ALMOST EVERYWHERE CONVERGENCE FOR MODIFIED BOCHNER RIESZ MEANS AT THE CRITICAL INDEX FOR $p \geq 2$

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by

MARCO ANNONI

Dr. Loukas Grafakos, Dissertation Supervisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by Marco Annoni,

a candidate for the degree of Doctor of Philosophy in Mathematics and hereby certify that, in their opinion, it is worthy of acceptance.

Professor Loukas Grafakos (Chair of the Committee)

Professor Fritz Gesztesy

Professor Steven Hofmann

Professor Carlo Morpurgo

Professor Giovanni Vignale (External member)

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BOCHNER RIESZ MEANS AT THE CRITICAL INDEX FOR $p \geq 2$

Marco Annoni

Dr. Loukas Grafakos, Dissertation Supervisor

ABSTRACT

For $\lambda, \gamma > 0$, let the function $m_{\lambda,\gamma} : \mathbf{R^n} \to [0, \infty)$ be defined by

$$m_{\lambda,\gamma}(\xi) = \frac{(1 - |\xi|^2)_+^\lambda}{(1 - \log(1 - |\xi|^2))^\gamma}.$$

For $R > 0$, we define the modified Bochner-Riesz mean $B_R^{\lambda,\gamma}$ by:

$$B_R^{\lambda,\gamma}(f)(x) = \int_{\mathbf{R^n}} m_{\lambda,\gamma}\left(\frac{\xi}{R}\right) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

We have the following theorem concerning it.

Theorem 1.1 *For every $\lambda > 0$ such that $1 + 2\lambda < n$, let $p_\lambda = \frac{2n}{n-2\lambda-1}$. If $\gamma > \frac{1}{p'_\lambda} + \frac{1}{2}$ (where $\frac{1}{p_\lambda} + \frac{1}{p'_\lambda} = 1$) and $f \in L^{p_\lambda}(\mathbf{R^n})$, then we have:*

$$\lim_{R \to \infty} B_R^{\lambda,\gamma}(f)(x) = f(x),$$

for almost every $x \in \mathbf{R^n}$.

Chapter 1

Introduction

A fundamental problem in harmonic analysis is whether for a given class of functions on $\mathbf{R}^n$ (such as integrable functions to a given power), Fourier inversion holds with respect to certain means. Examples of such means provide the Bochner-Riesz operators B_R^λ defined for $\lambda > 0$ and $R > 0$ by

$$B_R^\lambda(f)(x) = \int_{\mathbf{R}^n} \widehat{f}(\xi) m_R^\lambda(\xi) e^{2\pi i \langle \xi, x\rangle} d\xi$$

where $m_R^\lambda : \mathbf{R}^n \to [0, \infty)$ denotes the Fourier multiplier

$$m_R^\lambda(\xi) = \left(1 - |\xi|^2\right)_+^\lambda .$$

Here $\langle \cdot , \cdot \rangle$ denotes the inner product in $\mathbf{R}^n$ and g_+ denotes the positive part of a function g. The study of Bochner-Riesz means originated in the article of Bochner [2], who proved that they are bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for every p, $1 \leq p \leq \infty$, if $\lambda > \frac{n-1}{2}$ and thus Fourier inversion holds in the sense that for a given $f \in L^p(\mathbf{R}^n)$, $B_R^\lambda(f)$ converge to f in L^p

as $R \to \infty$ for $\lambda > \frac{n-1}{2}$. Carbery [3] showed that the maximal operator $B_*^\lambda(f) := \sup_{R>0} |B_R^\lambda(f)|$ is bounded on $L^p(\mathbf{R}^2)$ when $\lambda > 0$ and $2 \le p < \frac{4}{1-2\lambda}$, obtaining the convergence $B_R^\lambda(f) \to f$ almost everywhere for $f \in L^p(\mathbf{R}^2)$. For $n \ge 3$, $2 \le p < \frac{2n}{n-2\lambda-1}$, and $\lambda \ge \frac{n-1}{2(n+1)}$ the same result was obtained by Christ [5]. Carbery, Rubio de Francia and Vega [4] obtained the almost everywhere convergence of the Bochner-Riesz means in the range $2 \le p < \frac{2n}{n-2\lambda-1}$ and $\lambda > 0$. Tao obtained both boundedness and unboundedness results for the maximal operator B_*^λ on L^p when $1 < p < 2$. In [12], he proved that B_*^λ does not map $L^p(\mathbf{R}^n)$ to weak $L^p(\mathbf{R}^n)$ if $1 < p < 2$, $n \ge 2$ and $0 < \lambda < \frac{2n-1}{2p} - \frac{n}{2}$. In [11], he obtained boundedness for B_*^λ on $L^p(\mathbf{R}^2)$ whenever $1 < p < 2$ for an open range of pairs $(1/p, \lambda)$ that lie below the line $\lambda = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$.

Here is the precise formulation of the result contained in [4]:

Theorem A. (Carbery, Rubio de Francia, Vega) *Let $\lambda > 0$, and $n \ge 2$. Then for all $f \in L^p(\mathbf{R}^n)$ with $2 \le p < \frac{2n}{n-2\lambda-1}$ we have:*

$$\lim_{R\to\infty} B_R^\lambda(f)(x) = f(x)$$

for almost all $x \in \mathbf{R}^n$.

In this paper, we consider the following modified Bochner-Riesz multipliers introduced by A. Seeger in [9]:

$$m_{\lambda,\gamma}(\xi) = \frac{(1-|\xi|^2)_+^\lambda}{(1-\log(1-|\xi|^2))^\gamma}, \tag{1.1}$$

for $\gamma > 0$, and the corresponding modified Bochner-Riesz means

$$B_R^{\lambda,\gamma}(f)(x) = \int_{\mathbf{R}^n} m_{\lambda,\gamma}\left(\frac{\xi}{R}\right)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi. \tag{1.2}$$

Observe that $m_{\lambda,\gamma}$ is just slightly smoother than m_1^λ; that is, it's smoother than m_1^λ, but less smooth than $m_1^{\lambda+\varepsilon}$, for any $\varepsilon > 0$. Indeed, while we know that m_1^λ is an L^p multiplier in $\mathbf{R}^2$ if and only if $\frac{4}{3+2\lambda} < p < \frac{4}{1-2\lambda} =: p_\lambda$, A. Seeger ([9],[10]) proved that $m_{\lambda,\gamma}$ is also a L^{p_λ} multiplier in $\mathbf{R}^2$ if and only if $\gamma > \frac{1}{p_\lambda'}$. In fact, we can use duality to rephrase his result as follows:

Theorem B. (A. Seeger) *Suppose that* $0 < \lambda < \frac{1}{2}$ *and* $p_\lambda = \frac{4}{1-2\lambda}$. *Then* $m_{\lambda,\gamma}$ *is a Fourier multiplier of* $L^{p_\lambda}(\mathbf{R}^2)$ *if and only if* $\gamma > \frac{1}{p_\lambda'}$.

Likewise, one expects the modified means to behave better than the Bochner-Riesz means, that is, to converge almost everywhere, i.e.,

$$\lim_{R\to\infty} B_R^{\lambda,\gamma}(f)(x) = f(x)$$

for almost every $x \in \mathbf{R}^n$ even on the critical line $p = \frac{2n}{n-2\lambda-1}$, for some values of γ and all f in $L^p(\mathbf{R}^n)$.

The goal of the present paper is to extend Theorem A for the modified

Bochner-Riesz means $B_R^{\lambda,\gamma}(f)$ to the critical index $p = p_\lambda := \frac{2n}{n-2\lambda-1}$. The following theorem is the main result of this work:

Theorem 1.0.1. *For every $\lambda > 0$ such that $1 + 2\lambda < n$, let $p_\lambda = \frac{2n}{n-2\lambda-1}$. Then for every $f \in L^{p_\lambda}(\mathbf{R}^n)$ and every $\gamma > \frac{1}{p'_\lambda} + \frac{1}{2}$ (where $\frac{1}{p_\lambda} + \frac{1}{p'_\lambda} = 1$), we have that*

$$\lim_{R\to\infty} B_R^{\lambda,\gamma}(f)(x) = f(x)$$

for almost every $x \in \mathbf{R}^n$.

The proof of Theorem 1.0.1 follows closely the idea developed in [4], but deviates from that in certain points, in view of technical issues arising, at the critical index, from the need to work with weights that are not Riesz potentials. The dilation property (that is, the homogeneity) of the Riesz potentials has been used in [4] to justify few identities that significantly simplify the computations through the whole proof.

In the following paragraphs of this introduction, we are going to explain why we need to work with such weights, and which properties of them will be proved and used in the proof of Theorem 1.0.1.

Inspired by the strategy used in [4], we will look for a weight w on $\mathbf{R}^n$ (in the paper, $w_{\lambda\mu}$) that decays fast enough at infinity to have

$$L^{p_\lambda} \subseteq L^2 + L^2(w) \tag{1.3}$$

but slowly enough to have also

$$\|B_*^{\lambda,\gamma}(f)\|_{L^2(w)} \leq C\|f\|_{L^2(w)} \tag{1.4}$$

for some positive constant C independent of f, where the maximal operator $B_*^{\lambda,\gamma}$ is (naturally) defined by:

$$B_*^{\lambda,\gamma}(f)(x) = \sup_{R>0} |B_R^{\lambda,\gamma}(f)(x)| = \sup_{R>0} \left| \int_{\mathbf{R}^n} m_{\lambda,\gamma}\left(\frac{\xi}{R}\right) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \right|. \tag{1.5}$$

If we can find a weight w satisfying (1.4), then it's enough to observe that the proof in [4] can be recasted to obtain

$$\|B_*^{\lambda,\gamma}(f)\|_{L^2(dx)} \leq C\|f\|_{L^2(dx)}, \tag{1.6}$$

(where dx denotes the Lebesgue measure) in order to conclude that $B_R^{\lambda,\gamma}(f)$ converges almost everywhere to f for every $f \in L^2(w)$ and also for every $f \in L^2$. If w satisfies the condition in (1.3) too, then this guarantees almost everywhere convergence for every $f \in L^p$. Unfortunately, no weight of the form $w(x) = |x|^{-\alpha}$ satisfies both (1.3) and (1.4), in that we would need $\alpha > n\left(1 - \frac{2}{p_\lambda}\right)$ in order for w to satisfy (1.3), and $\alpha < n\left(1 - \frac{2}{p_\lambda}\right)$ in order for w to satisfy (1.4). So, we will have to work with a weight that doesn't dilate as nicely.

In order to prove that the Fourier transform is bounded between certain weighted L^2 spaces, we need w to be comparable to another weight w', with the property that w' is smooth on $\mathbf{R}^n \setminus \{0\}$ and $|\widehat{w'}|$ is bounded above by another suitable weight. After introducing the weights w and w', we will show that they have the required properties by making use of the formula for the Fourier transform of radial functions, some accurate asymptotic estimates of the Bessel functions (see [13]), some simple estimate from calculus (see Lemma (3.1.9, page 47) and Lemma (3.1.10, page 50)), iterated integration by parts, and an analytic continuation argument.

The next property we need on w is that there exists a function $u : \mathbf{R}^n \setminus \{0\} \to [0, \infty)$ (in the paper, $u_{\lambda,\mu}$) such that the function defined by

$$x \mapsto \int_{\mathbf{R}^n} |e^{i\langle x,y\rangle} - 1|^N \; u(y)dy$$

is comparable to $\frac{1}{w}$ for some positive even integer N. After defining u, we will show that it has the desired properties by using an appropriate splitting of $\mathbf{R}^n$ into slabs and by using comparability estimates on each of these.

Also, we need to prove that the following inequality holds for every $t > 0$:

$$\int_{||t\xi|-1|<\varepsilon} |\widehat{f}(\xi)|^2 d\xi \leq C_{n,\lambda,\mu} \; \omega_{\lambda\mu}(t)\varepsilon \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w(x)} \tag{1.7}$$

for every function f in the appropriate space, every $\varepsilon \in (0, 2)$, $w(x) = \omega(|x|)$ $(x \in \mathbf{R}^n \setminus \{0\})$ and for some positive constant $C_{n,\lambda,\mu}$. It's in the proof of this inequality that the smooth weight w' introduced above plays a role.

At last, we need to prove this other inequality:

$$\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2 \frac{d\xi}{(1+|t\xi|)^M} \leq C_{n,\lambda,\mu,M}\ \omega(t) \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w(x)}, \tag{1.8}$$

for all f in the appropriate space, all $t > 0$, every $M \geq 2n$ and some constant $C_{n,\lambda,\mu,M}$ that also depends on M. An attempt of using the results of [1] to prove this last inequality fails to provide the crucial factor $\omega(t)$ on the right hand side. So, instead we will use (1.7) and an appropriate partition of $\mathbf{R}^n$ into annuli in order to prove (1.8).

Chapter 2

The maximal operator is well defined

Let $w_{\lambda\mu}$ be defined as in (3.6, page 29). In this section we will find an upperbound for $|\widehat{m_{\lambda,\gamma}}|$ and we will use it to prove that, for every $R > 0$, $0 < \lambda < \frac{n-1}{2}$, $2\gamma - 1 > \mu > 0$ and every $f \in L^2(w_{\lambda\mu})$, the function $B_R^{\lambda,\gamma}(f) : \mathbf{R}^n \to \mathbf{C}$ is well defined everywhere (in fact, it's continuous on $\mathbf{R}^n$).

Therefore, the evaluation $B_*^{\lambda,\gamma}(f)$ of the maximal operator $B_*^{\lambda,\gamma}$ (introduced in (1.5, page 5)) at the function $f : \mathbf{R}^n \to \mathbf{C}$ is well defined if $f \in L^2(w_{\lambda\mu})$.

Additionally, for any $x \in \mathbf{R}^n$ and any function $f \in L^2(w_{\lambda\mu})$, the map $R \mapsto B_R^{\lambda,\gamma}(f)(x)$ is continuous from $\mathbf{R}^+$ to $\mathbf{C}$.

We begin by introducing the Bessel functions and stating a lemma on their asymptotics. This lemma (2.0.5, page 9) can be found in [13] and will be used here and in section 3.2.

Definition 2.0.2. *Let* $\nu \in \mathbf{C}$ *satisfy* $\operatorname{Re}\nu > -\frac{1}{2}$. *The Bessel function* J_ν *of order* ν *can be defined via its Poisson representation formula:*

$$J_\nu(t) = \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_{-1}^{+1} e^{its}(1-s^2)^\nu \frac{ds}{\sqrt{1-s^2}}$$

where $t \geq 0$.

Theorem 2.0.3. *For* $\nu \in \mathbf{Z}^+$ *we have the following equalities:*

$$\begin{aligned} J_\nu(t) &= \frac{1}{2\pi} \int_0^{2\pi} e^{it\sin\theta} e^{i\nu\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(t\sin\theta - \nu\theta) d\theta \\ &= \frac{(t/2)^\nu}{\Gamma(1/2)} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(j+1/2)}{\Gamma(j+\nu+1)} \frac{t^{2j}}{(2j)!} \end{aligned}$$

and $J_\nu(t)$ *is equal to the last expression also in the range* $\operatorname{Re}\nu > -\frac{1}{2}$.

Theorem 2.0.4. *Let* $f(x) = f_0(|x|)$ *be a radial function defined on* $\mathbf{R}^n$, *where* f_0 *is defined on* $[0,\infty)$. *Then the Fourier transform of* f, *when defined, is given by the formula*

$$\widehat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty f_0(r) J_{\frac{n-2}{2}}(2\pi|\xi|r) r^{\frac{n}{2}} dr. \tag{2.1}$$

The reader interested in the proofs of the theorems above, can find them, for example, in appendix B of [7]. We're ready to state the following lemma.

Lemma 2.0.5. *For every $k \geq -\frac{1}{2}$, there are real constants $\{c_{k,j}\}_{j=0}^{\infty}$ and c_k such that, for every $N \in \mathbf{Z}^+$, the following holds:*

$$J_k(x) = R_{k,N}\left(\frac{1}{x^{2N+\frac{5}{2}}}\right) + \sum_{j=0}^{N}\left(\frac{c_{k,2j}}{x^{2j+\frac{1}{2}}}\cos(x - c_k) - \frac{c_{k,2j+1}}{x^{2j+\frac{3}{2}}}\sin(x - c_k)\right)$$

where $R_{k,N}$ is controlled by:

$$\left|R_{k,N}\left(\frac{1}{x^{2N+\frac{5}{2}}}\right)\right| \leq \frac{C_{k,N}}{x^{2N+\frac{5}{2}}}$$

for every $x \geq 2\pi$ and some constant $C_{k,N}$.

Theorem 2.0.6. *Let $m_{\lambda,\gamma} : \mathbf{R}^n \to [0, \infty)$ be defined as in (1.1, page 3). For every $0 < \lambda < \frac{n-1}{2}$ and $\gamma \geq 0$ there exists a constant $C_{n,\lambda,\gamma} > 0$ such that the following estimate holds for all $x \in \mathbf{R}^n$ satisfying $|x| \geq 1$:*

$$|\widehat{m_{\lambda,\gamma}}(x)| \leq \frac{C_{n,\lambda,\gamma}}{|x|^{\frac{n+1}{2}+\lambda}(\log(e|x|))^{\gamma}}. \tag{2.2}$$

Proof. If $\gamma = 0$, the conclusion follows from the explicit computation of $\widehat{m_{\lambda,0}}(x) = \widehat{m_\lambda}(x) = \frac{\Gamma(\lambda+1)}{\pi^\lambda}\frac{J_{\frac{n}{2}+\lambda}(2\pi|x|)}{|x|^{\frac{n}{2}+\lambda}}$ (see, for instance, page 429 of [7]). This explicit computation also shows that the estimate in (2.2, page 10) is sharp at least when $\gamma = 0$. If $\gamma > 0$, we decompose the multiplier $m_{\lambda,\gamma}$ as an infinite sum of smooth bumps supported in small concentric annuli in the interior of the sphere $|\xi| = 1$. We pick a smooth function φ supported in $[-\frac{1}{2}, \frac{1}{2}]$ and a smooth function ψ supported in $[\frac{1}{8}, \frac{5}{8}]$ and with values in $[0, 1]$ that satisfies

$$\varphi(t) + \sum_{k=0}^{\infty}\psi\Big(\frac{1-t}{2^{-k}}\Big) = 1$$

for all $t \in [0,1)$. We decompose the multiplier $m_{\lambda,\gamma}$ as:

$$m_{\lambda,\gamma}(\xi) = m_{\lambda,\gamma}(\xi) \cdot 1 = m_{\lambda,\gamma}(\xi) \left(\varphi(t) + \sum_{k=0}^{\infty} \psi\Big(\frac{1-t}{2^{-k}}\Big) \right) \tag{2.3}$$

for all $\xi \in \mathbf{R}^n$ and $0 \leq t < 1$. In particular:

$$\begin{aligned} m_{\lambda,\gamma}(\xi) &= m_{\lambda,\gamma}(\xi) \left(\varphi(|\xi|) + \sum_{k=0}^{\infty} \psi\Big(\frac{1-|\xi|}{2^{-k}}\Big) \right) \\ &= m_{\lambda,\gamma}(\xi)\, \varphi(|\xi|) + \sum_{k=0}^{\infty} 2^{-k\lambda} 2^{k\lambda} m_{\lambda,\gamma}(\xi) \psi\Big(\frac{1-|\xi|}{2^{-k}}\Big) \end{aligned} \tag{2.4}$$

for all $\xi \in \mathbf{R}^n$ such that $|\xi| < 1$. If $e_1 = (1, 0, \ldots, 0) \in \mathbf{R}^n$ and we define $m_{\lambda,\gamma,00}$ and $m_{\lambda,\gamma,k}$ by

$$m_{\lambda,\gamma,00}(t) = m_{\lambda,\gamma}(te_1)\, \varphi(t)$$

and

$$m_{\lambda,\gamma,k}(t) = 2^{k\lambda}\, m_{\lambda,\gamma}(te_1)\, \psi\Big(\frac{1-t}{2^{-k}}\Big) \tag{2.5}$$

for all $t \geq 0$, then, equation (2.4) can be rewritten as follows:

$$m_{\lambda,\gamma}(\xi) = m_{\lambda,\gamma,00}(|\xi|) + \sum_{k=0}^{\infty} 2^{-k\lambda} m_{\lambda,\gamma,k}(|\xi|). \tag{2.6}$$

Observe that each multiplier $m_{\lambda,\gamma,k}$ is supported on the interval $[1-5\, 2^{-(k+3)}, 1-2^{-(k+3)}]$, takes values in the interval $[0,1]$ and satisfies:

$$\sup_{0 \leq t \leq 1} \left| \frac{d^\ell}{dt^\ell} m_{\lambda,\gamma,k}(t) \right| \leq C_{\lambda,\gamma,\ell}\, \frac{2^{k\ell}}{k^\gamma} \tag{2.7}$$

for all $\ell \in \mathbf{Z}^+ \cup \{0\}$. It's convenient to introduce additional notation. Define $m_k^{\lambda,\gamma}(\xi) := 2^{-k\lambda} m_{\lambda,\gamma,k}(|\xi|)$. Now we can rewrite (2.6, page 11) by:

$$m_{\lambda,\gamma}(\xi) = m_{\lambda,\gamma,00}(|\xi|) + \sum_{k=0}^{\infty} m_k^{\lambda,\gamma}(\xi).$$

Therefore:

$$\widehat{m_{\lambda,\gamma}}(x) = \widehat{m_{\lambda,\gamma,00}(|\cdot|)}(x) + \sum_{k=0}^{\infty} \widehat{m_k^{\lambda,\gamma}}(x)$$

and

$$|\widehat{m_{\lambda,\gamma}}(x)| \leq |\widehat{m_{\lambda,\gamma,00}(|\cdot|)}(x)| + \sum_{k=0}^{\infty} |\widehat{m_k^{\lambda,\gamma}}(x)|. \tag{2.8}$$

It will be enough to find suitable estimates for each multiplier $|\widehat{m_k^{\lambda,\gamma}}|$. Let's use equation (2.1, page 9):

$$\begin{aligned} \widehat{m_k^{\lambda,\gamma}}(x) &= \frac{2\pi}{|x|^{\frac{n-2}{2}}} \int_0^{\infty} 2^{-k\lambda} m_{\lambda,\gamma,k}(r) J_{\frac{n-2}{2}}(2\pi|x|r) r^{\frac{n}{2}}\, dr \\ &= \frac{2\pi}{|x|^{\frac{n-2}{2}}} \int_{1-5\,2^{-(k+3)}}^{1-2^{-(k+3)}} 2^{-k\lambda} m_{\lambda,\gamma,k}(r) J_{\frac{n-2}{2}}(2\pi|x|r) r^{\frac{n}{2}}\, dr \end{aligned} \tag{2.9}$$

where the second equality follows from the support of $m_{\lambda,\gamma,k}$. It's time to use Lemma (2.0.5, page 9). By setting $k = \frac{n-2}{2}$ and choosing $N := N(\lambda) = \lceil \frac{\lambda-2}{2} \rceil + 1$ in the lemma, we can rewrite the last term in (2.9, page 12) as:

$$\frac{2\pi\, 2^{-k\lambda}}{|x|^{\frac{n-2}{2}}} \left(R_{n,\lambda,\gamma,k}(|x|) + \sum_{j=0}^{\lceil \frac{\lambda-2}{2} \rceil + 1} I_{n,\lambda,\gamma,k,j,1}(|x|) + \sum_{j=0}^{\lceil \frac{\lambda-2}{2} \rceil + 1} I_{n,\lambda,\gamma,k,j,2}(|x|) \right) \tag{2.10}$$

where

$$R_{n,\lambda,\gamma,k}(|x|) = \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} m_{\lambda,\gamma,k}(r)\, R_{\frac{n-2}{2},N(\lambda)}\!\left(\frac{1}{(2\pi|x|r)^{2N(\lambda)+\frac{5}{2}}} \right) r^{\frac{n}{2}}\, dr$$

and, for each $j = 0, ..., N(\lambda)$, we define

$$I_{n,\lambda,\gamma,k,j,1}(|x|) = \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} m_{\lambda,\gamma,k}(r)\, \frac{c_{n,2j}}{(2\pi|x|r)^{2j+\frac{1}{2}}} \cos(2\pi|x|r - c_n)\, r^{\frac{n}{2}}\, dr$$

$$I_{n,\lambda,\gamma,k,j,2}(|x|) = \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} m_{\lambda,\gamma,k}(r)\, \frac{c_{n,2j+1}}{(2\pi|x|r)^{2j+\frac{3}{2}}} \sin(2\pi|x|r - c_n)\, r^{\frac{n}{2}}\, dr$$

for different constants c_n and $c_{n,j}$'s than in Lemma (2.0.5, page 9). Let's estimate $R_{n,\lambda,\gamma,k}$:

$$\begin{aligned}
|R_{n,\lambda,\gamma,k}(|x|)| &\leq \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} |m_{\lambda,\gamma,k}(r)| \left| R_{\frac{n-2}{2},N(\lambda)}\left(\frac{1}{(2\pi|x|r)^{2N(\lambda)+\frac{5}{2}}} \right) \right| r^{\frac{n}{2}}\, dr \\
&\leq \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} |m_{\lambda,\gamma,k}(r)|\, \frac{C_{n,N(\lambda)}}{(2\pi|x|r)^{2N(\lambda)+\frac{5}{2}}}\, r^{\frac{n}{2}}\, dr \\
&\leq \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} \frac{C_{\lambda,\gamma,0}}{k^\gamma}\, \frac{C_{n,N(\lambda)}}{(2\pi|x|r)^{2N(\lambda)+\frac{5}{2}}}\, r^{\frac{n}{2}}\, dr \\
&= \frac{C_{n,\lambda,\gamma}}{k^\gamma |x|^{2N(\lambda)+\frac{5}{2}}} \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} r^{\frac{n}{2}-2N(\lambda)-\frac{5}{2}}\, dr \\
&= \frac{C_{n,\lambda,\gamma}}{k^\gamma |x|^{2N(\lambda)+\frac{5}{2}}} C_{n,\lambda}\, 2^{-k} \\
&= \frac{C'_{n,\lambda,\gamma}}{k^\gamma |x|^{2N(\lambda)+\frac{5}{2}}}\, 2^{-k} \\
&= \frac{C'_{n,\lambda,\gamma}}{k^\gamma |x|^{2+2\left\lceil \frac{\lambda-2}{2} \right\rceil+\frac{5}{2}}}\, 2^{-k} \\
&\leq \frac{C'_{n,\lambda,\gamma}}{k^\gamma |x|^{2+2\frac{\lambda-2}{2}+\frac{5}{2}}}\, 2^{-k} \\
&\leq \frac{C'_{n,\lambda,\gamma}}{k^\gamma |x|^{\lambda+\frac{5}{2}}}\, 2^{-k} \\
&\leq \frac{C'_{n,\lambda,\gamma}}{k^\gamma |x|^{\lambda+\frac{3}{2}} (\log(e|x|))^\gamma}\, 2^{-k}
\end{aligned}$$

if $|x| \geq C_{\lambda,\gamma}$. In the computation above, $C_{\lambda,\gamma}$, $C'_{n,\lambda,\gamma}$, $C_{n,\lambda,\gamma}$, $C_{\lambda,\gamma,0}$ are suitable constants. In order to estimate the terms $I_{n,\lambda,\gamma,k,j,1}$ and $I_{n,\lambda,\gamma,k,j,2}$ we will use integration by parts. Since the way to estimate $I_{n,\lambda,\gamma,k,0,1}$ is very similar to the way to estimate any other term $I_{n,\lambda,\gamma,k,j,1}$ or $I_{n,\lambda,\gamma,k,j,2}$, and the worst term is $I_{n,\lambda,\gamma,k,0,1}$, and there are only $2N(\lambda)+2$ such terms, where $N(\lambda)$ depends on λ only, we will perform the computation only in the case $j=0$, for the term $I_{n,\lambda,\gamma,k,0,1}$. We have:

$$\begin{aligned} I_{n,\lambda,\gamma,k,0,1}(|x|) &= \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} m_{\lambda,\gamma,k}(r)\,\frac{c'_n}{(2\pi|x|r)^{\frac{1}{2}}}\cos(2\pi|x|r-c_n)\,r^{\frac{n}{2}}\,dr \\ &= \frac{c''_n}{|x|^{\frac{1}{2}}}\int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} m_{\lambda,\gamma,k}(r)\,\cos(2\pi|x|r-c_n)\,r^{\frac{n}{2}-\frac{1}{2}}\,dr. \end{aligned} \tag{2.11}$$

Observe that the function $\widetilde{m}_{\lambda,\gamma,k}(r) := m_{\lambda,\gamma,k}(r)r^{\frac{n}{2}-\frac{1}{2}}$ satisfies the same estimates as in (2.7, page 11), with the constants $C_{\lambda,\gamma,\ell}$ replaced by other constants $C_{n,\lambda,\gamma,\ell}$. Then, for any $\ell \in \mathbf{Z}^+$ we have:

$$\begin{aligned} I_{n,\lambda,\gamma,k,0,1}(|x|) &= \frac{c''_n}{|x|^{\frac{1}{2}}}\int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} \widetilde{m}_{\lambda,\gamma,k}(r)\,\cos(2\pi|x|r-c_n)dr \\ &= (-1)^{\ell}\frac{c''_n}{|x|^{\frac{1}{2}}}\int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} \frac{d^{\ell}}{dr^{\ell}}\widetilde{m}_{\lambda,\gamma,k}(r)\,\frac{\frac{d^{-\ell}}{dr^{-\ell}}(\cos)(2\pi|x|r-c_n)}{(2\pi|x|)^{\ell}}dr, \end{aligned} \tag{2.12}$$

where we wrote $\frac{d^{-\ell}}{dr^{-\ell}}(\cos)(2\pi|x|r-c_n)$ to denote the ℓth antiderivative of cosine evaluated at the point $2\pi|x|r-c_n$. So, we have:

$$|I_{n,\lambda,\gamma,k,0,1}(|x|)| \;\leq\; \frac{c_{n,\ell}}{|x|^{\frac{1}{2}+\ell}}\int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} \left|\frac{d^{\ell}}{dr^{\ell}}\widetilde{m}_{\lambda,\gamma,k}(r)\right. \cdot$$

$$
\begin{aligned}
& \cdot \ \frac{d^{-\ell}}{dr^{-\ell}}(\cos)(2\pi|x|r - c_n)\Big|\, dr \\
= \ & \frac{c_{n,\ell}}{|x|^{\frac{1}{2}+\ell}} \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} \left| \frac{d^\ell}{dr^\ell} \widetilde{m}_{\lambda,\gamma,k}(r) \right| dr \\
\leq \ & \frac{c_{n,\ell}}{|x|^{\frac{1}{2}+\ell}} \int_{1-\frac{5/8}{2^k}}^{1-\frac{1/8}{2^k}} C_{n,\lambda,\gamma,\ell} \frac{2^{k\ell}}{k^\gamma} dr \\
= \ & \frac{c'_{n,\ell}}{|x|^{\frac{1}{2}+\ell}}\, 2^{-k}\, C_{n,\lambda,\gamma,\ell} \frac{2^{k\ell}}{k^\gamma} \\
= \ & \frac{C'_{n,\lambda,\gamma,\ell}}{|x|^{\frac{1}{2}+\ell}}\, 2^{-k}\, \frac{2^{k\ell}}{k^\gamma} \\
= \ & \frac{C'_{n,\lambda,\gamma,\ell}}{k^\gamma |x|^{\frac{1}{2}}}\, 2^{-k} \left(\frac{2^k}{|x|} \right)^\ell .
\end{aligned}
$$

The estimate above was proved under the assumption that ℓ is a non negative integer. The fact that it holds in fact for every $\ell \in [0, \infty)$ is just a consequence of the fact that, if $\alpha \leq \beta \leq \gamma$, then we have $y^\beta \geq \min\{y^\alpha, y^\gamma\}$ for every $y > 0$ (apply this to $y := \frac{2^k}{|x|}$, $\alpha := \lfloor \ell \rfloor$, $\beta := \ell$, $\gamma := \lceil \ell \rceil$). Summarizing, we proved that

$$
|I_{n,\lambda,\gamma,k,0,1}(|x|)| \leq \frac{C'_{n,\lambda,\gamma,\ell}}{k^\gamma |x|^{\frac{1}{2}}}\, 2^{-k} \left(\frac{2^k}{|x|} \right)^\ell
$$

for all $\ell \in \mathbf{R}^+$. A similar computation would show that there exist also positive constants $C'n, \lambda, \gamma, \ell, j, 1$ and $C'n, \lambda, \gamma, \ell, j, 1$, $j = 0, ..., N(\lambda)$, such that:

$$
|I_{n,\lambda,\gamma,k,j,1}(|x|)| \leq \frac{C_{n,\lambda,\gamma,\ell,j,1}}{k^\gamma |x|^{\frac{1}{2}+2j}}\, 2^{-k} \left(\frac{2^k}{|x|} \right)^\ell
$$

and

$$|I_{n,\lambda,\gamma,k,j,2}(|x|)| \leq \frac{C_{n,\lambda,\gamma,\ell,j,2}}{k^\gamma |x|^{\frac{3}{2}+2j}}\, 2^{-k} \left(\frac{2^k}{|x|}\right)^\ell$$

which shows in which sense the term $I_{n,\lambda,\gamma,k,0,1}$ is the worst one. Now recall the estimate we got for $R_{n,\lambda,\gamma,k}$:

$$R_{n,\lambda,\gamma,k}(|x|) \leq \frac{C'_{n,\lambda,\gamma}}{k^\gamma |x|^{\lambda+\frac{3}{2}} (\log(e|x|))^\gamma}\, 2^{-k}.$$

Also recall the fact that $\widehat{m_k^{\lambda,\gamma}}(x)$ can be rewritten as in (2.10, page 12):

$$\frac{2\pi\, 2^{-k\lambda}}{|x|^{\frac{n-2}{2}}} \left(R_{n,\lambda,\gamma,k}(|x|) + \sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1} I_{n,\lambda,\gamma,k,j,1}(|x|) + \sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1} I_{n,\lambda,\gamma,k,j,2}(|x|) \right)$$

and the inequality (2.8, page 12):

$$|\widehat{m_{\lambda,\gamma}}(x)| \leq |\widehat{m_{\lambda,\gamma,00}(|\cdot|)}(x)| + \sum_{k=0}^{\infty} |\widehat{m_k^{\lambda,\gamma}}(x)|.$$

With this in mind, we can write:

$$\begin{aligned} |\widehat{m_{\lambda,\gamma}}(x)| \;\leq\; & |\widehat{m_{\lambda,\gamma,00}(|\cdot|)}(x)| \\ & + \sum_{k=0}^{\infty} \left| \frac{2\pi\, 2^{-k\lambda}}{|x|^{\frac{n-2}{2}}} R_{n,\lambda,\gamma,k}(|x|) \right| \\ & + \sum_{k=0}^{\infty} \left| \frac{2\pi\, 2^{-k\lambda}}{|x|^{\frac{n-2}{2}}} \sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1} I_{n,\lambda,\gamma,k,j,1}(|x|) \right| \\ & + \sum_{k=0}^{\infty} \left| \frac{2\pi\, 2^{-k\lambda}}{|x|^{\frac{n-2}{2}}} \sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1} I_{n,\lambda,\gamma,k,j,2}(|x|) \right| \\ \leq\; & |\widehat{m_{\lambda,\gamma,00}(|\cdot|)}(x)| \end{aligned}$$

$$
\begin{aligned}
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\infty}2^{-k\lambda}\ |R_{n,\lambda,\gamma,k}(|x|)| \\
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\infty}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}|I_{n,\lambda,\gamma,k,j,1}(|x|)| \\
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\infty}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}|I_{n,\lambda,\gamma,k,j,2}(|x|)| \\
\leq\ & |\widehat{m_{\lambda,\gamma,00}(|\cdot|)}(x)| \\
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\infty}2^{-k\lambda}\frac{C'_{n,\lambda,\gamma}}{k^{\gamma}|x|^{\lambda+\frac{3}{2}}(\log(e|x|))^{\gamma}}\,2^{-k} \\
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\log(e|x|)}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}\frac{C_{n,\lambda,\gamma,\ell,j,1}}{k^{\gamma}|x|^{\frac{1}{2}+2j}}\,2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell} \\
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\log(e|x|)}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}\frac{C_{n,\lambda,\gamma,\ell,j,2}}{k^{\gamma}|x|^{\frac{3}{2}+2j}}\,2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell} \\
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=\log(e|x|)+1}^{\infty}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}\frac{C_{n,\lambda,\gamma,\ell',j,1}}{k^{\gamma}|x|^{\frac{1}{2}+2j}}\,2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell'} \\
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=\log(e|x|)+1}^{\infty}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}\frac{C_{n,\lambda,\gamma,\ell',j,2}}{k^{\gamma}|x|^{\frac{3}{2}+2j}}\,2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell'} \\
\leq\ & |\widehat{m_{\lambda,\gamma,00}(|\cdot|)}(x)| \\
&+\frac{C_{n,\lambda,\gamma}}{|x|^{\frac{n+1}{2}+\lambda}(\log(e|x|))^{\gamma}} \\
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\log(e|x|)}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}\frac{C_{n,\lambda,\gamma,\ell,j,1}}{k^{\gamma}|x|^{\frac{1}{2}}}\,2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell} \\
&+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\log(e|x|)}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}\frac{C_{n,\lambda,\gamma,\ell,j,2}}{k^{\gamma}|x|^{\frac{3}{2}}}\,2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell}
\end{aligned}
$$

$$+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=\log(e|x|)+1}^{\infty}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}\frac{C_{n,\lambda,\gamma,\ell',j,1}}{k^\gamma|x|^{\frac{1}{2}}}2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell'}$$

$$+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=\log(e|x|)+1}^{\infty}2^{-k\lambda}\sum_{j=0}^{\lceil\frac{\lambda-2}{2}\rceil+1}\frac{C_{n,\lambda,\gamma,\ell',j,2}}{k^\gamma|x|^{\frac{3}{2}}}2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell'}$$

$$\leq |\widehat{m_{\lambda,\gamma,00}(|}\cdot|)(x)|$$

$$+\frac{C_{n,\lambda,\gamma}}{|x|^{\frac{n+1}{2}+\lambda}(\log(e|x|))^\gamma}$$

$$+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\log(e|x|)}2^{-k\lambda}\frac{C'_{n,\lambda,\gamma,\ell}}{k^\gamma|x|^{\frac{1}{2}}}2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell}$$

$$+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\log(e|x|)}2^{-k\lambda}\frac{C'_{n,\lambda,\gamma,\ell}}{k^\gamma|x|^{\frac{3}{2}}}2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell}$$

$$+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=\log(e|x|)+1}^{\infty}2^{-k\lambda}\frac{C'_{n,\lambda,\gamma,\ell'}}{k^\gamma|x|^{\frac{1}{2}}}2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell'}$$

$$+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=\log(e|x|)+1}^{\infty}2^{-k\lambda}\frac{C'_{n,\lambda,\gamma,\ell'}}{k^\gamma|x|^{\frac{3}{2}}}2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell'}$$

$$\leq |\widehat{m_{\lambda,\gamma,00}(|}\cdot|)(x)|$$

$$+\frac{C_{n,\lambda,\gamma}}{|x|^{\frac{n+1}{2}+\lambda}(\log(e|x|))^\gamma}$$

$$+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=0}^{\log(e|x|)}2^{-k\lambda}\frac{C''_{n,\lambda,\gamma,\ell}}{k^\gamma|x|^{\frac{1}{2}}}2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell}$$

$$+\frac{2\pi}{|x|^{\frac{n-2}{2}}}\sum_{k=\log(e|x|)+1}^{\infty}2^{-k\lambda}\frac{C''_{n,\lambda,\gamma,\ell'}}{k^\gamma|x|^{\frac{1}{2}}}2^{-k}\left(\frac{2^k}{|x|}\right)^{\ell'}$$

$$\leq |\widehat{m_{\lambda,\gamma,00}(|}\cdot|)(x)|$$

$$+\frac{C_{n,\lambda,\gamma}}{|x|^{\frac{n+1}{2}+\lambda}(\log(e|x|))^\gamma}$$

$$
\begin{aligned}
&+\frac{C_{n,\lambda,\gamma,\ell}}{|x|^{\frac{n-1}{2}}}\sum_{k=0}^{\log(e|x|)}\frac{2^{-k(\lambda+1-\ell)}}{k^{\gamma}|x|^{\ell}}\\
&+\frac{C_{n,\lambda,\gamma,\ell'}}{|x|^{\frac{n-1}{2}}}\sum_{k=\log(e|x|)+1}^{\infty}\frac{2^{-k(\lambda+1-\ell')}}{k^{\gamma}|x|^{\ell'}}\\
\leq\ & |\widehat{m_{\lambda,\gamma,00}(|\cdot|)}(x)|\\
&+\frac{C_{n,\lambda,\gamma}}{|x|^{\frac{n+1}{2}+\lambda}(\log(e|x|))^{\gamma}}\\
&+\frac{C'_{n,\lambda,\gamma,\ell}}{|x|^{\frac{n-1}{2}}}\frac{2^{-\log(e|x|)(\lambda+1-\ell)}}{(\log(e|x|))^{\gamma}|x|^{\ell}}\\
&+\frac{C'_{n,\lambda,\gamma,\ell'}}{|x|^{\frac{n-1}{2}}}\frac{2^{-\log(e|x|)(\lambda+1-\ell')}}{(\log(e|x|))^{\gamma}|x|^{\ell'}}\\
\leq\ & |\widehat{m_{\lambda,\gamma,00}(|\cdot|)}(x)|\\
&+\frac{C'_{n,\lambda,\gamma}}{|x|^{\frac{n+1}{2}+\lambda}(\log(e|x|))^{\gamma}}\\
\leq\ & \frac{C_{n,\lambda,\gamma}}{|x|^{\frac{n+1}{2}+\lambda}(\log(e|x|))^{\gamma}}
\end{aligned}
$$

for any choice of $\ell' < \lambda+1$ and $\ell > \lambda+1$. In the last step, we used the fact that $m_{\lambda,\gamma,00}(|\cdot|)$ is the Schwartz class. The theorem is proved. □

Theorem 2.0.7. *Let $w_{\lambda\mu}$ be defined as in (3.5, page 29), and let $B_R^{\lambda,\gamma}$: $L^2(\mathbf{R}^n, w_{\lambda\mu}) \to C(\mathbf{R}^n)$ be defined as in (1.2, page 3). If $0 < \lambda < \frac{n-1}{2}$, $n \geq 2$, $2\gamma - 1 > \mu > 0$ and $R > 0$, then $B_R^{\lambda,\gamma}$ is well defined from $L^2(\mathbf{R}^n, w_{\lambda\mu})$ to $C(\mathbf{R}^n)$. In particular, under these hypotheses, $B_R^{\lambda,\gamma}(f)$ is defined everywhere.*

Proof. First observe that for n, λ, γ, f, R as in the hypothesis, we have:

$$\begin{aligned} B_R^{\lambda,\gamma}(f)(x) &= \left(\widehat{f}(\cdot) m_{\lambda,\gamma}\left(\frac{\cdot}{R}\right)\right)^{\vee}(x) = R^n\left(f * \delta^R((m_{\lambda,\gamma})^{\vee})\right)(x) \\ &= R^n\left(f * \delta^R(\widehat{m_{\lambda,\gamma}})\right)(x) = \left(\delta^{\frac{1}{R}}(f) * \widehat{m_{\lambda,\gamma}}\right)(Rx) \end{aligned} \tag{2.13}$$

where we set $\delta^a(f)(x) = f(ax)$, whenever $a \in \mathbf{R}$, $x \in \mathbf{R}^n$ and f is a function defined on $\mathbf{R}^n$. We used the fact that $m_{\lambda,\gamma}$ is radial, in order to write $m_{\lambda,\gamma}^{\vee} = \widehat{m_{\lambda,\gamma}}$. Since $\delta^{\frac{1}{R}}(f) \in L^2(\mathbf{R}^n, w_{\lambda\mu})$ whenever $f \in L^2(\mathbf{R}^n, w_{\lambda\mu})$ and $R \neq 0$, we see from (2.13, page 20) that the continuity of $B_R^{\lambda,\gamma}(f)$ for $R > 0$ reduces to the case $R = 1$. So, we will prove that the function

$$\widehat{m_{\lambda,\gamma}} * f$$

is continuous if f, λ, γ, n satisfy the hypotheses of the theorem. The continuity of $\widehat{m_{\lambda,\gamma}} * f$ at a point $x_0 \in \mathbf{R}^n$ can be verified by using the dominated convergence theorem. It's enough to show that there exists $\varepsilon = \varepsilon(x_0, n, \lambda, \gamma)$ such that:

$$\int_{\mathbf{R}^n} \sup_{x \in B(x_0,\varepsilon)} |f(y)\widehat{m_{\lambda,\gamma}}(x-y)|\, dy < \infty \tag{2.14}$$

where $B(x_0, \varepsilon) \subset \mathbf{R}^n$ denotes the ball of center x_0 and radius ε. In fact, we can choose $\varepsilon = 1$. In view of Theorem (2.0.6, page 10) we have:

$$\int_{\mathbf{R}^n} \sup_{x \in B(x_0,1)} |f(y)\widehat{m_{\lambda,\gamma}}(x-y)|\, dy$$

$$
\begin{aligned}
&= \int_{|y-x_0|\le 2} \sup_{x\in B(x_0,1)} |f(y)\widehat{m_{\lambda,\gamma}}(x-y)|\,dy \\
&\quad + \int_{|y-x_0|\ge 2} \sup_{x\in B(x_0,1)} |f(y)\widehat{m_{\lambda,\gamma}}(x-y)|\,dy \\
&\le \|\widehat{m_{\lambda,\gamma}}\|_{L^\infty} \int_{|y-x_0|\le 2} |f(y)|\,dy \\
&\quad + \int_{|y-x_0|\ge 2} \left| f(y) \frac{C_{n,\lambda,\gamma}}{(|x_0-y|-1)^{\frac{n+1}{2}+\lambda}(\log(e(|x_0-y|-1)))^{\gamma}} \right| dy \\
&=: I + II,
\end{aligned}
$$

where

$$
I = \|\widehat{m_{\lambda,\gamma}}\|_{L^\infty} \int_{|y-x_0|\le 2} |f(y)|\,dy
$$

and

$$
II = \int_{|y-x_0|\ge 2} \left| f(y) \frac{C_{n,\lambda,\gamma}}{(|x_0-y|-1)^{\frac{n+1}{2}+\lambda}(\log(e(|x_0-y|-1)))^{\gamma}} \right| dy.
$$

Let's estimate I.

$$
\begin{aligned}
I &= \|\widehat{m_{\lambda,\gamma}}\|_{L^\infty} \int_{|y-x_0|\le 2} |f(y)|\,dy \\
&= \|\widehat{m_{\lambda,\gamma}}\|_{L^\infty} \int_{|y-x_0|\le 2} \frac{|f(y)|}{\sqrt{w_{\lambda\mu}(y)}} \sqrt{w_{\lambda\mu}(y)}dy \\
&\le \|\widehat{m_{\lambda,\gamma}}\|_{L^\infty} \sqrt{\int_{|y-x_0|\le 2} \frac{1}{w_{\lambda\mu}(y)}dy} \sqrt{\int_{|y-x_0|\le 2} |f(y)|^2 w_{\lambda\mu}(y)dy} \\
&\le \|\widehat{m_{\lambda,\gamma}}\|_{L^\infty} \sqrt{\frac{|B(x_0,2)|}{\omega_{\lambda\mu}(|x_0|+2)}} \sqrt{\int_{\mathbf{R}^n} |f(y)|^2 w_{\lambda\mu}(y)dy} \\
&= \|\widehat{m_{\lambda,\gamma}}\|_{L^\infty} \sqrt{\frac{|B(x_0,2)|}{\omega_{\lambda\mu}(|x_0|+2)}}\ \|f\|_{L^2(w_{\lambda\mu})} < \infty.
\end{aligned}
$$

