.2382977535763354228389 $\times 10^{-2}$
	0.1	4.0685074743197002029314	0.012253757537545963352730
	0.2	3.8984300901231444351442	0.023904709073400002011422
	0.3	3.7047126789561378877330	0.034224684845318922239037
	0.4	3.4802612633397051099485	0.042051508723962217440118
	0.5	3.2147204558103717283401	0.045475151592640011200830
	0.6	2.8922066101814184892296	0.041231220581314363882191

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Table 4.7 – Continued

Mixture	x_1	$d_1^{(70)\star} \times 10^2$	$d_{-1}^{(70)\star} \times 10^2$
	0.7	2.4867737197312108955953	0.0234349123140045550111124
	0.8	1.9523371575492170270755	-0.0193445187086994743187 31
	0.9	1.1968073272560080237081	-0.1118646036722760384734 8
	$1\text{-}10^{-3}$	1.5644185550670944681238 $\times 10^{-2}$	-0.3146355183145986128308 6
	$1\text{-}10^{-6}$	1.5694328017042747304561 $\times 10^{-5}$	-0.3176995991516618792341 1
	$1\text{-}10^{-9}$	1.5694378337668949736862 $\times 10^{-8}$	-0.3177026774147966926253 2
	$1\text{-}10^{-12}$	1.5694378387989754752915 $\times 10^{-11}$	-0.3177026804930740649670 0
	$1\text{-}10^{-100}$	1.5694378388040125929286 $\times 10^{-99}$	-0.3177026804961554237123 3
He:Xe	10^{-100}	3.4673186962665970744656	0.5937629666465030834817 9 $\times 10^{-99}$
	10^{-12}	3.4673186962656295647936	0.5937629666466471813509 1 $\times 10^{-11}$
	10^{-9}	3.4673186952990874019362	0.5937629667906009524482 2 $\times 10^{-8}$
	10^{-6}	3.4673177287563621145681	0.5937631107442226884220 9 $\times 10^{-5}$
	10^{-3}	3.4663506231837592181431	0.5939069147470213359841 9 $\times 10^{-2}$
	0.1	3.3644874076394762909512	0.0060638731601952336791200
	0.2	3.2473820810559992356527	0.0122791066128538259572 60
	0.3	3.1119162390641822820440	0.0183966967774790961874 57
	0.4	2.9522233048447136950036	0.0239521635747361065169 57
	0.5	2.7594972740168623722644	0.0280706923531464205213 50
	0.6	2.5197567768990965439503	0.0290537406631204965365 88
	0.7	2.2091059232089267519782	0.0234222416625769292776 81
	0.8	1.7823244353264621330658	0.0034200960048293677801596
	0.9	1.1398682451462442845056	-0.0507481920410073518283 15
	$1\text{-}10^{-3}$	1.6092867801281210588153 $\times 10^{-2}$	-2.0042654034354475370831
	$1\text{-}10^{-6}$	1.6162530382470888970178 $\times 10^{-5}$	-2.0297275153200616597052
	$1\text{-}10^{-9}$	1.6162600377374446302420 $\times 10^{-8}$	-2.0297531370138195988555
	$1\text{-}10^{-12}$	1.6162600447369683821691 $\times 10^{-11}$	-2.0297531626356737874865

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Table 4.7 – Continued

Mixture	x_1	$d_1^{(70)\star} \times 10^2$	$d_{-1}^{(70)\star} \times 10^2$
	$1\text{-}10^{-100}$	1.6162600447439749124847 $\times 10^{-99}$	-2.0297531626613212893374
Ne:Ar	10^{-100}	6.8051361546355200044767	-0.041879022254985669544761 $\times 10^{-99}$
	10^{-12}	6.8051361546325033885029	-0.041879022255127279143110 $\times 10^{-11}$
	10^{-9}	6.8051361516189040287427	-0.041879022396595267956871 $\times 10^{-8}$
	10^{-6}	6.8051331380175448445693	-0.041879163864646731259484 $\times 10^{-5}$
	10^{-3}	6.8021175363202510788528	-0.042020694597133796444143 $\times 10^{-2}$
	0.1	6.4824989441155266206228	-0.056700242944306218401378
	0.2	6.1136511017878787861433	-0.14597822768854675958116
	0.3	5.6911118788934794162800	-0.27301468281165250150124
	0.4	5.2055903330013921709977	-0.44428434727904906823195
	0.5	4.6453964413757067200473	-0.66798405442534408636952
	0.6	3.9956032964753233998547	-0.95464412580444685801376
	0.7	3.2368271824742410412939	-1.3180221071244563909314
	0.8	2.3434035465154609200701	-1.7764443620440708951122
	0.9	1.2805751103401287998939	-2.3548824696586706358720
	$1\text{-}10^{-3}$	1.4088846937865979604187 $\times 10^{-2}$	-3.0800332932392395381407
	$1\text{-}10^{-6}$	1.4103040150235432286031 $\times 10^{-5}$	-3.0882751396489053573493
	$1\text{-}10^{-9}$	1.4103054357132201046204 $\times 10^{-8}$	-3.0882833916466172523173
	$1\text{-}10^{-12}$	1.4103054371339111512870 $\times 10^{-11}$	-3.0882833998986251254943
	$1\text{-}10^{-100}$	1.4103054371353332644482 $\times 10^{-99}$	-3.0882833999068853936457
Ne:Kr	10^{-100}	6.9842020765439073950255	0.062365862237413899643744 $\times 10^{-99}$
	10^{-12}	6.9842020765413375846353	0.062365862237332075645615 $\times 10^{-11}$
	10^{-9}	6.9842020739740970031577	0.062365862155589901470987 $\times 10^{-8}$
	10^{-6}	6.9841995067318558157922	0.062365780413371533494848 $\times 10^{-5}$
	10^{-3}	6.9816306037978778801498	0.062283993970421463905658 $\times 10^{-2}$

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Table 4.7 – Continued

Mixture	x_1	$d_1^{(70)\star} \times 10^2$	$d_{-1}^{(70)\star} \times 10^2$
	0.1	6.7095576492982443716883	0.053707652236362391080685
	0.2	6.3947799779458224255944	0.087887725516225984546447
	0.3	6.0309654229517121893296	0.098359920984713534694866
	0.4	5.6061665096434919221748	0.079378255554048440181948
	0.5	5.1039329268707377900418	0.022853483753923600516486
	0.6	4.5009037487499275840346	-0.082944862849220122033484
	0.7	3.7626313957536795448026	-0.25566505926163533829738
	0.8	2.8358847838916118426785	-0.52311569402626402459159
	0.9	1.6333414488500147316851	-0.93179482271187317896422
	$1\text{-}10^{-3}$	1.9272272690270590436144 $\times 10^{-2}$	-1.5579017229222113302483
	$1\text{-}10^{-6}$	1.9307630519126044396654 $\times 10^{-5}$	-1.5657921299257655997544
	$1\text{-}10^{-9}$	1.9307665945270664023636 $\times 10^{-8}$	-1.5658000392129835050973
	$1\text{-}10^{-12}$	1.9307665980696877094628 $\times 10^{-11}$	-1.5658000471222896411730
	$1\text{-}10^{-100}$	1.9307665980732338769442 $\times 10^{-99}$	-1.5658000471302068645514
Ne:Xe	10^{-100}	6.3958166684793323688788	0.047194107980113576412401 $\times 10^{-99}$
	10^{-12}	6.3958166684773221560625	0.047194107980084438455537 $\times 10^{-11}$
	10^{-9}	6.3958166664691195513166	0.047194107950975619519709 $\times 10^{-8}$
	10^{-6}	6.3958146582652322876147	0.047194078842128931684690 $\times 10^{-5}$
	10^{-3}	6.3938051709955833039664	0.047164942221252511683547 $\times 10^{-2}$
	0.1	6.1810275840988564933328	0.043978498565561695569163
	0.2	5.9343775783820148793993	0.080091915544046163568457
	0.3	5.6476403124391831343909	0.10552906614673417712796
	0.4	5.3093313009561560326037	0.11621466919795972869582
	0.5	4.9028519977486896854760	0.10603867123828231923456
	0.6	4.4031626922541217059131	0.065446244792417791337443
	0.7	3.7703828389522436213371	-0.021330278175829250078049

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Table 4.7 – Continued

Mixture	x_1	$d_1^{(70)\star} \times 10^2$	$d_{-1}^{(70)\star} \times 10^2$
	0.8	2.9363179165871817704242	-0.1822190036136378599268 7
	0.9	1.7724656952757744504112	-0.4717198201985205184759 7
	$1\text{-}10^{-3}$	2.2465919348690653612256 $\times 10^{-2}$	-1.0048342498425198538760
	$1\text{-}10^{-6}$	2.2527811028973598295375 $\times 10^{-5}$	-1.0122501820500217826111
	$1\text{-}10^{-9}$	2.2527873103750732377271 $\times 10^{-8}$	-1.0122576247287958014642
	$1\text{-}10^{-12}$	2.2527873165825693165437 $\times 10^{-11}$	-1.0122576321715014057 85
	$1\text{-}10^{-100}$	2.2527873165887830263507 $\times 10^{-99}$	-1.0122576321789515615 72
Ar:Kr	10^{-100}	6.8864331123359307496360	-0.0280577517047405241980 31 $\times 10^{-99}$
	10^{-12}	6.8864331123321623837286	-0.0280577517049506916662 28 $\times 10^{-11}$
	10^{-9}	6.8864331085675648400445	-0.0280577519149079924168 12 $\times 10^{-8}$
	10^{-6}	6.8864293439678160216837	-0.0280579618722307938266 53 $\times 10^{-5}$
	10^{-3}	6.8826625385772440895402	-0.0282679412702446053939 47 $\times 10^{-2}$
	0.1	6.4869875415864928602251	-0.0493201201986795507557 00
	0.2	6.0399450066454203599592	-0.1423554735900808892779 3
	0.3	5.5411491375622558541078	-0.2813958013960662244759 9
	0.4	4.9855418872107064971994	-0.4694831403613106317787 1
	0.5	4.3669780784525989503508	-0.7105427492384186033343 1
	0.6	3.6779813148105787355294	-1.0095603209563401732183
	0.7	2.9094205785266462477925	-1.3728211133955258876623
	0.8	2.0500775713646046010223	-1.8082338282506004027788
	0.9	1.0860607443852140908906	-2.3257729950463662109105
	$1\text{-}10^{-3}$	1.1532363902408193104251 $\times 10^{-2}$	-2.9314485925908138948735
	$1\text{-}10^{-6}$	1.1539528064420931408134 $\times 10^{-5}$	-2.9380839643612146143970
	$1\text{-}10^{-9}$	1.1539535232677021301728 $\times 10^{-8}$	-2.9380906052532037802664
	$1\text{-}10^{-12}$	1.1539535239845281488232 $\times 10^{-11}$	-2.9380906118941012926805

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Table 4.7 – Continued

Mixture	x_1	$d_1^{(70)\star} \times 10^2$	$d_{-1}^{(70)\star} \times 10^2$
	$1\text{-}10^{-100}$	1.1539535239852456923859 $\times 10^{-99}$	-2.9380906119007488377435
Ar:Xe	10^{-100}	7.1171539072908300562494	0.061404836848073246984621 $\times 10^{-99}$
	10^{-12}	7.1171539072876513414605	0.061404836847925100979350 $\times 10^{-11}$
	10^{-9}	7.1171539041121152653678	0.061404836699927241674274 $\times 10^{-8}$
	10^{-6}	7.1171507285740849409840	0.061404688702028495058243 $\times 10^{-5}$
	10^{-3}	7.1139732354292251822722	0.061256651328757379602117 $\times 10^{-2}$
	0.1	6.7787841495671010700819	0.046160802251555233571530
	0.2	6.3951967203208228140819	0.059809270292339096313736
	0.3	5.9587860703505247767649	0.037097959065458145339902
	0.4	5.4597818473535817268260	-0.027183504749816013892827
	0.5	4.8854414731032027318782	-0.14010244977329128532683
	0.6	4.2188382477928903070849	-0.31133337939330762992877
	0.7	3.4369859494010128912307	-0.55432945136440111049190
	0.8	2.5078304550395923161791	-0.88821561096545807091737
	0.9	1.3852136913732217770068	-1.3409778992621535820615
	$1\text{-}10^{-3}$	1.5463803327197854968631 $\times 10^{-2}$	-1.9479814271398349369849
	$1\text{-}10^{-6}$	1.5482044672106161501217 $\times 10^{-5}$	-1.9551077863482633735572
	$1\text{-}10^{-9}$	1.5482062935722524752971 $\times 10^{-8}$	-1.9551149239518924436738
	$1\text{-}10^{-12}$	1.5482062953986163415740 $\times 10^{-11}$	-1.9551149310895073310806
	$1\text{-}10^{-100}$	1.5482062954004445336346 $\times 10^{-99}$	-1.9551149310966520907390
Kr:Xe	10^{-100}	6.3625549899831895174248	-0.15782386164498711599790 $\times 10^{-99}$
	10^{-12}	6.3625549899793902582758	-0.15782386164514644035730 $\times 10^{-11}$
	10^{-9}	6.3625549861839303665803	-0.15782386180431147543115 $\times 10^{-8}$
	10^{-6}	6.3625511907221944273805	-0.15782402096938441078490 $\times 10^{-5}$
	10^{-3}	6.3587538842579964679683	-0.15798322391708178446438 $\times 10^{-2}$

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Table 4.7 – Continued

Mixture	x_1	$d_1^{(70)\star} \times 10^2$	$d_{-1}^{(70)\star} \times 10^2$
	0.1	5.9636667836211603015102	-0.17414898516874150763973
	0.2	5.5247078393066594356673	-0.38263570760581984327531
	0.3	5.0421099846124873843107	-0.62828095028057410539187
	0.4	4.5117735202624057340845	-0.91434643679732453115280
	0.5	3.9289686172591868368514	-1.2446166097655892116625
	0.6	3.2882133118674370980733	-1.6234999959438491858183
	0.7	2.5831213417498324688684	-2.0561558346532122883624
	0.8	1.8062107283217263199401	-2.5486536171737218121736
	0.9	0.94866071849567548092291	-3.1081759619179639052944
	$1\text{-}10^{-3}$	0.99767932353274393356936 $\times 10^{-2}$	-3.7365219612687759475510
	$1\text{-}10^{-6}$	0.99819569574019978138763 $\times 10^{-5}$	-3.7432724554502830258084
	$1\text{-}10^{-9}$	0.99819621233987190871689 $\times 10^{-8}$	-3.7432792102218728542465
	$1\text{-}10^{-12}$	0.99819621285647180841529 $\times 10^{-11}$	-3.7432792169766487233506
	$1\text{-}10^{-100}$	0.99819621285698892543223 $\times 10^{-99}$	-3.7432792169834102607614

Table 4.8: Parameters at STP† of the gases considered in this work.

Gas	Molecular weight (amu) [42]	Molecular diameter $\times 10^8$ (cm) [10]
He	4.002602	2.193
Ne	20.1797	2.602
Ar	39.948	3.659
Kr	83.798	4.199
Xe	131.293	4.939

†STP signifies a temperature of 0 °C and a pressure of 1 atm.

As a result of obtaining normalized values of the transport coefficients for each order of approximation up to 70, there exists for each combination of parameters considered sequences of values that, at least for the thermal conductivities and the diffusion coefficients, are known to be increasing monotonically to fixed limiting values. As a part of this work, the *Mathematica*® function `SequenceLimit` has been applied to each such sequence of values in order to determine the limiting values associated with the order 70 results. The `SequenceLimit` function extrapolates to the limit using the Wynn epsilon algorithm. The extrapolated results, which are included in Tables 9-11, are truncated at the number of digits that appear to be common to most of the extrapolations done for each point using Wynn orders from 14 to 30 which is the range of Wynn orders that appears to give the most consistent extrapolated values. It appears that the extrapolated values presented in Tables 9-11 are likely correct to the precision shown excepting, perhaps, only the last digit and should be considered the best rigid-sphere, real gas mixture benchmark values available for the diffusion, thermal diffusion, and thermal conductivity coefficients. Even with the last digit considered to be unreliable, the extrapolated values of Tables 9-11 are more than double the precision of order 70 results and are, effectively, equivalent to about order 150 results. This approximate equivalency has been determined by examination of the results for simple gases where error free results to order 200 have been previously generated (Chapter 1 [15]).

Table 4.9: Limiting values of the normalized thermal conductivity coefficients, $[\lambda]^{\star}_{\infty}$, for binary mixtures of the noble gases He, Ne, Ar, Kr, Xe as functions of the mixture fractions of the lighter constituents obtained by applying the *Mathematica®* function `SequenceLimit` to the sequences of values from each of the first 70 orders of the Sonine polynomial expansion employed.

Mixture	x_1	$[\lambda]^{\star}_{\infty}$
He:Ne	10^{-100}	1.02521816832345231527552527441069478080
	10^{-12}	1.025218168323499299880470880940
	10^{-9}	1.0252181683704369202345801375
	10^{-6}	1.02521821530807072253945728
	10^{-3}	1.0252651662654557016332385
	0.1	1.029933612220840988938363
	0.2	1.0342711214919028713242832
	0.3	1.0377441997218291839800727
	0.4	1.04003097861405774384584220
	0.5	1.040950801811300863349658061
	0.6	1.0404352117351041030692180291
	0.7	1.03850009640026216045285796666
	0.8	1.035223945266113854349410283007
	0.9	1.03073594856984348311376456581078
	1-10^{-3}	1.025277679084433960233188771656782747
	1-10^{-6}	1.025218227873304858096118255809071649
	1-10^{-9}	1.025218168383002206858838206800993458
	1-10^{-12}	1.02521816832351186516714762778430180623
	1-10^{-100}	1.02521816832345231527552527441069478080
He:Ar	10^{-100}	1.02521816832345231527552527441069478080
	10^{-12}	1.02521816832365617160484992
	10^{-9}	1.02521816852730864427479
	10^{-6}	1.0252183721794559382770
	10^{-3}	1.025421699194638091083
	0.1	1.0425705726479693427
	0.2	1.05461121996223147659
	0.3	1.062218693886722936702
	0.4	1.0660150645757235901741
	0.5	1.06643530478454629852699
	0.6	1.063771773293864765106500
	0.7	1.0582057049808153810804488
	0.8	1.0498390730905283184266099
	0.9	1.038756513000525715854592297
	1-10^{-3}	1.025363391654560791069261181841521
	1-10^{-6}	1.025218313623317737765759496046063828
	1-10^{-9}	1.02521816846875225691070124977657560893
	1-10^{-12}	1.02521816832359761521723666274892113525
	1-10^{-100}	1.02521816832345231527552527441069478080
He:Kr	10^{-100}	1.02521816832345231527552527441069478080

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Table 4.9 – Continued

Mixture	x_1	$[\lambda]^{\star}_{\infty}$
	10^{-12}	1.0252181683238784202953\u0035
	10^{-9}	1.025218168749557333842
	10^{-6}	1.025218594427210104
	10^{-3}	1.02564301420239973
	0.1	1.0575547191488317
	0.2	1.07567732424800302
	0.3	1.085065924232252734
	0.4	1.088431043922897363
	0.5	1.087188691594638633
	0.6	1.08205916244733634099
	0.7	1.07333564521470226453
	0.8	1.06101944207578693495\u0039\u0033
	0.9	1.04494760814120670234615
	1-10^{-3}	1.0254296331624707681354071\u0038\u0031
	1-10^{-6}	1.02521837987723151956140390189689
	1-10^{-9}	1.0252181685350061825403934983961960
	1-10^{-12}	1.02521816832366386914287820233248078\u0036
	1-10^{-100}	1.0252181683234523152755252744106947808\u0030
He:Xe	10^{-100}	1.0252181683234523152755252744106947808\u0030
	10^{-12}	1.025218168324098341179
	10^{-9}	1.025218168969478216
	10^{-6}	1.02521881434660117
	10^{-3}	1.0258614492348593
	0.1	1.069718856501851
	0.2	1.090946448927887
	0.3	1.1007030665750814
	0.4	1.1035362533662975
	0.5	1.10142618839514720
	0.6	1.095192857402818518
	0.7	1.0849874866969934562
	0.8	1.07045329268151107\u0030\u0039
	0.9	1.050781014791557230636
	1-10^{-3}	1.025499334139315095601675\u0030\u0036
	1-10^{-6}	1.0252184496491771302082875061
	1-10^{-9}	1.025218168604778198062515204861
	1-10^{-12}	1.025218168323733641158470234445752\u0037\u0033
	1-10^{-100}	1.0252181683234523152755252744106947808\u0030
Ne:Ar	10^{-100}	1.0252181683234523152755252744106947808\u0030
	10^{-12}	1.0252181683234700489231542701012045897\u0030
	10^{-9}	1.02521816834118596289102939010339\u0036
	10^{-6}	1.0252181860570864392008946665369619
	10^{-3}	1.025235888462068563158802630877\u0038\u0032
	0.1	1.026852569867625674149660252005\u0036
	0.2	1.028193471448891518368343828610\u0036

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Table 4.9 – Continued

Mixture	x_1	$[\lambda]^\star_\infty$
	0.3	1.02921738020793234181956907354269
	0.4	1.029900081229369943276682339282927
	0.5	1.030215949156119081346441326073007
	0.6	1.030137261963999160207270661128510
	0.7	1.0296335518883510773651644338145078
	0.8	1.02867108161706677529731721407818457
	0.9	1.02721264826807272602467866135220343
	1-10^{-3}	1.0252409015804517672381685310852867758
	1-10^{-6}	1.0252181910855254369764296644096108038
	1-10^{-9}	1.0252181683462144172200007537833810390
	1-10^{-12}	1.02521816832347507737749857267129285887
	1-10^{-100}	1.02521816832345231527552527441069478080
Ne:Kr	10^{-100}	1.02521816832345231527552527441069478080
	10^{-12}	1.025218168323523547392339438998
	10^{-9}	1.0252181683946844320333305207
	10^{-6}	1.025218239555512713713449248
	10^{-3}	1.02528934399912160083787507
	0.1	1.0317538034466545185316162
	0.2	1.03703457196302089223279512
	0.3	1.04096603112360821703647572
	0.4	1.043476855355736301929171
	0.5	1.044507169432041590784811356
	0.6	1.0439985955546553411655331714
	0.7	1.04188656854128634230706704345
	0.8	1.0380969495951095201363274227354
	0.9	1.032553557996932673245215478638602
	1-10^{-3}	1.02530021186447871442519071043054617643
	1-10^{-6}	1.02521825045157134245835927428867355946
	1-10^{-9}	1.02521816840558051883188160515601329246
	1-10^{-12}	1.02521816832353444347916615986588178461
	1-10^{-100}	1.02521816832345231527552527441069478080
Ne:Xe	10^{-100}	1.02521816832345231527552527441069478080
	10^{-12}	1.0252181683236047194183992965
	10^{-9}	1.02521816847585645794287594
	10^{-6}	1.0252183207273883113155541
	10^{-3}	1.02537036571341900031626
	0.1	1.0385045279376701883941
	0.2	1.04826640158926140711657
	0.3	1.05493898195335992241080
	0.4	1.058807426601329051621293
	0.5	1.060035867583680514869077
	0.6	1.05868150110502293201057481
	0.7	1.054698447614547763530683129
	0.8	1.047937993669552804899377751

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Table 4.9 – Continued

Mixture	x_1	$[\lambda]^{\star}_{\infty}$
	0.9	1.03816797675763872317592017410
	1-10^{-3}	1.025362762347695018467799084958151203
	1-10^{-6}	1.0252183130551717891830055836918280423
	1-10^{-9}	1.02521816846818417218901301890637760402
	1-10^{-12}	1.02521816832359704713257620147825784345
	1-10^{-100}	1.02521816832345231527552527441069478080
Ar:Kr	10^{-100}	1.02521816832345231527552527441069478080
	10^{-12}	1.0252181683234604898878520487048307634
	10^{-9}	1.0252181683316269276043398068280595
	10^{-6}	1.025218176498066684313818961845019
	10^{-3}	1.02522634496352952470958983472354
	0.1	1.026041972182759490707888549975
	0.2	1.026825787427664537509867678628
	0.3	1.0274987585311430617256149396016
	0.4	1.02800063601515905754262048516934
	0.5	1.028281222566677577947753161795400
	0.6	1.0282996460522037136913352605455207
	0.7	1.0280235657720448767911156993876201
	0.8	1.027428417724990778299321259240671492
	0.9	1.026496807098092139301042953855765482
	1-10^{-3}	1.0252326904595083265683349253177422434
	1-10^{-6}	1.02521818286313630178105785597594494536
	1-10^{-9}	1.02521816833799201680998481789246269899
	1-10^{-12}	1.02521816832346685497707728190820785950
	1-10^{-100}	1.02521816832345231527552527441069478080
Ar:Xe	10^{-100}	1.02521816832345231527552527441069478080
	10^{-12}	1.02521816832348692620494043777517
	10^{-9}	1.025218168358063244681772981865
	10^{-6}	1.02521820293437280582300208881
	10^{-3}	1.02525277028506053372446985041
	0.1	1.028548878021145605382296325
	0.2	1.031470814648072957453159551
	0.3	1.0337974098516747615261165571
	0.4	1.03538502096898526302270981041
	0.5	1.03612646824127711187251486888
	0.6	1.035944324866904389858333631599
	0.7	1.0347850603709656719589487065358
	0.8	1.03261523758285882681906185711368
	0.9	1.0294215485323289043326862477042739
	1-10^{-3}	1.02526505027991585376463091515257001157
	1-10^{-6}	1.02521821525305620028804988210716454591
	1-10^{-9}	1.02521816837038196679180026926889461062
	1-10^{-12}	1.02521816832349924492708918065182461465

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Table 4.9 – Continued

Mixture	x_1	$[\lambda]^\star_\infty$
	1-10^{-100}	1.02521816832345231527552527441069478080
Kr:Xe	10^{-100}	1.02521816832345231527552527441069478080
	10^{-12}	1.0252181683234577755530802668810185826
	10^{-9}	1.0252181683289125928269119215716213766
	10^{-6}	1.0252181737837262608368657364551519212
	10^{-3}	1.02522362498942909609132319308034432
	0.1	1.0257259902623583823512265551450869
	0.2	1.0261492215722242213005229749095566
	0.3	1.02647616377549971563271556296998718
	0.4	1.02669582320929315566985027582703316
	0.5	1.02679784277496595713255591531988119
	0.6	1.026772437570126674141014390632986689
	0.7	1.026610338499758148153216018067542602
	0.8	1.026302749083486114139961655010977551
	0.9	1.0258413224542803786373749695417121610
	1-10^{-3}	1.02522522512197468093706711520692696675
	1-10^{-6}	1.0252181753886992495056356565557644633
	1-10^{-9}	1.0252181683305175706593512688929871061
	1-10^{-12}	1.02521816832345938053091755000224493999
	1-10^{-100}	1.02521816832345231527552527441069478080

Table 4.10: Limiting values of the normalized diffusion coefficients, $[D_{12}]^\star_\infty$, for binary mixtures of the noble gases He, Ne, Ar, Kr, Xe as functions of the mixture fractions of the lighter constituents obtained by applying the *Mathematica*® function `SequenceLimit` to the sequences of values from each of the first 70 orders of the Sonine polynomial expansion employed.

