

**Formal framework for a nonlocal generalization of Einstein's theory of gravitation**

Friedrich W. Hehl\*

*Institute for Theoretical Physics, University of Cologne, 50923 Köln, Germany  
and Department of Physics and Astronomy, University of Missouri, Columbia, Missouri 65211, USA*

Bahram Mashhoon†

*Department of Physics and Astronomy, University of Missouri, Columbia, Missouri 65211, USA  
(Received 3 February 2009; published 26 March 2009)*

The analogy between electrodynamics and the translational gauge theory of gravity is employed in this paper to develop an ansatz for a nonlocal generalization of Einstein's theory of gravitation. Working in the linear approximation, we show that the resulting nonlocal theory is equivalent to general relativity with "dark matter." The nature of the predicted dark matter, which is the manifestation of the nonlocal character of gravity in our model, is briefly discussed. It is demonstrated that this approach can provide a basis for the Tohline-Kuhn treatment of the astrophysical evidence for dark matter.

DOI: [10.1103/PhysRevD.79.064028](https://doi.org/10.1103/PhysRevD.79.064028)

PACS numbers: 04.50.Kd, 04.20.Cv, 11.10.Lm

**I. INTRODUCTION**

In special relativity theory, the principle of locality has been the subject of a detailed critical analysis, and nonlocal correction terms have been proposed that are induced by sufficiently high *accelerations* [1]. This nonlocal special relativity has been applied to electrodynamics and the Dirac equation. For a discussion of possible experimental tests of the theory, we refer to [1] and the references cited therein.

The next subject to be addressed from this nonlocal point of view is the theory of gravitation. Einstein's theory of general relativity (GR) is, by means of the equivalence principle, heuristically deduced from special relativity. Therefore, if special relativity is generalized to a nonlocal theory, then general relativity theory cannot be exempt from this generalization process. But how should such a generalization be implemented in Einstein's theory? Should one, for example, require the connection to be a nonlocal expression in terms of the metric? No obvious method suggests itself for a direct nonlocal generalization of Einstein's theory. The underlying reason for this is that the precise mathematical form of Einstein's principle of equivalence in GR is *strictly* local; therefore, it cannot be employed together with nonlocal special relativity to arrive at nonlocal gravitation. On the other hand, various heuristic arguments have been advanced in favor of a nonlocal classical theory of gravitation (see, for instance, [2]).

The idea behind this paper exploits the analogy between *gauge theories*. On the one hand we know that electrodynamics, in the framework of a gauged Dirac system, can be understood as a gauge theory of the  $U(1)$  group; see O'Raifeartaigh [3]. On the other hand it is known that a gauge theory of the *translation* group, for spinless matter,

yields a teleparallelism theory of gravity that, for a suitably chosen Lagrangian, is equivalent to Einstein's theory; see, for instance, Nitsch *et al.* [4,5]. Consequently, if we cannot recognize a direct method of generalizing Einstein's theory, it may be helpful to start from this so-called teleparallel equivalent of general relativity (GR<sub>||</sub>) instead. This will indeed provide a way to proceed to a nonlocal extension.

The analogy between these groups is pointed out in the rest of this section and the resulting ansatz for nonlocal gravitation is explained in Sec. II. To simplify matters, we work in the linear approximation and work out the linearized nonlocal gravitational field equations in Sec. III. In Sec. IV, some subtleties are discussed in connection with the use of variational principles in nonlocal field theories. Section V is devoted to an interpretation of these nonlocal equations in terms of the standard linearized general relativity theory but in the presence of "dark matter". The properties of the resulting dark matter are briefly pointed out. A discussion of our results is contained in Sec. VI. Various mathematical details are relegated to the appendixes. In this paper, spacetime indices run from 0 to 3 and units are chosen such that  $c = 1$ . The Minkowski metric tensor is given by  $\text{diag}(1, -1, -1, -1)$ . As indices, Latin letters indicate holonomic coordinate indices, while Greek letters indicate anholonomic Lorentz-frame (tetrad) indices.

**A. Electrodynamics**

The electromagnetic *excitation*  $\mathcal{H}^{ij} = (D, H) = -\mathcal{H}^{ji}$ , a contravariant tensor density, and the *field strength*  $F_{ij} = (E, B) = -F_{ji}$ , a covariant tensor, fulfill the Maxwell equations

$$\partial_j \mathcal{H}^{ij} = \mathcal{J}^i, \quad \partial_{[i} F_{jk]} = 0, \quad (1)$$

with  $\mathcal{J}^i$  as electric current density. Here  $i, j, \dots = 0, 1, 2, 3$

\*hehl@thp.uni-koeln.de

†mashhoonb@missouri.edu

are (holonomic) coordinate indices. The homogeneous part of equations (1) can be solved by the potential ansatz

$$F_{ij} = 2\partial_{[i}A_{j]}. \quad (2)$$

The two Maxwell equations have to be supplemented by a constitutive law that relates the excitation to the field strength. In conventional vacuum electrodynamics, one takes the local, linear, and isotropic ‘‘constitutive law’’

$$\mathcal{H}^{ij} = \langle \sqrt{-g}g^{kl}g^{[i}g^{j]l} \rangle F_{kl} = \sqrt{-g}F^{ij}. \quad (3)$$

This law is equally valid for the flat Minkowski spacetime of special relativity as well as for the curved Riemannian spacetime of general relativity. In the former case, in Cartesian coordinates, the metric  $g_{ij}$  is constant, in the latter case it becomes a field governed by the Einstein equations. If we substitute Eq. (3) into Eq. (1), then the Maxwell equations are expressed in terms of the field strength  $F_{ij}$  alone.

In a nonlocal theory, on the other hand, Eq. (3) is generalized such that the nonlocal constitutive law expresses  $\mathcal{H}^{ij}$  as a nonlocal spacetime relation in terms of  $F_{kl}$ . In fact, it is possible to give an effective interpretation of Mashhoon’s nonlocal electrodynamics in this way [6]. That is, one refers the excitation  $\mathcal{H}^{\alpha\beta} = e_i^\alpha e_j^\beta \mathcal{H}^{ij}$ , the field strength  $F_{\alpha\beta} = e_i^\alpha e_j^\beta F_{ij}$ , and the metric  $g_{\alpha\beta} = e_i^\alpha e_j^\beta g_{ij}$  to noninertial frames  $e_\alpha = e_i^\alpha \partial_i$  that characterize the tetrads of accelerated observers. With respect to such frames, the Maxwell equations read [6,7]

$$\partial_\beta \mathcal{H}^{\alpha\beta} - \frac{1}{2} C_{\beta\gamma}^\alpha \mathcal{H}^{\gamma\beta} + \frac{1}{2} C_{\beta\gamma}^\gamma \mathcal{H}^{\alpha\beta} = \mathcal{J}^\alpha, \quad (4)$$

$$\partial_{[\alpha} F_{\beta\gamma]} - C_{[\alpha\beta}^\delta F_{\gamma]\delta} = 0. \quad (5)$$

Here  $C_{\alpha\beta}^\gamma := 2e_i^\alpha e_j^\beta \partial_{[i} e_{j]}^\gamma$  is the object of anholonomy (‘‘structure constants’’) measuring the noninertial nature of the frame. Thus in this approach to nonlocal electrodynamics, the nonlocal expression that relates the excitation to the field strength is given by [6]

$$\begin{aligned} \mathcal{H}^{\alpha\beta}(x) &= \sqrt{-g} g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta}(x) \\ &+ \int \chi^{\alpha\beta\gamma\delta}(x, x') F_{\gamma\delta}(x') d^4 x'. \end{aligned} \quad (6)$$

Equation (6) represents a new axiom that, together with the Maxwell equations (1), determines a nonlocal theory of electrodynamics.

## B. Translational gauge theory of gravity

Let us now turn to the gauge theory of the translation group, which includes  $\text{GR}_{\parallel}$ . We take the field equations from [8,9]; however, we change some conventions in order to conform with [7,10]. For the tensor calculus background one should compare Schouten [11]. In Appendix A, we provide a brief sketch of translational gauge theory formu-

lated in terms of the calculus of exterior differential forms. Recent works on teleparallelism can be found, amongst others, in the articles of Aldrovandi *et al.* [12], Itin [13,14], Maluf *et al.* [15,16], Mielke [17], Obukhov *et al.* [18–20], So *et al.* [21], and Tung *et al.* [22]; moreover, the two monographs of Blagojević [23] and Ortín [24] should be consulted as well as the references cited therein. The inhomogeneous and the homogeneous gravitational field equations are, respectively, given by

$$\partial_j \mathcal{H}^{ij}{}_\alpha - \mathcal{E}_\alpha^i{}^* = \mathcal{T}_\alpha^i, \quad \partial_{[i} C_{jk]}^\alpha{}^* = 0. \quad (7)$$

The star over the equal sign means that the corresponding equation is only valid in a suitable frame; this will be discussed in Sec. II. The nonlinear correction terms in the inhomogeneous part of Eqs. (7) represent the energy-momentum tensor density of the gravitational gauge field,

$$\mathcal{E}_\alpha^i := -\frac{1}{4} e_i^\alpha (C_{jk}^\beta \mathcal{H}^{jk}{}_\beta) + C_{\alpha k}^\beta \mathcal{H}^{ik}{}_\beta. \quad (8)$$

As source, we have on the right-hand side of the inhomogeneous field equation the energy-momentum tensor density of matter. In analogy with Eq. (2), the field strength  $C_{ij}^\alpha$  is defined in terms of the potential  $e_i^\alpha$  by

$$C_{ij}^\alpha = 2\partial_{[i} e_{j]}^\alpha, \quad (9)$$

so that the homogeneous gravitational field equations (7) are thus identically satisfied.

