Continuum elasticity theory of edge waves in a two-dimensional electron liquid with finite-range interactions

Irene D'Amico and Giovanni Vignale

Department of Physics, University of Missouri, Columbia, Missouri 65211

(Received 21 January 1999)

We make use of continuum elasticity theory to investigate the collective modes that propagate along the edge of a two-dimensional electron liquid or crystal in a magnetic field. An exact solution of the equations of motion is obtained with the following simplifying assumptions: (i) The system is *macroscopically* homogeneous and isotropic in the half-plane delimited by the edge. (ii) The electron-electron interaction is of finite range due to screening by external electrodes. (iii) The system is nearly incompressible. At sufficiently small wave vector q we find a universal dispersion curve $\omega \sim q$ independent of the shear modulus. At larger wave vectors the dispersion can change its form in a manner dependent on the comparison of various length scales. We obtain analytical formulas for the dispersion and damping of the modes in various physical regimes. [S0163-1829(99)07927-8]

I. INTRODUCTION

The dynamical behavior of the edge of a two-dimensional electron gas (2DEG) in a strong magnetic field is receiving considerable attention. The interest has been spurred by recent experiments[1,2] in which an unusual tunneling current-voltage relation of the form $I = V^{1/\nu_0}$, where V is the bias voltage and ν_0 is the bulk filling factor, has been observed. Such a distinctly non-Fermi-liquid-like behavior can be explained[3–5] on the assumption that there is only one branch of collective edge waves (the so-called charged mode), and that the tunneling electron is initially accommodated as a coherent superposition of such waves spreading in time according to hydrodynamic equations of motion.

Unfortunately, a complete theory of edge waves is not yet available. For example, there is considerable controversy about the existence of additional ''neutral modes,'' and whether they contribute to the tunneling characteristics or not.[6–8]

Even the more conventional charged mode has not been fully analyzed yet. The two most successful theories so far[9,10] focus exclusively on the long-range part of the Coulomb interaction, and should therefore be viewed as macroscopic versions of the random-phase approximation (RPA). A complete description of the long-wave dynamics requires, however, the inclusion of short-range forces that arise from the relative motion of adjacent parts of the fluid. These can be pressure forces arising from the bulk modulus of the quantum-mechanical fluid, as well as shear forces arising from more subtle positional correlations. Shear forces are, of course, essential in the crystalline phase, but they are also present in the liquid phase at nonzero frequency.[11] In addition, there are viscous forces—completely ignored in the RPA—that cause damping of the collective modes.

These post-RPA effects are well established in the theory of the dynamics of the uniform electron gas, where they are usually described in terms of ''local field corrections'' (see Ref. 11). For purely longitudinal bulk modes these corrections are not too important, as the direct Coulomb interaction controls the physics of long-wavelength longitudinal fluctuations. The situation can be quite different for edge modes, or even for bulk modes in a magnetic field, because in these cases longitudinal and transverse channels are strongly coupled, and one cannot talk of purely longitudinal or transverse modes.

In a recent study,[11] we showed that, in the absence of a magnetic field, the response of an interacting electron liquid to an external potential can be described as the response of a continuum viscoelastic medium when we consider the collective regime, that is, long wavelengths ($q \ll q_F$, where q_F is the Fermi wave vector) and frequencies ω higher than the electron-hole excitation energies ($q v_F \ll \omega \ll E_F/\hbar$, where v_F and E_F are the Fermi velocity and energy).

We believe that this description is even more appropriate for the collective dynamic of electrons in a strong magnetic field (the physically most interesting case), since in this case the electrons are strongly correlated and seem to form an elastic network that is locally similar to a Wigner crystal (in this regime, low-energy electron-hole excitations are almost entirely suppressed).

Indeed, we have recently shown[4] that the collective dynamics of a *uniform* electron gas, in the limit of infinite magnetic field (i.e., in the lowest Landau level), must necessarily include a shear force term, otherwise the frequencies of the collective modes vanish. This led us to a description of the electron gas as a continuum elastic medium, characterized by elastic constants (the bulk and the shear modulus) which control the dynamical response, and by viscosity coefficients, which control dissipation. Encouraged by the qualitative success of the continuum elasticity approach to the dynamics of the *uniform* electron gas, in this paper we present its application to the problem of *edge dynamics* in a magnetic field.

On a macroscopic scale, our system is modeled as a uniform continuum in the half-plane delimited by the edge. The external potential, that confines the electrons to the half-plane, is constant in the bulk of the system, and rises sharply at the edge. A great mathematical simplification follows

from the assumption that the electron-electron interaction is of finite (microscopic) range. This assumption is justified for structures in which the long-range Coulomb interaction is screened by metallic electrodes. The short range of the interaction results in equations of motion for the elastic displacement field that are linear differential equations, as opposed to integral equations in the approach of Refs. 9 and 10.

The distinctive feature of our theory is that the presence of the edge is taken into account via *boundary conditions*. In an infinite system, the solutions of the equations of motion are purely longitudinal or transverse plane waves characterized by a *real* wave vector. The presence of the edge allows the existence of solutions with an *imaginary* wave vector perpendicular to the edge, which therefore vanish exponentially as one moves into the bulk. It is evident that these "bound" solutions exist independently of the nature of the bulk, in particular, irrespective of whether the bulk is incompressible or not. In addition, these solutions must have the property that the elastic stress vanish at the edge. Because the boundary conditions are linear, the problem can be solved exactly by elementary techniques, and analytic results for the dispersion and damping of the collective modes can be obtained. (The qualitative changes brought about by the long range of the Coulomb interaction will be discussed in the Appendix.) A particularly elegant solution can be obtained for nearly incompressible systems, in which a large electrostatic charging energy strongly opposes density fluctuations.

