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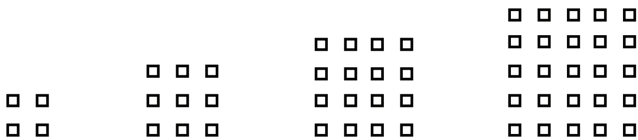
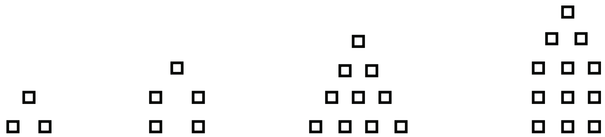
Polygonal Numbers

In mathematics we use square numbers often. In elementary school we memorize the squares of the natural numbers 1 through 10, and later use them to find the radius of a circle whose area is 9π, or to figure out that $\sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}\sqrt{}$. So we become fairly comfortable with square numbers. We may even have seen that they can be drawn as actual squares.



4 9 16 25

Square numbers, however, are not the only numbers that form geometric shapes. Numbers can also be triangular, pentagonal, or hexagonal, etc. These numbers that can be represented in geometric form, or polygonal numbers, have interested people for millennia, being traced back to the time of Pythagoras and the Pythagorean school (c. 572 - 497 B.C.) (Heath 1921, vol.I, p.67). It's easy to see how people who probably represented numbers in a strictly visual way, as quantities of pebbles in the sand, or dots arranged in a geometric pattern, could classify numbers as triangular, square, or pentagonal, etc., according to the shapes that were created by the arrangement of the objects (Burton 2003, p.90; Heath 1921, vol.I, p.76). For example, if we visualize the numbers 3, 5, 10, and 12, we see that the numbers 3 and 10 can be arranged to form equilateral triangles, while 5 and 12 make equilateral pentagons. Thus, 3 and 10 are examples of triangular numbers, while 5 and 12 are examples of pentagonal numbers.



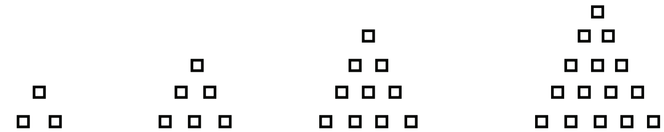
3 5 10 12

The triangular number $1 + 2 + 3 + 4 = 10$ was especially important to the Pythagoreans – it was the symbol on which they swore their oaths. Called the "tetractys" or "holy fourfoldness," it stood for the "four elements: fire, water, air, and earth" (Burton 2003, p. 187).

About 600 years after Pythagoras, two students of Pythagorean number philosophy, Nicomachus of Gerasa (c. 100 A.D.) and Theon of Smyrna (c. 130 A.D.), each included discussions of polygonal numbers in their collections of writings related to Plato's basic curriculum (Heath 1910, p.2; Katz 1998, p.171). Nicomachus' discussion of polygonal numbers is contained in his *Introduction to Arithmetic*, book II, chapters VII - XVIII. Theon's discussion is in the section on Arithmetic in his book *Mathematics Useful for Understanding Plato*. Our introduction to the properties of polygonal numbers will be guided by the writings of these two men.

An Introduction to the Properties of Polygonal Numbers

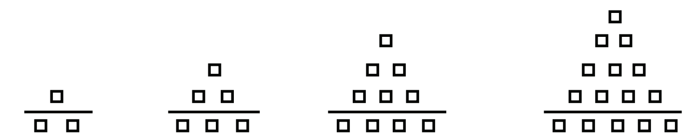
We begin with a look at the triangular numbers. Nicomachus tells us in chapter VIII, book II of his *Introduction to Arithmetic* that "a triangular number is one which, when it is analyzed into units, shapes into triangular form the equilateral placement of its parts in a plane" (meaning that a triangular number can always be made to form an equilateral triangle when it is visualized using objects arranged in a plane). Examples of this are 3, 6, 10, and 15.



3 6 10 15

Looking at the diagrams, we see that each new triangular number is obtained from the previous number by adding another row containing one more small square than the previous row had. Thus, the number added each time increases by one, and so we also notice that the triangular numbers are formed from successive additions of the natural numbers, beginning with 1.