Clearly, there is a similarity between the electrodynamic and gravitational cases, as indicated in Table I. In electrodynamics we have one potential  $A_i$  as covector, corresponding to the one-parameter  $U(1)$  group; in the gravitational case, because we have four linearly independent translations in spacetime, we have four covectors as potentials  $e_i^\alpha$ , the tetrad components with  $\alpha = \hat{0}, \hat{1}, \hat{2}, \hat{3}$ . Correspondingly, whereas we have six components of the electromagnetic field strength  $F_{ij} = -F_{ji}$ , we have  $4 \times 6$  components of the *gravitational field strength*  $C_{ij}^\alpha = -C_{ji}^\alpha$ . If one compares Eq. (1) with Eq. (7), one recognizes the quasi-Maxwellian nature of the gravitational case, only that we have 4 times more field equations in Eq. (7) and that the gravitational energy-momentum tensor density  $\mathcal{E}_\alpha^i$  emerges because all physical fields including

TABLE I. Analogy between electrodynamics and gravitodynamics. The approximation sign employed in the last column indicates validity only in the *linear* regime.

Expression for	Electro-dynamics	Translational gauge theory
Conservation law	$\partial_i \mathcal{J}^i = 0$	$\partial_i \mathcal{T}_\alpha^i \sim 0$
Inhomogeneous field equations	$\partial_j \mathcal{H}^{ij} = \mathcal{J}^i$	$\partial_j \mathcal{H}^{ij}{}_\alpha \sim \mathcal{T}_\alpha^i$
Force density	$f_i = F_{ij} \mathcal{J}^j$	$f_i \sim C_{ij}^\alpha \mathcal{T}_\alpha^j$
Homogeneous field equations	$\partial_{[i} F_{jk]} = 0$	$\partial_{[i} C_{jk]}^\alpha = 0$
Potentials	$F_{ij} = 2\partial_{[i} A_{j]}$	$C_{ij}^\alpha = 2\partial_{[i} e_{j]}^\alpha$

gravity are gravitationally “charged,” that is, all fields carry energy-momentum.

The analog of Eq. (3), the local, linear, and isotropic constitutive law in  $\text{GR}_{\parallel}$  is [5,8]

$$\mathcal{H}^{ij}{}_{\alpha} \stackrel{*}{=} \frac{e}{\kappa} \left( \frac{1}{2} C_{ij}{}^{\alpha} - C_{\alpha}{}^{[ij]} + 2e^{[i}{}_{\alpha} C^{j]\gamma}{}_{\gamma} \right), \quad (10)$$

where  $e := \det(e_i^{\alpha})$ . Here  $\kappa = 8\pi G$ , where  $G$  is Newton’s constant of gravitation. If Eq. (10) is substituted into the inhomogeneous field equations (7), the resulting equations have been shown to be equivalent to the Einstein equations [8]. Taking the electromagnetic case (6) as a prototype, we tentatively expect that a suitable nonlocal generalization of the constitutive relation (10) would lead to a nonlocal generalization of Einstein’s theory of gravitation. It is interesting to contemplate the nature of this analogy, which is the basis of the present paper. The *linear* and possibly *nonlocal* electrodynamic constitutive law is generally valid for sufficiently weak electromagnetic fields, since the constitutive tensor is assumed to be independent of the field strength. Therefore, we expect that the same holds in the linear approximation for the nonlocal theory of gravitation that we develop below on the basis of this analogy.

Finally, we must mention that this approach to nonlocal gravitation is essentially different from other nonlocal modifications of general relativity; see, for example, [25,26], and the references cited therein.

## II. ANSATZ FOR A NONLOCAL THEORY OF GRAVITY

The arena for teleparallelism is the Weitzenböck spacetime in which we have the orthonormal tetrad (coframe)  $\vartheta^{\alpha} = e_i^{\alpha} dx^i$ . Then the local (anholonomic) metric is  $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$  and we can determine the (holonomic) coordinate components of the metric by

$$g_{ij} = \eta_{\alpha\beta} e_i^{\alpha} e_j^{\beta}. \quad (11)$$

Simple algebra yields  $e := \det(e_i^{\alpha}) = \sqrt{-g}$ .

If a coframe is a coordinate frame, then the object of anholonomy

$$C^{\alpha} := d\vartheta^{\alpha} \quad (12)$$

vanishes; in general,  $C^{\alpha}$  is a 2-form which decomposes according to  $C^{\alpha} = \frac{1}{2} C_{ij}{}^{\alpha} dx^i \wedge dx^j$  such that Eq. (12) in components results in Eq. (9).

The connection 1-form  $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha} = \Gamma_i{}^{\alpha\beta} dx^i$  in a Weitzenböck spacetime is teleparallel. That is, we can introduce a suitable global Cartesian tetrad frame with respect to which the connection vanishes

$$\Gamma^{\alpha\beta} \stackrel{*}{=} 0. \quad (13)$$

Throughout the rest of our paper we will work in a global frame that obeys Eq. (13). Then all covariant derivatives reduce to partial derivatives and we can simplify our work

considerably. In particular, the torsion of the Weitzenböck spacetime becomes

$$T^{\alpha} := D\vartheta^{\alpha} = d\vartheta^{\alpha} + \Gamma_{\beta}{}^{\alpha} \wedge \vartheta^{\beta} \stackrel{*}{=} d\vartheta^{\alpha} = C^{\alpha}, \quad (14)$$

or, equivalently,

$$\begin{aligned} T_{ij}{}^{\alpha} &= 2D_{[i}e_{j]}{}^{\alpha} = 2\partial_{[i}e_{j]}{}^{\alpha} - 2\Gamma_{[ij]}{}^k e_k{}^{\alpha} \stackrel{*}{=} 2\partial_{[i}e_{j]}{}^{\alpha} \\ &= C_{ij}{}^{\alpha}. \end{aligned} \quad (15)$$

Accordingly, the *gravitational field strength*, in the special “gauge” (13), is represented by the object of anholonomy  $C^{\alpha}$ . This concludes the geometrical setup of a Weitzenböck spacetime and we will henceforth drop the star over the equal sign.

We now turn to physics in  $\text{GR}_{\parallel}$ , which is determined via the variation of the action  $\delta S = 0$ , where

$$S = \int (\mathcal{L}_g + \mathcal{L}_m) d^4x. \quad (16)$$

Here  $\mathcal{L}_g$  and  $\mathcal{L}_m$  are, respectively, the gravitational and matter Lagrangian densities. Out of the field strength  $C^{\alpha}$ , we can construct a gravitational Lagrangian  $\mathcal{L}_g$  that is—in analogy with electrodynamics—*quadratic* in the field strength. The explicit form of  $\mathcal{L}_g$  can be left open for the moment. However, we can introduce quite generally the components of the *excitation*  $\mathcal{H}^{ij}{}_{\alpha}$  that are related to those of the field strength by

$$\mathcal{H}^{ij}{}_{\alpha} := -2 \frac{\partial \mathcal{L}_g}{\partial C_{ij}{}^{\alpha}}. \quad (17)$$

In terms of differential forms we have  $H_{\alpha} = \frac{1}{2} H_{ij\alpha} dx^i \wedge dx^j$ , with  $\mathcal{H}^{ij}{}_{\alpha} = \frac{1}{2} \epsilon^{ijkl} H_{kl\alpha}$ , where  $\epsilon^{ijkl}$  is the totally antisymmetric Levi-Civita symbol with  $\epsilon^{0123} = 1$ . Because of the presumed quadratic nature of  $\mathcal{L}_g$  and the Euler theorem on homogeneous functions, we find

$$\mathcal{L}_g = \frac{1}{2} C_{ij}{}^{\alpha} \frac{\partial \mathcal{L}_g}{\partial C_{ij}{}^{\alpha}} = -\frac{1}{4} \mathcal{H}^{ij}{}_{\alpha} C_{ij}{}^{\alpha}, \quad (18)$$

which expresses the gravitational Lagrangian in terms of the excitation  $\mathcal{H}^{ij}{}_{\alpha}$  and the field strength  $C_{ij}{}^{\alpha}$  such that  $\mathcal{H}^{ij}{}_{\alpha}$  is linear in  $C_{ij}{}^{\alpha}$ .

The field equations of  $\text{GR}_{\parallel}$  given in Eq. (7) follow from the variational principle of stationary action (16) with  $\mathcal{T}^{\alpha}{}^i := \delta \mathcal{L}_m / \delta e_i^{\alpha}$  as the source. We note that, in linear approximation, the gravitational energy-momentum complex (8) will vanish and only the first term of the left-hand side of the inhomogeneous part of Eqs. (7) will survive.