Our main result is that, at sufficiently long wavelengths and high magnetic field, there is only *one* low-frequency edge mode, whose dispersion depends on the strengths of the confining electric and magnetic fields, but not on the bulk and shear moduli of the system. The nature of this solution does not change when the system has crystalline order. At shorter wavelengths, the dispersion can change its form in a manner dependent on the comparison of various length scales. In particular, we point out the possibility of a crossover from a linear dispersion, controlled by the confining electric field, to a quadratic dispersion, controlled by the shear modulus of the electronic system, if the latter is sufficiently large. Our result is at variance with a previous work[12] in which, for a crystal and in the long wavelengths regime, Monarkha, Peeters, and Sokolov predicted a quadratic edge wave dispersion. The discrepancy is due to the fact that these authors did not include the effect of the confining electric potential in the boundary conditions.

Our results support the idea that the dynamics of a sharp edge in a 2DEG is completely dominated by a *single* charged mode, with any additional structure associated with peculiar characteristics of the system (such as the fractional quantum Hall effect) becoming irrelevant at long wavelengths.

II. MODEL

We consider a two-dimensional electron fluid on a uniform background of positive charges, in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The system extends indefinitely for $x < 0$ while presenting a sharp edge at $x \sim 0$, parallel to the y axis. Since the width a of the edge is much smaller than the characteristic wavelength of the modes under consideration, we model the edge setting $a \simeq 0$. The *mass* density, at equilibrium, is a

constant ρ_0 for $x < 0$, and vanishes for $x > 0$. Consistent with this assumption, the external electrostatic potential $V_{ext}(x)$, created by the background of positive charges and by nearby gates, is taken to be constant for $x < 0$, and to rise sharply at $x = 0$ with a derivative

$$g \equiv \frac{e}{m} \left(\frac{dV_{ext}}{dx} \right)_{x=0}. \tag{1}$$

Here $-e$ is the *effective* electron charge, i.e., the bare charge divided by the square root of the static dielectric constant of the host semiconductor, and m is the *effective* mass of the electron in the host semiconductor. The quantity g is the acceleration imparted by the external potential to an electron at the edge. The effect of this acceleration on the collective edge modes is analogous to the effect of gravity on surface waves in a liquid—hence the notation.

To describe the dynamics of the system we introduce the displacement field $\mathbf{u}(x,y,t)$ of the infinitesimal volume element at point (x,y) from its equilibrium position. This obeys the linearized equation of motion of continuum elasticity theory:[13]

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{1}{\rho_0} \sum_j \partial_j \sigma_{ij} + \omega_c \left(\hat{\mathbf{z}} \times \frac{\partial \mathbf{u}}{\partial t} \right)_i, \tag{2}$$

where i and j are Cartesian indices, ∂_i is the derivative with respect to r_i, $\omega_c = eB/mc$ is the cyclotron frequency, and

$$\sigma_{ij} = K \nabla \cdot \mathbf{u} \delta_{ij} + \mu (\partial_i u_j + \partial_j u_i - \nabla \cdot \mathbf{u} \delta_{ij}) \tag{3}$$

is the elastic stress tensor with bulk modulus K and shear modulus μ. The first term on the right-hand side of Eq. (2) is the force exerted on the volume element by the surrounding medium, the second term is the Lorenz force. In writing these equations, we have assumed that the electron-electron interaction $v(\mathbf{r} - \mathbf{r}')$ is of finite range in space. This means that $v(\mathbf{q}=0) \equiv \int v(\mathbf{r}) d\mathbf{r}$ is finite, and is included as part of the bulk modulus

$$K = v(\mathbf{q}=0) n_0^2 + \widetilde{K}, \tag{4}$$

where $\widetilde{K}$ is the "proper" contribution arising from the kinetic and exchange-correlation energy and $n_0 \equiv \rho_0/m$ is the particle density. This procedure is, of course, only justified when the spatial variation of the density is small over distances of the order of the range of the interaction d, so that $qd \ll 1$, where q is the wave vector. We must also have $qa \ll 1$ for the "sharp edge" description to be valid. Clearly, both conditions are satisfied at sufficiently long wavelengths.

Substituting Eq. (3) into Eq. (2), we obtain the standard form of the equation of motion

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = C^2 \nabla (\nabla \cdot \mathbf{u}) + C_t^2 \nabla^2 \mathbf{u} + \omega_c \hat{\mathbf{z}} \times \dot{\mathbf{u}}, \tag{5}$$

where

$$C_t^2 = \frac{\mu}{\rho_0} \tag{6}$$

is the square of the transverse sound velocity in the absence of the magnetic field, and

$$C^2 = \frac{K}{\rho_0} = C_l^2 - C_t^2, \tag{7}$$

where C_l^2 is the square of the longitudinal sound velocity in the absence of a magnetic field. As stated in Sec. I, we assume that the system is nearly incompressible, in the sense that the charging energy $v(q=0)n_0$ gives the dominant contribution to the bulk modulus, and is much larger than any other energy scale in the problem, such as μ/n_0 or $\hbar\omega_c$. This implies that $C_l \gg C_t$, so that the difference between C^2 and C_l^2 can and will be ignored in the following. We emphasize that this assumption (which is expected to be reasonable for an electrically charged system) simplifies the calculations, but is not otherwise essential.

To complete the definition of the model we must now specify the boundary conditions on the solution of Eq. (5). In elasticity theory, the normal form of the boundary condition at a *free* surface is[13]

$$\sum_j \sigma_{ij}|_{edge}\hat{n}_j = 0, \tag{8}$$

where $i=x$ or y, $\hat{n}$ ($=\hat{x}$ in this case) is the unit vector perpendicular to the edge, and the subscript "edge" means that the quantity on the left hand side must be evaluated at the position of the *moving* edge, that is, at the point of coordi-

nates $[u_x(0,y),y]$. The physical significance of this condition is that there is no matter beyond the edge to exert a force on the system.