Mixture	x_1	$[D_{12}]^\star_\infty$
He:Ne	10^{-100}	1.01831682648121609784786
	10^{-12}	1.01831682648119560983426
	10^{-9}	1.01831682646072808424613
	10^{-6}	1.01831680599320382230743
	10^{-3}	1.01829633979477629789585
	0.1	1.016281021072661370258670
	0.2	1.0142707324702496526939133
	0.3	1.01228683607421585193499622
	0.4	1.010332698268826747835270929
	0.5	1.0084148710655480589782700635
	0.6	1.00654415323265631405947916301
	0.7	1.004737182772263130903084967507

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Table 4.10 – Continued

Mixture	x_1	$[D_{12}]^{\star}_{\infty}$
	0.8	1.0030188170227353370556218820082
	0.9	1.00142568730722360518393794135887
	1-10^{-3}	1.000024497300712106170916674011284060
	1-10^{-6}	1.0000114859182961044521136949612643382
	1-10^{-9}	1.00001147291960385779735239058755497366559
	1-10^{-12}	1.000011472906605178256319026501910294670773
	1-10^{-100}	1.000011472906592166565099479240132121638603
He:Ar	10^{-100}	1.0276991547067152383
	10^{-12}	1.0276991547066927650
	10^{-9}	1.0276991546842419330
	10^{-6}	1.0276991322334054092
	10^{-3}	1.0276766769121204747
	0.1	1.02540499352297683991
	0.2	1.023009202582916541116
	0.3	1.020499688093753658257
	0.4	1.0178647622219440512661
	0.5	1.01509465999213917520683
	0.6	1.012184834346976090481746
	0.7	1.0091430858762757264156966
	0.8	1.00600538003401512105042407
	0.9	1.0028725699954631378588986255
	1-10^{-3}	1.0000267165334709955682543273498
	1-10^{-6}	1.000000911918490880405152993252869
	1-10^{-9}	1.000000886155339534988925177659325799
	1-10^{-12}	1.00000088612957642526052597534537288044
	1-10^{-100}	1.00000088612955063636194060798808437619
He:Kr	10^{-100}	1.0352327170099902
	10^{-12}	1.0352327170099647
	10^{-9}	1.035232716984525
	10^{-6}	1.0352326915449476
	10^{-3}	1.035207245709673
	0.1	1.032619726507292
	0.2	1.02985741037526330
	0.3	1.026919155610518715
	0.4	1.0237753359212331549
	0.5	1.0203937341131676153
	0.6	1.016742003941597094644
	0.7	1.012795632171047187358
	0.8	1.0085614271055387795673
	0.9	1.00414838673627370252006
	1-10^{-3}	1.000036748907021914294188059
	1-10^{-6}	1.000000086478966164652556705109
	1-10^{-9}	1.00000004989590494281299912570769
	1-10^{-12}	1.0000000498593219613757736426133272

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Table 4.10 – Continued

Mixture	x_1	$[D_{12}]^{\star}_{\infty}$
	1-10^{-100}	1.0000000498592853417748152438444619
He:Xe	10^{-100}	1.038293259238067
	10^{-12}	1.038293259238044
	10^{-9}	1.038293259215191
	10^{-6}	1.03829323636272
	10^{-3}	1.038270375794141
	0.1	1.035919533616143
	0.2	1.033350957713110
	0.3	1.0305484178041635
	0.4	1.02746380710207373
	0.5	1.02403760817321869
	0.6	1.020196992910507289
	0.7	1.0158577112681115963
	0.8	1.0109426505517480600
	0.9	1.005471867492565516266
	1-10^{-3}	1.0000486313855238742296529
	1-10^{-6}	1.0000000569977426721219231625
	1-10^{-9}	1.00000000855092410500665729557
	1-10^{-12}	1.0000000085024774149929165575420
	1-10^{-100}	1.0000000085024289198078463010392
Ne:Ar	10^{-100}	1.0062522721826473052154086523687
	10^{-12}	1.0062522721826426354630508485802
	10^{-9}	1.0062522721779775528564358261868
	10^{-6}	1.00625226751289377720470846062277
	10^{-3}	1.00624760125985963374572214752383
	0.1	1.00577337211012093229293285762775
	0.2	1.00526973142610319753698679538908
	0.3	1.004740041047841454305180663449064
	0.4	1.004183096030722742935609059620682
	0.5	1.003597923048795965703625156672474
	0.6	1.0029839853027986999944035311278366
	0.7	1.00234151232935847811963989963576840
	0.8	1.001672034386448497412468581494630662
	0.9	1.0009792579461158030780948154969897966
	1-10^{-3}	1.0002776505333489962426871183921656793087
	1-10^{-6}	1.00027052864851041671352224684840589341298
	1-10^{-9}	1.00027052152645519982593896136683784476896359
	1-10^{-12}	1.00027052151933314444154168227579680690715288
	1-10^{-100}	1.00027052151932601525697254835579515866261690
Ne:Kr	10^{-100}	1.015505686574438721016146
	10^{-12}	1.0155056865744269102667173
	10^{-9}	1.0155056865626279715837807
	10^{-6}	1.0155056747636860408777936

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Table 4.10 – Continued

Mixture	x_1	$[D_{12}]^\star_\infty$
	10^{-3}	1.0154938725730782322298528
	0.1	1.0142911781379681969777048
	0.2	1.01300601350130224961773321
	0.3	1.01164440680785693696580514
	0.4	1.0102007770730447192901925180
	0.5	1.0086705244075581641826085930
	0.6	1.00705155999943675839168118215
	0.7	1.005347331897098044881893037255
	0.8	1.0035729243629650568174301785987
	0.9	1.001767749490295972878172662105549
	$1\text{-}10^{-3}$	1.00003969457203763023007325590065271801
	$1\text{-}10^{-6}$	1.000023092957050471927647553569757522954
	$1\text{-}10^{-9}$	1.00002307636769105096409953058212176261660
	$1\text{-}10^{-12}$	1.000023076351101703845851287115961922233499
	$1\text{-}10^{-100}$	1.000023076351085097892792282642874938329948
Ne:Xe	10^{-100}	1.021992270374199610512
	10^{-12}	1.021992270374185128839
	10^{-9}	1.02199227035971793708
	10^{-6}	1.021992255892520901500
	10^{-3}	1.021977783414322344989
	0.1	1.0204890430707023341454
	0.2	1.01886613675375711528625
	0.3	1.01710757344347706810780
	0.4	1.015195167849895726846184
	0.5	1.0131089292344308255887553
	0.6	1.0108286726193813454896806
	0.7	1.008338740357427117273032259
	0.8	1.0056409566784892847078258504
	0.9	1.002790746805246591579925273898
	$1\text{-}10^{-3}$	1.00003040719651646667805835366715134
	$1\text{-}10^{-6}$	1.0000045033143773098857599568661710146
	$1\text{-}10^{-9}$	1.000004477443935650895636036801379081496
	$1\text{-}10^{-12}$	1.000004477418065242860499454790227197878407
	$1\text{-}10^{-100}$	1.000004477418039346556193636460977064828465
Ar:Kr	10^{-100}	1.0068324667203317164537282110923
	10^{-12}	1.0068324667203251605675533016056
	10^{-9}	1.0068324667137758302784702472518
	10^{-6}	1.0068324601644451927187955202597
	10^{-3}	1.0068259104855023554461421208507
	0.1	1.00617356742366568743679879765311
	0.2	1.00550881496124196295677072971303
	0.3	1.004839524242155165457467110883483
	0.4	1.004167291669056657458540126045356
	0.5	1.0034940900571199643452001401556915

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Table 4.10 – Continued

Mixture	x_1	$[D_{12}]^{\star}_{\infty}$
	0.6	1.00282238574888487655572579088052585
	0.7	1.00215528563176738427247865729331590
	0.8	1.001496724076594829779736134376633777
	0.9	1.00085170289122998366340371750084555853
	1-10^{-3}	1.000232728674352654341060809980374440233
	1-10^{-6}	1.00022660746498123437495542712683411270398
	1-10^{-9}	1.0002266013451633539675371204877387427614731
	1-10^{-12}	1.00022660133904353748014518509949508113488256
	1-10^{-100}	1.00022660133903741153771675775516385998098386
Ar:Xe	10^{-100}	1.01223484970167383677219424
	10^{-12}	1.01223484970166278750984580
	10^{-9}	1.01223484969062457442219133
	10^{-6}	1.01223483865240991801245043
	10^{-3}	1.01222379886889776967460649
	0.1	1.011114134478969456323504791
	0.2	1.0099616503843936792566159600
	0.3	1.008775859371242079125200981
	0.4	1.007563075262406250005323804825
	0.5	1.006320755541292932953026566509
	0.6	1.0050556258810593802641532873928
	0.7	1.003776393622580599577078058614114
	0.8	1.002497623115601267751291674575991
	0.9	1.0012432298262261108163629364210728
	1-10^{-3}	1.0000636769289761947250006744868289196
	1-10^{-6}	1.0000522751552412447238033510082675917118
	1-10^{-9}	1.000052263759647887517974282109716834663749
	1-10^{-12}	1.0000522637482523003543345185006418351409890
	1-10^{-100}	1.0000522637482408933601829207247205649924320
Kr:Xe	10^{-100}	1.0042215316500266071664437767945052
	10^{-12}	1.0042215316500232467643361718352947
	10^{-9}	1.0042215316466662050584638993587417
	10^{-6}	1.0042215282896241242683548020370305
	10^{-3}	1.0042181708726392933168578256652282
	0.1	1.0038817543497835431662276659373973
	0.2	1.00353458457110902809594400165189914
	0.3	1.00318019315073501914154303723700679
	0.4	1.00281882992486308810499074754102546
	0.5	1.002450845821952800579697546097343598
	0.6	1.002076720631906145202910082737402712
	0.7	1.001697098064084212035542127394146072
	0.8	1.001312830206529645525750737174279678
	0.9	1.00092503416692994054791734845180412038
	1-10^{-3}	1.00053906775222710541632636038292290024681
	1-10^{-6}	1.000535168474396855869010808452547697882641

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Table 4.10 – Continued

Mixture	x_1	$[D_{12}]^{\star}_{\infty}$
	1-10^{-9}	1.0005351645751069970858033432533635642858984
	1-10^{-12}	1.000535164571207707215345513570109945808477288
	1-10^{-100}	1.000535164571203804022281980533109938163373981

Table 4.11: Limiting values of the normalized thermal diffusion coefficients, $[D_T]^{\star}_{\infty}$, for binary mixtures of the noble gases He, Ne, Ar, Kr, Xe as functions of the mixture fractions of the lighter constituents obtained by applying the *Mathematica*® function `SequenceLimit` to the sequences of values from each of the first 70 orders of the Sonine polynomial expansion employed.

Mixture	x_1	$[D_T]^{\star}_{\infty}$
He:Ne	10^{-100}	1.1727641524477615750680
	10^{-12}	1.1727641524476594232243
	10^{-9}	1.1727641523456097313347
	10^{-6}	1.1727640502959042014150
	10^{-3}	1.172661986942640152255
	0.1	1.16240535092409546437959
	0.2	1.151733450258779078165309
	0.3	1.1407145035474531018333009
	0.4	1.1293204418648855917929027
	0.5	1.117529248706176910562749919
	0.6	1.1053264335987840714281299953
	0.7	1.09270834717810430552186919012
	0.8	1.079688431796450137406436318265
	0.9	1.0663085295817499537119031119613
	1-10^{-3}	1.0527967781237656598798134621924725
	1-10^{-6}	1.0526595763539228477890814796077799
	1-10^{-9}	1.0526594391469324381230441964468612
	1-10^{-12}	1.0526594390097254425317572145192112
	1-10^{-100}	1.0526594390095880981918208095451919
He:Ar	10^{-100}	1.243619954924111046
	10^{-12}	1.243619954923981229
	10^{-9}	1.243619954794293805
	10^{-6}	1.243619825106835715
	10^{-3}	1.24349010330504121
	0.1	1.230275721403607946
	0.2	1.21612494592888583799
	0.3	1.20103005324014952888
	0.4	1.1848290004479504991591
	0.5	1.1673297089742302244567

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Table 4.11 – Continued

Mixture	x_1	$[D_T]^{\star}_{\infty}$
	0.6	1.14830465721066002919153
	0.7	1.127488770472888639366010
	0.8	1.1045900958372877153005938
	0.9	1.07934489266614582793961747
	1-10^{-3}	1.052016360439920344186688592327
	1-10^{-6}	1.051730438837696051382546211154301
	1-10^{-9}	1.051730152835598667135568915260216
	1-10^{-12}	1.0517301525495964895710084979199832
	1-10^{-100}	1.0517301525493102011048976465738973
He:Kr	10^{-100}	1.29680196405877
	10^{-12}	1.296801964058623
	10^{-9}	1.296801963904337
	10^{-6}	1.296801809618220
	10^{-3}	1.29664748223939
	0.1	1.2809161416620045
	0.2	1.26402097767581563
	0.3	1.24590062648835273
	0.4	1.226291891983980966
	0.5	1.20486665657155927
	0.6	1.1812088971226820728
	0.7	1.15478693860466367423
	0.8	1.124931596578372005971
	0.9	1.0908788523228468198851
	1-10^{-3}	1.0525993069657142893516012
	1-10^{-6}	1.052192515499378178727343546
	1-10^{-9}	1.05219210855331770199397386348
	1-10^{-12}	1.0521921081463714877940226731957
	1-10^{-100}	1.0521921081459641342261010779148
He:Xe	10^{-100}	1.3178691333862
	10^{-12}	1.3178691333860
	10^{-9}	1.3178691332459
	10^{-6}	1.3178689931225
	10^{-3}	1.3177288186014
	0.1	1.30329652274027
	0.2	1.28747601991466
	0.3	1.270129607629226
	0.4	1.25089914161468
	0.5	1.2293107547548959
	0.6	1.20472031166527301
	0.7	1.1762289206590653
	0.8	1.1425582112572240534
	0.9	1.10192587943654879158
	1-10^{-3}	1.05301318598849681760809
	1-10^{-6}	1.052475658775180015170604

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Table 4.11 – Continued

Mixture	x_1	$[D_T]^\star_\infty$
	1-10^{-9}	1.05247512089332216205029483
	1-10^{-12}	1.05247512035543995141906715
	1-10^{-100}	1.052475120354901530787451533319
Ne:Ar	10^{-100}	1.104424264658880648063125033914
	10^{-12}	1.10442426465884983840637602236
	10^{-9}	1.104424264628070991303709857187
	10^{-6}	1.104424233849213485017672747982
	10^{-3}	1.104393444584386609628102746372
	0.1	1.1012353408148875923061748574008
	0.2	1.0978142571966465663469601564817
	0.3	1.09413425465296295860153907786531
	0.4	1.09016498091869324769376661413760
	0.5	1.08587193805873406347512090948164
	0.6	1.081215884026707059173859701090421
	0.7	1.0761522347761726487875468867188601
	0.8	1.07063057736215325876313463941277386
	0.9	1.064594539496125430083881737040787143
	1-10^{-3}	1.0580517320233088747005463840769884866
	1-10^{-6}	1.0579826198431211967553689876805395059
	1-10^{-9}	1.0579825506989729518317911493031019743
	1-10^{-12}	1.0579825506298287716081377494905320933
	1-10^{-100}	1.0579825506297595582144884680367939403
Ne:Kr	10^{-100}	1.161800743578499574398635
	10^{-12}	1.1618007435784344801684
	10^{-9}	1.161800743513405344156502
	10^{-6}	1.161800678484242984537320
	10^{-3}	1.161735622962522524908284
	0.1	1.1550154735938151215065108
	0.2	1.1476262724829947443892895
	0.3	1.1395450966970005990700234
	0.4	1.1306687705987887699774956
	0.5	1.120875885894715737122506156
	0.6	1.11002358180145268724267663397
	0.7	1.09794518631475976604907006328
	0.8	1.084451617104771075175724013747
	0.9	1.06934497075866507169521913328204
	1-10^{-3}	1.052647548994092161853691449705390930
	1-10^{-6}	1.052470071795041580745340339418072555
	1-10^{-9}	1.052469894229782001469282456841852676
	1-10^{-12}	1.052469894052216653867810053118711888
	1-10^{-100}	1.052469894052038910777029778194288725
Ne:Xe	10^{-100}	1.206591368318376992471
	10^{-12}	1.206591368318294888289

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Table 4.11 – Continued

Mixture	x_1	$[D_T]^\star_\infty$
	10^{-9}	1.206591368236272810274
	10^{-6}	1.206591286214157592526
	10^{-3}	1.206509226876900658260
	0.1	1.197988800775016129400
	0.2	1.1885143309443707460948
	0.3	1.17801377231849665003442
	0.4	1.16629519418879548954900
	0.5	1.153117039500250634668426
	0.6	1.1381723588451261503194806
	0.7	1.12106926337988198140680085
	0.8	1.101312415019739045037403540
	0.9	1.07831138998805696602964402438
	1-10^{-3}	1.0518275585004210003473225245884968
	1-10^{-6}	1.0515411132567119738187386724680335
	1-10^{-9}	1.0515408266304189902749185982412299
	1-10^{-12}	1.0515408263437925164734181878960175
	1-10^{-100}	1.0515408263435056030860484998932885
Ar:Kr	10^{-100}	1.104062881151317652060940072939
	10^{-12}	1.104062881151281102187118148227
	10^{-9}	1.10406288111476777823184399742
	10^{-6}	1.104062844601436651599672229690
	10^{-3}	1.104026324096864608327527283293
	0.1	1.100334059863211185728864060327
	0.2	1.096449686979424259495619612619
	0.3	1.0923985781039877152926844162867
	0.4	1.0881703394320983283902951116336
	0.5	1.083755347661942718463651616354892
	0.6	1.07914481297165405089442122186016
	0.7	1.0743309456609715880234728930453344
	0.8	1.0693072600999595292311868190697444
	0.9	1.06406906888288642039531858764821071
	1-10^{-3}	1.0586698686117168269858245029823834600
	1-10^{-6}	1.0586143051789703141721902618237003407
	1-10^{-9}	1.0586142496047743957435167007499634389
	1-10^{-12}	1.0586142495492001890625511570299147781
	1-10^{-100}	1.0586142495491445592260229006828522345
Ar:Xe	10^{-100}	1.13808388977261100994414206
	10^{-12}	1.13808388977255209164191001
	10^{-9}	1.1380838897136927076942554
	10^{-6}	1.13808383085429092596802237
	10^{-3}	1.13802495361183838442178211
	0.1	1.13200686548347424797285530
	0.2	1.125532129205771134717307468
	0.3	1.1186170876383055906905991751

Continued on Next Page...

Table 4.11 – Continued

Mixture	x_1	$[D_T]^\star_\infty$
	0.4	1.1112167411098725901551497142
	0.5	1.10328358858029748008634871761
	0.6	1.094768024231321759910818909698
	0.7	1.0856196775951888772122932957870
	0.8	1.0757905940661633018790112097551
	0.9	1.065242083736639387711989556987402
	$1\text{-}10^{-3}$	1.054075521178631407617748040254408639
	$1\text{-}10^{-6}$	1.053959232063619076516133556536700361
	$1\text{-}10^{-9}$	1.053959115739301430297605846954604125
	$1\text{-}10^{-12}$	1.053959115622977077461454942987405991
	$1\text{-}10^{-100}$	1.053959115622860636667789937122754732
Kr:Xe	10^{-100}	1.09039022057540624845151718863659
	10^{-12}	1.09039022057538364066792063565779
	10^{-9}	1.090390220552798464850614541722420
	10^{-6}	1.090390197967618297879840388789254
	10^{-3}	1.09036760843687480649313001746969
	0.1	1.08808497952320662811731879518331
	0.2	1.08568704284418618085254412801513
	0.3	1.083190568819536128167891414455644
	0.4	1.0805894594508077610409633432603989
	0.5	1.0778773551066907050969170764086505
	0.6	1.07504763668425875061826263211448991
	0.7	1.07209343887218877393585763638799538
	0.8	1.069007679672099447814970894586439270
	0.9	1.0657831136199245401981045459119579001
	$1\text{-}10^{-3}$	1.0624468737640365743874394573232253460
	$1\text{-}10^{-6}$	1.0624124539748200894912581259552910657
	$1\text{-}10^{-9}$	1.0624124195473693182368638116721450222
	$1\text{-}10^{-12}$	1.0624124195129418598028580838886527486
	$1\text{-}10^{-100}$	1.0624124195129073978824960533701580748

4.9 Discussion and conclusions

The purpose in this work has been to explore the use of Sonine polynomial expansions to obtain error free results for the transport coefficients and related Chapman-Enskog functions for simple gases and gas mixtures. In Chapter 2 [15], the case of simple gases was explored. In Chapter 3 [16], the initial work was extended to viscosity in binary, real gas mixtures. Here, this work has been extended further to include an exploration of diffusion, thermal diffusion, and thermal conductivity and the related Chapman-Enskog solutions for the same set of binary, real gas mixtures. For specific results, the focus in this work, as in the previous work, has been on rigid-sphere molecules, since the omega integrals are readily available in a simple analytical form. However, it should be emphasized that the expressions obtained are completely general and results for any potential model can now readily be obtained subject only to omega integral values for the given potential model being available to sufficient precision.

Sonine polynomial expansions only up to order 70 have been considered in this chapter but the computational tools and programs that have been constructed apply to expansions of arbitrary order and the only limiting factors that have been encountered involve the speed and memory capacity of the computational resources available. Nevertheless, because of the high precision of the computations, excellent convergence has been obtained in all of the results and, via extrapolation, transport coefficient values to a precision of more than 14 significant figures have been obtained for all of the rigid-sphere gas mixtures considered. Note that for the case of viscosity (Chapter 3 [16]), a comparison of the numerical results with those of Garcia and Siewert [1] was made. Similar comparisons have not been made here as it appears that their computations would need to be recast in a different form for a direct comparison with the present results.

Since even in a binary-gas mixture the parameter space is very large (involving mole fractions, mass ratios, size ratios, temperature, etc.), it is always useful to have techniques or expressions such as those obtained in the current work that make rapid and precise computations possible. It is because the development of a very general set of such computational tools and techniques has been emphasized in this work, comparisons with the results of Takata et al. [2] have been relatively straightforward and quick to obtain. It is this same generality that has allowed the extension of this work in a straightforward manner to additional real gas mixtures and which will allow for a much more substantial parametric study in the near future as well as for additional study of realistic potential models. The attractiveness of this work is obvious with respect to ternary and multiple gas mixtures as no new basic expressions need to be computed and the computations would be straightforward. Efforts in this direction are currently underway.

Chapter 5

General expressions for the viscosity-related bracket integrals to order 5

5.1 Introduction

The Chapman-Enskog solutions of the Boltzmann equations provide a basis for the computation of important transport coefficients for both simple gases and gas mixtures [10–12, 20–26, 28–31, 39]. The use of Sonine polynomial expansions for the Chapman-Enskog solutions was first suggested by Burnett [13] and has become the general method for obtaining the transport coefficients due to the relatively rapid convergence of this series [10–13, 20–24]. While it has been found that relatively, low-order expansions (of order 4) can provide reasonable accuracy in computations of the transport coefficients (to about 1 part in 1,000), most existing computer codes do not use these solutions beyond order 2 or order 3 as the relevant expressions become increasingly complex with great rapidity and have not been available as general, explicit expressions in terms of arbitrary potential model (via the omega integrals) in the past. The present investigations of simple gases and gas mixtures described in the previous chapters [15–17] has permitted the exploration of the Chapman-Enskog solutions to relatively high orders computationally using *Mathematica*® and, thus, accurate, completely general, expressions have become available up to order 60 for the viscosity-related bracket integrals and up to order

70 for the diffusion- and thermal conductivity-related bracket integrals. The complexity of the general expressions that have been obtained for the bracket integrals for gas mixtures does make them impractical to report in the open literature beyond the lowest orders even when organized into compact form. However, insofar as such expressions can be explicitly reported in the literature, they are valuable from the point of view of having them available for immediate use in existing computer codes. Further, having such expressions explicitly reported in the literature, even to limited order, has a certain archival value in the field. Thus, in this chapter, a set of completely general expressions for the bracket integrals necessary to complete the Chapman-Enskog viscosity solutions up to order 5 is reported. In the following sections the basic relationships relevant to the Chapman-Enskog solutions for viscosity, the role of the bracket integrals in these solutions, the explicit expressions for the various bracket integrals, and some basic relationships that occur among the various bracket integrals that make them more tractable to generate and manipulate in the context of the Chapman-Enskog solutions are described.

5.2 The basic relationships

Following the work and notations of Chapman and Cowling [10] as was done in the previous chapters [15–17], the viscosity for binary gas mixtures the viscosity may be expressed to some order of approximation, m, (in terms of Sonine polynomial expansions) as:

$$[\mu]_m = p\left(x_1 b_1^{(m)} + x_2 b_{-1}^{(m)}\right) \ , \tag{5.1}$$

where $x_1 = n_1/n$ and $x_2 = n_2/n$ are the component mixture fractions, n_1 and n_2 are the component number densities with $n = n_1 + n_2$ being the total number density of the mixture, p is the total hydrostatic pressure of the system, m_1 and m_2 are the component molecular masses with $m_0 = m_1 + m_2$, k is Boltzmann's constant, T is the temperature, $M_1 = m_1/m_0$, $M_2 = m_2/m_0$, and in which the quantities

$b_{-1}^{(m)}$ and $b_1^{(m)}$ are expansion coefficients determined by solving the following system of algebraic equations:

$$\sum_{\substack{p=-m \\ p\neq 0}}^{+m} b_p b_{pq} = \beta_q \quad (q = \pm 1, \pm 2, \cdots, \pm m) \ , \tag{5.2}$$

in which:

$$\beta_1 = \tfrac{5}{2}\frac{n_1}{n^2} \ , \quad \beta_{-1} = \tfrac{5}{2}\frac{n_2}{n^2} \ , \quad \beta_q = 0 \ \ (q \neq \pm 1) \ . \tag{5.3}$$

Expressed as matrices, this system of equations may be written as:

$$\begin{bmatrix} b_{-m-m} & \cdots & b_{-m-1} & b_{-m1} & \cdots & b_{-mm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{-1-m} & \cdots & b_{-1-1} & b_{-11} & \cdots & b_{-1m} \\ b_{1-m} & \cdots & b_{1-1} & b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{m-m} & \cdots & b_{m-1} & b_{m1} & \cdots & b_{mm} \end{bmatrix} \begin{bmatrix} b_{-m} \\ \vdots \\ b_{-1} \\ b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \beta_{-1} \\ \beta_1 \\ \vdots \\ 0 \end{bmatrix} \ , \tag{5.4}$$

where, as the order of the expansion increases, the matrices build outward from their centers in the manner indicated. Thus, to obtain the needed expansion coefficients, $b_{-1}^{(m)}$ and $b_1^{(m)}$, and hence the viscosity coefficient for a given order of the expansion, m, one need only generate the $[(2m) \times (2m)]$ matrix of Eq. (5.4) and invert it.

The matrix elements, b_{pq}, in Eq. (5.4) are constructed from combinations of bracket integrals containing the appropriate Sonine polynomials from the expansions used. Specifically, since it is straightforward to show for any (p, q) that $b_{pq} = b_{qp}$, one has that:

$$\begin{aligned} b_{pq} = b_{qp} =& x_1^2 \big[S_{5/2}^{(p-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_1 \overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_1 \overset{\circ}{\boldsymbol{\mathscr{C}}}_1 \big]_1 \\ &+ x_1 x_2 \big[S_{5/2}^{(p-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_1 \overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_1 \overset{\circ}{\boldsymbol{\mathscr{C}}}_1 \big]_{12} \ , \end{aligned} \tag{5.5}$$

$$b_{p-q} = b_{-qp} = x_1 x_2 \big[S_{5/2}^{(p-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_1 \overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_2 \overset{\circ}{\boldsymbol{\mathscr{C}}}_2 \big]_{12} \ , \tag{5.6}$$

$$b_{-pq} = b_{q-p} = x_1 x_2 \big[S_{5/2}^{(p-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_2 \overset{\circ}{\boldsymbol{\mathscr{C}}}_2, S_{5/2}^{(q-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_1 \overset{\circ}{\boldsymbol{\mathscr{C}}}_1 \big]_{21} \ , \tag{5.7}$$

$$\begin{aligned} b_{-p-q} = b_{-q-p} =& x_2^2 \big[S_{5/2}^{(p-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_2 \overset{\circ}{\boldsymbol{\mathscr{C}}}_2, S_{5/2}^{(q-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_2 \overset{\circ}{\boldsymbol{\mathscr{C}}}_2 \big]_2 \\ &+ x_1 x_2 \big[S_{5/2}^{(p-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_2 \overset{\circ}{\boldsymbol{\mathscr{C}}}_2, S_{5/2}^{(q-1)}(\mathscr{C})\, \boldsymbol{\mathscr{C}}_2 \overset{\circ}{\boldsymbol{\mathscr{C}}}_2 \big]_{21} \ , \end{aligned} \tag{5.8}$$

where:

$$\begin{aligned} S_m^{(n)}(x) &= \sum_{p=0}^{n} \frac{(m+n)_{n-p}}{(p)!\,(n-p)!}(-x)^p \\ &= \sum_{p=0}^{n} \frac{(m+n)!}{(p)!\,(n-p)!\,(m+p)!}(-x)^p \ , \end{aligned} \tag{5.9}$$

(with $S_m^{(0)}(x) = 1$ and $S_m^{(1)}(x) = m+1-x$) are numerical multiples (un-normalized) of the Sonine polynomials originally used in the kinetic theory of gases by Burnett [13]. From the definitions used in the bracket integral notation, it follows that Eqs. (5.8) and (5.7) are essentially identical to Eqs. (5.5) and (5.6), respectively, with the only difference being the interchange of the subscripts 1 and 2 representing the different components of the mixture. Thus, in general, the complete Chapman-Enskog solutions for viscosity only require evaluation of the bracket integrals:

$$\left[S_{5/2}^{(p)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(q)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right]_1 , \tag{5.10}$$

$$\left[S_{5/2}^{(p)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(q)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right]_{12} , \tag{5.11}$$

and:

$$\left[S_{5/2}^{(p)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(q)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12} . \tag{5.12}$$

All of the above expressions for binary gas mixtures are readily generalized for use with arbitrary mixtures by replacing the 1 and 2 labeling scheme associated specifically with binary mixtures to a more general i and j labeling scheme. In the following sections, using this (i, j) labeling scheme, completely general expressions for all of the bracket integrals in Eqs. (5.10)-(5.12) are given up to order 5 of the expansion. Also shown in all of these expressions are the dependence of each on the molecular masses and the omega integrals, $\Omega_{ij}^{(\ell)}(r)$, which are defined as:

$$\Omega_{ij}^{(\ell)}(r) \equiv \left(\frac{kT}{2\pi m_0 M_i M_j}\right)^{1/2} \int_0^\infty \exp\left(-g^2\right) g^{2r+3} \phi_{ij}^{(\ell)} \mathrm{d}g \ , \tag{5.13}$$

with:

$$\phi_{ij}^{(\ell)} \equiv 2\pi \int_0^\pi \left(1 - \cos^\ell(\chi)\right) b \mathrm{d}b \ . \tag{5.14}$$

The omega integrals contain all of the dependencies relating to the specific intermolecular potential model that is employed. It is often considered convenient to define the omega integrals in terms of a simple scaling factor, $\sigma_{ij} = \frac{1}{2}(\sigma_i + \sigma_j)$, which is, generally, only a convenient, arbitrarily chosen length within some range where the impact parameter, b, is significant. Expressed in this manner, the omega integrals are then [10]:

$$\Omega_{ij}^{(\ell)}(r) = \tfrac{1}{2}\sigma_{ij}^2 \left(\frac{2\pi kT}{m_0 M_i M_j}\right)^{1/2} W_{ij}^{(\ell)}(r) \ , \tag{5.15}$$

where:

$$W_{ij}^{(\ell)}(r) \equiv 2\int_0^\infty \exp\left(-g^2\right) g^{2r+3} \int_0^\pi \left(1 - \cos^\ell(\chi)\right) (b/\sigma_{ij})\, \mathrm{d}\,(b/\sigma_{ij})\, \mathrm{d}g \ . \tag{5.16}$$

Note that when $i = j$, Eqs. (5.15) and (5.16) reduce to the following simple gas expressions:

$$\Omega_i^{(\ell)}(r) = \sigma_i^2 \left(\frac{\pi kT}{m_i}\right)^{1/2} W_i^{(\ell)}(r) \ , \tag{5.17}$$

with:

$$W_i^{(\ell)}(r) = 2\int_0^\infty \exp\left(-g^2\right) g^{2r+3} \int_0^\pi \left(1 - \cos^\ell(\chi)\right) \left(b/\sigma_i\right) \mathrm{d}\left(b/\sigma_i\right) \mathrm{d}g \ . \tag{5.18}$$

In Eqs. (5.15)-(5.18), σ_i is an arbitrary scale length associated with collisions between like molecules of type i while σ_{ij} is associated with collisions between unlike molecules of types i and j. These scale lengths are commonly associated with some concept of the molecular diameters depending upon the specific details of the intermolecular potential model that is employed.