Having thus established the geometry and the Lagrange-Noether machinery for the teleparallel theory of gravitation, we turn next to the explicit choice of the gravitational Lagrangian. General relativity is recovered via Eq. (10), which may be written as

$$\mathcal{H}^{ij}{}_{\alpha} = \left\langle \frac{e}{\kappa} g^{kl} g^{jl} \eta_{\alpha\beta} \right\rangle \mathfrak{C}_{kl}{}^{\beta}, \quad (19)$$

with the *modified* field strength

$$\mathfrak{C}_{ij}{}^{\alpha} := \frac{1}{2} C_{ij}{}^{\alpha} - C^{\alpha}{}_{[ij]} + 2e_{[i}{}^{\alpha} C_{j]\gamma}{}^{\gamma}. \quad (20)$$

In particular, along the same line of thought as in [6], we wish to consider a nonlocal constitutive law for gravitation. To prepare the ground, let us introduce the invariant proper infinitesimal distance  $ds$  between two neighboring events in Weitzenböck spacetime

$$ds^2 = g_{ij} dx^i \otimes dx^j \quad (21)$$

and define a geodesic between two fixed events  $P'$  and  $P$  to be the path that is an extremum of the spacetime distance between  $P'$  and  $P$ ,

$$\delta \int_{P'}^P ds = 0. \quad (22)$$

This path is given by the geodesic equations

$$\frac{d^2 x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \quad (23)$$

where

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l}) \quad (24)$$

are the Christoffel symbols.

We assume that two causally separated events are connected by a *unique* timelike or null geodesic; more generally, in the spacetime region under consideration, there exists a unique geodesic joining every pair of events. It then proves useful to employ the *world function*  $\Omega$ , which denotes half the square of the proper distance from  $P':x' = \xi(\zeta_0)$  to  $P:x = \xi(\zeta_1)$  along the geodesic path  $x^i = \xi^i(\zeta)$ . That is, we define [27]

$$\Omega(x, x') = \frac{1}{2} (\zeta_1 - \zeta_0) \int_{\zeta_0}^{\zeta_1} g_{ij} \frac{d\xi^i}{d\zeta} \frac{d\xi^j}{d\zeta} d\zeta. \quad (25)$$

It turns out that  $\Omega$  is independent of the affine parameter  $\zeta$ ; moreover, the integrand in Eq. (25) is constant by Eq. (23). The main properties of  $\Omega(x, x')$  are summarized in Appendix B.

To distinguish coordinate indices that refer to  $x$  from those that refer to  $x'$ , we henceforth use indices  $a, b, c, \dots$  at  $x$  and  $i, j, k, \dots$  at  $x'$ . Thus we define

$$\Omega_a(x, x') = \frac{\partial \Omega}{\partial x^a}, \quad \Omega_i(x, x') = \frac{\partial \Omega}{\partial x'^i}, \quad (26)$$

and note that covariant derivatives at  $x$  and  $x'$  commute for any bitensor. It follows from the results of Appendix B that

$$2\Omega = g^{ab} \Omega_a \Omega_b = g^{ij} \Omega_i \Omega_j. \quad (27)$$

Differentiating this equation, we find that  $\Omega_{ai}(x, x') =$

$\Omega_{ia}(x, x')$  are smooth dimensionless two-point tensors such that

$$\lim_{x' \rightarrow x} \Omega_{ai}(x, x') = -g_{ai}(x). \quad (28)$$

Thus a possible nonlocal generalization of Eq. (19) is given by

$$\begin{aligned} \mathcal{H}^{ab}{}_{c}(x) = & -\frac{1}{\kappa} \sqrt{-g(x)} \int U(x, x') \Omega^{ai} \Omega^{bj} \Omega_{ck} \chi(x, x') \\ & \times \mathfrak{C}_{ij}{}^k(x') \sqrt{-g(x')} d^4 x', \end{aligned} \quad (29)$$

where  $U(x, x')$  is unity except when  $x'$  is in the future of  $x$ , in which case  $U$  vanishes; in Minkowski spacetime, this means that events with  $x'^0 > x^0$  are excluded from the domain of integration in Eq. (29). Moreover,  $\chi(x, x')$  is a scalar given, for instance, by

$$\chi(x, x') = \delta(x - x') + \hat{K}(x, x'), \quad (30)$$

where the Dirac delta function  $\delta(x - x')$  is defined via

$$\int \delta(x - x') \varphi(x') \sqrt{-g(x')} d^4 x' = \varphi(x) \quad (31)$$

for any smooth function  $\varphi(x)$ . The scalar kernel  $\hat{K}(x, x')$  in Eq. (30) denotes the nonlocal deviation from a local constitutive law. The main nonlocal constitutive relation (29) can therefore be expressed as

$$\begin{aligned} \mathcal{H}^{ab}{}_{c}(x) = & \frac{1}{\kappa} \sqrt{-g(x)} \left[ \mathfrak{C}^{ab}{}_{c}(x) - \int U(x, x') \Omega^{ai} \Omega^{bj} \Omega_{ck} \right. \\ & \left. \times \hat{K}(x, x') \mathfrak{C}_{ij}{}^k(x') \sqrt{-g(x')} d^4 x' \right], \end{aligned} \quad (32)$$

where the nonlocal deviation from general relativity is made explicit. More complicated nonlocal constitutive relations are certainly possible; however, Eq. (32) is the simplest one involving an unknown scalar kernel  $\hat{K}$ .

We emphasize that the domain of applicability of Eq. (32) is not physically restricted; in particular, a nonlocal gravity theory of this type is not *directly* related to nonlocal special relativity [1]. That is, if the whole heuristic machinery of Einstein's principle of equivalence could be employed in this case, one would have to conclude—on the basis of nonlocal special relativity—that gravity should be nonlocal for sufficiently high gravitational “accelerations.” However, Einstein's principle of equivalence cannot be used in this way, as its precise formulation in GR is the very embodiment of locality. Thus nonlocal gravity is in this sense *decoupled* from nonlocal special relativity.

It is important to note that our ansatz (32) is *nonlinear* as well as nonlocal, since, among other things, the metric tensor (11) is quadratic in the gravitational potentials  $e_i{}^{\alpha}$ . The scalar kernel  $\hat{K}(x, x')$  could be given, for instance, by  $\hat{K}^{-1} = \Omega^2 + L_0^4$ , where  $L_0$  is a constant characteristic length. It could also involve the Weitzenböck invariants [8]

$$C_{ijk}C^{ijk}, \quad C_{kji}C^{ijk}, \quad C_{ij}{}^j C_{k}{}^{ik} \quad (33)$$

at  $x$  and  $x'$  as well as scalars formed from the covariant derivatives of  $\Omega(x, x')$ . To illustrate the latter, consider, for example,  $\Omega_a$  and  $\Omega_i$  in Eq. (26). The vector  $\Omega^a$  is tangent at  $x$  to the geodesic connecting  $x'$  to  $x$  and the length of  $\Omega^a$  is equal to that of this geodesic by Eq. (27); in Minkowski spacetime,  $\Omega^a = x^a - x'^a$ . Similarly, the vector  $\Omega^i$  has the same length, is tangent to the geodesic at  $x'$ , and is directed away from  $x$ ; in Minkowski spacetime,  $\Omega^i = x'^i - x^i$ . Hence one can form coordinate scalars  $\Omega^a e_a^\alpha(x)$  and  $\Omega^i e_i^\alpha(x')$ , which turn out to be of particular significance in Sec. V. This brief discussion indicates that many options indeed exist for generating a scalar kernel  $\hat{K}(x, x')$  in our ansatz (32).

In the following section, the nonlocal theory of gravity is treated in the linear approximation; therefore, Eq. (32) will be employed in linearized form. In particular,  $U(x, x')\hat{K}(x, x')$  will be evaluated in Minkowski spacetime, since Weitzenböck spacetime in the special gauge (13) reduces to Minkowski spacetime when the gravitational potentials  $e_i^\alpha$  reduce to  $\delta_i^\alpha$ . In this *limit*, we define

$$K(x, x') = U(x, x')\hat{K}(x, x'). \quad (34)$$

### III. GRAVITATIONAL FIELD EQUATIONS IN LINEAR APPROXIMATION

The gravitational field equations are obtained from the variation of the action,  $\int \mathcal{L} d^4x$ , where  $\mathcal{L}$  is the Lagrangian density given by the sum of the corresponding quantities for gravitation  $\mathcal{L}_g$  and matter  $\mathcal{L}_m$ , namely,  $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m$ , such that

$$\frac{\delta \mathcal{L}_m}{\delta e_i^\alpha} = \sqrt{-g} T_\alpha^i. \quad (35)$$

Here  $T_\alpha^i$  is the energy-momentum tensor of matter, so that  $\sqrt{-g} T_\alpha^i = \mathcal{T}_\alpha^i$  given in Eq. (7). The gravitational part  $\mathcal{L}_g$  is given by Eq. (18) in analogy with electrodynamics [7]. For the nonlocal theory under consideration,  $\mathcal{H}^{ij}_\alpha$  is now given by Eq. (32); however, to simplify matters, we will work in the linear approximation. That is, we assume that

$$e_i^\alpha = \delta_i^\alpha - \psi^i_\alpha, \quad (36)$$

where the nonzero components of  $\psi^i_\alpha$  are such that  $|\psi^i_\alpha| \ll 1$ . In this approximation, the holonomic coordinate indices and the anholonomic tetrad indices are indistinguishable. It then follows from Eq. (32) and Appendix B that

$$\begin{aligned} \mathcal{L}_g = & -\frac{1}{4\kappa} \mathfrak{G}^{ij}_k(x) C_{ij}{}^k(x) - \frac{1}{4\kappa} C_{ij}{}^k(x) \\ & \times \int K(x, y) \mathfrak{G}^{ij}_k(y) d^4y, \end{aligned} \quad (37)$$

where  $K$  is the scalar kernel in the weak-field approxima-

tion. A discussion of action principles in nonlocal field theories is contained in Edelen [28]. The expressions for  $C_{ij}{}^k$  and  $\mathfrak{G}^{ij}_k$  in terms of  $\psi^i_j$  can be easily determined. We find from Eq. (36) that

$$e_i^\alpha = \delta_i^\alpha + \psi^\alpha_i, \quad (38)$$

so that from (9),

$$C_{ij}{}^k = 2\psi^k_{[i,j]}. \quad (39)$$

Moreover,

$$g_{ij} = \eta_{ij} + h_{ij}, \quad h_{ij} = 2\psi_{(ij)}. \quad (40)$$

It follows from (20) that

$$\begin{aligned} \mathfrak{G}^{ij}_k = & -\frac{1}{2}(h_k{}^i{}^j - h_k{}^j{}^i) + \psi^{[ij]}{}_{,k} + \delta_k^i(\psi^{,j} - \psi_l{}^j{}^l) \\ & - \delta_k^j(\psi^{,i} - \psi_l{}^i{}^l), \end{aligned} \quad (41)$$

where  $\psi = \eta_{ij}\psi^{ij}$ .