This boundary condition must be slightly modified here because the external potential produces a stress in the edge region even when the system is in equilibrium. Let us denote this equilibrium stress by $\sigma_{ij}^{(0)}$. Then the free boundary conditions take the form

$$\sum_j [\sigma_{ij}^{(0)}(u_x,y) + \sigma_{ij}(u_x,y)]\hat{n}_j = 0. \tag{9}$$

From the symmetry of the problem one sees that only the xx component of the equilibrium stress tensor is nonzero, and from the equilibrium condition $d\sigma_{xx}^{(0)}(x)/dx + e(\rho_0/m)dV_{ext}(x)/dx = 0$, one sees that

$$\left(\frac{d\sigma_{xx}^{(0)}}{dx}\right)_{x=0} = g\rho_0, \tag{10}$$

with g defined in Eq. (1). Finally, one is free to choose $\sigma_{xx}^{(0)}(x=0,y)=0$ so as to satisfy the free boundary conditions at equilibrium.

Expanding Eq. (9) to first order in $\mathbf{u}$, and making use of Eq. (10), we finally obtain explicit forms for the two boundary conditions:

$$C_l^2\nabla\cdot\mathbf{u}(0,y) + C_t^2\left(\frac{\partial u_x(0,y)}{\partial x} - \frac{\partial u_y(0,y)}{\partial y}\right) + gu_x(0,y) = 0, \tag{11}$$

$$\frac{\partial u_y(0,y)}{\partial x} + \frac{\partial u_x(0,y)}{\partial y} = 0. \tag{12}$$

Together with the condition that the displacement field vanishes for $x\to-\infty$, Eq. (5) and Eqs. (11) and (12) completely define the mathematical problem under consideration.

III. EDGE WAVES IN A MAGNETIC FIELD

Ignoring the boundary conditions at first, it is easy to see that the solutions of Eq. (5) that vanish for $x\to-\infty$ can be chosen to have the form

$$\mathbf{u}(x,y,t) = \tilde{\mathbf{u}}e^{iqy+\lambda x-i\omega t}, \tag{13}$$

where q is a real wave vector parallel to the edge, and λ is, in general, a complex number, whose real part must be positive in order to ensure decay in the interior of the system.

Taking the divergence and the curl of the equations of motion (5) and making use of Eq. (13), we obtain

$$[-\omega^2 - C_l^2(\lambda^2 - q^2)]\nabla\cdot\mathbf{u} - i\omega\omega_c(\nabla\times\mathbf{u})_z = 0, \tag{14}$$

$$i\omega\omega_c\nabla\cdot\mathbf{u} + [-\omega^2 - C_t^2(\lambda^2 - q^2)](\nabla\times\mathbf{u})_z = 0. \tag{15}$$

These two linear homogeneous equations are compatible if and only if the square of the wave vector λ^2 has one of the two values

$$\lambda^2 = q^2 - \frac{\omega^2(C_t^2 + C_l^2)}{2C_t^2C_l^2} \pm \sqrt{\left(\frac{\omega^2(C_t^2 + C_l^2)}{2C_t^2C_l^2}\right)^2 - \frac{\omega^2(\omega^2 - \omega_c^2)}{C_l^2C_t^2}}. \tag{16}$$

In the limit $C_l \gg C_t$ the two solutions for λ^2 (which we denote λ_l^2 and λ_t^2) take the simple forms

$$\lambda_l^2 = q^2 + \frac{\omega_c^2 - \omega^2}{C_l^2}, \tag{17}$$

$$\lambda_t^2 = q^2 - \frac{\omega^2}{C_t^2} - \frac{\omega_c^2}{C_l^2}. \tag{18}$$

With a little algebra, it is possible to calculate the corresponding eigenfunctions in the limit $C_l \rightarrow \infty$:

$$\mathbf{u}_l \propto [\hat{\mathbf{x}} + i\, \text{sgn}(q)\hat{\mathbf{y}}]e^{iqy + |q|x - i\omega t}, \tag{19}$$

$$\mathbf{u}_t \propto [\hat{\mathbf{x}} + i\, \text{sgn}(q)\sqrt{1 - \omega^2/C_t^2 q^2}\,\hat{\mathbf{y}}]e^{iqy + |q|\sqrt{1 - \omega^2/C_t^2 q^2}\,x - i\omega t}. \tag{20}$$

Let us now turn to the problem of satisfying the boundary conditions (11) and (12) at $x = 0$. This can be done by forming a suitable superposition of the two independent solutions $\mathbf{u}_l$ and $\mathbf{u}_t$, namely,

$$\mathbf{u} = a\mathbf{u}_l + b\mathbf{u}_t, \tag{21}$$

where a and b are complex coefficients. In order to implement the boundary conditions of Eq. (11) we must calculate the limit for $C_l \rightarrow \infty$ of the products $C_l^2 \nabla \cdot \mathbf{u}_{l(t)}$. To accomplish this, we need to refine our calculation of the eigenfunctions $\mathbf{u}_{l(t)}$ by including terms of order $1/C_l^2$. This can be done straightforwardly, with the help of Eqs. (17) and (18), and the results are

$$\lim_{C_l \rightarrow \infty} C_l^2 \nabla \cdot \mathbf{u}_l(0, y) = \frac{\omega[\omega_c - \omega\, \text{sgn}(q)]}{q} e^{iqy} \tag{22}$$

and

$$\lim_{C_l \rightarrow \infty} C_l^2 \nabla \cdot \mathbf{u}_t(0, y) = \frac{\omega \omega_c}{q} e^{iqy}. \tag{23}$$