Now, let's estimate II.

$$\begin{aligned}
II &= \int_{|y-x_0|\geq 2} |f(y)| \frac{C_{n,\lambda,\gamma}}{(|x_0-y|-1)^{\frac{n+1}{2}+\lambda}(\log(e(|x_0-y|-1)))^{\gamma}} dy \\
&= \int_{|y-x_0|\geq 2} \frac{C_{n,\lambda,\gamma}|f(y)|\sqrt{w_{\lambda\mu}(y)}}{\sqrt{w_{\lambda\mu}(y)}(|x_0-y|-1)^{\frac{n+1}{2}+\lambda}(\log(e(|x_0-y|-1)))^{\gamma}} dy \\
&\leq C_{n,\lambda,\gamma}\sqrt{\int_{|y-x_0|\geq 2}|f(y)|^2 w_{\lambda\mu}(y)dy} \cdot \\
&\quad \cdot \sqrt{\int_{|y-x_0|\geq 2} \frac{\frac{1}{w_{\lambda\mu}(y)}}{(|x_0-y|-1)^{n+1+2\lambda}(\log(e(|x_0-y|-1)))^{2\gamma}} dy} \\
&\leq C_{n,\lambda,\gamma}\|f\|_{L^2(w_{\lambda\mu})} \cdot \\
&\quad \cdot \sqrt{\int_{|y-x_0|\geq 2} \frac{\frac{1}{w_{\lambda\mu}(y)}}{(|x_0-y|-1)^{n+1+2\lambda}(\log(e(|x_0-y|-1)))^{2\gamma}} dy} \\
&= C_{n,\lambda,\gamma}\|f\|_{L^2(w_{\lambda\mu})} \cdot \sqrt{III}
\end{aligned}$$

where

$$III = \int_{|y-x_0|\geq 2} \frac{\frac{1}{w_{\lambda\mu}(y)}}{(|x_0-y|-1)^{n+1+2\lambda}(\log(e(|x_0-y|-1)))^{2\gamma}} dy.$$

Then we use a splitting to estimate III as follows:

(2.15)

$$III = \int_{\substack{|y-x_0|\geq 2 \\ |y|\leq 2(|x_0|+2)}} \frac{\frac{1}{w_{\lambda\mu}(y)}}{(|x_0-y|-1)^{n+1+2\lambda}(\log(e(|x_0-y|-1)))^{2\gamma}} dy$$

$$+\int_{\substack{|y-x_0|\geq 2\\ |y|\geq 2(|x_0|+2)}} \frac{\frac{1}{w_{\lambda\mu}(y)}}{(|x_0-y|-1)^{n+1+2\lambda}(\log(e(|x_0-y|-1)))^{2\gamma}}dy$$

$$\leq \int_{\substack{|y-x_0|\geq 2\\ |y|\leq 2(|x_0|+2)}} \frac{\frac{1}{\omega_{\lambda\mu}(2(|x_0|+2))}}{(2-1)^{n+1+2\lambda}(\log(e(2-1)))^{2\gamma}}dy$$

$$+\int_{|y|\geq 2(|x_0|+2)} \frac{\frac{1}{w_{\lambda\mu}(y)}}{(|x_0-y|-1)^{n+1+2\lambda}(\log(e(|x_0-y|-1)))^{2\gamma}}dy$$

$$\leq \frac{1}{\omega_{\lambda\mu}(2(|x_0|+2))}|B(0,2(|x_0|+2))|$$

$$+\int_{|y|\geq 2(|x_0|+2)} \frac{\frac{1}{w_{\lambda\mu}(y)}}{\left(\frac{|y|}{2}\right)^{n+1+2\lambda}\left(\log\left(\frac{e|y|}{2}\right)\right)^{2\gamma}}dy,$$

where the last step is justified by the fact that, for $|y|\geq 2(|x_0|+2)$, we have $|y|-2(|x_0|+2)\geq 0$, that is $\frac{|y|}{2}-(|x_0|+2)\geq 0$, therefore $\frac{|y|}{2}+\frac{|y|}{2}-(|x_0|+2)\geq\frac{|y|}{2}$, that is $|y|-(|x_0|+2)\geq\frac{|y|}{2}$, which implies $|x_0-y|-1\geq |y|-|x_0|-1\geq |y|-(|x_0|+2)\geq\frac{|y|}{2}$. As $|y|\geq 2(|x_0|+2)$, we also have:

$$\begin{aligned}\log\left(\frac{e|y|}{2}\right) &= \frac{\log(e|y|)-\log(2)}{\log(e|y|)}\log(e|y|)\\ &= \log(e|y|)\left(1-\frac{\log(2)}{\log(e|y|)}\right)\\ &\geq \log(e|y|)\left(1-\frac{\log(2)}{\log(e(2|x_0|+4))}\right).\end{aligned}$$

Set $C_{n,\lambda,\gamma,x_0}=\frac{2^{n+1+2\lambda}}{\left(1-\frac{\log(2)}{\log(e(2|x_0|+4))}\right)^{2\gamma}}$. Then, by using the computation above and the estimate in (2.15, page 22) we get:

$$\begin{aligned}III &\leq \frac{1}{\omega_{\lambda\mu}(2(|x_0|+2))}|B(0,2(|x_0|+2))|\\ &\quad+\int_{|y|\geq 2(|x_0|+2)} \frac{\frac{1}{w_{\lambda\mu}(y)}}{\left(\frac{|y|}{2}\right)^{n+1+2\lambda}\left(\log\left(\frac{e|y|}{2}\right)\right)^{2\gamma}}dy\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\omega_{\lambda\mu}(2(|x_0|+2))}|B(0,2(|x_0|+2))| \\
&\quad + C_{n,\lambda,\gamma,x_0}\int_{|y|\ge 2(|x_0|+2)} \frac{\frac{1}{w_{\lambda\mu}(y)}}{|y|^{n+1+2\lambda}(\log(e|y|))^{2\gamma}}dy \\
&= \frac{1}{\omega_{\lambda\mu}(2(|x_0|+2))}|B(0,2(|x_0|+2))| \\
&\quad + C_{n,\lambda,\gamma,x_0}\int_{|y|\ge 2(|x_0|+2)} \frac{|y|^{2\lambda+1}(\log(e|y|))^{\mu}}{|y|^{n+1+2\lambda}(\log(e|y|))^{2\gamma}}dy \\
&= \frac{1}{\omega_{\lambda\mu}(2(|x_0|+2))}|B(0,2(|x_0|+2))| \\
&\quad + C_{n,\lambda,\gamma,x_0}\int_{|y|\ge 2(|x_0|+2)} \frac{1}{|y|^{n}(\log(e|y|))^{2\gamma-\mu}}dy,
\end{aligned}
$$

which is finite in view of the assumption $2\gamma-1>\mu$ (that is, $2\gamma-\mu>1$). So, $III<\infty$, which implies that $II<\infty$, which shows that (2.14, page 20) is satisfied. this proves that $B_R^{\lambda,\gamma}(f)$ is continuous at x_0 and, by the arbitrariety of $x_0\in\mathbf{R}^n$, we proved our claim. □

The theorem above implies that the maximal operator $B_*^{\lambda,\gamma}$ introduced in (1.5, page 5) is well defined on $L^2(\mathbf{R}^n, w_{\lambda\mu})$, because $B_R^{\lambda,\gamma}(f)$ is defined everywhere instead of just almost everywhere. Furhter evidence of the well-posedness of the definition of $B_*^{\lambda,\gamma}$ on $L^2(\mathbf{R}^n, w_{\lambda\mu})$ is provided by the following theorem, which implies that taking the supremum over $R>0$ in (1.5, page 5) is equivalent to taking the supremum over $R>0, R\in\mathbf{Q}$.

Theorem 2.0.8. *Let $w_{\lambda\mu}$ be defined as in (3.6, page 29), let $B_R^{\lambda,\gamma}: L^2(\mathbf{R}^n, w_{\lambda\mu})\to C(\mathbf{R}^n)$ be defined as in (1.2, page 3), let $f\in L^2(\mathbf{R}^n, w_{\lambda\mu})$*

and let $x \in \mathbf{R}^n$. Then the map $R \mapsto B_R^{\lambda,\gamma}(f)(x)$ defined from $(0, +\infty)$ to $\mathbf{C}$ is continuous.

Proof. As in the proof of Theorem (2.0.7, page 19), we use the identity:

$$B_R^{\lambda,\gamma}(f)(x) = R^n \left(f * \delta^R(\widehat{m_{\lambda,\gamma}})\right)(x).$$

Again, in order to prove that the map $R \mapsto B_R^{\lambda,\gamma}(f)(x)$ is continuous at a point $R_0 \in \mathbf{R}^+$, we can use the Lebesgue dominated convergence theorem. The condition to verify in this situation looks similar to the condition in (2.14, page 20): given $R_0 \in \mathbf{R}^+$, we need to find $\varepsilon = \varepsilon(R_0, n, \lambda, \gamma) > 0$ such that the following holds:

$$\int_{\mathbf{R}^n} \sup_{R \in (R_0 - \varepsilon, R_0 + \varepsilon)} |f(y)\widehat{m_{\lambda,\gamma}}(R(x-y))|\, dy < \infty. \tag{2.16}$$

The conclusion follows in a way that is very similar to the proof of Theorem (2.0.7, page 19), and is therefore left to the reader. □

Remark 2.0.9. *The condition on γ we required in theorem (2.0.7, page 19), links with condition (3.9, page 31) needed to have (3.7, page 30). So, we need:*

$$2\gamma - 1 > \mu > \frac{2\lambda + 1}{n}$$

which implies $\gamma > \frac{1}{p'_\lambda}$, that is the necessary and sufficient condition on γ for $m_{\lambda,\gamma}$ to be a p_λ multiplier (see [10]).

Chapter 3

Weighted L^2 estimates

3.1 Idea of the proof.

Let us recall the definition of the multiplier introduced in [10, p.544]:

$$m_{\lambda,\gamma}(\xi) = \frac{(1-|\xi|^2)_+^\lambda}{(1-\log(1-|\xi|^2))^\gamma},$$

then define the operators $B_R^{\lambda,\gamma}$ by:

$$B_R^{\lambda,\gamma}(f)(x) = \int_{\mathbf{R}^n} m_{\lambda,\gamma}\left(\frac{\xi}{R}\right)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi,$$

and define the maximal operator $B_*^{\lambda,\gamma}$ by:

$$B_*^{\lambda,\gamma}(f)(x) = \sup_{R>0}|B_R^{\lambda,\gamma}(f)(x)| = \sup_{R>0}\left|\int_{\mathbf{R}^n} m_{\lambda,\gamma}\left(\frac{\xi}{R}\right)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi\right|.$$

For every $\lambda, \gamma > 0$, the multiplier $m_{\lambda,\gamma}$ is smoother than the Bochner-Riesz multiplier with exponent λ. Therefore the maximal operator $B_*^{\lambda,\gamma}$ behaves better than the maximal Bochner-Riesz operator B_*^{λ}, and the proof in [4] can be used to show that the equality

$$\lim_{R\to\infty} B_R^{\lambda,\gamma}(f)(x) = f(x)$$

holds for almost every $x \in \mathbf{R}^n$, if $\lambda > 0, \gamma > 0$ and $f \in L^p(\mathbf{R}^n)$ with $2 \leq p < p_\lambda := \frac{2n}{n-2\lambda-1}$.

The main result of this work is the following:

Theorem 3.1.1. *For every $\lambda > 0$ such that $1 + 2\lambda < n$, let $p_\lambda = \frac{2n}{n-2\lambda-1}$. Then for every $f \in L^{p_\lambda}(\mathbf{R}^n)$ and every $\gamma > \frac{1}{p'_\lambda} + \frac{1}{2}$, we have that*

$$\lim_{R\to\infty} B_R^{\lambda,\gamma}(f)(x) = f(x)$$

for almost every $x \in \mathbf{R}^n$.

The proof of this result uses the following proposition.

Proposition 3.1.2 (A). *Let $0 < p < \infty$, $0 < q < \infty$, $\{T_\varepsilon\}_{\varepsilon>0}$ a family of linear operators defined on $L^p(X, \nu_1)$ and valued on the space of the measurable functions on (Y, ν_2), for two mesurable spaces (X, ν_1) and (Y, ν_2). Let*

T_* be defined by:

$$T_*(f)(x) = \sup_{\varepsilon>0} |T_\varepsilon(f)(x)|.$$

Suppose that for some $B > 0$ and all $f \in L^p(X, \nu_1)$ we have:

$$\|T_*(f)\|_{L^{q,\infty}} \leq B\|f\|_{L^p},$$

and for all f in a dense subspace $\mathcal{D}$ of $L^p(X, \nu_1)$ we have:

$$T(f)(x) := \lim_{\varepsilon\to 0} T_\varepsilon(f)(x) \tag{3.1}$$

exists and is finite for ν_1-almost all $x \in (X, \nu_1)$. That is, assume that equation (3.1) defines a linear operator T on $\mathcal{D}$. Then for all functions f in $L^p(X, \nu_1)$ the limit (3.1) exists and is finite ν_1-a.e., and defines a linear operator T on $L^p(X, \nu_1)$ (uniquely extending T defined on $\mathcal{D}$) that satisfies:

$$\|T(f)\|_{L^{q,\infty}} \leq B\|f\|_{L^p}. \tag{3.2}$$

For the proof of the proposition, see page 86 of [7].

Since the almost everywhere convergence is obvious for functions in the Schwartz class, in order to be able to use Proposition (3.1.2) to derive almost everywhere convergence for general L^p functions, it suffices to know a weak type (p, p) estimate for $B_*^{\lambda,\gamma}$. However, instead of proving a weak type (p, p) estimate, we prove an L^2 and a weighted L^2 estimate for $B_*^{\lambda,\gamma}$. Precisely, we prove the following result.

Proposition 3.1.3. *Let $\lambda > 0$ be such that $1 + 2\lambda < n$ and let $\gamma > \frac{1}{p'_\lambda} + \frac{1}{2}$. Then we have that $\frac{2\lambda+1}{n} < 2\gamma - 2$, and for every μ that satisfies $\frac{2\lambda+1}{n} < \mu < \min\{2\gamma - 2, 1\}$ there is a constant $C = C(n, \lambda, \gamma, \mu)$ such that*

$$\int_{\mathbf{R}^n} |B_*^{\lambda,\gamma}(f)(x)|^2 \, dx \leq C \int_{\mathbf{R}^n} |f(x)|^2 \, dx \tag{3.3}$$

for all functions $f \in L^2(\mathbf{R}^n, dx)$, and

$$\int_{\mathbf{R}^n} |B_*^{\lambda,\gamma}(f)(x)|^2 w_{\lambda\mu}(x) \, dx \leq C \int_{\mathbf{R}^n} |f(x)|^2 w_{\lambda\mu}(x) \, dx \tag{3.4}$$

where the weight $w_{\lambda\mu}$ is defined by:

$$\omega_{\lambda\mu}(t) = \begin{cases} \frac{1}{t^{2\lambda+1}} & \text{if} \quad 0 < t \leq 1, \\ \frac{1}{t^{2\lambda+1}(\log(et))^\mu} & \text{if} \quad t > 1. \end{cases} \tag{3.5}$$

and

$$w_{\lambda\mu}(x) = \omega_{\lambda\mu}(|x|). \tag{3.6}$$

We will only need to show the second part of the proposition, as the reader can find the proof of the first assertion in ([8]).

Remark 3.1.4. *When we apply Proposition (3.1.2, page 27), we have to choose weights ν_1 and ν_2. The choice $\nu_1 := w_{\lambda\mu}$ in the right hand side of (3.4, page 29) is given by the inclusion in (3.7, page 30), which we will see soon. The choice $\nu_2 := w_{\lambda\mu}$ is not random either. It follows from (3.1, page 28) and (3.2, page 28), as we apply Proposition (3.1.2, page 27) with $T_{\frac{1}{R}} = B_R^{\lambda,\gamma}$ and, consequently, T equal to the identity operator.*

Assuming the result of Proposition 3.1.3, given λ and γ such that $0 < 1 + 2\lambda < n$ and $\gamma > \frac{1}{p'_\lambda} + \frac{1}{2}$, the maximal operator $B_*^{\lambda,\gamma}$ is bounded on L^2 and also on $L^2(\omega_{\lambda\mu}(|x|)dx)$ for some μ satisfying $\frac{2\lambda+1}{n} < \mu < 2\gamma - 2$. Hence the almost everywhere convergence of the family $\{B_R^\lambda(f)\}_R$ holds on L^2 and also on $L^2(\omega_{\lambda\mu}(|x|)dx)$. Then we use that

$$L^{p_\lambda} \subsetneqq L^2 + L^2(\omega_{\lambda\mu}(|x|)dx)\,, \tag{3.7}$$

and thus $B_R^{\lambda,\gamma}(f)$ converges almost everywhere for functions $f \in L^{p_\lambda}(\mathbf{R}^n)$. To see this, let's write $f = f_1 + f_2 + f_3$ where $f_1 = f \cdot \chi_{\{|x|\leq 1\}}$, $f_2 = f \cdot \chi_{\{|x|>1, |f|\geq 1\}}$, and $f_3 = f \cdot \chi_{\{|x|>1, |f|<1\}}$. The fact that $f \in L^{p_\lambda}$ implies that $f_i \in L^{p_\lambda}$ $\forall i = 1, 2, 3$. Because of this and since $p_\lambda > 2$, we have that $f_1, f_2 \in L^2$, because f_1 is supported on a set of finite measure and $f_2(x) \notin (0,1)$ for any $x \in \mathbf{R}^n$. On the other hand, Holder's inequality (with exponents $q = p_\lambda/2$ and $q' =$

$\frac{p_\lambda/2}{p_\lambda/2-1}$) implies that $f_3 \in L^2(\omega_{\lambda\mu}(|x|)dx)$:

$$\begin{aligned}
&\int_{\mathbf{R}^n} |f_3(x)|^2 \omega_{\lambda\mu}(|x|)dx \\
=&\int_{\mathbf{R}^n\setminus B(0,1)} |f_3(x)|^2 \omega_{\lambda\mu}(|x|)dx \\
\leq&\|f_3^2\|_{L^{p_\lambda/2}(\mathbf{R}^n\setminus B(0,1),\, dx)} \|\omega_{\lambda\mu}\|_{L^{\frac{p_\lambda/2}{p_\lambda/2-1}}(\mathbf{R}^n\setminus B(0,1),\, dx)} \\
=&\left(\int_{\mathbf{R}^n\setminus B(0,1)} (|f_3(x)|^2)^{p_\lambda/2} dx\right)^{2/p_\lambda} \left(\int_{\mathbf{R}^n\setminus B(0,1)} \omega_{\lambda\mu}(|x|)^{\frac{p_\lambda/2}{p_\lambda/2-1}} dx\right)^{\frac{p_\lambda/2-1}{p_\lambda/2}} \\
=&\|f_3\|^2_{L^{p_\lambda}(\mathbf{R}^n\setminus B(0,1),dx)} |S^{n-1}| \left(\int_1^\infty \omega_{\lambda\mu}(r)^{\frac{p_\lambda/2}{p_\lambda/2-1}} r^{n-1} dr\right)^{\frac{p_\lambda/2-1}{p_\lambda/2}} \\
=&|S^{n-1}|\, \|f_3\|^2_{L^{p_\lambda}(\mathbf{R}^n\setminus B(0,1),\, dx)} \left(\int_1^\infty \frac{r^{n-1} dr}{\left(r^{2\lambda+1}(\log(er))^\mu\right)^{\frac{p_\lambda/2}{p_\lambda/2-1}}}\right)^{\frac{p_\lambda/2-1}{p_\lambda/2}} \\
=&|S^{n-1}|\, \|f_3\|^2_{L^{p_\lambda}(\mathbf{R}^n\setminus B(0,1),dx)} \left(\int_1^\infty \frac{dr}{r(\log(er))^{\frac{n\,\mu}{2\lambda+1}}}\right)^{\frac{p_\lambda/2-1}{p_\lambda/2}} < \infty
\end{aligned} \tag{3.8}$$

where the very last inequality is true provided that:

$$\mu > \frac{2\lambda+1}{n}. \tag{3.9}$$

To prove Proposition 3.1.3, we decompose the multiplier $m_{\lambda,\gamma}$ as in the proof of Theorem (2.0.6, page 10) so that, for all $t \geq 0$, we have:

$$m_{\lambda,\gamma}(\xi) = m_{\lambda,\gamma,00}(|\xi|) + \sum_{k=0}^{\infty} 2^{-k\lambda} m_{\lambda,\gamma,k}(|\xi|).$$

Now, for $t > 0$, $k \in \mathbf{Z}^+$, $f \in \mathcal{S}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, define:

$$(S_{\lambda,\gamma,k})_t(f)(x) = (\widehat{f}(\xi) m_{\lambda,\gamma,k}(t|\xi|))^\vee(x) \tag{3.10}$$

and

$$(S_{\lambda,\gamma,k})_*(f)(x) = \sup_{t>0} |(S_{\lambda,\gamma,k})_t(f)(x)|. \tag{3.11}$$

Similarly we define:

$$(S_{\lambda,\gamma,00})_*(f)(x) = \sup_{t>0} |(\widehat{f}(\xi) m_{\lambda,\gamma,00}(t\xi))^{\vee}(x)|.$$

With the notation just introduced we can write:

$$B_*^{\lambda,\gamma}(f) \leq (S_{\lambda,\gamma,00})_*(f) + \sum_{k=0}^{\infty} 2^{-k\lambda} (S_{\lambda,\gamma,k})_*(f). \tag{3.12}$$

Clearly this implies:

$$\begin{aligned} &\|B_*^{\lambda,\gamma}(f)\|_{L^2(w_{\lambda,\mu})\to L^2(w_{\lambda,\mu})} \\ \leq &\|(S_{\lambda,\gamma,00})_*\|_{L^2(w_{\lambda,\mu})\to L^2(w_{\lambda,\mu})} \\ &\qquad + \sum_{k=0}^{\infty} 2^{-k\lambda} \|(S_{\lambda,\gamma,k})_*\|_{L^2(w_{\lambda,\mu})\to L^2(w_{\lambda,\mu})}. \end{aligned} \tag{3.13}$$

Therefore, Proposition (3.1.3, page 29) can be proved by showing that:

$$\sum_{k=0}^{\infty} 2^{-k\lambda} \|(S_{\lambda,\gamma,k})_*\|_{L^2(w_{\lambda,\mu})\to L^2(w_{\lambda,\mu})} < \infty. \tag{3.14}$$

Now we shall prove that $\omega_{\lambda\mu}$ is an A_2 weight, that is, we need to check that:

$$\sup_B \left(\frac{1}{|B|} \int_B \omega_{\lambda\mu}(|x|) dx \right) \left(\frac{1}{|B|} \int_B \frac{1}{\omega_{\lambda\mu}(|x|)} dx \right) < \infty,$$

where the supremum is taken over the balls $B \in \mathbf{R}^n$. For the purpose of proving this, we will say that a ball $B(x_0, R)$ is **of type I** if $R < \frac{1}{4}|x_0|$, and that $B(x_0, R)$ is **of type II** otherwise. We're going to show that the supremum taken over all the ball of type I is finite. Let's assume that $B(x_0, R)$ is of type I, and consider any two points $y, z \in B(x_0, R)$. We can assume without loss of generality that $|y| \leq |z|$. It follows that $1 \leq \frac{|z|}{|y|} \leq \frac{5}{3}$. As $\omega_{\lambda\mu}$ is a decreasing function (because it's continuous, it's piecewise differentiable and the derivative is negative when defined), we have:

$$\omega_{\lambda\mu}(|z|) \leq \omega_{\lambda\mu}(|y|) \leq \omega_{\lambda\mu}\left(\frac{3}{5}|z|\right),$$

that is

$$1 \leq \frac{\omega_{\lambda\mu}(|y|)}{\omega_{\lambda\mu}(|z|)} \leq \frac{\omega_{\lambda\mu}(\frac{3}{5}|z|)}{\omega_{\lambda\mu}(|z|)}.$$

In order to estimate the latter, we study three cases:

Case 1: $|z| \leq 1$.

Then

$$\begin{aligned} \frac{\omega_{\lambda\mu}(\frac{3}{5}|z|)}{\omega_{\lambda\mu}(|z|)} &\leq \sup_{0<t\leq 1} \frac{\omega_{\lambda\mu}(\frac{3}{5}t)}{\omega_{\lambda\mu}(t)} \\ &= \frac{\frac{1}{(\frac{3}{5}t)^{2\lambda+1}}}{\frac{1}{t^{2\lambda+1}}} \\ &= \frac{t^{2\lambda+1}}{(\frac{3}{5}t)^{2\lambda+1}} = \left(\frac{5}{3}\right)^{2\lambda+1}. \end{aligned}$$

Case 2: $\frac{3}{5}|z| \leq 1 < |z|$.

Then

$$\begin{aligned}
\frac{\omega_{\lambda\mu}(\frac{3}{5}|z|)}{\omega_{\lambda\mu}(|z|)} &\leq \sup_{1<t\leq\frac{5}{3}} \frac{\omega_{\lambda\mu}(\frac{3}{5}t)}{\omega_{\lambda\mu}(t)} \\
&= \sup_{1<t\leq\frac{5}{3}} \frac{\frac{1}{(\frac{3}{5}t)^{2\lambda+1}}}{\frac{1}{t^{2\lambda+1}(\log(et))^{\mu}}} \\
&= \sup_{1<t\leq\frac{5}{3}} \frac{t^{2\lambda+1}(\log(et))^{\mu}}{(\frac{3}{5}t)^{2\lambda+1}} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \sup_{1<t\leq\frac{5}{3}} (\log(et))^{\mu} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \left(\log\left(e\frac{5}{3}\right)\right)^{\mu}.
\end{aligned}$$

Case 3: $1 < \frac{3}{5}|z|$.

Then

$$\begin{aligned}
\frac{\omega_{\lambda\mu}(\frac{3}{5}|z|)}{\omega_{\lambda,\mu}(|z|)} &\leq \sup_{\frac{5}{3}<t} \frac{\omega_{\lambda\mu}(\frac{3}{5}t)}{\omega_{\lambda\mu}(t)} \\
&= \sup_{\frac{5}{3}<t} \frac{\frac{1}{(\frac{3}{5}t)^{2\lambda+1}\left(\log\left(e\frac{3}{5}t\right)\right)^{\mu}}}{\frac{1}{t^{2\lambda+1}(\log(et))^{\mu}}} \\
&= \sup_{\frac{5}{3}<t} \frac{t^{2\lambda+1}(\log(et))^{\mu}}{(\frac{3}{5}t)^{2\lambda+1}\left(\log\left(e\frac{3}{5}t\right)\right)^{\mu}} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \sup_{\frac{5}{3}<t} \frac{(\log(et))^{\mu}}{\left(\log\left(e\frac{3}{5}t\right)\right)^{\mu}} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \left(\sup_{\frac{5}{3}<t} \frac{\log(et)}{\log\left(e\frac{3}{5}t\right)}\right)^{\mu} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \left(\sup_{\frac{5}{3}<t} \frac{\log\left(e\frac{3}{5}t\frac{5}{3}\right)}{\log\left(e\frac{3}{5}t\right)}\right)^{\mu}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{5}{3}\right)^{2\lambda+1} \left(\sup_{\frac{5}{3}<t} \frac{\log\left(e\frac{3}{5}t\right) + \log\left(\frac{5}{3}\right)}{\log\left(e\frac{3}{5}t\right)} \right)^{\mu} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \left(\sup_{\frac{5}{3}<t} \left(1 + \frac{\log\left(\frac{5}{3}\right)}{\log\left(e\frac{3}{5}t\right)}\right) \right)^{\mu} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \left(1 + \sup_{\frac{5}{3}<t} \frac{\log\left(\frac{5}{3}\right)}{\log\left(e\frac{3}{5}t\right)} \right)^{\mu} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \left(1 + \log\left(\frac{5}{3}\right) \right)^{\mu} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \left(\log\left(e\frac{5}{3}\right) \right)^{\mu}.
\end{aligned}
$$

As those three cases cover any possible situation, eventually we have:

$$
\begin{aligned}
\frac{\omega_{\lambda\mu}(\frac{3}{5}|z|)}{\omega_{\lambda\mu}(|z|)} &\leq \max\left\{ \left(\frac{5}{3}\right)^{2\lambda+1}, \left(\frac{5}{3}\right)^{2\lambda+1} \left(\log\left(e\frac{5}{3}\right)\right)^{\mu} \right\} \\
&= \left(\frac{5}{3}\right)^{2\lambda+1} \left(\log\left(e\frac{5}{3}\right)\right)^{\mu},
\end{aligned}
$$

which leads to the inequality:

$$
1 \leq \frac{\omega_{\lambda\mu}(|y|)}{\omega_{\lambda\mu}(|z|)} \leq \left(\frac{5}{3}\right)^{2\lambda+1} \left(\log\left(e\frac{5}{3}\right)\right)^{\mu},
$$

for any $y, z \in B(x_0, R)$. This implies that there exist positive constants $C_{1,\lambda\mu}$, $C_{2\lambda\mu}$ (namely, $C_{2,\lambda\mu} := \left(\frac{5}{3}\right)^{2\lambda+1} \log\left(e\frac{5}{3}\right)$ and $C_{1,\lambda\mu} := \frac{1}{C_{2,\lambda\mu}}$), independent of the choice of the ball (as long as it is of type I) such that:

$$
C_{1,\lambda\mu}\, \omega_{\lambda\mu}(|x_0|) \leq \left(\frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} \omega_{\lambda\mu}(|x|) dx \right) \leq C_{2\lambda\mu}\, \omega_{\lambda\mu}(|x_0|)
$$

and

$$C_{1,\lambda\mu}\,\frac{1}{\omega_{\lambda\mu}(|x_0|)} \leq \left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)}\frac{1}{\omega_{\lambda\mu}(|x|)}dx\right) \leq C_{2,\lambda\mu}\,\frac{1}{\omega_{\lambda\mu}(|x_0|)}$$

for any ball $B(x_0, R)$ of type I.

Therefore:

$$\begin{aligned}
&\sup_{B(x_0,R)}\left(\frac{1}{|B(x_0,R)|^2}\int_{B(x_0,R)}\omega_{\lambda\mu}(|x|)dx\right)\left(\int_{B(x_0,R)}\frac{dx}{\omega_{\lambda\mu}(|x|)}\right)\\
&\leq C_{2,\lambda\mu}\,\omega_{\lambda\mu}(|x_0|)\,C_{2,\lambda\mu}\frac{1}{\omega_{\lambda\mu}(|x_0|)}\\
&= C_{2,\lambda\mu}^2 < \infty
\end{aligned}$$

where the supremum is taken over all the ball of type I (i.e., such that $R < \frac{1}{4}|x_0|$).

If $B(x_0, R)$ is of type II (that is, not of type I), then we have that $B(x_0, R) \subset B(0, 5R)$. Therefore:

$$\begin{aligned}
&\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)}\omega_{\lambda\mu}(|x|)dx\right)\\
&\leq\left(\frac{|B(0,5R)|}{|B(x_0,R)||B(0,5R)|}\int_{B(0,5R)}\omega_{\lambda\mu}(|x|)dx\right)\\
&=\left(\frac{5^n}{|B(0,5R)|}\int_{B(0,5R)}\omega_{\lambda\mu}(|x|)dx\right)\\
&=\frac{C_n}{R^n}\int_0^{5R}\omega_{\lambda\mu}(r)r^{n-1}dr,
\end{aligned} \tag{3.15}$$

for a purely dimensional constant C_n. Similarly:

$$\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)}\frac{1}{\omega_{\lambda\mu}(|x|)}dx\right) \leq \frac{C_n}{R^n}\int_0^{5R}\frac{1}{\omega_{\lambda\mu}(r)}r^{n-1}dr.$$

Therefore:

$$
\begin{aligned}
&\sup_{B(x_0,R) \text{ of type II}} \left(\frac{1}{|B(x_0,R)|^2}\int_{B(x_0,R)} \omega_{\lambda\mu}(|x|)dx\right)\left(\int_{B(x_0,R)} \frac{dx}{\omega_{\lambda\mu}(|x|)}\right)\\
&\leq \sup_{R>0} \frac{C_n^2}{R^{2n}} \left(\int_0^{5R} \omega_{\lambda\mu}(r) r^{n-1} dr\right)\left(\int_0^{5R} \frac{1}{\omega_{\lambda\mu}(r)} r^{n-1} dr\right)\\
&= C_n^2 \sup_{R>0} F_{n,\lambda,\mu}(R),
\end{aligned}
\tag{3.16}
$$

where

$$
F_{n,\lambda,\mu}(R) = \frac{1}{R^{2n}} \left(\int_0^{5R} \omega_{\lambda\mu}(r) r^{n-1} dr\right)\left(\int_0^{5R} \frac{1}{\omega_{\lambda\mu}(r)} r^{n-1} dr\right).
$$

$F_{n,\lambda,\mu}$ is clearly a positive valued, differentiable function of variable $R > 0$.

For $0 < R \leq \frac{1}{5}$ we have:

$$
\begin{aligned}
F_{n,\lambda,\mu}(R) &= \frac{1}{R^{2n}} \left(\int_0^{5R} \frac{1}{r^{2\lambda+1}} r^{n-1} dr\right)\left(\int_0^{5R} r^{2\lambda+1} r^{n-1} dr\right)\\
&= \frac{1}{R^{2n}} \left(\int_0^{5R} r^{n-1-2\lambda-1} dr\right)\left(\int_0^{5R} r^{n+2\lambda} dr\right)\\
&= \frac{1}{R^{2n}} \frac{(5R)^{n-1-2\lambda}}{(n-1-2\lambda)} \frac{(5R)^{n+2\lambda+1}}{(n+2\lambda+1)}\\
&= C_{\lambda,n} R^{-2n+n-1-2\lambda+n+2\lambda+1} = C_{\lambda,n},
\end{aligned}
$$

with $C_{\lambda,n} = \frac{25^n}{(n-1-2\lambda)(n+1+2\lambda)}$.

For $R > \frac{1}{5}$ we have:

$$
\begin{aligned}
F_{n,\lambda,\mu}(R) &= \frac{1}{R^{2n}} \left(C_{\lambda,n,1} + \int_1^{5R} \frac{r^{n-2\lambda-2}}{(\log(er))^{\mu}} dr\right)\\
&\quad \cdot \left(C_{\lambda,n,2} + \int_1^{5R} (\log(er))^{\mu} r^{n+2\lambda} dr\right).
\end{aligned}
$$

In view of Lemma (3.1.10, page 50), there exist constants $C_{n,\lambda,\mu,1}$, $C_{n,\lambda,\mu,2}$ and $C_{n,\lambda,\mu,3}$ such that:

$$\begin{aligned} F_{n,\lambda,\mu}(R) &\leq \frac{1}{R^{2n}}\left(C_{\lambda,n,1} + C_{n,\lambda,\mu,1}\frac{R^{n-2\lambda-1}}{(\log(eR))^{\mu}}\right) \\ &\quad \cdot \left(C_{\lambda,n,2} + C_{n,\lambda,\mu,2}(\log(er))^{\mu} R^{n+2\lambda+1} dr\right) \end{aligned}$$

for all $R \geq C_{n,\lambda,\mu,3}$. As we're assuming that $2\lambda + 1 < n$ (cf. Proposition 3.1.3, page 29), this implies that:

$$\sup_{R \geq \max\{1/5,\, C_{n,\lambda,\mu,3}\}} F_{n,\lambda,\mu}(R) < \infty.$$

If $C_{n,\lambda,\mu,3} > \frac{1}{5}$, we still need to show that $\sup_{1/5 \leq R \leq C_{n,\lambda,\mu,3}} F_{n,\lambda,\mu}(R) < \infty$. But this is obvious because $F_{n,\lambda,\mu}$ is continuous on $[1/5, \infty)$ and the interval $[1/5, C_{n,\lambda,\mu,3}]$ is compact.

This concludes the proof that $\omega_{\lambda\mu} \in A_2$.

Since $(S_{\lambda,\gamma,00})_*$ and any sum of finitely many operators of the family $\{(S_{\lambda,\gamma,k})_*\}_{k=0}^{\infty}$ is pointwise controlled by the Hardy–Littlewood maximal operator, which is bounded on $L^2(\omega_{\lambda\mu})$ (because we just proved that $\omega_{\lambda\mu}$ is an A_2 weight), we focus attention on $(S_{\lambda,\gamma,k})_*$ for k big enough.

Recall that the multipliers $m_{\lambda,\gamma}$ satisfy the inequalities in (2.7, page 11). We define related functions

$$\widetilde{m}_{\lambda,\gamma,k}(t) = 2^{-k}\, t\, \frac{d}{dt} m_{\lambda,\gamma,k}(t)\,,$$

which obviously satisfies estimates (2.7) with other constants $\widetilde{C}_\ell$ in place of C_ℓ.

Next we introduce the multiplier operator $(\widetilde{S}_{\lambda,\gamma,k})_t$ which is the analogous of $(S_{\lambda,\gamma,k})_t$:

$$(\widetilde{S}_{\lambda,\gamma,k})_t(f)(x) = (\widehat{f}(\xi)\widetilde{m}_{\lambda,\gamma,k}(t|\xi|))^\vee(x)$$

and the $L^2(\omega_{\lambda\mu})$-bounded maximal multiplier operator

$$(\widetilde{S}_{\lambda,\gamma,k})_*(f)(x) = \sup_{t>0} |(\widetilde{S}_{\lambda,\gamma,k})_t(f)(x)|,$$

as well as the continuous square functions

$$G_{\lambda,\gamma,k}(f)(x) = \left(\int_0^\infty |(S_{\lambda,\gamma,k})_t(f)(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

and

$$\widetilde{G}_{\lambda,\gamma,k}(f)(x) = \left(\int_0^\infty |(\widetilde{S}_{\lambda,\gamma,k})_t(f)(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

The operators $(S_{\lambda,\gamma,k})_t$ and $(\widetilde{S}_{\lambda,\gamma,k})_t$ are related. For $f \in L^2(\omega_{\lambda\mu})$ and $t > 0$ we have

$$\frac{d}{dt}\left((S_{\lambda,\gamma,k})_t(f)(x)\right) = \frac{2^k}{t}(\widetilde{S}_{\lambda,\gamma,k})_t(f)(x).$$

Indeed, this operator identity is obvious for Schwartz functions f by the Lebesgue dominated convergence theorem, and thus it holds for $f \in L^2(\omega_{\lambda\mu})$ by density.

The quadratic operators $G_{\lambda,\gamma,k}$ and $\widetilde{G}_{\lambda,\gamma,k}$, as well as $\widetilde{m}_{\lambda,\gamma,k}$ and $(\widetilde{S}_{\lambda,\gamma,k})_t$, make their appearance in the application of the fundamental theorem of calculus in the following context:

$$\begin{aligned}
|(S_{\lambda,\gamma,k})_t(f)(x)|^2 &= 2\,\mathrm{Re}\int_0^t \overline{(S_{\lambda,\gamma,k})_u(f)(x)}\,\frac{d}{du}(S_{\lambda,\gamma,k})_u(f)(x)\,du \\
&= \frac{2}{2^{-k}}\,\mathrm{Re}\int_0^t \overline{(S_{\lambda,\gamma,k})_u(f)(x)}\,(\widetilde{S}_{\lambda,\gamma,k})_u(f)(x)\,\frac{du}{u}\,,
\end{aligned}$$

which is valid for all functions f in $L^2(\omega_{\lambda\mu})$ and almost all $x \in \mathbf{R}^n$. This identity uses the fact that for almost all $x \in \mathbf{R}^n$ we have

$$\lim_{t\to 0}(S_{\lambda,\gamma,k})_t(f)(x) = 0 \tag{3.17}$$

when $f \in L^2(\omega_{\lambda\mu})$. To see this, we observe that for Schwartz functions, (3.17) is trivial by the Lebesgue dominated convergence theorem, while for general f in $L^2(\omega_{\lambda\mu})$ it is a consequence of Proposition (3.1.2, page 27), since $(S_{\lambda,\gamma,k})_*(f) \le C_k\,M(f)$, where M is the Hardy–Littlewood maximal operator. Consequently,

$$\begin{aligned}
|(S_{\lambda,\gamma,k})_t(f)(x)|^2 &\le 2^{k+1}\int_0^t |(S_{\lambda,\gamma,k})_u(f)(x)|\,|(\widetilde{S}_{\lambda,\gamma,k})_u(f)(x)|\,\frac{du}{u} \\
&\le 2^{k+1}\,|G_{\lambda,\gamma,k}(f)(x)|\,|\widetilde{G}_{\lambda,\gamma,k}(f)(x)|
\end{aligned} \tag{3.18}$$

for all $t > 0$, for $f \in L^2(\omega_{\lambda\mu})$ and for almost all $x \in \mathbf{R}^n$. It follows that

$$\big\|(S_{\lambda,\gamma,k})_*(f)\big\|^2_{L^2(\omega_{\lambda\mu})} \le 2^{k+1}\,\big\|G_{\lambda,\gamma,k}(f)\big\|_{L^2(\omega_{\lambda\mu})}\big\|\widetilde{G}_{\lambda,\gamma,k}(f)\big\|_{L^2(\omega_{\lambda\mu})}\,, \tag{3.19}$$

and the asserted boundedness of $(S_{\lambda,\gamma,k})_*$ reduces to that of the continuous square functions $G_{\lambda,\gamma,k}$ and $\widetilde{G}_{\lambda,\gamma,k}$ on weighted L^2 spaces with suitable constants depending on k. The proof of the boundedness of $\widetilde{G}_{\lambda,\gamma,k}$ can be obtained by replacing $m_{\lambda,\gamma,k}$ with $\widetilde{m}_{\lambda,\gamma,k}$ in the proof of the boundedness of $G_{\lambda,\gamma,k}$.

The boundedness of $G_{\lambda,\gamma,k}$ on $L^2(\omega_{\lambda\mu})$ is a consequence of the following lemma.

Lemma 3.1.5. *For $k > 4$ we have:*

$$\begin{aligned}\int_{\mathbf{R}^n}\int_1^2 |(S_{\lambda,\gamma,k})_{a\,t}(f)(x)|^2\frac{dt}{t}\,\omega_{\lambda\mu}(|x|)dx \\ \leq C_{n,\lambda,\mu,\gamma}\frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}\int_{\mathbf{R}^n}|f(x)|^2\,\omega_{\lambda\mu}(|x|)dx\end{aligned} \tag{3.20}$$

for all $a > 0$ and for all functions f in $L^2(\omega_{\lambda\mu})$.

Assuming the statement of the lemma, we conclude the proof of Proposition 3.1.3 as follows. We take a Schwartz function ψ such that $\widehat{\psi}$ is supported in an annulus of radii $\frac{1}{4}$ and 4, with $\widehat{\psi}(\xi) = 1$ whenever $1/2 \leq |\xi| \leq 2$ and we let $\psi_{2^l}(x) = 2^{-nl}\psi(2^{-l}x)$. We make the observation that if $1 - 5\,2^{-k} \leq t|\xi| \leq 1 - 2^{-k}$ and $2^{l-1} \leq t \leq 2^l$, then $1/2 \leq 2^l|\xi| \leq 2$, since $2^{-k} < 1/10$. This implies that $\widehat{\psi}(2^l\xi) = 1$ on the support of the function $\xi \mapsto m_{\lambda,\gamma,k}(t|\xi|)$. Hence

$$(S_{\lambda,\gamma,k})_t(f) = (S_{\lambda,\gamma,k})_t(\psi_{2^l} * f)$$

whenever $2^{l-1} \le t \le 2^l$. The previous lemma, together with last observation, will allow us to control the operators $G_{\lambda,\gamma,k}$ (hence, the maximal operators $(S_{\lambda,\gamma,k})_*$):

$$\begin{aligned}
\|G_{\lambda,\gamma,k}(f)\|^2_{L^2(\omega_{\lambda\mu})} &= \int_{\mathbf{R}^n} |G_{\lambda,\gamma,k}(f)(x)|^2 \omega_{\lambda\mu}(x)dx \\
&= \int_{\mathbf{R}^n} \int_0^\infty |(S_{\lambda,\gamma,k})_t(f)(x)|^2 \frac{dt}{t} \omega_{\lambda\mu}(x)dx \\
&= \int_{\mathbf{R}^n} \sum_{l\in\mathbf{Z}} \int_{2^{l-1}}^{2^l} |(S_{\lambda,\gamma,k})_t(f)(x)|^2 \frac{dt}{t} \omega_{\lambda\mu}(x)dx \\
&= \int_{\mathbf{R}^n} \sum_{l\in\mathbf{Z}} \int_{2^{l-1}}^{2^l} |(S_{\lambda,\gamma,k})_t(f * \psi_{2^l})(x)|^2 \frac{dt}{t} \omega_{\lambda\mu}(x)dx \\
&= \int_{\mathbf{R}^n} \sum_{l\in\mathbf{Z}} \int_1^2 |(S_{\lambda,\gamma,k})_{2^{l-1}t}(f * \psi_{2^l})(x)|^2 \frac{dt}{t} \omega_{\lambda\mu}(x)dx \\
&= \sum_{l\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_1^2 |(S_{\lambda,\gamma,k})_{2^{l-1}t}(f * \psi_{2^l})(x)|^2 \frac{dt}{t} \omega_{\lambda\mu}(x)dx.
\end{aligned}$$

Because of Lemma 3.1.5 at page 41, with $a := 2^{l-1}$ and $f * \psi_{2^l}$ playing the role of f, we now have:

$$\begin{aligned}
\|G_{\lambda,\gamma,k}(f)\|^2_{L^2(\omega_{\lambda\mu})} &\le \sum_{l\in\mathbf{Z}} C_{n,\lambda,\gamma} \frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}} \int_{\mathbf{R}^n} |f * \psi_{2^l}(x)|^2 \omega_{\lambda\mu}(x)dx \\
&= C_{n,\lambda,\gamma} \frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}} \int_{\mathbf{R}^n} \sum_{l\in\mathbf{Z}} |f * \psi_{2^l}(x)|^2 \omega_{\lambda\mu}(x)dx \\
&= C_{n,\lambda,\gamma} \frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}} \left\| \left(\sum_{l\in\mathbf{Z}} |f * \psi_{2^l}|^2 \right)^{\frac{1}{2}} \right\|^2_{L^2(\mathbf{R}^n, \omega_{\lambda\mu})}.
\end{aligned}$$

A randomization argument relates the weighted L^2 norm of the square function to the L^2 norm of a linear expression involving the Rademacher functions as in

$$\Big\|\Big(\sum_{k\in\mathbf{Z}}|\psi_{2^k}*f|^2\Big)^{\frac{1}{2}}\Big\|^2_{L^2(\omega_{\lambda,\mu})}=\int_0^1\Big\|\sum_{k\in\mathbf{Z}}r_k(t)(\psi_{2^k}*f)\Big\|^2_{L^2(\omega_{\lambda,\mu})}dt\,,$$

where r_k denotes a renumbering of the Rademacher functions indexed by the entire set of integers. For each $t\in[0,1]$ the operator

$$M_t(f)=\sum_{k\in\mathbf{Z}}r_k(t)(\psi_{2^k}*f)$$

is associated with a multiplier $m=m_t$ that satisfies Mihlin's condition uniformly in t. At this point, we should recall the Mihlin-Hormander multiplier theorem.

