The world of Mathematics in the 17th century was rife with rivalries. Isaac Newton and Gottfried Wilhelm von Leibniz both claimed to have developed the methods of Calculus, and their competing claims split the society of mathematicians into factions. Newton’s supporters were primarily British mathematicians. Among those in the Leibniz camp was the Swiss mathematician Johann Bernoulli. Another famous rivalry was one between Johann and his older brother Jakob. One of them would develop some mathematical problem and challenge the other to solve it. Sometimes, money was staked on the outcome of the challenge, and the challenges were often rather public.

In the June 1696 Acta Eruditorum, one of the world’s first scientific journals, Johann proposed the following mathematical problem [1, p.645]:

“If two points $A$ and $B$ are given in a vertical plane, to assign to a mobile particle $M$ the path $AMB$ along which, descending under its own weight, it passes from the point $A$ to the point $B$ in the briefest time”.

This problem became known as the Problem of the Brachistochrone (from the Greek words brachistos and chronos, meaning “shortest” and “time”), and Bernoulli pledged to reveal the name of the solution curve in six months if none were able to find it within that time. Only Leibniz was able to determine the solution within that period, and at his urging,
Bernoulli extended his time limit an additional five months, “in order...that no one might have cause to complain of the shortness of time allotted” [1, p.647]. Bernoulli also restated the problem publicly in January 1697 at the University of Groningen, where he was a Professor of Mathematics, so that “those to whom the...Acta is not available” could be given a chance to solve it. Further, he specified that the two given points not be on a single vertical line (since in that special case, the vertical line would be the solution curve), and that the path be frictionless [1, p.647].

This problem was not a new one. In his 1638 publication, Two New Sciences, Galileo demonstrated that the Brachistochrone curve was not the straight line between the endpoints, despite being the curve of the shortest distance, and posited that an arc of a circle seemed to be the solution. As it turns out, his supposition was incorrect, but the true answer, the inversion of a curve called the Cycloid, is closely tied to the circle. If a circle is “rolled” along a straight line, the Cycloid is the curve generated by the path of a single point on that rolling circle. The inversion is the same curve, just upside-down.

The Cycloid has many interesting properties. The area under one arch of a cycloid is three times that of its generating circle. The arc length of one arch of a cycloid is eight times the diameter of its generating circle. Christiaan Huygens discovered a particularly remarkable physical property of the inverted Cycloid: that the time it takes for a particle to descend from rest at some point of the curve to the lowest point of the curve is the same, regardless of the starting point (so long as the starting point isn’t the lowest point of the curve). Particles starting higher up on the curve accelerate more quickly down
the steeper incline, but must travel a greater distance, while particles starting closer to the lowest point have less distance to travel, but are accelerated less quickly. Because of this property, the inverted Cycloid is often called the “Isochrone” or “Tautochrone” (from the Greek words *isos* and *tauto*, meaning “equal” and “the same”).

In the proclamation in which Bernoulli restated the Brachistochrone problem, he remarked on the fact that so few had theretofore solved it, and while he didn’t explicitly state any names, a challenge to Isaac Newton can be readily inferred from the phrasing [1, p.648]. Shortly thereafter, the problem came to Newton’s attention. Within twelve hours of receiving it, he solved it, and the result was published in the January 1697 issue of *Philosophical Transactions*, a publication of the Royal Society. In the *Acta Eruditorum* of May 1697, Bernoulli submits his own proof of the solution, along with those of his brother Jakob and the Marquis de L’Hospital, as well as the excerpt from *Philosophical Transactions* containing Newton’s solution. He also acknowledges the solutions of Leibniz and Ehrenfried Walther von Tschirnhaus. Of the six mathematicians whose solutions are mentioned or included, only L’Hospital’s is incorrect.

Johann’s proof is rather innovative. Operating under Fermat’s principle—that light always travels from one point to another along the path that takes the least time—he demonstrates that a beam of light traveling through (and being refracted by) differentially thin layers of varying transparent materials will travel along a Cycloid path. While this does effectively validate Bernoulli’s claim that the problem was a useful one to science, and not merely hypothetical speculation, his method cannot be generalized to apply to other fields or circumstances. His brother Jakob’s method, on the other hand, is generalizable, and was later developed into what is known as the Calculus of Variations.

Jakob precedes his proof with the following Lemma—a sort of preliminary proof, the result of which he uses to help complete his primary proof. My following analysis appears in square brackets.
Lemma:

Let $ACEDB$ be the desired curve along which a heavy point falls from $A$ to $B$ in the shortest time, and let $C$ and $D$ be two points on it as close together as we like.