Substituting Eqs. (22) and (23) in boundary conditions (11) and (12) yields the following set of linear homogeneous equation for the coefficients a and b:

$$\left[g + \frac{\omega \omega_c - \omega^2\, \text{sgn}(q)}{q} + 2C_t^2|q| \right]a + \left(g + \frac{\omega \omega_c}{q} + 2C_t^2\sqrt{1 - \omega^2/C_t^2 q^2}|q| \right)b = 0,$$

$$2a + (2 - \omega^2/C_t^2 q^2)b = 0. \tag{24}$$

These two equations are compatible if and only if the frequency ω satisfies the algebraic equation

$$(2 - \xi^2)^2 - \xi^2(\xi Z + X) = 4\sqrt{1 - \xi^2}, \tag{25}$$

where

$$X = \frac{g}{|q|C_t^2}, \tag{26}$$

$$Z = \frac{\omega_c}{|q|C_t}, \tag{27}$$

$$\xi = \frac{\omega}{qC_t}. \tag{28}$$

The complete solution for the displacement field is

$$\mathbf{u}(x, y, t) = \{[(2 - \xi^2)e^{|q|x} - 2e^{|q|\sqrt{1-\xi^2}x}]\hat{\mathbf{x}} + i\, \text{sgn}(q)[(2 - \xi^2)e^{|q|x} - 2\sqrt{1 - \xi^2}e^{|q|\sqrt{1-\xi^2}x}]\hat{\mathbf{y}}\}e^{i(qy - \omega t)}. \tag{29}$$

Equations (25) and (29) are the central results of this paper. In particular, Eq. (25) is the generalization of the classical equation for the dispersion of surface waves in an elastic medium—the so called ''Rayleigh waves'' (see Ref. 13)—to which it reduces when the dimensionless parameters X and Z, measuring the strength of the external electric and magnetic fields relative to the shear elastic forces, are equal to zero.

The solutions of Eq. (25) will be discussed in Sec. IV for various physical regimes. Before doing that, however, it is necessary to clarify a delicate point which arises when one attempts to take the limit $C_t \rightarrow 0$ of the above theory. Physically this corresponds to the very relevant case of a genuine liquid system, which is expected to have a vanishing shear modulus at low frequency. It is evident from Eq. (29) that this limit is singular: the variable ξ tends to infinity, imply-

ing that only the $\mathbf{u}_l$ component of the solution [that is, the part proportional to ξ^2 in Eq. (29)] survives. Nevertheless, the presence of a rapidly oscillating component of the solution with wave vector $\lambda \sim iq|\xi|$, which does not vanish for $x \to -\infty$, is disturbing. This difficulty becomes more evident when one tries to solve the equation for the dispersion: in the limit $C_t \to 0$, $\xi \to \infty$, the argument of the square root lies on the negative real axis, where the presence of the branch cut prevents us from finding a solution.

The resolution of these difficulties lies in the following physical considerations. Every system, liquid or solid, has a nonvanishing *shear viscosity* η at finite frequency. The shear viscosity contributes an additional term to the stress tensor (see Ref. 14), which can be accommodated within our formalism simply through the replacement $\mu \to \mu - i\omega\eta$ or, equivalently,

$$C_t^2 \to \tilde{C}_t^2 \equiv C_t^2 - i\omega\nu, \qquad (30)$$

where $\nu \equiv \eta/\rho_0$ is the so-called kinematic viscosity.[14] In a solid, C_t remains finite for $\omega \to 0$, and therefore ν can be safely neglected. In a liquid, however, C_t tends to 0 faster than ω, while ν remains finite: the low-frequency limit is therefore dominated by the viscosity. It is easy to see that the inclusion of viscosity eliminates the singularity of the solution in the limit $C_t \to 0$ and $\nu \to 0$. This is because, after the replacement indicated in Eq. (30), the wave vector $q\sqrt{1-\xi^2}$ in the limit $C_t \to 0$ reduces to $\sqrt{-i\omega/\nu} = \sqrt{\omega/2\nu}(1-i)$, which, for small ν, has a very large positive real part, and therefore vanishes very rapidly away from the edge. Correspondingly, a solution of Eq. (25) can always be found (at small q) if the viscosity is added according to the substitution indicated in Eq. (30). In practice, the viscosity is expected to be small. In Sec. IV we shall present our results treating ν, C_t^2, g, and ω_c as formal parameters. The actual values of these parameters will be discussed in Sec. V.

IV. DISPERSION RELATION IN VARIOUS PHYSICAL REGIMES

The behavior of the solutions of Eq. (25) as functions of the dimensionless parameters X and Z is rather complicated. It is convenient to distinguish three different regimes according to whether the magnetic field, the edge electric field (g), or the shear modulus dominates.

(a) *Strong magnetic field.* This regime is characterized by $|q| \ll |\omega_c/\tilde{C}_t|$ and $|q| \ll |\omega_c^2/g|$. Therefore, $|Z| \gg 1$ and $|X| \ll |Z|^2$. It is easy to see that Eq. (25) has two solutions in this limiting case. The first solution is

$$\xi = -\frac{X}{Z} + \frac{1}{Z}\left(\frac{X^2}{Z^2} - 4\frac{\sqrt{1-X^2/Z^2}-1}{X^2/Z^2} - 4\right). \qquad (31)$$

The explicit form of the dispersion depends on the value of X. The most interesting case is $|X| \gg |Z|$, which corresponds to long wavelengths $|q| \ll g/C_t^2$ and vanishing shear modulus $\omega_c C_t \ll g$, and is appropriate for the liquid state. In this case, substituting the definitions (26), (27), and (28) in Eq. (31), one obtains, after simple manipulations,

$$\omega = -\frac{gq}{\omega_c} - \frac{4C_t^2 q|q|}{\omega_c} + \frac{g^2|q|q}{\omega_c^3} - 4i\nu\frac{g|q|^3}{\omega_c^2}, \qquad (32)$$

where we have used Eq. (30) to obtain the imaginary part of the frequency. Thus, at sufficiently long wavelength, the dispersion is linear and independent of the visco-elastic constants.