5.3 General expressions for the bracket integrals

Of the three needed bracket integrals, the bracket integrals of Eq. (5.11) prove to have the most complicated dependence upon the molecular masses of the mixture constituents. At the same time, having general expressions for these bracket integrals proves to be the most useful as expressions for the remaining two bracket integrals can be generated from them with minimal effort. Thus, the general expressions for the bracket integrals of Eq. (5.11) which, in Chapman and Cowling [10], are associated with the functions $L_1(\chi)$ are reported first. For order 5 viscosity-related Chapman-Enskog solutions, one requires $p, q \in (0, 1, 2, 3, 4)$ and, hence, one has for the bracket integrals of Eq. (5.11) the following general expressions:

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(0)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_{ij} \\
&= \tfrac{80}{3}\left(M_iM_j\right)\Omega_{ij}^{(1)}(1) + 8\left(M_j^2\right)\Omega_{ij}^{(2)}(2)\ ,
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(1)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_{ij} \\
&= \tfrac{280}{3}\left(M_iM_j^2\right)\Omega_{ij}^{(1)}(1) - \tfrac{112}{3}\left(M_iM_j^2\right)\Omega_{ij}^{(1)}(2) + 28\left(M_j^3\right)\Omega_{ij}^{(2)}(2) \\
&- 8\left(M_j^3\right)\Omega_{ij}^{(2)}(3)\ ,
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(2)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_{ij} \\
&= 210\left(M_iM_j^3\right)\Omega_{ij}^{(1)}(1) - 168\left(M_iM_j^3\right)\Omega_{ij}^{(1)}(2) + 24\left(M_iM_j^3\right)\Omega_{ij}^{(1)}(3) \\
&+ 63\left(M_j^4\right)\Omega_{ij}^{(2)}(2) - 36\left(M_j^4\right)\Omega_{ij}^{(2)}(3) + 4\left(M_j^4\right)\Omega_{ij}^{(2)}(4)\ ,
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(3)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_{ij} \\
&= 385\left(M_iM_j^4\right)\Omega_{ij}^{(1)}(1) - 462\left(M_iM_j^4\right)\Omega_{ij}^{(1)}(2) + 132\left(M_iM_j^4\right)\Omega_{ij}^{(1)}(3) \\
&- \tfrac{88}{9}\left(M_iM_j^4\right)\Omega_{ij}^{(1)}(4) + \tfrac{231}{2}\left(M_j^5\right)\Omega_{ij}^{(2)}(2) - 99\left(M_j^5\right)\Omega_{ij}^{(2)}(3) \\
&+ 22\left(M_j^5\right)\Omega_{ij}^{(2)}(4) - \tfrac{4}{3}\left(M_j^5\right)\Omega_{ij}^{(2)}(5)\ ,
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(4)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_{ij} \\
&= \tfrac{5005}{8}\left(M_iM_j^5\right)\Omega_{ij}^{(1)}(1) - 1001\left(M_iM_j^5\right)\Omega_{ij}^{(1)}(2) \\
&+ 429\left(M_iM_j^5\right)\Omega_{ij}^{(1)}(3) - \tfrac{572}{9}\left(M_iM_j^5\right)\Omega_{ij}^{(1)}(4) + \tfrac{26}{9}\left(M_iM_j^5\right)\Omega_{ij}^{(1)}(5) \\
&+ \tfrac{3003}{16}\left(M_j^6\right)\Omega_{ij}^{(2)}(2) - \tfrac{429}{2}\left(M_j^6\right)\Omega_{ij}^{(2)}(3) + \tfrac{143}{2}\left(M_j^6\right)\Omega_{ij}^{(2)}(4)
\end{aligned}$$

$$- \tfrac{26}{3} \left(M_j^6\right) \Omega_{ij}^{(2)}(5) + \tfrac{1}{3} \left(M_j^6\right) \Omega_{ij}^{(2)}(6) \; , \tag{5.23}$$

$$\begin{aligned}
&\left[S_{5/2}^{(1)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i , S_{5/2}^{(1)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i \right]_{ij} \\
&= \tfrac{1540}{3} \left(\tfrac{4}{11} M_i^3 M_j + \tfrac{7}{11} M_i M_j^3\right) \Omega_{ij}^{(1)}(1) - \tfrac{784}{3} \left(M_i M_j^3\right) \Omega_{ij}^{(1)}(2) \\
&+ \tfrac{128}{3} \left(M_i M_j^3\right) \Omega_{ij}^{(1)}(3) + \tfrac{602}{3} \left(\tfrac{22}{43} M_i^2 M_j^2 + \tfrac{21}{43} M_j^4\right) \Omega_{ij}^{(2)}(2) \\
&- 56 \left(M_j^4\right) \Omega_{ij}^{(2)}(3) + 8 \left(M_j^4\right) \Omega_{ij}^{(2)}(4) + 16 \left(M_i M_j^3\right) \Omega_{ij}^{(3)}(3) \; ,
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
&\left[S_{5/2}^{(1)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i , S_{5/2}^{(2)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i \right]_{ij} \\
&= 1575 \left(\tfrac{8}{15} M_i^3 M_j^2 + \tfrac{7}{15} M_i M_j^4\right) \Omega_{ij}^{(1)}(1) - 1218 \left(\tfrac{8}{29} M_i^3 M_j^2\right. \\
&\left.+ \tfrac{21}{29} M_i M_j^4\right) \Omega_{ij}^{(1)}(2) + 276 \left(M_i M_j^4\right) \Omega_{ij}^{(1)}(3) - 24 \left(M_i M_j^4\right) \Omega_{ij}^{(1)}(4) \\
&+ \tfrac{1365}{2} \left(\tfrac{44}{65} M_i^2 M_j^3 + \tfrac{21}{65} M_j^5\right) \Omega_{ij}^{(2)}(2) - 321 \left(\tfrac{44}{107} M_i^2 M_j^3\right. \\
&\left.+ \tfrac{63}{107} M_j^5\right) \Omega_{ij}^{(2)}(3) + 50 \left(M_j^5\right) \Omega_{ij}^{(2)}(4) - 4 \left(M_j^5\right) \Omega_{ij}^{(2)}(5) \\
&+ 72 \left(M_i M_j^4\right) \Omega_{ij}^{(3)}(3) - 16 \left(M_i M_j^4\right) \Omega_{ij}^{(3)}(4) \; ,
\end{aligned} \tag{5.25}$$

$$\begin{aligned}
&\left[S_{5/2}^{(1)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i , S_{5/2}^{(3)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i \right]_{ij} \\
&= \tfrac{7315}{2} \left(\tfrac{12}{19} M_i^3 M_j^3 + \tfrac{7}{19} M_i M_j^5\right) \Omega_{ij}^{(1)}(1) - 4004 \left(\tfrac{6}{13} M_i^3 M_j^3\right. \\
&\left.+ \tfrac{7}{13} M_i M_j^5\right) \Omega_{ij}^{(1)}(2) + 1254 \left(\tfrac{4}{19} M_i^3 M_j^3 + \tfrac{15}{19} M_i M_j^5\right) \Omega_{ij}^{(1)}(3) \\
&- \tfrac{1496}{9} \left(M_i M_j^5\right) \Omega_{ij}^{(1)}(4) + \tfrac{80}{9} \left(M_i M_j^5\right) \Omega_{ij}^{(1)}(5) + \tfrac{6699}{4} \left(\tfrac{22}{29} M_i^2 M_j^4\right. \\
&\left.+ \tfrac{7}{29} M_j^6\right) \Omega_{ij}^{(2)}(2) - 1188 \left(\tfrac{11}{18} M_i^2 M_j^4 + \tfrac{7}{18} M_j^6\right) \Omega_{ij}^{(2)}(3) + \tfrac{770}{3} \left(\tfrac{11}{35} M_i^2 M_j^4\right. \\
&\left.+ \tfrac{24}{35} M_j^6\right) \Omega_{ij}^{(2)}(4) - \tfrac{80}{3} \left(M_j^6\right) \Omega_{ij}^{(2)}(5) + \tfrac{4}{3} \left(M_j^6\right) \Omega_{ij}^{(2)}(6) \\
&+ 198 \left(M_i M_j^5\right) \Omega_{ij}^{(3)}(3) - 88 \left(M_i M_j^5\right) \Omega_{ij}^{(3)}(4) + 8 \left(M_i M_j^5\right) \Omega_{ij}^{(3)}(5) \; ,
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
&\left[S_{5/2}^{(1)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i , S_{5/2}^{(4)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i \right]_{ij} \\
&= \tfrac{115115}{16} \left(\tfrac{16}{23} M_i^3 M_j^4 + \tfrac{7}{23} M_i M_j^6\right) \Omega_{ij}^{(1)}(1) - \tfrac{83083}{8} \left(\tfrac{48}{83} M_i^3 M_j^4\right. \\
&\left.+ \tfrac{35}{83} M_i M_j^6\right) \Omega_{ij}^{(1)}(2) + \tfrac{8723}{2} \left(\tfrac{24}{61} M_i^3 M_j^4 + \tfrac{37}{61} M_i M_j^6\right) \Omega_{ij}^{(1)}(3) \\
&- \tfrac{7007}{9} \left(\tfrac{8}{49} M_i^3 M_j^4 + \tfrac{41}{49} M_i M_j^6\right) \Omega_{ij}^{(1)}(4) + \tfrac{611}{9} \left(M_i M_j^6\right) \Omega_{ij}^{(1)}(5)
\end{aligned}$$

$$
\begin{aligned}
&- \tfrac{22}{9} \left(M_i M_j^6\right) \Omega_{ij}^{(1)}(6) + \tfrac{109109}{32} \left(\tfrac{88}{109} M_i^2 M_j^5 + \tfrac{21}{109} M_j^7\right) \Omega_{ij}^{(2)}(2) \\
&- \tfrac{52767}{16} \left(\tfrac{88}{123} M_i^2 M_j^5 + \tfrac{35}{123} M_j^7\right) \Omega_{ij}^{(2)}(3) + \tfrac{11869}{12} \left(\tfrac{44}{83} M_i^2 M_j^5\right. \\
&\left.+ \tfrac{39}{83} M_j^7\right) \Omega_{ij}^{(2)}(4) - \tfrac{2405}{18} \left(\tfrac{44}{185} M_i^2 M_j^5 + \tfrac{141}{185} M_j^7\right) \Omega_{ij}^{(2)}(5) \\
&+ \tfrac{59}{6} \left(M_j^7\right) \Omega_{ij}^{(2)}(6) - \tfrac{1}{3} \left(M_j^7\right) \Omega_{ij}^{(2)}(7) + 429 \left(M_i M_j^6\right) \Omega_{ij}^{(3)}(3) \\
&- 286 \left(M_i M_j^6\right) \Omega_{ij}^{(3)}(4) + 52 \left(M_i M_j^6\right) \Omega_{ij}^{(3)}(5) - \tfrac{8}{3} \left(M_i M_j^6\right) \Omega_{ij}^{(3)}(6) \,, \qquad (5.27)
\end{aligned}
$$

$$
\begin{aligned}
&\left[S_{5/2}^{(2)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S_{5/2}^{(2)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i\right]_{ij} \\
&= \tfrac{24255}{4} \left(\tfrac{8}{77} M_i^5 M_j + \tfrac{48}{77} M_i^3 M_j^3 + \tfrac{21}{77} M_i M_j^5\right) \Omega_{ij}^{(1)}(1) - 5670 \left(\tfrac{8}{15} M_i^3 M_j^3\right. \\
&\left.+ \tfrac{7}{15} M_i M_j^5\right) \Omega_{ij}^{(1)}(2) + 1746 \left(\tfrac{28}{97} M_i^3 M_j^3 + \tfrac{69}{97} M_i M_j^5\right) \Omega_{ij}^{(1)}(3) \\
&- 216 \left(M_i M_j^5\right) \Omega_{ij}^{(1)}(4) + 12 \left(M_i M_j^5\right) \Omega_{ij}^{(1)}(5) + \tfrac{25137}{8} \left(\tfrac{24}{133} M_i^4 M_j^2\right. \\
&\left.+ \tfrac{88}{133} M_i^2 M_j^4 + \tfrac{21}{133} M_j^6\right) \Omega_{ij}^{(2)}(2) - 1755 \left(\tfrac{44}{65} M_i^2 M_j^4 + \tfrac{21}{65} M_j^6\right) \Omega_{ij}^{(2)}(3) \\
&+ 381 \left(\tfrac{52}{127} M_i^2 M_j^4 + \tfrac{75}{127} M_j^6\right) \Omega_{ij}^{(2)}(4) - 36 \left(M_j^6\right) \Omega_{ij}^{(2)}(5) \\
&+ 2 \left(M_j^6\right) \Omega_{ij}^{(2)}(6) + 492 \left(\tfrac{14}{41} M_i^3 M_j^3 + \tfrac{27}{41} M_i M_j^5\right) \Omega_{ij}^{(3)}(3) \\
&- 144 \left(M_i M_j^5\right) \Omega_{ij}^{(3)}(4) + 16 \left(M_i M_j^5\right) \Omega_{ij}^{(3)}(5) + 16 \left(M_i^2 M_j^4\right) \Omega_{ij}^{(4)}(4) \,, \qquad (5.28)
\end{aligned}
$$

$$
\begin{aligned}
&\left[S_{5/2}^{(2)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S_{5/2}^{(3)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \overset{\circ}{\boldsymbol{\mathscr{C}}}_i\right]_{ij} \\
&= \tfrac{135135}{8} \left(\tfrac{8}{39} M_i^5 M_j^2 + \tfrac{24}{39} M_i^3 M_j^4 + \tfrac{7}{39} M_i M_j^6\right) \Omega_{ij}^{(1)}(1) \\
&- \tfrac{79695}{4} \left(\tfrac{8}{115} M_i^5 M_j^2 + \tfrac{72}{115} M_i^3 M_j^4 + \tfrac{35}{115} M_i M_j^6\right) \Omega_{ij}^{(1)}(2) \\
&+ 7722 \left(\tfrac{20}{39} M_i^3 M_j^4 + \tfrac{19}{39} M_i M_j^6\right) \Omega_{ij}^{(1)}(3) - 1320 \left(\tfrac{4}{15} M_i^3 M_j^4\right. \\
&\left.+ \tfrac{11}{15} M_i M_j^6\right) \Omega_{ij}^{(1)}(4) + 106 \left(M_i M_j^6\right) \Omega_{ij}^{(1)}(5) - 4 \left(M_i M_j^6\right) \Omega_{ij}^{(1)}(6) \\
&+ \tfrac{155925}{16} \left(\tfrac{24}{75} M_i^4 M_j^3 + \tfrac{44}{75} M_i^2 M_j^5 + \tfrac{7}{75} M_j^7\right) \Omega_{ij}^{(2)}(2) - \tfrac{56727}{8} \left(\tfrac{24}{191} M_i^4 M_j^3\right. \\
&\left.+ \tfrac{132}{191} M_i^2 M_j^5 + \tfrac{35}{191} M_j^7\right) \Omega_{ij}^{(2)}(3) + \tfrac{3795}{2} \left(\tfrac{74}{115} M_i^2 M_j^5 + \tfrac{41}{115} M_j^7\right) \Omega_{ij}^{(2)}(4) \\
&- 249 \left(\tfrac{30}{83} M_i^2 M_j^5 + \tfrac{53}{83} M_j^7\right) \Omega_{ij}^{(2)}(5) + 17 \left(M_j^7\right) \Omega_{ij}^{(2)}(6) \\
&- \tfrac{2}{3} \left(M_j^7\right) \Omega_{ij}^{(2)}(7) + 1815 \left(\tfrac{924}{1815} M_i^3 M_j^4 + \tfrac{891}{1815} M_i M_j^6\right) \Omega_{ij}^{(3)}(3) \\
&- \tfrac{2398}{3} \left(\tfrac{28}{109} M_i^3 M_j^4 + \tfrac{81}{109} M_i M_j^6\right) \Omega_{ij}^{(3)}(4) + 124 \left(M_i M_j^6\right) \Omega_{ij}^{(3)}(5) \\
&- 8 \left(M_i M_j^6\right) \Omega_{ij}^{(3)}(6) + 88 \left(M_i^2 M_j^5\right) \Omega_{ij}^{(4)}(4) - 16 \left(M_i^2 M_j^5\right) \Omega_{ij}^{(4)}(5) \,, \qquad (5.29)
\end{aligned}
$$

$$
\begin{aligned}
&\left[S_{5/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S_{5/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i\right]_{ij} \\
&= \tfrac{2477475}{64}\left(\tfrac{16}{55}M_i^5M_j^3 + \tfrac{32}{55}M_i^3M_j^5 + \tfrac{7}{55}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(1) \\
&- \tfrac{909909}{16}\left(\tfrac{16}{101}M_i^5M_j^3 + \tfrac{64}{101}M_i^3M_j^5 + \tfrac{21}{101}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(2) \\
&+ \tfrac{433719}{16}\left(\tfrac{16}{337}M_i^5M_j^3 + \tfrac{208}{337}M_i^3M_j^5 + \tfrac{113}{337}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(3) \\
&- \tfrac{11869}{2}\left(\tfrac{40}{83}M_i^3M_j^5 + \tfrac{43}{83}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(4) + \tfrac{2613}{4}\left(\tfrac{16}{67}M_i^3M_j^5\right. \\
&\left.+\tfrac{51}{67}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(5) - 37\left(M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(6) + \left(M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(7) \\
&+ \tfrac{3072069}{128}\left(\tfrac{144}{341}M_i^4M_j^4 + \tfrac{176}{341}M_i^2M_j^6 + \tfrac{21}{341}M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(2) \\
&- \tfrac{719433}{32}\left(\tfrac{144}{559}M_i^4M_j^4 + \tfrac{352}{559}M_i^2M_j^6 + \tfrac{63}{559}M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(3) \\
&+ \tfrac{237237}{32}\left(\tfrac{48}{553}M_i^4M_j^4 + \tfrac{384}{553}M_i^2M_j^6 + \tfrac{121}{553}M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(4) - \tfrac{4901}{4}\left(\tfrac{224}{377}M_i^2M_j^6\right. \\
&\left.+\tfrac{153}{377}M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(5) + \tfrac{891}{8}\left(\tfrac{272}{891}M_i^2M_j^6 + \tfrac{619}{891}M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(6) \\
&- \tfrac{35}{6}\left(M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(7) + \tfrac{1}{6}\left(M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(8) + \tfrac{9867}{2}\left(\tfrac{14}{23}M_i^3M_j^5\right. \\
&\left.+\tfrac{9}{23}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(3)}(3) - \tfrac{9152}{3}\left(\tfrac{7}{16}M_i^3M_j^5 + \tfrac{9}{16}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(3)}(4) \\
&+ \tfrac{1924}{3}\left(\tfrac{7}{37}M_i^3M_j^5 + \tfrac{30}{37}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(3)}(5) - 64\left(M_iM_j^7\right)\mathit{\Omega}_{ij}^{(3)}(6) \\
&+ \tfrac{8}{3}\left(M_iM_j^7\right)\mathit{\Omega}_{ij}^{(3)}(7) + 286\left(M_i^2M_j^6\right)\mathit{\Omega}_{ij}^{(4)}(4) - 104\left(M_i^2M_j^6\right)\mathit{\Omega}_{ij}^{(4)}(5) \\
&+ 8\left(M_i^2M_j^6\right)\mathit{\Omega}_{ij}^{(4)}(6)\,, \qquad (5.30)
\end{aligned}
$$

$$
\begin{aligned}
&\left[S_{5/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S_{5/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i\right]_{ij} \\
&= \tfrac{875875}{16}\left(\tfrac{64}{2275}M_i^7M_j + \tfrac{792}{2275}M_i^5M_j^3 + \tfrac{1188}{2275}M_i^3M_j^5 + \tfrac{231}{2275}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(1) \\
&- \tfrac{297297}{4}\left(\tfrac{8}{39}M_i^5M_j^3 + \tfrac{24}{39}M_i^3M_j^5 + \tfrac{7}{39}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(2) \\
&+ \tfrac{69267}{2}\left(\tfrac{152}{2099}M_i^5M_j^3 + \tfrac{1320}{2099}M_i^3M_j^5 + \tfrac{627}{2099}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(3) \\
&- \tfrac{22264}{3}\left(\tfrac{12}{23}M_i^3M_j^5 + \tfrac{11}{23}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(4) + \tfrac{2425}{3}\left(\tfrac{676}{2425}M_i^3M_j^5\right. \\
&\left.+\tfrac{1749}{2425}M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(5) - 44\left(M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(6) + \tfrac{32}{27}\left(M_iM_j^7\right)\mathit{\Omega}_{ij}^{(1)}(7) \\
&+ \tfrac{1168167}{32}\left(\tfrac{272}{5057}M_i^6M_j^2 + \tfrac{2376}{5057}M_i^4M_j^4 + \tfrac{2178}{5057}M_i^2M_j^6 + \tfrac{231}{5057}M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(2) \\
&- \tfrac{245025}{8}\left(\tfrac{24}{75}M_i^4M_j^4 + \tfrac{44}{75}M_i^2M_j^6 + \tfrac{7}{75}M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(3) + \tfrac{236005}{24}\left(\tfrac{2744}{21455}M_i^4M_j^4\right. \\
&+\tfrac{14652}{21455}M_i^2M_j^6 + \left.\tfrac{4059}{21455}M_j^8\right)\mathit{\Omega}_{ij}^{(2)}(4) - 1573\left(\tfrac{90}{143}M_i^2M_j^6\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{53}{143}M_j^8\big)\,\Omega_{ij}^{(2)}(5)+\tfrac{859}{6}\left(\tfrac{298}{859}M_i^2M_j^6+\tfrac{561}{859}M_j^8\right)\Omega_{ij}^{(2)}(6)\\
&-\tfrac{22}{3}\left(M_j^8\right)\Omega_{ij}^{(2)}(7)+\tfrac{2}{9}\left(M_j^8\right)\Omega_{ij}^{(2)}(8)+\tfrac{33737}{4}\big(\tfrac{328}{3067}M_i^5M_j^3+\tfrac{1848}{3067}M_i^3M_j^5\\
&+\tfrac{891}{3067}M_iM_j^7\big)\,\Omega_{ij}^{(3)}(3)-\tfrac{13310}{3}\left(\tfrac{28}{55}M_i^3M_j^5+\tfrac{27}{55}M_iM_j^7\right)\Omega_{ij}^{(3)}(4)\\
&+\tfrac{8290}{9}\left(\tfrac{1076}{4145}M_i^3M_j^5+\tfrac{3069}{4145}M_iM_j^7\right)\Omega_{ij}^{(3)}(5)-88\left(M_iM_j^7\right)\Omega_{ij}^{(3)}(6)\\
&+4\left(M_iM_j^7\right)\Omega_{ij}^{(3)}(7)+\tfrac{5852}{9}\left(\tfrac{34}{133}M_i^4M_j^4+\tfrac{99}{133}M_i^2M_j^6\right)\Omega_{ij}^{(4)}(4)\\
&-176\left(M_i^2M_j^6\right)\Omega_{ij}^{(4)}(5)+16\left(M_i^2M_j^6\right)\Omega_{ij}^{(4)}(6)\\
&+\tfrac{32}{3}\left(M_i^3M_j^5\right)\Omega_{ij}^{(5)}(5)\,,
\end{aligned}
\tag{5.31}
$$

$$
\begin{aligned}
&\left[S_{5/2}^{(3)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}}_i,S_{5/2}^{(4)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}}_i\right]_{ij}\\
&=\tfrac{18293275}{128}\big(\tfrac{256}{3655}M_i^7M_j^2+\tfrac{1584}{3655}M_i^5M_j^4+\tfrac{1584}{3655}M_i^3M_j^6\\
&+\tfrac{231}{3655}M_iM_j^8\big)\,\Omega_{ij}^{(1)}(1)-\tfrac{14559545}{64}\big(\tfrac{256}{14545}M_i^7M_j^2+\tfrac{4752}{14545}M_i^5M_j^4\\
&+\tfrac{7920}{14545}M_i^3M_j^6+\tfrac{1617}{14545}M_iM_j^8\big)\,\Omega_{ij}^{(1)}(2)+\tfrac{3990129}{32}\big(\tfrac{1744}{9301}M_i^5M_j^4\\
&+\tfrac{5808}{9301}M_i^3M_j^6+\tfrac{1749}{9301}M_iM_j^8\big)\,\Omega_{ij}^{(1)}(3)-\tfrac{1559701}{48}\big(\tfrac{688}{10907}M_i^5M_j^4\\
&+\tfrac{6864}{10907}M_i^3M_j^6+\tfrac{3355}{10907}M_iM_j^8\big)\,\Omega_{ij}^{(1)}(4)+\tfrac{107939}{24}\big(\tfrac{4288}{8303}M_i^3M_j^6\\
&+\tfrac{4015}{8303}M_iM_j^8\big)\,\Omega_{ij}^{(1)}(5)-\tfrac{4057}{12}\left(\tfrac{1120}{4057}M_i^3M_j^6+\tfrac{2937}{4057}M_iM_j^8\right)\Omega_{ij}^{(1)}(6)\\
&+\tfrac{713}{54}\left(M_iM_j^8\right)\Omega_{ij}^{(1)}(7)-\tfrac{7}{27}\left(M_iM_j^8\right)\Omega_{ij}^{(1)}(8)+\tfrac{26951925}{256}\big(\tfrac{1088}{8975}M_i^6M_j^3\\
&+\tfrac{4752}{8975}M_i^4M_j^5+\tfrac{2904}{8975}M_i^2M_j^7+\tfrac{231}{8975}M_j^9\big)\,\Omega_{ij}^{(2)}(2)-\tfrac{13505349}{128}\big(\tfrac{1088}{31481}M_i^6M_j^3\\
&+\tfrac{14256}{31481}M_i^4M_j^5+\tfrac{14520}{31481}M_i^2M_j^7+\tfrac{1617}{31481}M_j^9\big)\,\Omega_{ij}^{(2)}(3)\\
&+\tfrac{7661797}{192}\left(\tfrac{15728}{53579}M_i^4M_j^5+\tfrac{32208}{53579}M_i^2M_j^7+\tfrac{5643}{53579}M_j^9\right)\Omega_{ij}^{(2)}(4)\\
&-\tfrac{738569}{96}\left(\tfrac{6224}{56813}M_i^4M_j^5+\tfrac{38544}{56813}M_i^2M_j^7+\tfrac{12045}{56813}M_j^9\right)\Omega_{ij}^{(2)}(5)\\
&+\tfrac{41005}{48}\left(\tfrac{24472}{41005}M_i^2M_j^7+\tfrac{16533}{41005}M_j^9\right)\Omega_{ij}^{(2)}(6)-\tfrac{4175}{72}\big(\tfrac{1304}{4175}M_i^2M_j^7\\
&+\tfrac{2871}{4175}M_j^9\big)\,\Omega_{ij}^{(2)}(7)+\tfrac{85}{36}\left(M_j^9\right)\Omega_{ij}^{(2)}(8)-\tfrac{1}{18}\left(M_j^9\right)\Omega_{ij}^{(2)}(9)\\
&+\tfrac{221507}{8}\left(\tfrac{328}{1549}M_i^5M_j^4+\tfrac{924}{1549}M_i^3M_j^6+\tfrac{297}{1549}M_iM_j^8\right)\Omega_{ij}^{(3)}(3)\\
&-\tfrac{655655}{36}\left(\tfrac{328}{4585}M_i^5M_j^4+\tfrac{2772}{4585}M_i^3M_j^6+\tfrac{1485}{4585}M_iM_j^8\right)\Omega_{ij}^{(3)}(4)\\
&+\tfrac{41873}{9}\left(\tfrac{1538}{3221}M_i^3M_j^6+\tfrac{1683}{3221}M_iM_j^8\right)\Omega_{ij}^{(3)}(5)-\tfrac{5386}{9}\big(\tfrac{614}{2693}M_i^3M_j^6
\end{aligned}
$$

$$+\tfrac{2079}{2693}M_iM_j^8\big)\,\mathit{\Omega}_{ij}^{(3)}(6)+\tfrac{122}{3}\left(M_iM_j^8\right)\mathit{\Omega}_{ij}^{(3)}(7)-\tfrac{4}{3}\left(M_iM_j^8\right)\mathit{\Omega}_{ij}^{(3)}(8)$$

$$+\tfrac{23881}{9}\left(\tfrac{68}{167}M_i^4M_j^5+\tfrac{99}{167}M_i^2M_j^7\right)\mathit{\Omega}_{ij}^{(4)}(4)-\tfrac{9490}{9}\big(\tfrac{68}{365}M_i^4M_j^5$$

$$+\tfrac{297}{365}M_i^2M_j^7\big)\,\mathit{\Omega}_{ij}^{(4)}(5)+148\left(M_i^2M_j^7\right)\mathit{\Omega}_{ij}^{(4)}(6)-8\left(M_i^2M_j^7\right)\mathit{\Omega}_{ij}^{(4)}(7)$$

$$+\tfrac{208}{3}\left(M_i^3M_j^6\right)\mathit{\Omega}_{ij}^{(5)}(5)-\tfrac{32}{3}\left(M_i^3M_j^6\right)\mathit{\Omega}_{ij}^{(5)}(6)\,, \tag{5.32}$$

and:

$$\left[S_{5/2}^{(4)}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i,S_{5/2}^{(4)}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i\right]_{ij}$$

$$=\tfrac{428402975}{1024}\big(\tfrac{640}{85595}M_i^9M_j+\tfrac{13312}{85595}M_i^7M_j^3+\tfrac{41184}{85595}M_i^5M_j^5+\tfrac{27456}{85595}M_i^3M_j^7$$

$$+\tfrac{3003}{85595}M_iM_j^9\big)\,\mathit{\Omega}_{ij}^{(1)}(1)-\tfrac{47562515}{64}\big(\tfrac{256}{3655}M_i^7M_j^3+\tfrac{1584}{3655}M_i^5M_j^5$$

$$+\tfrac{1584}{3655}M_i^3M_j^7+\tfrac{231}{3655}M_iM_j^9\big)\,\mathit{\Omega}_{ij}^{(1)}(2)+\tfrac{29713255}{64}\big(\tfrac{3776}{207785}M_i^7M_j^3$$

$$+\tfrac{68016}{207785}M_i^5M_j^5+\tfrac{113256}{207785}M_i^3M_j^7+\tfrac{22737}{207785}M_iM_j^9\big)\,\mathit{\Omega}_{ij}^{(1)}(3)$$

$$-\tfrac{6779773}{48}\left(\tfrac{688}{3647}M_i^5M_j^5+\tfrac{2288}{3647}M_i^3M_j^7+\tfrac{671}{3647}M_iM_j^9\right)\mathit{\Omega}_{ij}^{(1)}(4)$$

$$+\tfrac{758173}{32}\left(\tfrac{11280}{174963}M_i^5M_j^5+\tfrac{111488}{174963}M_i^3M_j^7+\tfrac{52195}{174963}M_iM_j^9\right)\mathit{\Omega}_{ij}^{(1)}(5)$$

$$-\tfrac{27287}{12}\left(\tfrac{1120}{2099}M_i^3M_j^7+\tfrac{979}{2099}M_iM_j^9\right)\mathit{\Omega}_{ij}^{(1)}(6)+\tfrac{13213}{108}\big(\tfrac{3944}{13213}M_i^3M_j^7$$

$$+\tfrac{9269}{13213}M_iM_j^9\big)\,\mathit{\Omega}_{ij}^{(1)}(7)-\tfrac{91}{27}\left(M_iM_j^9\right)\mathit{\Omega}_{ij}^{(1)}(8)+\tfrac{5}{108}\left(M_iM_j^9\right)\mathit{\Omega}_{ij}^{(1)}(9)$$

$$+\tfrac{711736025}{2048}\big(\tfrac{10624}{711025}M_i^8M_j^2+\tfrac{169728}{711025}M_i^6M_j^4+\tfrac{370656}{711025}M_i^4M_j^6$$

$$+\tfrac{151008}{711025}M_i^2M_j^8+\tfrac{9009}{711025}M_j^{10}\big)\,\mathit{\Omega}_{ij}^{(2)}(2)-\tfrac{50053575}{128}\big(\tfrac{1088}{8975}M_i^6M_j^4$$

$$+\tfrac{4752}{8975}M_i^4M_j^6+\tfrac{2904}{8975}M_i^2M_j^8+\tfrac{231}{8975}M_j^{10}\big)\,\mathit{\Omega}_{ij}^{(2)}(3)$$

$$+\tfrac{64969619}{384}\big(\tfrac{16064}{454333}M_i^6M_j^4+\tfrac{204464}{454333}M_i^4M_j^6+\tfrac{209352}{454333}M_i^2M_j^8$$

$$+\tfrac{24453}{454333}M_j^{10}\big)\,\mathit{\Omega}_{ij}^{(2)}(4)-\tfrac{3630289}{96}\big(\tfrac{6224}{21481}M_i^4M_j^6+\tfrac{12848}{21481}M_i^2M_j^8$$

$$+\tfrac{2409}{21481}M_j^{10}\big)\,\mathit{\Omega}_{ij}^{(2)}(5)+\tfrac{953681}{192}\big(\tfrac{102480}{953681}M_i^4M_j^6+\tfrac{636272}{953681}M_i^2M_j^8$$

$$+\tfrac{214929}{953681}M_j^{10}\big)\,\mathit{\Omega}_{ij}^{(2)}(6)-\tfrac{29393}{72}\left(\tfrac{1304}{2261}M_i^2M_j^8+\tfrac{957}{2261}M_j^{10}\right)\mathit{\Omega}_{ij}^{(2)}(7)$$

$$+\tfrac{4699}{216}\left(\tfrac{1384}{4699}M_i^2M_j^8+\tfrac{3315}{4699}M_j^{10}\right)\mathit{\Omega}_{ij}^{(2)}(8)-\tfrac{13}{18}\left(M_j^{10}\right)\mathit{\Omega}_{ij}^{(2)}(9)$$

$$+\tfrac{1}{72}\left(M_j^{10}\right)\mathit{\Omega}_{ij}^{(2)}(10)+\tfrac{1705275}{16}\big(\tfrac{368}{11925}M_i^7M_j^3+\tfrac{4264}{11925}M_i^5M_j^5$$

$$+\tfrac{6006}{11925}M_i^3M_j^7+\tfrac{1287}{11925}M_iM_j^9\big)\,\mathit{\Omega}_{ij}^{(3)}(3)-\tfrac{2879591}{36}\big(\tfrac{328}{1549}M_i^5M_j^5$$

$$+\tfrac{924}{1549}M_i^3M_j^7+\tfrac{297}{1549}M_iM_j^9\big)\,\mathit{\Omega}_{ij}^{(3)}(4)+\tfrac{289133}{12}\big(\tfrac{4856}{66723}M_i^5M_j^5$$

$$
\begin{aligned}
&+\tfrac{39988}{66723}M_i^3M_j^7+\tfrac{21879}{66723}M_iM_j^9\big)\,\Omega_{ij}^{(3)}(5)-\tfrac{33982}{9}\big(\tfrac{614}{1307}M_i^3M_j^7\\
&+\tfrac{693}{1307}M_iM_j^9\big)\,\Omega_{ij}^{(3)}(6)+\tfrac{3061}{9}\big(\tfrac{682}{3061}M_i^3M_j^7+\tfrac{2379}{3061}M_iM_j^9\big)\,\Omega_{ij}^{(3)}(7)\\
&-\tfrac{52}{3}\big(M_iM_j^9\big)\,\Omega_{ij}^{(3)}(8)+\tfrac{4}{9}\big(M_iM_j^9\big)\,\Omega_{ij}^{(3)}(9)+\tfrac{470041}{36}\big(\tfrac{232}{3287}M_i^6M_j^4\\
&+\tfrac{1768}{3287}M_i^4M_j^6+\tfrac{1287}{3287}M_i^2M_j^8\big)\,\Omega_{ij}^{(4)}(4)-\tfrac{56446}{9}\big(\tfrac{68}{167}M_i^4M_j^6\\
&+\tfrac{99}{167}M_i^2M_j^8\big)\,\Omega_{ij}^{(4)}(5)+\tfrac{10682}{9}\big(\tfrac{1012}{5341}M_i^4M_j^6+\tfrac{4329}{5341}M_i^2M_j^8\big)\,\Omega_{ij}^{(4)}(6)\\
&-104\big(M_i^2M_j^8\big)\,\Omega_{ij}^{(4)}(7)+4\big(M_i^2M_j^8\big)\,\Omega_{ij}^{(4)}(8)+\tfrac{5096}{9}\big(\tfrac{10}{49}M_i^5M_j^5\\
&+\tfrac{39}{49}M_i^3M_j^7\big)\,\Omega_{ij}^{(5)}(5)-\tfrac{416}{3}\big(M_i^3M_j^7\big)\,\Omega_{ij}^{(5)}(6)+\tfrac{32}{3}\big(M_i^3M_j^7\big)\,\Omega_{ij}^{(5)}(7)\\
&+\tfrac{16}{3}\big(M_i^4M_j^6\big)\,\Omega_{ij}^{(6)}(6)\,. \qquad (5.33)
\end{aligned}
$$