Then the segment of arc $CED$ is among all segments of arc with $C$ and $D$ as end points the segment that a heavy point falling from $A$ traverses in the shortest time. [In other words, there are no other curve segments with endpoints $C$ and $D$ through which a heavy point falling from $A$ would pass in less time than it would through $CED$. Demonstrating this fact is the purpose of the Lemma, as it allows him (in his subsequent proof) to focus on any segment of the curve, rather than the curve in its entirety.]

[Proof of Lemma:] Indeed, if another segment of arc $CFD$ were traversed in a shorter time, then the point would move along $ACFDB$ in a shorter time than along $ACEDB$, which is contrary to our supposition.
[End of Proof of Lemma.]

[Proof of the Brachistochrone Problem:] Hence in a plane arbitrarily inclined to the horizon (the plane need not be [vertical]), take $ACB$ as the required curve, on which a heavy point from $A$ reaches $B$ in a shorter time than on any other curve in this plane.

Take on it two points $C$ and $D$ infinitesimally close together [practically the same point, in other words. Bear in mind that, even though (for the sake of clarity) $C$ and $D$ don’t look like they are infinitesimally close in the following diagrams, it is something he is taking for granted] and draw the horizontal line $AH$, the vertical $CH$, and $DF$ [perpendicular] to it.

Take $E$ halfway between $C$ and $F$ and complete [rectangle $EIDF$] by means of the line $EI$ [and $DI$].
On $EI$ we now must determine point $G$ such that the time of fall through $CG$ + the time of fall through $GD$ [which is denoted by $t_{CG} + t_{GD}$, being sure to keep in mind that the fall begins at point $A$] is a minimum. [In other words, given points $C$ and $D$, infinitesimally close, on $ACB$, we want to find a formula that describes how to place point $G$ between them.]

If we now take on the line $EI$ another point $L$ such that $GL$ is incomparably small as compared to $EG$ [so he is assuming that $G$ and $L$ are nearly the same point. He has already assumed that $CD$ is infinitesimally small, so he is taking $GL$ to be infinitesimally small compared to that], and if we draw $CL$ and $DL$,

then, [since he is assuming that $CGD$ and $CLD$ are effectively the same path:]

$$t_{CL} + t_{DL} = t_{CG} + t_{GD}$$

[here, “=” is technically only a very close approximation, and should be
taken to mean that the difference between the two things being
“equated” is negligibly small] and hence [by subtracting $t_{CL}$ and $t_{GD}$
from both sides]

$$t_{DL} - t_{GD} = t_{CG} - t_{CL}.$$ 

I now reason as follows. According to the nature of the fall of heavy
bodies [see Appendix 1 for explanation],

$$\frac{CG}{CE} = \frac{t_{CG}}{t_{CE}}, \quad \text{and} \quad \frac{CL}{CE} = \frac{t_{CL}}{t_{CE}},$$

hence [subtracting the right from the left, we have]

$$\frac{CG - CL}{CE} = \frac{t_{CG} - t_{CL}}{t_{CE}}.$$ 

If we take a point $M$ on $CG$ such that $CG - CL = GM$,

then we have, because of the similarity of the [“infinitesimal”] triangles
$LMG$ and $CEG$, [that $GL \over CG = GM \over EG$. Then, multiplying by $CG$ and dividing by
$CE$, we see that $GL \over CE = CG \ast GM \over CE \ast EG$. Since $GM = CG - CL$, we have that

$$\frac{GL}{CE} = \frac{CG \ast (CG - CL)}{CE \ast EG} = \frac{CG \ast (CG - CL)}{CE} \ast \frac{t_{CG} - t_{CL}}{t_{CE}}.$$ 

Thus, since

$$\frac{CG - CL}{CE} = \frac{t_{CG} - t_{CL}}{t_{CE}},$$

we have]

$$\frac{GL}{CE} \ast \frac{EG \ast t_{CE}}{t_{CE}} = \frac{EG \ast t_{CE}}{t_{CE}}.$$ 

(#{})

In the same way, we find, according to the nature of the fall of heavy
bodies,

$$\frac{GD}{EF} = \frac{t_{GD}}{t_{EF}}, \quad \text{and} \quad \frac{DL}{EF} = \frac{t_{DL}}{t_{EF}}, \quad \text{hence} \quad \frac{DL - GD}{EF} = \frac{t_{DL} - t_{GD}}{t_{EF}}.$$ 

If we take on $DL$ the point $N$ such that $DL - GD = LN$,

then we have, because of the similarity of [“infinitesimal”] triangles
LNG and GID, [that \( \frac{GL}{CE} = \frac{DG}{GI} \). Then, multiplying by LN and dividing by EF, we see that \( \frac{GL}{EF} = \frac{DG*LN}{EF*GI} = \frac{DG*(DL-GD)}{EF*GI} = \frac{DG*DL-GD}{EF} \), recalling that LN = DL - GD. Thus, since \( \frac{DL-GD}{EF} = \frac{t_{DL}-t_{GD}}{t_{EF}} \), and recalling that EF = CE by construction, we have]

\[
\frac{GL}{CE} = \frac{DG*(t_{DL}-t_{GD})}{GI*t_{EF}}. \tag{##}
\]