Let us first examine the local part of the Lagrangian (37). It follows from

$$\delta \mathcal{L}_g(\text{local}) = -\frac{1}{4\kappa} \mathfrak{G}^{ij}_k(\delta C_{ij}{}^k) - \frac{1}{4\kappa} (\delta \mathfrak{G}^{ij}_k) C_{ij}{}^k \quad (42)$$

that the first term in Eq. (42) is in effect given by

$$-\frac{1}{2\kappa} \mathfrak{G}^{ij}_{k,j}(\delta \psi^k_i) \quad (43)$$

via Eq. (39) and integration by parts, assuming that  $\delta \psi^k_i$  vanishes on the boundary, and hence neglecting such boundary terms in Eq. (43). Applying this same procedure to the second term in Eq. (42), we find after a detailed calculation that the result is again given by

$$-\frac{1}{2\kappa} \mathfrak{G}^{ij}_{k,j}(\delta \psi^k_i), \quad (44)$$

so that in the *absence* of the nonlocal term ( $K = 0$ ) the variation of the action can be expressed in the linear approximation as

$$\int \left( -\frac{1}{\kappa} \mathfrak{G}^{ij}_{k,j} + T_k{}^i \right) (\delta \psi^k_i) d^4x = 0. \quad (45)$$

Thus for  $K = 0$ , the gravitational field equations are  $\partial_j \mathfrak{G}^{ij}_k = \kappa T_k{}^i$ ; moreover, we prove in Appendix C that these are precisely Einstein's field equations according to the standard general linear approximation scheme. A general proof, valid in the nonlinear regime, is already contained in [8]. We note that  $\partial_i T_k{}^i = 0$  is implied by the field equations.

We now turn to the nonlocal part of the Lagrangian. In this case, the corresponding two terms in the variation of the gravitational action are

$$\begin{aligned} \delta \mathcal{L}_g(\text{nonlocal}) = & -\frac{1}{4\kappa} [\delta C_{ij}{}^k(x)] \int K(x, y) \mathfrak{G}^{ij}{}_k(y) d^4y \\ & - \frac{1}{4\kappa} C_{ij}{}^k(x) \int K(x, y) [\delta \mathfrak{G}^{ij}{}_k(y)] d^4y. \end{aligned} \quad (46)$$

As before, the first term in Eq. (46) is easily shown to lead to

$$-\frac{1}{2\kappa} [\delta \psi^k{}_i(x)] \int \frac{\partial K(x, y)}{\partial x^j} \mathfrak{G}^{ij}{}_k(y) d^4y. \quad (47)$$

For the second term in Eq. (46), we can write its contribution to the variation of the total action as

$$-\frac{1}{4\kappa} \iint K(x, y) C_{ij}{}^k(x) [\delta \mathfrak{G}^{ij}{}_k(y)] d^4y d^4x. \quad (48)$$

The domains of integration in the double integral are the same; therefore, it is possible to switch  $x$  and  $y$  in Eq. (48) to get

$$-\frac{1}{4\kappa} \iint K(y, x) C_{ij}{}^k(y) [\delta \mathfrak{G}^{ij}{}_k(x)] d^4y d^4x. \quad (49)$$

Hence the second term in the nonlocal part of the Lagrangian is equivalent to

$$-\frac{1}{4\kappa} [\delta \mathfrak{G}^{ij}{}_k(x)] \int K(y, x) C_{ij}{}^k(y) d^4y. \quad (50)$$

Now applying to Eq. (50) the same procedure we used in the derivation of Eq. (44), we find after a detailed calculation that the result is

$$-\frac{1}{2\kappa} [\delta \psi^k{}_i(x)] \int \frac{\partial K(y, x)}{\partial x^j} \mathfrak{G}^{ij}{}_k(y) d^4y. \quad (51)$$

Let us define the symmetric kernel  $\bar{K}$  by

$$\bar{K}(x, y) = \frac{1}{2} [K(x, y) + K(y, x)]. \quad (52)$$

Then, combining Eqs. (47) and (51), we find

$$\delta \mathcal{L}_g(\text{nonlocal}) = -\frac{1}{\kappa} [\delta \psi^k{}_i(x)] \int \frac{\partial \bar{K}(x, y)}{\partial x^j} \mathfrak{G}^{ij}{}_k(y) d^4y. \quad (53)$$

This means that the field equations of the nonlocal theory in the linear approximation are given by

$$\partial_j \mathfrak{G}^{ij}{}_k + \int \frac{\partial \bar{K}(x, y)}{\partial x^j} \mathfrak{G}^{ij}{}_k(y) d^4y = \kappa T_k{}^i. \quad (54)$$

We note that  $\partial_i T_k{}^i = 0$  is still identically satisfied in the nonlocal theory. For a general symmetric kernel  $\bar{K}$ , the energy-momentum tensor  $T_{ij}$  is not symmetric in general, since the nonlocal part in Eq. (54) is not in general symmetric. This poses no basic difficulty as  $T_{ij}$ , given in general by Eq. (35), is not symmetric in general. On the other hand, the requirement that  $T_{ij}$  be symmetric (as in

GR, for instance) would impose restrictions on the kernel  $\bar{K}$ .

A natural interpretation of the basic nonlocal gravitational field equations (54) can be obtained by considering the nonlocal term to be an effective source for dark matter and hence moving it to the right-hand side of Eq. (54). In this way, one can recover Einstein's theory but with dark matter. We explore this possibility in the rest of this paper.

Consider, for instance, the possibility that  $\bar{K}$  is an even function of  $x - y$ ; that is,

$$\bar{K}(x, y) = F(x - y), \quad (55)$$

where  $F(z) = F(-z)$ . Then,  $\partial \bar{K} / \partial x^j = -\partial \bar{K} / \partial y^j$  and Eq. (54) can be written as

$$\partial_j \mathfrak{G}^{ij}{}_k(x) + \int \bar{K}(x, y) \frac{\partial \mathfrak{G}^{ij}{}_k(y)}{\partial y^j} d^4y = \kappa T_k{}^i, \quad (56)$$

which implies, via Appendix C, that  $T_{ki} = T_{ik}$ . Here we have assumed that the derivatives of  $\psi^k{}_i$  vanish on the boundary of the spacetime region under consideration.

#### IV. NONLOCAL FIELD EQUATIONS

The treatment of the previous section has been based on the assumption that nonlocal gravitational field equations should be obtained from an action principle that incorporates the nonlocal constitutive relation (32). Alternatively, this relation could be employed in the field equations of teleparallelism. The purpose of this section is to address certain subtleties involved in these possibilities.

##### A. The ‘‘general’’ field equations of teleparallelism

In electrodynamics the Maxwell equations in (1) can be derived from electric charge and magnetic flux conservation; see [7]. No action principle is necessary. If one considers, for example, local and linear magnetoelectric matter, the constitutive law is  $\mathcal{H}^{ij} = \frac{1}{2} \chi^{ijkl} F_{kl}$ , with a constitutive tensor density  $\chi^{ijkl} = -\chi^{jikl} = -\chi^{ijlk}$  that has 36 independent components. This model can be derived from a Lagrangian, provided the constitutive tensor obeys additionally the Onsager-type relation  $\chi^{ijkl} = \chi^{klij}$ ; that is, if 15 of its 36 components vanish. Nevertheless, even if this condition is not fulfilled, one can still find reasonable applications in physics, though in this case irreversible processes have to be taken into account.

The situation is similar for the energy-momentum tensor density of the electromagnetic field. Starting from the Lorentz force density, substitution of the inhomogeneous Maxwell equations and partial integration result in the (canonical) Minkowski energy-momentum tensor density

$$\mathcal{T}{}^j{}_i = -\frac{1}{4} \delta_i^j F_{kl} \mathcal{H}^{kl} + F_{ik} \mathcal{H}^{jk}. \quad (57)$$

Apparently no Lagrangian is necessary for the derivation of this expression [7,29].

