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George Berkeley's Mathematical Philosophy And The Calculus

Berkeley's Life

As noted by Ronald Calinger (Calinger 1982), George Berkeley was born (1685) in County Kilkenny, Ireland; but, because of his ancestry, always considered himself an Englishman. Young Berkeley was enrolled in Kilkenny College (1696) and later Trinity College (1700) where he earned his Bachelor and Masters of Arts in 1704 and 1707 respectively. By 1710, Berkeley was ordained an Anglican priest and was lecturing at Trinity College on divinity, Greek and Hebrew. Throughout his years as a teacher and later an administrator, Berkeley wrote many influential philosophical texts, the most notable of which are *An Essay towards a New Theory of Vision* (hereafter referred to as *Theory*) and *Of the Principles of Human Knowledge* (hereafter referred to as *Principles*). In 1728 Berkeley married and shortly thereafter sailed for America where he and his new family resided in Rhode Island where he began a study group and set afoot the American philosophical movement with Samuel Johnson. While in the states, he wrote *Alciphron: or, the Minute Philosopher* (1732) and began, the subject of this essay, *The Analyst; or, A Discourse Addressed to an Infidel Mathematician* (hereafter referred to as *The Analyst*) which wouldn't be published until he moved back to England. The Berkeley family returned to England in 1734, the same year in which he was consecrated Anglican Bishop of Cloyne in Dublin and in which he published *The Analyst*. Together with his wife and six children, Berkeley lived in Cloyne until 1752 when

he went to live in Oxford until his death the next year, 1753.

Berkeley's Mathematical Beliefs / Writings

The Analyst was not the first work of Berkeley's to consider mathematics and its falsities. Berkeley became interested in mathematics while still at Trinity College and this attraction appears in many of his greatest texts. In particular, Berkeley's *Principles* contains ten pages regarding mathematics and its principles. Therein, he makes an argument regarding infinity, which we will consider in more detail for it relates greatly to *The Analyst* and its arguments.

Within *Principles*, Berkeley proceeds to discuss the "infinite divisibility of finite extension." To imagine the notion of extension, he asks the reader in *Theory* to envision a line free from any distinguishing characteristics—no color, shape, magnitude, nor lack thereof; this he states is perfectly incomprehensible. However in *Principles*, he discusses finite extension, such as lines and shapes, and how the human mind might conceive of this idea given the fact that,

> though [infinite divisibility of finite extension], is not expressly laid down either as an axiom or theorem… [it is] thought to have so inseparable and essential a connexion with the principles and demonstrations in Geometry that mathematicians never admit it into doubt, or make the least question of it (Berkeley 1901).

Berkeley believes that any idea, if not explicitly perceived by the human mind or senses is not possible and should not provide a base for science. In the remainder of *Principles*, he eventually reasons that, "when we say that a line is *infinitely divisible*, we mean (if we mean anything) *a line which is infinitely great*" thus,

> upon a thorough examination it will be found that in any instance it is necessary to make use of or conceive infinitesimal parts of finite lines… nay, it will be evident this is never done, it being impossible (Berkeley 1901).

So, Berkeley does not believe in any aspect of infinity—whether infinitely large or infinitely small, even saying that geometry is illegitimate if that science bases all understanding on the concept of infinity. This idea will be expressed more thoroughly in *The Analyst* as Berkeley's main backlash at the theoretical basis of the infinitesimal / differential calculus.

Newton's Mathematical Beliefs / Writings

Sir Isaac Newton was a gifted youngster, able to teach himself Euclid's *Elements* and other legendary mathematical texts, and he excelled quickly, taking a keen interest in infinite sums. It was through these infinite sums that the Englishman developed his theories of infinitesimals, which he referred to as "fluxions". These fluxions were the basis for Newton's calculus. Fluxional notation is most fully discussed in Newton's published posthumous works (Newton 1737), where he defines fluents and fluxions.

> Now those quantities which I consider as gradually and indefinitely increasing, I shall hereafter call *Fluents*, or Flowing Quantities, and shall represent them by the final letters of the alphabet v, x, y, and z …And the velocities by which every Fluent is increased by its generating motion (which I may call *Fluxions*, or simply Velocities, or Celerities,) I shall represent by the same letters pointed thus, y, x, y, and z; …(1737)

Fluents can be thought of as functions of time and their fluxions as derivatives of the fluents with respect to time.