Next, the bracket integrals of Eq. (5.12) that are related to the $L_{12}(\chi)$ functions in Chapman and Cowling [10] are considered. For each (p, q), these may be generated from the corresponding $L_1(\chi)$ expression in Eqs. (5.19)-(5.33). In general, Eqs. (5.19)-(5.33) consist of a series of terms each of which is associated with a specific omega integral. Each of these omega integral terms also contains a sign, a constant factor, and some function of M_i and M_j. The conversion process from the $L_1(\chi)$ expressions to the $L_{12}(\chi)$ expressions is quite straightforward as the magnitude of the numerical coefficients associated with each omega integral term are the same from the $L_1(\chi)$ expressions to the $L_{12}(\chi)$ expressions. The differences are in the distributions of the constituent masses and the signs of the terms. Since the same omega integrals must occur in both the $L_1(\chi)$ expressions and the $L_{12}(\chi)$ expressions, the total number of omega integral terms is the same in each and there is, in effect, a one-to-one correspondence between terms. The sign difference between corresponding terms is dependent only upon ℓ and the appropriate sign transformation is to multiply each term in the $L_1(\chi)$ expressions by a factor of $(-1)^\ell$ to yield the corresponding $L_{12}(\chi)$ signs. The distribution of constituent masses is much simpler in the $L_{12}(\chi)$ expressions than in the $L_1(\chi)$ expressions. In the $L_{12}(\chi)$ expressions given below the distribution of the constituent masses is exactly the same in every omega integral term for a given (p, q) and amounts to nothing more than a common factor of $(M_i^{q+1}M_j^{p+1})$ in each expression. Thus, the appropriate overall transformation from the $L_1(\chi)$ expressions to the corresponding $L_{12}(\chi)$ expressions is to first set $M_i = M_j = 1$ in each term in the $L_1(\chi)$ expressions, second to multiply each $L_1(\chi)$ expression with the appropriate common factor of $(M_i^{q+1}M_j^{p+1})$, and third to adjust the sign of each term as needed for the odd values of ℓ. When the above prescription is followed, one obtains the following expressions for the $L_{12}(\chi)$ bracket integrals where the common mass factor in each expression has been explicitly factored out for clarity:

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\,\mathscr{C}_i\overset{\circ}{}\mathscr{C}_i, S_{5/2}^{(0)}(\mathscr{C})\,\mathscr{C}_j\overset{\circ}{}\mathscr{C}_j\right]_{ij}\left(M_iM_j\right)^{-1}\\
&= -\tfrac{80}{3}\Omega_{ij}^{(1)}(1) + 8\Omega_{ij}^{(2)}(2)\;,
\end{aligned}\tag{5.34}$$

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\,\mathscr{C}_i\overset{\circ}{}\mathscr{C}_i, S_{5/2}^{(1)}(\mathscr{C})\,\mathscr{C}_j\overset{\circ}{}\mathscr{C}_j\right]_{ij}\left(M_i^2M_j\right)^{-1}\\
&= -\tfrac{280}{3}\Omega_{ij}^{(1)}(1) + \tfrac{112}{3}\Omega_{ij}^{(1)}(2) + 28\Omega_{ij}^{(2)}(2) - 8\Omega_{ij}^{(2)}(3)\;,
\end{aligned}\tag{5.35}$$

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\,\mathscr{C}_i\overset{\circ}{}\mathscr{C}_i, S_{5/2}^{(2)}(\mathscr{C})\,\mathscr{C}_j\overset{\circ}{}\mathscr{C}_j\right]_{ij}\left(M_i^3M_j\right)^{-1}\\
&= -210\Omega_{ij}^{(1)}(1) + 168\Omega_{ij}^{(1)}(2) - 24\Omega_{ij}^{(1)}(3) + 63\Omega_{ij}^{(2)}(2)\\
&- 36\Omega_{ij}^{(2)}(3) + 4\Omega_{ij}^{(2)}(4)\;,
\end{aligned}\tag{5.36}$$

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\,\mathscr{C}_i\overset{\circ}{}\mathscr{C}_i, S_{5/2}^{(3)}(\mathscr{C})\,\mathscr{C}_j\overset{\circ}{}\mathscr{C}_j\right]_{ij}\left(M_i^4M_j\right)^{-1}\\
&= -385\Omega_{ij}^{(1)}(1) + 462\Omega_{ij}^{(1)}(2) - 132\Omega_{ij}^{(1)}(3) + \tfrac{88}{9}\Omega_{ij}^{(1)}(4)\\
&+ \tfrac{231}{2}\Omega_{ij}^{(2)}(2) - 99\Omega_{ij}^{(2)}(3) + 22\Omega_{ij}^{(2)}(4) - \tfrac{4}{3}\Omega_{ij}^{(2)}(5)\;,
\end{aligned}\tag{5.37}$$

$$\begin{aligned}
&\left[S_{5/2}^{(0)}(\mathscr{C})\,\mathscr{C}_i\overset{\circ}{}\mathscr{C}_i, S_{5/2}^{(4)}(\mathscr{C})\,\mathscr{C}_j\overset{\circ}{}\mathscr{C}_j\right]_{ij}\left(M_i^5M_j\right)^{-1}\\
&= -\tfrac{5005}{8}\Omega_{ij}^{(1)}(1) + 1001\Omega_{ij}^{(1)}(2) - 429\Omega_{ij}^{(1)}(3) + \tfrac{572}{9}\Omega_{ij}^{(1)}(4)\\
&- \tfrac{26}{9}\Omega_{ij}^{(1)}(5) + \tfrac{3003}{16}\Omega_{ij}^{(2)}(2) - \tfrac{429}{2}\Omega_{ij}^{(2)}(3) + \tfrac{143}{2}\Omega_{ij}^{(2)}(4) - \tfrac{26}{3}\Omega_{ij}^{(2)}(5)\\
&+ \tfrac{1}{3}\Omega_{ij}^{(2)}(6)\;,
\end{aligned}\tag{5.38}$$

$$
\begin{aligned}
&\left[S^{(1)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(1)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_j\mathring{\boldsymbol{\mathscr{C}}}_j\right]_{ij}\left(M_i^2M_j^2\right)^{-1} \\
&= -\tfrac{1540}{3}\mathit{\Omega}^{(1)}_{ij}(1) + \tfrac{784}{3}\mathit{\Omega}^{(1)}_{ij}(2) - \tfrac{128}{3}\mathit{\Omega}^{(1)}_{ij}(3) + \tfrac{602}{3}\mathit{\Omega}^{(2)}_{ij}(2) \\
&- 56\mathit{\Omega}^{(2)}_{ij}(3) + 8\mathit{\Omega}^{(2)}_{ij}(4) - 16\mathit{\Omega}^{(3)}_{ij}(3)\ ,
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
&\left[S^{(1)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(2)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_j\mathring{\boldsymbol{\mathscr{C}}}_j\right]_{ij}\left(M_i^3M_j^2\right)^{-1} \\
&= -1575\mathit{\Omega}^{(1)}_{ij}(1) + 1218\mathit{\Omega}^{(1)}_{ij}(2) - 276\mathit{\Omega}^{(1)}_{ij}(3) + 24\mathit{\Omega}^{(1)}_{ij}(4) \\
&+ \tfrac{1365}{2}\mathit{\Omega}^{(2)}_{ij}(2) - 321\mathit{\Omega}^{(2)}_{ij}(3) + 50\mathit{\Omega}^{(2)}_{ij}(4) - 4\mathit{\Omega}^{(2)}_{ij}(5) - 72\mathit{\Omega}^{(3)}_{ij}(3) \\
&+ 16\mathit{\Omega}^{(3)}_{ij}(4)\ ,
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
&\left[S^{(1)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(3)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_j\mathring{\boldsymbol{\mathscr{C}}}_j\right]_{ij}\left(M_i^4M_j^2\right)^{-1} \\
&= -\tfrac{7315}{2}\mathit{\Omega}^{(1)}_{ij}(1) + 4004\mathit{\Omega}^{(1)}_{ij}(2) - 1254\mathit{\Omega}^{(1)}_{ij}(3) + \tfrac{1496}{9}\mathit{\Omega}^{(1)}_{ij}(4) \\
&- \tfrac{80}{9}\mathit{\Omega}^{(1)}_{ij}(5) + \tfrac{6699}{4}\mathit{\Omega}^{(2)}_{ij}(2) - 1188\mathit{\Omega}^{(2)}_{ij}(3) + \tfrac{770}{3}\mathit{\Omega}^{(2)}_{ij}(4) - \tfrac{80}{3}\mathit{\Omega}^{(2)}_{ij}(5) \\
&+ \tfrac{4}{3}\mathit{\Omega}^{(2)}_{ij}(6) - 198\mathit{\Omega}^{(3)}_{ij}(3) + 88\mathit{\Omega}^{(3)}_{ij}(4) - 8\mathit{\Omega}^{(3)}_{ij}(5)\ ,
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
&\left[S^{(1)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(4)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_j\mathring{\boldsymbol{\mathscr{C}}}_j\right]_{ij}\left(M_i^5M_j^2\right)^{-1} \\
&= -\tfrac{115115}{16}\mathit{\Omega}^{(1)}_{ij}(1) + \tfrac{83083}{8}\mathit{\Omega}^{(1)}_{ij}(2) - \tfrac{8723}{2}\mathit{\Omega}^{(1)}_{ij}(3) + \tfrac{7007}{9}\mathit{\Omega}^{(1)}_{ij}(4) \\
&- \tfrac{611}{9}\mathit{\Omega}^{(1)}_{ij}(5) + \tfrac{22}{9}\mathit{\Omega}^{(1)}_{ij}(6) + \tfrac{109109}{32}\mathit{\Omega}^{(2)}_{ij}(2) - \tfrac{52767}{16}\mathit{\Omega}^{(2)}_{ij}(3) \\
&+ \tfrac{11869}{12}\mathit{\Omega}^{(2)}_{ij}(4) - \tfrac{2405}{18}\mathit{\Omega}^{(2)}_{ij}(5) + \tfrac{59}{6}\mathit{\Omega}^{(2)}_{ij}(6) - \tfrac{1}{3}\mathit{\Omega}^{(2)}_{ij}(7) \\
&- 429\mathit{\Omega}^{(3)}_{ij}(3) + 286\mathit{\Omega}^{(3)}_{ij}(4) - 52\mathit{\Omega}^{(3)}_{ij}(5) + \tfrac{8}{3}\mathit{\Omega}^{(3)}_{ij}(6)\ ,
\end{aligned}
\tag{5.42}
$$

$$
\begin{aligned}
&\left[S^{(2)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(2)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_j\mathring{\boldsymbol{\mathscr{C}}}_j\right]_{ij}\left(M_i^3M_j^3\right)^{-1} \\
&= -\tfrac{24255}{4}\mathit{\Omega}^{(1)}_{ij}(1) + 5670\mathit{\Omega}^{(1)}_{ij}(2) - 1746\mathit{\Omega}^{(1)}_{ij}(3) + 216\mathit{\Omega}^{(1)}_{ij}(4) \\
&- 12\mathit{\Omega}^{(1)}_{ij}(5) + \tfrac{25137}{8}\mathit{\Omega}^{(2)}_{ij}(2) - 1755\mathit{\Omega}^{(2)}_{ij}(3) + 381\mathit{\Omega}^{(2)}_{ij}(4) - 36\mathit{\Omega}^{(2)}_{ij}(5) \\
&+ 2\mathit{\Omega}^{(2)}_{ij}(6) - 492\mathit{\Omega}^{(3)}_{ij}(3) + 144\mathit{\Omega}^{(3)}_{ij}(4) - 16\mathit{\Omega}^{(3)}_{ij}(5) + 16\mathit{\Omega}^{(4)}_{ij}(4)\ ,
\end{aligned}
\tag{5.43}
$$

$$
\left[S^{(2)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(3)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_j\mathring{\boldsymbol{\mathscr{C}}}_j\right]_{ij}\left(M_i^4M_j^3\right)^{-1}
$$

$$
\begin{aligned}
&= -\tfrac{135135}{8}\Omega^{(1)}_{ij}(1) + \tfrac{79695}{4}\Omega^{(1)}_{ij}(2) - 7722\Omega^{(1)}_{ij}(3) + 1320\Omega^{(1)}_{ij}(4) \\
&- 106\Omega^{(1)}_{ij}(5) + 4\Omega^{(1)}_{ij}(6) + \tfrac{155925}{16}\Omega^{(2)}_{ij}(2) - \tfrac{56727}{8}\Omega^{(2)}_{ij}(3) + \tfrac{3795}{2}\Omega^{(2)}_{ij}(4) \\
&- 249\Omega^{(2)}_{ij}(5) + 17\Omega^{(2)}_{ij}(6) - \tfrac{2}{3}\Omega^{(2)}_{ij}(7) - 1815\Omega^{(3)}_{ij}(3) + \tfrac{2398}{3}\Omega^{(3)}_{ij}(4) \\
&- 124\Omega^{(3)}_{ij}(5) + 8\Omega^{(3)}_{ij}(6) + 88\Omega^{(4)}_{ij}(4) - 16\Omega^{(4)}_{ij}(5)\ ,
\end{aligned}
\tag{5.44}
$$

$$
\begin{aligned}
&\left[S^{(2)}_{5/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S^{(4)}_{5/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j\right]_{ij}\left(M_i^5 M_j^3\right)^{-1} \\
&= -\tfrac{2477475}{64}\Omega^{(1)}_{ij}(1) + \tfrac{909909}{16}\Omega^{(1)}_{ij}(2) - \tfrac{433719}{16}\Omega^{(1)}_{ij}(3) + \tfrac{11869}{2}\Omega^{(1)}_{ij}(4) \\
&- \tfrac{2613}{4}\Omega^{(1)}_{ij}(5) + 37\Omega^{(1)}_{ij}(6) - \Omega^{(1)}_{ij}(7) + \tfrac{3072069}{128}\Omega^{(2)}_{ij}(2) - \tfrac{719433}{32}\Omega^{(2)}_{ij}(3) \\
&+ \tfrac{237237}{32}\Omega^{(2)}_{ij}(4) - \tfrac{4901}{4}\Omega^{(2)}_{ij}(5) + \tfrac{891}{8}\Omega^{(2)}_{ij}(6) - \tfrac{35}{6}\Omega^{(2)}_{ij}(7) + \tfrac{1}{6}\Omega^{(2)}_{ij}(8) \\
&- \tfrac{9867}{2}\Omega^{(3)}_{ij}(3) + \tfrac{9152}{3}\Omega^{(3)}_{ij}(4) - \tfrac{1924}{3}\Omega^{(3)}_{ij}(5) + 64\Omega^{(3)}_{ij}(6) - \tfrac{8}{3}\Omega^{(3)}_{ij}(7) \\
&+ 286\Omega^{(4)}_{ij}(4) - 104\Omega^{(4)}_{ij}(5) + 8\Omega^{(4)}_{ij}(6)\ ,
\end{aligned}
\tag{5.45}
$$

$$
\begin{aligned}
&\left[S^{(3)}_{5/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S^{(3)}_{5/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j\right]_{ij}\left(M_i^4 M_j^4\right)^{-1} \\
&= -\tfrac{875875}{16}\Omega^{(1)}_{ij}(1) + \tfrac{297297}{4}\Omega^{(1)}_{ij}(2) - \tfrac{69267}{2}\Omega^{(1)}_{ij}(3) + \tfrac{22264}{3}\Omega^{(1)}_{ij}(4) \\
&- \tfrac{2425}{3}\Omega^{(1)}_{ij}(5) + 44\Omega^{(1)}_{ij}(6) - \tfrac{32}{27}\Omega^{(1)}_{ij}(7) + \tfrac{1168167}{32}\Omega^{(2)}_{ij}(2) \\
&- \tfrac{245025}{8}\Omega^{(2)}_{ij}(3) + \tfrac{236005}{24}\Omega^{(2)}_{ij}(4) - 1573\Omega^{(2)}_{ij}(5) + \tfrac{859}{6}\Omega^{(2)}_{ij}(6) \\
&- \tfrac{22}{3}\Omega^{(2)}_{ij}(7) + \tfrac{2}{9}\Omega^{(2)}_{ij}(8) - \tfrac{33737}{4}\Omega^{(3)}_{ij}(3) + \tfrac{13310}{3}\Omega^{(3)}_{ij}(4) \\
&- \tfrac{8290}{9}\Omega^{(3)}_{ij}(5) + 88\Omega^{(3)}_{ij}(6) - 4\Omega^{(3)}_{ij}(7) + \tfrac{5852}{9}\Omega^{(4)}_{ij}(4) - 176\Omega^{(4)}_{ij}(5) \\
&+ 16\Omega^{(4)}_{ij}(6) - \tfrac{32}{3}\Omega^{(5)}_{ij}(5)\ ,
\end{aligned}
\tag{5.46}
$$

$$
\begin{aligned}
&\left[S^{(3)}_{5/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S^{(4)}_{5/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j\right]_{ij}\left(M_i^5 M_j^4\right)^{-1} \\
&= -\tfrac{18293275}{128}\Omega^{(1)}_{ij}(1) + \tfrac{14559545}{64}\Omega^{(1)}_{ij}(2) - \tfrac{3990129}{32}\Omega^{(1)}_{ij}(3) + \tfrac{1559701}{48}\Omega^{(1)}_{ij}(4) \\
&- \tfrac{107939}{24}\Omega^{(1)}_{ij}(5) + \tfrac{4057}{12}\Omega^{(1)}_{ij}(6) - \tfrac{713}{54}\Omega^{(1)}_{ij}(7) + \tfrac{7}{27}\Omega^{(1)}_{ij}(8) \\
&+ \tfrac{26951925}{256}\Omega^{(2)}_{ij}(2) - \tfrac{13505349}{128}\Omega^{(2)}_{ij}(3) + \tfrac{7661797}{192}\Omega^{(2)}_{ij}(4) - \tfrac{738569}{96}\Omega^{(2)}_{ij}(5) \\
&+ \tfrac{41005}{48}\Omega^{(2)}_{ij}(6) - \tfrac{4175}{72}\Omega^{(2)}_{ij}(7) + \tfrac{85}{36}\Omega^{(2)}_{ij}(8) - \tfrac{1}{18}\Omega^{(2)}_{ij}(9) \\
&- \tfrac{221507}{8}\Omega^{(3)}_{ij}(3) + \tfrac{655655}{36}\Omega^{(3)}_{ij}(4) - \tfrac{41873}{9}\Omega^{(3)}_{ij}(5) + \tfrac{5386}{9}\Omega^{(3)}_{ij}(6)
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{122}{3}\Omega_{ij}^{(3)}(7) + \tfrac{4}{3}\Omega_{ij}^{(3)}(8) + \tfrac{23881}{9}\Omega_{ij}^{(4)}(4) - \tfrac{9490}{9}\Omega_{ij}^{(4)}(5) + 148\Omega_{ij}^{(4)}(6) \\
&- 8\Omega_{ij}^{(4)}(7) - \tfrac{208}{3}\Omega_{ij}^{(5)}(5) + \tfrac{32}{3}\Omega_{ij}^{(5)}(6) \; , \qquad (5.47)
\end{aligned}
$$

and:

$$
\begin{aligned}
&\left[S_{5/2}^{(4)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(4)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_j\boldsymbol{\mathscr{C}}_j}\right]_{ij}\left(M_i^5 M_j^5\right)^{-1} \\
&= -\tfrac{428402975}{1024}\Omega_{ij}^{(1)}(1) + \tfrac{47562515}{64}\Omega_{ij}^{(1)}(2) - \tfrac{29713255}{64}\Omega_{ij}^{(1)}(3) \\
&+ \tfrac{6779773}{48}\Omega_{ij}^{(1)}(4) - \tfrac{758173}{32}\Omega_{ij}^{(1)}(5) + \tfrac{27287}{12}\Omega_{ij}^{(1)}(6) - \tfrac{13213}{108}\Omega_{ij}^{(1)}(7) \\
&+ \tfrac{91}{27}\Omega_{ij}^{(1)}(8) - \tfrac{5}{108}\Omega_{ij}^{(1)}(9) + \tfrac{711736025}{2048}\Omega_{ij}^{(2)}(2) - \tfrac{50053575}{128}\Omega_{ij}^{(2)}(3) \\
&+ \tfrac{64969619}{384}\Omega_{ij}^{(2)}(4) - \tfrac{3630289}{96}\Omega_{ij}^{(2)}(5) + \tfrac{953681}{192}\Omega_{ij}^{(2)}(6) - \tfrac{29393}{72}\Omega_{ij}^{(2)}(7) \\
&+ \tfrac{4699}{216}\Omega_{ij}^{(2)}(8) - \tfrac{13}{18}\Omega_{ij}^{(2)}(9) + \tfrac{1}{72}\Omega_{ij}^{(2)}(10) - \tfrac{1705275}{16}\Omega_{ij}^{(3)}(3) \\
&+ \tfrac{2879591}{36}\Omega_{ij}^{(3)}(4) - \tfrac{289133}{12}\Omega_{ij}^{(3)}(5) + \tfrac{33982}{9}\Omega_{ij}^{(3)}(6) - \tfrac{3061}{9}\Omega_{ij}^{(3)}(7) \\
&+ \tfrac{52}{3}\Omega_{ij}^{(3)}(8) - \tfrac{4}{9}\Omega_{ij}^{(3)}(9) + \tfrac{470041}{36}\Omega_{ij}^{(4)}(4) - \tfrac{56446}{9}\Omega_{ij}^{(4)}(5) + \tfrac{10682}{9}\Omega_{ij}^{(4)}(6) \\
&- 104\Omega_{ij}^{(4)}(7) + 4\Omega_{ij}^{(4)}(8) - \tfrac{5096}{9}\Omega_{ij}^{(5)}(5) + \tfrac{416}{3}\Omega_{ij}^{(5)}(6) - \tfrac{32}{3}\Omega_{ij}^{(5)}(7) \\
&+ \tfrac{16}{3}\Omega_{ij}^{(6)}(6) \; . \qquad (5.48)
\end{aligned}
$$

Now that the $L_1(\chi)$ and $L_{12}(\chi)$ bracket integrals of Eqs. (5.11) and (5.12) have been obtained, they can be used to generate the simple gas bracket integrals of Eq. (5.10). As noted in Chapman and Cowling [10], in the limit of a simple (single) gas where $m_i = m_j$, and $n_i = n_j$, one can write:

$$
\begin{aligned}
&\left[S_{5/2}^{(p)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(q)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_i \\
&= \Bigg(\left[S_{5/2}^{(p)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(q)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_{ij} \\
&\qquad + \left[S_{5/2}^{(p)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(q)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_j\boldsymbol{\mathscr{C}}_j}\right]_{ij}\Bigg)\Bigg|_{\substack{m_i=m_j \\ n_i=n_j}} \; . \qquad (5.49)
\end{aligned}
$$

