Acceleration-induced nonlocality: kinetic memory versus dynamic memory

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Abstract

The characteristics of the memory of accelerated motion in Minkowski spacetime are discussed within the framework of the nonlocal theory of accelerated observers. Two types of memory are distinguished: kinetic and dynamic. We show that only kinetic memory is acceptable, since dynamic memory leads to divergences for nonuniform accelerated motion.

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1 Introduction

The special theory of relativity is based on two basic postulates: Lorentz invariance and the hypothesis of locality. Lorentz invariance refers to a fundamental symmetry principle, namely, the invariance of basic physical laws under inhomogeneous Lorentz transformations. In practice these laws of nature involve physical quantities measured by inertial observers in Minkowski spacetime. An inertial observer always moves uniformly and refers its observations to the fixed spatial axes of an inertial frame; it can be depicted by a straight line in the Minkowski diagram and represents an ideal; in fact, physical observers are all effectively accelerated. For instance, one can imagine the influence of radiation pressure on the path of a cosmic particle. In general, the acceleration of an observer consists of the translational acceleration of its path as well as the rotation of its spatial frame. Observers with translational acceleration are therefore represented by curved lines in the Minkowski diagram. As an example of a rotating observer, consider a uniformly moving observer that refers its observations to spatial axes that rotate with respect to the spatial frame of the underlying inertial coordinate system. The hypothesis of locality refers to the measurements of realistic (i.e. accelerated) observers: such an observer is postulated to be equivalent, at each event along its worldline, to a momentarily comoving inertial observer. The origin of this assumption can be traced back to the work of Lorentz in the context of his classical electron theory [1]; later, it was simply adopted as a general rule in relativity theory [2].

Along its worldline, the accelerated observer passes through a continuous infinity of hypothetical momentarily comoving inertial observers. Stated mathematically, the translationally accelerated observer's curved worldline is the *envelope* of the straight worldlines of this class of hypothetical inertial observers. Therefore, the hypothesis of locality has two components: (i) the assumption that the measurements of the accelerated observer must be somehow connected to the measurements of the hypothetical class of momentarily comoving inertial observers along its worldline and (ii) that this connection is postulated to be the pointwise equivalence of the accelerated observer and the momentarily comoving inertial observer. The latter means that the acceleration of the observer does not *directly* affect the result of its measurement; devices that obey this rule are called "standard". Thus the hypothesis of locality is a simple generalization of the assumption that the rods and clocks of special relativity theory are not directly affected by

acceleration [2].

What is the physical basis for the hypothesis of locality? It is difficult to argue with part (i) of this hypothesis, since the fundamental laws of (nongravitational) physics have been formulated with respect to inertial observers and hence the measurements of accelerated observers should be in some way related to those of inertial observers. On the other hand, part (ii) can only be valid if the measurement process occurs instantaneously and in a pointwise manner. That is, (ii) is appropriate for phenomena involving coincidences of classical point particles and null rays. Classical waves, on the other hand, are extended in time and space with a characteristic wave period T and a corresponding wavelength λ, respectively. Imagine, for example, the measurement of the frequency of an incident electromagnetic wave by an accelerated observer; at least a few periods of the wave must be received by the observer before an adequate determination of the frequency would become possible. Thus this measurement process is nonlocal and extends over the worldline of the observer. The observer's acceleration can be characterized by certain *acceleration lengths* L given by c^2/g and c/Ω for translational acceleration g and rotational frequency Ω, respectively. The nonlocality of the external radiation is thus expected to couple with the intrinsic scales associated with the acceleration of the observer.

Classical wave phenomena are expected to violate the hypothesis of locality. The scale of such violation would be given by $\lambda/L = T/(L/c)$, where L/c is the *acceleration time*. The hypothesis of locality will hold if λ is so small that the incident radiation behaves like a ray, i.e. in the eikonal (or JWKB) limit such that $\lambda/L \to 0$; alternatively, L can be so large that deviations of the form λ/L would be below the sensitivity threshold of the detectors available at present. Consider, e.g., laboratory experiments on the Earth; typical acceleration lengths would be $c^2/g_\oplus \simeq 1\,\mathrm{lyr}$ and $c/\Omega_\oplus \simeq 28\,\mathrm{AU}$, so that for essentially all practical purposes one can ignore any possible deviations from locality at the present time. In this way, we can account for the fact that the standard theory of relativity is in agreement with all observational data available at present. As a matter of principle, however, it is necessary to contemplate generalizations of the hypothesis of locality in order to take due account of intrinsic wave phenomena for realistic (accelerated) observers.