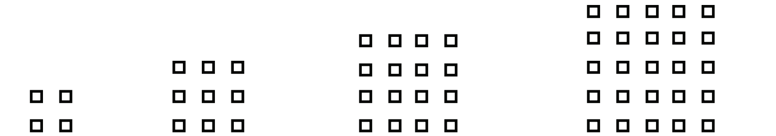
Theon uses the term "gnomon" to describe the numbers that are added to one polygonal number to get the next, and he notes that the gnomons of triangular numbers have a special property; "the sides of any triangle always have as many units as are contained in the last gnomon added to it" [Theon 1979, Arith.XXIIII].



$3 = 1 + 2$ $6 = 3 + 3$ $10 = 6 + 4$ $15 = 10 + 5$

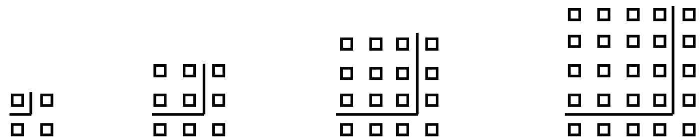
It is interesting to note that both Nicomachus and Theon call the number 1 a "potentially triangular number". While it was common for the ancients to consider the number 1 (or unity) not to be a number in itself, but rather that which measures other things, to Theon of Smyrna the number 1 is considered the "Mystical Monad" -- "much more than the quantified unit: it is comparable to God and is the seed or seminal essence of everything which exists." Theon tells us that 1 "is not a triangle in fact… but in power, for in being as the seed of all numbers, unity also possesses the faculty of engendering the triangle. When it is added to the number 2, it gives birth to the triangle…[Theon 1979, Arith.XXIII].

Squares can also be a result of adding numbers together. The square numbers are formed when, instead of adding up every natural number in succession, we add up just the odd numbers.



4 = 1 + 3 9 = 1+ 3 + 3 16 = 1 + 3 + 5 + 7 25 = 1 + 3 + 5 + 7 + 9

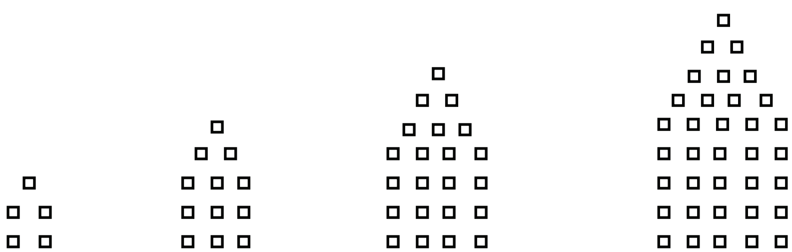
The gnomons that enlarge the squares are usually added around the outside of the previous square in an L-shape.



4 = 1 + 3 9 = 4 + 5 16 = 9 + 7 25 = 16 + 9

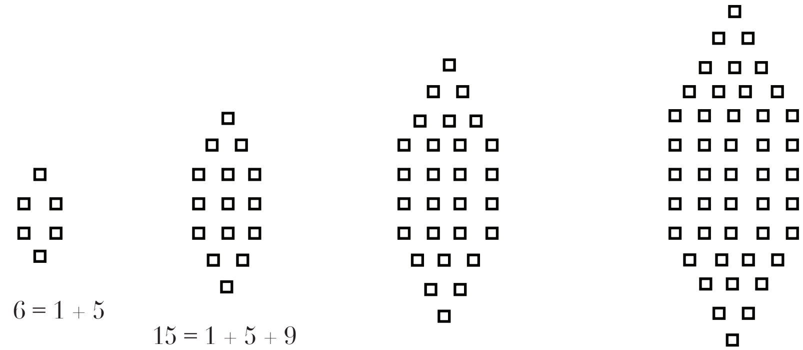
In chapter IX, book II of his *Introduction*, Nicomachus notices that for square numbers, "the side of each consists of as many units as there are numbers taken into the sum to produce it" (for example, 1 + 3 + 5 + 7 + 9 is the sum of five terms, and the number 25 has 5 units on a side). He later comments that this is true for all the polygonal numbers. Theon observes that the squares are alternately odd and even [9, Arith.XX], an interesting result of adding up odd numbers.

To get the pentagonal numbers, we add up every third number in the sequence of natural numbers,



5 = 1 + 4 12 = 1 + 4 + 7 22 = 1 + 4 + 7 + 10 35 = 1 + 4 + 7 + 10 + 13

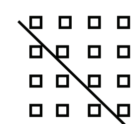
and for the hexagonal numbers we add up every fourth number.