Recalling the definition of the ''gravity acceleration'' in Eq. (1), we see that the phase velocity for long wavelength ω/q coincides with the classical drift velocity $v = cE/B$, where E is the magnitude of the electric field at the edge. Other cases, compatible with our definition of the strong-field regime, can also be calculated from Eq. (31). For example, in the limit of vanishing electric field $g \to 0$ or large shear modulus, such that $|X| \ll 1$, $|g/C_t^2| \ll |q| \ll |\omega_c/C_t|$ we obtain

$$\omega = -\frac{gq}{\omega_c} - \frac{2C_t^2 q|q|}{\omega_c} - 4i\frac{C_t^2\nu q^4}{\omega_c^2}. \qquad (33)$$

Setting $g = 0$ in this expression, we obtain the dispersion of surface elastic waves (''Rayleigh waves'')[13] in the presence of a strong magnetic field. Notice that, due to the presence of the strong magnetic field, the dispersion is quadratic rather than linear in q.

The second solution is

$$\xi \simeq Z + (X+4)/Z, \qquad (34)$$

which implies

$$\omega = \omega_c \,\text{sgn}(q) + \frac{gq}{\omega_c} + \frac{4C_t^2 q^2}{\omega_c}\,\text{sgn}(q) - 4i\nu q^2. \qquad (35)$$

Notice that the two solutions obtained in this section describe waves propagating in opposite directions (i.e., the sign of the ratio ω/q is opposite in the two cases), and that the second solution, for $q \to 0$, is simply the manifestation of the uniform Kohn mode[15] at the edge of the system.

(b) *Strong electric field (gravity waves).* Let us now assume that the edge electric field is large, in the sense that $\omega_c^2/g \ll |q| \ll g/C_t^2$. This implies $|X| \gg 1$ and $|X| \gg |Z|^2$. Notice that q cannot tend to zero in this regime, unless $\omega_c = 0$. The solutions of Eq. (25) are then found to be

$$\xi \sim \pm\sqrt{X \pm Z\sqrt{X+4}}, \qquad (36)$$

$$\omega = \pm\sqrt{g|q|}\left(1 + \frac{2C_t^2|q|}{g}\right) + \frac{\omega_c}{2} - 2i\nu q^2 \qquad (37)$$

It is comforting to notice that, for zero magnetic field and shear modulus, we recover the classical results for the dispersion and damping of ''gravity waves'' on the surface of a liquid.[14]

(c) *Large shear modulus (Rayleigh waves).* Next, consider the case that the shear modulus dominates, in the sense that $|q| \gg \omega_c/C_t$ and $|q| \gg g/C_t^2$. This implies $|X| \ll 1$ and $|Z| \ll 1$. Again, this definition does not allow one to take the limit $q \to 0$ unless g and ω_c are both zero.

The doubly degenerate solution of Eq. (25) in this case is $\xi = \pm 0.955$, which implies (for $q \ll \sqrt{C_t^2/\nu}$)
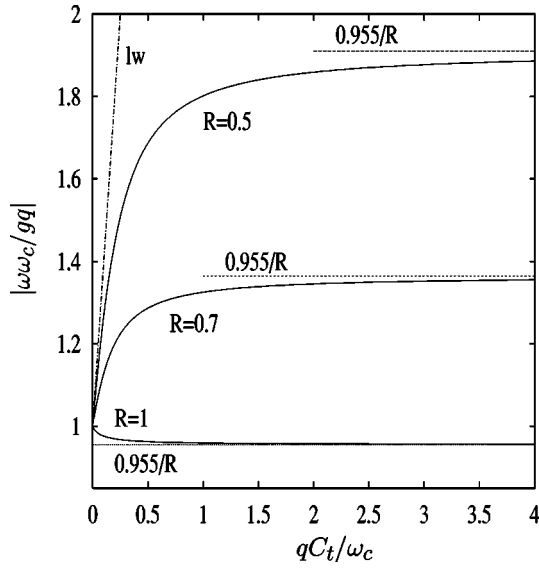
FIG. 1. Real part of the dimensionless phase velocity $|\omega\omega_c/gq|$ vs qC_t/ω_c for three values of $R=g/C_t\omega_c$. For each curve the short-wavelength limit $|\omega\omega_c/gq|=0.955/R$—coinciding with the dispersion of classical Rayleigh waves—is plotted for comparison. For the curve with $R=0.5$, we also plot the long-wavelength limit $|\omega\omega_c/gq|=1+2C_t^2|q|/g$ ("lw" curve).

$$\omega = \pm 0.955 C_t|q| - i(0.456)\nu q^2. \qquad (38)$$

This is nothing but the familiar Rayleigh wave on the surface of an elastic solid,[13] with a small damping due to viscosity, and no correction from external electric and magnetic fields.

(d) *Zero shear modulus limit (liquid state).* In this case (and for zero viscosity) the equations of motion can be solved exactly. The expression for the two branches of the dispersion relation is

$$\omega_\pm = \frac{\omega_c}{2}\operatorname{sgn}(q) \pm \sqrt{\left(\frac{\omega_c}{2}\right)^2 + g|q|}. \qquad (39)$$

For $q\ll\omega_c^2/g$ and $q\gg\omega_c^2/g$ we recover expressions (32) and (37), respectively. The corresponding eigenfunctions are given by u_l in Eq. (19).