Theorem 3.1.6. *Let m be a complex-valued bounded function on $\mathbf{R}^n\setminus\{0\}$ that satisfies Mihlin's condition, i.e., there exists a constant $A>0$ such that, for all miltiindices $\alpha=(\alpha_1,...,\alpha_n)$ with $|\alpha|\le[\frac{n}{2}]+1$, we have:*

$$|\partial^\alpha m(\xi)|\le A|\xi|^{-|\alpha|}\tag{3.21}$$

Then, for all $1<p<\infty$, m lies in $\mathcal{M}_p(\mathbf{R}^n)$ and the following estimate is valid:

$$\|m\|_{\mathcal{M}_p}\le C_n\max(p,(p-1)^{-1})(A+\|m\|_{L^\infty})$$

A proof of this theorem can be found at page 367 of [7]. It's a nice exercise to show that M_t is given by convolution with a standard kernel and, in view of theorem (3.1.6, page 43), it's also a Calderón-Zygmund operator (that is, it's in $CZO(\delta, A, B)$) with constants δ, A, B independent of t. Moreover, in view of the following theorem:

Theorem 3.1.7. *Let T be a Calderón-Zygmund operator with constants δ, A, B. Then for all $1 < p < \infty$ and for every weight $\omega \in A_p$ there is a constant $C_p = C_p(n, [\omega]_{A_\infty}, \delta, A + B)$ such that*

$$\|T(f)\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)}$$

for all smooth functions with compact support.

(applied to $p := 2, \omega := w_{\lambda\mu}$ and $T := M_t$) M_t's are also bounded on $L^2(w_{\lambda\mu})$ with a constant independent of t (a proof of the theorem above can be found at page 320 of [8]). We deduce that

$$\left\|G_{\lambda,\gamma,k}(f)\right\|_{L^2(\omega_{\lambda\mu})} + \left\|\widetilde{G}_{\lambda,\gamma,k}(f)\right\|_{L^2(\omega_{\lambda\mu})} \leq C'_{n,\lambda,\gamma} \left(\frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}\right)^{\frac{1}{2}} \left\|f\right\|_{L^2(\omega_{\lambda\mu})}.$$

We now recall estimate (3.19, page 40) to obtain:

$$\left\|(S_{\lambda,\gamma,k})_*(f)\right\|_{L^2(\omega_{\lambda\mu})} \leq C'(n, \lambda, \gamma) \left(\frac{2^{2k\lambda}}{k^{2\gamma-\mu}}\right)^{1/2} \left\|f\right\|_{L^2(\omega_{\lambda\mu})}. \tag{3.22}$$

that is:

$$\left\|(S_{\lambda,\gamma,k})_*(f)\right\|_{L^2(\omega_{\lambda\mu}) \to L^2(\omega_{\lambda\mu})} \leq C'(n, \lambda, \gamma) \left(\frac{2^{2k\lambda}}{k^{2\gamma-\mu}}\right)^{1/2}. \tag{3.23}$$

The estimate above shows that inequality (3.14, page 32) holds if $\gamma - \frac{\mu}{2} > 1$, that is, if:

$$\mu < 2\gamma - 2. \tag{3.24}$$

That is the second condition on μ required in Proposition (3.1.3, page 29). In turn, inequality (3.14) implies Proposition (3.1.3, page 29). Observe that the condition $\gamma - \frac{\mu}{2} > 1$ is written in the form $\mu < 2\gamma - 2$ in Proposition (3.1.3, page 29). Proposition 3.1.3 is now proved modulo the proof of Lemma (3.1.5, page 41) and the fact that the following means:

$$m_t(\xi) = \sum_{k \in \mathbf{Z}} r_k(t)\, \widehat{\psi}(2^k \xi) \tag{3.25}$$

satisfy condition (3.21), page 43, uniformly in t, a fact that we prove in the next lemma.

Lemma 3.1.8. *Let m_t be defined as in 3.25. Then there exists a constant $A = A(n, \psi) > 0$ such that*

$$|\partial^\alpha m_t(\xi)| \leq A|\xi|^{-|\alpha|}$$

for all $t > 0$

Proof. Let us observe that, due to the support condition on $\widehat{\psi}$, if $\widehat{\psi}(2^k\xi) \neq 0$ then we have:

$$\frac{1}{4} \leq 2^k|\xi| \leq 4$$

that is equivalent to

$$-2 - \log_2(|\xi|) = \log_2\left(\frac{1/4}{|\xi|}\right) \le k \le \log_2\left(\frac{4}{|\xi|}\right) = 2 - \log_2(|\xi|)$$

that is, for any fixed ξ, only 4 terms (at most) in the sum (3.25) are non-zero. Indeed we have:

$$m_t(\xi) = \sum_{\lceil k=-2-\log_2(|\xi|)\rceil}^{\lfloor 2-\log_2(|\xi|)\rfloor} r_k(t)\, \widehat{\psi}(2^k\xi) \tag{3.26}$$

where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a real number a, and $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a real number a. Then, for a multiindex $\alpha \in \mathbf{Z}^+ \times ... \times \mathbf{Z}^+$ (n times), we can write the α-derivative with respect to ξ of m_t as follows:

$$(\partial^\alpha m_t)(\xi) = \sum_{\lceil k=-2-\log_2(|\xi|)\rceil}^{\lfloor 2-\log_2(|\xi|)\rfloor} r_k(t)\, \partial^\alpha(\widehat{\psi})(2^k\xi)\, 2^{k\,|\alpha|} \tag{3.27}$$

Therefore:

$$\begin{aligned}
|(\partial^\alpha m_t)(\xi)| &\le \sum_{\lceil k=-2-\log_2(|\xi|)\rceil}^{\lfloor 2-\log_2(|\xi|)\rfloor} |r_k(t)| \left\|\partial^\alpha(\widehat{\psi})\right\|_\infty 2^{k\,|\alpha|} \\
&\le \sum_{\lceil k=-2-\log_2(|\xi|)\rceil}^{\lfloor 2-\log_2(|\xi|)\rfloor} 1 \left\|\partial^\alpha(\widehat{\psi})\right\|_\infty \left(\frac{4}{|\xi|}\right)^{|\alpha|} \\
&= \frac{1}{|\xi|^{|\alpha|}} \left(\sum_{\lceil k=-2-\log_2(|\xi|)\rceil}^{\lfloor 2-\log_2(|\xi|)\rfloor} \left\|\partial^\alpha(\widehat{\psi})\right\|_\infty 4^{|\alpha|} \right) \\
&= \frac{1}{|\xi|^{|\alpha|}} \left(4 \left\|\partial^\alpha(\widehat{\psi})\right\|_\infty 4^{|\alpha|} \right)
\end{aligned}$$

The last equality holds for almost every ξ. Since $(\partial^\alpha m_t)$ is smooth, the inequality holds for every $\xi \in \mathbf{R}^n$. So, we checked (3.21), page 43, for all multiindexes α with $A = \sup_{|\alpha|\leq[\frac{n}{2}]+1}\left(4\left\|\partial^\alpha(\widehat{\psi})\right\|_\infty 4^{|\alpha|}\right)$. Clearly, $A = A(n,\psi)$ doesn't depend on ξ nor on t. □

Now we can apply Theorem (3.1.6, page 43) with $p = 2$ to prove that $\widehat{m_t}$ is a Calderón-Zygmund operator with constant $B = B(n,\psi)$ independent of t. Then we apply Theorem (3.1.7, page 44) with T defined via the multiplier $\widehat{m_t}$ (that is, $T = M_t$).

This concludes the proof of Proposition (3.1.3, page 29), modulo Lemma (3.1.5, page 41).

In the next sections we will need asympthotic estimates for functions of the form:

$$f(x) = \int_{D_x} \frac{t^\alpha}{\log(t)^\gamma}dt,$$

where the domain of integration D_x is a certain subset of $\mathbf{R}$. We are going to state and prove such estimates in the following two lemmas. In the first one we will study the case $D_x = [x,\infty)$.

Lemma 3.1.9. *Let $x > e, \gamma > 0, \alpha < -1$. Then there are constants $C_{\alpha,\gamma} > 0$ and $C_\alpha > 0$ (namely, $C_{\alpha,\gamma} = \frac{1}{(\gamma-(\alpha+1))}$ and $C_\alpha = \frac{1}{(-(\alpha+1))}$) such that:*

$$C_{\alpha,\gamma}\frac{x^{\alpha+1}}{(\log(x))^\gamma} \leq \int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt \leq C_\alpha\frac{x^{\alpha+1}}{(\log(x))^\gamma}$$

for all $x > e$.

Furthermore, if $\gamma < 0$, then there exists constants C'_α, C''_α and $C'_{\alpha,\gamma}$ (namely,

$C'_{\alpha,\gamma} = \max\{1, e^{\frac{2\gamma}{\alpha+1}}\}$) *such that:*

$$C'_\alpha \frac{x^{\alpha+1}}{(\log(x))^\gamma} \leq \int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma} dt \leq C''_\alpha \frac{x^{\alpha+1}}{(\log(x))^\gamma}$$

for all $x > C'_{\alpha,\gamma}$.

Proof. As $\gamma \neq 0$, the fundamental theorem of calculus gives:

$$\begin{aligned} -\frac{x^{\alpha+1}}{(\log(x))^\gamma} &= \frac{t^{\alpha+1}}{(\log(t))^\gamma}\Big|_{t=x}^{t=\infty} \\ &= \int_x^\infty \frac{(\alpha+1)t^\alpha(\log(t))^\gamma - t^\alpha\gamma(\log(t))^{\gamma-1}}{(\log(t))^{2\gamma}} dt \\ &= (\alpha+1)\int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma} dt - \gamma\int_x^\infty \frac{t^\alpha}{(\log(t))^{\gamma+1}} dt \\ &= \int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}\left(\alpha + 1 - \frac{\gamma}{\log(t)}\right) dt. \end{aligned}$$

Then

$$\frac{x^{\alpha+1}}{(\log(x))^\gamma} = \int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}\left(\frac{\gamma}{\log(t)} - (\alpha+1)\right) dt. \tag{3.28}$$

In both cases ($\gamma < 0$ or $\gamma > 0$) we can assume $x > 1$. Then we study the two cases:

Case 1: $\gamma > 0$.

Then:

$$0 < -(\alpha+1) < \left(\frac{\gamma}{\log(t)} - (\alpha+1)\right) \leq \left(\frac{\gamma}{\log(x)} - (\alpha+1)\right).$$

Therefore,

$$\begin{aligned}
-(\alpha+1)\int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt &\le \frac{x^{\alpha+1}}{(\log(x))^\gamma} \\
&\le \left(\frac{\gamma}{\log(x)}-(\alpha+1)\right)\int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt.
\end{aligned}$$

As we also assume $x > e$ we have

$$\left(\frac{\gamma}{\log(x)}-(\alpha+1)\right) < \left(\frac{\gamma}{\log(e)}-(\alpha+1)\right) = (\gamma-(\alpha+1)).$$

Therefore:

$$\begin{aligned}
-(\alpha+1)\int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt &\le \frac{x^{\alpha+1}}{(\log(x))^\gamma} \\
&\le (\gamma-(\alpha+1))\int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt.
\end{aligned}$$

So:

$$\frac{1}{(\gamma-(\alpha+1))}\frac{x^{\alpha+1}}{(\log(x))^\gamma} \le \int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt \le \frac{1}{(-(\alpha+1))}\frac{x^{\alpha+1}}{(\log(x))^\gamma}$$

that is:

$$C_{\alpha,\gamma}\frac{x^{\alpha+1}}{(\log(x))^\gamma} \le \int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt \le C_\alpha\frac{x^{\alpha+1}}{(\log(x))^\gamma}.$$

Case 2: $\gamma < 0$.

As we're also assuming $x > 1$, we have (for $t > x$, as in equation (3.28)):

$$0 < \frac{\gamma}{\log(x)}-(\alpha+1) < \frac{\gamma}{\log(t)}-(\alpha+1) < -(\alpha+1).$$

As we additionally assume $x > \max\{1, e^{\frac{2\gamma}{\alpha+1}}\} =: C'_{\alpha,\gamma}$, then we have:

$$0 < -\frac{\alpha+1}{2} < \frac{\gamma}{\log(x)} - (\alpha+1) < \frac{\gamma}{\log(t)} - (\alpha+1) < -(\alpha+1)$$

and, as a consequence (in view of equation (3.28)):

$$\begin{aligned} -\frac{(\alpha+1)}{2}\int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt &\leq \frac{x^{\alpha+1}}{(\log(x))^\gamma} \\ &\leq -(\alpha+1)\int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt. \end{aligned}$$

Equivalently:

$$\begin{aligned} -\left(\frac{1}{\alpha+1}\right)\frac{x^{\alpha+1}}{(\log(x))^\gamma} &\leq \int_x^\infty \frac{t^\alpha}{(\log(t))^\gamma}dt \\ &\leq -\left(\frac{2}{\alpha+1}\right)\frac{x^{\alpha+1}}{(\log(x))^\gamma} \end{aligned}$$

for all $x \geq C'_{\alpha,\gamma}$, that is, we got the statement of the lemma with $C'_\alpha = -\left(\frac{1}{\alpha+1}\right)$ and $C''_\alpha = -\left(\frac{2}{\alpha+1}\right)$. □

Now we study the case $D_x = [e, x]$.

Lemma 3.1.10. *Let $\alpha > -1$ and $\gamma \in \mathbf{R}$. Then there exist constants $C^{(1)}_{\alpha,\gamma}$, $C^{(2)}_{\alpha,\gamma}$ and $C^{(3)}_{\alpha,\gamma}$ such that*

$$C^{(1)}_{\alpha,\gamma}\frac{x^{\alpha+1}}{(\log(x))^\gamma} \leq \int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt \leq C^{(2)}_{\alpha,\gamma}\frac{x^{\alpha+1}}{(\log(x))^\gamma}$$

for all $x > C^{(3)}_{\alpha,\gamma}$.

Proof. Using the fundamental theorem of calculus as before, for every $x > 1$ we have:

$$\begin{aligned} \frac{x^{\alpha+1}}{(\log(x))^{\gamma}} - e^{\alpha+1} &= \left.\frac{t^{\alpha+1}}{(\log(t))^{\gamma}}\right|_{t=e}^{t=x} \\ &= \int_e^x \frac{t^{\alpha}}{(\log(t))^{\gamma}} \left(\alpha + 1 - \frac{\gamma}{\log(t)}\right) dt. \end{aligned} \tag{3.29}$$

As the case $\gamma = 0$ is obvious, let's treat the two cases $\gamma > 0$ and $\gamma < 0$.

Case 1: $\gamma > 0$.

Observe that, for $e < t$, we have that $\alpha+1-\frac{\gamma}{\log(t)} > \alpha+1-\frac{\gamma}{\log(e)} = \alpha+1-\gamma$.

Let's treat two subcases of case 1.

Case 1a: $\alpha + 1 - \gamma > 0$ and $\gamma > 0$.

Then

$$(\alpha + 1 - \gamma) \int_e^x \frac{t^{\alpha}}{(\log(t))^{\gamma}} dt \le \frac{x^{\alpha+1}}{(\log(x))^{\gamma}} - e^{\alpha+1} \le (\alpha + 1) \int_e^x \frac{t^{\alpha}}{(\log(t))^{\gamma}} dt$$

so

$$\begin{aligned} (\alpha + 1 - \gamma) \int_e^x \frac{t^{\alpha}}{(\log(t))^{\gamma}} dt + e^{\alpha+1} &\le \frac{x^{\alpha+1}}{(\log(x))^{\gamma}} \\ &\le (\alpha + 1) \int_e^x \frac{t^{\alpha}}{(\log(t))^{\gamma}} dt + e^{\alpha+1}. \end{aligned}$$

Of course:

$$
\begin{aligned}
&(\alpha+1)\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt + e^{\alpha+1} \\
= &\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt\left((\alpha+1)+\frac{e^{\alpha+1}}{\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt}\right) \\
\leq &\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt\left((\alpha+1)+\frac{e^{\alpha+1}}{\int_e^{2e} \frac{t^\alpha}{(\log(t))^\gamma}dt}\right),
\end{aligned}
$$

provided that $x \geq 2e$. On the other hand, we obviously have

$$
(\alpha+1-\gamma)\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt + e^{\alpha+1} \geq (\alpha+1-\gamma)\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt.
$$

The last three (chains of) inequalities imply

$$
\begin{aligned}
&(\alpha+1-\gamma)\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt \\
\leq &\frac{x^{\alpha+1}}{(\log(x))^\gamma} \\
\leq &\left((\alpha+1)+\frac{e^{\alpha+1}}{\int_e^{2e} \frac{t^\alpha}{(\log(t))^\gamma}dt}\right)\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt
\end{aligned}
\tag{3.30}
$$

for every $x \geq 2e$, that is the conclusion of the lemma for the case “1a” with constants

$$
\begin{aligned}
C^{(1)}_{\alpha,\gamma} &= \frac{1}{\left((\alpha+1)+\frac{e^{\alpha+1}}{\int_e^{2e} \frac{t^\alpha}{(\log(t))^\gamma}dt}\right)} \\
C^{(2)}_{\alpha,\gamma} &= \frac{1}{(\alpha+1-\gamma)} \\
C^{(3)}_{\alpha,\gamma} &= 2e.
\end{aligned}
$$

Case 1b: $\alpha + 1 - \gamma \leq 0$.

In this case, $\frac{\gamma}{\alpha+1} \geq 1$, therefore $e \leq e^{\frac{\gamma}{\alpha+1}}$ and we can split the integral

$$\begin{aligned} &\int_e^x \frac{t^\alpha}{(\log(t))^\gamma} \left(\alpha + 1 - \frac{\gamma}{\log(t)}\right) dt \\ &= \int_e^{e^{\frac{2\gamma}{\alpha+1}}} \frac{t^\alpha}{(\log(t))^\gamma} \left(\alpha + 1 - \frac{\gamma}{\log(t)}\right) dt \\ &\quad + \int_{e^{\frac{2\gamma}{\alpha+1}}}^x \frac{t^\alpha}{(\log(t))^\gamma} \left(\alpha + 1 - \frac{\gamma}{\log(t)}\right) dt \\ &=: C_{1,\alpha,\gamma} + \int_{e^{\frac{2\gamma}{\alpha+1}}}^x \frac{t^\alpha}{(\log(t))^\gamma} \left(\alpha + 1 - \frac{\gamma}{\log(t)}\right) dt \end{aligned} \tag{3.31}$$

for all $x \geq e^{\frac{2\gamma}{\alpha+1}} \geq e^2$. The constant $C_{1,\alpha,\gamma}$ defined in the last equality may be negative. But, if $t \geq e^{\frac{2\gamma}{\alpha+1}}$, we have $\left(\alpha + 1 - \frac{\gamma}{\log(t)}\right) \geq \left(\alpha + 1 - \frac{\gamma}{\log(e^{\frac{2\gamma}{\alpha+1}})}\right) = \frac{\alpha+1}{2} > 0$. Then

$$\int_{e^{\frac{2\gamma}{\alpha+1}}}^x \frac{t^\alpha}{(\log(t))^\gamma} \left(\alpha + 1 - \frac{\gamma}{\log(t)}\right) dt \geq \int_{e^{\frac{2\gamma}{\alpha+1}}}^x \frac{t^\alpha}{(\log(t))^\gamma} dt \left(\frac{\alpha+1}{2}\right).$$

As we're assuming $\alpha > -1$, the right hand side is a positive function of x increasing to infinity. Therefore there exists a constant $C_{2,\alpha,\gamma} \geq e^{\frac{2\gamma}{\alpha+1}}$ such that:

$$\int_{e^{\frac{2\gamma}{\alpha+1}}}^{C_{2,\alpha,\gamma}} \frac{t^\alpha}{(\log(t))^\gamma} dt \left(\frac{\alpha+1}{2}\right) > |C_{1,\alpha,\gamma}|.$$

It follows that

$$0 < \int_e^{C_{2,\alpha,\gamma}} \frac{t^\alpha}{(\log(t))^\gamma} \left(\alpha + 1 - \frac{\gamma}{\log(t)}\right) dt =: C_{3,\alpha,\gamma}.$$

So, for every $x \geq C_{2,\alpha,\gamma} \geq e^{\frac{2\gamma}{\alpha+1}}$, it makes sense to write

$$\begin{aligned}&\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}\left(\alpha+1-\frac{\gamma}{\log(t)}\right)dt\\=&C_{3,\alpha,\gamma}+\int_{C_{2,\alpha,\gamma}}^x \frac{t^\alpha}{(\log(t))^\gamma}\left(\alpha+1-\frac{\gamma}{\log(t)}\right)dt.\end{aligned} \tag{3.32}$$

Comparing with the first equation of the proof (equation 3.29, page 51), we get:

$$\frac{x^{\alpha+1}}{(\log(x))^\gamma}-e^{\alpha+1}=C_{3,\alpha,\gamma}+\int_{C_{2,\alpha,\gamma}}^x \frac{t^\alpha}{(\log(t))^\gamma}\left(\alpha+1-\frac{\gamma}{\log(t)}\right)dt$$

for constants $C_{2,\alpha,\gamma} \geq e^{\frac{2\gamma}{\alpha+1}} \geq e^2$, $C_{3,\alpha,\gamma} > 0$ and for every $x \geq C_{2,\alpha,\gamma}$. Then

$$\frac{x^{\alpha+1}}{(\log(x))^\gamma}=C_{4,\alpha,\gamma}+\int_{C_{2,\alpha,\gamma}}^x \frac{t^\alpha}{(\log(t))^\gamma}\left(\alpha+1-\frac{\gamma}{\log(t)}\right)dt$$

for constants $C_{2,\alpha,\gamma} \geq e^{\frac{2\gamma}{\alpha+1}}$, $C_{4,\alpha,\gamma} = C_{3,\alpha,\gamma} + e^{\alpha+1} > 0$ and for every $x \geq C_{2,\alpha,\gamma}$. It follows:

$$\begin{aligned}&C_{4,\alpha,\gamma}+\int_{C_{2,\alpha,\gamma}}^x \frac{t^\alpha}{(\log(t))^\gamma}dt\left(\frac{\alpha+1}{2}\right)\\\leq&\frac{x^{\alpha+1}}{(\log(x))^\gamma}\\\leq&C_{4,\alpha,\gamma}+\int_{C_{2,\alpha,\gamma}}^x \frac{t^\alpha}{(\log(t))^\gamma}dt\ (\alpha+1).\end{aligned} \tag{3.33}$$

The conclusion is similar to that of case 1a.

Case 2: $\gamma < 0$.

Observe that in this case, for $e < t$, we have:

$$0 < \alpha+1 \leq \alpha+1-\frac{\gamma}{\log(t)} < \alpha+1-\frac{\gamma}{\log(e)} = \alpha+1-\gamma.$$

Therefore:

$$(\alpha+1)\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt \le \int_e^x \frac{t^\alpha}{(\log(t))^\gamma}\left(\alpha+1-\frac{\gamma}{\log(t)}\right)dt$$

and

$$\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}\left(\alpha+1-\frac{\gamma}{\log(t)}\right)dt \le (\alpha+1-\gamma)\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt.$$

Since equation (3.29, page 51):

$$(\alpha+1)\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt \le \frac{x^{\alpha+1}}{(\log(x))^\gamma} - e^{\alpha+1} \le (\alpha+1-\gamma)\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt.$$

Since $\alpha > -1$ we have that $\lim_{x\to\infty}\int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt = \infty$, therefore there exists $C^{(3)}_{\alpha,\gamma}$ such that:

$$\frac{x^{\alpha+1}}{(\log(x))^\gamma} \approx_{\alpha,\gamma} \int_e^x \frac{t^\alpha}{(\log(t))^\gamma}dt$$

on $\{x\in\mathbf{R} : x > C^{(3)}_{\alpha,\gamma}\}$. $\square$

3.2 An upper bound for $|\widehat{w_{\lambda\mu}}|$

Let $\phi \in C^\infty(\mathbf{R})$ satisfy $0 \leq \phi \leq 1$, $\operatorname{supp}(\phi) \subset [\frac{9}{10}, \frac{11}{10}]$, $\phi \equiv 1$ on $[\frac{19}{20}, \frac{21}{20}]$. Now define:

$$\omega^{(1)}_{\lambda,\mu}(t) = \omega_{\lambda\mu}(t)(1-\phi(t)) + \phi(t). \tag{3.34}$$

where $\omega_{\lambda\mu}$ was defined in (3.5, page 29). Define also:

$$w^{(1)}_{\lambda,\mu}(x) = \omega^{(1)}_{\lambda,\mu}(|x|) \tag{3.35}$$

for all $x \in \mathbf{R}^n \setminus \{0\}$. It's straightforward to verify that $w^{(1)}_{\lambda,\mu} \approx_{\lambda,\mu} w_{\lambda\mu}$, that is, $w^{(1)}_{\lambda,\mu}(x)$ and $w_{\lambda\mu}(x)$ are comparable (with respect to x) with comparability constants depending on λ and μ, but independent of n. In addition, $w^{(1)}_{\lambda,\mu}$ is smooth on $\mathbf{R}^n \setminus \{0\}$. The goal of this section is to prove the following theorem:

Theorem 3.2.1. *Let $w_{\lambda\mu}$ and $w^{(1)}_{\lambda,\mu}$ be defined as in (3.6, page 29) and (3.35, page 56) respectively. Then for every λ satisfying $\frac{n-1}{4} < \lambda < \frac{n-1}{2}$ and every μ satisfying $\frac{2\lambda+1}{n} < \mu < 2\gamma - 2$ there exists a constant $C_{n,\lambda,\mu}$ such that*

$$|\widehat{w_{\lambda\mu}}(\xi)| \leq \Omega_{\lambda,\mu}(\xi) := \begin{cases} C_{n,\lambda,\mu} \frac{1}{|\xi|^{n-2\lambda-1}\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} & \text{if} \quad |\xi| \leq 1, \\ C_{n,\lambda,\mu} \frac{1}{|\xi|^{n-2\lambda-1}} & \text{if} \quad |\xi| \geq 1 \end{cases} \tag{3.36}$$

and, for all λ *satisfying* $0 < \lambda < \frac{n-1}{2}$*, there exists a constant* $C'_{n,\lambda,\mu}$ *such that*

$$|\widehat{w^{(1)}_{\lambda,\mu}}(\xi)| \leq C'_{n,\lambda,\mu}\, \Omega_{\lambda,\mu}(\xi) \tag{3.37}$$

for all $\xi \in \mathbf{R}^n \setminus \{0\}$.

Let's prove (3.36, page 56).

As $w_{\lambda\mu}$ is radial, its Fourier transform is given by (cf. for example [7, pp. 428, 429]):

$$\widehat{w_{\lambda\mu}}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty \omega_{\lambda\mu}(r)\, J_{\frac{n-2}{2}}(2\pi|\xi|r)\, r^{\frac{n}{2}}\, dr$$

where $J_k(z)$ denotes the evaluations of the k^{th} Bessel function J_k at the point z. We will use the following asymptotic estimates:

$$|J_k(r)| \leq C_k\, r^k,$$

useful when $r \leq 2\pi$ and

$$|J_k(r)| \leq C_k\, r^{-\frac{1}{2}},$$

useful when $r \geq 2\pi$. In order to use the estimates above in our integral we will need to rewrite the domain of integration: $(0, \infty) = \left(0, \frac{1}{|\xi|}\right) \cup \left[\frac{1}{|\xi|}, \infty\right)$.

As the function $\omega_{\lambda\mu}$ is defined piecewise as well, we will need to study two cases separately:

Case 1: $\frac{1}{|\xi|} \leq 1$ (that is, $|\xi| \geq 1$). Then:

$$\widehat{w_{\lambda\mu}}(\xi) \quad = \quad \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \left(\int_0^{\frac{1}{|\xi|}} \omega_{\lambda\mu}(r)\, J_{\frac{n-2}{2}}(2\pi|\xi|r)\, r^{\frac{n}{2}}\, dr \right)$$

$$+\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\left(\int_{\frac{1}{|\xi|}}^{1}\omega_{\lambda\mu}(r)\,J_{\frac{n-2}{2}}(2\pi|\xi|r)\,r^{\frac{n}{2}}\,dr\right)$$
$$+\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\left(\int_{1}^{\infty}\omega_{\lambda\mu}(r)\,J_{\frac{n-2}{2}}(2\pi|\xi|r)\,r^{\frac{n}{2}}\,dr\right),$$

therefore

$$\begin{aligned}
|\widehat{w_{\lambda\mu}}(\xi)| &\leq \frac{C_n}{|\xi|^{\frac{n-2}{2}}}\left(\int_0^{\frac{1}{|\xi|}}\frac{1}{r^{2\lambda+1}}\,(2\pi|\xi|r)^{\frac{n-2}{2}}\,r^{\frac{n}{2}}\,dr\right)\\
&\quad+\frac{C_n}{|\xi|^{\frac{n-2}{2}}}\left(\int_{\frac{1}{|\xi|}}^{1}\frac{1}{r^{2\lambda+1}}\,(2\pi|\xi|r)^{-\frac{1}{2}}\,r^{\frac{n}{2}}\,dr\right)\\
&\quad+\frac{C_n}{|\xi|^{\frac{n-2}{2}}}\left(\int_{1}^{\infty}\frac{1}{r^{2\lambda+1}(\log(er))^{\mu}}\,(2\pi|\xi|r)^{-\frac{1}{2}}\,r^{\frac{n}{2}}\,dr\right)\\
&\leq C_n\left(\int_0^{\frac{1}{|\xi|}}r^{-2\lambda-1+\frac{n-2}{2}+\frac{n}{2}}\,dr\right)\\
&\quad+\frac{C_n}{|\xi|^{\frac{n-1}{2}}}\left(\int_{\frac{1}{|\xi|}}^{1}r^{-2\lambda-1-\frac{1}{2}+\frac{n}{2}}\,dr\right)\\
&\quad+\frac{C_n}{|\xi|^{\frac{n-1}{2}}}\left(\int_{1}^{\infty}\frac{r^{-2\lambda-1-\frac{1}{2}+\frac{n}{2}}}{(\log(er))^{\mu}}dr\right).
\end{aligned}$$

In order for the last integral to converge, we need to assume that $-2\lambda-1-\frac{1}{2}+\frac{n}{2}<-1$, that is, $\lambda>\frac{n-1}{4}$. Then:

$$\begin{aligned}
|\widehat{w_{\lambda\mu}}(\xi)| &\leq C_{n,\lambda}\left(r^{-2\lambda-1+n}\Big|_{r=0}^{r=\frac{1}{|\xi|}}\right)\\
&\quad-\frac{C'_{n,\lambda}}{|\xi|^{\frac{n-1}{2}}}\left(r^{-2\lambda-\frac{1}{2}+\frac{n}{2}}\Big|_{r=\frac{1}{|\xi|}}^{r=1}\right)\\
&\quad+\frac{C_{n,\lambda}}{|\xi|^{\frac{n-1}{2}}}\left(\int_{e}^{\infty}\frac{r^{-2\lambda-1-\frac{1}{2}+\frac{n}{2}}}{(\log(r))^{\mu}}dr\right),
\end{aligned}$$

where the constant can change between any two inequalities, but all the constants are positive. If $\lambda < \frac{n-1}{2}$ the first term is defined and the chain of inequalities continues:

$$\begin{aligned}
|\widehat{w_{\lambda\mu}}(\xi)| &\leq C_{n,\lambda} \frac{1}{|\xi|^{n-2\lambda-1}} \\
&\quad + \frac{C'_{n,\lambda}}{|\xi|^{\frac{n-1}{2}}} \left(\left(\frac{1}{|\xi|^{-2\lambda-\frac{1}{2}+\frac{n}{2}}} \right) - 1 \right) \\
&\quad + \frac{C'_{n,\lambda}}{|\xi|^{\frac{n-1}{2}}} \\
&= C_{n,\lambda} \frac{1}{|\xi|^{n-2\lambda-1}} \\
&\quad + C_{n,\lambda} \frac{1}{|\xi|^{n-2\lambda-1}} - \frac{C_{n,\lambda}}{|\xi|^{\frac{n-1}{2}}} \\
&\quad + \frac{C_{n,\lambda}}{|\xi|^{\frac{n-1}{2}}}.
\end{aligned}$$

The first assumption on λ we introduced during the computation $\left(\lambda > \frac{n-1}{4}\right)$ is in fact equivalent to $\frac{n-1}{2} > n - 2\lambda - 1$. As we are treating the case $|\xi| \geq 1$, this implies that we can control the absolute value of the last two terms with positive constant multiples of the first one. So the chain of inequalities continues as follows:

$$|\widehat{w_{\lambda\mu}}(\xi)| \leq C_{n,\lambda} \frac{1}{|\xi|^{n-2\lambda-1}}$$

with a new constant $C_{n,\lambda}$.

Case 2: $\frac{1}{|\xi|} \geq 1$ (that is, $|\xi| \leq 1$). Then

$$
\begin{aligned}
\widehat{w_{\lambda\mu}}(\xi) &= \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \left(\int_0^1 \omega_{\lambda\mu}(r)\, J_{\frac{n-2}{2}}(2\pi|\xi|r)\, r^{\frac{n}{2}}\, dr \right) \\
&\quad + \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \left(\int_1^{\frac{1}{|\xi|}} \omega_{\lambda\mu}(r)\, J_{\frac{n-2}{2}}(2\pi|\xi|r)\, r^{\frac{n}{2}}\, dr \right) \\
&\quad + \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \left(\int_{\frac{1}{|\xi|}}^{\infty} \omega_{\lambda\mu}(r)\, J_{\frac{n-2}{2}}(2\pi|\xi|r)\, r^{\frac{n}{2}}\, dr \right).
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
|\widehat{w_{\lambda\mu}}(\xi)| &\leq \frac{C_n}{|\xi|^{\frac{n-2}{2}}} \left(\int_0^1 \frac{1}{r^{2\lambda+1}}\ (2\pi|\xi|r)^{\frac{n-2}{2}}\ r^{\frac{n}{2}}\, dr \right) \\
&\quad + \frac{C_n}{|\xi|^{\frac{n-2}{2}}} \left(\int_1^{\frac{1}{|\xi|}} \frac{1}{r^{2\lambda+1}(\log(er))^{\mu}}\ (2\pi|\xi|r)^{\frac{n-2}{2}}\ r^{\frac{n}{2}}\, dr \right) \\
&\quad + \frac{C_n}{|\xi|^{\frac{n-2}{2}}} \left(\int_{\frac{1}{|\xi|}}^{\infty} \frac{1}{r^{2\lambda+1}(\log(er))^{\mu}}\ (2\pi|\xi|r)^{-\frac{1}{2}}\ r^{\frac{n}{2}}\, dr \right) \\
&= C_n \left(\int_0^1 r^{\frac{n-2}{2}} r^{\frac{n}{2}} r^{-2\lambda-1} dr \right) \\
&\quad + C_n \left(\int_1^{\frac{1}{|\xi|}} \frac{1}{(\log(er))^{\mu}} r^{\frac{n-2}{2}} r^{\frac{n}{2}} r^{-2\lambda-1} dr \right) \\
&\quad + \frac{C_n}{|\xi|^{\frac{n-1}{2}}} \left(\int_{\frac{1}{|\xi|}}^{\infty} \frac{1}{(\log(er))^{\mu}} r^{-\frac{1}{2}} r^{\frac{n}{2}} r^{-2\lambda-1} dr \right) \\
&= C_n \left(\int_0^1 r^{\frac{n-2}{2}+\frac{n}{2}-2\lambda-1} dr \right) \\
&\quad + C_n \left(\int_1^{\frac{1}{|\xi|}} \frac{1}{(\log(er))^{\mu}} r^{\frac{n-2}{2}+\frac{n}{2}-2\lambda-1} dr \right) \\
&\quad + \frac{C_n}{|\xi|^{\frac{n-1}{2}}} \left(\int_{\frac{1}{|\xi|}}^{\infty} \frac{1}{(\log(er))^{\mu}} r^{-\frac{1}{2}+\frac{n}{2}-2\lambda-1} dr \right).
\end{aligned}
$$

The first term doesn't depend on ξ. The second term can be estimated by applying Lemma (3.1.10, page 50), with $x = \frac{1}{|\xi|}$, $\alpha = \frac{n-2}{2} + \frac{n}{2} - 2\lambda - 1$, $\gamma = \mu$ and $t = r$. The estimate will only hold for $\frac{1}{|\xi|} > C_{n,\lambda,\mu}$ and some constant $C_{n,\lambda,\mu}$. The third term can be controlled by applying Lemma (3.1.9, page 47), with $x = \frac{1}{|\xi|}$, $\alpha = -\frac{1}{2} + \frac{n}{2} - 2\lambda - 1$ and $\gamma = 1$ (and $t = r$). So, we can continue our chain of inequalities as follows:

$$
\begin{aligned}
|\widehat{w_{\lambda\mu}}(\xi)| &\leq C_{n,\lambda} + C_{n,\lambda,\mu} \frac{\left(\frac{1}{|\xi|}\right)^{\frac{n-2}{2}+\frac{n}{2}-2\lambda}}{\left(\log\left(e\frac{1}{|\xi|}\right)\right)^{\mu}} \\
&\quad + \frac{C_{n,\lambda,\mu}}{|\xi|^{\frac{n-1}{2}}} \left(\frac{1}{\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} \left(\frac{1}{|\xi|}\right)^{-\frac{1}{2}+\frac{n}{2}-2\lambda} \right) \\
&= C_{n,\lambda} + C_{n,\lambda,\mu} \frac{1}{|\xi|^{\frac{n-2}{2}+\frac{n}{2}-2\lambda}\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} \\
&\quad + \frac{C_{n,\lambda,\mu}}{|\xi|^{\frac{n-1}{2}}} \frac{1}{|\xi|^{-\frac{1}{2}+\frac{n}{2}-2\lambda}\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} \\
&= C_{n,\lambda} + C_{n,\lambda,\mu} \frac{1}{|\xi|^{n-2\lambda-1}\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} \\
&\quad + C_{n,\lambda,\mu} \frac{1}{|\xi|^{n-2\lambda-1}\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} \\
&\leq C'_{n,\lambda,\mu} \frac{1}{|\xi|^{n-2\lambda-1}\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}}
\end{aligned}
$$

for some constant $C'_{n,\lambda,\mu}$ in the last step, as we're assuming $|\xi| \leq \frac{1}{C_{n,\lambda,\mu}}$ for some other constant $C_{n,\lambda,\mu}$ big enough (according with Lemma 3.1.9, page 47).

Summarizing, so far we proved that (3.36, page 56) holds for $\lambda > \frac{n-1}{4}$ and

$\lambda < \frac{n-1}{2}$.

The same holds with $w_{\lambda\mu}$ replaced by $w^{(1)}_{\lambda,\mu}$ and the proof is almost identical. In order to prove that (3.37, page 57) holds as well in the bigger range $0 < \lambda < \frac{n-1}{2}$, we will use an analytic continuation argument (and that's where the smoothness of $w^{(1)}_{\lambda,\mu}$ will play a role). At this point we need to say two more words about the way this is done. First recall that the Fourier transform $\widehat{w}$ (in the sense of the tempered distributions) of a locally integrable function w that is controlled by a polynomial in a neighborhood of infinity, is itself identified with a function u if and only if the following holds:

$$\int_{\mathbf{R}^n} \widehat{\varphi}(x)w(x)dx = \int_{\mathbf{R}^n} \varphi(\xi)u(\xi)d\xi \tag{3.38}$$

for all φ in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$. When this happens, we write (with a slight abuse of notation) w instead of u. In view of this fact, we can prove (3.37, page 57) by showing a function $u^{(1)}_{\lambda,\mu}$ defined on $\mathbf{R}^n \setminus \{0\}$ (in particular, almost everywhere in $\mathbf{R}^n$) such that:

$$\int_{\mathbf{R}^n} \widehat{\varphi}(x)w^{(1)}_{\lambda,\mu}(x)dx = \int_{\mathbf{R}^n} \varphi(\xi)u^{(1)}_{\lambda,\mu}(\xi)d\xi, \tag{3.39}$$

and

$$|u^{(1)}_{\lambda,\mu}(\xi)| \leq C'_{n,\lambda}\,\Omega_{\lambda,\mu}(\xi) \tag{3.40}$$

for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$, $\lambda \in \left(0, \frac{n-1}{2}\right)$, $\xi \in \mathbf{R}^n \setminus \{0\}$ and every μ that satisfies the hypothesis in Proposition (3.1.3, page 29) ($\Omega_{\lambda,\mu}$ was introduced in (3.36, page 56)). Indeed (3.39, page 62) and (3.38, page 62) imply that $u^{(1)}_{\lambda,\mu} = \widehat{w^{(1)}_{\lambda,\mu}}$, therefore (3.40, page 62) and (3.37, page 57) are equivalent.

The analytic continuation argument has to be used once $u^{(1)}_{\lambda,\mu}$ is introduced, in order to show that (3.39, page 62) holds for every λ. Since $w^{(1)}_{\lambda,\mu}$ is radial, we have indeed that (3.39, page 62) holds with

$$u^{(1)}_{\lambda,\mu}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty r^{\frac{n}{2}} \omega^{(1)}_{\lambda,\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|)dr \tag{3.41}$$

whenever the right hand side of (3.41, page 63) is defined, that is for $\lambda \in \left(\frac{n-1}{4}, \frac{n-1}{2}\right)$. Then we regard the left hand side and the right hand side of (3.39, page 62) as functions of variable λ and we show that both of them are defined and analytic on an open complex neighborhood of the real interval $\left(0, \frac{n-1}{2}\right)$. Since they coincide on $\left(\frac{n-1}{4}, \frac{n-1}{2}\right)$, we conclude from basic complex analysis that they coincide as well on $\left(0, \frac{n-1}{2}\right)$.

Let's show the details of the proof. Working with $\omega_{\lambda\mu}$ is very similar to working with $\omega^{(1)}_{\lambda,\mu}$, and we can carry the proof with either one as long as we don't need to use the smoothness of $\omega^{(1)}_{\lambda,\mu}$. So, to begin with, we consider the following version of the right hand side of (3.39):

$$\int_{\mathbf{R}^n} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty r^{\frac{n}{2}} \omega_{\lambda\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) dr d\xi \tag{3.42}$$

where $\omega^{(1)}_{\lambda,\mu}$ is replaced by $\omega_{\lambda\mu}$. We can rewrite (3.42) as the sum of 5 terms:

$$\int_{\mathbf{R}^n} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty r^{\frac{n}{2}} \omega_{\lambda\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) dr d\xi = I^1 + I^2 + I^3 + II^1 + II^2, \tag{3.43}$$

where:

$$\begin{aligned}
I^1 &= \int_{\frac{1}{|\xi|} \geq 1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^1 r^{\frac{n}{2}} \omega_{\lambda\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) dr d\xi \\
I^2 &= \int_{\frac{1}{|\xi|} \geq 1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_1^{\frac{1}{|\xi|}} r^{\frac{n}{2}} \omega_{\lambda\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) dr d\xi \\
I^3 &= \int_{\frac{1}{|\xi|} \geq 1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_{\frac{1}{|\xi|}}^\infty r^{\frac{n}{2}} \omega_{\lambda\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) dr d\xi \\
II^1 &= \int_{\frac{1}{|\xi|} < 1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^{\frac{1}{|\xi|}} r^{\frac{n}{2}} \omega_{\lambda\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) dr d\xi \\
II^2 &= \int_{\frac{1}{|\xi|} < 1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_{\frac{1}{|\xi|}}^\infty r^{\frac{n}{2}} \omega_{\lambda\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) dr d\xi.
\end{aligned}$$

Each term turns out to have the analitycity and boundedness properties we need. We show that I^1 is analytic by showing that it's holomorphic, and we do this via Lebesgue dominated convergence theorem. In order to apply the theorem to prove that I^1 is holomorphic at a given complex number $\lambda = \lambda_0$, it's enough to show that there exists $\varepsilon = \varepsilon_{\lambda_0}(\varphi, n) > 0$ such that the following holds:

$$\int_{\frac{1}{|\xi|}\geq 1}\int_0^1 \sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\frac{d}{d\lambda}\left(\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}}\omega_{\lambda\mu}(r)J_{\frac{n-2}{2}}(2\pi r|\xi|)\right)\right|_{\lambda=\eta} drd\xi <\infty, \tag{3.44}$$

where $B(\lambda_0,\varepsilon)\subset \mathbf{C}$ denotes the complex ball of center λ_0 and radius ε. The following chain of inequalities shows that this is true if $\mathrm{Re}\,(\lambda_0)<\frac{n-1}{2}$:

$$\begin{aligned}
&\int_{\frac{1}{|\xi|}\geq 1}\int_0^1 \sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\frac{d}{d\lambda}\left(\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}}\omega_{\lambda\mu}(r)J_{\frac{n-2}{2}}(2\pi r|\xi|)\right)\right|_{\lambda=\eta} drd\xi\\
&=\int_{\frac{1}{|\xi|}\geq 1}\int_0^1 \sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}}J_{\frac{n-2}{2}}(2\pi r|\xi|)\frac{d}{d\lambda}\left(\omega_{\lambda\mu}(r)\right)\right|_{\lambda=\eta} drd\xi\\
&=\int_{\frac{1}{|\xi|}\geq 1}\int_0^1 \sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}}J_{\frac{n-2}{2}}(2\pi r|\xi|)\frac{d}{d\lambda}\left(\frac{1}{r^{2\lambda+1}}\right)\right|_{\lambda=\eta} drd\xi\\
&=\int_{\frac{1}{|\xi|}\geq 1}\int_0^1 \sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}}J_{\frac{n-2}{2}}(2\pi r|\xi|)\frac{d}{d\lambda}\left(r^{-2\lambda}r^{-1}\right)\right|_{\lambda=\eta} drd\xi\\
&=\int_{\frac{1}{|\xi|}\geq 1}\int_0^1 \sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}-1}J_{\frac{n-2}{2}}(2\pi r|\xi|)\frac{d}{d\lambda}\left(r^{-2\lambda}\right)\right|_{\lambda=\eta} drd\xi\\
&=\int_{\frac{1}{|\xi|}\geq 1}\int_0^1 \sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}-1}J_{\frac{n-2}{2}}(2\pi r|\xi|)\,\cdot\right.\\
&\qquad \left.\cdot r^{-2\lambda}(-2\log(r))\right|_{\lambda=\eta} drd\xi\\
&=\int_{\frac{1}{|\xi|}\geq 1}\int_0^1 \sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}-1}J_{\frac{n-2}{2}}(2\pi r|\xi|)\,\cdot\right.\\
&\qquad \left.\cdot r^{-2\eta}(-2\log(r))\right| drd\xi\\
&=4\pi\int_{\frac{1}{|\xi|}\geq 1}\int_0^1 \left(|\varphi(\xi)|\frac{1}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}-1}\left|J_{\frac{n-2}{2}}(2\pi r|\xi|)\right|\log\left(\frac{1}{r}\right)\right)\cdot
\end{aligned}$$

$$\cdot \left(\sup_{\eta \in B(\lambda_0, \varepsilon)} |r^{-2\eta}| \right) dr d\xi$$

$$\leq 4\pi C_n \int_{\frac{1}{|\xi|} \geq 1} \int_0^1 \left(\frac{|\varphi(\xi)|}{|\xi|^{\frac{n-2}{2}}} r^{\frac{n}{2}-1} \left| (2\pi r |\xi|)^{\frac{n-2}{2}} \right| \log\left(\frac{1}{r}\right) \right) \cdot$$

$$\cdot \left(\sup_{\eta \in B(\lambda_0, \varepsilon)} r^{(-2)\mathrm{Re}\,(\eta)} \right) dr d\xi$$

$$= C_n' \int_{\frac{1}{|\xi|} \geq 1} \int_0^1 \left(|\varphi(\xi)| r^{n-2} \log\left(\frac{1}{r}\right) \right) \; r^{(-2)(\mathrm{Re}\,(\lambda_0)+\varepsilon)} \; dr d\xi$$

$$= C_n' \int_{\frac{1}{|\xi|} \geq 1} |\varphi(\xi)| \int_0^1 r^{n-2+(-2)(\mathrm{Re}\,(\lambda_0)+\varepsilon)} \log\left(\frac{1}{r}\right) dr d\xi$$

$$= C_n' \int_0^1 r^{n-2+(-2)(\mathrm{Re}\,(\lambda_0)+\varepsilon)} \log\left(\frac{1}{r}\right) dr \int_{\frac{1}{|\xi|} \geq 1} |\varphi(\xi)| d\xi.$$

This last term is finite if and only if $n - 2 + (-2)(\mathrm{Re}\,(\lambda_0) + \varepsilon) > -1$, that is, if $n - 1 - 2\mathrm{Re}\,(\lambda_0) > 2\varepsilon > 0$.