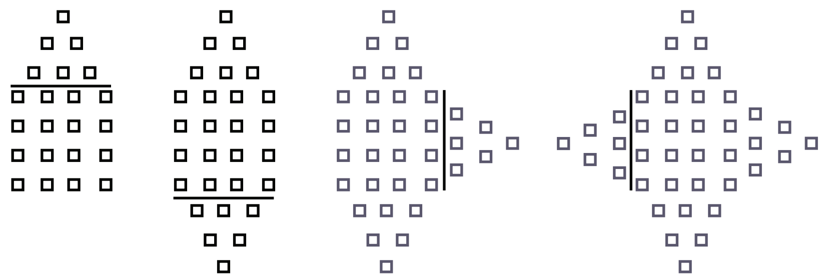
6 = 1 + 5 15 = 1 + 5 + 9 28 = 1 + 5 + 9 + 13 45 = 1 + 5 + 9 + 13 + 17

So in general, in order to form the triangular numbers, we added up successive natural numbers, for the square numbers we added had a difference of two, for the pentagonal numbers we added numbers with a difference of three, and so on, always starting with 1. Theon states this in section XXVII, "Thus generally in all the polygons, in removing two units from the number of angles, one will have the quantity by which the numbers serving to form the polygon must increase" (triangular numbers have a difference of 3 - 2 = 1, squares have a difference of 4 - 2 = 2, pentagonal numbers have a difference of 5 - 2 = 3, etc.).

But it isn't necessary to go all the way back to the natural numbers, beginning with 1, to form the polygonal numbers. We can use the triangular numbers as "building blocks" too. Another way to get a square number is to add two consecutive triangular numbers together. In fact, Theon tells us that the sum of two consecutive triangular numbers always equals the square number whose side is the same as the side of the larger of the two triangles [9, Arith.XXVIII].



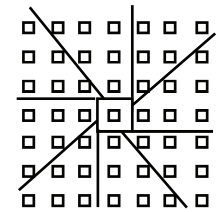
Nicomachus then extends this concept to another method for the formation of polygonal numbers in general. He notes that a "triangle joined with the next square makes a pentagon," and that the same is true for adding triangles to pentagons to produce hexagons, “and so on to infinity” (Nicomachus 1926, II.12).



He organizes this information in a table in which each new polygon is formed by adding the polygon immediately above it to the triangle in the previous column (Nicomachus 1926, II.12). For example, the hexagonal number 15 is created from the addition of the 12 above it, and the triangular number 3 at the top of the previous column.

Triangles	1	**3**	6	10	15	21	28	36	45	55
Squares	1	4	9	16	25	36	49	64	81	100
Pentagonals	1	5	**12**	22	35	51	70	92	117	145
Hexagonals	1	6	**15**	28	45	66	91	120	153	190
Hentagonals	1	7	18	34	55	81	112	148	189	235

One more way to obtain square numbers is found in the writings of Plutarch, a contemporary of Nicomachus of Gerasa and Theon of Smyrna. In his Platonic Questions, question V paragraph 2, he states that multiplying 8 times any triangular number and adding 1 results in a square.



Diophantus Extends the Study of Polygonal Numbers With His Treatise *On Polygonal Numbers*

Now that we’ve looked at the basic properties of polygonal numbers we are ready to explore a more proof-oriented approach to the study of polygonal numbers, as found in the writings of Diophantus. Although Diophantus is especially known for his more famous work, the Arithmetica, he also wrote about polygonal numbers.

About 150 years after Nicomachus and Theon, Diophantus (c. 250 A. D.) wrote his treatise *On Polygonal Numbers*, in which he proved that the sum of any arithmetic sequence is a polygonal number. In the process of doing this he also proved a generalization of Plutarch’s propition mentioned above, applying it not only to triangular numbers but to any polygonal number. Unlike Nicomachus of Gerasa and Theon of Smyrna, who present their statements without proof, Diophantus uses geometric proofs in which he represents the unknown numbers being discussed as lengths.

Unfortunately only a fragment of *On Polygonal Numbers* survives today. We focus our look at Diophantus’ work *On Polygonal Numbers* by studying the first and last complete proofs that are contained in the fragment we have today. The first proposition is a “preliminary,” which is later used in the last complete proof in the fragment, that the sum of any arithmetic sequence is a polygonal number. We begin with the first proof contained in the fragment.