By comparison [of (#) and (##)] we obtain

\[
\frac{CG*(t_{CG}-t_{CL})}{EG*t_{CE}} = \frac{DG*(t_{DL}-t_{GD})}{GI*t_{EF}},
\]

and [multiplying both sides of the equation by the right-hand denominator, dividing by the left-hand numerator, and recalling that \( t_{DL} - t_{GD} = t_{CG} - t_{CL} \), we see that]

\[
\frac{GI*t_{EF}}{EG*t_{CE}} = \frac{DG*(t_{DL}-t_{GD})}{CG*(t_{CG}-t_{CL})} = \frac{DG}{CG}.
\]

But [since C and E are infinitesimally close together, the acceleration due to gravity over CE is negligible, so we can treat the speed of the falling object at point C as the average speed, \( \bar{v} \). Then, because \( v_c = \bar{v} = \sqrt{2g*y_c} \) (see Appendix 1 for this extrapolation), and since \( \bar{v} = \frac{CE}{t_{CE}} \), we can see that \( t_{CE} = \frac{CE}{\sqrt{2g*y_c}} = \frac{CE}{\sqrt{2g*CH}} \). In the same way, since E and F are infinitesimally close together and \( CE = EF \) by construction, we have \( t_{EF} = \frac{CE}{\sqrt{2g*EH}} \). Then we see that \( \frac{t_{EF}}{t_{CE}} = \frac{CE*\sqrt{2g*CH}}{CE*\sqrt{2g*EH}} = \frac{\sqrt{CH}}{\sqrt{EH}} \), so] according to the law of gravity we have

\[
\frac{GI*t_{EF}}{EG*t_{CE}} = \frac{GI*\sqrt{CH}}{EG*\sqrt{EH}},
\]

and therefore finally:

\[
\frac{GI*\sqrt{CH}}{EG*\sqrt{EH}} = \frac{DG}{CG}.
\]
Now $EG$ and $GI$ are elements of the abscissa $AH$, $CG$ and $DG$ are elements of the curve $[ACB]$, $CH$ and $EH$ their ordinates, and $CE$ and $EF$ elements of the ordinate. [In the notation of modern differential Calculus, we would refer to $EG + GI$ as $dx$, where $AH$ is some nonnegative value of $x$, as on the axes above; we would refer to $CG + DG$ as $ds$; $CH$ and $EH$ are some nonnegative values of $y$, as on the axes above; and we would refer to $CE + EF$ as $dy$. Note: by these labels, we are assuming that the force of gravity is pulling downward in the positive $y$ direction. Further, since $GL$ is “incomparably small as compared to $EG$,” $GL$ would be referred to as $d^2x$, the second differential of $x$.] The problem can therefore be reduced to the purely geometric one of determining the curve of which the [curve] elements are directly proportional to the elements of the abscissa and indirectly proportional to the square roots of the ordinates. [In other words: $ds \sqrt{y} = k \cdot dx$, where $k$ is some nonnegative constant. Then, since $ds^2 = dx^2 + dy^2$, we can square both sides to get $(dx^2 + dy^2) \cdot y = k^2 \cdot dx^2$. Then, $y \cdot dy^2 = (k^2 - y) \cdot dx^2$. Dividing by the quantity $(k^2 - y)$ and taking the square root of both sides, we have $dy \cdot \sqrt{\frac{y}{k^2 - y}} = dx$, which is of identical form to the differential equation for the cycloid. Rather than stop at that, however, Jakob goes on to prove geometrically that the cycloid fits the equation he derived in the previous proof.] I find that this property belongs to the Isochrone of Huygens, which therefore is also the Oligochrone [Jakob’s name for the Brachistochrone, from the Greek oligo, meaning “scant”], namely the
cycloid [2, pp.396-398].

[Proof that the cycloid has the properties determined for the curve of least descent:] Let $ACP$ be a semicycloid; let $CM$ and $GN$ be tangents to the curve in $C$ and $G$; let $PQR$ be the [left] semicircle of the generating circle of $ACP$ [extend $PR$ to $M$. Drop a perpendicular from $H$ on $AR$ through $C$ to $E$ and draw lines $EI$ and $DI$ as before].