Fluents and their fluxions, as demonstrated, were defined in terms of the natural world—flowing quantities such as water and their velocities—which Newton was so interested in describing. It was due to their connections with the physical world that some pure mathematicians of the time struggled to comprehend Newton's definitions, and his notation was easily confused, leaving people to sometimes wonder if the small spot above a variable was to represent it as a fluxion or a stray pen mark. Nevertheless, his calculus of fluents and fluxions was taught in English schools for years.

Leibniz's Mathematical Beliefs / Writings

The mathematical beliefs of Leibniz were not widely known to the mathematical community, because like Berkeley, Leibniz was primarily a philosopher and his texts in philosophy were more widely read. Though mathematics was not his primary subject, Leibniz proved he was a brilliant mathematician by developing a simple system for naming and using infinitesimals.

Leibniz's calculus was based upon the idea of a difference of two values. Thus, his infinitesimals were referred to as differentials. Basically, as the difference between two values becomes less and less, as the two values become closer and closer, that difference tends toward a single value which he expressed as an infinitesimally small difference. This idea can also be thought of as a differential change in a quantity; using his notation, a differential change in the quantity x would be expressed as dx.

This simple, unique notation is now seen as much easier to deal with, and the idea of the difference is much more mathematically oriented and clear than that of Newton's fluxions. For this reason, calculus classes across the world are now taught using Leibniz's notation.

Calculus Controversies

Calculus was founded amongst two primary controversies. First was the debate regarding the true 'inventor' of the calculus. While Newton was the most prominent mathematician and physicist of his time, Leibniz (like Berkeley) grew a reputation as a brilliant philosopher. This led many people of the time, including the Royal Society (a very influential, independent scientific academy in England), to initially believe Newton when he claimed to be the creator of the calculus. The Royal Society gave credit to Newton for developing the calculus and decreed that Leibniz was influenced by his correspondence with Newton in developing his version of the calculus. This decree stood for many years until recently, when the Royal Society withdrew it. More recent research has shown that Leibniz's ideas regarding the calculus were printed by John Craig in 1685, while Newton's *Principia* and his notions of fluxions were not published until 1693 as a part of John Wallis's *Algebra* (Smith 1956, p.627).

Introduction to *The Analyst*

The second major debate regarding the 'early' calculus lies in the impreciseness of the calculus' foundations. Bernhard Nieuwentijt (Child 1920, p.145) wrote early criticisms of the concept of the infinitely small. He believed that neither Newton nor Leibniz had sufficiently defined the art of infinitesimals. One of his main arguments directed toward Leibniz included the lack of a distinction between zero and an infinitely small difference. As Nieuwentijt before him, Berkeley also lashed out at the foundations of Leibniz's infinitesimal calculus as well as Newton's fluxional calculus in his 1734 treatise *The Analyst*. Towards the beginning of *The Analyst*, Berkeley makes the statement:

> It hath been an old remark that Geometry is an excellent Logic. And it must be owned, that when the Definitions are clear; when the Postulata cannot be refused, nor the Axioms denied; ... there is acquired a habit of reasoning, close and exact and methodical: which habit strengthens and sharpens the Mind, and being transferred to other Subjects, is of general use in the inquiry after Truth. But how far this is the case of our Geometrical Analysts, it may be worth while to consider (Berkeley 1901).

Berkeley claims that the founding principles of the calculus are ambiguous and, while possibly on the right track, must be revised in order to obtain the exactness of the ancients in their geometrical proofs. Though not a mathematician, Berkeley made very relevant comments pertaining directly to the calculus of Leibniz and Newton and caused these systems to be revised and grounded more certainly by specific definitions.

The 'infidel mathematician' of *The Analyst*'s title is most likely the renowned astronomer, Sir Edmund Halley who was the financier of Newton's *Principia*. Halley also contributed

to *Principia*, proving himself a very clever mathematician. However, he is considered an "infidel" by Berkeley because he "persuaded a mutual friend that the doctrines of Christianity were inconceivable" (Katz 1998). In the first sentence of *The Analyst*, Berkeley discusses his knowledge of Halley's proficiency in mathematics, but states that Halley and

> too many more of the like Character are known to make such undue Authority, to the misleading of unwary Persons in matters of the highest Concernment, and whereof your mathematical Knowledge can by no means qualify you to be a competent Judge (Berkeley 1901).