The gravitational field equations of the translational gauge theory, as formulated in Eq. (7), should also have general validity within this framework, independently of the existence of the Lagrangian. For the homogeneous equations (7) this is apparent since they represent just the first Bianchi identities in a Weitzenböck spacetime. For the inhomogeneous equations (7) this can be seen as follows: We start with the *energy-momentum conservation law for matter* in the linear approximation  $\partial_i \mathcal{T}_\alpha^i \sim 0$ . We can “solve” it by  $\mathcal{T}_\alpha^i = \partial_j \tilde{\mathcal{H}}_\alpha^{ij} - \tilde{\mathcal{E}}_\alpha^i$  and  $\tilde{\mathcal{H}}_\alpha^{ij} = -\tilde{\mathcal{H}}_\alpha^{ji}$ , with some unknown correction term  $\tilde{\mathcal{E}}_\alpha^i$ . These are already the inhomogeneous field equations. We just have to determine  $\tilde{\mathcal{E}}_\alpha^i$  explicitly. Clearly it adds to  $\mathcal{T}_\alpha^i$ , namely,  $\partial_j \tilde{\mathcal{H}}_\alpha^{ij} = \mathcal{T}_\alpha^i + \tilde{\mathcal{E}}_\alpha^i$ ; then, it is natural to interpret  $\tilde{\mathcal{E}}_\alpha^i$  as the energy-momentum tensor density of the gravitational field. In analogy with Eq. (57) we expect it to have the form of Eq. (8); we now drop the tildes from  $\tilde{\mathcal{E}}_\alpha^i$  and  $\tilde{\mathcal{H}}_\alpha^{ij}$ . Therefore the inhomogeneous field equations extracted from the energy-momentum conservation law of matter in this heuristic manner have exactly the same form as (7) with (8). Thus we may postulate (7) as the general field equations valid independently of the existence of a Lagrangian.

By contrast, if we have a Lagrangian, the field equations turn out to be [8]

$$\partial_j \mathcal{H}^{ij}_\alpha - (e^i_\alpha \mathcal{L}_g + C_{\alpha k}^\beta \mathcal{H}^{ik}_\beta) = \mathcal{T}_\alpha^i. \quad (58)$$

If  $\mathcal{L}_g = -\frac{1}{4} C_{ij}^\alpha \mathcal{H}^{ij}_\alpha$  is substituted into (58), we recover the inhomogeneous part of (7). Consequently, we have shown that the general field equations (7) with (8) are correct if a Lagrangian exists. Accordingly, they are certainly one consistent and reasonable generalization of the Lagrangian-based equations (58).

### B. Ambiguity in the field equations

We now wish to address a certain ambiguity that is encountered in implementing a nonlocal constitutive law as in our present work. A *local* constitutive law relating the excitation  $\mathcal{H}^{ij}_k$  to the gravitational field strength  $C^{ij}_k$  can be used in the derivation of the field equations of the theory in either of two *equivalent* ways: it could be employed in the Lagrangian (18) that then results, via the variational principle of stationary action, in the desired field equations or, alternatively, it could be directly substituted in the field equations (7). These two methods in general produce different results for a *nonlocal* constitutive law, however. It is interesting to illustrate this point for the case at hand in the linear approximation. In this regime, we have the linear constitutive law

$$\kappa \mathcal{H}^{ij}_\alpha(x) = \mathfrak{G}^{ij}_\alpha(x) + \int K(x, y) \mathfrak{G}^{ij}_\alpha(y) d^4y, \quad (59)$$

which, when inserted in Eq. (18) results, as described in detail in Sec. III, in the nonlocal field equations (54).

Alternatively, however, we can equally well substitute the constitutive relation (59) in Eq. (7), which reduces in the linear regime to the new nonlocal field equations

$$\partial_j \mathfrak{G}^{ij}_k + \int \frac{\partial K(x, y)}{\partial x^j} \mathfrak{G}^{ij}_k(y) d^4y = \kappa T_k^i. \quad (60)$$

The two nonlocal field equations only differ in their kernels: Eq. (54) involves  $\bar{K}$ , while Eq. (60) involves  $K$ . The question of whether such nonlocal equations as (60) can be derived from a variational principle is beyond the scope of our investigation; for a related study in connection with acceleration-induced nonlocality see [30].

In a similar way as in Sec. III, one can recover a direct nonlocal generalization of Einstein’s theory with a symmetric energy-momentum tensor of matter by assuming that  $K(x, y)$  is a function of  $x - y$ . In this case, Eq. (60) can be written as

$$\partial_j \mathfrak{G}^{ij}_k(x) + \int K(x, y) \frac{\partial \mathfrak{G}^{ij}_k(y)}{\partial y^j} d^4y = \kappa T_k^i, \quad (61)$$

which should be compared and contrasted with Eq. (56). These inequivalent nonlocal field equations both admit a natural interpretation in terms of dark matter as described in detail in the following section; however, Eq. (61) has an advantage over Eq. (56) in that it involves a causal kernel  $K$ , while in the symmetric kernel  $\bar{K}$  of Eq. (56) past and future are treated in the same manner.

## V. DARK MATTER

Let us now consider an important consequence of our nonlocal equations for the gravitational field. To keep our discussion general, we start with an equation that has the same form as Eqs. (54) and (60), except for a kernel  $\mathcal{K}$ , which we take to be equal to  $\bar{K}$  or  $K$ , respectively, depending on whether one adopts the Lagrangian-based approach of Sec. III or the more direct approach of Sec. IV. We henceforth assume, for simplicity, that in the former case  $\bar{K}(x, y)$  is an even function of  $x - y$  and in the latter case  $K(x, y)$  is simply a function of  $x - y$ . Then, the arguments of the previous sections imply that the nonlocal modification of Einstein’s gravitational field equations for a symmetric energy-momentum tensor  $T_{ij}$  may be expressed as

$$G_{ij}(x) + \int \mathcal{K}(x, y) G_{ij}(y) d^4y = \kappa T_{ij}(x), \quad (62)$$

where  $G_{ij}$  is given by

$$G^i_j = \partial_k \mathfrak{G}^{ik}_j \quad (63)$$

and represents Einstein’s tensor in the linear approximation.

The nonlocal term in Eq. (62) can be interpreted in terms of an *effective* energy-momentum tensor for dark matter by writing Eq. (62) as

$$G_{ij}(x) = \kappa[T_{ij}(x) + \mathfrak{T}_{ij}(x)], \quad (64)$$

where the dark-matter component is given by the symmetric energy-momentum tensor

$$\mathfrak{T}_{ij}(x) = -\frac{1}{\kappa} \int \mathcal{K}(x, y) G_{ij}(y) d^4y. \quad (65)$$

Equation (65) may be written in a more transparent form by making use of the methods of Appendix D. That is, suppose that the integral equation (62) can be solved by the method of successive substitutions (cf. Appendix D) and that the resulting infinite series is uniformly convergent. Then, as shown in Appendix D, it is possible to introduce a reciprocal scalar kernel  $\mathcal{R}$  such that

$$G_{ij}(x) = \kappa T_{ij}(x) + \kappa \int \mathcal{R}(x, y) T_{ij}(y) d^4y. \quad (66)$$

Thus the dark-matter component in Eq. (64) is given by

$$\mathfrak{T}_{ij}(x) = \int \mathcal{R}(x, y) T_{ij}(y) d^4y, \quad (67)$$

which is the integral transform of matter  $T_{ij}$  by the kernel  $\mathcal{R}(x, y)$ . It is clear from Eq. (67) that in our model, dark matter should be quite similar in its characteristics to actual matter; for instance, for dust, the corresponding dark matter is pressure-free, while for radiation with  $T_k^k = 0$ , the “dark” energy-momentum tensor  $\mathfrak{T}_{ij}$  is traceless as well.

The reciprocal kernel  $\mathcal{R}$  is given as an infinite series in terms of iterated kernels constructed from  $\mathcal{K}(x, y)$ ; that is,

$$-\mathcal{R}(x, y) = \sum_{n=1}^{\infty} \mathcal{K}_n(x, y). \quad (68)$$

To ensure causality, it is useful to assume further that

$$\mathcal{K}(x, y) = \delta(x^0 - y^0) P(\mathbf{x}, \mathbf{y}), \quad (69)$$

where  $P$  is a function of  $\mathbf{x} - \mathbf{y}$ . Then, all iterated kernels as well as the reciprocal kernel are also of this general form; indeed,

$$\mathcal{K}_n(x, y) = \delta(x^0 - y^0) P_n(\mathbf{x}, \mathbf{y}), \quad (70)$$

where  $P_1(\mathbf{x}, \mathbf{y}) = P(\mathbf{x}, \mathbf{y})$  and

$$P_{n+1}(\mathbf{x}, \mathbf{y}) = - \int P(\mathbf{x}, \mathbf{z}) P_n(\mathbf{z}, \mathbf{y}) d^3z. \quad (71)$$

Therefore,

$$\mathcal{R}(x, y) = \delta(x^0 - y^0) Q(\mathbf{x}, \mathbf{y}), \quad (72)$$

where  $Q$  is reciprocal to  $P$ ,

$$-Q(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} P_n(\mathbf{x}, \mathbf{y}). \quad (73)$$

Moreover,  $P_n$  and  $Q$  are functions of  $\mathbf{x} - \mathbf{y}$ , since the integration in Eq. (71) extends over the entire three-dimensional Euclidean space. Thus dark matter is the

convolution of matter and the reciprocal kernel in this case. The mathematical implications of this fact are treated in Appendix E.

It is interesting to consider the confrontation of our nonlocal theory in the linear approximation, represented by Eqs. (62)–(67), with experimental data. Let us first determine the Newtonian limit of our nonlocal gravity. This follows directly from Eq. (64) in the standard manner and we find the Poisson equation

$$\nabla^2 \Phi = 4\pi G(\rho + \rho_D), \quad (74)$$

where  $\Phi$  is the Newtonian potential. Here  $\rho$  is the density of matter and  $\rho_D$ ,

$$\rho_D(t, \mathbf{x}) = \int Q(\mathbf{x}, \mathbf{y}) \rho(t, \mathbf{y}) d^3y, \quad (75)$$

is the corresponding density of dark matter. We note that in the general linear approximation of our theory the kernel  $\mathcal{K}(x, y)$  and its reciprocal  $\mathcal{R}(x, y)$  are given functions in Minkowski spacetime and are thus *independent* of any particular physical system. The solar-system tests of general relativity imply that our dark matter must be a rather small fraction of the actual matter in the Solar System. This can be simply arranged with a suitable choice of the universal reciprocal kernel  $Q$ .