The First Proof in the Existing Fragment of *On Polygonal Numbers* (Diophantus 1890, p. 298; Heath 1910, p. 247)

Diophantus says:

Given three numbers with a common difference, then 8 times the product of the greatest and middle, when added to the square of the least gives a square, the side of which is the sum of the greatest and twice the middle number.

[That is, given three numbers such that (greatest) - (middle) = (middle) - (least), then 8(greatest)(middle) + $(\text{least})^2$ = $(\text{greatest} + 2\text{middle})^2$.]

[Proof:]

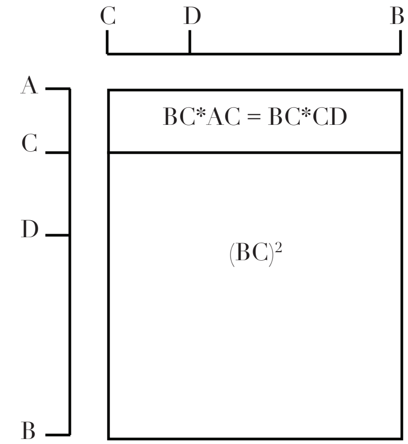
Let the three numbers that lie an equal distance apart [that have a common difference] be AB, BC, BD. *[1]

It will be shown that $8(AB\times BC) + (BD)^2 = (AB + 2BC)^2$.

By hypothesis, AC = CD, AB = BC+CD, and BD = BC - CD.

*[1] The diagrams used in this paper to illustrate Diophantus' proofs are based on the diagrams that Sir Thomas Heath uses in his translation.

[Now, notice that $AB\times BC = (BC)^2 + BC\times CD$:



]

So, we have $8(AB\times BC) = 8(BC)^2 + 8(BC\times CD)$.

And since $8(AB\times BC) = \underbrace{4[(BC)^2 + (BC\times CD)]} + 4(BC)^2 + 4(BC\times CD)$,

thus $8(AB\times BC) = 4(AB\times BC) + 4(BC)^2 + 4(BC\times CD)$.

Adding $(BD)^2$ to both sides of the equation gives

$$8(AB\times BC) + (BD)^2 = 4(AB\times BC) + 4(BC)^2 + \underbrace{4(BC\times CD) + (BD)^2},$$

[and from Euclid II.8 4(BC×CD) + (BD)2 = (AB)2], *[1]

so we have 8(AB×BC) + (BD)2 = 4(AB×BC) + 4(BC)2 + (AB)2. {1}

*[1] Euclid II.8 states that "If a straight line [BC] be cut at random, four times the rectangle contained by the whole and one of the segments [BC×CD] together with the square on the remaining segment [BD] is equal to the square described on the whole [BC] and the aforesaid segment as on the straight line" (Heath 1956, p. 389).

Draw AE = BC.

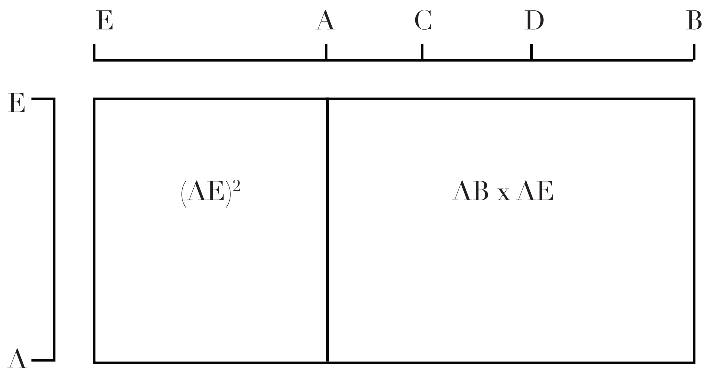
Now we focus on the $4(AB\times BC) + 4(BC)^2$, on the right side of this equation.

Since AE = BC, we get $4(AB\times BC) + 4(BC)^2 = 4(AB\times AE) + 4(AE)^2$.

So $\quad 8(AB\times BC) + (BD)^2 = 4(AB\times BC) + 4(BC)^2 + (AB)^2 \qquad \{1\}$

becomes $\quad 8(AB\times BC) + (BD)^2 = 4(AB\times AE) + 4(AE)^2 + (AB)^2. \quad \{2\}$

[Notice that $(AB\times AE) + (AE)^2 = BE\times AE$.