Under these conditions, one has that $M_i = M_j = \frac{1}{2}$ and the simple gas bracket integrals are then mass independent except for the presence of a single m_i in the simple gas omega integrals. Since, for any given (p, q) all of the terms in the pairs of corresponding $L_1(\chi)$ and $L_{12}(\chi)$ bracket integral expressions have the same total power of the constituent masses, substitution of $M_i = M_j = \frac{1}{2}$ simply yields an additional constant factor for the corresponding pairs of expressions of $(\frac{1}{2})^{p+q+2}$. The difference in the signs of the terms in the corresponding expressions due to the factor of $(-1)^\ell$ has the effect that all terms involving omega integrals with odd values of ℓ cancel exactly and all terms involving omega integrals with even values of ℓ add identically to produce an additional factor of 2 in each surviving term. Following this prescription, one obtains for the simple gas bracket integrals of Eq. (5.10):

$$\begin{aligned}&\big[S_{5/2}^{(0)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i, S_{5/2}^{(0)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i\big]_i\\&= 4\Omega_i^{(2)}(2)\ ,\end{aligned} \tag{5.50}$$

$$\begin{aligned}&\big[S_{5/2}^{(0)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i, S_{5/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i\big]_i\\&= 7\Omega_i^{(2)}(2) - 2\Omega_i^{(2)}(3)\ ,\end{aligned} \tag{5.51}$$

$$\begin{aligned}&\big[S_{5/2}^{(0)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i, S_{5/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i\big]_i\\&= \tfrac{63}{8}\Omega_i^{(2)}(2) - \tfrac{9}{2}\Omega_i^{(2)}(3) + \tfrac{1}{2}\Omega_i^{(2)}(4)\ ,\end{aligned} \tag{5.52}$$

$$\begin{aligned}&\big[S_{5/2}^{(0)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i, S_{5/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i\big]_i\\&= \tfrac{231}{32}\Omega_i^{(2)}(2) - \tfrac{99}{16}\Omega_i^{(2)}(3) + \tfrac{11}{8}\Omega_i^{(2)}(4) - \tfrac{1}{12}\Omega_i^{(2)}(5)\ ,\end{aligned} \tag{5.53}$$

$$\begin{aligned}&\big[S_{5/2}^{(0)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i, S_{5/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i\big]_i\\&= \tfrac{3003}{512}\Omega_i^{(2)}(2) - \tfrac{429}{64}\Omega_i^{(2)}(3) + \tfrac{143}{64}\Omega_i^{(2)}(4) - \tfrac{13}{48}\Omega_i^{(2)}(5) + \tfrac{1}{96}\Omega_i^{(2)}(6)\ ,\end{aligned} \tag{5.54}$$

$$\begin{aligned}&\big[S_{5/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i, S_{5/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\overset{\circ}{}\boldsymbol{\mathscr{C}}_i\big]_i\\&= \tfrac{301}{12}\Omega_i^{(2)}(2) - 7\Omega_i^{(2)}(3) + \Omega_i^{(2)}(4)\ ,\end{aligned} \tag{5.55}$$

$$\left[S^{(1)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(2)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i\right]_i$$

$$= \tfrac{1365}{32}\Omega^{(2)}_i(2) - \tfrac{321}{16}\Omega^{(2)}_i(3) + \tfrac{25}{8}\Omega^{(2)}_i(4) - \tfrac{1}{4}\Omega^{(2)}_i(5)\ , \tag{5.56}$$

$$\left[S^{(1)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(3)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i\right]_i$$

$$= \tfrac{6699}{128}\Omega^{(2)}_i(2) - \tfrac{297}{8}\Omega^{(2)}_i(3) + \tfrac{385}{48}\Omega^{(2)}_i(4) - \tfrac{5}{6}\Omega^{(2)}_i(5) + \tfrac{1}{24}\Omega^{(2)}_i(6)\ , \tag{5.57}$$

$$\left[S^{(1)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(4)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i\right]_i$$

$$= \tfrac{109109}{2048}\Omega^{(2)}_i(2) - \tfrac{52767}{1024}\Omega^{(2)}_i(3) + \tfrac{11869}{768}\Omega^{(2)}_i(4) - \tfrac{2405}{1152}\Omega^{(2)}_i(5)$$

$$+ \tfrac{59}{384}\Omega^{(2)}_i(6) - \tfrac{1}{192}\Omega^{(2)}_i(7)\ , \tag{5.58}$$

$$\left[S^{(2)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(2)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i\right]_i$$

$$= \tfrac{25137}{256}\Omega^{(2)}_i(2) - \tfrac{1755}{32}\Omega^{(2)}_i(3) + \tfrac{381}{32}\Omega^{(2)}_i(4) - \tfrac{9}{8}\Omega^{(2)}_i(5)$$

$$+ \tfrac{1}{16}\Omega^{(2)}_i(6) + \tfrac{1}{2}\Omega^{(4)}_i(4)\ , \tag{5.59}$$

$$\left[S^{(2)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(3)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i\right]_i$$

$$= \tfrac{155925}{1024}\Omega^{(2)}_i(2) - \tfrac{56727}{512}\Omega^{(2)}_i(3) + \tfrac{3795}{128}\Omega^{(2)}_i(4) - \tfrac{249}{64}\Omega^{(2)}_i(5)$$

$$+ \tfrac{17}{64}\Omega^{(2)}_i(6) - \tfrac{1}{96}\Omega^{(2)}_i(7) + \tfrac{11}{8}\Omega^{(4)}_i(4) - \tfrac{1}{4}\Omega^{(4)}_i(5)\ , \tag{5.60}$$

$$\left[S^{(2)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(4)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i\right]_i$$

$$= \tfrac{3072069}{16384}\Omega^{(2)}_i(2) - \tfrac{719433}{4096}\Omega^{(2)}_i(3) + \tfrac{237237}{4096}\Omega^{(2)}_i(4) - \tfrac{4901}{512}\Omega^{(2)}_i(5)$$

$$+ \tfrac{891}{1024}\Omega^{(2)}_i(6) - \tfrac{35}{768}\Omega^{(2)}_i(7) + \tfrac{1}{768}\Omega^{(2)}_i(8) + \tfrac{143}{64}\Omega^{(4)}_i(4) - \tfrac{13}{16}\Omega^{(4)}_i(5)$$

$$+ \tfrac{1}{16}\Omega^{(4)}_i(6)\ , \tag{5.61}$$

$$\left[S^{(3)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i, S^{(3)}_{5/2}(\mathscr{C})\,\boldsymbol{\mathscr{C}}_i\mathring{\boldsymbol{\mathscr{C}}}_i\right]_i$$

$$= \tfrac{1168167}{4096}\Omega^{(2)}_i(2) - \tfrac{245025}{1024}\Omega^{(2)}_i(3) + \tfrac{236005}{3072}\Omega^{(2)}_i(4) - \tfrac{1573}{128}\Omega^{(2)}_i(5)$$

$$+ \tfrac{859}{768}\Omega^{(2)}_i(6) - \tfrac{11}{192}\Omega^{(2)}_i(7) + \tfrac{1}{576}\Omega^{(2)}_i(8) + \tfrac{1463}{288}\Omega^{(4)}_i(4) - \tfrac{11}{8}\Omega^{(4)}_i(5)$$

$$+\ \tfrac{1}{8}\Omega_i^{(4)}(6)\ , \tag{5.62}$$

$$\begin{aligned}
&\left[S_{5/2}^{(3)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i},S_{5/2}^{(4)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_i\\
&=\tfrac{26951925}{65536}\Omega_i^{(2)}(2)-\tfrac{13505349}{32768}\Omega_i^{(2)}(3)+\tfrac{7661797}{49152}\Omega_i^{(2)}(4)-\tfrac{738569}{24576}\Omega_i^{(2)}(5)\\
&+\tfrac{41005}{12288}\Omega_i^{(2)}(6)-\tfrac{4175}{18432}\Omega_i^{(2)}(7)+\tfrac{85}{9216}\Omega_i^{(2)}(8)-\tfrac{1}{4608}\Omega_i^{(2)}(9)\\
&+\tfrac{23881}{2304}\Omega_i^{(4)}(4)-\tfrac{4745}{1152}\Omega_i^{(4)}(5)+\tfrac{37}{64}\Omega_i^{(4)}(6)-\tfrac{1}{32}\Omega_i^{(4)}(7)\ ,
\end{aligned}\tag{5.63}$$

and:

$$\begin{aligned}
&\left[S_{5/2}^{(4)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i},S_{5/2}^{(4)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_i\\
&=\tfrac{711736025}{1048576}\Omega_i^{(2)}(2)-\tfrac{50053575}{65536}\Omega_i^{(2)}(3)+\tfrac{64969619}{196608}\Omega_i^{(2)}(4)-\tfrac{3630289}{49152}\Omega_i^{(2)}(5)\\
&+\tfrac{953681}{98304}\Omega_i^{(2)}(6)-\tfrac{29393}{36864}\Omega_i^{(2)}(7)+\tfrac{4699}{110592}\Omega_i^{(2)}(8)-\tfrac{13}{9216}\Omega_i^{(2)}(9)\\
&+\tfrac{1}{36864}\Omega_i^{(2)}(10)+\tfrac{470041}{18432}\Omega_i^{(4)}(4)-\tfrac{28223}{2304}\Omega_i^{(4)}(5)+\tfrac{5341}{2304}\Omega_i^{(4)}(6)\\
&-\tfrac{13}{64}\Omega_i^{(4)}(7)+\tfrac{1}{128}\Omega_i^{(4)}(8)+\tfrac{1}{96}\Omega_i^{(6)}(6)\ .
\end{aligned}\tag{5.64}$$

Equations (5.19)-(5.64) constitute the complete set of all bracket integrals necessary to evaluate the Chapman-Enskog viscosity coefficients using order 5 Sonine polynomial expansions.

5.4 Symmetry and matrix element generation

There are several important symmetries to point out that are present among the bracket integral expressions and, hence, the b_{pq} matrix elements. Proper application of these symmetries aids in most efficiently constructing the appropriate expressions for the matrix elements, b_{pq}, from the bracket integral expressions presented in this work and those required for higher orders of approximation. First, it is clear from Eqs. (5.5)-(5.8) that the bracket integral expressions needed for arbitrary b_{pq} matrix elements are:

$$\left[S_{5/2}^{(p-1)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i},S_{5/2}^{(q-1)}(\mathscr{C})\,\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}\right]_i\ , \tag{5.65}$$

$$\left[S_{5/2}^{(p-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j, S_{5/2}^{(q-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j\right]_j , \tag{5.66}$$

$$\left[S_{5/2}^{(p-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S_{5/2}^{(q-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i\right]_{ij} , \tag{5.67}$$

$$\left[S_{5/2}^{(p-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j, S_{5/2}^{(q-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j\right]_{ji} , \tag{5.68}$$

$$\left[S_{5/2}^{(p-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S_{5/2}^{(q-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j\right]_{ij} , \tag{5.69}$$

and:

$$\left[S_{5/2}^{(p-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j, S_{5/2}^{(q-1)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i\right]_{ji} , \tag{5.70}$$

where the symmetries for the b_{pq} matrix elements: $b_{pq} = b_{qp}$, $b_{p-q} = b_{-qp}$, $b_{-pq} = b_{q-p}$, and $b_{-p-q} = b_{-q-p}$ have already been noted. For the bracket integral expressions, note that the bracket integrals of Eqs. (5.65) and (5.66) describe simple gas collision interactions and are identical in their general expression, excepting where the subscripts for the component gases differ. It follows from the definitions that $n_i^{-2}\left[S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i\right]_i = n_j^{-2}\left[S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j, S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\overset{\circ}{\boldsymbol{\mathscr{C}}}_j\right]_j$. The expressions for these simple gas bracket integrals can be constructed from appropriate combinations of the $L_1\left(\chi\right)$ and $L_{12}\left(\chi\right)$ bracket integral expressions as has been indicated in Eq. (5.49). However, in the practical construction of these expressions to orders much higher than 5, the direct generation of the expressions has proven to be computationally more economical than algebraic combination of the $L_1\left(\chi\right)$ and $L_{12}\left(\chi\right)$ bracket integral expressions.

The bracket integral expressions given in Eqs. (5.67) and (5.68) and Eqs. (5.69) and (5.70) are related to the $L_1\left(\chi\right)$ and $L_{12}\left(\chi\right)$ expressions, respectively, and exhibit rather useful symmetries. First, as previously indicated, the structures of the bracket integrals labeled with the indices, ji, Eqs. (5.68) and (5.70), are identical to the structures of those labeled with the indicies, ij, Eqs. (5.67) and (5.69), respectively, and only a complete interchange of the subscript indices is needed. Lastly, of particular use are the symmetries:

$$\left[S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i\right]_{ij} = \left[S_{5/2}^{(q)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i, S_{5/2}^{(p)}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\overset{\circ}{\boldsymbol{\mathscr{C}}}_i\right]_{ij} , \tag{5.71}$$

for the $L_1(\chi)$ related bracket integral expressions and:

$$\begin{aligned}\left[S_{5/2}^{(p)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(q)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_j\boldsymbol{\mathscr{C}}_j}\right]_{ij} \\ = \left[S_{5/2}^{(q)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_i\boldsymbol{\mathscr{C}}_i}, S_{5/2}^{(p)}(\mathscr{C})\overset{\circ}{\boldsymbol{\mathscr{C}}_j\boldsymbol{\mathscr{C}}_j}\right]_{ij}\left(\frac{M_i}{M_j}\right)^{q-p},\end{aligned} \tag{5.72}$$

for the $L_{12}(\chi)$ related bracket integral expressions.

5.5 Discussion and conclusions

The purpose in this work has been to explore the use of Sonine polynomial expansions to obtain error free results for the transport coefficients and the related Chapman-Enskog functions for simple gases and gas mixtures. In the current chapter, the relevant explicit bracket integral expressions needed to compute the viscosity-related Chapman-Enskog solutions up to order 5 based upon expansions in Sonine polynomials have been reported. A method for combining the given bracket integral expressions to generate the requisite matrix elements used to solve the system of equations in the Chapman-Enskog method, Eq. (5.2), for the viscosity solutions has also been described. Additionally, several of the associated symmetries between the matrix elements, and also the bracket integral expressions, have been clearly detailed which, in turn, one may use to improve and optimize the construction of the relevant expressions and computational programs based upon them. In the following chapter, a similar discussion with the appropriate expressions for the thermal conductivity- and diffusion-related, Chapman-Enskog solutions will be presented.

Chapter 6

General expressions for the diffusion- and thermal conductivity-related bracket integrals to order 5

6.1 Introduction

The Chapman-Enskog solutions of the Boltzmann equations provide a basis for the computation of important transport coefficients for both simple gases and gas mixtures [10–12, 20–26, 28–31, 39]. The use of Sonine polynomial expansions for the Chapman-Enskog solutions was first suggested by Burnett [13] and has become the general method for obtaining the transport coefficients due to the relatively rapid convergence of this series [10–13, 20–24]. While it has been found that relatively, low-order expansions (of order 4) can provide reasonable accuracy in computations of the transport coefficients (to about 1 part in 1,000), most existing computer codes do not use these solutions beyond order 2 or order 3 as the relevant expressions become increasingly complex with great rapidity and have not been available in the past. The present investigation of simple gases and gas mixtures [15–17] has permitted the exploration of the Chapman-Enskog solutions to relatively high orders computationally using *Mathematica*® and, thus, accurate, completely general, expressions have become available up to order 60 for the viscosity-related bracket integrals and up to order 70 for the diffusion- and thermal conductivity-related bracket integrals. The complexity of the general expressions that have been obtained for the bracket integrals for gas mixtures does make them impractical to report in the open literature beyond the lowest orders even when organized into compact form. However, insofar as such expressions can be explicitly reported in the literature, they are valuable from the point of view of having them available for general use in the existing computer codes. Further, having such expressions explicitly reported in the literature, even to limited order, has a certain archival value in the field. Thus, in this chapter, a set of completely general expressions for the bracket integrals necessary to complete the Chapman-Enskog diffusion and thermal conductivity solutions up to order 5 is reported. In the previous chapter [18] a set of completely general and accurate expressions for the bracket integrals necessary to complete the Chapman-Enskog viscosity solutions up to order 5 were similarly reported. In the following sections, the basic relationships relevant to the Chapman-Enskog solutions for diffusion, thermal diffusion, and thermal conductivity, the role of the bracket integrals in these solutions, the explicit expressions for the various bracket integrals, and some basic relationships that occur among the various bracket integrals that make them more tractable to generate and manipulate in the context of the Chapman-Enskog solutions are described.

6.2 The basic relationships

Following the work and notations of Chapman and Cowling [10], as was done in the previous chapters [15–18], the diffusion, the thermal diffusion, and the thermal conductivity for binary gas mixtures may be expressed to some order of approximation, m, (in terms of Sonine polynomial expansions) as:

$$[D_{12}]_m = \tfrac{1}{2} x_1 x_2 \left(2kT/m_0\right)^{1/2} d_0^{(m)} \ , \tag{6.1}$$

$$[D_T]_m = -\tfrac{5}{4} x_1 x_2 \left(2kT/m_0\right)^{1/2} \left(x_1 M_1^{-1/2} d_1^{(m)} + x_2 M_2^{-1/2} d_{-1}^{(m)}\right) \ , \tag{6.2}$$

$$[\lambda]_m = -\tfrac{5}{4}kn\left(2kT/m_0\right)^{1/2}\left(x_1 M_1^{-1/2} a_1^{(m)} + x_2 M_2^{-1/2} a_{-1}^{(m)}\right) \ , \tag{6.3}$$

respectively, where $x_1 = n_1/n$ and $x_2 = n_2/n$ are the component mixture fractions, n_1 and n_2 are the component number densities with $n = n_1 + n_2$ being the total number density of the mixture, m_1 and m_2 are the component molecular masses with $m_0 = m_1 + m_2$, k is Boltzmann's constant, T is the temperature, $M_1 = m_1/m_0$, $M_2 = m_2/m_0$, and in which the quantities $d_0^{(m)}$, $d_{-1}^{(m)}$, $d_1^{(m)}$, $a_{-1}^{(m)}$, and $a_1^{(m)}$ are determined by solving the following systems of algebraic equations:

$$\sum_{p=-m}^{+m} d_p a_{pq} = \delta_q \quad (q = 0, \pm 1, \pm 2, \cdots, \pm m) \ , \tag{6.4}$$

$$\sum_{\substack{p=-m \\ p\neq 0}}^{+m} a_p a_{pq} = \alpha_q \quad (q = \pm 1, \pm 2, \cdots, \pm m) \ , \tag{6.5}$$

in which:

$$\delta_0 = \frac{3}{2n}\left(\frac{2kT}{m_0}\right)^{1/2} \ , \quad \delta_q = 0 \ (q \neq 0) \ . \tag{6.6}$$

$$\begin{gathered} \alpha_1 = -\tfrac{15}{4}\frac{n_1}{n^2}\left(\frac{2kT}{m_1}\right)^{1/2} \ , \quad \alpha_{-1} = -\tfrac{15}{4}\frac{n_2}{n^2}\left(\frac{2kT}{m_2}\right)^{1/2} \ , \\ \alpha_q = 0 \ (q \neq \pm 1) \ . \end{gathered} \tag{6.7}$$

Expressed as matrices, these systems of equations may be written as:

$$\begin{bmatrix} a_{-m-m} & \cdots & a_{-m-1} & a_{-m0} & a_{-m1} & \cdots & a_{-mm} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{-1-m} & \cdots & a_{-1-1} & a_{-10} & a_{-11} & \cdots & a_{-1m} \\ a_{0-m} & \cdots & a_{0-1} & a_{00} & a_{01} & \cdots & a_{0m} \\ a_{1-m} & \cdots & a_{1-1} & a_{10} & a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m-m} & \cdots & a_{m-1} & a_{m0} & a_{m1} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} d_{-m} \\ \vdots \\ d_{-1} \\ d_0 \\ d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \delta_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ , \tag{6.8}$$

and:

$$\begin{bmatrix} a_{-m-m} & \cdots & a_{-m-1} & a_{-m1} & \cdots & a_{-mm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{-1-m} & \cdots & a_{-1-1} & a_{-11} & \cdots & a_{-1m} \\ a_{1-m} & \cdots & a_{1-1} & a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m-m} & \cdots & a_{m-1} & a_{m1} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} a_{-m} \\ \vdots \\ a_{-1} \\ a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \alpha_{-1} \\ \alpha_1 \\ \vdots \\ 0 \end{bmatrix} \ , \tag{6.9}$$

where, as the order of the expansion increases, the matrices build outward from their centers. Thus, to obtain the needed expansion coefficients, d_{-1}, d_0, and d_1, for the diffusion-related solutions (the diffusion and thermal diffusion coefficients), for a given order of the expansion, m, one need only generate the $[(2m+1)\times(2m+1)]$ matrix of Eq. (6.8) and invert it. Similarly, to obtain the needed expansion coefficients, a_{-1} and a_1, and hence the thermal conductivity coefficient, for a given order of the expansion, m, one need only generate the $[(2m)\times(2m)]$ matrix of Eq. (6.9) and invert it.

The matrix elements, a_{pq}, in Eqs. (6.8) and (6.9) are constructed from combinations of bracket integrals containing the appropriate Sonine polynomials from the expansions used. Specifically, since it is straightforward to show for any (p,q) that $a_{pq}=a_{qp}$, one has that:

$$\begin{aligned} a_{pq}=a_{qp}=&x_1^2\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1\right]_1 \\ &+x_1x_2\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1\right]_{12}\ , \end{aligned} \tag{6.10}$$

$$a_{p-q}=a_{-qp}=x_1x_2\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_2\right]_{12}\ , \tag{6.11}$$

$$a_{-pq}=a_{q-p}=x_1x_2\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1\right]_{21}\ , \tag{6.12}$$

$$\begin{aligned} a_{-p-q}=a_{-q-p}=&x_2^2\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_2\right]_2 \\ &+x_1x_2\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_2, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_2\right]_{21}\ , \end{aligned} \tag{6.13}$$

where:

$$\begin{aligned} S_m^{(n)}(x)&=\sum_{p=0}^{n}\frac{(m+n)_{n-p}}{(p)!\,(n-p)!}(-x)^p \\ &=\sum_{p=0}^{n}\frac{(m+n)!}{(p)!\,(n-p)!\,(m+p)!}(-x)^p\ , \end{aligned} \tag{6.14}$$

(with $S_m^{(0)}(x)=1$ and $S_m^{(1)}(x)=m+1-x$) are numerical multiples (un-normalized) of the Sonine polynomials originally used in the kinetic theory of gases by Burnett [13]. From the definitions used in the bracket integral notation, it follows that Eqs. (6.13) and (6.12) are essentially identical to Eqs. (6.10) and (6.11), respectively, with the only difference being the interchange of the subscripts 1 and 2 representing the different components of the mixture. Thus, in general, the complete Chapman-Enskog solutions for diffusion and thermal conductivity only require evaluation of the bracket integrals:

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1\right]_1 , \tag{6.15}$$

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1\right]_{12} , \tag{6.16}$$

and:

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_1, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_2\right]_{12} . \tag{6.17}$$

All of the above expressions for binary gas mixtures are readily generalized for use with arbitrary mixtures by replacing the 1 and 2 labeling scheme associated specifically with binary mixtures to a more general i and j labeling scheme. In the following sections, using this (i, j) labeling scheme, completely general expressions for all of the bracket integrals in Eqs. (6.15)-(6.17) are given up to order 5 of the expansion. Also shown in all of these expressions are the dependence of each on the molecular masses and the omega integrals, $\Omega_{ij}^{(\ell)}(r)$, which are defined as:

$$\Omega_{ij}^{(\ell)}(r) \equiv \left(\frac{kT}{2\pi m_0 M_i M_j}\right)^{1/2} \int_0^{\infty} \exp\left(-g^2\right) g^{2r+3} \phi_{ij}^{(\ell)} \mathrm{d}g , \tag{6.18}$$

with:

$$\phi_{ij}^{(\ell)} \equiv 2\pi \int_0^{\pi} \left(1 - \cos^{\ell}(\chi)\right) b\mathrm{d}b . \tag{6.19}$$

The omega integrals contain all of the dependencies relating to the specific intermolecular potential model that is employed. It is often considered convenient to define the omega integrals in terms of a simple scaling factor, $\sigma_{ij} = \frac{1}{2}(\sigma_i + \sigma_j)$, which is, generally, only a convenient, arbitrarily chosen length within some range where the impact parameter, b, is significant. Expressed in this manner, the omega integrals are then [10]:

$$\Omega_{ij}^{(\ell)}(r) = \tfrac{1}{2}\sigma_{ij}^2 \left(\frac{2\pi kT}{m_0 M_i M_j}\right)^{1/2} W_{ij}^{(\ell)}(r) , \tag{6.20}$$

where:

$$W_{ij}^{(\ell)}(r) \equiv 2 \int_0^\infty \exp\left(-g^2\right) g^{2r+3} \times \int_0^\pi \left(1-\cos^\ell(\chi)\right) (b/\sigma_{ij})\, \mathrm{d}\,(b/\sigma_{ij})\, \mathrm{d}g \; . \tag{6.21}$$

Note that when $i = j$, Eqs. (6.20) and (6.21) reduce to the following simple gas expressions:

$$\Omega_i^{(\ell)}(r) = \sigma_i^2 \left(\frac{\pi k T}{m_i}\right)^{1/2} W_i^{(\ell)}(r) \; , \tag{6.22}$$

with:

$$W_i^{(\ell)}(r) = 2 \int_0^\infty \exp\left(-g^2\right) g^{2r+3} \times \int_0^\pi \left(1-\cos^\ell(\chi)\right) \left(b/\sigma_i\right) \mathrm{d}\left(b/\sigma_i\right) \mathrm{d}g \; . \tag{6.23}$$

In Eqs. (6.20)-(6.23), σ_i is an arbitrary scale length associated with collisions between like molecules of type i while σ_{ij} is associated with collisions between unlike molecules of types i and j. These scale lengths are commonly associated with some concept of the molecular diameters depending upon the specific details of the intermolecular potential model that is employed.

6.3 General expressions for the bracket integrals

Of the three needed bracket integrals, the bracket integrals of Eq. (6.16) prove to have the most complicated dependence upon the molecular masses of the mixture constituents. At the same time, having general expressions for these bracket integrals proves to be the most useful as expressions for the remaining two bracket integrals can be generated from them with minimal effort. Thus, the general expressions for the bracket integrals of Eq. (6.16) which, in Chapman and Cowling [10], are associated with the functions $H_1(\chi)$ are reported first. For order 5 diffusion- and thermal conductivity-related Chapman-Enskog solutions, one requires $p, q \in (0, 1, 2, 3, 4, 5)$ and, hence, one has for the bracket integrals of Eq. (6.16) the following general expressions:

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\right]_{ij}$$
$$= 8M_j\varOmega^{(1)}_{ij}(1)\ , \tag{6.24}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(1)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\right]_{ij}$$
$$= 20M_j^2\varOmega^{(1)}_{ij}(1) - 8M_j^2\varOmega^{(1)}_{ij}(2)\ , \tag{6.25}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(2)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\right]_{ij}$$
$$= 35M_j^3\varOmega^{(1)}_{ij}(1) - 28M_j^3\varOmega^{(1)}_{ij}(2) + 4M_j^3\varOmega^{(1)}_{ij}(3)\ , \tag{6.26}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(3)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\right]_{ij}$$
$$= \tfrac{105}{2}M_j^4\varOmega^{(1)}_{ij}(1) - 63M_j^4\varOmega^{(1)}_{ij}(2) + 18M_j^4\varOmega^{(1)}_{ij}(3) - \tfrac{4}{3}M_j^4\varOmega^{(1)}_{ij}(4)\ , \tag{6.27}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(4)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\right]_{ij}$$
$$= \tfrac{1155}{16}M_j^5\varOmega^{(1)}_{ij}(1) - \tfrac{231}{2}M_j^5\varOmega^{(1)}_{ij}(2) + \tfrac{99}{2}M_j^5\varOmega^{(1)}_{ij}(3)$$
$$- \tfrac{22}{3}M_j^5\varOmega^{(1)}_{ij}(4) + \tfrac{1}{3}M_j^5\varOmega^{(1)}_{ij}(5)\ , \tag{6.28}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(5)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\right]_{ij}$$
$$= \tfrac{3003}{32}M_j^6\varOmega^{(1)}_{ij}(1) - \tfrac{3003}{16}M_j^6\varOmega^{(1)}_{ij}(2) + \tfrac{429}{4}M_j^6\varOmega^{(1)}_{ij}(3)$$
$$- \tfrac{143}{6}M_j^6\varOmega^{(1)}_{ij}(4) + \tfrac{13}{6}M_j^6\varOmega^{(1)}_{ij}(5) - \tfrac{1}{15}M_j^6\varOmega^{(1)}_{ij}(6)\ , \tag{6.29}$$

$$\left[S^{(1)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(1)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\right]_{ij}$$
$$= 110\left(\tfrac{6}{11}M_i^2M_j + \tfrac{5}{11}M_j^3\right)\varOmega^{(1)}_{ij}(1) - 40M_j^3\varOmega^{(1)}_{ij}(2) + 8M_j^3\varOmega^{(1)}_{ij}(3)$$
$$+ 16M_iM_j^2\varOmega^{(2)}_{ij}(2)\ , \tag{6.30}$$

$$\left[S^{(1)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(2)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i\right]_{ij}$$
$$= \tfrac{595}{2}\left(\tfrac{12}{17}M_i^2M_j^2 + \tfrac{5}{17}M_j^4\right)\varOmega^{(1)}_{ij}(1) - 189\left(\tfrac{4}{9}M_i^2M_j^2\right.$$

$$
\begin{aligned}
&+\tfrac{5}{9}M_j^4\big)\,\Omega_{ij}^{(1)}(2)+38M_j^4\Omega_{ij}^{(1)}(3)-4M_j^4\Omega_{ij}^{(1)}(4)\\
&+56M_iM_j^3\Omega_{ij}^{(2)}(2)-16M_iM_j^3\Omega_{ij}^{(2)}(3)\ ,
\end{aligned}
\tag{6.31}
$$

$$
\begin{aligned}
&\big[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\big]_{ij}\\
&=\tfrac{2415}{4}\big(\tfrac{18}{23}M_i^2M_j^3+\tfrac{5}{23}M_j^5\big)\,\Omega_{ij}^{(1)}(1)-588\big(\tfrac{9}{14}M_i^2M_j^3\\
&+\tfrac{5}{14}M_j^5\big)\,\Omega_{ij}^{(1)}(2)+162\big(\tfrac{1}{3}M_i^2M_j^3+\tfrac{2}{3}M_j^5\big)\,\Omega_{ij}^{(1)}(3)\\
&-\tfrac{64}{3}M_j^5\Omega_{ij}^{(1)}(4)+\tfrac{4}{3}M_j^5\Omega_{ij}^{(1)}(5)+126M_iM_j^4\Omega_{ij}^{(2)}(2)\\
&-72M_iM_j^4\Omega_{ij}^{(2)}(3)+8M_iM_j^4\Omega_{ij}^{(2)}(4)\ ,
\end{aligned}
\tag{6.32}
$$

$$
\begin{aligned}
&\big[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\big]_{ij}\\
&=\tfrac{33495}{32}\big(\tfrac{24}{29}M_i^2M_j^4+\tfrac{5}{29}M_j^6\big)\,\Omega_{ij}^{(1)}(1)-\tfrac{22407}{16}\big(\tfrac{72}{97}M_i^2M_j^4\\
&+\tfrac{25}{97}M_j^6\big)\,\Omega_{ij}^{(1)}(2)+\tfrac{2145}{4}\big(\tfrac{36}{65}M_i^2M_j^4+\tfrac{29}{65}M_j^6\big)\,\Omega_{ij}^{(1)}(3)\\
&-\tfrac{539}{6}\big(\tfrac{12}{49}M_i^2M_j^4+\tfrac{37}{49}M_j^6\big)\,\Omega_{ij}^{(1)}(4)+\tfrac{49}{6}M_j^6\Omega_{ij}^{(1)}(5)\\
&-\tfrac{1}{3}M_j^6\Omega_{ij}^{(1)}(6)+231M_iM_j^5\Omega_{ij}^{(2)}(2)-198M_iM_j^5\Omega_{ij}^{(2)}(3)\\
&+44M_iM_j^5\Omega_{ij}^{(2)}(4)-\tfrac{8}{3}M_iM_j^5\Omega_{ij}^{(2)}(5)\ ,
\end{aligned}
\tag{6.33}
$$

$$
\begin{aligned}
&\big[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\big]_{ij}\\
&=\tfrac{105105}{64}\big(\tfrac{6}{7}M_i^2M_j^5+\tfrac{1}{7}M_j^7\big)\,\Omega_{ij}^{(1)}(1)-\tfrac{45045}{16}\big(\tfrac{4}{5}M_i^2M_j^5+\tfrac{1}{5}M_j^7\big)\,\Omega_{ij}^{(1)}(2)\\
&+\tfrac{22737}{16}\big(\tfrac{36}{53}M_i^2M_j^5+\tfrac{17}{53}M_j^7\big)\,\Omega_{ij}^{(1)}(3)-\tfrac{1859}{6}\big(\tfrac{6}{13}M_i^2M_j^5+\tfrac{7}{13}M_j^7\big)\,\Omega_{ij}^{(1)}(4)\\
&+\tfrac{143}{4}\big(\tfrac{2}{11}M_i^2M_j^5+\tfrac{9}{11}M_j^7\big)\,\Omega_{ij}^{(1)}(5)-\tfrac{7}{3}M_j^7\Omega_{ij}^{(1)}(6)+\tfrac{1}{15}M_j^7\Omega_{ij}^{(1)}(7)\\
&+\tfrac{3003}{8}M_iM_j^6\Omega_{ij}^{(2)}(2)-429M_iM_j^6\Omega_{ij}^{(2)}(3)+143M_iM_j^6\Omega_{ij}^{(2)}(4)\\
&-\tfrac{52}{3}M_iM_j^6\Omega_{ij}^{(2)}(5)+\tfrac{2}{3}M_iM_j^6\Omega_{ij}^{(2)}(6)\ ,
\end{aligned}
\tag{6.34}
$$

$$
\begin{aligned}
&\big[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\big]_{ij}\\
&=\tfrac{8505}{8}\big(\tfrac{40}{243}M_i^4M_j+\tfrac{168}{243}M_i^2M_j^3+\tfrac{35}{243}M_j^5\big)\,\Omega_{ij}^{(1)}(1)-833\big(\tfrac{12}{17}M_i^2M_j^3\\
&+\tfrac{5}{17}M_j^5\big)\,\Omega_{ij}^{(1)}(2)+241\big(\tfrac{108}{241}M_i^2M_j^3+\tfrac{133}{241}M_j^5\big)\,\Omega_{ij}^{(1)}(3)-28M_j^5\Omega_{ij}^{(1)}(4)
\end{aligned}
$$

$$+ 2M_j^5 \mathit{\Omega}_{ij}^{(1)}(5) + 308 \left(\tfrac{4}{11} M_i^3 M_j^2 + \tfrac{7}{11} M_i M_j^4\right) \mathit{\Omega}_{ij}^{(2)}(2)$$