Clearly, all we need for such an ε to exist is that $n - 1 - 2\mathrm{Re}\,(\lambda_0) > 0$, that is $\mathrm{Re}\,(\lambda_0) < \frac{n-1}{2}$.

We just proved that I^1 is defined and holomorphic (hence analytic) with respect to λ on the set

$$\left\{ \lambda \in \mathbf{C} : \mathrm{Re}\,(\lambda) < \frac{n-1}{2} \right\},$$

which is open in $\mathbf{C}$ and clearly contains the real interval $\left(0, \frac{n-2}{2}\right)$.

Similarly, I^2 is holomorphic on the set of λ_0's such that there exists $\varepsilon =$

$\varepsilon_{\lambda_0}(\varphi, n) > 0$ satisfying the following condition similar to (3.44):

$$\int_{\frac{1}{|\xi|}\geq 1}\int_1^{\frac{1}{|\xi|}} \sup_{\eta\in B(\lambda_0,\varepsilon)} \left| \frac{d}{d\lambda}\left(\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}} r^{\frac{n}{2}}\omega_{\lambda,\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|)\right)\right|_{\lambda=\eta} dr d\xi < \infty$$

Following the same steps we made after (3.44), and recalling that $\omega_{\lambda,\mu}(r)$ is defined piecewise, we get:

$$\begin{aligned}
&\int_{\frac{1}{|\xi|}\geq 1}\int_1^{\frac{1}{|\xi|}} \sup_{\eta\in B(\lambda_0,\varepsilon)} \left| \frac{d}{d\lambda}\left(\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}} r^{\frac{n}{2}}\omega_{\lambda,\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|)\right)\right|_{\lambda=\eta} dr d\xi \\
&\leq C_n \int_{\frac{1}{|\xi|}\geq 1}\int_1^{\frac{1}{|\xi|}} \left(\frac{|\varphi(\xi)|}{|\xi|^{\frac{n-2}{2}}} r^{\frac{n}{2}-1} \left|(2\pi r|\xi|)^{\frac{n-2}{2}}\right| \left(\frac{\log(r)}{(\log(er))^{\mu}}\right)\right) \\
&\quad \cdot \left(\sup_{\eta\in B(\lambda_0,\varepsilon)} r^{(-2)\mathrm{Re}\,(\eta)}\right) dr d\xi \\
&= C_n' \int_{\frac{1}{|\xi|}\geq 1}\int_1^{\frac{1}{|\xi|}} \left(|\varphi(\xi)| r^{n-2} \left(\frac{\log(r)}{(\log(er))^{\mu}}\right)\right) \left(r^{(-2)(\mathrm{Re}\,(\lambda_0)-\varepsilon)}\right) dr d\xi \\
&= C_n' \int_{\frac{1}{|\xi|}\geq 1} |\varphi(\xi)| \int_1^{\frac{1}{|\xi|}} r^{n-2-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} \frac{\log(r)}{(\log(er))^{\mu}} dr d\xi. \qquad (3.45)
\end{aligned}$$

If, in addition, $\mu < 1$, then it will be convenient to control the latter by:

$$\begin{aligned}
& C_n' \int_{\frac{1}{|\xi|}\geq 1} |\varphi(\xi)| \int_1^{\frac{1}{|\xi|}} r^{n-2-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} \frac{\log(r)}{(\log(r))^{\mu}} dr d\xi \\
&= C_n' \int_{\frac{1}{|\xi|}\geq 1} |\varphi(\xi)| \int_1^{\frac{1}{|\xi|}} r^{n-2-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} (\log(r))^{1-\mu} dr d\xi \\
&\leq C_n' \int_{\frac{1}{|\xi|}\geq 1} |\varphi(\xi)| \int_1^{\frac{1}{|\xi|}} r^{n-2-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} \left(\log\left(\frac{1}{|\xi|}\right)\right)^{1-\mu} dr d\xi \\
&= C_n' \int_{\frac{1}{|\xi|}\geq 1} |\varphi(\xi)| \left(\log\left(\frac{1}{|\xi|}\right)\right)^{1-\mu} \int_1^{\frac{1}{|\xi|}} r^{n-2-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} dr d\xi \\
&= \frac{C_n'}{n-1-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} \cdot
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{\frac{1}{|\xi|}\geq 1} |\varphi(\xi)| \left(\log\left(\frac{1}{|\xi|}\right)\right)^{1-\mu} \left(r^{n-1-2\mathrm{Re}\,(\lambda_0)+2\varepsilon}\Big|_{r=1}^{\frac{1}{|\xi|}} \right) d\xi \\
= \ & \frac{C_n'}{n-1-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} \cdot \\
&\cdot \int_{\frac{1}{|\xi|}\geq 1} \frac{|\varphi(\xi)| \left(\log\left(\frac{1}{|\xi|}\right)\right)^{1-\mu}}{|\xi|^{n-1-2\mathrm{Re}\,(\lambda_0)+2\varepsilon}} - |\varphi(\xi)| \left(\log\left(\frac{1}{|\xi|}\right)\right)^{1-\mu} d\xi.
\end{aligned}
$$

Otherwise, if $\mu \geq 1$, we control (3.45, page 67) by:

$$
\begin{aligned}
& C_n' \int_{\frac{1}{|\xi|}\geq 1} |\varphi(\xi)| \int_1^{\frac{1}{|\xi|}} r^{n-2-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} \frac{\log(r)}{(\log(er))} dr d\xi \\
\leq \ & C_n' \int_{\frac{1}{|\xi|}\geq 1} |\varphi(\xi)| \int_1^{\frac{1}{|\xi|}} r^{n-2-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} dr d\xi \\
= \ & \frac{C_n'}{n-1-2\mathrm{Re}\,(\lambda_0)+2\varepsilon} \int_{\frac{1}{|\xi|}\geq 1} \frac{|\varphi(\xi)|}{|\xi|^{n-1-2\mathrm{Re}\,(\lambda_0)+2\varepsilon}} - |\varphi(\xi)| d\xi.
\end{aligned}
$$

In any case (either $\mu < 1$ or $\mu \geq 1$) the last integral if finite independently of φ if and only if $n - 1 - 2\mathrm{Re}\,(\lambda_0) + 2\varepsilon < n$, that is, if $2\varepsilon < 1 + 2\mathrm{Re}\,(\lambda_0)$. Clearly, such a positive ε exists if and only if $0 < 1 + 2\mathrm{Re}\,(\lambda_0)$, if and only if $-\frac{1}{2} < \mathrm{Re}\,(\lambda_0)$. Provided this is true, it's also clear that ε can be chosen small enough to have $n - 1 - 2\mathrm{Re}\,(\lambda_0) + 2\varepsilon \neq 0$.

We just proved that I^2 is holomorphic (hence analytic) on the following open subset of $\mathbf{C}$:

$$
\left\{ \lambda \in \mathbf{C} : \mathrm{Re}\,(\lambda) > -\frac{1}{2} \right\}.
$$

In order to rewrite I^3 in a form that is defined on an open subset of $\mathbf{C}$ that contains the real interval $\left(0, \frac{n-1}{2}\right)$, we will need to use Lemma 2.0.5 (for some

N to be determined soon). We write:

$$I^3 = \left(\sum_{j=0}^{N} I^3_{j,1}\right) - \left(\sum_{j=0}^{N} I^3_{j,2}\right) + I^3_{N+1},$$

where

$$I^3_{N+1} = \int_{\frac{1}{|\xi|}\geq 1} \varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}}\omega_{\lambda\mu}(r)R_{\frac{n-2}{2},N}\left(\frac{1}{(2\pi r|\xi|)^{2N+\frac{5}{2}}}\right)drd\xi$$

and

$$\begin{aligned} I^3_{j,1} \quad &= \quad \int_{\frac{1}{|\xi|}\geq 1} \varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}}\omega_{\lambda\mu}(r)\frac{c_{\frac{n-2}{2},2j}}{(2\pi r|\xi|)^{2j+\frac{1}{2}}} \cdot \\ &\quad \cdot\cos((2\pi r|\xi|) - c_{\frac{n-2}{2}})drd\xi \end{aligned}$$

and also

$$\begin{aligned} I^3_{j,2} \quad &= \quad \int_{\frac{1}{|\xi|}\geq 1} \varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}}\omega_{\lambda\mu}(r)\frac{c_{\frac{n-2}{2},2j+1}}{(2\pi r|\xi|)^{2j+\frac{3}{2}}} \cdot \\ &\quad \cdot\sin((2\pi r|\xi|) - c_{\frac{n-2}{2}})drd\xi \end{aligned}$$

for all $j = 0, ..., N$. The idea now is to choose N big enough for I^3_{N+1} to be well defined for all $\lambda \in (0, (n-1)/2)$, and to iterate integration by parts on each other term $I^3_{j,1}$ and $I^3_{j,2}$. Due to the estimate for $R_{k,N}$ in Lemma 2.0.5, we can see that I^3_{N+1} is well defined for all $\lambda \in (0, (n-1)/2)$ if and only if $\frac{n}{2} - 2\lambda - 1 - 2N - \frac{5}{2} < -1$ for every λ in such range, if and only if $\frac{n}{2} - 1 - 2N - \frac{5}{2} \leq -1$, if and only if $N \geq \frac{n-5}{4}$. So, we can just set $N = N_n := \lceil \frac{n-5}{4} \rceil$ (where $\lceil a \rceil$ represents the smallest integer greater than or equal to a real number a). We may also check, as we did before, that $I^3_{N_n+1}$ is analytic in a complex neighborhood of the real interval $\left(0, \frac{n-1}{2}\right)$. Furthermore, in view of (3.1.9, page 47) we have:

$$\begin{aligned}
&\left| \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}} \omega_{\lambda\mu}(r) R_{\frac{n-2}{2}, N_n} \left(\frac{1}{(2\pi r|\xi|)^{2N_n+\frac{5}{2}}} \right) dr \right| \\
\leq \quad & \frac{C_n}{|\xi|^{\frac{n-2}{2}+2N_n+\frac{5}{2}}} \int_{\frac{1}{|\xi|}}^{\infty} \frac{r^{\frac{n}{2}-2\lambda-1-2N_n-\frac{5}{2}}}{\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} dr \\
\leq \quad & \frac{C_{n,\lambda}}{|\xi|^{\frac{n-2}{2}+2N_n+\frac{5}{2}}} \left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-2\lambda-2N_n-\frac{5}{2}} \frac{1}{\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} \\
= \quad & \frac{C_{n,\lambda}}{|\xi|^{\frac{n-2}{2}+2N_n+\frac{5}{2}+\frac{n}{2}-2\lambda-2N_n-\frac{5}{2}}} \frac{1}{\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} \\
= \quad & \frac{C_{n,\lambda}}{|\xi|^{n-2\lambda-1}} \frac{1}{\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}},
\end{aligned}$$

that is, $I^3_{N_n+1} = \int_{\frac{1}{|\xi|}\geq 1} \varphi(\xi) u(\xi) d\xi$ with the function u satisfying the inequality claimed in (3.40, page 62):

$$|u(\xi)| \leq \frac{C_{n,\lambda}}{|\xi|^{n-2\lambda-1}\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} \tag{3.46}$$

when $\frac{1}{|\xi|} \geq 1$.

To prove the same properties of analiticity and "boundedness" for the terms $I^3_{j,1}$, $I^3_{j,2}$, $j = 0, ..., N_n$, it will be useful to use the following lemma.

Lemma 3.2.2. *Let* $-\infty \leq a < b \leq \infty$, $f \in C([a,b])$, $g \in C^{\infty}((a,b))$ *and assume that* $\frac{d^l}{dt^l} g$ *admits a continuous extension on* $[a,b]$ *for any* $l \in \mathbf{Z}^+$ *and* *for $l = 0$. Then, if we denote a k^{th} antiderivative of f by $\frac{d^{-k}}{dt^{-k}}f$, the following iterated integration by parts formula holds for every $N \in \mathbf{Z}^+$:*

$$\begin{aligned}\int_a^b f(t)g(t)dt &= (-1)^N \int_a^b \frac{d^{-N}}{dt^{-N}}f(t)\frac{d^N}{dt^N}g(t)dt \\ &\quad + \sum_{k=0}^{N-1} \left((-1)^k \frac{d^{-k-1}}{dt^{-k-1}}f(t)\frac{d^k}{dt^k}g(t)\right)\Bigg|_{t=a}^{b}.\end{aligned}$$

We can in fact just study the terms $I^3_{j,1}$, as the study of $I^3_{j,2}$ is almost identical. First, let's relabel and redefine the constants $c_{\frac{n-2}{2},2j}$ in order to rewrite $I^3_{j,1}$ in a form easier to read:

$$\begin{aligned}I^3_{j,1} &= \int_{\frac{1}{|\xi|}\geq 1} \varphi(\xi)\frac{1}{|\xi|^{\frac{n-2}{2}}}\int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}}\omega_{\lambda\mu}(r)\frac{c_{n,j}}{(r|\xi|)^{2j+\frac{1}{2}}}\cos((2\pi r|\xi|) - c_n)drd\xi \\ &= c_{n,j}\int_{\frac{1}{|\xi|}\geq 1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}}\int_{\frac{1}{|\xi|}}^{\infty} \frac{r^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}}}{(\log(er))^{\mu}}\cos((2\pi r|\xi|) - c_n)drd\xi\end{aligned}$$

Now, we want to apply Lemma 3.2.2 with $a = \frac{1}{|\xi|}$, $b = \infty$, $f(t) = \cos((2\pi t|\xi|) - c_n)$ and $g(t) = \frac{t^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}}}{(\log(et))^{\mu}}$. With this notation we can write:

$$I^3_{j,1} = c_{n,j}\int_{\frac{1}{|\xi|}\geq 1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}}\left(\int_a^b f(r)g(r)dr\right)d\xi$$

Since the definition of f, for $k \in \mathbf{Z}^+$ or $k = 0$ we also have:

$$\frac{d^{-k-1}}{dt^{-k-1}}f(t) = \frac{\frac{d^{-k-1}}{dt^{-k-1}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{k+1}}, \tag{3.47}$$

where $\frac{d^{-k-1}}{dt^{-k-1}}(\cos)((2\pi t|\xi|) - c_n)$ denotes the $(k+1)^{\text{th}}$ antiderivative of cos (that is, either cos, $-$cos, sin, or $-$sin) evaluated at $(2\pi t|\xi|) - c_n$. In particular we have:

$$\left|\frac{d^{-k-1}}{dt^{-k-1}}f(t)\right| \le \frac{C_k}{|\xi|^{k+1}}$$

for all $t \in \mathbf{R}$.

Since the definition of g and for $k \in \mathbf{Z}^+$, instead, we have:

$$\frac{d^k}{dt^k}g(t) = \frac{t^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}-k}}{(\log(et))^{\mu}}\, p_{k,\lambda,n,j}\left(\frac{1}{\log(et)}\right), \tag{3.48}$$

for some polynomials $p_{k,\lambda,n,j}$.

We finally apply Lemma 3.2.2, with $N = N'_n := \lceil \frac{n}{2} - \frac{1}{2} \rceil$, which is the smallest integer N indepent of j and λ that will make the integral $\int_a^b \frac{d^{-N}}{dt^{-N}}f(t)\frac{d^N}{dt^N}g(t)dt$ to converge, to write:

$$\begin{aligned}
&\int_{\frac{1}{|\xi|}}^{\infty} \frac{r^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}}}{(\log(er))^{\mu}} \cos((2\pi r|\xi|) - c_n)dr \\
=\ & \left(\int_a^b f(r)g(r)dr\right) \\
=\ & (-1)^N \int_a^b \frac{d^{-N}}{dt^{-N}}f(t)\frac{d^N}{dt^N}g(t)dt \\
& + \sum_{k=0}^{N-1}\left((-1)^k \frac{d^{-k-1}}{dt^{-k-1}}f(t)\frac{d^k}{dt^k}g(t)\right)\Big|_{t=a} \\
=\ & (-1)^{N'_n} \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N'_n}}{dt^{-N'_n}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N'_n}} \cdot \\
& \cdot \frac{t^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}-N'_n}}{(\log(et))^{\mu}}\, p_{N'_n,\lambda,n,j}\left(\frac{1}{\log(et)}\right) dt \\
& + \sum_{k=0}^{N'_n-1}\left((-1)^k \frac{\frac{d^{-k-1}}{dt^{-k-1}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{k+1}} \cdot \right.
\end{aligned}$$

$$\cdot \frac{t^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}-k}}{(\log(et))^{\mu}} p_{k,\lambda,n,j}\left(\frac{1}{\log(et)}\right)\Bigg)\Bigg|_{t=\frac{1}{|\xi|}}^{\infty}$$

If we first assume that $\lambda > \frac{n-3}{4}$ (or even that $\lambda > \frac{n-1}{4}$) then we get:

$$\begin{aligned}
& \int_{\frac{1}{|\xi|}}^{\infty} \frac{r^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}}}{(\log(et))^{\mu}} \cos((2\pi r|\xi|) - c_n)dr \\
= \quad & (-1)^{N_n'} \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N_n'}}{dt^{-N_n'}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N_n'}} \cdot \\
& \cdot \frac{t^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}-N_n'}}{(\log(et))^{\mu}} p_{N_n',\lambda,n,j}\left(\frac{1}{\log(et)}\right) dt \\
& + \sum_{k=0}^{N_n'-1} \Bigg((-1)^{k+1} \frac{\frac{d^{-k-1}}{dt^{-k-1}}(\cos)(2\pi - c_n)}{(2\pi|\xi|)^{k+1}} \cdot \\
& \cdot \frac{\left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}-k}}{\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} p_{k,\lambda,n,j}\left(\frac{1}{\log\left(\frac{e}{|\xi|}\right)}\right)\Bigg) \\
= \quad & (-1)^{N_n'} \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N_n'}}{dt^{-N_n'}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N_n'}} \cdot \\
& \cdot \frac{t^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}-N_n'}}{(\log(et))^{\mu}} p_{N_n',\lambda,n,j}\left(\frac{1}{\log(et)}\right) dt \\
& + \sum_{k=0}^{N_n'-1} \left(\frac{c_{k,n}}{|\xi|^{\frac{n}{2}-2\lambda-2j-\frac{1}{2}}} \frac{p_{k,\lambda,n,j}\left(\frac{1}{\log\left(\frac{e}{|\xi|}\right)}\right)}{\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} \right) \\
=: \quad & |\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}} u^{(1)}_{n,\lambda,\mu,j}(\xi).
\end{aligned}$$

As we now have

$$\left(\int_a^b f(r)g(r)dr\right) = |\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}} u^{(1)}_{n,\lambda,\mu,j}(\xi),$$

then we can write:

$$\begin{aligned}
I^3_{j,1} &= c_{n,j}\int_{\frac{1}{|\xi|}\geq 1}\frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}}\left(\int_a^b f(r)g(r)dr\right)d\xi\\
&= c_{n,j}\int_{\frac{1}{|\xi|}\geq 1}\frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}}|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}u^{(1)}_{n,\lambda,\mu,j}(\xi)d\xi\\
&= c_{n,j}\int_{\frac{1}{|\xi|}\geq 1}\varphi(\xi)u^{(1)}_{n,\lambda,\mu,j}(\xi)d\xi.
\end{aligned}$$

In view of Lemma (3.1.9, page 47), we see that there exist constants $C_{n,\lambda,j}$ and $C_{n,\lambda,\mu,j}$ such that

$$|u^{(1)}_{n,\lambda,\mu,j}(\xi)| \leq \frac{C_{n,\lambda,j}}{|\xi|^{n-2\lambda-1}\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}}, \tag{3.49}$$

whenever $|\xi| \leq C_{n,\lambda,\mu,j}$ (therefore, by continuity and compactness, whenever $|\xi| \leq 1$).

Furthermore, as the derivatives $\frac{d}{d\lambda}\left(p_{N'_n,\lambda,n,j}\right)(x)$ still happen to be polynomials with respect to the variable x, we can prove (by repeating the analogous steps made for I^1 and I^2) that the function:

$$\lambda \longmapsto c_{n,j}\int_{\frac{1}{|\xi|}\geq 1}\varphi(\xi)u^{(1)}_{n,\lambda,\mu,j}(\xi)d\xi = I^3_{j,1}$$

is defined and analytic on the set:

$$\left\{\lambda \in \mathbf{C} : \mathrm{Re}\,(\lambda) > \max\left\{-\frac{1}{2}, \left(\frac{n-1}{2}\right) - \left\lceil\frac{n-1}{2}\right\rceil\right\}\right\}$$

that obviously contains the real interval $\left(0, \frac{n-1}{2}\right)$ and is open in $\mathbf{C}$. Estimate (3.49) holds on such set.

The proof of analyticity and "boundedness" of II^1 doesn't show any complication. In fact II^1 is already defined and analytic with respect to λ on the set:

$$\left\{\lambda \in \mathbf{C} : \mathrm{Re}\,(\lambda) < \frac{n-1}{2}\right\}$$

and the fact that

$$II^1 = \int_{\frac{1}{|\xi|}\leq 1} \varphi(\xi) u^{(2)}_{n,\lambda}(\xi) d\xi$$

for a function $u^{(2)}_{n,\lambda}$ satisfying

$$|u^{(2)}_{n,\lambda}(\xi)| \leq \frac{C_{n,\lambda}}{|\xi|^{n-2\lambda-1}} \tag{3.50}$$

for some constant $C_{n,\lambda}$ is straightforward.

To deal with the term II^2, on the other hand, is different. As the integral involved only converges for $\lambda \in \left(\frac{n-1}{4}, \frac{n-1}{2}\right)$, we need to use an analytic continuation argument, as we did to treat the term I^3. Unfortunatly, as $\frac{1}{|\xi|} < 1$ in II^2, the piecewise-defined function $\omega_{\lambda\mu}$ is not of class C^∞ on $\left(\frac{1}{|\xi|}, \infty\right)$. This does have a consequence. As we integrate by parts as we did to treat I^3, we can still prove that II^2 has an analytic continuation with respect to λ on a complex open set that contains the real interval $\left(0, \frac{n-1}{2}\right)$, but with a different bound. Precisely, we can still prove that:

$$II^2 = \int_{\frac{1}{|\xi|}\leq 1} \varphi(\xi) u^{(3)}_{n,\lambda}(\xi) d\xi$$

but then, if $\lambda \leq \frac{n-1}{4}$, we can only prove that:

$$|u^{(3)}_{n,\lambda}(\xi)| \leq \frac{C_{n,\lambda}}{|\xi|^{\frac{n-1}{2}}}$$

for all $|\xi| \geq 1$.

This is why we introduced the new weights $\omega^{(1)}_{\lambda,\mu}$ and $w^{(1)}_{\lambda,\mu}$ at the beginning of this section (page 55).

Recall that:

$$\omega^{(1)}_{\lambda,\mu}(t) = \omega_{\lambda\mu}(t)(1-\phi(t)) + \phi(t), \tag{3.51}$$

and observe that $\omega^{(1)}_{\lambda,\mu}$ coincides with $\omega_{\lambda\mu}$ in $\left(0, \frac{9}{10}\right] \cup \left[\frac{11}{10}, \infty\right)$ and coincides with 1 in $\left[\frac{19}{20}, \frac{21}{20}\right]$.

Since $w^{(1)}_{\lambda,\mu}$ is comparable to $w_{\lambda,\mu}$, we can use our first computations to show that:

$$|\widehat{w^{(1)}_{\lambda,\mu}}(x)| \leq C_{n,\lambda} \begin{cases} \frac{1}{|x|^{n-2\lambda-1}\log\left(\frac{e}{|x|}\right)} & \text{if} \quad |x| \leq 1, \\ \frac{1}{|x|^{n-2\lambda-1}} & \text{if} \quad |x| > 1. \end{cases} \tag{3.52}$$

for all $\frac{n-1}{4} < \lambda < \frac{n-1}{2}$.

Then we can rewrite equation (3.43, page 64) with $\omega^{(1)}_{\lambda,\mu}$ instead of $\omega_{\lambda\mu}$ and redefine the terms I^1, I^2, I^3, II^1, II^2 accordingly. The new terms, from I^1 to II^1, can be treated the same way as the old ones and have the same properties we proved for the old ones. Plus, due to the differentiability of $\omega^{(1)}_{\lambda,\mu}$, we can treat the term II^2 as well.

Let's compare the new term II^2 with the old term I^3:

$$II^2 = \int_{\frac{1}{|\xi|}<1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}} \omega_{\lambda,\mu}^{(1)}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) dr d\xi, \tag{3.53}$$

$$I^3 = \int_{\frac{1}{|\xi|}\geq 1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}} \omega_{\lambda\mu}(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) dr d\xi.$$

Due to the similarity of these terms, we will need to apply the asymptotic estimate in Lemma 2.0.5 with $k = \frac{n-2}{2}$, $N = N_n = \lceil \frac{n-5}{4} \rceil$, and $x = 2\pi|\xi|r$, that is exactly as we already did to rewrite the old term I^3.

So, the estimate in Lemma 2.0.5 will take the form:

$$\begin{aligned} J_{\frac{n-2}{2}}(2\pi|\xi|r) &= R_{\frac{n-2}{2},N_n}\left(\frac{1}{(2\pi|\xi|r)^{2N_n+\frac{5}{2}}}\right) \\ &\quad + \sum_{j=0}^{N_n}\left(\frac{c_{\frac{n-2}{2},2j}}{(2\pi|\xi|r)^{2j+\frac{1}{2}}}\cos(2\pi|\xi|r - c_{\frac{n-2}{2}})\right. \\ &\quad \left. - \frac{c_{\frac{n-2}{2},2j+1}}{(2\pi|\xi|r)^{2j+\frac{3}{2}}}\sin(2\pi|\xi|r - c_{\frac{n-2}{2}})\right). \end{aligned} \tag{3.54}$$

By using equation (3.54) in (3.53), we can rewrite:

$$II^2 = \left(\sum_{j=0}^{N_n} II^2_{j,1}\right) - \left(\sum_{j=0}^{N_n} II^2_{j,2}\right) + II^2_{N_n+1}, \tag{3.55}$$

where:

$$II^2_{N_n+1} = \int_{\frac{1}{|\xi|}<1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}} \omega_{\lambda,\mu}^{(1)}(r) R_{\frac{n-2}{2},N_n}\left(\frac{1}{(2\pi r|\xi|)^{2N_n+\frac{5}{2}}}\right) dr d\xi$$

and

$$
\begin{aligned}
II_{j,1}^{2} &= \int_{\frac{1}{|\xi|}<1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}} \omega_{\lambda,\mu}^{(1)}(r) \frac{c_{\frac{n-2}{2},2j}}{(2\pi r|\xi|)^{2j+\frac{1}{2}}} \cdot \\
&\quad \cdot \cos((2\pi r|\xi|) - c_{\frac{n-2}{2}}) dr d\xi \\
II_{j,2}^{2} &= \int_{\frac{1}{|\xi|}<1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}} \omega_{\lambda,\mu}^{(1)}(r) \frac{c_{\frac{n-2}{2},2j+1}}{(2\pi r|\xi|)^{2j+\frac{3}{2}}} \cdot \\
&\quad \cdot \sin((2\pi r|\xi|) - c_{\frac{n-2}{2}}) dr d\xi
\end{aligned}
$$

for all $j = 0, ..., N_n$. Again, we won't need to study the terms $II_{j,2}^{2}$, because they are so similar to the terms $II_{j,1}^{2}$ and can be treated in the same way. As usual, we can prove that $II_{N_n+1}^{2}$ is holomorphic at a complex number λ_0 (via the Lebesgue dominated convergence theorem) if we can show that there exists $\varepsilon = \varepsilon(\lambda_0, n) > 0$ such that

$$
\begin{aligned}
&\int_{\frac{1}{|\xi|}<1} \int_{\frac{1}{|\xi|}}^{\infty} \sup_{\eta \in B(\lambda_0,\varepsilon)} \left| \frac{d}{d\lambda} \left(\varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} r^{\frac{n}{2}} \omega_{\lambda,\mu}^{(1)}(r) \cdot \right.\right. \\
&\qquad \left.\left. \cdot\ R_{\frac{n-2}{2},N_n} \left(\frac{1}{(2\pi r|\xi|)^{2N_n+\frac{5}{2}}} \right) \right) \right|_{\lambda=\eta} dr d\xi
\end{aligned}
\tag{3.56}
$$

is finite. From now on, we will just write R_n instead of $R_{\frac{n-2}{2},N_n}$. For given $\lambda_0 \in \mathbf{C}$ and $\varepsilon > 0$ we have:

$$
\begin{aligned}
&\int_{\frac{1}{|\xi|}<1} \int_{\frac{1}{|\xi|}}^{\infty} \sup_{\eta \in B(\lambda_0,\varepsilon)} \left| \frac{d}{d\lambda} \left(\varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} r^{\frac{n}{2}} \omega_{\lambda,\mu}^{(1)}(r) \cdot \right.\right. \\
&\qquad \left.\left. \cdot\ R_n \left(\frac{1}{(2\pi r|\xi|)^{2N_n+\frac{5}{2}}} \right) \right) \right|_{\lambda=\eta} dr d\xi \\
= &\int_{\frac{1}{|\xi|}<1} \int_{\frac{1}{|\xi|}}^{\infty} \sup_{\eta \in B(\lambda_0,\varepsilon)} \left| \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} r^{\frac{n}{2}} \frac{d}{d\lambda} \left(\omega_{\lambda,\mu}^{(1)}(r) \right) \cdot \right.
\end{aligned}
$$

$$\cdot R_n\left(\frac{1}{(2\pi r|\xi|)^{2N_n+\frac{5}{2}}}\right)\Bigg|_{\lambda=\eta} drd\xi$$

$$\leq 2\pi\int_{\frac{1}{|\xi|}<1}\int_{\frac{1}{|\xi|}}^{\infty}\left|\frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}}}\right| r^{\frac{n}{2}}\left|\left(\frac{C_n}{(2\pi r|\xi|)^{2N_n+\frac{5}{2}}}\right)\right|\cdot$$

$$\cdot\sup_{\eta\in B(\lambda_0,\varepsilon)}\left(\left|\frac{d}{d\lambda}\left(\omega^{(1)}_{\lambda,\mu}(r)\right)\right|_{\lambda=\eta}\right) drd\xi.$$

Now:

$$\begin{aligned}
\frac{d}{d\lambda}\left(\omega^{(1)}_{\lambda,\mu}(r)\right) &= \frac{d}{d\lambda}\left(\omega_{\lambda\mu}(r)(1-\phi(r))+\phi(r)\right)\\
&= \frac{d}{d\lambda}\left(\omega_{\lambda\mu}(r)(1-\phi(r))\right)+\frac{d}{d\lambda}(\phi(r))\\
&= \frac{d}{d\lambda}\left(\omega_{\lambda\mu}(r)\right)(1-\phi(r))\\
&= -2\log(r)\omega_{\lambda\mu}(r)(1-\phi(r)).
\end{aligned}$$

So:

$$\int_{\frac{1}{|\xi|}<1}\int_{\frac{1}{|\xi|}}^{\infty}\sup_{\eta\in B(\lambda_0,\varepsilon)}\left|\frac{d}{d\lambda}\left(\varphi(\xi)\frac{2\pi}{|\xi|^{\frac{n-2}{2}}}r^{\frac{n}{2}}\omega^{(1)}_{\lambda,\mu}(r)\cdot\right.\right.$$

$$\left.\left.\cdot R_n\left(\frac{1}{(2\pi r|\xi|)^{2N_n+\frac{5}{2}}}\right)\right)\right|_{\lambda=\eta} drd\xi$$

$$\leq C'_n\int_{\frac{1}{|\xi|}<1}\int_{\frac{1}{|\xi|}}^{\infty}\left|\frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}}}\right| r^{\frac{n}{2}-2N_n-\frac{5}{2}}\left|\left(\frac{|\log(r)|}{|\xi|^{2N_n+\frac{5}{2}}}\right)\right|\cdot$$

$$\cdot\sup_{\eta\in B(\lambda_0,\varepsilon)}|\omega_{\eta,\mu}(r)|\, drd\xi$$

$$= C'_n\int_{\frac{1}{|\xi|}<1}\int_{\frac{1}{|\xi|}}^{\infty}\left|\frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}}}\right| r^{\frac{n}{2}-2N_n-\frac{5}{2}}\left|\left(\frac{|\log(r)|}{|\xi|^{2N_n+\frac{5}{2}}}\right)\right|\cdot$$

$$\cdot\sup_{\eta\in B(\lambda_0,\varepsilon)}\left(\omega_{\mathrm{Re}\,(\eta),\mu}(r)\right) drd\xi$$

$$
\begin{aligned}
&= C_n' \int_{\frac{1}{|\xi|}<1} \int_{\frac{1}{|\xi|}}^{1} \left| \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}}} \right| r^{\frac{n}{2}-2N_n-\frac{5}{2}} \left| \left(\frac{|\log(r)|}{|\xi|^{2N_n+\frac{5}{2}}} \right) \right| \cdot \\
&\quad \cdot \sup_{\eta \in B(\lambda_0,\varepsilon)} \left(\omega_{\mathrm{Re}\,(\eta),\mu}(r) \right) dr d\xi \\
&\quad + C_n' \int_{\frac{1}{|\xi|}<1} \int_{1}^{\infty} \left| \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}}} \right| r^{\frac{n}{2}-2N_n-\frac{5}{2}} \left| \left(\frac{|\log(r)|}{|\xi|^{2N_n+\frac{5}{2}}} \right) \right| \cdot \\
&\quad \cdot \sup_{\eta \in B(\lambda_0,\varepsilon)} \left(\omega_{\mathrm{Re}\,(\eta),\mu}(r) \right) dr d\xi \\
&= C_n' \int_{\frac{1}{|\xi|}<1} \int_{\frac{1}{|\xi|}}^{1} \left| \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}}} \right| r^{\frac{n}{2}-2N_n-\frac{5}{2}} \left| \left(\frac{|\log(r)|}{|\xi|^{2N_n+\frac{5}{2}}} \right) \right| \cdot \\
&\quad \cdot \left(\omega_{\mathrm{Re}\,(\lambda_0)+\varepsilon,\mu}(r) \right) dr d\xi \\
&\quad + C_n' \int_{\frac{1}{|\xi|}<1} \int_{1}^{\infty} \left| \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}}} \right| r^{\frac{n}{2}-2N_n-\frac{5}{2}} \left| \left(\frac{|\log(r)|}{|\xi|^{2N_n+\frac{5}{2}}} \right) \right| \cdot \\
&\quad \cdot \left(\omega_{\mathrm{Re}\,(\lambda_0)-\varepsilon,\mu}(r) \right) dr d\xi \\
&= FIRST + SECOND
\end{aligned}
$$

where:

$$
\begin{aligned}
FIRST &= C_n' \int_{\frac{1}{|\xi|}<1} \int_{\frac{1}{|\xi|}}^{1} \left| \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}}} \right| r^{\frac{n}{2}-2N_n-\frac{5}{2}} \left| \left(\frac{|\log(r)|}{|\xi|^{2N_n+\frac{5}{2}}} \right) \right| \cdot \\
&\quad \cdot \left(\omega_{\mathrm{Re}\,(\lambda_0)+\varepsilon,\mu}(r) \right) dr d\xi \\
&= C_n' \int_{\frac{1}{|\xi|}<1} \left| \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2N_n+\frac{5}{2}}} \right| \int_{\frac{1}{|\xi|}}^{1} r^{\frac{n}{2}-2N_n-\frac{5}{2}-2\mathrm{Re}\,(\lambda_0)-2\varepsilon-1} \cdot \\
&\quad \cdot |\log(r)| dr d\xi \\
&\le C_n' \int_{\frac{1}{|\xi|}<1} \left| \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2N_n+\frac{5}{2}}} \right| \int_{\frac{1}{|\xi|}}^{1} r^{\frac{n}{2}-2N_n-\frac{5}{2}-2\mathrm{Re}\,(\lambda_0)-2\varepsilon-1} \frac{1}{r} dr d\xi \\
&\le C_n'' \left(\int_{\frac{1}{|\xi|}<1} \left| \frac{\varphi(\xi)}{|\xi|^{M_{n,\lambda_0,\varepsilon}}} \right| d\xi + \int_{\frac{1}{|\xi|}<1} \left| \frac{\varphi(\xi)}{|\xi|^{M'_{n,\lambda_0,\varepsilon}}} \right| d\xi \right),
\end{aligned}
$$

which converges independently of the exponents $M_{n,\lambda_0,\varepsilon}$ and $M'_{n,\lambda_0,\varepsilon}$ as we're assuming that φ is in the Schwartz class. Also:

$$\begin{aligned} SECOND &= C'_n \int_{\frac{1}{|\xi|}<1} \int_1^{\infty} \left| \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}}} \right| r^{\frac{n}{2}-2N_n-\frac{5}{2}} \left| \left(\frac{|\log(r)|}{|\xi|^{2N_n+\frac{5}{2}}} \right) \right| \cdot \\ &\quad \cdot \left(\omega_{\mathrm{Re}\,(\lambda_0)-\varepsilon,\mu}(r)\right) dr d\xi \\ &= C'_n \int_{\frac{1}{|\xi|}<1} \left| \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2N_n+\frac{5}{2}}} \right| \int_1^{\infty} r^{\frac{n}{2}-2N_n-\frac{5}{2}-2\mathrm{Re}\,(\lambda_0)+2\varepsilon-1} \cdot \\ &\quad \cdot |\log(r)| dr d\xi. \end{aligned}$$

All we need for this integral to converge is that

$$\frac{n}{2} - 2N_n - \frac{5}{2} - 2\mathrm{Re}\,(\lambda_0) + 2\varepsilon - 1 < -1.$$

Equivalently:

$$2\varepsilon < -\frac{n}{2} + 2N_n + \frac{5}{2} + 2\mathrm{Re}\,(\lambda_0).$$

Obviously, such a positive ε exists if and only if $0 < -\frac{n}{2} + 2N_n + \frac{5}{2} + 2\mathrm{Re}\,(\lambda_0)$, if and only if $\frac{n-5}{4} - \lceil \frac{n-5}{4} \rceil < \mathrm{Re}\,(\lambda_0)$. This shows that the new $II^2_{N_n+1}$ is holomorphic (hence analytic) on:

$$\left\{ \lambda \in \mathbf{C} : \mathrm{Re}\,(\lambda) > \frac{n-5}{4} - \left\lceil \frac{n-5}{4} \right\rceil \right\}.$$

Observe that this is an open subset of $\mathbf{C}$ that contains the real interval $\left(0, \frac{n-1}{2}\right)$. We can now write:

$$II^2_{N_n+1} = \int_{\frac{1}{|\xi|}<1} \varphi(\xi) u(\xi) d\xi$$

with

$$u(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}} \omega_{\lambda,\mu}^{(1)}(r) R_n \left(\frac{1}{(2\pi r|\xi|)^{2N_n+\frac{5}{2}}} \right) dr.$$

What we still need to show is that $|u(\xi)| \leq \frac{C_{n,\lambda}}{|\xi|^{n-2\lambda-1}}$. We have:

$$\begin{aligned} |u(\xi)| &\leq \frac{C_{n,\lambda}}{|\xi|^{\frac{n-2}{2}+2N_n+\frac{5}{2}}} \int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}-2N_n-\frac{5}{2}-2\lambda-1} dr \\ &= \frac{C'_{n,\lambda}}{|\xi|^{\frac{n-2}{2}+2N_n+\frac{5}{2}}} \left(\frac{1}{|\xi|} \right)^{\frac{n}{2}-2N_n-\frac{5}{2}-2\lambda} \\ &= \frac{C'_{n,\lambda}}{|\xi|^{\frac{n-2}{2}+2N_n+\frac{5}{2}+\frac{n}{2}-2N_n-\frac{5}{2}-2\lambda}} \\ &= \frac{C'_{n,\lambda}}{|\xi|^{n-2\lambda-1}}. \end{aligned} \tag{3.57}$$

Let us now take care of the terms $II_{j,1}^2$. To simplify the notation, we will write $c_{n,j}$ instead of $c_{\frac{n-2}{2},2j}$ and c_n instead of $c_{\frac{n-2}{2}}$. So, we can now express term $II_{j,1}^2$ as:

$$\begin{aligned} & \int_{\frac{1}{|\xi|}<1} \varphi(\xi) \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \left(\int_{\frac{1}{|\xi|}}^{\infty} r^{\frac{n}{2}} \omega_{\lambda,\mu}^{(1)}(r) \frac{c_{n,j}}{(2\pi r|\xi|)^{2j+\frac{1}{2}}} \cdot \right. \\ & \left. \cdot \cos((2\pi r|\xi|) - c_n) dr \right) d\xi \\ = \quad & c'_{n,j} \int_{\frac{1}{|\xi|}<1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \left(\int_{\frac{1}{|\xi|}}^{\infty} \cos((2\pi r|\xi|) - c_n)\, r^{\frac{n}{2}-2j-\frac{1}{2}}\, \omega_{\lambda,\mu}^{(1)}(r) dr \right) d\xi \\ = \quad & c'_{n,j} \int_{\frac{1}{|\xi|}<1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \left(\int_a^b f(r) g(r) dr \right) d\xi, \end{aligned}$$

where we set:

$$\begin{aligned} a &= \frac{1}{|\xi|} \\ b &= \infty \\ f(r) &= \cos((2\pi r|\xi|) - c_n) \\ g(r) &= r^{\frac{n}{2}-2j-\frac{1}{2}}\omega^{(1)}_{\lambda,\mu}(r). \end{aligned} \tag{3.58}$$

Hence

$$II^2_{j,1} = c'_{n,j}\int_{\frac{1}{|\xi|}<1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}}\left(\int_a^b f(r)g(r)dr\right)d\xi. \tag{3.59}$$

Let's observe that, for $r < \frac{9}{10}$, we have:

$$g(r) = r^{\frac{n}{2}-2j-\frac{1}{2}}\omega^{(1)}_{\lambda,\mu}(r) = r^{\frac{n}{2}-2j-\frac{1}{2}}\omega_{\lambda,\mu}(r) = r^{\frac{n}{2}-2j-\frac{1}{2}-2\lambda-1}.$$

Therefore, for $0 < r < \frac{9}{10}$ we have:

$$\frac{d^k}{dt^k}g(t) = c_{n,\lambda,j,k}\, t^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}-k}, \tag{3.60}$$

for all $k \in \mathbf{Z}^+$ or $k = 0$, where the constants $c_{n,\lambda,j,k}$ are polynomials with respect to λ. On the other hand, if $r > \frac{11}{10}$ we have:

$$g(r) = r^{\frac{n}{2}-2j-\frac{1}{2}}\omega^{(1)}_{\lambda,\mu}(r) = r^{\frac{n}{2}-2j-\frac{1}{2}}\omega_{\lambda,\mu}(r) = \frac{r^{\frac{n}{2}-2j-\frac{1}{2}-2\lambda-1}}{(\log(er))^{\mu}}, \tag{3.61}$$

for all $k \in \mathbf{Z}^+$ or $k = 0$. Therefore, for $r > \frac{11}{10}$, equation (3.48) holds for the new function g as well. The function f is the same we defined to use Lemma

3.2.2 in order to treat the old terms $I^3_{j,1}$. Therefore equation (3.47) still holds for the new f. We are now ready to use Lemma 3.2.2 consistently with the notation just introduced, and with constant N to choose later:

$$\begin{aligned}
\int_a^b f(t)g(t)dt &= (-1)^N \int_a^b \frac{d^{-N}}{dt^{-N}} f(t) \frac{d^N}{dt^N} g(t)dt \\
&\quad + \sum_{k=0}^{N-1} \left((-1)^k \frac{d^{-k-1}}{dt^{-k-1}} f(t) \frac{d^k}{dt^k} g(t) \right) \Bigg|_{t=a}^{b} \\
&= (-1)^N \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N}}{dt^{-N}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^N} \frac{d^N}{dt^N} g(t)dt \\
&\quad + \sum_{k=0}^{N-1} \left((-1)^k \frac{\frac{d^{-k-1}}{dt^{-k-1}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{k+1}} \cdot \right. \\
&\quad \left. \cdot \frac{d^k}{dt^k} g(t) \right) \Bigg|_{t=\frac{1}{|\xi|}}^{\infty} \\
&= (-1)^N \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N}}{dt^{-N}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^N} \frac{d^N}{dt^N} g(t)dt \qquad (3.62) \\
&\quad + \sum_{k=0}^{N-1} \left((-1)^{k+1} \frac{\frac{d^{-k-1}}{dt^{-k-1}}(\cos)((2\pi \frac{1}{|\xi|}|\xi|) - c_n)}{(2\pi|\xi|)^{k+1}} \cdot \right. \\
&\quad \left. \cdot \frac{d^k}{dt^k} g\left(\frac{1}{|\xi|}\right) \right).
\end{aligned}$$

Because of equation (3.61), we can see that the integral in (3.62, page 84) converges for all $\lambda > 0$ and all $j \geq 0$ if and only if $N > \frac{n-1}{2}$. So, we set $N := N_n := \left\lceil \frac{n-1}{2} \right\rceil + 1$. Furthermore, in view of equation (3.60), the last term (that is, the summatory) can be simplified if $\frac{1}{|\xi|} < \frac{9}{10}$. In such case we have:

$$
\begin{aligned}
\int_a^b f(t)g(t)dt &= (-1)^N \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N}}{dt^{-N}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^N} \frac{d^N}{dt^N} g(t)dt \\
&\quad + \sum_{k=0}^{N-1} \Big((-1)^{k+1} c_{n,\lambda,j,k} \cdot \\
&\quad \cdot \frac{\frac{d^{-k-1}}{dt^{-k-1}}(\cos)(2\pi - c_n)}{(2\pi|\xi|)^{k+1}|\xi|^{\frac{n}{2}-2\lambda-1-2j-\frac{1}{2}-k}}\Big) \\
&= (-1)^N \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N}}{dt^{-N}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^N} \frac{d^N}{dt^N} g(t)dt \\
&\quad + \sum_{k=0}^{N-1} \left(\left(-\frac{1}{2\pi}\right)^{k+1} c_{n,\lambda,j,k} \frac{\frac{d^{-k-1}}{dt^{-k-1}}(\cos)(c'_n)}{|\xi|^{\frac{n}{2}-2\lambda-2j-\frac{1}{2}}} \right) \\
&= (-1)^N \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N}}{dt^{-N}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^N} \frac{d^N}{dt^N} g(t)dt \\
&\quad + \sum_{k=0}^{N-1} \left(c_{n,\lambda,j,k} \frac{c''_{n,k}}{|\xi|^{\frac{n}{2}-2\lambda-2j-\frac{1}{2}}} \right) \\
&= (-1)^N \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N}}{dt^{-N}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^N} \frac{d^N}{dt^N} g(t)dt \\
&\quad + \frac{\sum_{k=0}^{N-1} \left(c_{n,\lambda,j,k}\, c''_{n,k} \right)}{|\xi|^{\frac{n}{2}-2\lambda-2j-\frac{1}{2}}} \\
&= (-1)^N \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N}}{dt^{-N}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^N} \frac{d^N}{dt^N} g(t)dt \\
&\quad + \frac{c''_{n,\lambda,j}}{|\xi|^{\frac{n}{2}-2\lambda-2j-\frac{1}{2}}},
\end{aligned} \tag{3.63}
$$

for all $\xi \in \mathbf{R}^n \setminus \{0\}$ such that $|\xi| > \frac{10}{9}$. Observe that the constant $c''_{n,\lambda,j}$ is a polynomial with respect to λ, as the constants $c_{n,\lambda,j,k}$ are polynomials with respect to λ. Equations (3.63) and (3.59) show that we can write:

$$II_{j,1}^2 = c'_{n,j} \int_{\frac{1}{|\xi|}<1} \varphi(\xi) u^{(4)}(\xi)\xi$$

with

$$u^{(4)}(\xi) = \frac{1}{|\xi|^{\frac{n}{2}-\frac{1}{2}+2j}} \left(\int_a^b f(t)g(t)dt \right)$$