It is interesting to illustrate these considerations in the case of the problem of dark matter in spiral galaxies [31–37]. Imagine, for instance, the circular motion of stars in the disk of a spiral galaxy. Beyond the galactic bulge, the Newtonian acceleration of gravity for each star at radius  $|\mathbf{x}|$  is given by  $v_0^2/|\mathbf{x}|$  toward the center of the galaxy. Here  $v_0$  is the constant “asymptotic” speed of stars in accordance with the observed rotation curves of spiral galaxies. It follows from Poisson’s equation that the effective density of dark matter is essentially given by  $v_0^2/(4\pi G|\mathbf{x}|^2)$ . Comparing this result with Eq. (75) and setting

$$\rho(t, \mathbf{y}) = M\delta(\mathbf{y}), \quad \rho_D(t, \mathbf{x}) = \frac{v_0^2}{4\pi G|\mathbf{x}|^2}, \quad (76)$$

where  $M$  is the effective mass of the galaxy and the dimensions of the galactic bulge have been ignored, we find

$$Q(\mathbf{x}, \mathbf{0}) = \frac{v_0^2}{4\pi GM|\mathbf{x}|^2}. \quad (77)$$

The reciprocal kernel  $Q$  is a function of  $\mathbf{x} - \mathbf{y}$ ; therefore, it follows from Eq. (77) that

$$Q(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi\lambda} \frac{1}{|\mathbf{x} - \mathbf{y}|^2} \quad (78)$$

with a universal length parameter  $\lambda = GM/v_0^2$ . Thus taking due account of the observed rotation curves of spiral galaxies, Eqs. (74) and (75) with kernel (78) imply

$$\nabla^2\Phi = 4\pi G \left[ \rho(t, \mathbf{x}) + \frac{1}{4\pi\lambda} \int \frac{\rho(t, \mathbf{y}) d^3y}{|\mathbf{x} - \mathbf{y}|^2} \right]. \quad (79)$$

It is important to point out in passing a defect of the specific form of Eq. (78): the total amount of dark matter associated with a nonzero matter density  $\rho$  is infinite (see Appendix E); therefore, kernel (78) is too simple to be quite adequate for the task at hand.

For a point mass  $m$  with  $\rho(t, \mathbf{x}) = m\delta(\mathbf{x})$ , the Newtonian potential given by Eq. (79) can be expressed as

$$\Phi(t, \mathbf{x}) = -\frac{Gm}{|\mathbf{x}|} + \frac{Gm}{\lambda} \ln\left(\frac{|\mathbf{x}|}{\lambda}\right). \quad (80)$$

The observational data for spiral galaxies indicate that  $\lambda$  is of the order of a kpc, so that the logarithmic term in Eq. (80) is essentially negligible in the Solar System. Moreover, the universality of the kernel in the linear approximation implies, via  $\lambda = GM/v_0^2$ , that for spiral galaxies  $M \propto v_0^2$ , since  $\lambda$  should be independent of any particular physical system.

It is remarkable that in the simple considerations regarding Eqs. (79) and (80)—based on our linear approximation scheme—we have recovered the significant proposal put forward by Tohline and further developed by Kuhn *et al.* [38,39] to solve the dark-matter problem by a natural modification of the Newtonian law of gravitation. An interesting discussion of the Tohline-Kuhn scheme is contained in the review paper of Bekenstein [39]. Despite various successes, the main drawback of this approach appears to be the violation of the Tully-Fisher relation, which implies that  $M \propto v_0^4$  [38,39]. To agree with the empirical Tully-Fisher law, it seems necessary to go beyond the linear approximation scheme.

Within the general framework of this work, but going beyond the linear approximation as well as the specific constitutive model employed thus far, it may be possible in principle to have a relation of the general form of Eq. (67) with a kernel that is highly dependent upon the particular system under consideration. This could provide a natural way to interpret the observational evidence for dark matter as the nonlocal manifestation of the gravitational interaction. It remains to elucidate the physical origin of the constitutive kernel that has been the starting point of our investigation.

## VI. CONCLUSION

To develop a nonlocal generalization of GR, it proves useful to approach Einstein's theory via its equivalent within teleparallelism gravity, namely, GR<sub>||</sub>. Therefore, we work in Weitzenböck spacetime with a *tetrad* field  $e_i^\alpha$  and its dual  $e^j_\beta$ , a *metric*  $g_{ij}$ , and a flat *connection*  $\Gamma_i^{\alpha\beta} = -\Gamma_i^{\beta\alpha}$  that is chosen to vanish globally ( $\Gamma_i^{\alpha\beta} = 0$ ). The gravitational field strength is given by  $C_{ij}^\alpha$  and the modified field strength by  $\mathfrak{C}_{ij}^\alpha$ . In this framework, which is

capable of nonlocal generalization, GR<sub>||</sub> corresponds to a specific gravitational Lagrangian. Working with a nonlocal “constitutive” kernel  $K(x, x')$  in the linear approximation, we construct an explicit nonlocal generalization of Einstein's theory of gravitation that is consistent with causality. This theory can be reformulated as linearized general relativity but with dark matter, which mimics the contribution of nonlocal gravity. We find that the effective energy-momentum tensor of dark matter is simply the integral transform of the energy-momentum tensor of matter.

The application of our nonlocal model in the linear approximation to the dark-matter problem in spiral galaxies is in conflict with the empirical Tully-Fisher relation. It is possible that the situation can be significantly improved with a suitable choice for the kernel in the nonlinear regime. However, a more basic theory is needed to determine the nonlocal constitutive kernel from first principles. This is a task for the future.

## ACKNOWLEDGMENTS

We are grateful to Jacob Bekenstein (Jerusalem), Yakov Itin (Jerusalem), José Maluf (Brasilia), Eckehard Mielke (Mexico City), and Yuri Obukhov (Moscow) for helpful comments.

## APPENDIX A: TRANSLATIONAL GAUGE THEORY IN EXTERIOR CALCULUS

The coframe 1-form  $\vartheta^\alpha = e_i^\alpha dx^i$  represents the gravitational potential; here  $i, j, \dots = 0, 1, 2, 3$  are (holonomic) coordinate indices and  $\alpha, \beta, \dots = \hat{0}, \hat{1}, \hat{2}, \hat{3}$  are (anholonomic) frame indices. The frame  $e_\beta = e^j_\beta \partial_j$  is dual to the coframe, that is,  $e_i^\alpha e^i_\beta = \delta_\beta^\alpha$  and  $e_i^\alpha e^j_\alpha = \delta_i^j$ . The field strength of gravitation is the object of anholonomy 2-form  $C^\alpha := d\vartheta^\alpha$ .

Spacetime is described by a Weitzenböck geometry with a teleparallel connection 1-form  $\Gamma_\alpha^\beta = \Gamma_{i\alpha}^\beta dx^i$  and with the curvature 2-form

$$R_\alpha^\beta := d\Gamma_\alpha^\beta - \Gamma_\alpha^\gamma \wedge \Gamma_\gamma^\beta = \frac{1}{2} R_{ij\alpha}^\beta dx^i \wedge dx^j \quad (A1)$$

that vanishes

$$R_\alpha^\beta = 0. \quad (A2)$$

We decompose the connection into a Riemannian (Levi-Civita) part and the contortion,

$$\Gamma_\alpha^\beta = \tilde{\Gamma}_\alpha^\beta - K_\alpha^\beta, \quad (A3)$$

with the following definitions of the torsion 2-form and the contortion 1-form, respectively,

$$T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta, \quad T^\alpha =: K^\alpha_\beta \wedge \vartheta^\beta. \quad (A4)$$

As in a Euclidean space, one can likewise choose in a Weitzenböck space suitable frames such that the connection vanishes everywhere:

$$\Gamma_{\alpha}{}^{\beta} = \Gamma_{i\alpha}{}^{\beta} dx^i \stackrel{*}{=} 0. \quad (\text{A5})$$

Equation (A5) should be understood as a choice of a certain gauge, which is possible since the curvature  $R_{\alpha}{}^{\beta}$  vanishes everywhere. Many calculations are simplified in this gauge.

The Lagrangian 4-form is the sum of a gravitational part and a matter part

$$L = L_g(\vartheta^\alpha, C^\alpha) + L_m(\psi, d\psi, \vartheta^a). \quad (\text{A6})$$

The field equation reads

$$dH_\alpha - E_\alpha = \frac{\delta L_m}{\delta \vartheta^\alpha} = \Sigma_\alpha. \quad (\text{A7})$$

Here we have the 2-form  $H_\alpha := -\partial L_g / \partial C^\alpha = \frac{1}{2} H_{ij\alpha} dx^i \wedge dx^j$  as the excitation and the 3-form  $\Sigma_\alpha$  as the energy-momentum density of matter, whereas the 3-form  $E_\alpha$  represents the energy-momentum density of the gravitational field [10]

$$E_\alpha := e_\alpha \rfloor L_g + (e_\alpha \rfloor C^\beta) \wedge H_\beta. \quad (\text{A8})$$

Equations (A7) with (A8) represent the Lagrangian-based form of the field equations. If we express the gravitational energy-momentum 3-form  $E_\alpha$  in terms of a Minkowski type current [or, equivalently, if we substitute  $L_g = -\frac{1}{2} C^\alpha \wedge \mathcal{H}_\alpha$  into Eq. (A8)], we find the general field equations

$$dH_\alpha - \frac{1}{2} [H_\beta \wedge (e_\alpha \rfloor C^\beta) - C^\beta \wedge (e_\alpha \rfloor H_\beta)] \stackrel{*}{=} \Sigma_\alpha. \quad (\text{A9})$$

This is the exterior-form version of the inhomogeneous field equations (7) with Eq. (8).