$$- 112 M_i M_j^4 \mathit{\Omega}_{ij}^{(2)}(3) + 16 M_i M_j^4 \mathit{\Omega}_{ij}^{(2)}(4) + 16 M_i^2 M_j^3 \mathit{\Omega}_{ij}^{(3)}(3) \ , \tag{6.35}$$

$$\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij}$$

$$= \tfrac{42735}{16} \left(\tfrac{120}{407} M_i^4 M_j^2 + \tfrac{252}{407} M_i^2 M_j^4 + \tfrac{35}{407} M_j^6\right) \mathit{\Omega}_{ij}^{(1)}(1) - \tfrac{22071}{8} \left(\tfrac{120}{1051} M_i^4 M_j^2\right.$$

$$\left. + \tfrac{756}{1051} M_i^2 M_j^4 + \tfrac{175}{1051} M_j^6\right) \mathit{\Omega}_{ij}^{(1)}(2) + \tfrac{2001}{2} \left(\tfrac{450}{667} M_i^2 M_j^4 + \tfrac{217}{667} M_j^6\right) \mathit{\Omega}_{ij}^{(1)}(3)$$

$$- \tfrac{499}{3} \left(\tfrac{198}{499} M_i^2 M_j^4 + \tfrac{301}{499} M_j^6\right) \mathit{\Omega}_{ij}^{(1)}(4) + \tfrac{41}{3} M_j^6 \mathit{\Omega}_{ij}^{(1)}(5) - \tfrac{2}{3} M_j^6 \mathit{\Omega}_{ij}^{(1)}(6)$$

$$+ 945 \left(\tfrac{8}{15} M_i^3 M_j^3 + \tfrac{7}{15} M_i M_j^5\right) \mathit{\Omega}_{ij}^{(2)}(2) - 522 \left(\tfrac{8}{29} M_i^3 M_j^3\right.$$

$$\left. + \tfrac{21}{29} M_i M_j^5\right) \mathit{\Omega}_{ij}^{(2)}(3) + 100 M_i M_j^5 \mathit{\Omega}_{ij}^{(2)}(4) - 8 M_i M_j^5 \mathit{\Omega}_{ij}^{(2)}(5)$$

$$+ 72 M_i^2 M_j^4 \mathit{\Omega}_{ij}^{(3)}(3) - 16 M_i^2 M_j^4 \mathit{\Omega}_{ij}^{(3)}(4) \ , \tag{6.36}$$

$$\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij}$$

$$= \tfrac{705705}{128} \left(\tfrac{240}{611} M_i^4 M_j^3 + \tfrac{336}{611} M_i^2 M_j^5 + \tfrac{35}{611} M_j^7\right) \mathit{\Omega}_{ij}^{(1)}(1)$$

$$- \tfrac{234927}{32} \left(\tfrac{80}{339} M_i^4 M_j^3 + \tfrac{224}{339} M_i^2 M_j^5 + \tfrac{35}{339} M_j^7\right) \mathit{\Omega}_{ij}^{(1)}(2)$$

$$+ \tfrac{104973}{32} \left(\tfrac{240}{3181} M_i^4 M_j^3 + \tfrac{2304}{3181} M_i^2 M_j^5 + \tfrac{637}{3181} M_j^7\right) \mathit{\Omega}_{ij}^{(1)}(3)$$

$$- \tfrac{8437}{12} \left(\tfrac{480}{767} M_i^2 M_j^5 + \tfrac{287}{767} M_j^7\right) \mathit{\Omega}_{ij}^{(1)}(4) + \tfrac{623}{8} \left(\tfrac{208}{623} M_i^2 M_j^5\right.$$

$$\left. + \tfrac{415}{623} M_j^7\right) \mathit{\Omega}_{ij}^{(1)}(5) - \tfrac{29}{6} M_j^7 \mathit{\Omega}_{ij}^{(1)}(6) + \tfrac{1}{6} M_j^7 \mathit{\Omega}_{ij}^{(1)}(7) + \tfrac{4389}{2} \left(\tfrac{12}{19} M_i^3 M_j^4\right.$$

$$\left. + \tfrac{7}{19} M_i M_j^6\right) \mathit{\Omega}_{ij}^{(2)}(2) - 1716 \left(\tfrac{6}{13} M_i^3 M_j^4 + \tfrac{7}{13} M_i M_j^6\right) \mathit{\Omega}_{ij}^{(2)}(3)$$

$$+ 440 \left(\tfrac{1}{5} M_i^3 M_j^4 + \tfrac{4}{5} M_i M_j^6\right) \mathit{\Omega}_{ij}^{(2)}(4) - \tfrac{160}{3} M_i M_j^6 \mathit{\Omega}_{ij}^{(2)}(5)$$

$$+ \tfrac{8}{3} M_i M_j^6 \mathit{\Omega}_{ij}^{(2)}(6) + 198 M_i^2 M_j^5 \mathit{\Omega}_{ij}^{(3)}(3)$$

$$- 88 M_i^2 M_j^5 \mathit{\Omega}_{ij}^{(3)}(4) + 8 M_i^2 M_j^5 \mathit{\Omega}_{ij}^{(3)}(5) \ , \tag{6.37}$$

$$\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij}$$

$$= \tfrac{2567565}{256} \left(\tfrac{80}{171} M_i^4 M_j^4 + \tfrac{84}{171} M_i^2 M_j^6 + \tfrac{7}{171} M_j^8\right) \mathit{\Omega}_{ij}^{(1)}(1)$$

$$- \tfrac{2129127}{128} \left(\tfrac{240}{709} M_i^4 M_j^4 + \tfrac{420}{709} M_i^2 M_j^6 + \tfrac{49}{709} M_j^8\right) \mathit{\Omega}_{ij}^{(1)}(2)$$

$$+ \tfrac{579579}{64} \left(\tfrac{240}{1351} M_i^4 M_j^4 + \tfrac{936}{1351} M_i^2 M_j^6 + \tfrac{175}{1351} M_j^8\right) \mathit{\Omega}_{ij}^{(1)}(3)$$

$$
\begin{aligned}
&- \tfrac{75933}{32} \left(\tfrac{80}{1593} M_i^4 M_j^4 + \tfrac{1128}{1593} M_i^2 M_j^6 + \tfrac{385}{1593} M_j^8 \right) \mathit{\Omega}_{ij}^{(1)}(4) \\
&+ \tfrac{16237}{48} \left(\tfrac{708}{1249} M_i^2 M_j^6 + \tfrac{541}{1249} M_j^8 \right) \mathit{\Omega}_{ij}^{(1)}(5) \\
&- \tfrac{655}{24} \left(\tfrac{36}{131} M_i^2 M_j^6 + \tfrac{95}{131} M_j^8 \right) \mathit{\Omega}_{ij}^{(1)}(6) + \tfrac{79}{60} M_j^8 \mathit{\Omega}_{ij}^{(1)}(7) \\
&- \tfrac{1}{30} M_j^8 \mathit{\Omega}_{ij}^{(1)}(8) + \tfrac{69069}{16} \left(\tfrac{16}{23} M_i^3 M_j^5 + \tfrac{7}{23} M_i M_j^7 \right) \mathit{\Omega}_{ij}^{(2)}(2) \\
&- \tfrac{35607}{8} \left(\tfrac{48}{83} M_i^3 M_j^5 + \tfrac{35}{83} M_i M_j^7 \right) \mathit{\Omega}_{ij}^{(2)}(3) + \tfrac{3003}{2} \left(\tfrac{8}{21} M_i^3 M_j^5 \right. \\
&\left. + \tfrac{13}{21} M_i M_j^7 \right) \mathit{\Omega}_{ij}^{(2)}(4) - \tfrac{715}{3} \left(\tfrac{8}{55} M_i^3 M_j^5 + \tfrac{47}{55} M_i M_j^7 \right) \mathit{\Omega}_{ij}^{(2)}(5) \\
&+ \tfrac{59}{3} M_i M_j^7 \mathit{\Omega}_{ij}^{(2)}(6) - \tfrac{2}{3} M_i M_j^7 \mathit{\Omega}_{ij}^{(2)}(7) + 429 M_i^2 M_j^6 \mathit{\Omega}_{ij}^{(3)}(3) \\
&- 286 M_i^2 M_j^6 \mathit{\Omega}_{ij}^{(3)}(4) + 52 M_i^2 M_j^6 \mathit{\Omega}_{ij}^{(3)}(5) - \tfrac{8}{3} M_i^2 M_j^6 \mathit{\Omega}_{ij}^{(3)}(6) \, ,
\end{aligned}
\tag{6.38}
$$

$$
\begin{aligned}
&\left[S_{3/2}^{(3)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i, S_{3/2}^{(3)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \right]_{ij} \\
&= \tfrac{255255}{32} \left(\tfrac{112}{2431} M_i^6 M_j + \tfrac{1080}{2431} M_i^4 M_j^3 + \tfrac{1134}{2431} M_i^2 M_j^5 + \tfrac{105}{2431} M_j^7 \right) \mathit{\Omega}_{ij}^{(1)}(1) \\
&- \tfrac{76923}{8} \left(\tfrac{120}{407} M_i^4 M_j^3 + \tfrac{252}{407} M_i^2 M_j^5 + \tfrac{35}{407} M_j^7 \right) \mathit{\Omega}_{ij}^{(1)}(2) + \tfrac{34119}{8} \left(\tfrac{440}{3791} M_i^4 M_j^3 \right. \\
&\left. + \tfrac{2700}{3791} M_i^2 M_j^5 + \tfrac{651}{3791} M_j^7 \right) \mathit{\Omega}_{ij}^{(1)}(3) - 895 \left(\tfrac{594}{895} M_i^2 M_j^5 + \tfrac{301}{895} M_j^7 \right) \mathit{\Omega}_{ij}^{(1)}(4) \\
&+ \tfrac{201}{2} \left(\tfrac{26}{67} M_i^2 M_j^5 + \tfrac{41}{67} M_j^7 \right) \mathit{\Omega}_{ij}^{(1)}(5) - 6 M_j^7 \mathit{\Omega}_{ij}^{(1)}(6) + \tfrac{2}{9} M_j^7 \mathit{\Omega}_{ij}^{(1)}(7) \\
&+ \tfrac{14553}{4} \left(\tfrac{8}{77} M_i^5 M_j^2 + \tfrac{48}{77} M_i^3 M_j^4 + \tfrac{21}{77} M_i M_j^6 \right) \mathit{\Omega}_{ij}^{(2)}(2) - 2430 \left(\tfrac{8}{15} M_i^3 M_j^4 \right. \\
&\left. + \tfrac{7}{15} M_i M_j^6 \right) \mathit{\Omega}_{ij}^{(2)}(3) + 626 \left(\tfrac{176}{626} M_i^3 M_j^4 + \tfrac{450}{626} M_i M_j^6 \right) \mathit{\Omega}_{ij}^{(2)}(4) \\
&- 72 M_i M_j^6 \mathit{\Omega}_{ij}^{(2)}(5) + 4 M_i M_j^6 \mathit{\Omega}_{ij}^{(2)}(6) + 444 \left(\tfrac{120}{444} M_i^4 M_j^3 \right. \\
&\left. + \tfrac{324}{444} M_i^2 M_j^5 \right) \mathit{\Omega}_{ij}^{(3)}(3) - 144 M_i^2 M_j^5 \mathit{\Omega}_{ij}^{(3)}(4) + 16 M_i^2 M_j^5 \mathit{\Omega}_{ij}^{(3)}(5) \\
&+ \tfrac{32}{3} M_i^3 M_j^4 \mathit{\Omega}_{ij}^{(4)}(4) \, ,
\end{aligned}
\tag{6.39}
$$

$$
\begin{aligned}
&\left[S_{3/2}^{(3)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i, S_{3/2}^{(4)}(\mathscr{C}) \boldsymbol{\mathscr{C}}_i \right]_{ij} \\
&= \tfrac{4879875}{256} \left(\tfrac{448}{4225} M_i^6 M_j^2 + \tfrac{2160}{4225} M_i^4 M_j^4 + \tfrac{1512}{4225} M_i^2 M_j^6 + \tfrac{105}{4225} M_j^8 \right) \mathit{\Omega}_{ij}^{(1)}(1) \\
&- \tfrac{3516513}{128} \left(\tfrac{448}{15223} M_i^6 M_j^2 + \tfrac{6480}{15223} M_i^4 M_j^4 + \tfrac{7560}{15223} M_i^2 M_j^6 \right. \\
&\left. + \tfrac{735}{15223} M_j^8 \right) \mathit{\Omega}_{ij}^{(1)}(2) + \tfrac{919413}{64} \left(\tfrac{2480}{9287} M_i^4 M_j^4 + \tfrac{5904}{9287} M_i^2 M_j^6 \right. \\
&\left. + \tfrac{903}{9287} M_j^8 \right) \mathit{\Omega}_{ij}^{(1)}(3) - \tfrac{118107}{32} \left(\tfrac{1040}{10737} M_i^4 M_j^4 + \tfrac{7632}{10737} M_i^2 M_j^6 \right. \\
&\left. + \tfrac{2065}{10737} M_j^8 \right) \mathit{\Omega}_{ij}^{(1)}(4) + \tfrac{8353}{16} \left(\tfrac{5304}{8353} M_i^2 M_j^6 + \tfrac{3049}{8353} M_j^8 \right) \mathit{\Omega}_{ij}^{(1)}(5)
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{339}{8}\left(\tfrac{40}{113}M_i^2M_j^6 + \tfrac{73}{113}M_j^8\right)\Omega_{ij}^{(1)}(6) + \tfrac{71}{36}M_j^8\Omega_{ij}^{(1)}(7) - \tfrac{1}{18}M_j^8\Omega_{ij}^{(1)}(8) \\
&+ \tfrac{81081}{8}\left(\tfrac{8}{39}M_i^5M_j^3 + \tfrac{24}{39}M_i^3M_j^5 + \tfrac{7}{39}M_iM_j^7\right)\Omega_{ij}^{(2)}(2) - \tfrac{34155}{4}\left(\tfrac{8}{115}M_i^5M_j^3\right. \\
&\left.+\tfrac{72}{115}M_i^3M_j^5 + \tfrac{35}{115}M_iM_j^7\right)\Omega_{ij}^{(2)}(3) + 2717\left(\tfrac{1364}{2717}M_i^3M_j^5\right. \\
&\left.+\tfrac{1353}{2717}M_iM_j^7\right)\Omega_{ij}^{(2)}(4) - 422\left(\tfrac{52}{211}M_i^3M_j^5 + \tfrac{159}{211}M_iM_j^7\right)\Omega_{ij}^{(2)}(5) \\
&+ 34M_iM_j^7\Omega_{ij}^{(2)}(6) - \tfrac{4}{3}M_iM_j^7\Omega_{ij}^{(2)}(7) + 1551\left(\tfrac{660}{1551}M_i^4M_j^4\right. \\
&\left.+\tfrac{891}{1551}M_i^2M_j^6\right)\Omega_{ij}^{(3)}(3) - \tfrac{2222}{3}\left(\tfrac{20}{101}M_i^4M_j^4 + \tfrac{81}{101}M_i^2M_j^6\right)\Omega_{ij}^{(3)}(4) \\
&+ 124M_i^2M_j^6\Omega_{ij}^{(3)}(5) - 8M_i^2M_j^6\Omega_{ij}^{(3)}(6) + \tfrac{176}{3}M_i^3M_j^5\Omega_{ij}^{(4)}(4) \\
&- \tfrac{32}{3}M_i^3M_j^5\Omega_{ij}^{(4)}(5)\ ,
\end{aligned} \tag{6.40}
$$

$$
\begin{aligned}
&\left[S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij} \\
&= \tfrac{20165145}{512}\left(\tfrac{224}{1343}M_i^6M_j^3 + \tfrac{720}{1343}M_i^4M_j^5 + \tfrac{378}{1343}M_i^2M_j^7 + \tfrac{21}{1343}M_j^9\right)\Omega_{ij}^{(1)}(1) \\
&- \tfrac{4327323}{64}\left(\tfrac{112}{1441}M_i^6M_j^3 + \tfrac{720}{1441}M_i^4M_j^5 + \tfrac{567}{1441}M_i^2M_j^7 + \tfrac{42}{1441}M_j^9\right)\Omega_{ij}^{(1)}(2) \\
&+ \tfrac{329043}{8}\left(\tfrac{112}{6136}M_i^6M_j^3 + \tfrac{2400}{6136}M_i^4M_j^5 + \tfrac{3267}{6136}M_i^2M_j^7 + \tfrac{357}{6136}M_j^9\right)\Omega_{ij}^{(1)}(3) \\
&- \tfrac{200057}{16}\left(\tfrac{320}{1399}M_i^4M_j^5 + \tfrac{918}{1399}M_i^2M_j^7 + \tfrac{161}{1399}M_j^9\right)\Omega_{ij}^{(1)}(4) \\
&+ \tfrac{34333}{16}\left(\tfrac{200}{2641}M_i^4M_j^5 + \tfrac{1857}{2641}M_i^2M_j^7 + \tfrac{584}{2641}M_j^9\right)\Omega_{ij}^{(1)}(5) \\
&- \tfrac{881}{4}\left(\tfrac{525}{881}M_i^2M_j^7 + \tfrac{356}{881}M_j^9\right)\Omega_{ij}^{(1)}(6) + \tfrac{613}{45}\left(\tfrac{765}{2452}M_i^2M_j^7\right. \\
&\left.+\tfrac{1687}{2452}M_j^9\right)\Omega_{ij}^{(1)}(7) - \tfrac{23}{45}M_j^9\Omega_{ij}^{(1)}(8) + \tfrac{1}{90}M_j^9\Omega_{ij}^{(1)}(9) + \tfrac{1486485}{64}\left(\tfrac{16}{55}M_i^5M_j^4\right. \\
&\left.+\tfrac{32}{55}M_i^3M_j^6 + \tfrac{7}{55}M_iM_j^8\right)\Omega_{ij}^{(2)}(2) - \tfrac{389961}{16}\left(\tfrac{16}{101}M_i^5M_j^4 + \tfrac{64}{101}M_i^3M_j^6\right. \\
&\left.+\tfrac{21}{101}M_iM_j^8\right)\Omega_{ij}^{(2)}(3) + \tfrac{150293}{16}\left(\tfrac{48}{1051}M_i^5M_j^4 + \tfrac{640}{1051}M_i^3M_j^6\right. \\
&\left.+\tfrac{363}{1051}M_iM_j^8\right)\Omega_{ij}^{(2)}(4) - \tfrac{3653}{2}\left(\tfrac{128}{281}M_i^3M_j^6 + \tfrac{153}{281}M_iM_j^8\right)\Omega_{ij}^{(2)}(5) \\
&+ \tfrac{779}{4}\left(\tfrac{160}{779}M_i^3M_j^6 + \tfrac{619}{779}M_iM_j^8\right)\Omega_{ij}^{(2)}(6) - \tfrac{35}{3}M_iM_j^8\Omega_{ij}^{(2)}(7) \\
&+ \tfrac{1}{3}M_iM_j^8\Omega_{ij}^{(2)}(8) + \tfrac{8151}{2}\left(\tfrac{10}{19}M_i^4M_j^5 + \tfrac{9}{19}M_i^2M_j^7\right)\Omega_{ij}^{(3)}(3) \\
&- \tfrac{8008}{3}\left(\tfrac{5}{14}M_i^4M_j^5 + \tfrac{9}{14}M_i^2M_j^7\right)\Omega_{ij}^{(3)}(4) + \tfrac{1820}{3}\left(\tfrac{1}{7}M_i^4M_j^5\right. \\
&\left.+\tfrac{6}{7}M_i^2M_j^7\right)\Omega_{ij}^{(3)}(5) - 64M_i^2M_j^7\Omega_{ij}^{(3)}(6) + \tfrac{8}{3}M_i^2M_j^7\Omega_{ij}^{(3)}(7) \\
&+ \tfrac{572}{3}M_i^3M_j^6\Omega_{ij}^{(4)}(4) - \tfrac{208}{3}M_i^3M_j^6\Omega_{ij}^{(4)}(5) + \tfrac{16}{3}M_i^3M_j^6\Omega_{ij}^{(4)}(6)\ ,
\end{aligned} \tag{6.41}
$$

$$
\begin{aligned}
&\left[S^{(4)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S^{(4)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij} \\
&= \tfrac{105930825}{2048}\left(\tfrac{1152}{91715}M_i^8 M_j + \tfrac{19712}{91715}M_i^6 M_j^3 + \tfrac{47520}{91715}M_i^4 M_j^5 + \tfrac{22176}{91715}M_i^2 M_j^7\right. \\
&\left.+\tfrac{1155}{91715}M_j^9\right)\mathit{\Omega}^{(1)}_{ij}(1) - \tfrac{10735725}{128}\left(\tfrac{448}{4225}M_i^6 M_j^3 + \tfrac{2160}{4225}M_i^4 M_j^5 + \tfrac{1512}{4225}M_i^2 M_j^7\right. \\
&\left.+\tfrac{105}{4225}M_j^9\right)\mathit{\Omega}^{(1)}_{ij}(2) + \tfrac{6435429}{128}\left(\tfrac{5824}{195013}M_i^6 M_j^3 + \tfrac{81840}{195013}M_i^4 M_j^5\right. \\
&\left.+\tfrac{97416}{195013}M_i^2 M_j^7 + \tfrac{9933}{195013}M_j^9\right)\mathit{\Omega}^{(1)}_{ij}(3) - \tfrac{483637}{32}\left(\tfrac{1040}{3997}M_i^4 M_j^5 + \tfrac{2544}{3997}M_i^2 M_j^7\right. \\
&\left.+\tfrac{413}{3997}M_j^9\right)\mathit{\Omega}^{(1)}_{ij}(4) + \tfrac{165827}{64}\left(\tfrac{15600}{165827}M_i^4 M_j^5 + \tfrac{116688}{165827}M_i^2 M_j^7\right. \\
&\left.+\tfrac{33539}{165827}M_j^9\right)\mathit{\Omega}^{(1)}_{ij}(5) - \tfrac{2123}{8}\left(\tfrac{120}{193}M_i^2 M_j^7 + \tfrac{73}{193}M_j^9\right)\mathit{\Omega}^{(1)}_{ij}(6) \\
&+ \tfrac{1189}{72}\left(\tfrac{408}{1189}M_i^2 M_j^7 + \tfrac{781}{1189}M_j^9\right)\mathit{\Omega}^{(1)}_{ij}(7) - \tfrac{11}{18}M_j^9\mathit{\Omega}^{(1)}_{ij}(8) + \tfrac{1}{72}M_j^9\mathit{\Omega}^{(1)}_{ij}(9) \\
&+ \tfrac{525525}{16}\left(\tfrac{64}{2275}M_i^7 M_j^2 + \tfrac{792}{2275}M_i^5 M_j^4 + \tfrac{1188}{2275}M_i^3 M_j^6 + \tfrac{231}{2275}M_i M_j^8\right)\mathit{\Omega}^{(2)}_{ij}(2) \\
&- \tfrac{127413}{4}\left(\tfrac{8}{39}M_i^5 M_j^4 + \tfrac{24}{39}M_i^3 M_j^6 + \tfrac{7}{39}M_i M_j^8\right)\mathit{\Omega}^{(2)}_{ij}(3) \\
&+ \tfrac{48323}{4}\left(\tfrac{312}{4393}M_i^5 M_j^4 + \tfrac{2728}{4393}M_i^3 M_j^6 + \tfrac{1353}{4393}M_i M_j^8\right)\mathit{\Omega}^{(2)}_{ij}(4) \\
&- 2310\left(\tfrac{52}{105}M_i^3 M_j^6 + \tfrac{53}{105}M_i M_j^8\right)\mathit{\Omega}^{(2)}_{ij}(5) + 247\left(\tfrac{60}{247}M_i^3 M_j^6\right. \\
&\left.+\tfrac{187}{247}M_i M_j^8\right)\mathit{\Omega}^{(2)}_{ij}(6) - \tfrac{44}{3}M_i M_j^8\mathit{\Omega}^{(2)}_{ij}(7) + \tfrac{4}{9}M_i M_j^8\mathit{\Omega}^{(2)}_{ij}(8) \\
&+ \tfrac{26169}{4}\left(\tfrac{56}{793}M_i^6 M_j^3 + \tfrac{440}{793}M_i^4 M_j^5 + \tfrac{297}{793}M_i^2 M_j^7\right)\mathit{\Omega}^{(3)}_{ij}(3) \\
&- \tfrac{11374}{3}\left(\tfrac{20}{47}M_i^4 M_j^5 + \tfrac{27}{47}M_i^2 M_j^7\right)\mathit{\Omega}^{(3)}_{ij}(4) + \tfrac{2566}{3}\left(\tfrac{260}{1283}M_i^4 M_j^5\right. \\
&\left.+\tfrac{1023}{1283}M_i^2 M_j^7\right)\mathit{\Omega}^{(3)}_{ij}(5) - 88M_i^2 M_j^7\mathit{\Omega}^{(3)}_{ij}(6) + 4M_i^2 M_j^7\mathit{\Omega}^{(3)}_{ij}(7) \\
&+ \tfrac{1232}{3}\left(\tfrac{3}{14}M_i^5 M_j^4 + \tfrac{11}{14}M_i^3 M_j^6\right)\mathit{\Omega}^{(4)}_{ij}(4) - \tfrac{352}{3}M_i^3 M_j^6\mathit{\Omega}^{(4)}_{ij}(5) \\
&+ \tfrac{32}{3}M_i^3 M_j^6\mathit{\Omega}^{(4)}_{ij}(6) + \tfrac{16}{3}M_i^4 M_j^5\mathit{\Omega}^{(5)}_{ij}(5)\ ,
\end{aligned} \tag{6.42}
$$

$$
\begin{aligned}
&\left[S^{(4)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S^{(5)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij} \\
&= \tfrac{489834345}{4096}\left(\tfrac{1152}{32623}M_i^8 M_j^2 + \tfrac{9856}{32623}M_i^6 M_j^4 + \tfrac{15840}{32623}M_i^4 M_j^6 + \tfrac{5544}{32623}M_i^2 M_j^8\right. \\
&\left.+\tfrac{231}{32623}M_j^{10}\right)\mathit{\Omega}^{(1)}_{ij}(1) - \tfrac{452873421}{2048}\left(\tfrac{384}{50269}M_i^8 M_j^2 + \tfrac{9856}{50269}M_i^6 M_j^4\right. \\
&\left.+\tfrac{26400}{50269}M_i^4 M_j^6 + \tfrac{12936}{50269}M_i^2 M_j^8 + \tfrac{693}{50269}M_j^{10}\right)\mathit{\Omega}^{(1)}_{ij}(2) \\
&+ \tfrac{38424243}{256}\left(\tfrac{8288}{89567}M_i^6 M_j^4 + \tfrac{44880}{89567}M_i^4 M_j^6 + \tfrac{33858}{89567}M_i^2 M_j^8\right. \\
&\left.+\tfrac{2541}{89567}M_j^{10}\right)\mathit{\Omega}^{(1)}_{ij}(3) - \tfrac{6633055}{128}\left(\tfrac{224}{9277}M_i^6 M_j^4 + \tfrac{3696}{9277}M_i^4 M_j^6 + \tfrac{4818}{9277}M_i^2 M_j^8\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{539}{9277}M_j^{10}\big)\,\Omega_{ij}^{(1)}(4)+\tfrac{1330927}{128}\big(\tfrac{24400}{102379}M_i^4M_j^6+\tfrac{66132}{102379}M_i^2M_j^8\\
&+\tfrac{11847}{102379}M_j^{10}\big)\,\Omega_{ij}^{(1)}(5)-\tfrac{82491}{64}\big(\tfrac{6800}{82491}M_i^4M_j^6+\tfrac{57420}{82491}M_i^2M_j^8\\
&+\tfrac{18271}{82491}M_j^{10}\big)\,\Omega_{ij}^{(1)}(6)+\tfrac{72643}{720}\big(\tfrac{43350}{72643}M_i^2M_j^8+\tfrac{29293}{72643}M_j^{10}\big)\,\Omega_{ij}^{(1)}(7)\\
&-\tfrac{199}{40}\big(\tfrac{190}{597}M_i^2M_j^8+\tfrac{407}{597}M_j^{10}\big)\,\Omega_{ij}^{(1)}(8)+\tfrac{109}{720}M_j^{10}\Omega_{ij}^{(1)}(9)\\
&-\tfrac{1}{360}M_j^{10}\Omega_{ij}^{(1)}(10)+\tfrac{10975965}{128}\big(\tfrac{256}{3655}M_i^7M_j^3+\tfrac{1584}{3655}M_i^5M_j^5+\tfrac{1584}{3655}M_i^3M_j^7\\
&+\tfrac{231}{3655}M_iM_j^9\big)\,\Omega_{ij}^{(2)}(2)-\tfrac{6239805}{64}\big(\tfrac{256}{14545}M_i^7M_j^3+\tfrac{4752}{14545}M_i^5M_j^5\\
&+\tfrac{7920}{14545}M_i^3M_j^7+\tfrac{1617}{14545}M_iM_j^9\big)\,\Omega_{ij}^{(2)}(3)+\tfrac{1378663}{32}\big(\tfrac{1776}{9641}M_i^5M_j^5\\
&+\tfrac{5984}{9641}M_i^3M_j^7+\tfrac{1881}{9641}M_iM_j^9\big)\,\Omega_{ij}^{(2)}(4)-\tfrac{157651}{16}\big(\tfrac{720}{12127}M_i^5M_j^5\\
&+\tfrac{7392}{12127}M_i^3M_j^7+\tfrac{4015}{12127}M_iM_j^9\big)\,\Omega_{ij}^{(2)}(5)+\tfrac{10391}{8}\big(\tfrac{4880}{10391}M_i^3M_j^7\\
&+\tfrac{5511}{10391}M_iM_j^9\big)\,\Omega_{ij}^{(2)}(6)-\tfrac{1229}{12}\big(\tfrac{272}{1229}M_i^3M_j^7+\tfrac{957}{1229}M_iM_j^9\big)\,\Omega_{ij}^{(2)}(7)\\
&+\tfrac{85}{18}M_iM_j^9\Omega_{ij}^{(2)}(8)-\tfrac{1}{9}M_iM_j^9\Omega_{ij}^{(2)}(9)+\tfrac{160875}{8}\big(\tfrac{56}{375}M_i^6M_j^4+\tfrac{220}{375}M_i^4M_j^6\\
&+\tfrac{99}{375}M_i^2M_j^8\big)\,\Omega_{ij}^{(3)}(3)-\tfrac{173173}{12}\big(\tfrac{56}{1211}M_i^6M_j^4+\tfrac{660}{1211}M_i^4M_j^6\\
&+\tfrac{495}{1211}M_i^2M_j^8\big)\,\Omega_{ij}^{(3)}(4)+\tfrac{12103}{3}\big(\tfrac{370}{931}M_i^4M_j^6+\tfrac{561}{931}M_i^2M_j^8\big)\,\Omega_{ij}^{(3)}(5)\\
&-562\big(\tfrac{50}{281}M_i^4M_j^6+\tfrac{231}{281}M_i^2M_j^8\big)\,\Omega_{ij}^{(3)}(6)+\tfrac{122}{3}M_i^2M_j^8\Omega_{ij}^{(3)}(7)\\
&-\tfrac{4}{3}M_i^2M_j^8\Omega_{ij}^{(3)}(8)+\tfrac{4862}{3}\big(\tfrac{6}{17}M_i^5M_j^5+\tfrac{11}{17}M_i^3M_j^7\big)\,\Omega_{ij}^{(4)}(4)\\
&-676\big(\tfrac{2}{13}M_i^5M_j^5+\tfrac{11}{13}M_i^3M_j^7\big)\,\Omega_{ij}^{(4)}(5)+\tfrac{296}{3}M_i^3M_j^7\Omega_{ij}^{(4)}(6)\\
&-\tfrac{16}{3}M_i^3M_j^7\Omega_{ij}^{(4)}(7)+\tfrac{104}{3}M_i^4M_j^6\Omega_{ij}^{(5)}(5)-\tfrac{16}{3}M_i^4M_j^6\Omega_{ij}^{(5)}(6)\;, \qquad (6.43)
\end{aligned}
$$