Therefore, $u^{(4)}$ is a function defined on $\mathbf{R}^n \setminus \{0\}$ that, in view of equation (3.63), satisfies the inequality:

$$|u^{(4)}(\xi)| \leq \frac{C_{n,\lambda,\mu,j}}{|\xi|^{n-2\lambda-1}} \tag{3.64}$$

for all $|\xi| > \frac{10}{9}$. Since $u^{(4)}$ is also continuous, the estimate above also holds for all $|\xi| \geq 1$, at least after replacing $C_{n,\lambda,\mu,j}$ with a bigger constant $C'_{n,\lambda,\mu,j}$. We still need to prove that $II_{j,1}^2$ is analytic with respect to λ on an open subset of $\mathbf{C}$ that contains the real interval $\left(0, \frac{n-1}{2}\right)$. To do so, we look back at equation (3.62) at page 84 and equation (3.59) at page 83 to rewrite:

$$II_{j,1}^2 = \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil + 1} II_{j,1,k}^2, \tag{3.65}$$

where

$$II_{j,1,k}^2 = c'_{n,j} \int_{\frac{1}{|\xi|}<1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}+k+1}} c''_{n,k} \left(\frac{d^k}{dt^k} g\left(\frac{1}{|\xi|}\right)\right) d\xi, \tag{3.66}$$

for all $k = 0, ..., \lceil \frac{n-1}{2} \rceil$ and

$$II^2_{j,1,N'_n} = c'_{n,j} \int_{\frac{1}{|\xi|}<1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \left((-1)^{N'_n} \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N'_n}}{dt^{-N'_n}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N'_n}} \frac{d^{N'_n}}{dt^{N'_n}} g(t)dt \right) d\xi, \tag{3.67}$$

where we set $N'_n = \lceil \frac{n-1}{2} \rceil + 1$ to simplify the notation (recall that we already set $N_n := \lceil \frac{n-5}{4} \rceil$ at page 85). We will now show that each term $II^2_{j,1,k}$ is analytic with respect to λ. We begin with $II^2_{j,1,k}$, $k = 0, ..., \lceil \frac{n-1}{2} \rceil$. Because of equations (3.58, page 83) and (3.34, page 56), we have:

$$g(r) = r^{\frac{n}{2}-2j-\frac{1}{2}}(\omega_{\lambda\mu}(r)(1-\phi(r)) + \phi(r)) = g_{\lambda\mu}(r) + g_1(r)$$

where

$$g_{\lambda\mu}(r) = r^{\frac{n}{2}-2j-\frac{1}{2}}\omega_{\lambda\mu}(r)(1-\phi(r)), \tag{3.68}$$

and

$$g_1(r) = r^{\frac{n}{2}-2j-\frac{1}{2}}\phi(r).$$

Therefore we can rewrite the term in (3.66, page 86) as follows:

$$\frac{d^k}{dt^k} g\left(\frac{1}{|\xi|}\right) = \frac{d^k}{dt^k} g_{\lambda\mu}\left(\frac{1}{|\xi|}\right) + \frac{d^k}{dt^k} g_1\left(\frac{1}{|\xi|}\right)$$

In view of this last equation and of (3.66, page 86), we have:

$$II^2_{j,1,k} = II^{2,1}_{j,1,k,\lambda,\mu} + II^{2,2}_{j,1,k,\lambda,\mu} + II^2_{j,1,k,1}, \tag{3.69}$$

where:

$$
\begin{aligned}
II^{2,1}_{j,1,k,\lambda,\mu} &= c'_{n,j}\int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}+k+1}} c''_{n,k}\left(\frac{d^k}{dt^k}g_{\lambda\mu}\left(\frac{1}{|\xi|}\right)\right)d\xi \\
&= c''_{n,j,k}\int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{\varphi(\xi)}{|\xi|^{\frac{n}{2}+2j+\frac{1}{2}+k}}\left(\frac{d^k}{dt^k}g_{\lambda\mu}\left(\frac{1}{|\xi|}\right)\right)d\xi, \qquad (3.70)
\end{aligned}
$$

for $c''_{n,j,k} := c''_{n,k}\, c'_{n,j}$, and

$$
II^{2,2}_{j,1,k,\lambda,\mu} = c''_{n,j,k}\int_{\frac{9}{10}\le\frac{1}{|\xi|}<1} \frac{\varphi(\xi)}{|\xi|^{\frac{n}{2}+2j+\frac{1}{2}+k}}\left(\frac{d^k}{dt^k}g_{\lambda\mu}\left(\frac{1}{|\xi|}\right)\right)d\xi, \quad (3.71)
$$

and

$$
II^{2}_{j,1,k,1} = c'_{n,j}\int_{\frac{1}{|\xi|}<1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}+k+1}} c''_{n,k}\left(\frac{d^k}{dt^k}g_{1}\left(\frac{1}{|\xi|}\right)\right)d\xi.
$$

Observe that $II^2_{j,1,k,1}$ doesn't depend on λ and it's defined for all $k = 0, ..., \lceil\frac{n-1}{2}\rceil$ and $j = 0, ..., N_n = \lceil\frac{n-5}{4}\rceil$. So, it's entire with respect to λ for all such j's and k's. Now, to show that $II^2_{j,1,k}$ is analytic, it's enough to show that $II^{2,1}_{j,1,k,\lambda,\mu}$ and $II^{2,2}_{j,1,k,\lambda,\mu}$ are analytic. We will need more decomposition. Because of the definition of $g_{\lambda\mu}$ (3.68, page 87), for $t < 1$ we can write:

$$
\begin{aligned}
\frac{d^k}{dt^k}g_{\lambda\mu}(t) &= \frac{d^k}{dt^k}\left(t^{\frac{n}{2}-2j-\frac{1}{2}}\omega_{\lambda\mu}(t)(1-\phi(t))\right) \\
&= \frac{d^k}{dt^k}\left(t^{\frac{n}{2}-2j-\frac{1}{2}-2\lambda-1}(1-\phi(t))\right) \\
&= \frac{d^k}{dt^k}\left(t^{\frac{n}{2}-2j-\frac{1}{2}-1}(1-\phi(t))t^{-2\lambda}\right) \\
&= \sum_{l=0}^{k} b_{k,l}\,\frac{d^{k-l}}{dt^{k-l}}\left(t^{\frac{n}{2}-2j-\frac{1}{2}-1}(1-\phi(t))\right)\frac{d^l}{dt^l}\left(t^{-2\lambda}\right),
\end{aligned}
$$

where $b_{k,l}$ denotes the binomials coefficients corresponding to the indexes k and l (that is, $(a+c)^k = \sum_{l=0}^{k} b_{k,l}\, a^l\, c^{k-l}$). The equalities continue:

$$\frac{d^k}{dt^k} g_{\lambda\mu}(t) = \sum_{l=0}^{k} b_{k,l}\, \frac{d^{k-l}}{dt^{k-l}} \left(t^{\frac{n}{2}-2j-\frac{1}{2}-1}(1-\phi(t)) \right) p_l(\lambda) \; \left(t^{-2\lambda-l} \right), \qquad (3.72)$$

for some polynomials p_l. If, in addition, we assume $t < \frac{9}{10}$, then we have:

$$\begin{aligned}
\frac{d^k}{dt^k} g_{\lambda\mu}(t) &= \sum_{l=0}^{k} b_{k,l}\, \frac{d^{k-l}}{dt^{k-l}} \left(t^{\frac{n}{2}-2j-\frac{1}{2}-1} \right) p_l(\lambda)\, t^{-2\lambda-l} \\
&= \sum_{l=0}^{k} C_{n,k,l,j} t^{\frac{n}{2}-2j-\frac{1}{2}-1-k+l}\, p_l(\lambda)\, t^{-2\lambda-l} \\
&= \sum_{l=0}^{k} C_{n,k,l,j} t^{\frac{n}{2}-2j-\frac{1}{2}-1-k}\, p_l(\lambda)\, t^{-2\lambda},
\end{aligned}$$

for some constants $C_{n,k,l,j}$. In this case (that is, if $t < \frac{9}{10}$) we also have:

$$\begin{aligned}
\frac{d}{d\lambda}\frac{d^k}{dt^k} g_{\lambda\mu}(t) &= \sum_{l=0}^{k} C_{n,k,l,j}\; t^{\frac{n}{2}-2j-\frac{1}{2}-1-k}\, \frac{d}{d\lambda} \left(p_l(\lambda)\, t^{-2\lambda} \right) \\
&= \sum_{l=0}^{k} C_{n,k,l,j}\; t^{\frac{n}{2}-2j-\frac{1}{2}-1-k} \,\cdot \\
&\quad \cdot \left(p_l'(\lambda)\, t^{-2\lambda} + p_l(\lambda)\, t^{-2\lambda} \log\left(\frac{1}{t^2}\right) \right) \\
&= \sum_{l=0}^{k} C_{n,k,l,j}\; t^{\frac{n}{2}-2j-\frac{1}{2}-1-k} \,\cdot \\
&\quad \cdot \left(p_l'(\lambda) + p_l(\lambda)\, \log\left(\frac{1}{t^2}\right) \right) t^{-2\lambda},
\end{aligned}$$

where p_l' just denotes the derivative of p_l (with respect to λ).

Therefore:

$$
\begin{aligned}
\left|\frac{d}{d\lambda}\frac{d^k}{dt^k}g_{\lambda\mu}(t)\right|\Bigg|_{\lambda=\eta} &\le \left(\sum_{l=0}^{k} C'_{n,k,l,j}\ t^{\frac{n}{2}-2j-\frac{1}{2}-1-k}\cdot\right.\\
&\quad \left.\cdot\left(|p_l'(\lambda)| + |p_l(\lambda)|\ \log\left(\frac{1}{t^2}\right)\right)|t^{-2\lambda}|\right)\Bigg|_{\lambda=\eta}\\
&= \sum_{l=0}^{k} C'_{n,k,l,j}\ t^{\frac{n}{2}-2j-\frac{1}{2}-1-k}\ \cdot\\
&\quad \cdot\left(|p_l'(\eta)| + |p_l(\eta)|\ \log\left(\frac{1}{t^2}\right)\right)t^{-2\mathrm{Re}\,(\eta)},
\end{aligned}
$$

where $C'_{n,k,l,j} = |C_{n,k,l,j}|$. Then, if $\lambda_0 \in \mathbf{C}$ and $\varepsilon > 0$, we have:

$$
\begin{aligned}
&\sup_{\eta\in B(\lambda_0,\varepsilon)}\left(\left|\frac{d}{d\lambda}\frac{d^k}{dt^k}g_{\lambda\mu}(t)\right|\Bigg|_{\lambda=\eta}\right) \qquad (3.73)\\
\le\ &\sup_{\eta\in B(\lambda_0,\varepsilon)}\left(\sum_{l=0}^{k} C'_{n,k,l,j}\ t^{\frac{n}{2}-2j-\frac{1}{2}-1-k}\cdot\right.\\
&\left.\cdot\left(|p_l'(\eta)| + |p_l(\eta)|\ \log\left(\frac{1}{t^2}\right)\right)t^{-2\mathrm{Re}\,(\eta)}\right)\\
\le\ &\sum_{l=0}^{k} C'_{n,k,l,j}\ t^{\frac{n}{2}-2j-\frac{1}{2}-1-k}\ \cdot\\
&\cdot \sup_{\eta\in B(\lambda_0,\varepsilon)}\left(\left(|p_l'(\eta)| + |p_l(\eta)|\ \log\left(\frac{1}{t^2}\right)\right)t^{-2\mathrm{Re}\,(\eta)}\right)\\
\le\ &\sum_{l=0}^{k} C'_{n,k,l,j}\ t^{\frac{n}{2}-2j-\frac{1}{2}-1-k}\ \cdot\\
&\cdot \sup_{\eta\in B(\lambda_0,\varepsilon)}\left(|p_l'(\eta)| + |p_l(\eta)|\ \log\left(\frac{1}{t^2}\right)\right)\sup_{\eta\in B(\lambda_0,\varepsilon)}\left(t^{-2\mathrm{Re}\,(\eta)}\right).
\end{aligned}
$$

As we are assuming $t < \frac{9}{10} < 1$, we have:

$$
\sup_{\eta\in B(\lambda_0,\varepsilon)}\left(t^{-2\mathrm{Re}\,(\eta)}\right) = t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon}
$$

So, (3.73) becomes:

$$
\begin{aligned}
&\sup_{\eta \in B(\lambda_0,\varepsilon)} \left(\left| \frac{d}{d\lambda} \frac{d^k}{dt^k} g_{\lambda,\mu}(t) \right| \Big|_{\lambda=\eta} \right) \\
\le &\sum_{l=0}^{k} C'_{n,k,l,j} \; t^{\frac{n}{2}-2j-\frac{1}{2}-1-k} \cdot \\
&\cdot \sup_{\eta \in B(\lambda_0,\varepsilon)} \left(|p'_l(\eta)| + |p_l(\eta)| \log\left(\frac{1}{t^2}\right) \right) t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} \\
= &\sum_{l=0}^{k} C'_{n,k,l,j} t^{\frac{n}{2}-2j-\frac{1}{2}-1-k} \left(C^{(1)}_{l,\lambda_0,\varepsilon} + C^{(2)}_{l,\lambda_0,\varepsilon} \cdot \log\left(\frac{1}{t^2}\right) \right) t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon},
\end{aligned}
\tag{3.74}
$$

where $0 < C^{(i)}_{l,\lambda_0,\varepsilon} < \infty$ for $i = 1, 2$. All of these computations are useful because we're going to prove that $II^{2,1}_{j,1,k,\lambda,\mu}$ is analytic with respect to λ in the same way we proved analitycity for the other terms. That is, we will prove that $II^{2,1}_{j,1,k,\lambda,\mu}$ is holomorphic at a point $\lambda_0 \in \mathbf{C}$ by showing that (cf. equation 3.70, page 88):

$$
\begin{aligned}
II^{2,1,*}_{j,1,k,\lambda} :=& c'''_{n,j,k} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j+\frac{1}{2}+k}} \\
&\cdot \sup_{\eta \in B(\lambda_0,\varepsilon)} \left(\left| \frac{d}{d\lambda} \frac{d^k}{dt^k} g_{\lambda\mu}\left(\frac{1}{|\xi|}\right) \right| \Big|_{\lambda=\eta} \right) d\xi < \infty
\end{aligned}
\tag{3.75}
$$

where $c'''_{n,j,k} = |c''_{n,j,k}|$. Observe that $II^{2,1,*}_{j,1,k,\lambda}$ really doesn't depend on μ, because $\frac{d}{d\lambda}\frac{d^k}{dt^k} g_{\lambda\mu}$ is only evaluated at $\frac{1}{|\xi|}$, which is less than $\frac{9}{10}$. As (3.74) holds for $0 < t < \frac{9}{10}$, we can use it in the equation above to write:

$$
II^{2,1,*}_{j,1,k,\lambda} \le \sum_{l=0}^{k} II^{2,1,*}_{j,1,k,\lambda,l},
$$

where

$$
\begin{aligned}
II^{2,1,*}_{j,1,k,\lambda,l} &= c'''_{n,j,k} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j+\frac{1}{2}+k}} C'_{n,k,l,j} \cdot \\
&\cdot \left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-2j-\frac{3}{2}-k-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} C^{(2)}_{l,\lambda_0,\varepsilon} \log\left(|\xi|^2\right) d\xi \\
&+ c'''_{n,j,k} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j+\frac{1}{2}+k}} C'_{n,k,l,j} \cdot \\
&\cdot \left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-2j-\frac{3}{2}-k-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} C^{(1)}_{l,\lambda_0,\varepsilon} d\xi \\
&= c'''_{n,j,k} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{n-1-2\mathrm{Re}\,(\lambda_0)-2\varepsilon}} C'_{n,k,l,j} C^{(2)}_{l,\lambda_0,\varepsilon} \log\left(|\xi|^2\right) d\xi \\
&+ c'''_{n,j,k} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{n-1-2\mathrm{Re}\,(\lambda_0)-2\varepsilon}} C'_{n,k,l,j} C^{(1)}_{l,\lambda_0,\varepsilon} d\xi < \infty,
\end{aligned}
$$

for any $\lambda_0 \in \mathbf{C}$, any $\varepsilon > 0$, any $j = 0, ..., N_n = \lceil \frac{n-5}{4} \rceil$, any $k = 0, ..., \lceil \frac{n-1}{2} \rceil$ and any $l = 0, ..., k$, because $\varphi \in \mathcal{S}$. This shows that each term $II^{2,1,*}_{j,1,k,\lambda}$ is finite. We just proved that each $II^{2,1}_{j,1,k,\lambda,\mu}$ is entire with respect to λ.

To prove that the terms $II^{2,2}_{j,1,k,\lambda,\mu}$ are analytic, we will argue in the same way. If $\frac{9}{10} \le t < 1$, we can only use that equation (3.72, page 89) holds for $0 < t < 1$ to write:

$$
\frac{d^k}{dt^k} g_\lambda(t) = \sum_{l=0}^{k} h_{n,j,k,l}(t)\, p_l(\lambda)\, t^{-2\lambda},
$$

for functions $h_{n,j,k,l} \in C^\infty((0,\infty))$. Then:

$$\begin{aligned}
\frac{d}{d\lambda}\frac{d^k}{dt^k} g_\lambda(t) &= \sum_{l=0}^{k} h_{n,j,k,l}(t)\, \frac{d}{d\lambda}\left(p_l(\lambda)\, t^{-2\lambda}\right) \\
&= \sum_{l=0}^{k} h_{n,j,k,l}(t)\, \left(p_l'(\lambda)\, t^{-2\lambda} + p_l(\lambda)\, t^{-2\lambda}\, \log\left(\frac{1}{t^2}\right)\right) \\
&= \sum_{l=0}^{k} h_{n,j,k,l}(t)\, t^{-2\lambda} \left(p_l'(\lambda) + p_l(\lambda)\, \log\left(\frac{1}{t^2}\right)\right).
\end{aligned}$$

Then:

$$\begin{aligned}
\left|\frac{d}{d\lambda}\frac{d^k}{dt^k} g_\lambda(t)\right|\Bigg|_{\lambda=\eta} &\leq \sum_{l=0}^{k} |h_{n,j,k,l}(t)|\, |t^{-2\eta}| \left(|p_l'(\eta)| + |p_l(\eta)|\, \log\left(\frac{1}{t^2}\right)\right) \\
&\leq \sum_{l=0}^{k} |h_{n,j,k,l}(t)|\, t^{-2\mathrm{Re}\,(\eta)} \cdot \\
&\quad \cdot \left(|p_l'(\eta)| + |p_l(\eta)|\, \log\left(\frac{1}{t^2}\right)\right).
\end{aligned}$$

Therefore, if $\lambda_0 \in \mathbf{C}$ and $\varepsilon > 0$:

$$\begin{aligned}
\sup_{\eta \in B(\lambda_0,\varepsilon)} \left(\left|\frac{d}{d\lambda}\frac{d^k}{dt^k} g_\lambda(t)\right|\Bigg|_{\lambda=\eta}\right) &\leq \sum_{l=0}^{k} |h_{n,j,k,l}(t)|\, t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon}\, C_{l,\lambda_0,\varepsilon} \cdot \\
&\quad \cdot \left(1 + \log\left(\frac{1}{t^2}\right)\right) \\
&= \sum_{l=0}^{k} \widetilde{h}_{n,j,k,l}(t)\, t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon}\, C_{l,\lambda_0,\varepsilon}, \quad (3.76)
\end{aligned}$$

because we're assuming $0 < t < 1$. In the equality above we set:

$$C_{l,\lambda_0,\varepsilon} = \max\left\{\sup_{\eta \in B(\lambda_0,\varepsilon)} (|p_l(\eta)|),\ \sup_{\eta \in B(\lambda_0,\varepsilon)} (|p_l'(\eta)|)\right\}$$

and

$$\widetilde{h}_{n,j,k,l}(t) = |h_{n,j,k,l}(t)| \left(1 + \log\left(\frac{1}{t^2}\right)\right).$$

Again, we have that $II^{2,2}_{j,1,k,\lambda,\mu}$ is holomorphic at $\lambda = \lambda_0 \in \mathbf{C}$ if there exists $\varepsilon > 0$ such that (cf. equation 3.71, page 88):

$$\begin{aligned} II^{2,2,*}_{j,1,k,\lambda} :=& c'''_{n,j,k} \int_{\frac{9}{10} \le \frac{1}{|\xi|} < 1} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j+\frac{1}{2}+k}} \cdot \\ & \cdot \sup_{\eta \in B(\lambda_0,\varepsilon)} \left(\left| \frac{d}{d\lambda} \frac{d^k}{dt^k} g_{\lambda\mu}\left(\frac{1}{\xi}\right) \right| \Big|_{\lambda=\eta} \right) d\xi < \infty \end{aligned} \tag{3.77}$$

where we set again $c'''_{n,j,k} = |c''_{n,j,k}|$. Now we use equation (3.76, page 93) to write:

$$II^{2,2,*}_{j,1,k,\lambda} = \sum_{l=0}^{k} II^{2,2,*}_{j,1,k,\lambda,l}$$

where:

$$\begin{aligned} II^{2,2,*}_{j,1,k,\lambda,l} &= c'''_{n,j,k} \int_{\frac{9}{10} \le \frac{1}{|\xi|} < 1} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j+\frac{1}{2}+k}} \widetilde{h}_{n,j,k,l}\left(\frac{1}{|\xi|}\right) \cdot \\ & \quad \cdot \left(\frac{1}{|\xi|}\right)^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} C_{l,\lambda_0,\varepsilon} d\xi \\ &= c'''_{n,j,k} \int_{\frac{9}{10} \le \frac{1}{|\xi|} < 1} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j+\frac{1}{2}+k-2\mathrm{Re}\,(\lambda_0)-2\varepsilon}} \widetilde{h}_{n,j,k,l}\left(\frac{1}{|\xi|}\right) \cdot \\ & \quad \cdot C_{l,\lambda_0,\varepsilon} d\xi < \infty. \end{aligned}$$

The inequality holds because all the functions are continuous on the closure of the domain of integration, that is compact, for any $\lambda_0 \in \mathbf{C}$, $\varepsilon > 0$ and any of the indexes j, k and l. So, $II^{2,2,*}_{j,1,k,\lambda} < \infty$ and $II^{2,2}_{j,1,k,\lambda}$ is an entire function with respect to the variable λ.

This concludes the proof that $II^2_{j,1,k}$ is an entire function with respect to λ for every $k = 0, ..., \lceil \frac{n-1}{2} \rceil$ and $j = 0, ..., N_n = \lceil \frac{n-5}{4} \rceil$ (cf. equation 3.69, page 87).

We still need to prove that $II^2_{j,1,N'_n}$ (cf. equation 3.67, page 87) is analytic with respect to λ on some open subset of $\mathbf{C}$ that contains our real interval. We use again the decomposition $g = g_{\lambda\mu} + g_1$ (cf. equation 3.68, page 87) to decompose $II^2_{j,1,N'_n}$ as we did in equation (3.69, page 87):

$$II^2_{j,1,N'_n} = II^{2,1}_{j,1,N'_n,\lambda,\mu} + II^{2,2}_{j,1,N'_n,\lambda,\mu} + II^2_{j,1,N'_n,1}, \tag{3.78}$$

where

$$\begin{aligned} II^{2,1}_{j,1,N'_n,\lambda,\mu} =& c'_{n,j} \int_{\frac{1}{|\xi|} < \frac{9}{10}} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot \\ & \cdot \left((-1)^{N'_n} \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N'_n}}{dt^{-N'_n}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N'_n}} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) dt \right) d\xi \end{aligned} \tag{3.79}$$

$$\begin{aligned} II^{2,2}_{j,1,N'_n,\lambda,\mu} =& c'_{n,j} \int_{\frac{9}{10} \le \frac{1}{|\xi|} < 1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot \\ & \cdot \left((-1)^{N'_n} \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N'_n}}{dt^{-N'_n}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N'_n}} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) dt \right) d\xi \end{aligned} \tag{3.80}$$

and

$$II^2_{j,1,N_n',1} = c'_{n,j} \int_{\frac{1}{|\xi|}<1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot \cdot \left((-1)^{N_n'} \int_{\frac{1}{|\xi|}}^{\infty} \frac{\frac{d^{-N_n'}}{dt^{-N_n'}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N_n'}} \frac{d^{N_n'}}{dt^{N_n'}} g_1(t) dt \right) d\xi \tag{3.81}$$

Again, $II^2_{j,1,N_n',1}$ doesn't depend on λ and is therefore entire with respect to λ. This term is defined for all $j = 0, ..., N_n = \lceil \frac{n-5}{4} \rceil$. To see that $II^{2,1}_{j,1,N_n',\lambda,\mu}$ is analytic, we further decompose it:

$$II^{2,1}_{j,1,N_n',\lambda,\mu} = II^{2,1,1}_{j,1,N_n',\lambda,\mu} + II^{2,1,2}_{j,1,N_n',\lambda,\mu} + II^{2,1,3}_{j,1,N_n',\lambda,\mu} + II^{2,1,4}_{j,1,N_n',\lambda,\mu}, \tag{3.82}$$

where

$$\begin{aligned}
II^{2,1,1}_{j,1,N_n',\lambda,\mu} &= c''_{n,j} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot \\
&\quad \cdot \left(\int_{\frac{1}{|\xi|}}^{\frac{9}{10}} \frac{\frac{d^{-N_n'}}{dt^{-N_n'}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N_n'}} \frac{d^{N_n'}}{dt^{N_n'}} g_{\lambda,\mu}(t) dt \right) d\xi \qquad (3.83) \\
II^{2,1,2}_{j,1,N_n',\lambda,\mu} &= c''_{n,j} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot \\
&\quad \cdot \left(\int_{\frac{9}{10}}^{1} \frac{\frac{d^{-N_n'}}{dt^{-N_n'}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N_n'}} \frac{d^{N_n'}}{dt^{N_n'}} g_{\lambda,\mu}(t) dt \right) d\xi \\
II^{2,1,3}_{j,1,N_n',\lambda,\mu} &= c''_{n,j} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot \\
&\quad \cdot \left(\int_{1}^{\frac{11}{10}} \frac{\frac{d^{-N_n'}}{dt^{-N_n'}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N_n'}} \frac{d^{N_n'}}{dt^{N_n'}} g_{\lambda,\mu}(t) dt \right) d\xi \\
II^{2,1,4}_{j,1,N_n',\lambda,\mu} &= c''_{n,j} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot
\end{aligned}$$

$$\cdot \left(\int_{\frac{11}{10}}^{\infty} \frac{\frac{d^{-N'_n}}{dt^{-N'_n}}(\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N'_n}} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t)dt \right) d\xi.$$

To treat $II^{2,1,1}_{j,1,N'_n,\lambda,\mu}$ recall (cf. equation 3.68, page 87) that, for $0 < t \leq \frac{9}{10}$, we have:

$$g_{\lambda\mu}(t) = t^{\frac{n}{2}-2j-\frac{1}{2}} \frac{1}{t^{2\lambda+1}} = t^{\frac{n}{2}-2j-\frac{3}{2}} t^{-2\lambda}.$$

Therefore:

$$\begin{aligned}
\frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) &= \sum_{l=0}^{N'_n} b_{N'_n,l} \frac{d^{N'_n-l}}{dt^{N'_n-l}} t^{\frac{n}{2}-2j-\frac{3}{2}} \frac{d^l}{dt^l} t^{-2\lambda} \\
&= \sum_{l=0}^{N'_n} b_{N'_n,l}\, c_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n+l}\, p_l(\lambda)\, t^{-2\lambda-l} \\
&= \sum_{l=0}^{N'_n} c'_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}\, p_l(\lambda)\, t^{-2\lambda},
\end{aligned}$$

for constants $c'_{n,j,l}$ and polynomials (with respect to λ) $p_l(\lambda)$. Then:

$$\begin{aligned}
\frac{d}{d\lambda} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) &= \sum_{l=0}^{N'_n} c'_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n} \frac{d}{d\lambda} \left(p_l(\lambda)\, t^{-2\lambda}\right) \\
&= \sum_{l=0}^{N'_n} c'_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n} \cdot \\
&\quad \cdot \left(p_l(\lambda)\, t^{-2\lambda} \log\left(\frac{1}{t^2}\right) + p'_l(\lambda) t^{-2\lambda} \right) \\
&= \sum_{l=0}^{N'_n} c'_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n} \cdot \\
&\quad \cdot \left(p_l(\lambda) \log\left(\frac{1}{t^2}\right) + p'_l(\lambda) \right) t^{-2\lambda}.
\end{aligned}$$

Then:

$$\left|\frac{d}{d\lambda}\frac{d^{N'_n}}{dt^{N'_n}}g_{\lambda\mu}(t)\right|\Big|_{\lambda=\eta} \leq \sum_{l=0}^{N'_n} c''_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}\, \log\left(\frac{1}{t^2}\right)\, |p_l(\eta)|\, t^{-2\mathrm{Re}\,(\eta)} + \sum_{l=0}^{N'_n} c''_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}\, |p'_l(\eta)|\, t^{-2\mathrm{Re}\,(\eta)},$$

where $c''_{n,j,l} = |c'_{n,j,l}|$. Now, for $\lambda_0 \in \mathbf{C}$ and $\varepsilon > 0$ we have:

$$\sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\frac{d}{d\lambda}\frac{d^{N'_n}}{dt^{N'_n}}g_{\lambda\mu}(t)\right|\Big|_{\lambda=\eta} \leq \sum_{l=0}^{N'_n}\Big(c''_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}\cdot \log\left(\frac{1}{t^2}\right)\, C_{l,\lambda_0,\varepsilon}\, t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon}\Big) + \sum_{l=0}^{N'_n} c''_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}\cdot C_{l,\lambda_0,\varepsilon}\, t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon}, \tag{3.84}$$

for constants $0 < C_{l,\lambda_0,\varepsilon} < \infty$ (as p_l and p'_l are polynomials, such constants are finite whenever $0 < \varepsilon < \infty$). Now, let's define (cf. equations 3.83, page 96):

$$II^{2,1,1,*}_{j,1,N'_n,\lambda_0,\varepsilon} = \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \int_{\frac{1}{|\xi|}}^{\frac{9}{10}} \frac{1}{|\xi|^{N'_n}}\cdot \sup_{\eta\in B(\lambda_0,\varepsilon)} \left|\frac{d}{d\lambda}\frac{d^{N'_n}}{dt^{N'_n}}g_{\lambda\mu}(t)\right|\Big|_{\lambda=\eta} dt d\xi. \tag{3.85}$$

In view of inequality (3.84, page 98), we can write:

$$II^{2,1,1,*}_{j,1,N'_n,\lambda_0,\varepsilon} \leq \sum_{l=0}^{N'_n}\left(II^{2,1,1,*,l,1}_{j,1,N'_n,\lambda_0,\varepsilon} + II^{2,1,1,*,l,2}_{j,1,N'_n,\lambda_0,\varepsilon}\right)$$

where

$$
\begin{aligned}
II^{2,1,1,*,l,1}_{j,1,N'_n,\lambda_0,\varepsilon} &= \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \int_{\frac{1}{|\xi|}}^{\frac{9}{10}} \frac{1}{|\xi|^{N'_n}} c''_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n} \cdot \\
&\quad \cdot \log\left(\frac{1}{t^2}\right) C_{l,\lambda_0,\varepsilon}\, t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} dt\, d\xi \\
&= C'_{n,j,l,\lambda_0,\varepsilon} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j-\frac{1}{2}+N'_n}} \cdot \\
&\quad \cdot \int_{\frac{1}{|\xi|}}^{\frac{9}{10}} t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} \log\left(\frac{1}{t^2}\right) dt\, d\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
II^{2,1,1,*,l,2}_{j,1,N'_n,\lambda_0,\varepsilon} &= \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \int_{\frac{1}{|\xi|}}^{\frac{9}{10}} \frac{1}{|\xi|^{N'_n}} c''_{n,j,l}\, t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n} C_{l,\lambda_0,\varepsilon} \cdot \\
&\qquad \cdot t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} dt d\xi \\
&= C'_{n,j,l,\lambda_0,\varepsilon} \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j-\frac{1}{2}+N'_n}} \cdot \\
&\quad \cdot \int_{\frac{1}{|\xi|}}^{\frac{9}{10}} t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} dt d\xi,
\end{aligned}
$$

for all $l = 0, ..., N'_n = \lceil \frac{n-1}{2} \rceil + 1$. As $\varphi \in \mathcal{S}(\mathbf{R}^n)$, it's immediate to check that

$$II^{2,1,1,*,l,1}_{j,1,N'_n,\lambda_0,\varepsilon} < \infty$$

and

$$II^{2,1,1,*,l,2}_{j,1,N'_n,\lambda_0,\varepsilon} < \infty$$

for all $l = 0, ..., N'_n$ and for any $\lambda_0 \in \mathbf{C}$ and $0 < \varepsilon < \infty$. Therefore, $II^{2,1,1,*}_{j,1,N'_n,\lambda_0,\varepsilon} < \infty$ for the same λ_0 and ε, which implies that $II^{2,1,1}_{j,1,N'_n,\lambda,\mu}$ is entire with respect to λ.

To treat $II^{2,1,2}_{j,1,N'_n,\lambda,\mu}$ recall (cf. equation 3.68, page 87) that, for $\frac{9}{10} \leq t \leq 1$, we have:

$$g_{\lambda\mu}(t) = t^{\frac{n}{2}-2j-\frac{1}{2}}\,(1-\phi(t))\,\frac{1}{t^{2\lambda+1}} = t^{\frac{n}{2}-2j-\frac{3}{2}}\,(1-\phi(t))\,t^{-2\lambda}$$

that is:

$$g_{\lambda\mu}(t) = h(t)\,t^{-2\lambda}$$

for a function $h \in C^\infty(\mathbf{R})$ (explicitely, $h(t) = t^{\frac{n}{2}-2j-\frac{3}{2}}\,(1-\phi(t))$). Then, for $\frac{9}{10} \leq t \leq 1$:

$$\begin{aligned} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) &= \sum_{l=0}^{N'_n} b_{N'_n,l}\,\frac{d^{N'_n-l}}{dt^{N'_n-l}} h(t)\,\frac{d^l}{dt^l} t^{-2\lambda} \\ &= \sum_{l=0}^{N'_n} h_{n,l}(t)\,p_l(\lambda)\,t^{-2\lambda}, \end{aligned}$$

where $b_{N'_n,l}$ is the binomial coefficient with indexes N'_n and l, $h_{n,l}(t) = b_{N'_n,l}\,\frac{d^{N'_n-l}}{dt^{N'_n-l}} h(t)$ for all $l = 0, ..., N'_n$ and $\frac{9}{10} \leq t \leq 1$, and $p_l(\lambda)$ is a polynomial with respect to λ for all l's. Then:

$$\frac{d}{d\lambda}\frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) = \sum_{l=0}^{N'_n} h_{n,l}(t)\,\left(p'_l(\lambda)\cdot t^{-2\lambda} + p_l(\lambda)\cdot t^{-2\lambda}\cdot \log\left(\frac{1}{t^2}\right)\right),$$

and

$$\begin{aligned} \left|\frac{d}{d\lambda}\frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t)\right|\Bigg|_{\lambda=\eta} \leq & \sum_{l=0}^{N'_n} |h_{n,l}(t)| \cdot \\ & \cdot \left(|p'_l(\eta)|\,t^{-2\mathrm{Re}\,(\eta)} + |p_l(\eta)|\,t^{-2\mathrm{Re}\,(\eta)}\,\log\left(\frac{1}{t^2}\right)\right), \end{aligned}$$

and, for $\lambda_0 \in \mathbf{C}$ and $0 < \varepsilon < \infty$:

$$\begin{aligned}
\sup_{\eta \in B(\lambda_0,\varepsilon)} \left| \frac{d}{d\lambda} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) \right|\Bigg|_{\lambda=\eta} &\leq \sum_{l=0}^{N'_n} |h_{n,l}(t)|\, C_{l,\lambda_0,\varepsilon} \cdot \\
&\quad \cdot \left(t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} + t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} \log\left(\frac{1}{t^2}\right)\right) \\
&= \sum_{l=0}^{N'_n} |h_{n,l}(t)|\, C_{l,\lambda_0,\varepsilon} \left(t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon}\right) \\
&\quad + \sum_{l=0}^{N'_n} |h_{n,l}(t)|\, C_{l,\lambda_0,\varepsilon} \cdot \\
&\quad \cdot \left(t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} \log\left(\frac{1}{t^2}\right)\right).
\end{aligned}$$

As before (cf. equation 3.85, page 98), we define:

$$\begin{aligned}
II^{2,1,2,*}_{j,1,N'_n,\lambda_0,\varepsilon} &= \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \int_{\frac{9}{10}}^{1} \frac{1}{|\xi|^{N'_n}} \cdot \\
&\quad \cdot \sup_{\eta \in B(\lambda_0,\varepsilon)} \left| \frac{d}{d\lambda} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda}(t) \right|\Bigg|_{\lambda=\eta} dt d\xi
\end{aligned}$$

which we can control by:

$$II^{2,1,2,*}_{j,1,N'_n,\lambda_0,\varepsilon} \leq \sum_{l=0}^{N'_n} \left(II^{2,1,2,*,l,1}_{j,1,N'_n,\lambda_0,\varepsilon} + II^{2,1,2,*,l,2}_{j,1,N'_n,\lambda_0,\varepsilon} \right)$$

where

$$II^{2,1,2,*,l,1}_{j,1,N'_n,\lambda_0,\varepsilon} = \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}+N'_n}} \int_{\frac{9}{10}}^{1} |h_{n,l}(t)| \cdot C_{l,\lambda_0,\varepsilon} \cdot t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} dt d\xi$$

and

$$
\begin{aligned}
II^{2,1,2,*,l,2}_{j,1,N'_n,\lambda_0,\varepsilon} &= \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}+N'_n}} \cdot \\
&\cdot \int_{\frac{9}{10}}^{1} |h_{n,l}(t)|\, C_{l,\lambda_0,\varepsilon}\, t^{-2\mathrm{Re}\,(\lambda_0)-2\varepsilon} \log\left(\frac{1}{t^2}\right) dtd\xi.
\end{aligned}
$$

As $|h_{n,l}|$ are continuous on $\mathbf{R}$ and therefore bounded on $\left[\frac{9}{10}, 1\right]$, and $\varphi \in \mathcal{S}(\mathbf{R}^n)$, we have that both $II^{2,1,2,*,l,1}_{j,1,N'_n,\lambda_0,\varepsilon}$ and $II^{2,1,2,*,l,2}_{j,1,N'_n,\lambda_0,\varepsilon}$ are finite for any l, λ_0 and ε. This shows that $II^{2,1,2,*}_{j,1,N'_n,\lambda_0,\varepsilon}$ is finite for every $\lambda_0 \in \mathbf{C}$ and every $0 < \varepsilon < \infty$, which implies that $II^{2,1,2}_{j,1,N'_n,\lambda,\mu}$ is entire with respect to λ.

To treat $II^{2,1,3}_{j,1,N'_n,\lambda,\mu}$ recall (cf. equation 3.68, page 87) that, for $1 \leq t \leq \frac{11}{10}$, we have:

$$
g_{\lambda\mu}(t) = t^{\frac{n}{2}-2j-\frac{1}{2}} \cdot (1-\phi(t)) \cdot \frac{1}{t^{2\lambda+1} \cdot (\log(et))^{\mu}} = \frac{t^{\frac{n}{2}-2j-\frac{3}{2}}}{(\log(et))^{\mu}} \cdot (1-\phi(t)) \cdot t^{-2\lambda}.
$$

It follows that this term can be treated exactly as the term $II^{2,1,2}_{j,1,N'_n,\lambda,\mu}$ and we can show that $II^{2,1,3}_{j,1,N'_n,\lambda,\mu}$ is entire with respect to λ.