He then proceeds to explain the object of his essay,

> I shall claim the privilege of a Free-Thinker; and take the Liberty to inquire into the Object, Principles, and Method of Demonstration admitted by the Mathematicians of the present Age, with the same freedom that you presume to treat the Principles and Mysteries of Religion (Berkeley 1901).

Berkeley essentially claims that mathematicians of the time blindly follow the thoughts and ideas of a select few on the basis of faith, just as Christians follow God on the basis of faith. However he claims, the foundation on which the ideas of the mathematician's faith lie is unstable and in need of repair.

Berkeley demonstrates his knowledge of fluxions by examining their definitions and pointing out a few questionable arguments, the most notable of which is that "the Velocities of the Velocities, the second, third, fourth, and fifth Velocities, &c. exceed, if I mistake not, all Humane Understanding."

The Analyst Explication

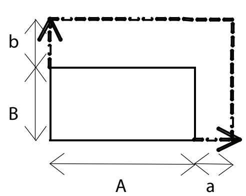
Below appear excerpts from *The Analyst* with my short introductions and explications. My comments will be in a smaller font within brackets so as to distinguish them from Berkeley's text which will be in 'normal' font. Also, the arguments of Berkeley begin with a proof in the manner of either Newton or Leibniz after which is Berkeley's revised proof of the same statement. These initial proofs are quoted from Berkeley's text, not the original text of its author.

Newton's Rectangle Problem

The following proof can be found as Case I in Newton's famous *Principia*, Book II, Lemma II. In his rebuttal of Newton's proof, Berkeley demonstrates the ease of his own solutions and the simplicity of his own mathematical ideas.

Original Newtonian Proof [Repeated by Berkeley]

Suppose the Product or Rectangle AB increased by continual Motion: and that the momentaneous Increments of the Sides A and B are a and b. When the Sides A and were deficient, or lesser by one half of their Moments, the Rectangle was: $\overline{A - \frac{1}{2}a} \times \overline{B - \frac{1}{2}b}$, [$(A - \frac{1}{2}a)(B - \frac{1}{2}b)$] i. e., $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$). And as soon as the Sides A and B are increased by the other two halves of their Moments, the Rectangle becomes: [$(A + \frac{1}{2}a)(B + \frac{1}{2}b)$] or $AB + \frac{1}{2}aB + \frac{1}{2}bA + 1/4ab$.
From the latter Rectangle subduct [subtract] the former, and the remaining Difference will be $aB + bA$. Therefore the Increment of the Rectangle generated by the entire Increments a and b is $aB + bA$. *Q.E.D.*



Berkeley's Rebuttal

But it is plain that the direct and true Method to obtain the Moment or Increment of the Rectangle AB, is to take the Sides as increased by their whole Increments, and so multiply them together, $A + a$ by $B + b$, the Product whereof $AB + aB + bA + ab$ is the augmented Rectangle; whence if we subduct AB, the Remainder $aB + bA + ab$ will be the true Increment of the Rectangle, exceeding that which was obtained by the former illegitimate and indirect Method by the Quantity ab. And this holds universally be the Quantities a and b what they will, big or little, Finite or Infinitesimal, Increments, Moments, or Velocities. Nor will it avail to say that ab is a Quantity exceeding small: Since we are told that *in rebus mathematicis errores quàm minimi non sunt contemnendi* [For errors, however small, are not to be neglected in Mathematics. This phrase is found in the introduction to Newton's *Quadrature of the Curve*.].

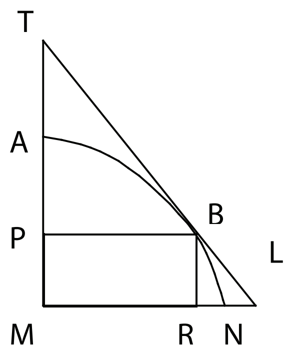
Triangle Problem

There were many ideas held as common knowledge by mathematicians of the time. In fact, this is one of Berkeley's complaints of mathematics as he mentioned the "men who pretend to believe no further than they can see." By this he makes reference to an earlier comment stating that many mathematicians when confronted with a greater mind will accept the ideas of this person without thinking about these ideas for themselves. This greater mind he refers to is Newton; however the following problem represents a commonly held belief in which Berkeley finds fault.

Original Problem [Repeated by Berkeley]

In order therefore to clear up this Point, we will suppose for instance that a Tangent is to be drawn to a Parabola, and examine the progress of this Affair, as it is performed by infinitesimal Differences.