We have thus exhausted all the physically different regimes. In cases (a)–(c), the form of the eigenfunction is obtained by substituting the appropriate value of ξ in Eq. (29). Results for the lower branch of the complete numerical solution of Eq. (25), exhibiting crossovers between the various regimes discussed in this section, are shown in Figs. 1 and 2. In Fig. 1, the real part of the phase velocity $|\omega\omega_c/gq|$ is plotted against qC_t/ω_c for decreasing values of the ratio $R \equiv g/C_t\omega_c$ ($R=1$, 0.7, and 0.5), that is, for increasing magnetic field or shear modulus or decreasing electric field. The figure clearly shows the universal long-wavelength behavior $|\omega\omega_c/gq|\approx 1$ for very small q. For $R=0.5$ (strong magnetic field and finite shear modulus) and small q ($q\ll\omega_c/C_t$), the exact curve agrees with the approximate expression Eq. (33) (labeled in the figure as "lw"). For $R=1$ (strong magnetic field and small shear modulus) the third term in Eq. (32) becomes predominant and the exact curve lies below 1. For large q the exact curves in all cases approach the "Rayleigh wave" regime, that is, the lines $|\omega\omega_c/gq|=0.955/R$.
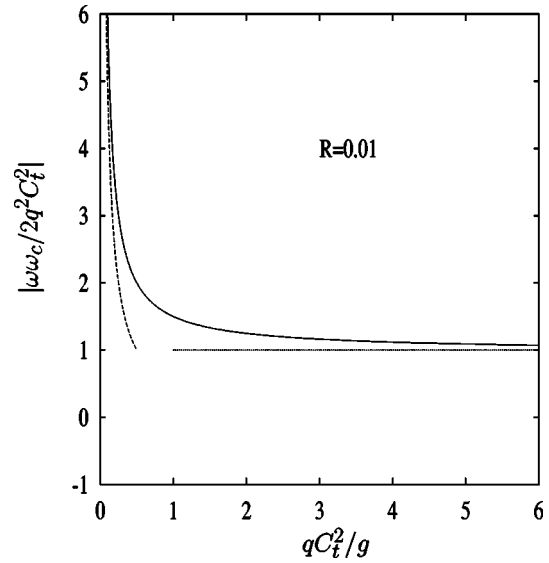

FIG. 2. Real part of $|\omega\omega_c/2q^2C_t^2|$ vs qC_t^2/g for $R=0.01$ (solid line). The dashed line is the approximate curve for the "linear regime" ($|q|\ll g/C_t^2$, $|\omega\omega_c/2q^2C_t^2|=|g/2qC_t^2|$), while the dotted line corresponds to the "quadratic regime" ($|g/C_t^2|\ll|q|\ll|\omega_c/C_t|$, $|\omega\omega_c/2q^2C_t^2|=1$).

In Fig. 2 we plot the real part of $|\omega\omega_c/2q^2C_t^2|$ versus qC_t^2/g for $|q|\ll|\omega_c/C_t|$ for $R=0.01$ (solid line). This figure clearly shows a crossover from the linear dispersion controlled by the electric field to the quadratic dispersion controlled by the shear modulus in the case of strong magnetic field [see Eq. (33)]. The dashed line is the approximate curve for the "linear regime" ($|q|\ll g/C_t^2$, $|\omega\omega_c/2q^2C_t^2|=|g/2qC_t^2|$), while the dotted line corresponds to the "quadratic regime" ($|g/C_t^2|\ll|q|\ll|\omega_c/C_t|$, $|\omega\omega_c/2q^2C_t^2|=1$).

In Fig. 3 we plot $|\omega/\omega_c|$ against qg/ω_c^2 for the case of zero shear modulus [Eq. (39)]. For the low-frequency branch
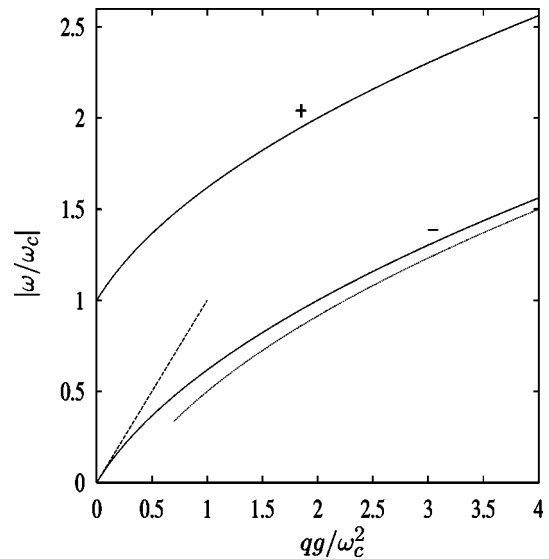

FIG. 3. Real part of the upper ("+") and lower ("−") branches of the dimensionless dispersion curve $|\omega/\omega_c|$ vs qg/ω_c^2 for vanishing shear modulus. The dashed line represents the long-wavelength limit ($|\omega/\omega_c|=qg/\omega_c^2$), and the dotted line the short-wavelength limit ($|\omega/\omega_c|=\sqrt{qg/\omega_c^2-1/2}$), of the lower branch.

(labeled with ''−''), we also plot the leading term for long-wavelength ($|\omega/\omega_c|=qg/\omega_c^2$, dashed line) and short-wavelength [Eq. (37), dotted line] approximate behaviors.

It is interesting to note that no acceptable solution is found in the transition region between regimes (b) and (c), that is, for $\omega_c/C_t \lesssim q \sim g/C_t^2$.

V. DISCUSSION

Up to this point our classification of different physical regimes has been purely formal: we have not yet specified the values of the parameters. We now wish to state the concrete predictions of our theory for typical systems at high magnetic field.

The value of the electric acceleration at the edge is easily estimated as

$$g = \frac{e^2 n_0}{m} \frac{d}{a}, \tag{40}$$

where $n_0 = \rho_0/m$ is the equilibrium density, d is the range of the interaction, and a is the width of the edge, which is typically of the order of the magnetic length $l = \sqrt{\hbar c/eB}$. In a magnetic field, it is convenient to introduce the filling factor $\nu_0 \equiv 2\pi n_0 l^2$. Then our estimate for g takes the form

$$g = \frac{e^2 \nu_0}{2\pi\hbar} \omega_c \frac{d}{a}. \tag{41}$$

The calculation of the shear viscosity is considerably more difficult. A mode-coupling calculation for the two-dimensional electron gas *at zero magnetic field*[16] yields numerical results that can be accurately described by the formula[11]

$$\nu \simeq (59 r_s^{-3/2} + c_1 r_s^{-1} + c_2 r_s^{-2/3} + c_3 r_s^{-1/3})^{-1} \frac{\hbar}{m}, \tag{42}$$

where $r_s = \sqrt{1/\pi n_0 a_0^2}$ is the usual electron-gas parameter, $a_0 = \hbar^2/me^2$ is the effective Bohr radius of the host semiconductor, and $c_0 = 0.25$, $c_1 = 20.6$, $c_2 = 22.7$, and $c_3 = 12.8$. We are not aware of any calculation of the viscosity in the presence of a magnetic field, but we expect Eq. (42) to give at least the right order of magnitude at a given density.