To treat $II^{2,1,4}_{j,1,N'_n,\lambda,\mu}$ recall (cf. equation 3.68, page 87) that, for $\frac{11}{10} \leq t$, we have:

$$
g_{\lambda\mu}(t) = t^{\frac{n}{2}-2j-\frac{1}{2}} \cdot \frac{1}{t^{2\lambda+1} \cdot (\log(et))^{\mu}} = \frac{t^{\frac{n}{2}-2j-\frac{3}{2}}}{(\log(et))^{\mu}} \cdot t^{-2\lambda}.
$$

Then:

$$\begin{aligned}
\frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) &= \sum_{l=0}^{N'_n} b_{N'_n,l} \cdot \frac{d^{N'_n-l}}{dt^{N'_n-l}} \left(\frac{t^{\frac{n}{2}-2j-\frac{3}{2}}}{(\log(et))^{\mu}} \right) \cdot \frac{d^l}{dt^l} \left(t^{-2\lambda} \right) \\
&= \sum_{l=0}^{N'_n} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n+l}}{(\log(et))^{\mu}} \cdot p_{n,j,l,\mu} \left(\frac{1}{\log(et)} \right) \cdot q_l(\lambda) \cdot t^{-2\lambda-l} \\
&= \sum_{l=0}^{N'_n} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}}{(\log(et))^{\mu}} \cdot p_{n,j,l,\mu} \left(\frac{1}{\log(et)} \right) \cdot q_l(\lambda) \cdot t^{-2\lambda},
\end{aligned}$$

for polynomials $p_{n,j,l,\mu}$ and q_l. Then:

$$\begin{aligned}
\frac{d}{d\lambda} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) &= \sum_{l=0}^{N'_n} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}}{(\log(et))^{\mu}} \, p_{n,j,l,\mu} \left(\frac{1}{\log(et)} \right) q'_l(\lambda) \, t^{-2\lambda} \\
&\quad + \sum_{l=0}^{N'_n} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}}{(\log(et))^{\mu}} \, p_{n,j,l,\mu} \left(\frac{1}{\log(et)} \right) \cdot \\
&\quad \cdot \, q_l(\lambda) \, t^{-2\lambda} \, \log(t^{-2}).
\end{aligned}$$

Then:

$$\begin{aligned}
\left| \frac{d}{d\lambda} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) \right| \Bigg|_{\lambda=\eta} &\leq \sum_{l=0}^{N'_n} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}}{(\log(et))^{\mu}} \cdot \\
&\quad \cdot \left| p_{n,j,l,\mu} \left(\frac{1}{\log(et)} \right) \right| \, |q'_l(\eta)| \, t^{-2\mathrm{Re}\,(\eta)} \\
&\quad + \sum_{l=0}^{N'_n} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}}{(\log(et))^{\mu}} \, \left| p_{n,j,l,\mu} \left(\frac{1}{\log(et)} \right) \right| \cdot \\
&\quad \cdot |q_l(\eta)| \, t^{-2\mathrm{Re}\,(\eta)} \, \log(t^2).
\end{aligned}$$

So, for $\lambda_0 \in \mathbf{C}$ and $0 < \varepsilon < \infty$:

$$\sup_{\eta\in B(\lambda_0,\varepsilon)} \left| \frac{d}{d\lambda} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) \right| \Bigg|_{\lambda=\eta} \le \sum_{l=0}^{N'_n} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}}{(\log(et))^{\mu}} \left| p_{n,j,l,\mu}\left(\frac{1}{\log(et)}\right)\right| \cdot$$
$$\cdot C_{l,\lambda_0,\varepsilon}\, t^{-2\mathrm{Re}\,(\lambda_0)+2\varepsilon}$$
$$+\sum_{l=0}^{N'_n} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n}}{(\log(et))^{\mu}} \left| p_{n,j,l,\mu}\left(\frac{1}{\log(et)}\right)\right| \cdot$$
$$\cdot C_{l,\lambda_0,\varepsilon}\, t^{-2\mathrm{Re}\,(\lambda_0)+2\varepsilon}\, \log(t^2).$$

As usual we define:

$$II^{2,1,4,*}_{j,1,N'_n,\lambda_0,\varepsilon,\mu} = \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot$$
$$\cdot \int_{\frac{11}{10}}^{\infty} \frac{1}{|\xi|^{N'_n}} \sup_{\eta\in B(\lambda_0,\varepsilon)} \left| \frac{d}{d\lambda} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) \right| \Bigg|_{\lambda=\eta} dt d\xi$$

and we use the last estimate obtained to control:

$$II^{2,1,4,*}_{j,1,N'_n,\lambda_0,\varepsilon,\mu} \le \sum_{l=0}^{N'_n} \left(II^{2,1,4,*,l,1}_{j,1,N'_n,\lambda_0,\varepsilon,\mu} + II^{2,1,4,*,l,2}_{j,1,N'_n,\lambda_0,\varepsilon,\mu} \right)$$

where

$$II^{2,1,4,*,l,1}_{j,1,N'_n,\lambda_0,\varepsilon,\mu} = \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j-\frac{1}{2}+N'_n}} \int_{\frac{11}{10}}^{\infty} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n-2\mathrm{Re}\,(\lambda_0)+2\varepsilon}}{(\log(et))^{\mu}} \cdot$$
$$\cdot \left| p_{n,j,l,\mu}\left(\frac{1}{\log(et)}\right)\right| C_{l,\lambda_0,\varepsilon} dt d\xi$$

and

$$II^{2,1,4,*,l,2}_{j,1,N'_n,\lambda_0,\varepsilon,\mu} = \int_{\frac{1}{|\xi|}<\frac{9}{10}} \frac{|\varphi(\xi)|}{|\xi|^{\frac{n}{2}+2j-\frac{1}{2}+N'_n}} \int_{\frac{11}{10}}^{\infty} \frac{t^{\frac{n}{2}-2j-\frac{3}{2}-N'_n-2\mathrm{Re}\,(\lambda_0)+2\varepsilon}}{(\log(et))^{\mu}} \cdot$$
$$\cdot \left| p_{n,j,l,\mu}\left(\frac{1}{\log(et)}\right)\right| C_{l,\lambda_0,\varepsilon}\, \log(t^2) dt d\xi$$

for all l. All we need for theese integrals to converge is that:

$$\frac{n}{2} - 2j - \frac{3}{2} - N'_n - 2\mathrm{Re}\,(\lambda_0) + 2\varepsilon < -1$$

that is

$$0 < 2\varepsilon < -\frac{n}{2} + 2j + \frac{1}{2} + N'_n + 2\mathrm{Re}\,(\lambda_0).$$

Clearly, such an ε exists if and only if:

$$0 < -\frac{n}{2} + 2j + \frac{1}{2} + N'_n + 2\mathrm{Re}\,(\lambda_0)$$

that is, if and only if:

$$\frac{n}{2} - 2j - \frac{1}{2} - N'_n < 2\mathrm{Re}\,(\lambda_0).$$

As we need this condition to be satisfied for all $j = 0, ..., N_n = \lceil \frac{n-5}{4} \rceil$ (cf. equation 3.55, page 77), we need:

$$\frac{n}{2} - \frac{1}{2} - N'_n < 2\mathrm{Re}\,(\lambda_0).$$

Since $N'_n = \lceil \frac{n-1}{2} \rceil + 1$, we have that $\frac{n}{2} - \frac{1}{2} - N'_n < 0$. Therefore, the set:

$$\left\{ \lambda \in \mathbf{C} : \mathrm{Re}\,(\lambda) > \frac{n}{4} - \frac{1}{4} - \frac{N'_n}{2} \right\},$$

where all the functions $II^{2,1,4}_{j,1,N'_n,\lambda,\mu}$ are analytic with respect to λ, contains the real interval $\left(0, \frac{n-1}{2}\right)$. This concludes the proof of the fact that $II^{2,1}_{j,1,N'_n,\lambda,\mu}$ ((cf. equation 3.82, page 96)) is analytic with respect to λ on the set:

$$\left\{ \lambda \in \mathbf{C} : \mathrm{Re}\,(\lambda) > \frac{n}{4} - \frac{1}{4} - \frac{N'_n}{2} \right\}.$$

In order to prove that $II^{2,2}_{j,1,N'_n,\lambda,\mu}$ is analytic, we split it as follows:

$$II^{2,2}_{j,1,N'_n,\lambda,\mu} = II^{2,2,1}_{j,1,N'_n,\lambda} + II^{2,2,2}_{j,1,N'_n,\lambda,\mu} + II^{2,2,3}_{j,1,N'_n,\lambda,\mu}$$

where

$$\begin{aligned} II^{2,2,1}_{j,1,N'_n,\lambda} &= c'_{n,j} \int_{\frac{9}{10} \le \frac{1}{|\xi|} < 1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot \\ &\quad \cdot \left((-1)^{N'_n} \int_{\frac{1}{|\xi|}}^{1} \frac{\frac{d^{-N'_n}}{dt^{-N'_n}} (\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N'_n}} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) dt \right) d\xi \end{aligned}$$

$$\begin{aligned} II^{2,2,2}_{j,1,N'_n,\lambda,\mu} &= c'_{n,j} \int_{\frac{9}{10} \le \frac{1}{|\xi|} < 1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot \\ &\quad \cdot \left((-1)^{N'_n} \int_{1}^{\frac{11}{10}} \frac{\frac{d^{-N'_n}}{dt^{-N'_n}} (\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N'_n}} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) dt \right) d\xi \end{aligned}$$

and

$$\begin{aligned} II^{2,2,3}_{j,1,N'_n,\lambda,\mu} &= c'_{n,j} \int_{\frac{9}{10} \le \frac{1}{|\xi|} < 1} \frac{\varphi(\xi)}{|\xi|^{\frac{n-2}{2}+2j+\frac{1}{2}}} \cdot \\ &\quad \cdot \left((-1)^{N'_n} \int_{\frac{11}{10}}^{\infty} \frac{\frac{d^{-N'_n}}{dt^{-N'_n}} (\cos)((2\pi t|\xi|) - c_n)}{(2\pi|\xi|)^{N'_n}} \frac{d^{N'_n}}{dt^{N'_n}} g_{\lambda\mu}(t) dt \right) d\xi. \end{aligned}$$

It follows that $II^{2,2,1}_{j,1,N'_n,\lambda}$ and $II^{2,2,2}_{j,1,N'_n,\lambda,\mu}$ are entire with respect to λ (just as $II^{2,1,2}_{j,1,N'_n,\lambda,\mu}$ and $II^{2,1,3}_{j,1,N'_n,\lambda,\mu}$) and that $II^{2,2,3}_{j,1,N'_n,\lambda,\mu}$ is analytic with respect to λ on the set:

$$\left\{ \lambda \in \mathbf{C} : \mathrm{Re}\,(\lambda) > \frac{n}{4} - \frac{1}{4} - \frac{N'_n}{2} \right\},$$

just as $II^{2,1,4}_{j,1,N'_n,\lambda,\mu}$. The proofs are very similar to those showed for the terms $II^{2,1,2}_{j,1,N'_n,\lambda,\mu}$, $II^{2,1,3}_{j,1,N'_n,\lambda,\mu}$ and $II^{2,1,4}_{j,1,N'_n,\lambda,\mu}$ respectively, and will be therefore omitted. This implies that $II^{2,2}_{j,1,N'_n,\lambda,\mu}$ is analytic on such set. We just proved (cf. equation 3.78, page 95) that $II^{2}_{j,1,N'_n}$ is analytic with respect to λ on the set described above. In turn (cf. equation 3.65, page 86), this was all we still needed to prove in order to show that $II^{2}_{j,1}$ is analytic. In turn again, this was all we still needed to prove in order to show that II^2 is analytic (cf. equation 3.55, page 77). In turn again, this was all we still needed to prove in order to show that $I^1 + I^2 + I^3 + II^1 + II^2$ is analytic (cf. equation 3.43, page 64). At this point, we proved that the right hand side in equation (3.39, page 62) admits an analytic extension with respect to λ on an open subset of $\mathbf{C}$ that contains the real interval $\left(0, \frac{n-1}{2}\right)$. Of course, the left hand side of the same equation is naturally defined and analytic on a complex neighborhood of the same interval as well. As both sides of (3.39) are defined for all $\lambda \in \left(\frac{n-1}{4}, \frac{n-1}{2}\right)$ and (3.39) holds for each of such λ's, by analyticity (3.39, page 62) holds for all $\lambda \in \left(0, \frac{n-1}{2}\right)$, when we replace the right hand side of it with its analytic extension. Through the proof we also took care to show that (cf. (3.46, page 70), (3.49, page 74), (3.50, page 75), (3.57, page 82) and (3.64, page 86)) such analytic extension can still be written as in the right hand side of (3.39, page 62) for a function $u^{(1)}_{\lambda,\mu}$ satisfying:

$$|u^{(1)}_{\lambda,\mu}(\xi)| \leq \begin{cases} C_{n,\lambda,\mu} \frac{1}{|\xi|^{n-2\lambda-1}\left(\log\left(\frac{e}{|\xi|}\right)\right)^{\mu}} & \text{if} \quad |\xi| \leq 1, \\ C_{n,\lambda,\mu} \frac{1}{|\xi|^{n-2\lambda-1}} & \text{if} \quad |\xi| \geq 1, \end{cases} \tag{3.86}$$

Therefore we have $\widehat{w^{(1)}_{\lambda,\mu}} = u^{(1)}_{\lambda,\mu}$ and $\widehat{w^{(1)}_{\lambda,\mu}}$ satisfies the inequality claimed in (3.37, page 57).

3.3 A useful weight comparable to $\frac{1}{w_{\lambda\mu}}$

In this section we consider the weight $w_{\lambda\mu}(x) = \omega_{\lambda\mu}(|x|)$, where

$$\omega_{\lambda\mu}(t) = \begin{cases} \frac{1}{t^{2\lambda+1}} & \text{if} \quad 0 < t \leq 1, \\ \frac{1}{t^{2\lambda+1}(\log(et))^{\mu}} & \text{if} \quad t > 1, \end{cases} \tag{3.87}$$

and we show that $1/w$ is comparable to another weight which can be written in a more useful way for our purposes. More precisely, let's define a function $u_{\lambda,\mu}$ by:

$$u_{\lambda,\mu}(y) = \begin{cases} |y|^{-n-2\lambda-1}\left(\log\left(\frac{e}{|y|}\right)\right)^{\mu} & \text{if} \quad |y| < 1, \\ |y|^{-n-2\lambda-1} & \text{if} \quad |y| \geq 1. \end{cases} \tag{3.88}$$

Let's define a new weight $\widetilde{w}_{N\lambda,\mu}$ by:

$$\widetilde{w}_{N\lambda,\mu}(x) = \int_{\mathbf{R}^n} |e^{i\langle x,y\rangle} - 1|^N \, u_{\lambda,\mu}(y)dy, \tag{3.89}$$

where $\langle x, y\rangle$ denotes the inner product between x and y and N is a large enough integer.

The goal of this section is to prove that then there exist constants $C_{1,n,\lambda,\mu,N}$ and $C_{2,n,\lambda,\mu,N}$ such that:

$$\frac{C_{1,n,\lambda,\mu,N}}{w_{\lambda,\mu}(x)} \leq \widetilde{w}_{N,\lambda,\mu}(x) \leq \frac{C_{2,n,\lambda,\mu,N}}{w_{\lambda,\mu}(x)} \tag{3.90}$$

for all $x \in \mathbf{R}^n \setminus \{0\}$. Let's also define:

$$\widetilde{w}_{N,\lambda,\mu,1}(x) = \int_{|y|\leq\frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N \, u_{\lambda,\mu}(y)dy \tag{3.91}$$

and

$$\widetilde{w}_{N,\lambda,\mu,2}(x) = \int_{|y|>\frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N \, u_{\lambda,\mu}(y)dy \tag{3.92}$$

so that:

$$\widetilde{w}_{N,\lambda,\mu} = \widetilde{w}_{N,\lambda,\mu,1} + \widetilde{w}_{N,\lambda,\mu,2}. \tag{3.93}$$

Observe that $|y| \leq \frac{1}{|x|}$ implies that $|\langle x,y\rangle| \leq |x|\;|y| \leq 1$ which implies

$$C_1\;|x|\;|y| \leq |e^{i\langle x,y\rangle} - 1| \leq |x|\;|y| \tag{3.94}$$

for a constant $0 < C_1 < 1$ that we don't compute for simplicity.

Now, we will prove that $\widetilde{w}_{N,\lambda,\mu} \approx 1/w_{\lambda,\mu}$. Let us estimate $\widetilde{w}_{N,\lambda,\mu,1}$.

Case 1: $\frac{1}{|x|} \leq 1$ (i.e., $|x| \geq 1$).

Then, since $|e^{ia}-1| \leq |a|$ for all $a \in \mathbf{R}$, we have:

$$\begin{aligned}
\widetilde{w}_{N,\lambda,\mu,1}(x) &= \int_{|y|\leq\frac{1}{|x|}} |e^{i\langle x,y\rangle}-1|^N \ |y|^{-n-2\lambda-1} \ \left(\log\left(\frac{e}{|y|}\right)\right)^\mu dy \\
&\leq \int_{|y|\leq\frac{1}{|x|}} |x|^N \ |y|^N \ |y|^{-n-2\lambda-1} \ \left(\log\left(\frac{e}{|y|}\right)\right)^\mu dy \\
&= |x|^N \ |S^{n-1}| \ \int_0^{\frac{1}{|x|}} r^N \ r^{-n-2\lambda-1} \ \left(\log\left(\frac{e}{r}\right)\right)^\mu \ r^{n-1} dr \\
&= |x|^N \ |S^{n-1}| \ \int_0^{\frac{1}{|x|}} r^{N-2\lambda-2} \ \left(\log\left(\frac{e}{r}\right)\right)^\mu dr \\
&= |x|^N \ |S^{n-1}| \ \int_\infty^{|x|} \left(\frac{1}{t}\right)^{N-2\lambda-2} \ (\log(et))^\mu \ \left(-\frac{1}{t^2}\right) dt \\
&= |x|^N \ |S^{n-1}| \ \int_{|x|}^\infty t^{2\lambda-N} \ (\log(et))^\mu \, dt \\
&= |x|^N \ |S^{n-1}| \ \int_{e|x|}^\infty \left(\frac{s}{e}\right)^{2\lambda-N} \ (\log(s))^\mu \ \frac{ds}{e} \\
&= |x|^N \ \frac{|S^{n-1}|}{e^{2\lambda+1-N}} \ \int_{e|x|}^\infty s^{2\lambda-N} \ (\log(s))^\mu \, ds. \qquad (3.95)
\end{aligned}$$

In order for this integral to converge, we need $2\lambda - N < -1$. As we're assuming $\lambda \in \left(0, \frac{n-1}{2}\right)$ to be given and $N \in \mathbf{Z}^+$ still to choose, we solve this inequality for N to get $N > 2\lambda + 1$. So far we may just set:

$$N = N_\lambda := \lceil 2\lambda + 1 \rceil + 1 \qquad (3.96)$$

In fact, later we will need N to be even. So, we rather set:

$$N = N_\lambda := 2\lceil 2\lambda + 1 \rceil. \qquad (3.97)$$

Observe that we may choose N independent of $\lambda \in \left(0, \frac{n-1}{2}\right)$ by setting $N := 2n$. We're working under the hypothesis of case 1, that is we're assuming

$|x| \geq 1$. Therefore, $e|x| \geq e$ and we can use the result in Lemma (3.1.9, page 47) with $t := s$, $\alpha := 2\lambda - N$, $\gamma := -\mu$ and $x := e|x|$ (the left hand sides are consistent with the notation in (3.1.9), while the right hand sides are consistent with the current notation). This, together with inequality (3.95, page 110), implies that there exist constants $C'_{\lambda,N}$ and $C_{\lambda,\mu,N}$ such that:

$$\begin{aligned}
\widetilde{w}_{N,\lambda,\mu,1}(x) &\leq |x|^N \frac{|S^{n-1}|}{e^{2\lambda+1-N}} \int_{e|x|}^{\infty} s^{2\lambda-N} \, (\log(s))^{\mu} \, ds \\
&\leq |x|^N \frac{|S^{n-1}|}{e^{2\lambda+1-N}} C'_{\lambda,N} \, |x|^{2\lambda-N+1} \, (\log(e|x|))^{\mu} \\
&= C''_{n,\lambda,N} \, |x|^{2\lambda+1} \, (\log(e|x|))^{\mu} \\
&= C''_{n,\lambda,N} \, 1/w_{\lambda\mu}(x) \qquad (3.98)
\end{aligned}$$

for all $x \in \mathbf{R}^n$ satisfying $|x| \geq C_{\lambda,\mu,N}$, where the last inequality follows from the assumption of this case. We proved that $\widetilde{w}_{N,\lambda,\mu,1}(x) \leq \frac{C''_{n,\lambda,N}}{w_{\lambda,\mu}(x)}$ if $|x| \geq \max\{1, C_{\lambda,\mu,N}\}$. Similarly:

$$\begin{aligned}
\widetilde{w}_{N,\lambda,\mu,1}(x) &= \int_{|y|\leq\frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N \, |y|^{-n-2\lambda-1} \left(\log\left(\frac{e}{|y|}\right)\right)^{\mu} dy \\
&\geq C_1^N \int_{|y|\leq\frac{1}{|x|}} |x|^N \, |y|^N \, |y|^{-n-2\lambda-1} \left(\log\left(\frac{e}{|y|}\right)\right)^{\mu} dy \\
&\ldots \\
&\ldots \\
&\ldots \\
&= C_1^N \, |x|^N \frac{|S^{n-1}|}{e^{2\lambda+1-N}} \int_{e|x|}^{\infty} s^{2\lambda-N} \, (\log(s))^{\mu} \, ds \qquad (3.99)
\end{aligned}$$

Now we apply another of the estimates in Lemma (3.1.9, page 47) with the same setting written after equation (3.97, page 110), this time to deduce:

$$\begin{aligned}
\widetilde{w}_{N,\lambda,\mu,1}(x) &\geq C_1^N\,|x|^N\,\frac{|S^{n-1}|}{e^{2\lambda+1-N}}\int_{e\,|x|}^{\infty} s^{2\lambda-N}\,(\log(s))^{\mu}ds \\
&\geq C_1^N\,|x|^N\,\frac{|S^{n-1}|}{e^{2\lambda+1-N}}\,C_{\lambda,N}\,(e|x|)^{2\lambda-N+1}\,(\log(e|x|))^{\mu} \\
&= C'_{n,\lambda,N}\,|x|^N\,|x|^{2\lambda-N+1}\,(\log(e|x|))^{\mu} \\
&= C'_{n,\lambda,N}\,|x|^{2\lambda+1}\,(\log(e|x|))^{\mu} \\
&= C'_{n,\lambda,N}\,1/w_{\lambda\mu}(x) \qquad (3.100)
\end{aligned}$$

for all $x \in \mathbf{R}^n$ satisfying $|x| \geq C_{\lambda,\mu,N}$. This estimate, together with inequality (3.98, page 111), shows that

$$\widetilde{w}_{N,\lambda,\mu,1} \approx_{n,\lambda,N} 1/w_{\lambda\mu} \qquad (3.101)$$

on the set:

$$\{x \in \mathbf{R}^n : |x| \geq \max\{1, C_{\lambda,\mu,N}\}\} \qquad (3.102)$$

Case 2: $\frac{1}{|x|} > 1$ (i.e., $|x| < 1$).

$$\widetilde{w}_{N,\lambda,\mu,1}(x) = I + II \qquad (3.103)$$

where

$$\begin{aligned}
I &= \int_{|y|\leq 1} |e^{i\langle x,y\rangle}-1|^N\, u_{\lambda,\mu}(y)dy \\
&= \int_{|y|\leq 1} |e^{i\langle x,y\rangle}-1|^N\, |y|^{-n-2\lambda-1}\, \left(\log\left(\frac{e}{|y|}\right)\right)^{\mu} dy \\
&\approx_{n,\lambda,\mu,N} \int_{|y|\leq 1} |x|^N\, |y|^N\, |y|^{-n-2\lambda-1}\, \left(\log\left(\frac{e}{|y|}\right)\right)^{\mu} dy \\
&\approx_{n,\lambda,\mu,N} |x|^N
\end{aligned}$$

and

$$\begin{aligned}
II &= \int_{1<|y|\leq \frac{1}{|x|}} |e^{i\langle x,y\rangle}-1|^N\, u_{\lambda,\mu}(y)dy \\
&= \int_{1<|y|\leq \frac{1}{|x|}} |e^{i\langle x,y\rangle}-1|^N\, |y|^{-n-2\lambda-1}dy \\
&\approx_{n,\lambda,N} \int_{1<|y|\leq \frac{1}{|x|}} |x|^N\, |y|^N\, |y|^{-n-2\lambda-1}dy \\
&= |x|^N\, |S^{n-1}| \int_1^{\frac{1}{|x|}} r^N\, r^{-n-2\lambda-1}\, r^{n-1}dr \\
&= |x|^N\, |S^{n-1}| \int_1^{\frac{1}{|x|}} r^{-2\lambda-2+N}dr \\
&= |x|^N\, \frac{|S^{n-1}|}{(-2\lambda-1+N)}\, \left(r^{-2\lambda-1+N}\right)\Big|_1^{\frac{1}{|x|}} \\
&= |x|^N\, \frac{|S^{n-1}|}{(-2\lambda-1+N)}\, \left(|x|^{2\lambda+1-N}-1\right) \\
&= \frac{|S^{n-1}|}{(-2\lambda-1+N)}\, \left(|x|^{2\lambda+1}-|x|^N\right) \\
&\approx_{n,\lambda,N} \frac{|S^{n-1}|}{(-2\lambda-1+N)}\, |x|^{2\lambda+1} \\
&\approx_{n,\lambda,N} |x|^{2\lambda+1},
\end{aligned}$$

where the notation $f(x) \approx_{n,\lambda,N} g(x)$ means that f is comparable to g and that the constants of comparability depend on n, λ and N at most. The 1st and 3rd "$\approx$" relations follow from the inequalities in (3.94, page 109). The 2nd one follows from the fact that the integral above it is finite, provided that we choose N_λ as in (3.97, page 110). The $4th$ one follows again from the choice of N_λ and from the hypothesis of case 2, that is $|x| < 1$.

Finally, equation (3.103, page 112) implies:

$$\widetilde{w}_{N,\lambda,\mu,1}(x) \approx_{n,\lambda,\mu,N} |x|^N + |x|^{2\lambda+1} \approx_{n,\lambda,N} |x|^{2\lambda+1} = \frac{1}{w_{\lambda\mu}(x)} \tag{3.104}$$

on $\{x \in \mathbf{R}^n : |x| \leq 1\}$. If $C_{\lambda,\mu,N} \leq 1$, then (cf. equation (3.102, page 112)) relations (3.101, page 112) and (3.104, page 114) imply that $\widetilde{w}_{N,\lambda,\mu,1} \approx_{n,\lambda,\mu,N} \frac{1}{w_{\lambda\mu}}$ on $\mathbf{R}^n$. Otherwise, in order to achieve the same conclusion, just observe that both functions $\widetilde{w}_{N,\lambda,\mu,1}$ and $\frac{1}{w_{\lambda\mu}}$ are positive and continuous on the compact annulus $1 \leq |x| \leq C_{\lambda,\mu,N}$. In view of (3.91, page 109), (3.92, page 109) and (3.93, page 109), it's enough to show that $\widetilde{w}_{N,\lambda,\mu,2} \approx_{n,\lambda,\mu,N} \frac{1}{w_{\lambda\mu}}$ in order to prove that $\widetilde{w}_{N,\lambda,\mu} \approx_{n,\lambda,\mu,N} \frac{1}{w_{\lambda\mu}}$. Let us define:

$$\widetilde{\widetilde{w}}_{\lambda\mu,2}(x) = \int_{|y|>\frac{1}{|x|}} u_{\lambda,\mu}(y)dy \tag{3.105}$$

Then:

$$\begin{aligned} \widetilde{w}_{N,\lambda,\mu,2}(x) &= \int_{|y|>\frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N u_{\lambda,\mu}(y)dy \leq 2^N \int_{|y|>\frac{1}{|x|}} u_{\lambda,\mu}(y)dy \\ &= 2^N \widetilde{\widetilde{w}}_{\lambda\mu,2}(x). \end{aligned} \tag{3.106}$$

We will prove that the inverse inequality also holds (with a constant different from 2^N), so that we have $\widetilde{\widetilde{w}}_{\lambda\mu,2} \approx_{N,\lambda,\mu,n} \widetilde{w}_{N,\lambda,\mu,2}$, and that $\widetilde{\widetilde{w}}_{\lambda\mu,2} \approx_{N,\lambda,\mu,n} 1/w_{\lambda\mu}$. This obviously means that $\widetilde{w}_{N,\lambda,\mu,2} \approx_{N,\lambda,\mu,n} 1/w_{\lambda\mu}$. Now, let's prove that $\widetilde{\widetilde{w}}_{\lambda\mu,2} \approx_{N,\lambda,\mu,n} 1/w_{\lambda\mu}$.

Case 1: $\frac{1}{|x|} > 1$. Then:

$$\begin{aligned}
\widetilde{\widetilde{w}}_{\lambda\mu,2}(x) &= \int_{|y|>\frac{1}{|x|}} u_{\lambda,\mu}(y)dy \\
&= \int_{|y|>\frac{1}{|x|}} |y|^{-n-2\lambda-1}dy \\
&= |S^{n-1}| \int_{\frac{1}{|x|}}^{\infty} r^{-n-2\lambda-1}\, r^{n-1}dr \\
&= |S^{n-1}| \int_{\frac{1}{|x|}}^{\infty} r^{-2\lambda-2}dr \\
&= \frac{|S^{n-1}|}{-2\lambda-1} \left(r^{-2\lambda-1}\right)\Big|_{\frac{1}{|x|}}^{\infty} \\
&= \frac{|S^{n-1}|}{-2\lambda-1} \left(0-\left(\frac{1}{|x|}\right)^{-2\lambda-1}\right) \\
&= \frac{|S^{n-1}|}{2\lambda+1}\, |x|^{2\lambda+1} \\
&\approx_{\lambda,n} |x|^{2\lambda+1} \\
&= \frac{1}{w_{\lambda\mu}(x)}
\end{aligned}$$

where the 2nd and the last equalities follow from the assumption of case 1 ($\frac{1}{|x|} > 1$).

Case 2: $\frac{1}{|x|} \leq 1$. Then:

$$\begin{aligned}
\widetilde{\widetilde{w}}_{\lambda\mu,2}(x) &= \int_{1\geq|y|>\frac{1}{|x|}} u_{\lambda,\mu}(y)dy + \int_{|y|>1} u_{\lambda,\mu}(y)dy \\
&= \int_{1\geq|y|>\frac{1}{|x|}} |y|^{-n-2\lambda-1} \left(\log\left(\frac{e}{|y|}\right)\right)^{\mu} dy \\
&\quad + \int_{|y|>1} |y|^{-n-2\lambda-1} dy \\
&= C_{n,\lambda} + \int_{1\geq|y|>\frac{1}{|x|}} |y|^{-n-2\lambda-1} \left(\log\left(\frac{e}{|y|}\right)\right)^{\mu} dy \\
&= C_{n,\lambda} + |S^{n-1}| \int_{\frac{1}{|x|}}^{1} r^{-n-2\lambda-1} \left(\log\left(\frac{e}{r}\right)\right)^{\mu} r^{n-1}dr \\
&= C_{n,\lambda} + |S^{n-1}| \int_{\frac{1}{|x|}}^{1} r^{-2\lambda-2} \left(\log\left(\frac{e}{r}\right)\right)^{\mu} dr \\
&= C_{n,\lambda} + |S^{n-1}| \int_{|x|}^{1} s^{2\lambda+2} \left(\log(es)\right)^{\mu} \left(-\frac{1}{s^2}\right) ds \\
&= C_{n,\lambda} + |S^{n-1}| \int_{1}^{|x|} s^{2\lambda+2} \left(\log(es)\right)^{\mu} \frac{1}{s^2} ds \\
&= C_{n,\lambda} + |S^{n-1}| \int_{1}^{|x|} s^{2\lambda} \left(\log(es)\right)^{\mu} ds \\
&= C_{n,\lambda} + |S^{n-1}| \int_{e}^{e|x|} \left(\frac{t}{e}\right)^{2\lambda} \left(\log(t)\right)^{\mu} \frac{dt}{e} \\
&= C_{n,\lambda} + \frac{|S^{n-1}|}{e^{2\lambda+1}} \int_{e}^{e|x|} t^{2\lambda} \left(\log(t)\right)^{\mu} dt\,.
\end{aligned}$$

Since the assumption of this case (i.e., $|x| \geq 1$) we have that $e|x| \geq e$. Also, we are assuming $\lambda > 0$. So, we can apply the Lemma (3.1.10, page 50) by setting:

$$\begin{aligned} e|x| &=: x \\ 2\lambda &=: \alpha \\ -1 &=: \gamma \end{aligned}$$

where the right hand sides are consistent with the notation in Lemma (3.1.10, page 50). This implies that there exists a constant $C_{\lambda,\mu}$ ($C^{(3)}_{\alpha,\gamma}$ in the section) such that:

$$\begin{aligned} \int_e^{e|x|} t^{2\lambda}\,(\log(t))^{\mu}\,dt &\approx_{\lambda,\mu} |x|^{2\lambda+1}\,(\log(e|x|))^{\mu}\,e^{2\lambda+1} \\ &\approx_{\lambda} |x|^{2\lambda+1}\,(\log(e|x|))^{\mu} \end{aligned}$$

on the set $\{x \in \mathbf{R}^n : |x| \geq C_{\lambda,\mu}\}$. So, for $|x| \geq \max\{1, C_\lambda\}$, we have:

$$\begin{aligned} \widetilde{\widetilde{w}}_{\lambda\mu,2}(x) &= C_{n,\lambda} + \frac{|S^{n-1}|}{e^{2\lambda+1}} \int_e^{e|x|} t^{2\lambda}\,\log(t)\,dt \\ &\approx_{\lambda,\mu} C_{n,\lambda} + |S^{n-1}|\,|x|^{2\lambda+1}\,\log(e|x|) \\ &\approx_{\lambda,\mu,n} 1 + |x|^{2\lambda+1} \cdot \log(e|x|) \\ &\approx_{\lambda,\mu} |x|^{2\lambda+1} \cdot \log(e|x|) \\ &= \frac{1}{w_{\lambda\mu}(x)}. \end{aligned}$$

If $C_{\lambda,\mu} > 1$, we still have to prove that $\widetilde{\widetilde{w}}_{\lambda\mu,2} \approx_{\lambda,n} \frac{1}{w_{\lambda\mu}}$ on $\{x \in \mathbf{R}^n : 1 \leq |x| \leq C_{\lambda,\mu}\}$. But this is obvious as both $\frac{1}{w_{\lambda\mu}}$ and $\widetilde{\widetilde{w}}_{\lambda\mu,2}$ are comparable to 1 on such annulus.

This concludes the proof that:

$$\widetilde{\widetilde{w}}_{\lambda\mu,2} \approx_{\lambda,\mu,n} \frac{1}{w_{\lambda\mu}} \tag{3.107}$$

on $\mathbf{R}^n \setminus \{0\}$. In view of (3.106, page 114), now we need to prove that there exists a constant $C_{N,\lambda,\mu,n}$ such that the inequality:

$$\widetilde{\widetilde{w}}_{\lambda\mu,2} \leq C_{N,\lambda,\mu,n} \cdot \widetilde{w}_{N,\lambda,\mu,2}$$

holds on $\mathbf{R}^n \setminus \{0\}$, in order to prove that $\widetilde{\widetilde{w}}_{\lambda\mu,2} \approx_{N,\lambda,\mu,n} \widetilde{w}_{N,\lambda,\mu,2}$ and therefore $\widetilde{w}_{N,\lambda,\mu,2} \approx_{N,\lambda,\mu,n} 1/w_{\lambda\mu}$. First, observe that $\widetilde{\widetilde{w}}_{\lambda\mu,2}$ is obviously radial and $\widetilde{w}_{N,\lambda,\mu,2}$ is radial as well. In fact, if $A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear orthonormal transformation, we have:

$$\begin{aligned}
\widetilde{w}_{N,\lambda,\mu,2}(Ax) &= \int_{|y|>\frac{1}{|Ax|}} |e^{i\langle Ax,y\rangle} - 1|^N \cdot u_{\lambda,\mu}(y)dy \\
&= \int_{|y|>\frac{1}{|x|}} |e^{i\left\langle x,A^T y\right\rangle} - 1|^N \cdot u_{\lambda,\mu}(y)dy \\
&= \int_{|y|>\frac{1}{|x|}} |e^{i\left\langle x,A^{-1} y\right\rangle} - 1|^N \cdot u_{\lambda,\mu}(y)dy \\
&= |\det(A)| \cdot \int_{|z|>\frac{1}{|x|}} |e^{i\langle x,z\rangle} - 1|^N \cdot u_{\lambda,\mu}(Az)dz
\end{aligned}$$

$$
\begin{aligned}
&= \int_{|y|>\frac{1}{|z|}} |e^{i\langle x,z\rangle} - 1|^N \cdot u_{\lambda,\mu}(Az)dz \\
&= \int_{|y|>\frac{1}{|z|}} |e^{i\langle x,z\rangle} - 1|^N \cdot u_{\lambda,\mu}(z)dz \\
&= \widetilde{w}_{N,\lambda,\mu,2}(x).
\end{aligned}
$$

The 2nd, 3rd and 5th equalities follow from the hypothesis on A and the 6th one follows from the fact that $u_{\lambda,\mu}$ is radial. Let us call $e_1, ..., e_n$ the vectors of the canonical basis of $\mathbf{R}^n$ (e.g., $e_1 = (1, 0, ..., 0)$). Since $\widetilde{w}_{N,\lambda,\mu,2}$ and $\widetilde{\widetilde{w}}_{\lambda\mu,2}$ are radial, it will be enough to prove that the functions $t \mapsto \widetilde{w}_{N,\lambda,\mu,2}(te_1)$ and $t \mapsto \widetilde{\widetilde{w}}_{\lambda\mu,2}(te_1)$ are comparable on $\mathbf{R}^+ := \{x \in \mathbf{R} : x > 0\}$. Recall the definition of $\widetilde{w}_{N,\lambda,\mu,2}$ and observe that:

$$
|e^{i\langle te_1,y\rangle} - 1| > \sqrt{2}
$$

if and only if

$$
t \cdot \langle e_1, y\rangle \in \left(\frac{(4k+1)\pi}{2}, \frac{(4k+3)\pi}{2}\right)
$$

for some $k \in \mathbf{Z}$. If $t > 0$ and $k \in \mathbf{Z}$, let's define:

$$
G_k^t := \left\{ y \in \mathbf{R}^n : \langle e_1, y\rangle \in \left[\frac{(4k+1)\pi}{2t}, \frac{(4k+3)\pi}{2t}\right) \right\}.
$$

If $t > 0$ and $k \in \mathbf{Z} \setminus \{0\}$ we define

$$
R_k^t := \left\{ y \in \mathbf{R}^n : \langle e_1, y\rangle \in \left[\frac{(4k-1)\pi}{2t}, \frac{(4k+1)\pi}{2t}\right) \right\}
$$

and

$$R_0^t := \left\{ y \in \mathbf{R}^n : \langle e_1, y \rangle \in \left[\frac{-\pi}{2t}, \frac{\pi}{2t} \right) \text{ and } |y| > \frac{1}{t} \right\}.$$

Clearly we have:

$$u_{\lambda,\mu}(y) \approx_N |e^{i\langle te_1, y\rangle} - 1|^N u_{\lambda,\mu}(y)$$

on

$$G^t := \bigcup_{k \in \mathbf{Z}} G_k^t.$$

In particular, there exists a constant C_N such that:

$$\int_{G^t} u_{\lambda,\mu}(y)dy \leq C_N \cdot \int_{G^t} |e^{i\langle te_1, y\rangle} - 1|^N u_{\lambda,\mu}(y)dy.$$

As $\int_{R_k^t} u_{\lambda,\mu}(y)dy \leq \int_{G_{k-1}^t} u_{\lambda,\mu}(y)dy$ for all $k \in \mathbf{Z}^+ := \{k \in \mathbf{Z} : k \geq 1\}$, and $\int_{R_k^t} u_{\lambda,\mu}(y)dy \leq \int_{G_k^t} u_{\lambda,\mu}(y)dy$ for all $k \in \mathbf{Z}^- := \{k \in \mathbf{Z} : k \leq 1\}$, we also have:

$$\int_{\bigcup_{k \in \mathbf{Z} \setminus \{0\}} R_k^t} u_{\lambda,\mu}(y)dy \leq \int_{G^t} u_{\lambda,\mu}(y)dy \leq C_N \cdot \int_{G^t} |e^{i\langle te_1, y\rangle} - 1|^N u_{\lambda,\mu}(y)dy.$$

Therefore:

$$\begin{aligned}
\int_{|\langle e_1,y\rangle|>\frac{\pi}{2t}} u_{\lambda,\mu}(y)dy &= \int_{\bigcup_{k\in\mathbf{Z}\setminus\{0\}} R_k^t} u_{\lambda,\mu}(y)dy + \int_{G^t} u_{\lambda,\mu}(y)dy \\
&\le 2C_N \cdot \int_{G^t} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)dy \\
&\le 2C_N \cdot \int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)dy.
\end{aligned}$$

Since $u_{\lambda,\mu}$ is radial, it follows that:

$$\int_{|\langle e_j,y\rangle|>\frac{\pi}{2t}} u_{\lambda,\mu}(y)dy \le 2C_N \cdot \int_{|y|>\frac{1}{t}} |e^{i\langle te_j,y\rangle}-1|^N u_{\lambda,\mu}(y)dy$$

for all $j = 1, ..., n$. Let's denote the ∞-norm of a vector $y \in \mathbf{R}^n$ by:

$$|y|_\infty := \sup_{1\le j\le n} |\langle e_j, y\rangle|.$$

Then we have:

$$\left\{y \in \mathbf{R}^n : |y|_\infty > \frac{\pi}{2t}\right\} = \bigcup_{1\le j\le n} \left\{y \in \mathbf{R}^n : |\langle e_j, y\rangle| > \frac{\pi}{2t}\right\}$$

and therefore:

$$
\begin{aligned}
\int_{\{y\in\mathbf{R}^n:|y|_\infty>\frac{\pi}{2t}\}} u_{\lambda,\mu}(y)dy &\leq \sum_{1\leq j\leq n}\int_{\{y\in\mathbf{R}^n:|\langle e_j,y\rangle|>\frac{\pi}{2t}\}} u_{\lambda,\mu}(y)dy \qquad (3.108)\\
&\leq \sum_{1\leq j\leq n} 2C_N\cdot\int_{|y|>\frac{1}{t}} |e^{i\langle te_j,y\rangle}-1|^N u_{\lambda,\mu}(y)dy\\
&= \sum_{1\leq j\leq n} 2C_N\cdot\int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)dy\\
&= 2\cdot C_N\cdot n\cdot\int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)dy.
\end{aligned}
$$

As we will see later, if we can prove that there exists a constant $C_{N,\lambda,\mu,n}$ such that:

$$
\int_{\{y\in\mathbf{R}^n:|y|>\frac{1}{t},\ |y|_\infty\leq\frac{\pi}{2t}\}} u_{\lambda,\mu}(y)dy \leq C_{N,\lambda,\mu,n}\int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)dy
$$

then we are done. To see this, let's use the following lemma.

Lemma 3.3.1. *Let $u_{\lambda,\mu}$ be as in definition (3.88, page 108). Then, for all $n\in\mathbf{Z}^+,\lambda\in\mathbf{R}$ and $C>1$ there exists constant $D=D(n,\lambda,C)\in\mathbf{R}$ such that:*

$$
u_{\lambda,\mu}\left(\frac{y}{C}\right)\leq D\cdot u_{\lambda,\mu}(y)
$$

for all $y\in\mathbf{R}^n\setminus\{0\}$. For $n\geq 2$ and $\lambda>0$, the inequality:

$$
u_{\lambda,\mu}(Cy)\leq u_{\lambda,\mu}(y)
$$

is obvious since the definition of $u_{\lambda,\mu}$.

Proof. All we need to show is that:

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}} \frac{u_{\lambda,\mu}\left(\frac{y}{C}\right)}{u_{\lambda,\mu}(y)} = \sup_{y\in\mathbf{R}^n\setminus\{0\}} \frac{u_{\lambda,\mu}(y)}{u_{\lambda,\mu}(Cy)} < \infty.$$

There are three cases.

Case 1: $|y| \geq 1,\ |Cy| \geq 1$ (that is, $|y| \geq 1$). Then:

$$\begin{aligned}\frac{u_{\lambda,\mu}(y)}{u_{\lambda,\mu}(Cy)} &= \frac{|y|^{-n-2\lambda-1}}{|Cy|^{-n-2\lambda-1}} \\ &= \frac{|y|^{-n-2\lambda-1}}{|y|^{-n-2\lambda-1}\cdot C^{-n-2\lambda-1}} \\ &= C^{n+2\lambda+1},\end{aligned}$$

therefore:

$$\sup_{|y|\geq 1} \frac{u_{\lambda,\mu}(y)}{u_{\lambda,\mu}(Cy)} = C^{n+2\lambda+1}. \tag{3.109}$$

Case 2: $|y| < 1 \leq |Cy|$ (that is, $\frac{1}{C} \leq |y| < 1$). Then:

$$\begin{aligned}\frac{u_{\lambda,\mu}(y)}{u_{\lambda,\mu}(Cy)} &= \frac{|y|^{-n-2\lambda-1}\cdot\left(\log\left(\frac{e}{|y|}\right)\right)^{\mu}}{|Cy|^{-n-2\lambda-1}} \\ &= \left(\log\left(\frac{e}{|y|}\right)\right)^{\mu}\cdot C^{n+2\lambda+1},\end{aligned}$$

therefore:

$$\sup_{\frac{1}{C}\leq|y|<1} \frac{u_{\lambda,\mu}(y)}{u_{\lambda,\mu}(Cy)} = C^{n+2\lambda+1}\cdot(\log(eC))^{\mu}. \tag{3.110}$$

Case 3: $|y| < |Cy| \leq 1$ (that is, $|y| \leq \frac{1}{C}$). Then:

$$
\begin{aligned}
\frac{u_{\lambda,\mu}(y)}{u_{\lambda,\mu}(Cy)} &= \frac{|y|^{-n-2\lambda-1} \cdot \log\left(\frac{e}{|y|}\right)}{|Cy|^{-n-2\lambda-1} \cdot \log\left(\frac{e}{|Cy|}\right)} \\
&= C^{n+2\lambda+1} \cdot \left(\frac{\log\left(\frac{e}{|y|}\right)}{\log\left(\frac{e}{|Cy|}\right)}\right)^{\mu} \\
&= C^{n+2\lambda+1} \cdot \left(\frac{\log(e) - \log(|y|)}{\log(e) - \log(|y|) - \log(C)}\right)^{\mu} \\
&= C^{n+2\lambda+1} \cdot \left(\frac{\log(e) - \log(|y|) - \log(C) + \log(C)}{\log(e) - \log(|y|) - \log(C)}\right)^{\mu} \\
&= C^{n+2\lambda+1} \cdot \left(1 + \frac{\log(C)}{\log(e) - \log(|y|) - \log(C)}\right)^{\mu} \\
&= C^{n+2\lambda+1} \cdot \left(1 + \frac{\log(C)}{\log(\frac{e}{C|y|})}\right)^{\mu},
\end{aligned}
$$

therefore:

$$
\begin{aligned}
\sup_{0<|y|\leq\frac{1}{C}} \frac{u_{\lambda,\mu}(y)}{u_{\lambda,\mu}(Cy)} &= C^{n+2\lambda+1} \cdot \left(1 + \frac{\log(C)}{\log(e)}\right)^{\mu} \\
&= C^{n+2\lambda+1} \cdot (1 + \log(C))^{\mu} \\
&= C^{n+2\lambda+1} \cdot (\log(e) + \log(C)^{\mu}) \qquad (3.111) \\
&= C^{n+2\lambda+1} \cdot (\log(eC))^{\mu}.
\end{aligned}
$$

Equations (3.109, page 123), (3.110, page 123) and (3.111, page 124) finally imply:

$$
\begin{aligned}
\sup_{y\in\mathbf{R}^n\setminus\{0\}} \frac{u_{\lambda,\mu}(y)}{u_{\lambda,\mu}(Cy)} &= \max\{C^{n+2\lambda+1}, C^{n+2\lambda+1} \cdot (\log(eC))^{\mu}\} \qquad (3.112) \\
&= C^{n+2\lambda+1} \cdot (\log(eC))^{\mu} =: D < \infty
\end{aligned}
$$

as claimed, with $D = C^{n+2\lambda+1} \cdot (\log(eC))^{\mu}$. $\square$

Since

$$\left\{y \in \mathbf{R}^n : |y| > \frac{1}{t},\ |y|_\infty \leq \frac{\pi}{2t}\right\} \subseteq \left\{y \in \mathbf{R}^n : \frac{1}{t} < |y| \leq \frac{\pi\sqrt{n}}{2t}\right\}$$

we have

$$\begin{aligned}
\int_{\substack{\{y\,\in\,\mathbf{R}^n\,:\,|y|\,>\,\frac{1}{t},\\ |y|_\infty\,\leq\,\frac{\pi}{2t}\}}} u_{\lambda,\mu}(y)dy &\leq \int_{\left\{y\in\mathbf{R}^n:\frac{1}{t}<|y|\leq\frac{\pi\sqrt{n}}{2t}\right\}} u_{\lambda,\mu}(y)dy \\
&= \frac{2^n}{(\pi\sqrt{n})^n} \int_{\frac{1}{t}<\frac{2|x|}{\pi\sqrt{n}}\leq\frac{\pi\sqrt{n}}{2t}} u_{\lambda,\mu}\left(\frac{2x}{\pi\sqrt{n}}\right)dx \\
&= \frac{2^n}{(\pi\sqrt{n})^n} \int_{\frac{\pi\sqrt{n}}{2t}<|x|\leq\frac{(\pi\sqrt{n})^2}{4t}} u_{\lambda,\mu}\left(\frac{2x}{\pi\sqrt{n}}\right)dx \\
&\leq (\pi\sqrt{n}/2)^{2\lambda+1}\,(\log(e\pi\sqrt{n}/2))^{\mu} \cdot \\
&\quad \cdot \int_{\frac{\pi\sqrt{n}}{2t}<|x|\leq\frac{(\pi\sqrt{n})^2}{4t}} u_{\lambda,\mu}(x)dx,
\end{aligned}$$

where the 4th step follows from Lemma (3.3.1, page 122), with $C = \pi\sqrt{n}/2$ and therefore:

$$D = C^{n+2\lambda+1}\,(\log(eC))^{\mu} = (\pi\sqrt{n}/2)^{n+2\lambda+1}\,(\log(e\pi\sqrt{n}/2))^{\mu}.$$

So, after setting $C_{n,\lambda,\mu} := (\pi\sqrt{n}/2)^{2\lambda+1}\,(\log(e\pi\sqrt{n}/2))^{\mu}$, we have:

$$\begin{aligned}\int_{\{y\in\mathbf{R}^n:|y|>\frac{1}{t},\ |y|_\infty\leq\frac{\pi}{2t}\}} u_{\lambda,\mu}(y)dy &\leq C_{n,\lambda,\mu}\int_{\frac{\pi\sqrt{n}}{2t}<|x|\leq\frac{(\pi\sqrt{n})^2}{4t}} u_{\lambda,\mu}(x)dx \\ &\leq C_{n,\lambda,\mu}\int_{\frac{\pi\sqrt{n}}{2t}<|y|} u_{\lambda,\mu}(y)dy \\ &\leq C_{n,\lambda,\mu}\int_{|y|_\infty>\frac{\pi}{2t}} u_{\lambda,\mu}(y)dy \qquad (3.113) \\ &\leq C_{n,\lambda,\mu}\,2\,C_N\,n\,\cdot \\ &\quad\cdot\int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)dy.\end{aligned}$$

The 4th step follows from (3.108, page 122). Finally, (3.108, page 122) and (3.113, page 126) together imply:

$$\int_{\{y\in\mathbf{R}^n:|y|>\frac{1}{t}\}} u_{\lambda,\mu}(y)dy \leq C_{n,\lambda,\mu,N}\cdot\int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)dy \qquad (3.114)$$

that is (cf. (3.92, page 109) and (3.105, page 114)):

$$\widetilde{\widetilde{w}}_{\lambda\mu,2}(te_1) \leq C_{n,\lambda,\mu,N}\cdot\widetilde{w}_{N,\lambda,\mu,2}(te_1) \qquad (3.115)$$

Inequalities (3.106, page 114) and (3.115, page 126) say that:

$$\widetilde{w}_{N,\lambda,\mu,2} \approx_{n,\lambda,\mu,N} \widetilde{\widetilde{w}}_{\lambda\mu,2} \qquad (3.116)$$

on $\mathbf{R}^n\setminus\{0\}$. The relations (3.107, page 118) and (3.116, page 126) obviously imply that:

$$\widetilde{w}_{N,\lambda,\mu,2} \approx_{n,\lambda,\mu,N} \frac{1}{w_{\lambda\mu}} \qquad (3.117)$$

on $\mathbf{R}^n \setminus \{0\}$. The relations (3.93, page 109), (3.101, page 112) and (3.117, page 126) conclude the proof of (3.90, page 109), that is the claim of this section.