[See the figure below. Extend $EI$ to $PR$, intersecting the semicircle in $V$ and its diameter in $X$. Draw $PV$ and $RV$. Recalling that $G$ and $D$ are assumed to be infinitesimally close together, we can treat $D$ as though it is a point on tangent $GN$. A property of cycloids that Jakob uses here is that Differential Triangle $DGI$ is similar to Triangles $PVX$ and $NGX$. A property of semicircles he uses is that Triangles $PVX$, $VRX$, and $PRV$ are]
Then [by the similarity of the aforementioned triangles] we have

\[ \frac{GI}{DG} = \frac{GX}{GN} = \frac{VX}{PV} = \frac{RX}{RV} \]

[Then, from the equation of the semicircle

\[ VX = \sqrt{\left(\frac{1}{2} PR\right)^2 - (RX - \frac{1}{2} PR)^2} = \sqrt{RX \cdot PR - RX^2}, \]

we can use the Pythagorean

Theorem on Triangle VRX to find that

\[ RV = \sqrt{RX^2 + VX^2} = \sqrt{RX^2 + RX \cdot PR - RX^2} = \sqrt{RX \cdot PR} \].

Thus, recalling that \( RX = EH \), we have]

\[ \frac{GI}{DG} = \frac{RX}{RV} = \frac{\sqrt{RX}^2}{\sqrt{RX} \cdot \sqrt{PR}} = \frac{\sqrt{RX}}{\sqrt{PR}} = \frac{\sqrt{EH}}{\sqrt{PR}}. \] (&) 

[Now we draw a perpendicular from \( C \) to \( PR \), intersecting the semicircle and its diameter at \( Q \) and \( S \), respectively. Because of the same properties of semicircles and cycloids from before, we have that

Differential Triangle \( CEG \) is similar to Triangles \( PSQ, MSC, QSR, \) and \( PQR \). Additionally, just as before, from the equation of the semicircle

\[ QS = \sqrt{\left(\frac{1}{2} PR\right)^2 - (RS - \frac{1}{2} PR)^2} = \sqrt{RS \cdot PR - RS^2}, \]

we can use the Pythagorean

Theorem on Triangle QSR to determine that \( QR = \sqrt{RS \cdot PR} \).]
Then we have [recalling that $RS = CH$]

$$\frac{CG}{EG} = \frac{CM}{CS} = \frac{PQ}{QS} = \frac{QR}{RS} = \frac{\sqrt{RS} \cdot \sqrt{PR}}{\sqrt{RS}} = \frac{\sqrt{PR}}{\sqrt{CH}}.$$

Therefore [multiplying (&) by (&&) and dividing $\frac{GI}{EG}$ by the result yields]

$$\frac{DG}{CG} = \frac{GI \cdot \sqrt{PR} \cdot \sqrt{CH}}{EG \cdot \sqrt{PR} \cdot \sqrt{EH}} = \frac{GI \cdot \sqrt{CH}}{EG \cdot \sqrt{EH}},$$

as was desired [3, p.213]. [End of Proof.]

This problem attracted some of the most famous European mathematicians of the period to attempt to uncover its solution. It was yet another outlet for the fierce competition so typical of that era, and contributed to many scientific advances, including the development of a new field of Calculus. Like Bernoulli, many were fascinated by the fact that the inverted Cycloid is the solution to both the Brachistochrone and Isochrone problems, and others are fascinated by it even today.
Appendix 1

The average speed of a moving body is the distance of travel divided by the time of travel. The statements (regarding “the nature of the fall of heavy bodies”) made by Jakob are equivalent to saying

\[
\frac{CG}{t_{CG}} = \frac{CE}{t_{CE}} = \frac{CL}{t_{CL}},
\]

which means that the average speed is the same over CG, CE, and CL.

Since C and G are infinitesimally close, we can treat the slope of the curve—and, consequently, the acceleration due to gravity on a body moving along the curve—as constant from C to G. Thus, by the Mean Speed Rule, the average speed, \( \bar{v} \), of a body moving from C to G is

\[
\bar{v} = \frac{1}{2}(v_C + v_G).
\]

The mean speed rule also holds true for a body moving from C to E or from C to L.

The law of conservation of energy states that the sum of kinetic energy (energy due to movement) and potential energy (energy due to position) remains constant. Assuming that gravity is pulling in the direction of positive \( y \), this can be expressed as

\[
\frac{1}{2}m(v_1)^2 + -m \cdot g \cdot y_1 = \frac{1}{2}m(v_2)^2 + -m \cdot g \cdot y_2.
\]

If we divide everything by \( m \), the mass, and take the initial velocity and \( y \) value to be zero, then

\[
\frac{1}{2}(v_1)^2 - g \cdot y_1 = 0 = \frac{1}{2}(v_2)^2 - g \cdot y_2.
\]

Rearranging and taking the square root of both sides shows

\[
v_2 = \sqrt{2g \cdot y_2},
\]

so an object’s speed at any point along the curve is directly proportional to the square root of its \( y \) value. Thus, since the \( y \) value is the same for
points \( E, L, \) and \( G \), the object’s speed would also be the same at those points. Then the mean speeds over \( CE, CL, \) and \( CG \) are equal, as was stated.
Works Cited