and:

$$
\begin{aligned}
&\big[S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\big]_{ij}\\
&=\tfrac{2505429927}{8192}\big(\tfrac{2816}{834309}M_i^{10}M_j+\tfrac{74880}{834309}M_i^8M_j^3+\tfrac{320320}{834309}M_i^6M_j^5\\
&+\tfrac{343200}{834309}M_i^4M_j^7+\tfrac{90090}{834309}M_i^2M_j^9+\tfrac{3003}{834309}M_j^{11}\big)\,\Omega_{ij}^{(1)}(1)\\
&-\tfrac{1273569297}{2048}\big(\tfrac{1152}{32623}M_i^8M_j^3+\tfrac{9856}{32623}M_i^6M_j^5+\tfrac{15840}{32623}M_i^4M_j^7+\tfrac{5544}{32623}M_i^2M_j^9\\
&+\tfrac{231}{32623}M_j^{11}\big)\,\Omega_{ij}^{(1)}(2)+\tfrac{958832589}{2048}\big(\tfrac{17280}{2235041}M_i^8M_j^3+\tfrac{430976}{2235041}M_i^6M_j^5\\
&+\tfrac{1166880}{2235041}M_i^4M_j^7+\tfrac{586872}{2235041}M_i^2M_j^9+\tfrac{33033}{2235041}M_j^{11}\big)\,\Omega_{ij}^{(1)}(3)\\
&-\tfrac{23205897}{128}\big(\tfrac{1120}{12483}M_i^6M_j^5+\tfrac{6160}{12483}M_i^4M_j^7+\tfrac{4818}{12483}M_i^2M_j^9
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{385}{12483}M_j^{11}\big)\,\Omega_{ij}^{(1)}(4)+\tfrac{10626655}{256}\big(\tfrac{19040}{817435}M_i^6M_j^5+\tfrac{317200}{817435}M_i^4M_j^7\\
&+\tfrac{429858}{817435}M_i^2M_j^9+\tfrac{51337}{817435}M_j^{11}\big)\,\Omega_{ij}^{(1)}(5)-\tfrac{1923623}{320}\big(\tfrac{34000}{147971}M_i^4M_j^7\\
&+\tfrac{95700}{147971}M_i^2M_j^9+\tfrac{18271}{147971}M_j^{11}\big)\,\Omega_{ij}^{(1)}(6)+\tfrac{181901}{320}\big(\tfrac{129200}{1637109}M_i^4M_j^7\\
&+\tfrac{1127100}{1637109}M_i^2M_j^9+\tfrac{380809}{1637109}M_j^{11}\big)\,\Omega_{ij}^{(1)}(7)-\tfrac{12701}{360}\big(\tfrac{570}{977}M_i^2M_j^9\\
&+\tfrac{407}{977}M_j^{11}\big)\,\Omega_{ij}^{(1)}(8)+\tfrac{2047}{1440}\big(\tfrac{630}{2047}M_i^2M_j^9+\tfrac{1417}{2047}M_j^{11}\big)\,\Omega_{ij}^{(1)}(9)\\
&-\tfrac{13}{360}M_j^{11}\Omega_{ij}^{(1)}(10)+\tfrac{1}{1800}M_j^{11}\Omega_{ij}^{(1)}(11)+\tfrac{257041785}{1024}\big(\tfrac{640}{85595}M_i^9M_j^2\\
&+\tfrac{13312}{85595}M_i^7M_j^4+\tfrac{41184}{85595}M_i^5M_j^6+\tfrac{27456}{85595}M_i^3M_j^8+\tfrac{3003}{85595}M_iM_j^{10}\big)\,\Omega_{ij}^{(2)}(2)\\
&-\tfrac{20383935}{64}\big(\tfrac{256}{3655}M_i^7M_j^4+\tfrac{1584}{3655}M_i^5M_j^6+\tfrac{1584}{3655}M_i^3M_j^8\\
&+\tfrac{231}{3655}M_iM_j^{10}\big)\,\Omega_{ij}^{(2)}(3)+\tfrac{10212345}{64}\big(\tfrac{1280}{71415}M_i^7M_j^4+\tfrac{23088}{71415}M_i^5M_j^6\\
&+\tfrac{38896}{71415}M_i^3M_j^8+\tfrac{8151}{71415}M_iM_j^{10}\big)\,\Omega_{ij}^{(2)}(4)-\tfrac{673803}{16}\big(\tfrac{720}{3987}M_i^5M_j^6\\
&+\tfrac{2464}{3987}M_i^3M_j^8+\tfrac{803}{3987}M_iM_j^{10}\big)\,\Omega_{ij}^{(2)}(5)+\tfrac{210763}{32}\big(\tfrac{12240}{210763}M_i^5M_j^6\\
&+\tfrac{126880}{210763}M_i^3M_j^8+\tfrac{71643}{210763}M_iM_j^{10}\big)\,\Omega_{ij}^{(2)}(6)-\tfrac{2561}{4}\big(\tfrac{272}{591}M_i^3M_j^8\\
&+\tfrac{319}{591}M_iM_j^{10}\big)\,\Omega_{ij}^{(2)}(7)+\tfrac{1409}{36}\big(\tfrac{304}{1409}M_i^3M_j^8+\tfrac{1105}{1409}M_iM_j^{10}\big)\,\Omega_{ij}^{(2)}(8)\\
&-\tfrac{13}{9}M_iM_j^{10}\Omega_{ij}^{(2)}(9)+\tfrac{1}{36}M_iM_j^{10}\Omega_{ij}^{(2)}(10)+\tfrac{1130415}{16}\big(\tfrac{48}{2635}M_i^8M_j^3\\
&+\tfrac{728}{2635}M_i^6M_j^5+\tfrac{1430}{2635}M_i^4M_j^7+\tfrac{429}{2635}M_i^2M_j^9\big)\,\Omega_{ij}^{(3)}(3)-\tfrac{232375}{4}\big(\tfrac{56}{375}M_i^6M_j^5\\
&+\tfrac{220}{375}M_i^4M_j^7+\tfrac{99}{375}M_i^2M_j^9\big)\,\Omega_{ij}^{(3)}(4)+\tfrac{230789}{12}\big(\tfrac{840}{17753}M_i^6M_j^5+\tfrac{9620}{17753}M_i^4M_j^7\\
&+\tfrac{7293}{17753}M_i^2M_j^9\big)\,\Omega_{ij}^{(3)}(5)-3302\big(\tfrac{50}{127}M_i^4M_j^7+\tfrac{77}{127}M_i^2M_j^9\big)\,\Omega_{ij}^{(3)}(6)\\
&+321\big(\tfrac{170}{963}M_i^4M_j^7+\tfrac{793}{963}M_i^2M_j^9\big)\,\Omega_{ij}^{(3)}(7)-\tfrac{52}{3}M_i^2M_j^9\Omega_{ij}^{(3)}(8)\\
&+\tfrac{4}{9}M_i^2M_j^9\Omega_{ij}^{(3)}(9)+\tfrac{15015}{2}\big(\tfrac{16}{315}M_i^7M_j^4+\tfrac{156}{315}M_i^5M_j^6\\
&+\tfrac{143}{315}M_i^3M_j^8\big)\,\Omega_{ij}^{(4)}(4)-\tfrac{11492}{3}\big(\tfrac{6}{17}M_i^5M_j^6+\tfrac{11}{17}M_i^3M_j^8\big)\,\Omega_{ij}^{(4)}(5)\\
&+\tfrac{2284}{3}\big(\tfrac{90}{571}M_i^5M_j^6+\tfrac{481}{571}M_i^3M_j^8\big)\,\Omega_{ij}^{(4)}(6)-\tfrac{208}{3}M_i^3M_j^8\Omega_{ij}^{(4)}(7)\\
&+\tfrac{8}{3}M_i^3M_j^8\Omega_{ij}^{(4)}(8)+\tfrac{4108}{15}\big(\tfrac{14}{79}M_i^6M_j^5+\tfrac{65}{79}M_i^4M_j^7\big)\,\Omega_{ij}^{(5)}(5)\\
&-\tfrac{208}{3}M_i^4M_j^7\Omega_{ij}^{(5)}(6)+\tfrac{16}{3}M_i^4M_j^7\Omega_{ij}^{(5)}(7)+\tfrac{32}{15}M_i^5M_j^6\Omega_{ij}^{(6)}(6)\;. \qquad (6.44)
\end{aligned}
$$

Next, the bracket integrals of Eq. (6.17) that are related to the $H_{12}(\chi)$ functions in Chapman and Cowling [10] are considered. For each (p, q), these may be generated from the corresponding $H_1(\chi)$ expression in Eqs. (6.24)-(6.44). In general, Eqs. (6.24)-(6.44) consist of a series of terms each of which is associated with a specific omega integral. Each of these omega integral terms also contains a sign, a constant factor, and some function of M_i and M_j. The conversion process from the $H_1(\chi)$ expressions to the $H_{12}(\chi)$ expressions is quite straightforward as the magnitude of the numerical coefficients associated with each omega integral term are the same from the $H_1(\chi)$ expressions to the $H_{12}(\chi)$ expressions. The differences are in the distributions of the constituent masses and the signs of the terms. Since the same omega integrals must occur in both the $H_1(\chi)$ expressions and the $H_{12}(\chi)$ expressions, the total number of omega integral terms is the same in each and there is, in effect, a one-to-one correspondence between terms. The sign difference between corresponding terms is dependent only upon ℓ and the appropriate sign transformation is to multiply each term in the $H_1(\chi)$ expressions by a factor of $(-1)^\ell$ to yield the corresponding $H_{12}(\chi)$ signs. The distribution of constituent masses is much simpler in the $H_{12}(\chi)$ expressions than in the $H_1(\chi)$ expressions. In the $H_{12}(\chi)$ expressions given below the distribution of the constituent masses is exactly the same in every omega integral term for a given (p, q) and amounts to nothing more than a common factor of $\left(M_i^{q+1/2} M_j^{p+1/2}\right)$ in each expression. Thus, the appropriate overall transformation from the $H_1(\chi)$ expressions to the corresponding $H_{12}(\chi)$ expressions is to first set $M_i = M_j = 1$ in each term in the $H_1(\chi)$ expressions, second to multiply each $H_1(\chi)$ expression with the appropriate common factor of $(M_i^{q+1/2} M_j^{p+1/2})$, and third to adjust the sign of each term as needed for the odd values of ℓ. When the above prescription is followed, one obtains the following expressions for the $H_{12}(\chi)$ bracket integrals where the common mass factor in each expression has been explicitly factored out for clarity:

$$\left[S_{3/2}^{(0)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(0)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij} \left(M_i^{1/2} M_j^{1/2}\right)^{-1}$$
$$= -8\Omega_{ij}^{(1)}(1) \;, \tag{6.45}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(1)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{3/2}M_j^{1/2}\right)^{-1}$$
$$= -20\varOmega^{(1)}_{ij}(1) + 8\varOmega^{(1)}_{ij}(2)\ , \tag{6.46}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(2)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{5/2}M_j^{1/2}\right)^{-1}$$
$$= -35\varOmega^{(1)}_{ij}(1) + 28\varOmega^{(1)}_{ij}(2) - 4\varOmega^{(1)}_{ij}(3)\ , \tag{6.47}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(3)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{7/2}M_j^{1/2}\right)^{-1}$$
$$= -\tfrac{105}{2}\varOmega^{(1)}_{ij}(1) + 63\varOmega^{(1)}_{ij}(2) - 18\varOmega^{(1)}_{ij}(3) + \tfrac{4}{3}\varOmega^{(1)}_{ij}(4)\ , \tag{6.48}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(4)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{9/2}M_j^{1/2}\right)^{-1}$$
$$= -\tfrac{1155}{16}\varOmega^{(1)}_{ij}(1) + \tfrac{231}{2}\varOmega^{(1)}_{ij}(2) - \tfrac{99}{2}\varOmega^{(1)}_{ij}(3) + \tfrac{22}{3}\varOmega^{(1)}_{ij}(4)$$
$$- \tfrac{1}{3}\varOmega^{(1)}_{ij}(5)\ , \tag{6.49}$$

$$\left[S^{(0)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(5)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{11/2}M_j^{1/2}\right)^{-1}$$
$$= -\tfrac{3003}{32}\varOmega^{(1)}_{ij}(1) + \tfrac{3003}{16}\varOmega^{(1)}_{ij}(2) - \tfrac{429}{4}\varOmega^{(1)}_{ij}(3) + \tfrac{143}{6}\varOmega^{(1)}_{ij}(4)$$
$$- \tfrac{13}{6}\varOmega^{(1)}_{ij}(5) + \tfrac{1}{15}\varOmega^{(1)}_{ij}(6)\ , \tag{6.50}$$

$$\left[S^{(1)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(1)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{3/2}M_j^{3/2}\right)^{-1}$$
$$= -110\varOmega^{(1)}_{ij}(1) + 40\varOmega^{(1)}_{ij}(2) - 8\varOmega^{(1)}_{ij}(3) + 16\varOmega^{(2)}_{ij}(2)\ , \tag{6.51}$$

$$\left[S^{(1)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(2)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{5/2}M_j^{3/2}\right)^{-1}$$
$$= -\tfrac{595}{2}\varOmega^{(1)}_{ij}(1) + 189\varOmega^{(1)}_{ij}(2) - 38\varOmega^{(1)}_{ij}(3) + 4\varOmega^{(1)}_{ij}(4)$$
$$+ 56\varOmega^{(2)}_{ij}(2) - 16\varOmega^{(2)}_{ij}(3)\ , \tag{6.52}$$

$$\left[S^{(1)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_i, S^{(3)}_{3/2}\left(\mathscr{C}\right)\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{7/2}M_j^{3/2}\right)^{-1}$$
$$= -\tfrac{2415}{4}\varOmega^{(1)}_{ij}(1) + 588\varOmega^{(1)}_{ij}(2) - 162\varOmega^{(1)}_{ij}(3) + \tfrac{64}{3}\varOmega^{(1)}_{ij}(4)$$

$$- \tfrac{4}{3}\Omega_{ij}^{(1)}(5) + 126\Omega_{ij}^{(2)}(2) - 72\Omega_{ij}^{(2)}(3) + 8\Omega_{ij}^{(2)}(4)\ , \tag{6.53}$$

$$\begin{aligned}
&\left[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{9/2}M_j^{3/2}\right)^{-1}\\
&= -\tfrac{33495}{32}\Omega_{ij}^{(1)}(1) + \tfrac{22407}{16}\Omega_{ij}^{(1)}(2) - \tfrac{2145}{4}\Omega_{ij}^{(1)}(3) + \tfrac{539}{6}\Omega_{ij}^{(1)}(4)\\
&- \tfrac{49}{6}\Omega_{ij}^{(1)}(5) + \tfrac{1}{3}\Omega_{ij}^{(1)}(6) + 231\Omega_{ij}^{(2)}(2) - 198\Omega_{ij}^{(2)}(3) + 44\Omega_{ij}^{(2)}(4)\\
&- \tfrac{8}{3}\Omega_{ij}^{(2)}(5)\ ,
\end{aligned} \tag{6.54}$$

$$\begin{aligned}
&\left[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{11/2}M_j^{3/2}\right)^{-1}\\
&= -\tfrac{105105}{64}\Omega_{ij}^{(1)}(1) + \tfrac{45045}{16}\Omega_{ij}^{(1)}(2) - \tfrac{22737}{16}\Omega_{ij}^{(1)}(3) + \tfrac{1859}{6}\Omega_{ij}^{(1)}(4)\\
&- \tfrac{143}{4}\Omega_{ij}^{(1)}(5) + \tfrac{7}{3}\Omega_{ij}^{(1)}(6) - \tfrac{1}{15}\Omega_{ij}^{(1)}(7) + \tfrac{3003}{8}\Omega_{ij}^{(2)}(2) - 429\Omega_{ij}^{(2)}(3)\\
&+ 143\Omega_{ij}^{(2)}(4) - \tfrac{52}{3}\Omega_{ij}^{(2)}(5) + \tfrac{2}{3}\Omega_{ij}^{(2)}(6)\ ,
\end{aligned} \tag{6.55}$$

$$\begin{aligned}
&\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{5/2}M_j^{5/2}\right)^{-1}\\
&= -\tfrac{8505}{8}\Omega_{ij}^{(1)}(1) + 833\Omega_{ij}^{(1)}(2) - 241\Omega_{ij}^{(1)}(3) + 28\Omega_{ij}^{(1)}(4)\\
&- 2\Omega_{ij}^{(1)}(5) + 308\Omega_{ij}^{(2)}(2) - 112\Omega_{ij}^{(2)}(3) + 16\Omega_{ij}^{(2)}(4)\\
&- 16\Omega_{ij}^{(3)}(3)\ ,
\end{aligned} \tag{6.56}$$

$$\begin{aligned}
&\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{7/2}M_j^{5/2}\right)^{-1}\\
&= -\tfrac{42735}{16}\Omega_{ij}^{(1)}(1) + \tfrac{22071}{8}\Omega_{ij}^{(1)}(2) - \tfrac{2001}{2}\Omega_{ij}^{(1)}(3) + \tfrac{499}{3}\Omega_{ij}^{(1)}(4)\\
&- \tfrac{41}{3}\Omega_{ij}^{(1)}(5) + \tfrac{2}{3}\Omega_{ij}^{(1)}(6) + 945\Omega_{ij}^{(2)}(2) - 522\Omega_{ij}^{(2)}(3) + 100\Omega_{ij}^{(2)}(4)\\
&- 8\Omega_{ij}^{(2)}(5) - 72\Omega_{ij}^{(3)}(3) + 16\Omega_{ij}^{(3)}(4)\ ,
\end{aligned} \tag{6.57}$$

$$\begin{aligned}
&\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{9/2}M_j^{5/2}\right)^{-1}\\
&= -\tfrac{705705}{128}\Omega_{ij}^{(1)}(1) + \tfrac{234927}{32}\Omega_{ij}^{(1)}(2) - \tfrac{104973}{32}\Omega_{ij}^{(1)}(3) + \tfrac{8437}{12}\Omega_{ij}^{(1)}(4)\\
&- \tfrac{623}{8}\Omega_{ij}^{(1)}(5) + \tfrac{29}{6}\Omega_{ij}^{(1)}(6) - \tfrac{1}{6}\Omega_{ij}^{(1)}(7) + \tfrac{4389}{2}\Omega_{ij}^{(2)}(2) - 1716\Omega_{ij}^{(2)}(3)\\
&+ 440\Omega_{ij}^{(2)}(4) - \tfrac{160}{3}\Omega_{ij}^{(2)}(5) + \tfrac{8}{3}\Omega_{ij}^{(2)}(6) - 198\Omega_{ij}^{(3)}(3) + 88\Omega_{ij}^{(3)}(4)
\end{aligned}$$

$$- 8\mathit{\Omega}^{(3)}_{ij}(5) \ , \tag{6.58}$$

$$\left[S^{(2)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S^{(5)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{11/2}M_j^{5/2}\right)^{-1}$$
$$= -\tfrac{2567565}{256}\mathit{\Omega}^{(1)}_{ij}(1) + \tfrac{2129127}{128}\mathit{\Omega}^{(1)}_{ij}(2) - \tfrac{579579}{64}\mathit{\Omega}^{(1)}_{ij}(3) + \tfrac{75933}{32}\mathit{\Omega}^{(1)}_{ij}(4)$$
$$- \tfrac{16237}{48}\mathit{\Omega}^{(1)}_{ij}(5) + \tfrac{655}{24}\mathit{\Omega}^{(1)}_{ij}(6) - \tfrac{79}{60}\mathit{\Omega}^{(1)}_{ij}(7) + \tfrac{1}{30}\mathit{\Omega}^{(1)}_{ij}(8) + \tfrac{69069}{16}\mathit{\Omega}^{(2)}_{ij}(2)$$
$$- \tfrac{35607}{8}\mathit{\Omega}^{(2)}_{ij}(3) + \tfrac{3003}{2}\mathit{\Omega}^{(2)}_{ij}(4) - \tfrac{715}{3}\mathit{\Omega}^{(2)}_{ij}(5) + \tfrac{59}{3}\mathit{\Omega}^{(2)}_{ij}(6) - \tfrac{2}{3}\mathit{\Omega}^{(2)}_{ij}(7)$$
$$- 429\mathit{\Omega}^{(3)}_{ij}(3) + 286\mathit{\Omega}^{(3)}_{ij}(4) - 52\mathit{\Omega}^{(3)}_{ij}(5) + \tfrac{8}{3}\mathit{\Omega}^{(3)}_{ij}(6) \ , \tag{6.59}$$

$$\left[S^{(3)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S^{(3)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{7/2}M_j^{7/2}\right)^{-1}$$
$$= -\tfrac{255255}{32}\mathit{\Omega}^{(1)}_{ij}(1) + \tfrac{76923}{8}\mathit{\Omega}^{(1)}_{ij}(2) - \tfrac{34119}{8}\mathit{\Omega}^{(1)}_{ij}(3) + 895\mathit{\Omega}^{(1)}_{ij}(4)$$
$$- \tfrac{201}{2}\mathit{\Omega}^{(1)}_{ij}(5) + 6\mathit{\Omega}^{(1)}_{ij}(6) - \tfrac{2}{9}\mathit{\Omega}^{(1)}_{ij}(7) + \tfrac{14553}{4}\mathit{\Omega}^{(2)}_{ij}(2) - 2430\mathit{\Omega}^{(2)}_{ij}(3)$$
$$+ 626\mathit{\Omega}^{(2)}_{ij}(4) - 72\mathit{\Omega}^{(2)}_{ij}(5) + 4\mathit{\Omega}^{(2)}_{ij}(6) - 444\mathit{\Omega}^{(3)}_{ij}(3) + 144\mathit{\Omega}^{(3)}_{ij}(4)$$
$$- 16\mathit{\Omega}^{(3)}_{ij}(5) + \tfrac{32}{3}\mathit{\Omega}^{(4)}_{ij}(4) \ , \tag{6.60}$$

$$\left[S^{(3)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S^{(4)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{9/2}M_j^{7/2}\right)^{-1}$$
$$= -\tfrac{4879875}{256}\mathit{\Omega}^{(1)}_{ij}(1) + \tfrac{3516513}{128}\mathit{\Omega}^{(1)}_{ij}(2) - \tfrac{919413}{64}\mathit{\Omega}^{(1)}_{ij}(3) + \tfrac{118107}{32}\mathit{\Omega}^{(1)}_{ij}(4)$$
$$- \tfrac{8353}{16}\mathit{\Omega}^{(1)}_{ij}(5) + \tfrac{339}{8}\mathit{\Omega}^{(1)}_{ij}(6) - \tfrac{71}{36}\mathit{\Omega}^{(1)}_{ij}(7) + \tfrac{1}{18}\mathit{\Omega}^{(1)}_{ij}(8) + \tfrac{81081}{8}\mathit{\Omega}^{(2)}_{ij}(2)$$
$$- \tfrac{34155}{4}\mathit{\Omega}^{(2)}_{ij}(3) + 2717\mathit{\Omega}^{(2)}_{ij}(4) - 422\mathit{\Omega}^{(2)}_{ij}(5) + 34\mathit{\Omega}^{(2)}_{ij}(6) - \tfrac{4}{3}\mathit{\Omega}^{(2)}_{ij}(7)$$
$$- 1551\mathit{\Omega}^{(3)}_{ij}(3) + \tfrac{2222}{3}\mathit{\Omega}^{(3)}_{ij}(4) - 124\mathit{\Omega}^{(3)}_{ij}(5) + 8\mathit{\Omega}^{(3)}_{ij}(6) + \tfrac{176}{3}\mathit{\Omega}^{(4)}_{ij}(4)$$
$$- \tfrac{32}{3}\mathit{\Omega}^{(4)}_{ij}(5) \ , \tag{6.61}$$

$$\left[S^{(3)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S^{(5)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{11/2}M_j^{7/2}\right)^{-1}$$
$$= -\tfrac{20165145}{512}\mathit{\Omega}^{(1)}_{ij}(1) + \tfrac{4327323}{64}\mathit{\Omega}^{(1)}_{ij}(2) - \tfrac{329043}{8}\mathit{\Omega}^{(1)}_{ij}(3) + \tfrac{200057}{16}\mathit{\Omega}^{(1)}_{ij}(4)$$
$$- \tfrac{34333}{16}\mathit{\Omega}^{(1)}_{ij}(5) + \tfrac{881}{4}\mathit{\Omega}^{(1)}_{ij}(6) - \tfrac{613}{45}\mathit{\Omega}^{(1)}_{ij}(7) + \tfrac{23}{45}\mathit{\Omega}^{(1)}_{ij}(8) - \tfrac{1}{90}\mathit{\Omega}^{(1)}_{ij}(9)$$
$$+ \tfrac{1486485}{64}\mathit{\Omega}^{(2)}_{ij}(2) - \tfrac{389961}{16}\mathit{\Omega}^{(2)}_{ij}(3) + \tfrac{150293}{16}\mathit{\Omega}^{(2)}_{ij}(4) - \tfrac{3653}{2}\mathit{\Omega}^{(2)}_{ij}(5)$$
$$+ \tfrac{779}{4}\mathit{\Omega}^{(2)}_{ij}(6) - \tfrac{35}{3}\mathit{\Omega}^{(2)}_{ij}(7) + \tfrac{1}{3}\mathit{\Omega}^{(2)}_{ij}(8) - \tfrac{8151}{2}\mathit{\Omega}^{(3)}_{ij}(3) + \tfrac{8008}{3}\mathit{\Omega}^{(3)}_{ij}(4)$$

$$
\begin{aligned}
&- \tfrac{1820}{3}\mathit{\Omega}^{(3)}_{ij}(5) + 64\mathit{\Omega}^{(3)}_{ij}(6) - \tfrac{8}{3}\mathit{\Omega}^{(3)}_{ij}(7) + \tfrac{572}{3}\mathit{\Omega}^{(4)}_{ij}(4) - \tfrac{208}{3}\mathit{\Omega}^{(4)}_{ij}(5) \\
&+ \tfrac{16}{3}\mathit{\Omega}^{(4)}_{ij}(6)\ , \qquad (6.62)
\end{aligned}
$$

$$
\begin{aligned}
&\left[S^{(4)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S^{(4)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{9/2}M_j^{9/2}\right)^{-1} \\
&= -\tfrac{105930825}{2048}\mathit{\Omega}^{(1)}_{ij}(1) + \tfrac{10735725}{128}\mathit{\Omega}^{(1)}_{ij}(2) - \tfrac{6435429}{128}\mathit{\Omega}^{(1)}_{ij}(3) + \tfrac{483637}{32}\mathit{\Omega}^{(1)}_{ij}(4) \\
&- \tfrac{165827}{64}\mathit{\Omega}^{(1)}_{ij}(5) + \tfrac{2123}{8}\mathit{\Omega}^{(1)}_{ij}(6) - \tfrac{1189}{72}\mathit{\Omega}^{(1)}_{ij}(7) + \tfrac{11}{18}\mathit{\Omega}^{(1)}_{ij}(8) - \tfrac{1}{72}\mathit{\Omega}^{(1)}_{ij}(9) \\
&+ \tfrac{525525}{16}\mathit{\Omega}^{(2)}_{ij}(2) - \tfrac{127413}{4}\mathit{\Omega}^{(2)}_{ij}(3) + \tfrac{48323}{4}\mathit{\Omega}^{(2)}_{ij}(4) - 2310\mathit{\Omega}^{(2)}_{ij}(5) \\
&+ 247\mathit{\Omega}^{(2)}_{ij}(6) - \tfrac{44}{3}\mathit{\Omega}^{(2)}_{ij}(7) + \tfrac{4}{9}\mathit{\Omega}^{(2)}_{ij}(8) - \tfrac{26169}{4}\mathit{\Omega}^{(3)}_{ij}(3) + \tfrac{11374}{3}\mathit{\Omega}^{(3)}_{ij}(4) \\
&- \tfrac{2566}{3}\mathit{\Omega}^{(3)}_{ij}(5) + 88\mathit{\Omega}^{(3)}_{ij}(6) - 4\mathit{\Omega}^{(3)}_{ij}(7) + \tfrac{1232}{3}\mathit{\Omega}^{(4)}_{ij}(4) - \tfrac{352}{3}\mathit{\Omega}^{(4)}_{ij}(5) \\
&+ \tfrac{32}{3}\mathit{\Omega}^{(4)}_{ij}(6) - \tfrac{16}{3}\mathit{\Omega}^{(5)}_{ij}(5)\ , \qquad (6.63)
\end{aligned}
$$

$$
\begin{aligned}
&\left[S^{(4)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S^{(5)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{11/2}M_j^{9/2}\right)^{-1} \\
&= -\tfrac{489834345}{4096}\mathit{\Omega}^{(1)}_{ij}(1) + \tfrac{452873421}{2048}\mathit{\Omega}^{(1)}_{ij}(2) - \tfrac{38424243}{256}\mathit{\Omega}^{(1)}_{ij}(3) \\
&+ \tfrac{6633055}{128}\mathit{\Omega}^{(1)}_{ij}(4) - \tfrac{1330927}{128}\mathit{\Omega}^{(1)}_{ij}(5) + \tfrac{82491}{64}\mathit{\Omega}^{(1)}_{ij}(6) - \tfrac{72643}{720}\mathit{\Omega}^{(1)}_{ij}(7) \\
&+ \tfrac{199}{40}\mathit{\Omega}^{(1)}_{ij}(8) - \tfrac{109}{720}\mathit{\Omega}^{(1)}_{ij}(9) + \tfrac{1}{360}\mathit{\Omega}^{(1)}_{ij}(10) + \tfrac{10975965}{128}\mathit{\Omega}^{(2)}_{ij}(2) \\
&- \tfrac{6239805}{64}\mathit{\Omega}^{(2)}_{ij}(3) + \tfrac{1378663}{32}\mathit{\Omega}^{(2)}_{ij}(4) - \tfrac{157651}{16}\mathit{\Omega}^{(2)}_{ij}(5) + \tfrac{10391}{8}\mathit{\Omega}^{(2)}_{ij}(6) \\
&- \tfrac{1229}{12}\mathit{\Omega}^{(2)}_{ij}(7) + \tfrac{85}{18}\mathit{\Omega}^{(2)}_{ij}(8) - \tfrac{1}{9}\mathit{\Omega}^{(2)}_{ij}(9) - \tfrac{160875}{8}\mathit{\Omega}^{(3)}_{ij}(3) + \tfrac{173173}{12}\mathit{\Omega}^{(3)}_{ij}(4) \\
&- \tfrac{12103}{3}\mathit{\Omega}^{(3)}_{ij}(5) + 562\mathit{\Omega}^{(3)}_{ij}(6) - \tfrac{122}{3}\mathit{\Omega}^{(3)}_{ij}(7) + \tfrac{4}{3}\mathit{\Omega}^{(3)}_{ij}(8) + \tfrac{4862}{3}\mathit{\Omega}^{(4)}_{ij}(4) \\
&- 676\mathit{\Omega}^{(4)}_{ij}(5) + \tfrac{296}{3}\mathit{\Omega}^{(4)}_{ij}(6) - \tfrac{16}{3}\mathit{\Omega}^{(4)}_{ij}(7) - \tfrac{104}{3}\mathit{\Omega}^{(5)}_{ij}(5) + \tfrac{16}{3}\mathit{\Omega}^{(5)}_{ij}(6)\ , \qquad (6.64)
\end{aligned}
$$

and:

$$
\begin{aligned}
&\left[S^{(5)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S^{(5)}_{3/2}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(M_i^{11/2}M_j^{11/2}\right)^{-1} \\
&= -\tfrac{2505429927}{8192}\mathit{\Omega}^{(1)}_{ij}(1) + \tfrac{1273569297}{2048}\mathit{\Omega}^{(1)}_{ij}(2) - \tfrac{958832589}{2048}\mathit{\Omega}^{(1)}_{ij}(3) \\
&+ \tfrac{23205897}{128}\mathit{\Omega}^{(1)}_{ij}(4) - \tfrac{10626655}{256}\mathit{\Omega}^{(1)}_{ij}(5) + \tfrac{1923623}{320}\mathit{\Omega}^{(1)}_{ij}(6) - \tfrac{181901}{320}\mathit{\Omega}^{(1)}_{ij}(7) \\
&+ \tfrac{12701}{360}\mathit{\Omega}^{(1)}_{ij}(8) - \tfrac{2047}{1440}\mathit{\Omega}^{(1)}_{ij}(9) + \tfrac{13}{360}\mathit{\Omega}^{(1)}_{ij}(10) - \tfrac{1}{1800}\mathit{\Omega}^{(1)}_{ij}(11) \\
&+ \tfrac{257041785}{1024}\mathit{\Omega}^{(2)}_{ij}(2) - \tfrac{20383935}{64}\mathit{\Omega}^{(2)}_{ij}(3) + \tfrac{10212345}{64}\mathit{\Omega}^{(2)}_{ij}(4) - \tfrac{673803}{16}\mathit{\Omega}^{(2)}_{ij}(5)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{210763}{32}\Omega_{ij}^{(2)}(6) - \tfrac{2561}{4}\Omega_{ij}^{(2)}(7) + \tfrac{1409}{36}\Omega_{ij}^{(2)}(8) - \tfrac{13}{9}\Omega_{ij}^{(2)}(9) \\
&+ \tfrac{1}{36}\Omega_{ij}^{(2)}(10) - \tfrac{1130415}{16}\Omega_{ij}^{(3)}(3) + \tfrac{232375}{4}\Omega_{ij}^{(3)}(4) - \tfrac{230789}{12}\Omega_{ij}^{(3)}(5) \\
&+ 3302\Omega_{ij}^{(3)}(6) - 321\Omega_{ij}^{(3)}(7) + \tfrac{52}{3}\Omega_{ij}^{(3)}(8) - \tfrac{4}{9}\Omega_{ij}^{(3)}(9) + \tfrac{15015}{2}\Omega_{ij}^{(4)}(4) \\
&- \tfrac{11492}{3}\Omega_{ij}^{(4)}(5) + \tfrac{2284}{3}\Omega_{ij}^{(4)}(6) - \tfrac{208}{3}\Omega_{ij}^{(4)}(7) + \tfrac{8}{3}\Omega_{ij}^{(4)}(8) - \tfrac{4108}{15}\Omega_{ij}^{(5)}(5) \\
&+ \tfrac{208}{3}\Omega_{ij}^{(5)}(6) - \tfrac{16}{3}\Omega_{ij}^{(5)}(7) + \tfrac{32}{15}\Omega_{ij}^{(6)}(6) \ . \qquad (6.65)
\end{aligned}
$$