Let AB be a Curve, the Abscisse [x-axis] $AP = x$, the Ordinate [y-axis] $PB = y$, the Difference of the Abscisse $PM = dx$, the Difference of the Ordinate $RN = dy$. Now by supposing the Curve to be a Polygon [the curve is assumed to be a polygon with infinitely many sides, each of which is of infinitely small length], and consequently BN, the Increment or Difference of the Curve [one side of the polygon], to be a straight Line coincident with the Tangent [TL], and the differential Triangle BRN to be similar to the triangle TPB



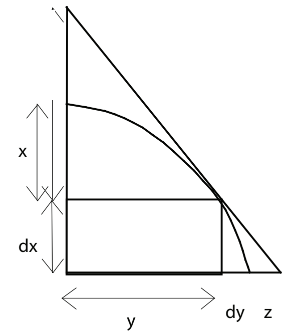
Berkeley's Illustration

[Now, according to the picture, triangle TPB is clearly not similar to BRN. In fact BRN is not even a triangle and triangle TPB clearly is similar to triangle BRL. This point Berkeley objects to as well, but allow me to clarify the reasoning behind Newton's proof. Remember, the text refers to a differential triangle BRN and we have assumed the curve to be part of a polygon with a side coincident (a member of) the line tangent to the curve. Thus at point B, the curve ABN is supposed to be a straight line (side of the polygon) of infinitesimal length which is equal to the line TL at point B; so on the infinitesimal scale, triangle BRN is indeed similar to triangle TPB.]

the Subtangent [the projection of the tangent upon the x-axis] PT is found a fourth Proportional to $RN : RB : PB$: that is to $dy : dx : y$. Hence the Subtangent will be

$$\frac{y \cdot dx}{dy}$$

[in modern notation, $\frac{RN}{RB} = \frac{PB}{PT}, \frac{dy}{dx} = \frac{y}{sub\tan gent}$, and subtangent = $\frac{y \cdot dx}{dy}$].



Clarifying Illustration

Berkeley's Rebuttal

But herein there is an error arising from the aforementioned false supposition [remember, Newton supposes the curve to be a polygon], whence the value of PT comes out greater than the Truth [as will be demonstrated]: for in reality it is not the Triangle RNB but RLB which is similar to PBT, and therefore (instead of RN) RL should have been the first term of the Proportion, i. e. RN + NL, i. e. dy + z: whence the true expression for the Subtangent should have been $\frac{y \cdot dx}{dy + z}$.

There was therefore an error of defect in making dy the divisor [the *first* error]: which error was equal to z, i. e. NL the Line comprehended between the Curve and the Tangent. Now by the nature of the Curve yy = px [because the curve is supposed to be a parabola], supposing p to be the Parameter [a constant], whence by the rule of Differences [Leibniz]

$$2y \cdot dy = p \cdot dx \text{ and } dy = \frac{p \cdot dx}{2y}.$$

[The Rule of Differences simply means taking the derivative; thus, by applying $\frac{d}{dy}$ to each side of our original equation, yy = px, we obtain 2y = p $\cdot \frac{dx}{dy}$, where 2y is the (modern) derivative of y^2 with respect to y. Algebra then takes us to the stated conclusion.]

But if you multiply $y + dy$ by itself, and retain the whole Product without rejecting the Square of the Difference, it will then come out, by substituting the augmented Quantities in the Equation of the Curve:

$$dy = \frac{p \cdot dx}{2y} - \frac{dy \cdot dy}{2y}$$

[So, by finding the moment of each side of the equation for the curve as above with Newton's rectangle problem $(y + dy)^2 = p(x + dx)$. Now through algebra, $y^2 + 2ydy + (dy)^2 = px + pdx$ and since $y^2 = px$, $2ydy + (dy)^2 = pdx$, $2ydy = pdx - (dy)^2$, and $dy = \frac{pdx}{2y} - \frac{(dy)^2}{2y}$]

There was therefore an error of excess [the *second* error] in making

$$dy = \frac{p \cdot dx}{2y},$$

which followed from the erroneous Rule of Differences. And the measure of this second error is:

$$\frac{dy \cdot dy}{2y}$$

[Berkeley will describe his procedure for this final conclusion shortly]. Therefore the two errors being equal and contrary destroy each other; the first error of defect being corrected by a second error of excess.