As for the shear modulus, we expect it to vanish, if the system is liquid, at low frequency, leaving us with $\tilde{C}_t^2 = -i\omega\nu$. If, instead, the system is a solid, then C_t^2 has a finite value which can be estimated from dimensional considerations:

$$C_t^2 \simeq \frac{e^2 n_0^{1/2}}{m}. \tag{43}$$

It is the viscosity that can be neglected in this case. For example, in the classical two-dimensional Wigner crystal with Coulomb interactions ($d=\infty$), the exact result is $C_t^2 \simeq 0.24 e^2 n_0^{1/2}/m$[11].

Let us first consider the dispersion of edge waves in a *liquid*. Because $C_t = 0$, and ν is very small we immediately see that $|X| \gg 1$ and $|Z| \gg 1$ at all realistic wave vectors $q \ll 1/a \sim 1/d$. The inverse length ω_c^2/g is given by

$$\frac{\omega_c^2}{g} = a_0^{-1} \frac{2\pi}{\nu_0} \frac{a}{d} \left(\frac{a_0}{l}\right)^2. \tag{44}$$

This is of the order of $1/a$ or larger for typical quantum Hall systems in which the density is $n_0 \sim 10^{10}$–10^{11} cm^{-2}, and the filling factor is less than 1. Therefore, these systems fall within the ''strong-magnetic-field regime'' of Sec. IV. Because the frequency vanishes linearly with q, the ratio $X/Z = g/\omega_c \sqrt{-i\omega\nu}$ tends to infinity, and the dispersion is therefore given by Eq. (32):

$$\omega = -\frac{gq}{\omega_c} + \frac{g^2 q|q|}{\omega_c^3} - 4i\nu \frac{g|q|^3}{\omega_c^2}. \tag{45}$$

The eigenfunction has the simple form

$$\mathbf{u} \propto [\hat{\mathbf{x}} + i \, \mathrm{sgn}(q)\hat{\mathbf{y}}] e^{|q||x + iqy}, \tag{46}$$

which describes a circular motion of each volume element. Then, making use of our estimate (41) for the electric acceleration, the phase velocity of the wave $v = g/\omega_c = (e^2 \nu_0/h)(d/a)$ is obtained at once.

Let us now consider the case that the low-frequency shear modulus does *not* vanish: this would happen, for example, if the electrons solidified in a Wigner crystal structure. According to our general discussion it might be possible, with increasing wave vector, to cross over to an ''intermediate wavelength regime,'' in which the dispersion is controlled by the shear modulus, and varies as q^2 [see Eq. (33)].

Unfortunately, this crossover is not likely to occur within the region of wave vectors in which our theory applies, namely, $q \ll 1/a \sim 1/d$. Indeed, from Eqs. (40), (41), and (43), we see that, for d comparable to, but somewhat larger than, the average distance between electrons, both the ratios g/C_t^2, and ω_c/C_t are of the order of the inverse of the interelectron distance. This implies that for $q \ll 1/a, 1/d$ both X and Z are $\gg 1$, and we clearly fall into the ''long-wavelength regime'' of Sec. IV. The only possibility to observe the crossover to shear-modulus-sustained waves in the present model, would arise if the range of the interaction d were much less than the typical interelectron distance—not an easily realizable situation.

Thus, in summary, we have shown that the two-dimensional electron gas on a neutralizing background of charge sustains only *one* macroscopically charged collective mode which decays exponentially as one moves away from the edge. *The scale of this exponential decay is the same as the scale of variation of the density along the edge.* We have derived analytical and numerical expressions for the dispersion relations and eigenfunctions in various physically distinct regimes. An important result of our study is that the character of the edge waves in this model is controlled almost exclusively by the strength of the electric field at the edge, and does not depend significantly on the shear modu-

lus: therefore, the dispersion is the same, to leading order in q, for the liquid and the solid state [see Eq. (31)]. These results suggest that the behavior of the I-V tunneling characteristics at low bias voltage would be the same at the edge of a liquid and of a Wigner crystal: in particular, the power law $I \sim V^{1/\nu_0}$, where ν_0 is the bulk filling factor, is expected in both cases.[3]

ACKNOWLEDGMENTS

APPENDIX: EXTENSION TO LONG-RANGE INTERACTION

In this paper we have taken advantage of some mathematical simplifications arising from the assumption that the electron-electron interaction is of finite range in space. Extending the theory to properly include Coulomb interactions is nontrivial. In this appendix we want to sketch an approximate method to do this extension, which entails minimal changes in the structure of the equations. The method is nonrigorous, yet it yields qualitatively correct results for the long-wavelength dispersion.

To begin, we observe that, in the equation of motion (5), the term $C^2 \nabla(\nabla \cdot \mathbf{u})$ should be replaced (ignoring the small contribution from $\tilde{K}$) by

$$\nabla \int \frac{e^2}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{u}(\mathbf{r}') d\mathbf{r}'. \tag{A1}$$

In an infinite system this term would pose no problem: by Fourier transformation it could be recast in the form of Eq. (5), with a q-dependent longitudinal ''sound velocity''

$$C^2(\mathbf{q}) = \frac{2\pi e^2 n_0}{mq}, \tag{A2}$$

which tends to infinity as q tends to zero. In a semi-infinite system, however, things are not so simple, and the Coulomb interaction cannot be simply absorbed in a q-dependent sound velocity.