Usually  $H_\alpha$  is, like in electrodynamics, linear in the field strength

$$H_\alpha = \frac{g_{\alpha\beta}}{\kappa} \star (a_1^{(1)} C^\beta + a_2^{(2)} C^\beta + a_3^{(3)} C^\beta). \quad (\text{A10})$$

Here  ${}^{(I)}C^\beta$  are the different irreducible pieces of  $C^\beta$  and  $\star$  represents the Hodge star. The ‘‘Einsteinian’’ choice for  ${}^{(I)}C^\beta$ ,  $I = 1, 2, 3$ , turns out to be

$$a_1 = -1, \quad a_2 = 2, \quad a_3 = \frac{1}{2}. \quad (\text{A11})$$

In this case, Eq. (A10) is distinguished from all linear Lagrangians in that it becomes *locally* Lorentz covariant.

## APPENDIX B: PROPERTIES OF THE WORLD FUNCTION $\Omega$

Consider a variation of Eq. (25) that changes the end-points, then

$$\delta\Omega(x, x') = (\zeta_1 - \zeta_0) \left[ g_{ij} \frac{d\xi^j}{d\zeta} \delta\xi^i \right]_{\zeta_0}^{\zeta_1}. \quad (\text{B1})$$

On the other hand,

$$\delta\Omega = \frac{\partial\Omega}{\partial x^a} \delta x^a + \frac{\partial\Omega}{\partial x'^i} \delta x'^i, \quad (\text{B2})$$

so that

$$\begin{aligned} \frac{\partial\Omega}{\partial x^a} &= (\zeta_1 - \zeta_0) g_{ab}(x) \frac{dx^b}{d\zeta}, \\ \frac{\partial\Omega}{\partial x'^i} &= -(\zeta_1 - \zeta_0) g_{ij}(x') \frac{dx'^j}{d\zeta}. \end{aligned} \quad (\text{B3})$$

It is possible to see from Eq. (23) that the integrand in Eq. (25) is indeed constant; therefore,

$$\begin{aligned} \Omega(x, x') &= \frac{1}{2} (\zeta_1 - \zeta_0)^2 g_{ab}(x) \frac{dx^a}{d\zeta} \frac{dx^b}{d\zeta} \\ &= \frac{1}{2} (\zeta_1 - \zeta_0)^2 g_{ij}(x') \frac{dx'^i}{d\zeta} \frac{dx'^j}{d\zeta}. \end{aligned} \quad (\text{B4})$$

It follows from Eqs. (B3) and (B4) that Eq. (27) is satisfied; moreover,  $\Omega = 0$  for a null geodesic,  $\Omega = \frac{1}{2} \tau^2$  for a time-like geodesic of length  $\tau$  and  $\Omega = -\frac{1}{2} \sigma^2$  for a spacelike geodesic of length  $\sigma$ .

In Minkowski spacetime  $\Omega$  is given by

$$\Omega(x, x') = \frac{1}{2} \eta_{ij} (x'^i - x^i) (x'^j - x^j). \quad (\text{B5})$$

According to our convention,  $\eta_{ij} = \text{diag}(1, -1, -1, -1)$ ; hence,

$$\Omega(x, x') = \frac{1}{2} [(t' - t)^2 - (\mathbf{x}' - \mathbf{x})^2]. \quad (\text{B6})$$

In this case, we find that

$$\Omega_{ai} = \frac{\partial^2 \Omega(x, x')}{\partial x^a \partial x'^i} = -\eta_{ai}, \quad (\text{B7})$$

while

$$\Omega_{ab} = \frac{\partial^2 \Omega}{\partial x^a \partial x^b} = \eta_{ab}, \quad \Omega_{ij} = \frac{\partial^2 \Omega}{\partial x'^i \partial x'^j} = \eta_{ij}. \quad (\text{B8})$$

## APPENDIX C: SYMMETRY OF THE TENSOR

$$\partial_j \mathcal{G}_i{}^j{}_k$$

The purpose of this appendix is to show that  $\partial_j \mathcal{G}_i{}^j{}_k$  is a symmetric tensor, so that in the field equations

$$\partial_j \mathcal{G}^{ij}{}_k = \kappa T_k{}^i \quad (\text{C1})$$

the energy-momentum tensor of the source is symmetric ( $T_{ik} = T_{ki}$ ). In fact Eq. (C1) is identical to Einstein’s field equations in the linear approximation.

It follows from Eq. (41) that

$$\partial_j \mathcal{G}^{ij}_k = -\frac{1}{2}(\square h_k^i - h_{k,j}^{ij}) + \frac{1}{2}h^{ij}_{,kj} - \frac{1}{2}\delta_k^i h^{jl}_{,jl} + \delta_k^i \square \psi - \psi_{,k}^i, \quad (\text{C2})$$

where we have used the relation  $2\psi^{jl}_{,jl} = h^{jl}_{,jl}$ . Define the trace-reversed gravitational potentials  $\bar{h}_{ik}$  as

$$\bar{h}_{ik} = h_{ik} - \frac{1}{2}\eta_{ik}h, \quad (\text{C3})$$

where  $h = \eta_{ij}h^{ij} = 2\psi$ . Then, replacing  $h_{ij}$  by  $\bar{h}_{ij} + \eta_{ij}\psi$  everywhere in Eq. (C2) results in

$$\partial_j \mathcal{G}^j_k = -\frac{1}{2}\square \bar{h}_k + \frac{1}{2}\bar{h}_{k,j}^j + \frac{1}{2}\bar{h}_{i,j}^j - \frac{1}{2}\eta_{ki}\bar{h}^{jl}_{,jl}, \quad (\text{C4})$$

which is manifestly symmetric in  $i$  and  $k$ . From Eqs. (C1) and (C4), we get

$$-\square \bar{h}_{ik} + \bar{h}_{i,j}^j + \bar{h}_{k,j}^j - \eta_{ik}\bar{h}^{jl}_{,jl} = 2\kappa T_{ik}. \quad (\text{C5})$$

These are exactly the same as Einstein's field equations in the linear approximation.

#### APPENDIX D: LIOUVILLE-NEUMANN METHOD OF SUCCESSIVE SUBSTITUTIONS

Consider a linear integral equation of the second kind given by

$$\phi(x) = f(x) + \int_a^b k(x, y)\phi(y)dy, \quad (\text{D1})$$

where  $a$  and  $b$  are constants. We seek a solution of this Fredholm equation by the method of successive substitutions. That is, we replace  $\phi$  in the integrand by its value given by Eq. (D1). Repeating this process eventually leads to an infinite series of the type

$$\phi(x) = f(x) + \int_a^b k(x, y)f(y)dy + \int_a^b \int_a^b k(x, z)k(z, y)f(y)dydz + \dots \quad (\text{D2})$$

If this series is uniformly convergent, we have a solution of the integral equation (D1). This solution is unique in the space of real continuous functions on the interval  $[a, b]$ ; a generalization of this result to the space of square-integrable functions is contained in Tricomi [40].

Let us define the iterated kernels  $k_n$ ,  $n = 1, 2, \dots$ , by

$$k_1(x, y) = k(x, y), \quad k_{n+1}(x, y) = \int_a^b k(x, z)k_n(z, y)dz. \quad (\text{D3})$$

These functions occur in the infinite series of Eq. (D2). We define the *reciprocal* kernel  $r(x, y)$  such that

$$-r(x, y) = \sum_{n=1}^{\infty} k_n(x, y). \quad (\text{D4})$$

Then, Eq. (D2) can be written as

$$f(x) = \phi(x) + \int_a^b r(x, y)f(y)dy. \quad (\text{D5})$$

It is clear from Eqs. (D1) and (D5) that the kernels  $k$  and  $r$  are reciprocal of each other.

Let us note here some properties of these kernels. It follows from Eq. (D3) that

$$k_{n+p}(x, y) = \int_a^b k_n(x, z)k_p(z, y)dz, \quad (\text{D6})$$

where  $p = 1, 2, \dots$ . Using Eqs. (D3) and (D4) we find that

$$\begin{aligned} k(x, y) + r(x, y) &= \int_a^b k(x, z)r(z, y)dz \\ &= \int_a^b r(x, z)k(z, y)dz. \end{aligned} \quad (\text{D7})$$

Moreover, if  $k$  is symmetric,  $k(x, y) = k(y, x)$ , then all iterated kernels as well as  $r(x, y)$  are likewise symmetric.