Now that the $H_1(\chi)$ and $H_{12}(\chi)$ bracket integrals of Eqs. (6.16) and (6.17) have been obtained, they can be used to generate the simple gas bracket integrals of Eq. (6.15). As noted in Chapman and Cowling [10], in the limit of a simple (single) gas where $m_i = m_j$, and $n_i = n_j$, one can write:

$$
\begin{aligned}
&\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i \\
&= \Bigg(\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij} \\
&\qquad\qquad + \left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij} \Bigg)\Bigg|_{\substack{m_i=m_j \\ n_i=n_j}} \ . \qquad (6.66)
\end{aligned}
$$

Under these conditions, one has that $M_i = M_j = \frac{1}{2}$ and the simple gas bracket integrals are then mass independent except for the presence of a single m_i in the simple gas omega integrals. Since, for any given (p, q) all of the terms in the pairs of corresponding $H_1(\chi)$ and $H_{12}(\chi)$ bracket integral expressions have the same total power of the constituent masses, substitution of $M_i = M_j = \frac{1}{2}$ simply yields an additional constant factor for the corresponding pairs of expressions of $(\frac{1}{2})^{p+q+1}$. The difference in the signs of the terms in the corresponding expressions due to the factor of $(-1)^\ell$ has the effect that all terms involving omega integrals with odd values of ℓ cancel exactly and all terms involving omega integrals with even values of ℓ add identically to produce an additional factor of 2 in each surviving term. Following this prescription, one obtains for the simple gas bracket integrals of Eq. (6.15):

$$
\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i = 0 \quad (p, q = 0) \ , \qquad (6.67)
$$

$$\left[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i$$
$$= 4\Omega_i^{(2)}(2)\,, \tag{6.68}$$

$$\left[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i$$
$$= 7\Omega_i^{(2)}(2) - 2\Omega_i^{(2)}(3)\,, \tag{6.69}$$

$$\left[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i$$
$$= \tfrac{63}{8}\Omega_i^{(2)}(2) - \tfrac{9}{2}\Omega_i^{(2)}(3) + \tfrac{1}{2}\Omega_i^{(2)}(4)\,, \tag{6.70}$$

$$\left[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i$$
$$= \tfrac{231}{32}\Omega_i^{(2)}(2) - \tfrac{99}{16}\Omega_i^{(2)}(3) + \tfrac{11}{8}\Omega_i^{(2)}(4) - \tfrac{1}{12}\Omega_i^{(2)}(5)\,, \tag{6.71}$$

$$\left[S_{3/2}^{(1)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i$$
$$= \tfrac{3003}{512}\Omega_i^{(2)}(2) - \tfrac{429}{64}\Omega_i^{(2)}(3) + \tfrac{143}{64}\Omega_i^{(2)}(4) - \tfrac{13}{48}\Omega_i^{(2)}(5) + \tfrac{1}{96}\Omega_i^{(2)}(6)\,, \tag{6.72}$$

$$\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i$$
$$= \tfrac{77}{4}\Omega_i^{(2)}(2) - 7\Omega_i^{(2)}(3) + \Omega_i^{(2)}(4)\,, \tag{6.73}$$

$$\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i$$
$$= \tfrac{945}{32}\Omega_i^{(2)}(2) - \tfrac{261}{16}\Omega_i^{(2)}(3) + \tfrac{25}{8}\Omega_i^{(2)}(4) - \tfrac{1}{4}\Omega_i^{(2)}(5)\,, \tag{6.74}$$

$$\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i$$
$$= \tfrac{4389}{128}\Omega_i^{(2)}(2) - \tfrac{429}{16}\Omega_i^{(2)}(3) + \tfrac{55}{8}\Omega_i^{(2)}(4) - \tfrac{5}{6}\Omega_i^{(2)}(5) + \tfrac{1}{24}\Omega_i^{(2)}(6)\,, \tag{6.75}$$

$$\left[S_{3/2}^{(2)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i$$
$$= \tfrac{69069}{2048}\Omega_i^{(2)}(2) - \tfrac{35607}{1024}\Omega_i^{(2)}(3) + \tfrac{3003}{256}\Omega_i^{(2)}(4) - \tfrac{715}{384}\Omega_i^{(2)}(5)$$

$$+ \tfrac{59}{384} \Omega_i^{(2)}(6) - \tfrac{1}{192} \Omega_i^{(2)}(7) \ , \tag{6.76}$$

$$\begin{aligned}
&\left[S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i \\
&= \tfrac{14553}{256} \Omega_i^{(2)}(2) - \tfrac{1215}{32} \Omega_i^{(2)}(3) + \tfrac{313}{32} \Omega_i^{(2)}(4) - \tfrac{9}{8} \Omega_i^{(2)}(5) \\
&+ \tfrac{1}{16} \Omega_i^{(2)}(6) + \tfrac{1}{6} \Omega_i^{(4)}(4) \ ,
\end{aligned} \tag{6.77}$$

$$\begin{aligned}
&\left[S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i \\
&= \tfrac{81081}{1024} \Omega_i^{(2)}(2) - \tfrac{34155}{512} \Omega_i^{(2)}(3) + \tfrac{2717}{128} \Omega_i^{(2)}(4) - \tfrac{211}{64} \Omega_i^{(2)}(5) \\
&+ \tfrac{17}{64} \Omega_i^{(2)}(6) - \tfrac{1}{96} \Omega_i^{(2)}(7) + \tfrac{11}{24} \Omega_i^{(4)}(4) - \tfrac{1}{12} \Omega_i^{(4)}(5) \ ,
\end{aligned} \tag{6.78}$$

$$\begin{aligned}
&\left[S_{3/2}^{(3)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i \\
&= \tfrac{1486485}{16384} \Omega_i^{(2)}(2) - \tfrac{389961}{4096} \Omega_i^{(2)}(3) + \tfrac{150293}{4096} \Omega_i^{(2)}(4) - \tfrac{3653}{512} \Omega_i^{(2)}(5) \\
&+ \tfrac{779}{1024} \Omega_i^{(2)}(6) - \tfrac{35}{768} \Omega_i^{(2)}(7) + \tfrac{1}{768} \Omega_i^{(2)}(8) + \tfrac{143}{192} \Omega_i^{(4)}(4) \\
&- \tfrac{13}{48} \Omega_i^{(4)}(5) + \tfrac{1}{48} \Omega_i^{(4)}(6) \ ,
\end{aligned} \tag{6.79}$$

$$\begin{aligned}
&\left[S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i \\
&= \tfrac{525525}{4096} \Omega_i^{(2)}(2) - \tfrac{127413}{1024} \Omega_i^{(2)}(3) + \tfrac{48323}{1024} \Omega_i^{(2)}(4) - \tfrac{1155}{128} \Omega_i^{(2)}(5) \\
&+ \tfrac{247}{256} \Omega_i^{(2)}(6) - \tfrac{11}{192} \Omega_i^{(2)}(7) + \tfrac{1}{576} \Omega_i^{(2)}(8) + \tfrac{77}{48} \Omega_i^{(4)}(4) \\
&- \tfrac{11}{24} \Omega_i^{(4)}(5) + \tfrac{1}{24} \Omega_i^{(4)}(6) \ ,
\end{aligned} \tag{6.80}$$

$$\begin{aligned}
&\left[S_{3/2}^{(4)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i \\
&= \tfrac{10975965}{65536} \Omega_i^{(2)}(2) - \tfrac{6239805}{32768} \Omega_i^{(2)}(3) + \tfrac{1378663}{16384} \Omega_i^{(2)}(4) - \tfrac{157651}{8192} \Omega_i^{(2)}(5) \\
&+ \tfrac{10391}{4096} \Omega_i^{(2)}(6) - \tfrac{1229}{6144} \Omega_i^{(2)}(7) + \tfrac{85}{9216} \Omega_i^{(2)}(8) - \tfrac{1}{4608} \Omega_i^{(2)}(9) \\
&+ \tfrac{2431}{768} \Omega_i^{(4)}(4) - \tfrac{169}{128} \Omega_i^{(4)}(5) + \tfrac{37}{192} \Omega_i^{(4)}(6) - \tfrac{1}{96} \Omega_i^{(4)}(7) \ ,
\end{aligned} \tag{6.81}$$

and:

$$
\begin{aligned}
&\left[S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(5)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i \\
&= \tfrac{257041785}{1048576}\Omega_i^{(2)}(2) - \tfrac{20383935}{65536}\Omega_i^{(2)}(3) + \tfrac{10212345}{65536}\Omega_i^{(2)}(4) - \tfrac{673803}{16384}\Omega_i^{(2)}(5) \\
&+ \tfrac{210763}{32768}\Omega_i^{(2)}(6) - \tfrac{2561}{4096}\Omega_i^{(2)}(7) + \tfrac{1409}{36864}\Omega_i^{(2)}(8) - \tfrac{13}{9216}\Omega_i^{(2)}(9) \\
&+ \tfrac{1}{36864}\Omega_i^{(2)}(10) + \tfrac{15015}{2048}\Omega_i^{(4)}(4) - \tfrac{2873}{768}\Omega_i^{(4)}(5) + \tfrac{571}{768}\Omega_i^{(4)}(6) \\
&- \tfrac{13}{192}\Omega_i^{(4)}(7) + \tfrac{1}{384}\Omega_i^{(4)}(8) + \tfrac{1}{480}\Omega_i^{(6)}(6)\ . \qquad (6.82)
\end{aligned}
$$

Equations (6.24)-(6.82) constitute the complete set of all bracket integrals necessary to evaluate the Chapman-Enskog diffusion, thermal conductivity, and thermal diffusion coefficients using order 5 Sonine polynomial expansions.

6.4 Symmetry and matrix element generation

There are several important symmetries to point out that are present among the bracket integral expressions and, hence, the a_{pq} matrix elements. Proper application of these symmetries aids in most efficiently constructing the appropriate expressions for the matrix elements, a_{pq}, from the bracket integral expressions presented in this work and those required for higher orders of approximation. First, it is clear from Eqs. (6.10)-(6.13) that the bracket integral expressions needed for arbitrary a_{pq} matrix elements are:

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i\ , \qquad (6.83)$$

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_j\ , \qquad (6.84)$$

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij}\ , \qquad (6.85)$$

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ji}\ , \qquad (6.86)$$

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\ , \qquad (6.87)$$

and:

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ji}\ , \qquad (6.88)$$

where the symmetries for the a_{pq} matrix elements: $a_{pq} = a_{qp}$, $a_{p-q} = a_{-qp}$, $a_{-pq} = a_{q-p}$, and $a_{-p-q} = a_{-q-p}$ have already been noted. For the bracket integral expressions, note that the integrals of Eqs. (6.83) and (6.84) describe simple gas collision interactions. As indicated in Eq. (6.67), the bracket integrals of this type are equal to zero for values of $p, q = 0$. For values of $p, q > 0$, note that the form of the general expressions are identical, excepting only where the subscripts for the component gases differ. It follows from the definitions then that one has $n_i^{-2}\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_i = n_j^{-2}\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_j$. The expressions for these simple gas bracket integrals can be constructed from appropriate combinations of the $H_1(\chi)$ and $H_{12}(\chi)$ bracket integral expressions as has been indicated in Eq. (6.66). However, in the practical construction of these expressions to orders much higher than 5, the direct generation of the expressions has proven to be computationally more economical than algebraic combination of the $H_1(\chi)$ and $H_{12}(\chi)$ bracket integral expressions.

The bracket integral expressions given in Eqs. (6.85) and (6.86) and Eqs. (6.87) and (6.88) are related to the $H_1(\chi)$ and $H_{12}(\chi)$ expressions, respectively, and exhibit rather useful symmetries. First, as previously indicated, the structures of the bracket integrals labeled with the indices, ji, Eqs. (6.86) and (6.88), are identical to the structures of those labeled with the indices, ij, Eqs. (6.85) and (6.87), respectively, and only a complete interchange of the subscript indices is needed. Lastly, of particular use are the symmetries:

$$\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij} = \left[S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i\right]_{ij}, \tag{6.89}$$

for the $H_1(\chi)$ related bracket integral expressions and:

$$\begin{aligned}\left[S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij} \\ = \left[S_{3/2}^{(q)}(\mathscr{C})\boldsymbol{\mathscr{C}}_i, S_{3/2}^{(p)}(\mathscr{C})\boldsymbol{\mathscr{C}}_j\right]_{ij}\left(\frac{M_i}{M_j}\right)^{q-p},\end{aligned} \tag{6.90}$$

for the $H_{12}(\chi)$ related bracket integral expressions.

6.5 Discussion and conclusions

The purpose in this work has been to explore the use of Sonine polynomial expansions to obtain error free results for the transport coefficients and the related Chapman-Enskog functions for simple gases and gas mixtures. In the previous chapter [18], the relevant, explicit bracket integral expressions needed to compute the viscosity-related Chapman-Enskog solutions up to order 5 based upon expansions in Sonine polynomials were presented. Here, in the current work, the bracket integral expressions needed to compute the diffusion- and thermal conductivity-related Chapman-Enskog solutions up to order 5 based upon expansions in Sonine polynomials have been similarly reported. Further, as was done for the viscosity-related bracket integral expressions in Chapter 5 [18], a method for how one can readily combine the reported bracket integral expressions to generate the requisite matrix elements used to solve the systems of equations in the Chapman-Enskog method, Eqs. (6.4) and (6.5), for the diffusion and thermal conductivity solutions has been described. Additionally, several of the associated symmetries between the matrix elements, and also the bracket integral expressions have been clearly detailed which, in turn, one may use to improve and optimize the construction of the relevant expressions and computational programs based upon them.

Chapter 7

Discussion and conclusions

The purpose of this dissertation has been to explore the use of high-order Sonine polynomial expansions to obtain high-precision results for the transport coefficients and the related Chapman-Enskog functions for simple gases and gas mixtures composed of rigid-sphere molecules. In this process, direct, fully-general expressions for the bracket integrals have been generated and archived up to order 150 for simple gases, up to order 60 for the viscosity-related bracket integrals for binary gas mixtures, and up to order 70 for the diffusion- and thermal conductivity-related bracket integrals for binary gas mixtures. Also, full, explicit bracket integral expressions needed to compute the viscosity-, diffusion-, and thermal conductivity-related Chapman-Enskog solutions up to order 5 based upon expansions in Sonine polynomials both simple gases and binary gas mixtures have been presented. It is important to note that in all of the expressions generated for this work, one could use any potential model expressible in the form of the omega-integrals; however, in this work, the focus has been on results for the rigid-sphere potential model as direct, analytical expressions are avaible for the omega-integrals such that no numerical errors are introduced during the integration process. The availability of the high-precision values tabulated in this work support the validity of previous works and provide a comprehensive set of benchmark values for future work in the area. Further, the availability of the archived general expressions for the bracket integrals and the methods developed during this work provide a basis for exploring a wide range of problems in future work.

7.1 Suggestions for future work

In the near future, it is apparent that the current work can be applied to a variety of problems of interest. First, the current work can be readily extended to model multi-component gas mixtures by an appropriate combination of the archived bracket integral expressions. Second, as the archived bracket integral expressions allow for the application of arbitrary potential models expressible in terms of omega integrals, one would be able to explore the use of more realistic potential models pending the difficult numerical computations required can be addressed to a sufficiently high precision. Other important areas of interest which merit exploration include the computation of slip and jump coefficients for gas and gas mixture flows and heat transfer to a greater degree of precision than has been obtained with the use of lower-order Chapman-Enskog expansions. Such computations would shed further light on the role of various models of gas-surface interactions and intermolecular potentials on the rarefied gas flows and would find applications in a host of areas where such flows are of interest and consequence.

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Appendix A

Additional higher-order bracket integrals for a simple, rigid-sphere gas

Reported below are additional, higher-order bracket integrals needed for the order $N = 5$ Sonine polynomial expansion which have not been previously reported in the literature. For rigid-sphere molecules, the corresponding $\hat{a}_{pq}$ and $\hat{b}_{pq}$ reduce to those values given in Tables 2.1 and 2.2, respectively.

$$a_{14} = \tfrac{1}{96}\Big(693\Omega_1^{(2)}(2) - 594\Omega_1^{(2)}(3) + 132\Omega_1^{(2)}(4) - 8\Omega_1^{(2)}(5)\Big)\ , \tag{A.1}$$

$$\begin{aligned} a_{24} = \tfrac{1}{384}\Big(&13167\Omega_1^{(2)}(2) - 8\Big(1287\Omega_1^{(2)}(3) - 330\Omega_1^{(2)}(4) \\ &+ 40\Omega_1^{(2)}(5) - 2\Omega_1^{(2)}(6)\Big)\Big)\ , \end{aligned} \tag{A.2}$$

$$\begin{aligned} a_{34} = \tfrac{1}{3072}\Big(&243243\Omega_1^{(2)}(2) - 204930\Omega_1^{(2)}(3) + 65208\Omega_1^{(2)}(4) \\ &- 10128\Omega_1^{(2)}(5) + 816\Omega_1^{(2)}(6) - 32\Omega_1^{(2)}(7) \\ &+ 1408\Omega_1^{(4)}(4) - 256\Omega_1^{(4)}(5)\Big)\ , \end{aligned} \tag{A.3}$$

$$\begin{aligned} a_{44} = \tfrac{1}{36864}\Big(&4729725\Omega_1^{(2)}(2) - 4\Big(1146717\Omega_1^{(2)}(3) \\ &- 434907\Omega_1^{(2)}(4) + 83160\Omega_1^{(2)}(5) - 8892\Omega_1^{(2)}(6) \\ &+ 528\Omega_1^{(2)}(7) - 16\Omega_1^{(2)}(8) - 14784\Omega_1^{(4)}(4) \\ &+ 4224\Omega_1^{(4)}(5) - 384\Omega_1^{(4)}(6)\Big)\Big)\ , \end{aligned} \tag{A.4}$$

$$\begin{aligned} a_{15} = \tfrac{1}{1536}\Big(&9009\Omega_1^{(2)}(2) - 8\Big(1287\Omega_1^{(2)}(3) - 429\Omega_1^{(2)}(4) \\ &+ 52\Omega_1^{(2)}(5) - 2\Omega_1^{(2)}(6)\Big)\Big)\ , \end{aligned} \tag{A.5}$$

$$
\begin{aligned}
a_{25} = \tfrac{1}{6144}\Big(& 207207\Omega_{1}^{(2)}(2) - 213642\Omega_{1}^{(2)}(3) + 8\Big(9009\Omega_{1}^{(2)}(4) \\
& - 1430\Omega_{1}^{(2)}(5) + 118\Omega_{1}^{(2)}(6) - 4\Omega_{1}^{(2)}(7)\Big)\Big) \ ,
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
a_{35} = \tfrac{1}{49152}\Big(& 4459455\Omega_{1}^{(2)}(2) - 4\Big(1169883\Omega_{1}^{(2)}(3) \\
& - 450879\Omega_{1}^{(2)}(4) + 87672\Omega_{1}^{(2)}(5) - 9348\Omega_{1}^{(2)}(6) \\
& + 560\Omega_{1}^{(2)}(7) - 16\Omega_{1}^{(2)}(8) - 9152\Omega_{1}^{(4)}(4) \\
& + 3328\Omega_{1}^{(4)}(5) - 256\Omega_{1}^{(4)}(6)\Big)\Big) \ ,
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
a_{45} = \tfrac{1}{589824}\Big(& 98783685\Omega_{1}^{(2)}(2) - 2\Big(56158245\Omega_{1}^{(2)}(3) \\
& - 24815934\Omega_{1}^{(2)}(4) + 5675436\Omega_{1}^{(2)}(5) \\
& - 748152\Omega_{1}^{(2)}(6) + 58992\Omega_{1}^{(2)}(7) - 2720\Omega_{1}^{(2)}(8) \\
& + 64\Omega_{1}^{(2)}(9) - 933504\Omega_{1}^{(4)}(4) + 389376\Omega_{1}^{(4)}(5) \\
& - 56832\Omega_{1}^{(4)}(6) + 3072\Omega_{1}^{(4)}(7)\Big)\Big) \ ,
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
a_{55} = \tfrac{1}{47185920}\Big(& 11566880325\Omega_{1}^{(2)}(2) - 16\Big(917277075\Omega_{1}^{(2)}(3) \\
& - 459555525\Omega_{1}^{(2)}(4) + 2\Big(60642270\Omega_{1}^{(2)}(5) \\
& - 9484335\Omega_{1}^{(2)}(6) + 8\Big(115245\Omega_{1}^{(2)}(7) \\
& - 7045\Omega_{1}^{(2)}(8) + 260\Omega_{1}^{(2)}(9) - 5\Omega_{1}^{(2)}(10) \\
& - 1351350\Omega_{1}^{(4)}(4) + 689520\Omega_{1}^{(4)}(5) \\
& - 137040\Omega_{1}^{(4)}(6) + 12480\Omega_{1}^{(4)}(7) \\
& - 480\Omega_{1}^{(4)}(8) - 384\Omega_{1}^{(6)}(6)\Big)\Big)\Big)\Big) \ ,
\end{aligned} \tag{A.9}
$$

$$
b_{14} = \tfrac{1}{96}\Big(693\Omega_{1}^{(2)}(2) - 594\Omega_{1}^{(2)}(3) + 132\Omega_{1}^{(2)}(4) - 8\Omega_{1}^{(2)}(5)\Big) \ , \tag{A.10}
$$

$$
\begin{aligned}
b_{24} = \tfrac{1}{384}\Big(& 20097\Omega_{1}^{(2)}(2) - 8\Big(1782\Omega_{1}^{(2)}(3) - 385\Omega_{1}^{(2)}(4) \\
& + 40\Omega_{1}^{(2)}(5) - 2\Omega_{1}^{(2)}(6)\Big)\Big) \ ,
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
b_{34} = \tfrac{1}{3072}\Big(& 467775\Omega_{1}^{(2)}(2) - 340362\Omega_{1}^{(2)}(3) + 8\Big(11385\Omega_{1}^{(2)}(4) \\
& - 1494\Omega_{1}^{(2)}(5) + 102\Omega_{1}^{(2)}(6) - 4\Omega_{1}^{(2)}(7) \\
& + 528\Omega_{1}^{(4)}(4) - 96\Omega_{1}^{(4)}(5)\Big)\Big) \ ,
\end{aligned} \tag{A.12}
$$

$$
\begin{aligned}
b_{44} = \tfrac{1}{36864}\Big(& 10513503\Omega_1^{(2)}(2) - 4\Big(2205225\Omega_1^{(2)}(3) \\
& - 708015\Omega_1^{(2)}(4) + 113256\Omega_1^{(2)}(5) - 10308\Omega_1^{(2)}(6) \\
& + 528\Omega_1^{(2)}(7) - 16\Omega_1^{(2)}(8) - 46816\Omega_1^{(4)}(4) \\
& + 12672\Omega_1^{(4)}(5) - 1152\Omega_1^{(4)}(6)\Big)\Big) ,
\end{aligned} \tag{A.13}
$$

$$
\begin{aligned}
b_{15} = \tfrac{1}{1536}\Big(& 9009\Omega_1^{(2)}(2) - 8\Big(1287\Omega_1^{(2)}(3) - 429\Omega_1^{(2)}(4) \\
& + 52\Omega_1^{(2)}(5) - 2\Omega_1^{(2)}(6)\Big)\Big) ,
\end{aligned} \tag{A.14}
$$

$$
\begin{aligned}
b_{25} = \tfrac{1}{18432}\Big(& 981981\Omega_1^{(2)}(2) - 949806\Omega_1^{(2)}(3) + 8\Big(35607\Omega_1^{(2)}(4) \\
& - 4810\Omega_1^{(2)}(5) + 354\Omega_1^{(2)}(6) - 12\Omega_1^{(2)}(7)\Big)\Big) ,
\end{aligned} \tag{A.15}
$$

$$
\begin{aligned}
b_{35} = \tfrac{1}{49152}\Big(& 9216207\Omega_1^{(2)}(2) - 4\Big(2158299\Omega_1^{(2)}(3) \\
& - 711711\Omega_1^{(2)}(4) + 117624\Omega_1^{(2)}(5) - 10692\Omega_1^{(2)}(6) \\
& + 560\Omega_1^{(2)}(7) - 16\Omega_1^{(2)}(8) - 27456\Omega_1^{(4)}(4) \\
& + 9984\Omega_1^{(4)}(5) - 768\Omega_1^{(4)}(6)\Big)\Big) ,
\end{aligned} \tag{A.16}
$$

$$
\begin{aligned}
b_{45} = \tfrac{1}{589824}\Big(& 242567325\Omega_1^{(2)}(2) - 2\Big(121548141\Omega_1^{(2)}(3) \\
& - 45970782\Omega_1^{(2)}(4) + 4\Big(2215707\Omega_1^{(2)}(5) \\
& - 246030\Omega_1^{(2)}(6) + 4\Big(4175\Omega_1^{(2)}(7) - 170\Omega_1^{(2)}(8) \\
& + 4\Omega_1^{(2)}(9) - 191048\Omega_1^{(4)}(4) + 75920\Omega_1^{(4)}(5) \\
& - 10656\Omega_1^{(4)}(6) + 576\Omega_1^{(4)}(7)\Big)\Big)\Big)\Big) ,
\end{aligned} \tag{A.17}
$$

$$
\begin{aligned}
b_{55} = \tfrac{1}{28311552}\Big(& 19216872675\Omega_1^{(2)}(2) - 16\Big(1351446525\Omega_1^{(2)}(3) \\
& - 584726571\Omega_1^{(2)}(4) + 2\Big(65345202\Omega_1^{(2)}(5) \\
& - 8583129\Omega_1^{(2)}(6) + 8\Big(88179\Omega_1^{(2)}(7) \\
& - 4699\Omega_1^{(2)}(8) + 3\Big(52\Omega_1^{(2)}(9) - \Omega_1^{(2)}(10) \\
& - 940082\Omega_1^{(4)}(4) + 451568\Omega_1^{(4)}(5) - 85456\Omega_1^{(4)}(6) \\
& + 7488\Omega_1^{(4)}(7) - 288\Omega_1^{(4)}(8) - 384\Omega_1^{(6)}(6)\Big)\Big)\Big)\Big)\Big) .
\end{aligned} \tag{A.18}
$$

VITA

Earl Lynn Tipton was born in Cape Girardeau, Missouri, United States of America, on March 26, 1979. He attended public schools in Marble Hill, Missouri. He was further educated at the University of Missouri-Columbia, where he received a Bachelor of Science degree in Chemical Engineering in December 2002, a Master of Science degree in Nuclear Engineering in May 2005, and a Doctor of Philosophy degree in Nuclear Engineering in August 2008. He is married to the former Yu Ning (Annie) Hsu.