Chapter 4

Proofs of lemmas

4.1 Proof of Lemma 3.1.5, page 41

Let's recall equation (3.20, page 41):

$$\begin{aligned}\int_{\mathbf{R}^n}\int_1^2 |(S_{\lambda,\gamma,k})_{at}(f)(x)|^2 \frac{dt}{t}\,\omega_{\lambda\mu}(|x|)dx \\ \le C^2_{n,\lambda,\mu,\gamma,k}\int_{\mathbf{R}^n}|f(x)|^2\,\omega_{\lambda\mu}(|x|)dx\end{aligned} \tag{4.1}$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$. This is equivalent to saying that the operator $(S_{\lambda,\gamma,k})_{a(\cdot)}$: $L^2(\mathbf{R}^n, \omega_{\lambda\mu}(|\cdot|)) \to L^2(\mathbf{R}^n \times [1,2], \frac{dt}{t}\omega_{\lambda\mu}(|\cdot|))$ is bounded, that is:

$$\|(S_{\lambda,\gamma,k})_{a\,t}(f)((x))\|_{L^2(\frac{dt}{t}\omega_{\lambda\mu}(|x|)dx)} \le C_{n,\lambda,\mu,\gamma,k}\|f\|_{L^2(\omega_{\lambda\mu}(|x|)dx)}.$$

Since the operators $(S_{\lambda,\gamma,k})_{at}$ are defined via real and radial multipliers, they are self-adjoint.

In order to use this fact, we rewrite the condition (4.1, page 128) by duality:

$$\begin{aligned} &\langle\, g(t,x)\,,\, (S_{\lambda,\gamma,k})_{a\,t}(f)(x)\,\rangle_{L^2(\frac{dt}{t}\omega_{\lambda\mu}(|x|)dx)} \\ \le& C_{n,\lambda,\mu,\gamma,k}\|g(t,x)\|_{L^2(\frac{dt}{t}\omega_{\lambda\mu}(|x|)dx)}\|f\|_{L^2(\omega_{\lambda\mu}(|x|)dx)}. \end{aligned} \tag{4.2}$$

Now we use the self-adjointness of $(S_{\lambda,\gamma,k})_{at}$ to rewrite the first line of (4.2, page 129):

$$\begin{aligned} &\langle\, g(t,x)\,,\, (S_{\lambda,\gamma,k})_{a\,t}(f)(x)\,\rangle_{L^2(\frac{dt}{t}\omega_{\lambda\mu}(|x|)dx)} \\ =& \int_{\mathbf{R}^n}\int_1^2 (S_{\lambda,\gamma,k})_{a\,t}(f)(x)\; g(t,x)\,\frac{dt}{t}\,\omega_{\lambda\mu}(|x|)dx \\ =& \int_1^2\int_{\mathbf{R}^n} (S_{\lambda,\gamma,k})_{a\,t}(f)(x)\; g(t,x)\,\omega_{\lambda\mu}(|x|)dx\,\frac{dt}{t} \\ =& \int_1^2\int_{\mathbf{R}^n} f(x)\;(S_{\lambda,\gamma,k})_{a\,t}(g(t,\,\cdot\,)\omega_{\lambda\mu}(|\,\cdot\,|))(x)\,dx\,\frac{dt}{t} \\ =& \int_{\mathbf{R}^n}\int_1^2 f(x)\;(S_{\lambda,\gamma,k})_{a\,t}(g(t,\,\cdot\,)\omega_{\lambda\mu}(|\,\cdot\,|))(x)\,\frac{dt}{t}\,dx \\ =& \int_{\mathbf{R}^n} f(x)\left(\int_1^2 (S_{\lambda,\gamma,k})_{a\,t}(g(t,\,\cdot\,)\omega_{\lambda\mu}(|\,\cdot\,|))(x)\,\frac{dt}{t}\right)dx \\ =& \left\langle\, f(x)\,,\,\left(\int_1^2 (S_{\lambda,\gamma,k})_{at}(g(t,\,\cdot\,)\omega_{\lambda\mu}(|\,\cdot\,|))(x)\,\frac{dt}{t}\right)\right\rangle_{L^2(dx)}. \end{aligned} \tag{4.3}$$

Since, in the last line, the operator $(S_{\lambda,\gamma,k})_{a\,t}$ applies to the function $g(t,\,\cdot\,)\omega_{\lambda\mu}(|\,\cdot\,|)$, we want to rewrite the factor $\|g(t,x)\|_{L^2(\frac{dt}{t}\omega_{\lambda\mu}(|x|)dx)}$ in the second line of (4.2, page 129) as a norm of the function $g(t,\,\cdot\,)\omega_{\lambda\mu}(|\,\cdot\,|)$:

$$\begin{aligned} \|g(t,x)\|^2_{L^2(\frac{dt}{t}\omega_{\lambda\mu}(|x|)dx)} &= \int_{\mathbf{R}^n}\int_1^2 |g(t,x)|^2\frac{dt}{t}\,\omega_{\lambda\mu}(|x|)dx \\ &= \int_{\mathbf{R}^n}\int_1^2 |g(t,x)\omega_{\lambda\mu}(|x|)|^2\frac{dt}{t}\,\frac{dx}{\omega_{\lambda\mu}(|x|)} \end{aligned} \tag{4.4}$$

$$= \quad \|g(t,x)\omega_{\lambda\mu}(|x|)\|^2_{L^2(\frac{dt}{t}\frac{dx}{\omega_{\lambda\mu}(|x|)})}.$$

We can summarize the last steps by setting $h(t,x) := g(t,x)\omega_{\lambda\mu}(|x|)$ for all $t \in [1,2]$ and $x \in \mathbf{R}^n \setminus \{0\}$, and by using (4.3, page 129) and (4.4, page 129) in (4.2, page 129) to rewrite this last condition as follows:

$$\begin{aligned} &\left\langle f(x)\,,\,\left(\int_1^2 (S_{\lambda,\gamma,k})_{a\,t}(h(t,\,\cdot\,))(x)\,\frac{dt}{t}\right)\right\rangle_{L^2(dx)} \\ \leq & C_{n,\lambda,\mu,\gamma,k}\|h(t,x)\|_{L^2(\frac{dt}{t}\frac{dx}{\omega_{\lambda\mu}(|x|)})}\|f\|_{L^2(\omega_{\lambda\mu}(|x|)dx)} \end{aligned} \tag{4.5}$$

for all functions h, f such that the previous expressions make sense. As we're trying to use (4.5, page 130) to obtain an alternative formulation of the boundedness of the operators $(S_{\lambda,\gamma,k})_{at}$, we should view f as a test function and (4.5, page 130) as a version (by duality) of such boundedness. We can achieve this by rewriting two terms in the following way:

$$\begin{aligned} &\left\langle f(x)\,,\,\left(\int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\,\cdot\,))(x)\,\frac{dt}{t}\right)\right\rangle_{L^2(dx)} \\ = \quad & \int_{\mathbf{R}^n} f(x)\left(\int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\,\cdot\,))(x)\,\frac{dt}{t}\right)\,dx \\ = \quad & \int_{\mathbf{R}^n} \frac{f(x)}{\mu(x)}\left(\int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\,\cdot\,))(x)\,\frac{dt}{t}\right)\,\mu(x)dx \\ = \quad & \left\langle \frac{f(x)}{\mu(x)}\,,\,\left(\int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\,\cdot\,))(x)\,\frac{dt}{t}\right)\right\rangle_{L^2(\mu(x)dx)} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\|f\|^2_{L^2(\omega_{\lambda,\mu}(|x|)dx)} &= \int_{\mathbf{R}^n} |f(x)|^2 \omega_{\lambda,\mu}(|x|)dx \\ &= \int_{\mathbf{R}^n} \left|\frac{f(x)}{\mu(x)}\right|^2 \mu(x)^2 \omega_{\lambda,\mu}(|x|)dx \\ &= \left\|\frac{f}{\mu}\right\|^2_{L^2(\mu(x)^2\omega_{\lambda,\mu}(|x|)dx)},\end{aligned} \tag{4.7}$$

where μ is a function satisfying:

$$\mu(x) = \mu(x)^2 \omega_{\lambda,\mu}(|x|),$$

that is, $\mu(x) = \frac{1}{\omega_{\lambda,\mu}(|x|)}$ for all $x \in \mathbf{R}^n \setminus \{0\}$. So, we can rewrite (4.5, page 130) as:

$$\begin{aligned}&\left\langle \frac{f(x)}{\mu(x)}, \left(\int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\,\cdot\,))(x)\,\frac{dt}{t}\right) \right\rangle_{L^2\left(\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)} \\ &\leq C_{n,\lambda,\mu,\gamma,k}\|h(t,x)\|_{L^2(\frac{dt}{t}\frac{dx}{\omega_{\lambda,\mu}(|x|)})} \left\|\frac{f}{\mu}\right\|_{L^2\left(\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)}\end{aligned} \tag{4.8}$$

for all functions f, h, that is:

$$\begin{aligned}&\left\langle \varphi(x), \left(\int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\,\cdot\,))(x)\,\frac{dt}{t}\right) \right\rangle_{L^2\left(\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)} \\ &\leq C_{n,\lambda,\mu,\gamma,k}\|h(t,x)\|_{L^2\left(\frac{dt}{t}\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)} \|\varphi\|_{L^2\left(\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)}\end{aligned} \tag{4.9}$$

for all functions φ, h in the appropriate spaces. By duality, this means that:

$$\begin{aligned}
&\left\| \int_1^2 (S_{\lambda,\gamma,k})_{a\,t}(h(t,\,\cdot\,))(x)\, \frac{dt}{t} \right\|_{L^2\left(\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)} \\
&\leq C_{n,\lambda,\mu,\gamma,k} \|h(t,x)\|_{L^2\left(\frac{dt}{t}\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)} \\
&= \sqrt{C_{n,\lambda,\mu,\gamma} \frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}} \|h(t,x)\|_{L^2\left(\frac{dt}{t}\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)}
\end{aligned} \tag{4.10}$$

for all functions $h(t,x)$ in the appropriate spaces.

So far, we have reduced (3.20, page 41) to (4.10, page 132). (We haven't proved, yet, that $C_{n,\lambda,\mu,\gamma,k} = C_{n,\lambda,\mu,\gamma}\frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}$.) Now we will use (3.90, page 109), that is:

$$\frac{C_{1,n,\lambda,\mu,N}}{w_{\lambda,\mu}(x)} \leq \widetilde{w}_{N,\lambda,\mu}(x) \leq \frac{C_{2,n,\lambda,\mu,N}}{w_{\lambda,\mu}(x)}$$

where $u_{\lambda,\mu}, \widetilde{w}_{N,\lambda,\mu}$ and N were defined in (3.88, page 108), (3.89, page 108) and (3.97, page 110) respectively as follows:

$$u_{\lambda,\mu}(y) = \begin{cases} |y|^{-n-2\lambda-1}\left(\log\left(\frac{e}{|y|}\right)\right)^{\mu} & \text{if} \quad |y| < 1, \\ |y|^{-n-2\lambda-1} & \text{if} \quad |y| \geq 1, \end{cases}$$

$$\widetilde{w}_{N,\lambda,\mu}(x) = \int_{\mathbf{R}^n} |e^{i\langle x,y\rangle} - 1|^N\, u_{\lambda,\mu}(y) dy,$$

$$N = N_\lambda := 2\left\lceil 2\lambda + 1\right\rceil,$$

and $C_{1,n,\lambda,\mu,N_\lambda}$, $C_{1,n,\lambda,\mu,N_\lambda}$ are constants that depend only on n, λ and μ.

Therefore, for every $f \in L^2\left(\mathbf{R}^n, \frac{1}{w_{\lambda\mu}}\right)$, we have:

$$
\begin{aligned}
&\|f\|^2_{L^2\left(\frac{dx}{\omega_{\lambda\mu}(|x|)}\right)} && (4.11)\\
= & \int_{\mathbf{R}^n} |f(x)|^2 \frac{1}{w_{\lambda\mu}(x)} dx \\
\approx_{n,\lambda,\mu} & \int_{\mathbf{R}^n} |f(x)|^2 \left(\int_{\mathbf{R}^n} |e^{i\langle x,y\rangle} - 1|^{N_\lambda} u_{\lambda,\mu}(y) dy \right) dx \\
= & \int_{\mathbf{R}^n} u_{\lambda,\mu}(y) \int_{\mathbf{R}^n} |f(x)|^2 |e^{i\langle x,y\rangle} - 1|^{N_\lambda} dx dy \\
= & \int_{\mathbf{R}^n} u_{\lambda,\mu}(y) \int_{\mathbf{R}^n} \left| f(x) |e^{i\langle x,y\rangle} - 1|^{\frac{N_\lambda}{2}} \right|^2 dx dy \\
= & \int_{\mathbf{R}^n} u_{\lambda,\mu}(y) \int_{\mathbf{R}^n} \left| f(x) \left(\sum_{j=0}^{\frac{N_\lambda}{2}} (-1)^j e^{i\langle x,y\rangle\left(\frac{N_\lambda}{2} - j\right)} \; b_{\frac{N_\lambda}{2},j} \right) \right|^2 dx dy \\
= & \int_{\mathbf{R}^n} u_{\lambda,\mu}(y) \int_{\mathbf{R}^n} \left| f(x) \left(\sum_{j=0}^{\frac{N_\lambda}{2}} (-1)^j e^{2\pi i\langle x,y\rangle\left(\frac{N_\lambda/2 - j}{2\pi}\right)} \; b_{\frac{N_\lambda}{2},j} \right) \right|^2 dx dy \\
= & \int_{\mathbf{R}^n} u_{\lambda,\mu}(y) \int_{\mathbf{R}^n} \left| \left(\sum_{j=0}^{\frac{N_\lambda}{2}} f(x) e^{2\pi i\langle x,y\rangle\left(\frac{N_\lambda/2 - j}{2\pi}\right)} (-1)^j \; b_{\frac{N_\lambda}{2},j} \right) \right|^2 dx dy \\
= & \int_{\mathbf{R}^n} u_{\lambda,\mu}(y) \int_{\mathbf{R}^n} \left| \left(\sum_{j=0}^{\frac{N_\lambda}{2}} f(x) e^{2\pi i\langle (x),y\rangle\left(\frac{N_\lambda/2 - j}{2\pi}\right)} (-1)^j \; b_{\frac{N_\lambda}{2},j} \right)^{\wedge} (\xi) \right|^2 d\xi dy \\
= & \int_{\mathbf{R}^n} u_{\lambda,\mu}(y) \int_{\mathbf{R}^n} \left| \left(\sum_{j=0}^{\frac{N_\lambda}{2}} \widehat{f}\left(\xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right) (-1)^j \; b_{\frac{N_\lambda}{2},j} \right) \right|^2 d\xi dy
\end{aligned}
$$

$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} u_{\lambda,\mu}(y) \left| \left(\sum_{j=0}^{\frac{N_\lambda}{2}} \widehat{f}\left(\xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right) (-1)^j \; b_{\frac{N_\lambda}{2},j} \right) \right|^2 dy d\xi,$$

where $b_{\frac{N_\lambda}{2},j}$ denotes the binomial coefficients with indexes $\frac{N_\lambda}{2}$ and j. We want to apply equation (4.11, page 133) to the function f defined by:

$$f(x) = \int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\,\cdot\,))(x)\,\frac{dt}{t}. \tag{4.12}$$

Observe that, in view of the definition of the operators $(S_{\lambda,\gamma,k})_t$ (equation 3.10, page 31), we have:

$$\widehat{f}(\xi) = \int_1^2 \widehat{h}(t,\xi) m_{\lambda,\gamma,k}(a\,t|\xi|)\frac{dt}{t}, \tag{4.13}$$

where the Fourier transform $\widehat{h}$ of h is computed with respect to the second variable.

Therefore (after evaluating at $\left(\xi - \left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)$ instead of at ξ):

$$\begin{aligned} &\widehat{f}\left(\xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right) \\ &= \int_1^2 \widehat{h}\left(t, \xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right) m_{\lambda,\gamma,k}\left(a\,t \left| \xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right| \right) \frac{dt}{t} \end{aligned} \tag{4.14}$$

Then:

$$\begin{aligned} &\left| \left(\sum_{j=0}^{\frac{N_\lambda}{2}} \widehat{f}\left(\xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right) (-1)^j \; b_{\frac{N_\lambda}{2},j} \right) \right|^2 \\ &= \left| \left(\sum_{j=0}^{\frac{N_\lambda}{2}} \int_1^2 \widehat{h}\left(t, \xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right) \cdot \right. \right. \end{aligned} \tag{4.15}$$

$$\cdot\ m_{\lambda,\gamma,k}\!\left(a\,t\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|\right)\frac{dt}{t}(-1)^j\ b_{\frac{N_\lambda}{2},j}\Bigg)\Bigg|^2$$

$$= \left|\int_1^2\left(\sum_{j=0}^{\frac{N_\lambda}{2}}\widehat{h}\left(t,\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\cdot\right.\right.$$

$$\left.\left.\cdot\ m_{\lambda,\gamma,k}\!\left(a\,t\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|\right)(-1)^j\ b_{\frac{N_\lambda}{2},j}\right)\frac{dt}{t}\right|^2.$$

In view of definition in equation (2.5, page 11) and the fact that ψ is supported in $\left[\frac{1}{8},\frac{5}{8}\right]$, we have:

$$\mathrm{supp}(m_{\lambda,\gamma,k})\subset\left[1-\frac{5}{8\cdot 2^k},1-\frac{1}{8\cdot 2^k}\right]. \tag{4.16}$$

Therefore, the integrand in (4.15, page 134) is non-zero at a point $t\in[1,2]$ only if there exists $j\in\{0,1,...,N_\lambda/2\}$ such that:

$$1-\frac{5}{8\cdot 2^k}\le a\ t\ \left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|\le 1-\frac{1}{8\cdot 2^k} \tag{4.17}$$

that is, only if:

$$\begin{aligned} t\ &\in\ [1,2]\cap\bigcup_{j=0}^{N_\lambda/2}\left[\frac{1-\frac{5}{8\cdot 2^k}}{a\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|},\frac{1-\frac{1}{8\cdot 2^k}}{a\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|}\right]\\ &=\ \bigcup_{j=0}^{N_\lambda/2}\left(\left[\frac{1-\frac{5}{8\cdot 2^k}}{a\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|},\frac{1-\frac{1}{8\cdot 2^k}}{a\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|}\right]\cap[1,2]\right) \qquad (4.18)\\ &=\ \bigcup_{j=0}^{N_\lambda/2}A_j^{\xi,\lambda,y,k,a}\end{aligned}$$

where the sets $A_j^{\xi,\lambda,y,k,a}$ are defined by:

$$A_j^{\xi,\lambda,y,k,a}:=\left(\left[\frac{1-\frac{5}{8\cdot 2^k}}{a\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|},\frac{1-\frac{1}{8\cdot 2^k}}{a\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|}\right]\cap[1,2]\right). \tag{4.19}$$

Now, for every ξ, y, λ, k, a, let's define the set of indeces $J^{\xi,\lambda,y,k,a} = J^{\xi,\lambda,y,k,a,1} \cap J^{\xi,\lambda,y,k,a,2}$, where

$$J^{\xi,\lambda,y,k,a,1} = \left\{ j \in \mathbf{Z} : \frac{1 - \frac{1}{8 \cdot 2^k}}{a \left| \xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right|} \geq 1 \right\} \tag{4.20}$$

and

$$J^{\xi,\lambda,y,k,a,2} = \left\{ j \in \mathbf{Z} : \frac{1 - \frac{5}{8 \cdot 2^k}}{a \left| \xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right|} \leq 2 \right\}. \tag{4.21}$$

With this notation, we can prove that the dt-measure (hence, the $\frac{dt}{t}$-measure) of the support of the integrand in (4.15, page 134) is at most:

$$\begin{aligned}
& \left| \bigcup_{\substack{0 \leq j \leq \frac{N_\lambda}{2} \\ j \in J^{\xi,\lambda,y,k,a}}} A_j^{\xi,\lambda,y,k,a} \cup \bigcup_{\substack{0 \leq j \leq \frac{N_\lambda}{2} \\ j \notin J^{\xi,\lambda,y,k,a}}} A_j^{\xi,\lambda,y,k,a} \right| \qquad (4.22) \\
\leq & \sum_{\substack{0 \leq j \leq \frac{N_\lambda}{2} \\ j \in J^{\xi,\lambda,y,k,a}}} |A_j^{\xi,\lambda,y,k,a}| + \sum_{\substack{0 \leq j \leq \frac{N_\lambda}{2} \\ j \notin J^{\xi,\lambda,y,k,a}}} |A_j^{\xi,\lambda,y,k,a}| \\
= & \sum_{\substack{0 \leq j \leq \frac{N_\lambda}{2} \\ j \in J^{\xi,\lambda,y,k,a}}} |A_j^{\xi,\lambda,y,k,a}| \\
\leq & \sum_{\substack{0 \leq j \leq \frac{N_\lambda}{2} \\ j \in J^{\xi,\lambda,y,k,a}}} \left| \left[\frac{1 - \frac{5}{8 \cdot 2^k}}{a \left| \xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right|}, \frac{1 - \frac{1}{8 \cdot 2^k}}{a \left| \xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right|} \right] \right| \\
= & \sum_{\substack{0 \leq j \leq \frac{N_\lambda}{2} \\ j \in J^{\xi,\lambda,y,k,a}}} \frac{\frac{1}{2^{k+1}}}{a \left| \xi - \left(\frac{N_\lambda/2 - j}{2\pi} \right) y \right|}
\end{aligned}$$

$$\leq \sum_{\substack{0\leq j\leq \frac{N_\lambda}{2} \\ j\in J^{\xi,\lambda,y,k,a}}} \frac{\frac{1}{2^{k+1}}}{\frac{2}{5}}$$

$$= \sum_{\substack{0\leq j\leq \frac{N_\lambda}{2} \\ j\in J^{\xi,\lambda,y,k,a}}} \frac{5}{2^{k+2}}$$

$$\leq \frac{5}{2^{k+2}}\left(\frac{N_\lambda}{2}+1\right) = \frac{5}{2^{k+2}}\lceil 2\lambda+2\rceil \leq \frac{5(n+1)}{2^{k+2}}.$$

This observation is useful when we apply the Cauchy-Schwarz inequality to the integral in (4.15, page 134):

$$\begin{aligned} &\left|\left(\sum_{j=0}^{N_\lambda/2} \widehat{f}\left(\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)(-1)^j\, b_{\frac{N_\lambda}{2},j}\right)\right|^2 \\ &\leq \frac{5}{2^{k+2}}\lceil 2\lambda+2\rceil \int_1^2\left|\sum_{j=0}^{N_\lambda/2}\widehat{h}\left(t,\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\cdot\right. \\ &\left.\cdot m_{\lambda,\gamma,k}\left(a\,t\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|\right)(-1)^j\, b_{\frac{N_\lambda}{2},j}\right|^2\frac{dt}{t}. \end{aligned} \tag{4.23}$$

Before using (4.23), let's observe that, by (4.11, page 133) with f as in (4.12, page 134), we get the following comparability estimate for the left hand side of (4.10, page 132):

$$\begin{aligned} &\left\|\int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\,\cdot\,))(x)\,\frac{dt}{t}\right\|^2_{L^2\left(\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)} \\ \approx_{n,\lambda,\mu}&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} u_{\lambda,\mu}(y)\left|\left(\sum_{j=0}^{\frac{N_\lambda}{2}}\int_1^2 \widehat{h}\left(t,\left(\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\right)\cdot\right.\right. \\ &\left.\left.\cdot m_{\lambda,\gamma,k}\left(a\,t\left|\left(\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\right|\right)\frac{dt}{t}(-1)^j\, b_{\frac{N_\lambda}{2},j}\right)\right|^2 dyd\xi. \end{aligned} \tag{4.24}$$

Therefore not only (3.20, page 41) is equivalent to (4.10, page 132), but it's also equivalent to:

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} u_{\lambda,\mu}(y)\Big|\Big(\sum_{j=0}^{\frac{N_\lambda}{2}}\int_1^2 \widehat{h}\left(t,\left(\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\right)\cdot$$
$$\cdot m_{\lambda,\gamma,k}\left(a\,t\left|\left(\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\right|\right)\frac{dt}{t}(-1)^j\ b_{\frac{N_\lambda}{2},j}\Big)\Big|^2 dyd\xi \tag{4.25}$$
$$\leq C_{n,\lambda,\mu,\gamma}\frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}\|h(t,x)\|^2_{L^2\left(\frac{dt}{t}\frac{dx}{\omega_{\lambda\mu}(|x|)}\right)},$$

for all functions h in the appropriate spaces (the constant $C_{n,\lambda,\mu,\gamma}$ may not be the same as in (4.10, page 132), due to the fact that (4.24, page 137) is not an equation).

After recalling that f was set in (4.12, page 134), we can use (4.23, page 137) to see that (4.25, page 138) is a consequence of:

$$\frac{5}{2^{k+2}}\lceil 2\lambda+2\rceil\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} u_{\lambda,\mu}(y)\int_1^2\left|\sum_{j=0}^{N_\lambda/2}\widehat{h}\left(t,\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\cdot\right.$$
$$\left.\cdot m_{\lambda,\gamma,k}\left(a\,t\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|\right)(-1)^j\ b_{\frac{N_\lambda}{2},j}\right|^2\frac{dt}{t}dyd\xi \tag{4.26}$$
$$\leq C_{n,\lambda,\mu,\gamma}\frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}\|h(t,x)\|^2_{L^2\left(\frac{dt}{t}\frac{dx}{\omega_{\lambda\mu}(|x|)}\right)}$$

which we can rewrite (with a different constant $C_{n,\lambda,\mu,\gamma}$):

$$
\begin{aligned}
&\int_{\mathbf{R}^n}\!\!\int_{\mathbf{R}^n} u_{\lambda,\mu}(y)\int_1^2\left|\sum_{j=0}^{N_\lambda/2}\widehat{h}\left(t,\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\cdot\right.\\
&\left.\cdot m_{\lambda,\gamma,k}\left(a\,t\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|\right)(-1)^j\ b_{\frac{N_\lambda}{2},j}\right|^2\frac{dt}{t}dyd\xi\\
&\le C_{n,\lambda,\mu,\gamma}\frac{2^{k(2\lambda-1)}2^{k+1}}{k^{2\gamma-\mu}}\|h(t,x)\|^2_{L^2\left(\frac{dt}{t}\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)}\\
&=C'_{n,\lambda,\mu,\gamma}\frac{2^{2\,k\,\lambda}}{k^{2\gamma-\mu}}\|h(t,x)\|^2_{L^2\left(\frac{dt}{t}\frac{dx}{\omega_{\lambda,\mu}(|x|)}\right)}\\
&=C'_{n,\lambda,\mu,\gamma}\frac{2^{2\,k\,\lambda}}{k^{2\gamma-\mu}}\int_1^2\int_{\mathbf{R}^n}|h(t,x)|^2\frac{dx}{w_{\lambda\mu}(x)}\frac{dt}{t},
\end{aligned}
\tag{4.27}
$$

which in turn follows from the fact (which we still have to prove) that the following inequality holds for every $t\in[1,2]$ and for every $a>0$:

$$
\begin{aligned}
&\int_{\mathbf{R}^n}\!\!\int_{\mathbf{R}^n} u_{\lambda,\mu}(y)\left|\sum_{j=0}^{N_\lambda/2}\widehat{h}\left(t,\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\cdot\right.\\
&\left.\cdot m_{\lambda,\gamma,k}\left(a\,t\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|\right)(-1)^j\ b_{\frac{N_\lambda}{2},j}\right|^2 dyd\xi\\
&\le C'_{n,\lambda,\mu,\gamma}\frac{2^{2\,k\,\lambda}}{k^{2\gamma-\mu}}\int_{\mathbf{R}^n}|h(t,x)|^2\frac{dx}{w_{\lambda\mu}(x)}.
\end{aligned}
\tag{4.28}
$$

for every h in the appropriate space. At this point we can simplify the notation, as (4.28, page 139) is in fact equivalent to:

$$
\begin{aligned}
&\int_{\mathbf{R}^n}\!\!\int_{\mathbf{R}^n} u_{\lambda,\mu}(y)\left|\sum_{j=0}^{N_\lambda/2}\widehat{h}\left(\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right)\cdot\right.\\
&\left.\cdot m_{\lambda,\gamma,k}\left(t\left|\xi-\left(\frac{N_\lambda/2-j}{2\pi}\right)y\right|\right)(-1)^j\ b_{\frac{N_\lambda}{2},j}\right|^2 dyd\xi\\
&\le C'_{n,\lambda,\mu,\gamma}\frac{2^{2\,k\,\lambda}}{k^{2\gamma-\mu}}\int_{\mathbf{R}^n}|h(x)|^2\frac{dx}{w_{\lambda\mu}(x)}.
\end{aligned}
\tag{4.29}
$$

if it holds for every $h \in L^2\left(\mathbf{R}^n, \frac{dx}{w_{\lambda\mu}(x)}\right)$ and every $t > 0$ (observe in fact that the t variable of h no longer played a role in (4.28, page 139), therefore we replaced $h\,(t,\,\cdot\,)$ with h and the product "$a\,t$" with just t).

Because of (4.11, page 133) applied to the function f defined by:

$$\widehat{f}(\xi) = \widehat{h}(\xi) m_{\lambda,\gamma,k}(t|\xi|) \tag{4.30}$$

that is, by:

$$f(x) = \left(\widehat{h}(\,\cdot\,) m_{\lambda,\gamma,k}(t|\,\cdot\,|)\right)^{\wedge}(x) = (S_{\lambda,\gamma,k})_t(h)(x), \tag{4.31}$$

we have the following comparability relation for the left hand side of (4.29, page 139):

$$\begin{aligned} &\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} u_{\lambda,\mu}(y) \Bigg| \sum_{j=0}^{N_\lambda/2} \widehat{h}\left(\xi - \left(\frac{N_\lambda/2 - j}{2\pi}\right) y\right) \cdot \\ &\cdot m_{\lambda,\gamma,k}\left(t \left|\xi - \left(\frac{N_\lambda/2 - j}{2\pi}\right) y\right|\right) (-1)^j \; b_{\frac{N_\lambda}{2},j} \Bigg|^2 dy d\xi \\ \approx_{n,\lambda,\mu} &\left\|(S_{\lambda,\gamma,k})_t(h)(x)\right\|^2_{L^2\left(\frac{dx}{\omega_{\lambda\mu}(|x|)}\right)}, \end{aligned} \tag{4.32}$$

that is, (4.29, page 139) is equivalent to:

$$\begin{aligned} \left\|(S_{\lambda,\gamma,k})_t(h)(x)\right\|^2_{L^2\left(\frac{dx}{\omega_{\lambda\mu}(|x|)}\right)} &\leq C_{n,\lambda,\mu,\gamma} \frac{2^{2\,k\,\lambda}}{k^{2\gamma-\mu}} \int_{\mathbf{R}^n} |h(x)|^2 \frac{dx}{w_{\lambda\mu}(x)} \\ &= C_{n,\lambda,\mu,\gamma} \frac{2^{2\,k\,\lambda}}{k^{2\gamma-\mu}} \|h\|^2_{L^2\left(\frac{dx}{\omega_{\lambda\mu}(|x|)}\right)} \end{aligned} \tag{4.33}$$

for another constant $C_{n,\lambda,\mu,\gamma}$, for every $t>0$ and $h \in L^2\left(\mathbf{R}^n, \frac{dx}{w_{\lambda\mu}(x)}\right)$. By duality, (4.33, page 140) is equivalent to:

$$\|(S_{\lambda,\gamma,k})_t(h)\|^2_{L^2\left(w_{\lambda\mu}\right)} \le C_{n,\lambda,\mu,\gamma}\frac{2^{2\,k\,\lambda}}{k^{2\gamma-\mu}}\|h\|^2_{L^2(w_{\lambda\mu})} \tag{4.34}$$

for all $h \in L^2(w_{\lambda\mu}(x)dx)$, $t>0$.

So, it's enough to prove the latter in order to prove that (3.20, page 41) holds for every f in the appropriate space and every $a>0$.

We denote by $(K_{\lambda,\gamma,k})_t(x)$ the kernel of the operator $(S_{\lambda,\gamma,k})_t$, i.e., the inverse Fourier transform of the multiplier $m_{\lambda,\gamma,k}(t\,|\cdot|)$. Certainly $(K_{\lambda,\gamma,k})_t$ is a radial kernel on $\mathbf{R}^n$, and it is convenient to decompose it radially as

$$(K_{\lambda,\gamma,k})_t = (K_{\lambda,\gamma,k})^{(0)}_t + \sum_{j=1}^{\infty}(K_{\lambda,\gamma,k})^{(j)}_t\,,$$

where $(K_{\lambda,\gamma,k})^{(0)}_t(x) = (K_{\lambda,\gamma,k})_t(x)\phi(2^{-(k+3)}x/t)$ and $(K_{\lambda,\gamma,k})^{(j)}_t(x)$
$= (K_{\lambda,\gamma,k})_t(x)\left(\phi(2^{-(j+k+3)}\,x/t) - \phi(2^{-(k+2+j)}\,x/t)\right)$, for some radial smooth function ϕ supported in the ball $B(0,2)$ and equal to one on $B(0,1)$.

Therefore we have

$$\operatorname{supp}(K_{\lambda,\gamma,k})^{(j)}_t \subseteq B(0, 2^{j+k+4}\,t)\,. \tag{4.35}$$

Observe also that, with this definition, we have:

$$\begin{aligned}
(K_{\lambda,\gamma,k})_t^{(j)}(x) &= (K_{\lambda,\gamma,k})_t(x)\left(\phi(2^{-(j+k+3)}\,x/t) - \phi(2^{-(k+2+j)}\,x/t)\right) \\
&= \frac{1}{t^n}(K_{\lambda,\gamma,k})_1(x/t)\left(\phi(2^{-(j+k+3)}\,x/t)\right. \\
&\qquad\qquad \left. -\phi(2^{-(k+2+j)}\,x/t)\right) \\
&= \frac{1}{t^n}(K_{\lambda,\gamma,k})_1^{(j)}(x/t)
\end{aligned}$$

which can be stated as follows:

$$\widehat{(K_{\lambda,\gamma,k})}_t^{(j)}(\xi) = \widehat{(K_{\lambda,\gamma,k})}_1^{(j)}(t\,\xi) \tag{4.36}$$

To prove estimate (4.34) we make use of the subsequent lemmas.

Lemma 4.1.1. *For all $M \geq 2n$ there is a constant $C_{\lambda,\gamma,k,M} = C_{\lambda,\gamma,k,M}(n,\phi)$ such that for all $j = 0,1,2,\ldots$ we have:*

$$\sup_{\xi\in\mathbf{R}^n} |\widehat{(K_{\lambda,\gamma,k})}_t^{(j)}(\xi)| \leq C_{\lambda,\gamma,M}\,\frac{2^{-jM}}{k^\gamma} \tag{4.37}$$

and also

$$|\widehat{(K_{\lambda,\gamma,k})}_t^{(j)}(\xi)| \leq C_{\lambda,\gamma,M}\,\frac{2^{-(j+l)M}}{k^\gamma} \tag{4.38}$$

whenever $|\,t|\xi| - 1| \geq 2^{l-k-3}$ and $l \geq 4$. Also

$$|\widehat{(K_{\lambda,\gamma,k})}_t^{(j)}(\xi)| \leq C_{\lambda,\gamma,M}\,\frac{2^{-(j+k+3)M}}{k^\gamma}\,(1+t|\xi|)^{-M} \tag{4.39}$$

whenever $|t\,\xi| \leq 1/8$ or $|t\,\xi| \geq 15/8$.

Proof.

The proof for $t = 1$ follows the lines of the proof of Lemma 10.5.5 in [8, p. 413] (even if the proof itself is only at page 416 of the same book). Just observe that estimate (10.5.9) at page 409 of [8] is now replaced by (2.7, page 11), because of which the factor $\frac{1}{k^\gamma}$ appears.

The general case (any $t > 0$) is straightforward in view of (4.36, page 142).

□

Lemma 4.1.2. *There is a constant $C_{n,\lambda,\mu}$ such that for all Schwartz functions f, all $t > 0$ and all $0 < \varepsilon < 2$, we have:*

$$\int_{|\,|t\,\xi|-1|\leq\varepsilon} |\widehat{f}(\xi)|^2 \, d\xi \leq C_{n,\lambda,\mu} \; \omega_{\lambda\mu}(t) \, \varepsilon \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)} \tag{4.40}$$

and also for $M \geq 2n$ there is a constant $C_{n,\lambda,\mu,M}$ such that:

$$\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2 \, \frac{1}{(1+|t\,\xi|)^M} \, d\xi \leq C_{n,\lambda,\mu,M} \; \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)} \,. \tag{4.41}$$

Proof.

Postponed until section 4.2.

□

Assuming Lemmas 4.1.1 and 4.1.2 we prove estimate (4.34) as follows.

Using Plancherel's theorem we write

$$\begin{aligned}\int_{\mathbf{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f)(x)|^2\,dx &= \int_{\mathbf{R}^n} |\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)|^2 |\widehat{f}(\xi)|^2\,d\xi \\ &\le I + II + III\,,\end{aligned}$$

where

$$\begin{aligned} I &= \int_{|t\,\xi|\le\frac{1}{8}\,,\,|t\,\xi|\ge\frac{15}{8}} |\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)|^2 |\widehat{f}(\xi)|^2\,d\xi\,, \\ II &= \sum_{l=4}^{[\log_2 \frac{7}{2} 2^k]+1} \int_{2^{l-k-3}\le|\,|t\,\xi|-1|\le 2^{l-k-2}} |\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)|^2 |\widehat{f}(\xi)|^2\,d\xi\,, \\ III &= \int_{|\,|t\,\xi|-1|\le 2^{-k+1}} |\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)|^2 |\widehat{f}(\xi)|^2\,d\xi\,. \end{aligned}$$

Using (4.39) and (4.41) we obtain that:

$$\begin{aligned} I &\le C^2_{\lambda,\gamma,M}\,\frac{2^{-2(j+k+3)M}}{k^{2\gamma}} \int_{\mathbf{R}^n} \frac{|\widehat{f}(\xi)|^2}{(1+t|\xi|)^{2M}}\,d\xi \\ &= C^2_{\lambda,\gamma,M}\,\frac{2^{-2(j+k+3)M}}{k^{2\gamma}}\, C_{n,\lambda,\mu,2M}\;\omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2\,\frac{dx}{w_{\lambda\mu}(x)} \\ &= C_{n,\lambda,\mu,\gamma,M}\,\frac{2^{-2j\,M}}{2^{2\,k\,M}\,k^{2\gamma}}\;\omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2\,\frac{dx}{w_{\lambda\mu}(x)} \end{aligned}$$

In view of (4.38) and (4.40) we have:

$$
\begin{aligned}
II &= \sum_{l=4}^{[\log_2 \frac{7}{2} 2^k]+1} \int_{2^{l-k-3} \le |\,|t\,\xi|-1| \le 2^{l-k-2}} |\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)|^2 |\widehat{f}(\xi)|^2 \, d\xi \\
&\le 2^{-2\,j\,M} \frac{C^2_{\lambda,\gamma,M}}{k^{2\gamma}} \sum_{l=4}^{[\log_2 \frac{7}{2} 2^k]+1} \left(2^{-2\,l\,M} \int_{2^{l-k-3} \le |\,|t\,\xi|-1| \le 2^{l-k-2}} |\widehat{f}(\xi)|^2 \, d\xi \right) \\
&\le 2^{-2\,j\,M} \frac{C^2_{\lambda,\gamma,M}}{k^{2\gamma}} \cdot \\
&\quad \cdot \sum_{l=4}^{[\log_2 \frac{7}{2} 2^k]+1} \left(2^{-2\,l\,M} C_{n,\lambda,\mu} \; \omega_{\lambda\mu}(t) \, 2^{l-k-2} \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)} \right) \\
&= 2^{-2\,j\,M} \, \frac{C_{n,\lambda,\mu,\gamma,M}}{2^k \, k^{2\gamma}} \, \omega_{\lambda\mu}(t) \left(\int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)} \right) \cdot \\
&\quad \cdot \sum_{l=4}^{[\log_2 \frac{7}{2} 2^k]+1} \left(2^{l(1-2M)} \right) \\
&\le C'_{n,\lambda,\mu,\gamma,M} \; \frac{2^{-2\,j\,M}}{2^k \, k^{2\gamma}} \; \omega_{\lambda\mu}(t) \; \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)}.
\end{aligned}
$$

Finally, (4.37) and (4.40) yield:

$$
\begin{aligned}
III &= \int_{|\,|t\,\xi|-1| \le 2^{-k+1}} |\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)|^2 |\widehat{f}(\xi)|^2 \, d\xi \\
&\le C^2_{\lambda,\gamma,M} \frac{2^{-2\,j\,M}}{k^{2\gamma}} \int_{|\,|t\,\xi|-1| \le 2^{-k+1}} |\widehat{f}(\xi)|^2 \, d\xi \\
&\le C_{n,\lambda,\mu,\gamma,M} \; \frac{2^{-2\,j\,M}}{2^k \, k^{2\gamma}} \; \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)}.
\end{aligned}
$$

Summing the estimates for I, II, and III we deduce:

$$
\begin{aligned}
&\int_{\mathbf{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f)(x)|^2 \, dx \\
\le & C_{n,\lambda,\mu,\gamma,M} \; \frac{2^{-2\,j\,M}}{2^k \, k^{2\gamma}} \; \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)}
\end{aligned}
\tag{4.42}
$$

for some other constant $C_{n,\lambda,\mu,\gamma,M}$.