#### APPENDIX E. CONVOLUTION KERNELS

Suppose that the spatial kernels in Sec. V are such that

$$P(\mathbf{x}, \mathbf{y}) = p(\mathbf{x} - \mathbf{y}), \quad Q(\mathbf{x}, \mathbf{y}) = q(\mathbf{x} - \mathbf{y}). \quad (\text{E1})$$

The purpose of this appendix is to point out the consequences of the convolution theorem for a system of density  $\rho$  such that the density of dark matter  $\rho_D$  is a convolution of  $\rho$  and  $q$ . In general, the spatial kernels  $p$  and  $q$  could depend on the characteristics of the particular system under consideration. The main results of interest here are Eqs. (69)–(75). Let us note that the particular solution of Poisson's equation (74) is given by

$$\Phi(t, \mathbf{x}) = -G \int \frac{\rho(t, \mathbf{y}) + \rho_D(t, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d^3y, \quad (\text{E2})$$

provided  $\Phi$  vanishes sufficiently fast at spatial infinity. It follows from Eq. (75) that this Newtonian potential can also be expressed as

$$\Phi(t, \mathbf{x}) = -G \int S(\mathbf{x} - \mathbf{y})\rho(t, \mathbf{y})d^3y, \quad (\text{E3})$$

where

$$S(\mathbf{r}) = \frac{1}{|\mathbf{r}|} + \int \frac{q(\mathbf{z})d^3z}{|\mathbf{r} - \mathbf{z}|}. \quad (\text{E4})$$

Let us assume that the functions of interest here can be expressed as Fourier integrals; that is, for a function  $f$ ,

$$\nu f(\mathbf{r}) = \int \hat{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}d^3k, \quad (\text{E5})$$

where  $\hat{f}(\mathbf{k})$  is given by

$$\nu \hat{f}(\mathbf{k}) = \int f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r. \quad (\text{E6})$$

For the sake of simplicity, we have introduced here a constant parameter  $\nu$ ,

$$\nu = (2\pi)^{3/2}. \quad (\text{E7})$$

We are interested in Fourier integral transforms of *real* functions; therefore,  $\hat{f}^*(\mathbf{k}) = \hat{f}(-\mathbf{k})$ .

It follows from the convolution theorem for Fourier integrals that Eq. (75) implies

$$\hat{\rho}_D(\mathbf{k}) = \nu \hat{\rho}(\mathbf{k}) \hat{q}(\mathbf{k}). \quad (\text{E8})$$

Here  $\hat{q}(\mathbf{k})$  is a dimensionless function such that

$$\nu \hat{q}(\mathbf{0}) = \frac{M_D}{M}, \quad (\text{E9})$$

where  $M = \nu \hat{\rho}(\mathbf{0})$  is the mass of the system under consideration and  $M_D = \nu \hat{\rho}_D(\mathbf{0})$  is the corresponding dark mass. Consider, for instance, kernel (78) that is associated with the discussion of the rotation curves of spiral galaxies in Sec. V; that is,

$$q(\mathbf{r}) = \frac{1}{4\pi\lambda} \frac{1}{|\mathbf{r}|^2}. \quad (\text{E10})$$

It follows that

$$\hat{q}(\mathbf{k}) = \frac{1}{4(2\pi)^{1/2}\lambda} \frac{1}{|\mathbf{k}|}. \quad (\text{E11})$$

Thus for any  $M > 0$ ,  $M_D = \infty$ .

Once  $q$  is determined from Eq. (E8), it is possible to work out its reciprocal kernel. In fact, it follows from Eqs. (71) and (73) that  $\hat{p}_1(\mathbf{k}) = \hat{p}(\mathbf{k})$ ,

$$\hat{p}_{n+1}(\mathbf{k}) = -\nu \hat{p}(\mathbf{k}) \hat{p}_n(\mathbf{k}), \quad (\text{E12})$$

and

$$-\hat{q}(\mathbf{k}) = \sum_{n=1}^{\infty} \hat{p}_n(\mathbf{k}). \quad (\text{E13})$$

It is then straightforward to show that

$$-\hat{q}(\mathbf{k}) = \frac{\hat{p}(\mathbf{k})}{1 + \nu \hat{p}(\mathbf{k})}. \quad (\text{E14})$$

Therefore,

$$-\hat{p}(\mathbf{k}) = \frac{\hat{q}(\mathbf{k})}{1 + \nu \hat{q}(\mathbf{k})}. \quad (\text{E15})$$

These results are consistent with the fact that  $p$  and  $q$  are reciprocal of each other.

- 
- [1] B. Mashhoon, *Ann. Phys. (Berlin)* **17**, 705 (2008).
  - [2] B. Mashhoon, *Ann. Phys. (Berlin)* **16**, 57 (2007).
  - [3] L. O’Raifeartaigh, *The Dawning of Gauge Theory* (Princeton University Press, Princeton, NJ, 1997).
  - [4] J. Nitsch, in *Proceedings of the 6th Course of the School of Cosmology and Gravitation on Spin, Torsion, Rotation, and Supergravity, Erice, Italy, 1979*, edited by P.G. Bergmann and V. de Sabbata (Plenum, New York, 1980), p. 63.
  - [5] J. Nitsch and F. W. Hehl, *Phys. Lett.* **90B**, 98 (1980).
  - [6] U. Muench, F. W. Hehl, and B. Mashhoon, *Phys. Lett. A* **271**, 8 (2000).
  - [7] F. W. Hehl and Yu. N. Obukhov, *Foundations of Classical Electrodynamics: Charge, Flux, and Metric* (Birkhäuser, Boston, MA, 2003).
  - [8] F. W. Hehl, J. Nitsch, and P. Von der Heyde, in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980), Vol. 1, p. 329; see, in particular, the Appendix.
  - [9] F. W. Hehl, in *Proceedings of the 6th Course of the School of Cosmology and Gravitation on Spin, Torsion, Rotation, and Supergravity, Erice, Italy, 1979*, edited by P.G. Bergmann and V. de Sabbata (Plenum, New York, 1980), p. 5.
  - [10] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne’eman, *Phys. Rep.* **258**, 1 (1995).
  - [11] J. A. Schouten, *Tensor Analysis for Physicists* (Dover, Mineola, NY, 1989), 2nd ed., reprinted.
  - [12] R. Aldrovandi, J. G. Pereira, and K. H. Vu, *Gen. Relativ. Gravit.* **36**, 101 (2004).
  - [13] Y. Itin, *Gen. Relativ. Gravit.* **34**, 1819 (2002).
  - [14] Y. Itin, in *Classical and Quantum Gravity Research Progress*, edited by M. N. Christiansen and T. K. Rasmussen (Nova Science Publishers, Hauppauge, NY, 2008).
  - [15] J. W. Maluf, M. V. O. Veiga, and J. F. da Rocha-Neto, *Gen. Relativ. Gravit.* **39**, 227 (2007).
  - [16] J. W. Maluf and S. C. Ulhoa, *Phys. Rev. D* **78**, 047502 (2008).
  - [17] E. W. Mielke, *Ann. Phys. (N.Y.)* **219**, 78 (1992).
  - [18] Yu. N. Obukhov and J. G. Pereira, *Phys. Rev. D* **67**, 044016 (2003).
  - [19] Y. N. Obukhov and G. F. Rubilar, *Phys. Rev. D* **73**, 124017 (2006).
  - [20] Y. N. Obukhov, G. F. Rubilar, and J. G. Pereira, *Phys. Rev. D* **74**, 104007 (2006).
  - [21] L. L. So and J. M. Nester, in *Proceedings of the 10th Marcel Grossmann Meeting, Rio de Janeiro, Brazil, 2003*, edited by M. Novello *et al.* (World Scientific, Hackensack, NJ, 2005), Part B, p. 1498.
  - [22] R. S. Tung and J. M. Nester, *Phys. Rev. D* **60**, 021501 (1999).

- [23] M. Blagojević, *Gravitation and Gauge Symmetries* (IoP, Bristol, UK, 2002).
- [24] T. Ortín, *Gravity and Strings* (Cambridge University Press, Cambridge, UK, 2004).
- [25] M.E. Soussa and R.P. Woodard, *Classical Quantum Gravity* **20**, 2737 (2003).
- [26] A. O. Barvinsky, *Phys. Lett. B* **572**, 109 (2003).
- [27] J.L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1971).
- [28] D.G.B. Edelen, *Nonlocal Variations and Local Invariance of Fields* (American Elsevier, New York, 1969).
- [29] H. Minkowski, *Königliche Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse. Nachrichten* (1908), p. 53.
- [30] C. Chicone and B. Mashhoon, *Ann. Phys. (Berlin)* **16**, 811 (2007).
- [31] V. C. Rubin and W. K. Ford, *Astrophys. J.* **159**, 379 (1970).
- [32] M. S. Roberts and R. N. Whitehurst, *Astrophys. J.* **201**, 327 (1975).
- [33] Y. Sofue and V. Rubin, *Annu. Rev. Astron. Astrophys.* **39**, 137 (2001).
- [34] C. Carignan, L. Chemin, W.K. Huchtmeier, and F.J. Lockman, *Astrophys. J. Lett.* **641**, L109 (2006); P. Salucci *et al.*, *Mon. Not. R. Astron. Soc.* **378**, 41 (2007).
- [35] J.D. Bekenstein, *Phys. Rev. D* **70**, 083509 (2004); **71**, 069901(E) (2005).
- [36] J.D. Bekenstein, *Contemp. Phys.* **47**, 387 (2006).
- [37] J.P. Bruneton, S. Liberati, L. Sindoni, and B. Famaey, arXiv:0811.3143.
- [38] J.E. Tohline, in *IAU Symposium 100, Internal Kinematics and Dynamics of Galaxies*, edited by E. Athanassoula (Reidel, Dordrecht, 1983), p. 205; J.R. Kuhn and L. Kruglyak, *Astrophys. J.* **313**, 1 (1987)
- [39] J.D. Bekenstein, in *Second Canadian Conference on General Relativity and Relativistic Astrophysics*, edited by A. Coley, C. Dyer, and T. Tupper (World Scientific, Singapore, 1988), p. 68.
- [40] F. G. Tricomi, *Integral Equations* (Interscience, New York, 1957).