So $\quad 8(AB\times BC) + (BD)^2 = \underbrace{4(AB\times AE) + 4(AE)^2} + (AB)^2. \quad \{2\}$

becomes $\quad 8(AB\times BC) + (BD)^2 = \underbrace{4(BE\times AE) + (AB)^2}.$

Thus [by Euclid II.8] $8(AB\times BC) + (BD)^2 = (BE + AE)^2. \qquad \{3\}$

Now, [since AE = BC, and A is between B and E], notice that

$$(BE) + AE = (AB + AE) + AE = AB + 2BC.$$

[So we have $8(AB\times BC) + (BD)^2 = (AB + 2BC)^2$, as desired.

Since AB, BC, and BD were the three numbers with common difference, we have that

$$8(\text{greatest})(\text{middle}) + (\text{least})^2 = (\text{greatest} + 2\text{middle})^2.]$$

Q.E.D.

Diophantus uses the above proposition in his proof that the sum of any arithmetic sequence is a polygonal number. In the next section we will explore this proof, which is the last complete proof in the fragment of *On Polygonal Numbers* (1890 p.307; Heath 1910, p.251).

Diophantus uses the following pieces of information as "preliminaries":

i) The proposition proved above, that given three numbers with a common difference [such that (greatest) - (middle) = (middle) - (least)], we have 8(greatest)(middle) + (least)2 = (greatest + 2middle)2.

ii) His generalization of the proposition from Plutarch that given any number of terms of an arithmetic sequence beginning with 1, then (sum of all the terms)(8)(common difference) + (common difference $-2)^2 = [$(common difference) $(2 \times$ number of terms $-1) + 2]^2$.

iii) The first statement contained in the fragment of *On Polygonal Numbers*, that "all [natural] numbers from 3 upwards are polygonal numbers, containing as many angles as they have units. . .thus 3 is a triangular number, 4 a square number, 5 a pentagonal number, etc."

iv) A definition of a polygonal number that he attributes to Hypsicles (c. 180 B.C.), which is essentially the same as what we stated earlier as a general rule for forming the triangular, square, and pentagonal numbers, etc. Hypsicles' definition is, "If as many numbers as you please be set out at equal interval from 1, and the interval is 1, their sum is a triangular number; if the interval is 2, a square; if 3, a pentagonal; and generally the number of angles is greater by 2 than the interval" (Nicomachus 1926, p.246).

The Last Proof in the Existing Fragment of *On Polygonal Numbers* (Diophantus 1890 p.307; Heath 1910, p.251)

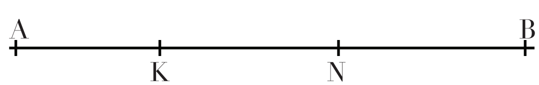
Diophantus says:
The above being premised, I say that if there be as many terms as we please in an arithmetic sequence, beginning with 1, the sum of the terms is polygonal.

[Proof:]

Let AB be any [natural] number [other than 1] that is a member of an arithmetic sequence that begins with 1.

[Say the arithmetic sequence is
$1, 1 + (AB - 1), 1 + 2(AB -1), \ldots.$]

Let AK = 1 unit, and KN = 2 units.



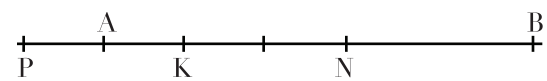
Applying Diophantus' generalization of the proposition from Plutarch (ii above), that

$$(\text{sum of all the terms})(8)(\text{common difference}) + (\text{common difference} - 2)^2 = [(\text{common difference})(2 \times \text{number of terms} - 1) + 2]^2,$$

[and noticing that AB - 1 = BK = the common difference in this arithmetic sequence,] we have

$$(\text{sum of terms})\times 8(BK) + (BN)^2 = [(BK)(2\times\text{number of terms} - 1) + 2]2. \quad \{1\}$$

Draw AP = AK = 1 unit.



This gives us KP = 2, and KN = 2,

and thus PB, BK, BN have a common difference [PB - BK = BK - BN = 2].

So we can put PB, BK, and BN as "greatest," "middle," and "least," respectively, into the equation stated in (i) above,

that $$8(\text{greatest})(\text{middle}) + (\text{least})^2 = (\text{greatest} + \text{middle})^2,$$

and we get $$8(PB)(BK) + (BN)^2 = [PB + 2(BK)]^2. \quad \{2\}$$

[Now Diophantus compares equations {1} and {2}. In order to make the right sides of the equations more similar he makes the following remark relating to equation {2}.]

Since PB - PK = BK, we have PB + 2(BK) = PK + 3(BK) = 2 + 3(BK).