If you had committed only one error, you would not have come at a true Solution of the Problem. But by virtue of a twofold mistake you arrive, though not at Science, yet at Truth. For Science it cannot be called, when you proceed blindfold, and arrive at the Truth not knowing how or by what means. To demonstrate that z is equal to ,

$$\frac{dy \cdot dy}{2y}$$

let *BR* or dx be m and *RN* or dy be n. By the thirty-third Proposition of the first Book of the Conics of Apollonius, and from similar Triangles, as $2x$ to y so is m to $n + z = \frac{my}{2x}$.

Likewise from the Nature of the Parabola $yy + 2yn + nn = xp + mp$ [$(y + dy)^2 = p(x + dx)$], and $2yn + nn = mp$

[found by substituting from the equation $y^2 = px$, expressed earlier as the equation for the curve ABN]: wherefore

$$\frac{2yn + nn}{p} = m$$

and because $yy = px$, will be equal to x. Therefore substituting these values instead of m and x we shall have

$$n + z = \frac{my}{2x} = \frac{2yynp + ynnp}{2yyp}$$

i. e. $n + z = \frac{2ny + nn}{2y}$: which being reduced gives $z = \frac{nn}{2y} = \frac{dy \cdot dy}{2y}$

Q.E.D.

Response to *The Analyst*

After *The Analyst*'s publication, responses began pouring in to the local papers and publishing companies. Many people took offence to Berkeley's suggestion that members of the mathematical community followed Newton in blind faith and did not verify his conclusions for themselves. Many of these rebuttals did not fully answer the questions posed by Berkeley and in the end showed that the writers did not even comprehend Berkeley's propositions.

None of the papers trying to refute the claims of Berkeley made any great jumps in the theory of calculus except Benjamin Robin's *Discourse Concerning the Nature and Certainty of Sir Isaac Newton's Method of Fluxions and of Prime and Ultimate Ratios* in 1735 and Colin Maclaurin's *Treatise of Fluxions* in 1742. Maclaurin spent more time developing his thoughts and created a very intricate view of and argument for fluxions.

Maclaurin took offence to Berkeley's claim of infidelity even taking it as a personal insult, thinking Berkeley charged all mathematicians with infidelity. His *Treatise of Fluxions* aimed "to show that 'infinitesimals' in the arguments of Newton can always be replaced by finite quantities" (Katz 1998). Maclaurin made a great effort to take the ideas of fluxions (he was a fan of Newton's calculus) and define them with great rigor in the manner of "the ancients" like in Euclid's *Elements* or Apollonius' *Conics*. Many of his proofs entailed single and double *reductio ad absurdum*. This proof style involves assuming a 'false' statement to be true, then by the usual methods of proof generating a contradictory statement. Many dislike Maclaurin's proofs, but his work is the first correct method of proving calculus and was the first step toward a rigorous derivation of the calculus.

Validity of *The Analyst*

Berkeley does not criticize the conclusions drawn by the great mathematicians who developed the calculus, but as he states repeatedly, merely criticizes the method by which the conclusions were obtained. These methods appear to be a little suspect, but did they really require such a strong criticism? Consider Newton's rectangle problem. Berkeley provides a logical and more straightforward derivation of the moment of a rectangle and the only difference between his and Newton's answers is the product ab (deficient in Newton's). Since this product is so small, Newton treated it as negligible. Berkeley did not believe in neglecting any amount for the very fact that it is a real amount; therefore its absence creates a real error. Today when calculus is taught, one of the first topics (missing from these early developers) is the idea of a "limit," which is how this difference between these arguments can be settled.

Newton (as indicated above) thought of the "momentaneous increment" of the rectangle as being squeezed between the two values of $\overline{A - \frac{1}{2}a} \times \overline{B - \frac{1}{2}b}$ and $\overline{A + \frac{1}{2}a} \times \overline{B + \frac{1}{2}b}$. Today we might say: "the limit of $A \pm \frac{1}{2}a$ as a approaches 0 is A", (similarly for B), which is what Newton lacked in his definitions. But the rectangle problem shows that he had an idea of what needed to be done, albeit without the proper vocabulary or tools to complete a rigorous definition.

Now, with the idea of a limit having been determined, were Berkeley's analysis and treatise still necessary? We will never know of course, but it is my belief that the calculus as we know it would have advanced less quickly and taken longer to attain a solid foundation if Berkeley had not published his objections. Though his main objection was with the idea of infinity and not the mathematics in general, he caused an uproar in the mathematical community of the time and helped to lead the way toward a fully-developed theory of infinitesimals, differentials, and hence, the calculus.

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