The idea of our approximation scheme is to neglect the effect of the edge on the bulk equation of motion, which therefore retains the form of Eq. (5), with C^2 tending to infinity in the long-wavelength limit. The effect of the edge will be taken into account only via the boundary conditions (11) and (12), which force the solution to be a certain superposition of bulk waves.

So far the theory is formally identical to the short-range case. However, observe that the key quantity g, which enters the boundary conditions, is ill defined in the case of the Coulomb interaction. The external potential in this case is simply the electrostatic potential created by a uniform distribution of positive charge precisely compensating for the electronic charge in the half-plane. The electric field produced by this charge tends to infinity (logarithmically) at the edge of the half-plane, and therefore g is infinite [see Eq. (1)].

In order to obtain the correct form of the boundary conditions in the Coulomb case we return to Eq. (9), and note that from the equilibrium condition $\sigma_{xx}^{(0)}(\mathbf{r})$ is given by

$$\sigma_{xx}^{(0)}(\mathbf{r}) = -n_0^2 \int \int_{x'<0} \frac{e^2}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}', \tag{A3}$$

where the integral is restricted to the half-plane $x' < 0$.

Consider a point in the vicinity of the geometric edge with $\mathbf{r} = (\epsilon + u_x(0,y), y)$, where ϵ is a length of the order of the physical width of the edge a. Similarly to what we did in the short-range case in Sec. II we now expand $\sigma_{xx}^{(0)}$ to first order in u_x and discard the constant zero-order contribution. After integrating with respect to x', we obtain

$$\sigma_{xx}^{(0)}(u_x, y) \simeq n_0^2 \int_{-\infty}^{\infty} \frac{e^2}{\sqrt{\epsilon^2 + (y - y')^2}} u_x(0, y') dy'. \tag{A4}$$

Specializing to solutions with definite wave vector along the edge [i.e., $u_x(0,y) \propto e^{iqy}$], we see that Eq. (A4) takes the simpler form

$$\sigma_{xx}^{(0)}(u_x, y) \simeq n_0^2 2 e^2 K_0(|q\epsilon|) u_x(0, y), \tag{A5}$$

where K_0 is the modified Bessel function. Unfortunately, the $\epsilon \to 0$ limit of this expression does not exist, due to the logarithmic divergence of the Bessel function $K_0(x) \sim -ln(x)$ for $x \to 0$. However, the divergence is very weak, and, in view of the fact that the position of the edge is defined within an uncertainty of the order of a (~ 0 in our theory) it is legitimate (with logarithmic accuracy) to replace $K_0(|q\epsilon|)$ by $K_0(|qa|)$.

Substitution of this expansion in Eq. (9) leads to our boundary conditions: these can still be written in the form of Eqs. (24), but now both the longitudinal sound velocity, and the electrical acceleration g are functions of q: the former is given by Eq. (A2), while for the latter

$$g \to g(q) = 2 e^2 \frac{n_0}{m} K_0(|qa|). \tag{A6}$$

From this point on, all calculations proceed as in the short-range case. In particular, the logarithmic divergence of the effective g causes the dispersion of the edge waves to vary as $-(2\nu_0 e^2/h) q \ln(qa)$ in a magnetic field, and as $\sqrt{2 e^2 n_0} \sqrt{q \ln(qa)}$ without a magnetic field. It is amusing to observe that in the three-dimensional case the effective g at the surface would be $g(q) = 2\pi \rho_0 e^2/m^2 q$, leading, in the absence of a magnetic field, to the well-known result for the frequency of surface plasmons, namely, $\omega = \sqrt{gq} = \sqrt{2\pi e^2 n_0/m}$.

[1] M. Grayson, D. C. Tsui, L. N. Pfeiffer, K. W. West, and A. M. Chang, Phys. Rev. Lett. **80**, 1062 (1998).

[2] A. M. Chang, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **77**, 2538 (1996).

[3] S. Conti and G. Vignale, Phys. Rev. B **54**, 14 309 (1996); Phys. Rev. E **1**, 101 (1997).

[4] S. Conti and G. Vignale, J. Phys.: Condens. Matter **10**, L779 (1998).

[5] U. Zülicke and A. H. MacDonald, cond-mat/9802019 (unpublished).

[6] A. V. Shytov, L. S. Levitov, and B. I. Halperin, Phys. Rev. Lett. **80**, 141 (1998).

[7] D. H. Lee and X. G. Wen, cond-mat/9809160 (unpublished).

[8] A. Lopez and E. Fradkin, cond-mat/9810168 (unpublished).

[9] V. A. Volkov and S. A. Mikhailov, Zh. Éksp. Teor. Fiz. **94**, 217 (1988) [Sov. Phys. JETP **67**, 1639 (1988)].

[10] I. L. Aleiner and L. I. Glazman, Phys. Rev. Lett. **72**, 2935 (1994).

[11] S. Conti and G. Vignale, cond-mat/9811214 (unpublished). For a review of post-RPA effects in the uniform electron gas, see K. S. Singwi and M. P. Tosi, in *Solid State Physics* edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1981), Vol. 36, p. 177.

[12] Y. S. Monarkha, F. M. Peeters, and S. S. Sokolov, J. Phys.: Condens. Matter **9**, 1537 (1997).

[13] L. D. Landau and E. Lifshitz, *Theory of Elasticity*, 3rd ed., Course of Theoretical Physics, Vol. 7 (Pergamon Press, Oxford, 1986).

[14] L. D. Landau and E. Lifshitz, *Mechanics of Fluids*, 2nd ed., Course of Theoretical Physics, Vol. 6 (Pergamon Press, Oxford, 1987).

[15] W. Kohn, Phys. Rev. **123**, 1242 (1961).

[16] R. Nifosí, S. Conti, and M. P. Tosi, Phys. Rev. B **58**, 12 758 (1998).