So equation {2} becomes $8(PB)(BK) + (BN)^2 = [3(BK) + 2]^2$.

Focusing on PB, we remember that PB = 1 + AB, so PB is the sum of the first two terms of the arithmetic sequence
1, 1 + (AB - 1), 1 + 2(AB -1), . . .

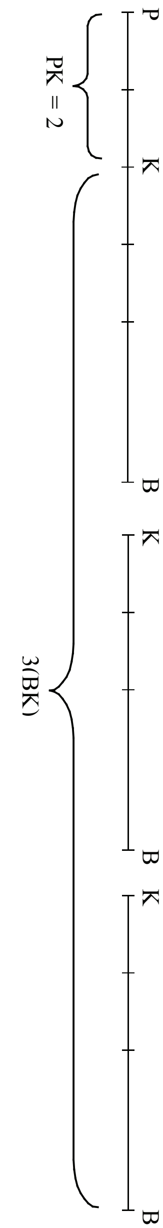
Thus, PB corresponds to the "(sum of terms)" in equation {1} when the number of terms is 2.

So $(\text{sum of terms})\times 8(BK) + (BN)^2 = [(BK)(2\times\text{number of terms} - 1) + 2]^2$ {1}

becomes $(PB)\times 8(BK) + (BN)^2 = [(BK)(2\times 2 - 1) + 2]^2$.

[PB is a polygonal number since AB is a natural number other than 1 (so at least 2), and AP = 1,
so PB ≥ 3, and by the fact stated in (iii) above that "all [natural] numbers from 3 upwards are polygonal numbers, containing as many angles as they have units."]

Now, the number of angles in PB = the number of units in PB
= BK + PK = BK + 2
= (common difference) + 2.



Therefore, since

1) PB is *any* polygonal number, and also the sum of the first two terms of any arithmetic sequence beginning with 1 [since AB is arbitrary],

2) and the sum of all the terms of the arithmetic progression is "subject to the same law" as PB [from the fact that PB corresponds to the (sum of terms) in equation {1} when the number of terms is 2],

we have that
the sum of all the terms of the arithmetic sequence must also be a polygonal number. And it must have the same number of angles as PB has [(common difference) + 2].

So we have shown that if there be as many terms as we please in an arithmetic sequence beginning with 1, the sum of the terms is polygonal.

Q.E.D.

Recent Writings Concerning Polygonal Numbers

Polygonal numbers have continued to interest people throughout history. Fermat, Lagrange, Gauss, and Cauchy have all explored polygonal numbers. For example, over thirteen centuries after Diophantus, Pierre de Fermat (1601-1635) made this note in the margin of his copy of a translation of Diophantus' writings (1890):

> I have been the first to discover a most beautiful theorem of the greatest generality, namely this: Every [natural] number is either a triangular number or the sum of two or three triangular numbers; every [natural] number is a square or the sum of two, three, or four squares; every [natural] number is a pentagonal number or the sum of two, three, four, or five pentagonal numbers; and so on ad infinitum, for hexagons, heptagons, and any polygons whatever, the enunciation of this general and wonderful theorem being varied according to the number of angles. (p. 188)

The part of the theorem relating to squares was first proved by Lagrange in 1770, and the part relating to triangular numbers was proved by Gauss in 1801 (Burton 2003 p.93). Cauchy finally proved the theorem in its entirety in the 1800's (Heath 1910 p.188).

Polygonal numbers continue to interest people even today. Researchers are exploring the study of polygonal numbers in the classroom as a way to introduce students to number theory (Abramovich, Fuji, & Wilson). And journals such as the *American Mathematical Monthly*, *The Joy of Mathematics*, and *Quantum* have recently published articles dealing with polygonal numbers. The authors of these articles are continuing the study of a subject that has interested people for over two millennia, and has included such famous mathematicians as the Pythagoreans, Nicomachus of Gerasa, Theon of Smyrna, Diophantus, Fermat, Lagrange, Gauss and Cauchy.

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The citations for the translations of the writings of Nicomachus of Gerasa and Theon of Smyrna appear in the body of this paper as follows: [8, II.18] refers to section 18 of book II of Nicomachus of Gerasa's *Introduction to Arithmetic* [9, Arith. XIII] refers to the 23rd section of the chapter on Arithmetic in Theon of Smyrna's *Mathematics Useful for Understanding Plato*.