By duality, this estimate can be written as:

$$\int_{\mathbf{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f)(x)|^2 \, w_{\lambda\mu}(x)dx \\ \leq C_{n,\lambda,\mu,\gamma,M} \, \frac{2^{-2\,j\,M}}{2^k \, k^{2\gamma}} \, \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \, dx. \tag{4.43}$$

Given a Schwartz function f, we write $f_0 = f\chi_{Q_0^{(n,k,j,t)}}$, where $Q_0^{(n,k,j,t)}$ is a cube centered at the origin of side length $C_n \, 2^{j+k+4} \, t$ (cf. the support of $(K_{\lambda,\gamma,k})_t^{(j)}$ (4.35, page 141)) for a purely dimensional constant C_n (for example, $C_n = 10n$ is large enough for our purpose). Then for $x \in Q_0^{(n,k,j,t)}$ we have $|x| \leq \sqrt{n}C_n \, 2^{j+k+4} \, t$, hence (4.43, page 146) implies:

$$\begin{aligned} & \int_{\mathbf{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f_0)(x)|^2 \, w_{\lambda\mu}(x)dx \\ \leq \; & C_{n,\lambda,\mu,\gamma,M} \, \frac{2^{-2\,j\,M}}{2^k \, k^{2\gamma}} \, \omega_{\lambda\mu}(t) \int_{Q_0^{(n,k,j,t)}} |f_0(x)|^2 \, dx \\ \leq \; & C_{n,\lambda,\mu,\gamma,M} \, \frac{2^{-2\,j\,M}}{2^k \, k^{2\gamma}} \, \frac{\omega_{\lambda\mu}(t)}{\omega_{\lambda\mu}(\sqrt{n}C_n \, 2^{j+k+4} \, t)} \int_{Q_0^{(n,k,j,t)}} |f_0(x)|^2 \, w_{\lambda\mu}(x)dx \end{aligned} \tag{4.44}$$

because the function $\frac{1}{\omega_{\lambda\mu}}$ is increasing. A computation similar to the one started at page 33, with $\frac{3}{5}$ replaced by $a > 0$, shows that:

$$\sup_{t>0} \frac{\omega_{\lambda\mu}(a\,t)}{\omega_{\lambda\mu}(t)} = \frac{1}{a^{2\lambda+1}}$$

if $a > 1$ and that:

$$\sup_{t>0} \frac{\omega_{\lambda\mu}(a\,t)}{\omega_{\lambda\mu}(t)} = \frac{(\log(e/a))^\mu}{a^{2\lambda+1}}$$

if $a \leq 1$. Therefore, for all j and k such that $j+k \geq C'_n$ for a suitable purely dimensional constant C'_n:

$$\begin{aligned}
\sup_{t>0} \frac{\omega_{\lambda\mu}(t)}{\omega_{\lambda\mu}(\sqrt{n}C_n\, 2^{j+k+4}\, t)} &= \sup_{t>0} \frac{\omega_{\lambda\mu}(t/(\sqrt{n}C_n\, 2^{j+k+4}))}{\omega_{\lambda\mu}(t)} \\
&= \left(\sqrt{n}C_n\, 2^{j+k+4}\right)^{2\lambda+1} \left(\log\left(e\sqrt{n}C_n\, 2^{j+k+4}\right)\right)^\mu \\
&= C_{n,\lambda}\, 2^{(j+k)(2\lambda+1)} \left(\log\left(e\sqrt{n}C_n\, 2^{j+k+4}\right)\right)^\mu \\
&\leq C'_{n,\lambda}\, 2^{(j+k)(2\lambda+1)}\, (j+k)^\mu \qquad (4.45)
\end{aligned}$$

where we used the hypothesis on j and k in the last equality. Now observe that, if $0 < \mu < 1$, we have:

$$\begin{aligned}
1 &= \frac{j}{j+k} + \frac{k}{j+k} \\
&\leq \left(\frac{j}{j+k}\right)^\mu + \left(\frac{k}{j+k}\right)^\mu \\
&= \frac{j^\mu}{(j+k)^\mu} + \frac{k^\mu}{(j+k)^\mu} \\
&= \frac{j^\mu + k^\mu}{(j+k)^\mu},
\end{aligned}$$

that is:

$$(j+k)^\mu \leq j^\mu + k^\mu.$$

On the other hand, for $\mu \geq 1$, the function $x \mapsto x^\mu$ is convex, which implies:

$$\left(\frac{j+k}{2}\right)^{\mu} \leq \frac{j^{\mu}+k^{\mu}}{2}$$

that is:

$$(j+k)^{\mu} \leq 2^{\mu-1}(j^{\mu}+k^{\mu}).$$

We conclude that for every $\mu > 0$ there exists a constant C_{μ} such that:

$$(j+k)^{\mu} \leq C_{\mu}(j^{\mu}+k^{\mu}). \tag{4.46}$$

(In fact, the same tricks can be used to show that $(j+k)^{\mu} \approx_{\mu} (j^{\mu}+k^{\mu})$, for all $j, k, \mu > 0$.) The last inequality allows us to go a step further in (4.45, page 147) and get:

$$\sup_{t>0} \frac{\omega_{\lambda\mu}(t)}{\omega_{\lambda\mu}(\sqrt{n}C_n\, 2^{j+k+4}\, t)} \leq C'_{n,\lambda,\mu}\, 2^{(j+k)(2\lambda+1)}\, (j^{\mu}+k^{\mu}). \tag{4.47}$$

Then we can use (4.47) and (4.44, page 146) to obtain the following estimate:

$$\begin{aligned} & \int_{\mathbf{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f_0)(x)|^2\, w_{\lambda\mu}(x)dx \qquad (4.48) \\ \leq \;& C_{n,\lambda,\gamma,M}\, \frac{2^{-2\,j\,M}}{2^k\, k^{2\gamma}}\, C'_{n,\lambda,\mu}\, 2^{(j+k)(2\lambda+1)}\, (j^{\mu}+k^{\mu}) \cdot \\ & \cdot \int_{Q_0^{(n,k,j,t)}} |f_0(x)|^2\, w_{\lambda\mu}(x)dx \\ = \;& C'_{n,\lambda,\mu,\gamma,M}\, 2^{j(2\lambda+1-2M)}\, \frac{2^{2\,k\lambda}}{k^{2\gamma}}\, (j^{\mu}+k^{\mu}) \int_{Q_0^{(n,k,j,t)}} |f_0(x)|^2\, w_{\lambda\mu}(x)dx, \end{aligned}$$

provided that

$$j+k \geq C'_n. \tag{4.49}$$

Now write $\mathbf{R}^n \setminus Q_0^{(n,k,j,t)}$ as a mesh of cubes $Q_i^{(n,k,j,t)}$, indexed by $i \in \mathbf{Z} \setminus \{0\}$, of side lengths $C_n\, 2^{j+k+4}\, t$ (the same side length of $Q_0^{(n,k,j,t)}$) and centers c_{Q_i}. Since $(K_{\lambda,\gamma,k})_t^{(j)}$ is supported in a ball centered at the origin, of radius $2^{j+k+4}\, t$, if f_i is supported in Q_i, then $f_i * (K_{\lambda,\gamma,k})_t^{(j)}$ is supported in the cube $2\sqrt{n}\, Q_i$. As the constant C_n is large enough (recall that we set $C_n = 10\, n$) then for any $x \in Q_i^{(n,k,j,t)}$ and $x' \in 2\sqrt{n}\, Q_i^{(n,k,j,t)}$ we have

$$|x| \approx_n |c_{Q_i}| \approx_n |x'|\,,$$

which says that the moduli of x and x' are comparable in the following inequality:

$$\begin{aligned}&\int_{2\sqrt{n}\, Q_i^{(n,k,j,t)}} |((K_{\lambda,\gamma,k})_t^{(j)} * f_i)(x')|^2\, w_{\lambda\mu}(x')dx' \\ \leq & C_{\lambda,\mu,\gamma,M}\, \frac{2^{-2jM}}{k^{2\gamma}} \int_{Q_i^{(n,k,j,t)}} |f_i(x)|^2\, w_{\lambda\mu}(x)dx\,.\end{aligned} \tag{4.50}$$

Thus (4.50) (with a different constant $C_{\lambda,\mu,\gamma,M}$) is a consequence of

$$\begin{aligned}&\int_{2\sqrt{n}\, Q_i^{(n,k,j,t)}} |((K_{\lambda,\gamma,k})_t^{(j)} * f_i)(x')|^2\, dx' \\ &\qquad \leq C_{\lambda,\mu,\gamma,M}^2\, \frac{2^{-2jM}}{k^{2\gamma}} \int_{Q_i^{(n,k,j,t)}} |f_i(x)|^2\, dx\,,\end{aligned} \tag{4.51}$$

which is certainly satisfied, as seen by applying Plancherel's theorem and using (4.37, page 142). Since for $j \geq 1, k \geq 1$ we have $2^{2\,k\lambda}\,(j^\mu + k^\mu) \geq 1$, it follows from (4.50, page 149) that

$$\begin{aligned}&\int_{\mathbf{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f_i)(x')|^2 \, w_{\lambda\mu}(x')dx' \\ \le& C_{\lambda,\mu,\gamma,M}\, 2^{-2jM} \frac{2^{2\,k\lambda}}{k^{2\gamma}} \, (j^\mu + k^\mu) \int_{Q_i^{(n,k,j,t)}} |f_i(x)|^2 \, w_{\lambda\mu}(x)dx\end{aligned} \tag{4.52}$$

whenever f_i is supported in $Q_i^{(n,k,j,t)}$. Given a general f in the Schwartz class, write

$$f = \sum_{i\in\mathbf{Z}} f_i \,, \qquad \text{where} \qquad f_i = f\chi_{Q_i^{(n,k,j,t)}} \,.$$

Then, in view of (4.48, page 148) and (4.52, page 150):

$$\begin{aligned}&\big\|(K_{\lambda,\gamma,k})_t^{(j)} * f\big\|^2_{L^2(w_{\lambda\mu})} \\ \le& 2\big\|(K_{\lambda,\gamma,k})_t^{(j)} * f_0\big\|^2_{L^2(w_{\lambda\mu})} + 2\big\|\sum_{i\neq 0}(K_{\lambda,\gamma,k})_t^{(j)} * f_i\big\|^2_{L^2(w_{\lambda\mu})} \\ \le& 2\big\|(K_{\lambda,\gamma,k})_t^{(j)} * f_0\big\|^2_{L^2(w_{\lambda\mu})} + 2C_n \sum_{i\neq 0} \big\|(K_{\lambda,\gamma,k})_t^{(j)} * f_i\big\|^2_{L^2(w_{\lambda\mu})} \\ \le& C_{n,\lambda,\mu,\gamma,M}\; 2^{j(2\lambda+1-2M)} \;\frac{2^{2\,k\lambda}}{k^{2\gamma}}\,(j^\mu + k^\mu)\Big[\big\|f_0\big\|^2_{L^2(w_{\lambda\mu})} + \sum_{i\neq 0}\big\|f_i\big\|^2_{L^2(w_{\lambda\mu})}\Big] \\ =& C_{n,\lambda,\mu,\gamma,M}\; 2^{j(2\lambda+1-2M)} \;\frac{2^{2\,k\lambda}}{k^{2\gamma}}\,(j^\mu + k^\mu)\,\big\|f\big\|^2_{L^2(w_{\lambda\mu})}\,,\end{aligned} \tag{4.53}$$

where we used the bounded overlap of the family $\{K_j * f_i\}_{i\neq 0}$ in the second displayed inequality. Then we take square roots in (4.53, page 150) to get:

$$\begin{aligned}&\big\|(K_{\lambda,\gamma,k})_t^{(j)} * f\big\|_{L^2(w_{\lambda\mu})} \\ \le& C'_{n,\lambda,\mu,\gamma,M}\; 2^{j(\lambda+\frac{1}{2}-M)} \;\frac{2^{k\lambda}}{k^{\gamma}}\,\sqrt{j^\mu + k^\mu}\,\big\|f\big\|_{L^2(w_{\lambda\mu})} \\ \le& C''_{n,\lambda,\mu,\gamma,M}\; 2^{j(\lambda+\frac{1}{2}-M)} \;\frac{2^{k\lambda}}{k^{\gamma}}\,(j^{\frac{\mu}{2}} + k^{\frac{\mu}{2}})\,\big\|f\big\|_{L^2(w_{\lambda\mu})}\end{aligned} \tag{4.54}$$

where the last inequality follows by another application of (4.46, page 148). Observe that condition (4.49, page 148) is satisfied if we assume $k \geq C'_n$, which we can as the convergence of (3.13, page 32) only depends on the estimates we have for k big enough. So, for $k \geq C'_n$, by using (4.54, page 150) and summing over $j = 0, 1, 2, \ldots$, we deduce (4.34, page 141):

$$\begin{aligned}
& \|(S_{\lambda,\gamma,k})_t(h)\|_{L^2\left(w_{\lambda,\mu}\right)} \qquad (4.55) \\
= & \|(K_{\lambda,\gamma,k})_t * h\|_{L^2\left(w_{\lambda,\mu}\right)} \\
= & \left\| \left(\sum_{j=0}^{\infty} (K_{\lambda,\gamma,k})_t^{(j)} \right) * h \right\|_{L^2\left(w_{\lambda,\mu}\right)} \\
\leq & C''_{n,\lambda,\mu,\gamma,M} \frac{2^{k\lambda}}{k^{\gamma}} \sum_{j=0}^{\infty} 2^{j(\lambda+\frac{1}{2}-M)} \; (j^{\frac{\mu}{2}} + k^{\frac{\mu}{2}}) \|h\|_{L^2\left(w_{\lambda,\mu}\right)} \\
= & C''_{n,\lambda,\mu,\gamma,M} \frac{2^{k\lambda}}{k^{\gamma-\frac{\mu}{2}}} \|h\|_{L^2\left(w_{\lambda,\mu}\right)} \sum_{j=0}^{\infty} 2^{j(\lambda+\frac{1}{2}-M)} \\
& + C''_{n,\lambda,\mu,\gamma,M} \frac{2^{k\lambda}}{k^{\gamma}} \|h\|_{L^2\left(w_{\lambda,\mu}\right)} \sum_{j=0}^{\infty} 2^{j(\lambda+\frac{1}{2}-M)} \; j^{\frac{\mu}{2}} \\
\leq & C'''_{n,\lambda,\mu,\gamma} \frac{2^{k\lambda}}{k^{\gamma-\frac{\mu}{2}}} \|h\|_{L^2\left(w_{\lambda,\mu}\right)}
\end{aligned}$$

as claimed, if we just choose $M > \frac{n}{2}$ before the last two equalities (recall that $n > 2\lambda + 1$). This proves (4.34, page 141), that is equivalent to (4.33, page 140), that is equivalent to (4.29, page 139), that is equivalent to (4.28, page 139), which implies (4.27, page 139), that is equivalent to (4.26, page 138), which implies (4.25, page 138), which is equivalent to (3.20, page

41). Therefore, this completes the proof of Lemma (3.1.5, page 41), modulo Lemma (4.1.2, page 143).

4.2 Proof of Lemma 4.1.2, page 143

We reduce estimate (4.40, page 143) by duality to

$$\int_{\mathbf{R}^n} |\widehat{g}(\xi)|^2\, w_{\lambda\mu}(\xi) d\xi \leq C_{n,\lambda,\mu}\, \omega_{\lambda\mu}(t)\, \varepsilon \int_{|\,|t\,x|-1|\leq\varepsilon} |g(x)|^2\, dx \tag{4.56}$$

for functions g supported in the annulus $|\,|t\,x| - 1| \leq \varepsilon$. In section (3.2, page 56) we proved that $w_{\lambda\mu} \approx_{\lambda,\mu} w^{(1)}_{\lambda,\mu}$, that $\widehat{w^{(1)}_{\lambda,\mu}}$ is represented by a locally integrable and polynomially increasing function, and that the function $|\widehat{w^{(1)}_{\lambda,\mu}}|$ is bounded by a scalar multiple of $\Omega_{\lambda,\mu}$ (cf. (3.37, page 57)) in the whole range $\lambda \in \big(0, \frac{n-1}{2}\big)$. Therefore, we can start to prove (4.56, page 152) as follows:

$$\tag{4.57}$$

$$\begin{aligned}
\int_{\mathbf{R}^n} |\widehat{g}(\xi)|^2\, w_{\lambda\mu}(\xi) d\xi \quad &\approx_{\lambda,\mu} \int_{\mathbf{R}^n} |\widehat{g}(\xi)|^2\, w^{(1)}_{\lambda,\mu}(\xi) d\xi \\
&= \int_{\mathbf{R}^n} \widehat{g}(\xi)\, \overline{\widehat{g}(\xi)}\, w^{(1)}_{\lambda,\mu}(\xi) d\xi \\
&= \int_{\mathbf{R}^n} \big(\widehat{g}\,\overline{\widehat{g}}\big)^{\vee}(x)\, \widehat{w^{(1)}_{\lambda,\mu}}(x) dx \\
&= \int_{\mathbf{R}^n} (g * \widetilde{\overline{g}})(x)\, \widehat{w^{(1)}_{\lambda,\mu}}(x) dx \\
&\leq C_{n,\lambda\mu} \int_{\mathbf{R}^n} (|g| * |\widetilde{\overline{g}}|)(x)\, \Omega_{\lambda,\mu}(x) dx
\end{aligned}$$

$$
\begin{aligned}
&= C_{n,\lambda,\mu} \iint_{\substack{|\,|t\,y|-1| \le \varepsilon \\ |\,|t\,x|-1| \le \varepsilon}} |g(x)|\,|\widetilde{g}(y)| \cdot \\
&\qquad \cdot \Omega_{\lambda,\mu}(x-y)\,dx\,dy \\
&\le C_{n,\lambda,\mu} B(n,\lambda,\mu,\varepsilon,t)\, \|g\|_{L^2}^2\,,
\end{aligned}
$$

where $\widetilde{g}(x) = g(-x)$ and

$$
\begin{aligned}
B(n,\lambda,\mu,\varepsilon,t) &= \sup_{\{x:|\,|t\,x|-1|\le\varepsilon\}} \int_{|\,|t\,y|-1|\le\varepsilon} \Omega_{\lambda,\mu}(y-x)\,dy \\
&= \sup_{\{x:|\,|x|-1|\le\varepsilon\}} \int_{|\,|t\,y|-1|\le\varepsilon} \Omega_{\lambda,\mu}(y-x/t)\,dy \\
&= \sup_{\{x:|\,|x|-1|\le\varepsilon\}} \frac{1}{t^n} \int_{|\,|y|-1|\le\varepsilon} \Omega_{\lambda,\mu}(y/t-x/t)\,dy \\
&=: \frac{1}{t^n} \sup_{\{x:|\,|x|-1|\le\varepsilon\}} \int_{|\,|y|-1|\le\varepsilon} \Omega^t_{\lambda,\mu}(y-x)\,dy
\end{aligned}
$$

where $\Omega^t_{\lambda,\mu}(x) := \Omega_{\lambda,\mu}(x/t)$. The last inequality of (4.57, page 152) is proved by interpolating between the norm

$$
L^1(\{x \in \mathbf{R}^n : |\,|t\,x|-1| \le \varepsilon\}) \to L^1(\{x \in \mathbf{R}^n : |\,|t\,x|-1| \le \varepsilon\})
$$

and the norm

$$
L^\infty(\{x \in \mathbf{R}^n : |\,|t\,x|-1| \le \varepsilon\}) \to L^\infty(\{x \in \mathbf{R}^n : |\,|t\,x|-1| \le \varepsilon\})
$$

of the linear operator

$$
\begin{aligned}
L_{\lambda,\mu,t,\varepsilon}(g)(x) &= \int_{\mathbf{R}^n} g(y)\,\Omega_{\lambda,\mu}(y-x)\,dy \\
&= \int_{|\,|t\,y|-1|\le\varepsilon} g(y)\,\Omega_{\lambda,\mu}(y-x)\,dy
\end{aligned}
$$

(recall the hypothesis on the support of the function g), and by using the Cauchy-Schwarz inequality. It remains to establish that

$$B(n,\lambda,\mu,\varepsilon,t) \le C_{n,\lambda,\mu}\ \omega_{\lambda\mu}(t)\,\varepsilon\,.$$

Applying a rotation and a change of variables, matters reduce to proving that:

$$\frac{1}{t^n}\sup_{|\,|x|-1|\le\varepsilon}\int_{|\,|y-|x|e_1|-1|\le\varepsilon}\Omega^t_{\lambda,\mu}(x-y)\,dy \le C'_{n,\lambda,\mu}\ \varepsilon\ \omega_{\lambda\mu}(t),$$

where $e_1 = (1,0,\dots,0)$. This, in turn, is a consequence of

$$\frac{1}{t^n}\int_{|\,|y-e_1|-1|\le 2\varepsilon}\Omega^t_{\lambda,\mu}(y)\,dy \le C''_{n,\lambda,\mu}\ \varepsilon\ \omega_{\lambda\mu}(t), \tag{4.58}$$

since $|\,|y-e_1|x|\,|-1| \le \varepsilon$ and $|\,|x|-1| \le \varepsilon$ imply $|\,|y-e_1|-1| \le 2\varepsilon$. In proving (4.58), it suffices to assume that $\varepsilon < \frac{1}{100}$; otherwise, the left-hand side of (4.58) is bounded from above by a constant, and the right-hand side of (4.58) is bounded from below by another constant. The region of integration in (4.58) is a ring centered at e_1 and width 4ε. We estimate the integral in (4.58) by the sum of the integrals of the function $\frac{1}{t^n}\Omega^t_{\lambda,\mu}$ over the sets:

$$\begin{aligned} S_0 &= \{y\in\mathbf{R}^n:\ |y|\le\varepsilon,\quad |\,|y-e_1|-1|\le 2\varepsilon\}\,,\\ S_\ell &= \{y\in\mathbf{R}^n:\ \ell\varepsilon\le|y|\le(\ell+1)\varepsilon,\quad |\,|y-e_1|-1|\le 2\varepsilon\}\,,\\ S_\infty &= \{y\in\mathbf{R}^n:\ |y|\ge 1,\quad |\,|y-e_1|-1|\le 2\varepsilon\}\,, \end{aligned} \tag{4.59}$$

where $\ell = 1,\dots,[\frac{1}{\varepsilon}]+1$. The volume of each S_ℓ is comparable to

$$\varepsilon\left[((\ell+1)\varepsilon)^{n-1}-(\ell\varepsilon)^{n-1}\right]\approx_n \varepsilon^n\ell^{n-2}\,. \tag{4.60}$$

Now we estimate the integral in (4.58, page 154) a piece at a time:

$$
\begin{aligned}
\int_{S_0} \Omega^t_{\lambda,\mu}(y)dy &= \int_{S_0} \Omega_{\lambda,\mu}(y/t)dy \\
&\leq C'_{n,\lambda}\, t^{n-(2\lambda+1)} \int_0^{\varepsilon} \frac{r^{n-1}}{r^{n-(2\lambda+1)}}\, dr, \\
&= C'_{n,\lambda}\, t^{n-(2\lambda+1)} \int_0^{\varepsilon} r^{2\lambda} dr, \\
&= C'_{n,\lambda}\, t^{n-(2\lambda+1)} \frac{1}{(2\lambda+1)}\, (r^{2\lambda+1})\Big|_0^{\varepsilon} \\
&= C'_{n,\lambda}\, t^{n-(2\lambda+1)} \frac{1}{(2\lambda+1)} \varepsilon^{2\lambda+1} \\
&= C''_{n,\lambda}\, t^{n-(2\lambda+1)} \varepsilon^{2\lambda+1}.
\end{aligned}
\tag{4.61}
$$

The estimate above holds for all $t > 0$. In addition, for $t \geq \max\{2, C_{n,\lambda,\mu}\}$ (where $C_{n,\lambda,\mu}$ is a suitable constant that comes from using Lemma 3.1.9, page 47) we have that $t > \varepsilon$ (because $0 < \varepsilon < 2$) and that $\log(e\,t) > 0$. Therefore:

(4.62)

$$
\begin{aligned}
\int_{S_0} \Omega^t_{\lambda,\mu}(y)dy &= \int_{S_0} \Omega_{\lambda,\mu}(y/t)dy \\
&\leq C'_{n,\lambda,\mu}\, t^{n-(2\lambda+1)} \int_0^{\varepsilon} \frac{r^{n-1}}{r^{n-(2\lambda+1)} \left(\log\left(\frac{e\,t}{r}\right)\right)^{\mu}}\, dr \\
&= -C'_{n,\lambda,\mu}\, t^{n-(2\lambda+1)} \cdot \\
&\quad \cdot \int_{\infty}^{\frac{1}{\varepsilon}} \frac{(1/s)^{n-1}}{(1/s)^{n-(2\lambda+1)} \left(\log\left(\frac{e\,t}{(1/s)}\right)\right)^{\mu}} \frac{ds}{s^2} \\
&= C'_{n,\lambda,\mu}\, t^{n-(2\lambda+1)} \int_{\frac{1}{\varepsilon}}^{\infty} \frac{s^{-2\lambda-2}}{(\log(e\,t\,s))^{\mu}}\, ds \\
&= C'_{n,\lambda,\mu}\, t^{n-(2\lambda+1)} \frac{1}{t} \int_{\frac{t}{\varepsilon}}^{\infty} \frac{(r/t)^{-2\lambda-2}}{(\log(e\,r))^{\mu}}\, dr
\end{aligned}
$$

$$
\begin{aligned}
&= \quad C'_{n,\lambda\mu}\, t^n \int_{\frac{t}{\varepsilon}}^{\infty} \frac{r^{-2\lambda-2}}{(\log{(e\,r)})^{\mu}}\, dr \\
&\approx_{n,\lambda,\mu} \quad t^n \frac{(t/\varepsilon)^{-2\lambda-1}}{(\log{(e\,t/\varepsilon)})^{\mu}} \\
&= \quad \frac{t^n}{t^{2\lambda+1}\,(\log{(e\,t/\varepsilon)})^{\mu}}\, \varepsilon^{2\lambda+1} \\
&\leq \quad C_{\mu} \frac{t^n}{t^{2\lambda+1}\,(\log{(e\,t)})^{\mu}}\, \varepsilon^{2\lambda+1} \\
&= \quad C_{\mu}\, t^n\, \omega_{\lambda\mu}(t)\, \varepsilon^{2\lambda+1}.
\end{aligned}
$$

The second, ninth and tenth steps follow from the assumption that $t \geq 2$, the third and the fifth follow from a change of variable while the seventh follows from Lemma 3.1.9, page 47. The inequalities (4.61, page 155) and (4.62, page 155) imply that:

$$
\int_{S_0} \Omega^t_{\lambda,\mu}(y)dy \leq C_{n,\lambda\mu}\, t^n\, \omega_{\lambda\mu}(t)\, \varepsilon^{2\lambda+1}.
$$

Let's now estimate the second piece of the integral in (4.58, page 154). Because of (4.60, page 154) we have:

$$
\begin{aligned}
\sum_{\ell=1}^{[\frac{1}{\varepsilon}]+1} \int_{S_\ell} \Omega^t_{\lambda,\mu}(y)dy &\leq C_{n,\lambda,\mu} \sum_{\ell=1}^{2/\varepsilon} \varepsilon^n \ell^{n-2} \Omega^t_{\lambda,\mu}(\ell\,\varepsilon\,e_1) \\
&\approx_{n,\lambda,\mu} \sum_{\ell=1}^{2/\varepsilon} \varepsilon^n \ell^{n-2} \frac{t^n}{\ell^n\,\varepsilon^n} \omega_{\lambda\mu}\left(\frac{t}{\ell\,\varepsilon}\right).
\end{aligned} \tag{4.63}
$$

The second relation in (4.63, page 156) follows from the identity:

$$\Omega_{\lambda,\mu}(t\ e_1) = C_{n,\lambda,\mu}\frac{1}{t^n}\,\omega_{\lambda\mu}\left(\frac{1}{t}\right) \tag{4.64}$$

(cf. definitions (3.5, page 29) and (3.36, page 56)). We can assume without loss of generality that $t \geq 2$, because the other case is an immediate consequence of Lemma 10.5.6 at page 414 of [8]. Therefore $\frac{t}{\varepsilon} \geq \frac{2}{\varepsilon}$ and the index ℓ in (4.63, page 156) satisfies $\ell \leq \frac{t}{\varepsilon}$, that is, $\frac{t}{\ell\,\varepsilon} \geq 1$. Because of the piecewise definition of $\omega_{\lambda\mu}$ we can continue (4.63, page 156) as follows:

$$\begin{aligned}\sum_{\ell=1}^{[\frac{1}{\varepsilon}]+1}\int_{S_\ell}\Omega^t_{\lambda,\mu}(y)dy &\approx_{n,\lambda,\mu} \sum_{\ell=1}^{2/\varepsilon}\varepsilon^n\ell^{n-2}\frac{t^n}{\ell^n\,\varepsilon^n}\frac{1}{\left(\frac{t}{\ell\,\varepsilon}\right)^{2\lambda+1}\left(\log\left(\frac{e\,t}{\ell\,\varepsilon}\right)\right)^\mu} \\ &= \varepsilon^{2\lambda+1}t^{n-2\lambda-1}\sum_{\ell=1}^{2/\varepsilon}\frac{\ell^{2\lambda-1}}{\left(\log\left(\frac{e\,t}{\ell\,\varepsilon}\right)\right)^\mu}.\end{aligned} \tag{4.65}$$

Now, observe that we can estimate the summation in (4.65, page 157) with an integral. In fact we have:

$$\begin{aligned}\left.\frac{\ell^{2\lambda-1}}{\left(\log\left(\frac{e\,t}{\ell\,\varepsilon}\right)\right)^\mu}\right|_{\ell=1} + \left.\frac{\ell^{2\lambda-1}}{\left(\log\left(\frac{e\,t}{\ell\,\varepsilon}\right)\right)^\mu}\right|_{\ell=2/\varepsilon} &= \frac{1^{2\lambda-1}}{\left(\log\left(\frac{e\,t}{1\,\varepsilon}\right)\right)^\mu} \\ &\quad + \frac{(2/\varepsilon)^{2\lambda-1}}{\left(\log\left(\frac{e\,t}{(2/\varepsilon)\,\varepsilon}\right)\right)^\mu} \\ &= \frac{1}{\left(\log\left(\frac{e\,t}{\varepsilon}\right)\right)^\mu} \\ &\quad + \frac{2^{2\lambda-1}}{\varepsilon^{2\lambda-1}\left(\log\left(\frac{e\,t}{2}\right)\right)^\mu} \\ &\leq C_\lambda\,\frac{\max\{1,\varepsilon^{1-2\lambda}\}}{\left(\log\left(\frac{e\,t}{2}\right)\right)^\mu} \\ &\leq C_{\lambda,\mu}\,\frac{1}{\varepsilon^{2\lambda}\left(\log\left(e\,t\right)\right)^\mu},\end{aligned}$$

where the last term will turn out to be equal (up to constant only depending on λ and μ) to the integral corresponding to the summation in (4.65, page 157). Also, the function $f(x) = \frac{x^{2\lambda-1}}{\left(\log\left(\frac{e\,t}{x\,\varepsilon}\right)\right)^{\mu}}$ corresponding to the argument $g(\ell)$ of the summation (that is, $g(\ell) = \frac{\ell^{2\lambda-1}}{\left(\log\left(\frac{e\,t}{\ell\,\varepsilon}\right)\right)^{\mu}}$) is positive and monotone on $[1, 2/\varepsilon]$. This proves that we can estimate the sum with the integral. Therefore we continue the steps in (4.65, page 157) as follows:

$$
\begin{aligned}
\sum_{\ell=1}^{[\frac{1}{\varepsilon}]+1} \int_{S_\ell} \Omega^t_{\lambda,\mu}(y)dy \quad &\approx_{n,\lambda,\mu} \quad \varepsilon^{2\lambda+1}\, t^{n-2\lambda-1} \sum_{\ell=1}^{2/\varepsilon} \frac{\ell^{2\lambda-1}}{\left(\log\left(\frac{e\,t}{\ell\,\varepsilon}\right)\right)^{\mu}} \\
&\approx_{\lambda,\mu} \quad \varepsilon^{2\lambda+1}\, t^{n-2\lambda-1} \int_1^{2/\varepsilon} \frac{x^{2\lambda-1}}{\left(\log\left(\frac{e\,t}{x\,\varepsilon}\right)\right)^{\mu}} dx \\
&= \quad \varepsilon^{2\lambda+1}\, t^{n-2\lambda-1} \int_{\varepsilon/2}^{1} \frac{r^{-2\lambda-1}}{\left(\log\left(\frac{e\,t\,r}{\varepsilon}\right)\right)^{\mu}} dr \\
&= \quad \varepsilon^{2\lambda+1}\, t^{n-2\lambda-1} \int_{t/2}^{t/\varepsilon} \frac{\left(\frac{\varepsilon\, s}{t}\right)^{-2\lambda-1}}{\left(\log\left(\frac{e\,t\left(\frac{\varepsilon\, s}{t}\right)}{\varepsilon}\right)\right)^{\mu}} \frac{\varepsilon}{t} ds \\
&= \quad t^{n-1}\,\varepsilon \int_{t/2}^{t/\varepsilon} \frac{s^{-2\lambda-1}}{(\log(e\,s))^{\mu}} ds \\
&\approx_{\lambda,\mu} \quad t^{n-1}\,\varepsilon \left(C^{(2)}_{\lambda,\mu} \frac{1}{(t/2)^{2\lambda}\,(\log(e\,t/2))^{\mu}} \right. \\
&\qquad \left. -C^{(1)}_{\lambda,\mu} \frac{1}{(t/\varepsilon)^{2\lambda}\,(\log(e\,t/\varepsilon))^{\mu}} \right) \\
&\leq \quad C_{\lambda,\mu}\, t^{n-1}\,\varepsilon \left(\frac{1}{t^{2\lambda}\,(\log(e\,t/2))^{\mu}} \right) \\
&\leq \quad C_{\lambda,\mu}\, t^{n-1}\,\varepsilon \left(\frac{1}{t^{2\lambda}\,(\log(e\,t))^{\mu}} \right)
\end{aligned}
\tag{4.66}
$$

$$
\begin{aligned}
&= C_{\lambda,\mu}\, t^n\, \varepsilon \left(\frac{1}{t^{2\lambda+1}\,(\log(e\,t))^{\mu}} \right) \\
&= C_{\lambda,\mu}\, t^n\, \varepsilon\, \omega_{\lambda\mu}(t).
\end{aligned}
$$

In the second step we used the consideration we just made, the third and fourth are changes of variable, the 6th and the 8th hold provided that $t > C_{\lambda,\mu}$ for a suitable constant (that comes from Lemma 3.1.10, page 50) and the last one holds since we're also assuming that $t \geq 2$. Finally, the volume of S_∞ is about ε (cf. (4.59, page 154)); in addition, if $y \in S_\infty$, then $1 \leq |y| \leq 1 + 2\varepsilon$. Because of these considerations, and in view of (4.64, page 157), we have:

$$
\begin{aligned}
\int_{S_\infty} \Omega^t_{\lambda,\mu}(y) dy &= \int_{S_\infty} \Omega_{\lambda,\mu}(y/t) dy \\
&\approx_{n,\lambda,\mu} \int_{S_\infty} \omega_{\lambda\mu}(t/|y|) \frac{t^n}{|y|^n} dy \\
&\leq C_{n,\lambda,\mu} |S_\infty|\, \omega_{\lambda\mu}(t)\, t^n \leq C_n\, \varepsilon\, \omega_{\lambda\mu}(t)\, t^n\,.
\end{aligned} \tag{4.67}
$$

Combining estimates (4.62, page 155), (4.66, page 158) and (4.67, page 159), we obtain (4.58, page 154). This concludes the proof of (4.56, page 152), that is equivalent to (4.40, page 143) by duality.

Let's move to the proof of (4.41, page 143):

$$
\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi \leq C_{n,\lambda,\mu,M}\ \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)}\,.
$$

for all f in the appropriate spaces. This estimate is already known for $t \leq 1$ (see for instance equation 10.5.22 in [8]). Indeed, if $0 < t \leq 1$ then $\omega_{\lambda\mu}(t) = \frac{1}{t^{2\lambda+1}}$, and (4.41, page 143) follows by dilation from the case $t = 1$, that is shown in [8]. For $t > 1$ define:

$$
\begin{aligned}
A_1^t &= \left\{ \xi \in \mathbf{R}^n : |\xi| \leq \frac{1}{t} \right\} \\
A_2^t &= \left\{ \xi \in \mathbf{R}^n : \frac{1}{t} < |\xi| \leq \frac{2+\sqrt{t}}{t} \right\} \\
A_3^t &= \left\{ \xi \in \mathbf{R}^n : \frac{2+\sqrt{t}}{t} < |\xi| \leq \frac{2+t}{t} \right\} \\
A_4^t &= \left\{ \xi \in \mathbf{R}^n : \frac{2+t}{t} < |\xi| \right\},
\end{aligned}
$$

therefore:

$$
\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi = I + II + III + IV
$$

where:

$$
\begin{aligned}
I &= \int_{A_1^t} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi \\
II &= \int_{A_2^t} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi \\
III &= \int_{A_3^t} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi \\
IV &= \int_{A_4^t} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi.
\end{aligned}
$$

We're going to show that

$$\int_{A_j^t} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi \le C_{n,\lambda,\mu,M}\ \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)} \tag{4.68}$$

for all $j \in \{1,2,3,4\}$. In order to show that (4.68, page 161) holds for $j=1$, first observe that $\frac{1}{(1+|t\,\xi|)^M} \approx_M 1$ on A_1^t and then argue as in the proof of (4.40, page 143) at the beginning of this section. By duality, we reduce (4.68, page 161) with $j=1$ to:

$$\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2\, w_{\lambda\mu}(\xi)\, d\xi \le C_{n,\lambda,\mu}\ \omega_{\lambda\mu}(t) \int_{A_1^t} |f(x)|^2\, dx \tag{4.69}$$

for all functions f supported in the ball A_1^t. By proceeding as in (4.57, page 152), we can prove that

$$\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2\, w_{\lambda\mu}(\xi) d\xi \le B'(n,\lambda,\mu,t)\, \left\|f\right\|_{L^2}^2$$

for every f supported in A_1^t, where $B'(n,\lambda,\mu,t)$, now, is defined by:

$$\begin{aligned} B'(n,\lambda,\mu,t) &= \sup_{\left\{x:|x|\le \frac{1}{t}\right\}} \int_{|y|\le\frac{1}{t}} \Omega_{\lambda,\mu}(y-x)\, dy \\ &= \frac{1}{t^n} \sup_{\{x:|x|\le 1\}} \int_{|y|\le 1} \Omega_{\lambda,\mu}\left(\frac{y-x}{t}\right) dy \\ &= \frac{1}{t^n} \sup_{\{x:|x|\le 1\}} \int_{|y+x|\le 1} \Omega_{\lambda,\mu}\left(\frac{y}{t}\right) dy \end{aligned} \tag{4.70}$$

and all we still need to show is that:

$$B'(n,\lambda,\mu,t) \le C_{n,\lambda\mu}\, \omega_{\lambda\mu}(t). \tag{4.71}$$

Since $|x| \leq 1$ and $|x+y| \leq 1$ we have $|y| \leq 2$. So, (4.71, page 161) is a consequence of:

$$\frac{1}{t^n} \int_{|y|\leq 2} \Omega_{\lambda,\mu}\left(\frac{y}{t}\right)\, dy \leq C_{n,\lambda\mu}\, \omega_{\lambda\mu}(t). \tag{4.72}$$

which can be proved by following the same steps performed in (4.62, page 155).

Next, we are going to show that (4.68, page 161) holds for $j = 2$, that is:

$$\int_{\frac{1}{t}<|\xi|\leq \frac{2+\sqrt{t}}{t}} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi \leq C_{n,\lambda,\mu,M}\ \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)}. \tag{4.73}$$

To prove this, we split further:

$$\begin{aligned}
&\int_{\frac{1}{t}<|\xi|\leq \frac{2+\sqrt{t}}{t}} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi \\
\leq &\int_{\frac{1}{t}<|\xi|\leq \frac{2+\sqrt{\lceil t\rceil}}{t}} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\, d\xi \\
= &\sum_{\ell=0}^{\lceil\sqrt{t}\rceil} \int_{\frac{1+\ell}{t}<|\xi|\leq \frac{2+\ell}{t}} |\widehat{f}(\xi)|^2 \frac{1}{(1+t\,|\xi|)^M}\, d\xi \\
\leq &\sum_{\ell=0}^{\lceil\sqrt{t}\rceil} \int_{\frac{1+\ell}{t}<|\xi|\leq \frac{2+\ell}{t}} |\widehat{f}(\xi)|^2 \frac{1}{(2+\ell)^M}\, d\xi \\
\leq &\sum_{\ell=0}^{\lceil\sqrt{t}\rceil} \frac{1}{(2+\ell)^M} \int_{\frac{1+\ell}{t}<|\xi|\leq \frac{2+\ell}{t}} |\widehat{f}(\xi)|^2\, d\xi.
\end{aligned} \tag{4.74}$$

Next, we apply estimate (4.40, page 143), which we already proved at the beginning of this section, on each of the latter integrals. Rewrite (4.40, page

143) as follows:

$$\int_{|\,|\tilde{t}\,\xi|-1|\leq\tilde{\varepsilon}} |\widehat{f}(\xi)|^2\, d\xi \leq C_{n,\lambda}\ \omega_{\lambda\mu}(\tilde{t})\,\tilde{\varepsilon}\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{w_{\lambda\mu}(x)}. \tag{4.75}$$

The inequality $|\,|\tilde{t}\,\xi|-1|\leq\tilde{\varepsilon}$ is equivalent to $\frac{1-\tilde{\varepsilon}}{\tilde{t}}\leq|\xi|\leq\frac{1+\tilde{\varepsilon}}{\tilde{t}}$. Since we want to use (4.75, page 163) for each of the latter integrals in (4.74, page 162), now we set:

$$\frac{1-\tilde{\varepsilon}}{\tilde{t}}=\frac{1+\ell}{t}$$

and

$$\frac{1+\tilde{\varepsilon}}{\tilde{t}}=\frac{2+\ell}{t}$$

It follows that

$$\tilde{\varepsilon}=\frac{1}{3+2\ell}$$

and

$$\tilde{t}=\frac{2t}{3+2\ell}$$

with this setting, (4.75, page 163) becomes:

$$\int_{\frac{1+\ell}{t}<|\xi|\leq\frac{2+\ell}{t}} |\widehat{f}(\xi)|^2\, d\xi \leq C_{n,\lambda,\mu}\ \omega_{\lambda\mu}\left(\frac{2t}{3+2\ell}\right)\ \frac{1}{(3+2\ell)}\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{w_{\lambda\mu}(x)}. \tag{4.76}$$

As the proof of (4.41, page 143) for $0 < t \leq 1$ follows from (10.5.22) in [8], and since $\omega_{\lambda\mu}(t) \approx_{\lambda,\mu} 1$ on any compact subinterval of $(0,\infty)$, we can in fact assume $t \geq 3$. In this case, as $\ell \in \{0,1,...,\lceil\sqrt{t}\rceil\}$, we have that

$$\begin{aligned}
\omega_{\lambda\mu}\left(\frac{2t}{3+2\ell}\right) &= \frac{1}{\left(\frac{2t}{3+2\ell}\right)^{2\lambda+1}\left(\log\left(\frac{2t}{3+2\ell}\right)\right)^{\mu}} \\
&= C_\lambda \frac{(3+2\ell)^{2\lambda+1}}{t^{2\lambda+1}\left(\log\left(\frac{2t}{3+2\ell}\right)\right)^{\mu}} \qquad (4.77)\\
&\leq C_{\lambda,\mu}\frac{(3+2\ell)^{2\lambda+1}}{t^{2\lambda+1}\left(\log(e\,t)\right)^{\mu}}.
\end{aligned}$$

In view of (4.76, page 163) and (4.77, page 164), we can continue (4.74, page 162) as follows:

$$\begin{aligned}
&\int_{\frac{1}{t}<|\xi|\leq\frac{2+\sqrt{t}}{t}} |\widehat{f}(\xi)|^2\,\frac{1}{(1+|t\,\xi|)^M}\,d\xi \qquad (4.78)\\
\leq\;& \sum_{\ell=0}^{\lceil\sqrt{t}\rceil}\frac{1}{(2+\ell)^M}\int_{\frac{1+\ell}{t}<|\xi|\leq\frac{2+\ell}{t}}|\widehat{f}(\xi)|^2\,d\xi \\
\leq\;& C_{n,\lambda,\mu}\sum_{\ell=0}^{\lceil\sqrt{t}\rceil}\frac{1}{(2+\ell)^M}\;\omega_{\lambda\mu}\left(\frac{2t}{3+2\ell}\right)\frac{1}{(3+2\ell)}\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{w_{\lambda\mu}(x)} \\
\leq\;& C'_{n,\lambda,\mu}\sum_{\ell=0}^{\lceil\sqrt{t}\rceil}\frac{1}{(2+\ell)^M}\;\frac{(3+2\ell)^{2\lambda+1}}{t^{2\lambda+1}\left(\log(e\,t)\right)^{\mu}}\;\frac{1}{(3+2\ell)}\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{w_{\lambda\mu}(x)} \\
\leq\;& C''_{n,\lambda,\mu}\sum_{\ell=0}^{\lceil\sqrt{t}\rceil}\frac{1}{(2+\ell)^{M-2\lambda}}\;\frac{1}{t^{2\lambda+1}\left(\log(e\,t)\right)^{\mu}}\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{w_{\lambda\mu}(x)} \\
=\;& C''_{n,\lambda,\mu}\sum_{\ell=0}^{\lceil\sqrt{t}\rceil}\frac{1}{(2+\ell)^{M-2\lambda}}\;\omega_{\lambda\mu}(t)\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{w_{\lambda\mu}(x)} \\
=\;& C''_{n,\lambda,\mu}\,\omega_{\lambda\mu}(t)\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{w_{\lambda\mu}(x)}\sum_{\ell=0}^{\lceil\sqrt{t}\rceil}\frac{1}{(2+\ell)^{M-2\lambda}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad C''_{n,\lambda,\mu}\, \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)} \sum_{\ell=0}^{\infty} \frac{1}{(2+\ell)^{M-2\lambda}} \\
&= \quad C_{n,\lambda,\mu,M}\, \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)},
\end{aligned}
$$

provided $M > 2\lambda + 1$. We proved that (4.68, page 161) holds for $j = 2$.

If $j = 3$, then (4.68, page 161) becomes:

$$
\int_{\frac{2+\sqrt{t}}{t} < |\xi| \leq \frac{2+t}{t}} |\widehat{f}(\xi)|^2 \, \frac{1}{(1+|t\,\xi|)^M} \, d\xi \leq C_{n,\lambda,\mu,M} \; \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)}. \tag{4.79}
$$

First observe that $\frac{1}{(1+|t\,\xi|)^M} \leq \frac{1}{(3+\sqrt{t})^M}$ if $\frac{2+\sqrt{t}}{t} < |\xi|$, and in particular if $\frac{2+\sqrt{t}}{t} < |\xi| \leq \frac{2+t}{t}$, as in (4.79, page 165). Then apply (4.75, page 163) with:

$$
\frac{1-\tilde{\varepsilon}}{\tilde{t}} = \frac{2+\sqrt{t}}{t}
$$

and

$$
\frac{1+\tilde{\varepsilon}}{\tilde{t}} = \frac{2+t}{t}
$$

that is:

$$
\tilde{t} = \frac{2t}{4+\sqrt{t}+t}
$$

and

$$\tilde{\varepsilon} = \frac{t - \sqrt{t}}{4 + \sqrt{t} + t}.$$

Observe that, as long as $t > 1$, we have that $\tilde{t}$ is bounded above and below by absolute constants, so $\omega_{\lambda\mu}(\tilde{t}) \approx_{\lambda,\mu} 1$. In addition, for t in the same range, we have $\tilde{\varepsilon} \leq 1$.

These considerations together imply that:

$$\begin{aligned}
& \int_{\frac{2+\sqrt{t}}{t} < |\xi| \leq \frac{2+t}{t}} |\widehat{f}(\xi)|^2 \, \frac{1}{(1 + |t\,\xi|)^M} \, d\xi \qquad (4.80) \\
\leq \; & \frac{1}{(3+\sqrt{t})^M} \int_{\frac{2+\sqrt{t}}{t} < |\xi| \leq \frac{2+t}{t}} |\widehat{f}(\xi)|^2 \, d\xi \\
= \; & \frac{1}{(3+\sqrt{t})^M} \int_{\frac{1-\tilde{\varepsilon}}{\tilde{t}} < |\xi| \leq \frac{1+\tilde{\varepsilon}}{\tilde{t}}} |\widehat{f}(\xi)|^2 \, d\xi \\
\leq \; & \frac{1}{(3+\sqrt{t})^M} C_{n,\lambda,\mu} \; \omega_{\lambda\mu}(\tilde{t}) \, \tilde{\varepsilon} \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)} \\
\leq \; & C'_{n,\lambda,\mu} \, \frac{1}{(3+\sqrt{t})^M} \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)} \\
\leq \; & C''_{n,\lambda,\mu,M} \, \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)},
\end{aligned}$$

where the last inequality holds for a suitable constant $C'''_{n,\lambda,\mu,M}$ provided that $M > 4\lambda + 2$.

It only remains to prove (4.68, page 161) with $j = 4$, that is:

$$\int_{\frac{2+t}{t} < |\xi|} |\widehat{f}(\xi)|^2 \, \frac{1}{(1 + |t\,\xi|)^M} \, d\xi \leq C_{n,\lambda,\mu,M} \; \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)} \qquad (4.81)$$

We have:

$$\begin{aligned}
&\int_{\frac{2+t}{t}<|\xi|} |\widehat{f}(\xi)|^2 \, \frac{1}{(1+|t\,\xi|)^M} \, d\xi \\
&\leq \int_{\frac{2+\lfloor t\rfloor}{t}<|\xi|} |\widehat{f}(\xi)|^2 \, \frac{1}{(1+|t\,\xi|)^M} \, d\xi \\
&= \sum_{\ell=\lfloor t\rfloor+1}^{\infty} \int_{\frac{1+\ell}{t}<|\xi|\leq\frac{2+\ell}{t}} |\widehat{f}(\xi)|^2 \, \frac{1}{(1+|t\,\xi|)^M} \, d\xi
\end{aligned} \tag{4.82}$$

Next, we want to apply (4.75, page 163) to each integral in the last term of (4.82, page 167) by setting:

$$\tilde{\varepsilon} = \frac{1}{3+2\ell}$$

and

$$\tilde{t} = \frac{2t}{3+2\ell}$$

as in the case $j=2$. Now we can continue (4.82, page 167) as follows:

$$\begin{aligned}
&\int_{\frac{2+t}{t}<|\xi|} |\widehat{f}(\xi)|^2 \, \frac{1}{(1+|t\,\xi|)^M} \, d\xi \\
\leq\ & \sum_{\ell=\lfloor t\rfloor+1}^{\infty} \int_{\frac{1+\ell}{t}<|\xi|\leq\frac{2+\ell}{t}} |\widehat{f}(\xi)|^2 \, \frac{1}{(1+|t\,\xi|)^M} \, d\xi \\
\leq\ & \sum_{\ell=\lfloor t\rfloor+1}^{\infty} \int_{\frac{1+\ell}{t}<|\xi|\leq\frac{2+\ell}{t}} |\widehat{f}(\xi)|^2 \, \frac{1}{(2+\ell)^M} \, d\xi \\
\leq\ & C_{n,\lambda,\mu} \sum_{\ell=\lfloor t\rfloor+1}^{\infty} \frac{1}{(2+\ell)^M} \, \frac{1}{\left(\frac{2t}{3+2\ell}\right)^{2\lambda+1}} \, \frac{1}{(3+2\ell)} \int_{\mathbf{R}^n} |f(x)|^2 \, \frac{dx}{w_{\lambda\mu}(x)}
\end{aligned} \tag{4.83}$$

$$
\begin{aligned}
&\le C'_{n,\lambda,\mu} \sum_{\ell=\lfloor t\rfloor+1}^{\infty} \frac{1}{(2+\ell)^{M-2\lambda}} \frac{1}{t^{2\lambda+1}} \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)} \\
&= C'_{n,\lambda,\mu} \frac{1}{t^{2\lambda+1}} \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)} \sum_{\ell=\lfloor t\rfloor+1}^{\infty} \frac{1}{(2+\ell)^{M-2\lambda}} \\
&\le C'_{n,\lambda,\mu} \frac{1}{t^{2\lambda+1}} \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)} \int_{\lfloor t\rfloor}^{\infty} \frac{1}{(2+s)^{M-2\lambda}} ds \\
&\le C'_{n,\lambda,\mu} \frac{1}{t^{2\lambda+1}} \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)} \int_{t-1}^{\infty} \frac{1}{(2+s)^{M-2\lambda}} ds \\
&= C_{n,\lambda,\mu,M} \frac{1}{t^{2\lambda+1}} \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)} \frac{1}{(1+t)^{M-2\lambda-1}} \\
&\le C_{n,\lambda,\mu,M} \frac{1}{t^{M}} \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)} \\
&\le C'_{n,\lambda,\mu,M}\, \omega_{\lambda\mu}(t) \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda\mu}(x)},
\end{aligned}
$$

where the last inequality holds for a suitable constant $C'_{n,\lambda,\mu,M}$, provided that $M > 2\lambda + 1$. By choosing any $M > 4\lambda + 2$ (as required after (4.80, page 166)), we conclude the proof of (4.41, page 143), and of the claimed statement.

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Vita

Marco Annoni was born January 29, 1981 in Como, Italy. After completing his early education through high school at "Liceo Scientifico Statale Paolo Giovio" in Como (Italy, 2000), he gained a "Laurea in matematica" at "Università degli Studi dell'Insubria" (2003, Como, Italy), a maitrise in Mathematics at "Université de Toulon et du Var" (Toulon, France, 2004), and a "Laurea specialistica in matematica" at "Università degli Studi dell'Insubria" (Como, Italy, 2005). Marco Annoni is currently a graduate student at the University of Missouri at Columbia. He expects to obtain his Ph.D. in mathematics in May 2010.