All of our considerations in this paper are within the framework of classical field theory; nevertheless, it is necessary to remark that quantum theory is based on the notion of wave-particle duality, and so an adequate treatment of classical wave phenomena is a necessary prelude to a satisfactory quantum

theory.

To proceed, we consider the most general extension of the hypothesis of locality that is consistent with causality and the superposition principle. A nonlocal Lorentz-invariant theory of accelerated observers has been developed along these lines [3, 4, 5, 6] and is presented in Section 2. The theory involves a kernel that depends primarily on the acceleration of the observer; that is, the measurements of the observer depend on its past history of acceleration. The main physical principle that is employed in the nonlocal theory for the determination of the kernel is the assumption that an intrinsic radiation field can never stand completely still with respect to an accelerated observer; this statement involves a simple generalization of a property of inertial observers to all observers. Thus the accelerated observer is endowed with *memory*, and the past affects the present through an averaging process, where the weight function is proportional to the kernel $K(\tau, \tau')$. It turns out that the kernel K cannot be completely determined by the theory presented in Section 2. An additional simplifying assumption is therefore introduced in Section 3: $K(\tau, \tau')$ must be a function of a single variable. Two cases are then considered: (1) $K(\tau, \tau') = k_0(\tau')$ and (2) $K(\tau, \tau') = k(\tau - \tau')$. We show that case (1)—i.e. the *kinetic* memory case—has acceptable properties that are described in Section 3. Case (2), i.e. the *dynamic* memory case, is treated in detail in Sections 3 and 4, where it is shown that the kernel function k can be unbounded even if the observer's past history has constant velocity except for one episode of smooth translational acceleration with finite duration. Specifically, we study the measurement of electromagnetic radiation fields by an observer that undergoes translational or rotational acceleration that lasts for only a finite interval of its proper time. After the acceleration is turned off, the observer measures in addition to the regular field a residual field that contains the memory of its past acceleration. This leftover piece is a finite *constant* field (*kinetic* memory) in case (1); however, it is time dependent (*dynamic* memory) in case (2). We rule out the latter case, since we prove that the measured field could diverge under certain reasonable circumstances. We are thus left with a unique theory that involves kinetic memory. An important aspect of our nonlocal ansatz is that the kernel induced by the acceleration of the observer depends on the spin of the radiation field under consideration. In particular, the kernel vanishes for an intrinsic scalar field, i.e. such a field is always *local*. As discussed in Section 5, our theory therefore rules out the possibility that a pure scalar (or pseudoscalar) field exists in nature. This conclusion is in agreement with available experimental data.

The nonlocal theory therefore predicts that any scalar particle would have to be a composite. Section 5 contains a brief discussion and our conclusions. A detailed discussion of the observational consequences of the nonlocal theory is beyond the scope of this work. In the following, we use units such that $c = 1$, i.e. the speed of light in vacuum is unity.

2 Accelerated observers and nonlocality

The measurement of length by accelerated observers involves subtle issues in relativity theory that have been investigated in detail [7, 8, 9]; for our present purpose, the main result of such studies is that an accelerated frame of reference, i.e. an extended coordinate system set up in the neighborhood of an accelerated observer, is of rather limited theoretical significance. We shall therefore refer all measurements to an inertial reference frame in Minkowski spacetime.

Imagine a global inertial frame with coordinates $x = (t, \mathbf{x})$ and the standard class of static inertial observers with their orthonormal tetrad frame $\lambda^\mu_{(\alpha)} = \delta^\mu_\alpha$, where $\lambda^\mu_{(0)}$ is the temporal direction at each event and $\lambda^\mu_{(i)}$, $i = 1, 2, 3$, are the spatial directions. The hypothesis of locality implies that an accelerated observer is also endowed with a tetrad frame $\hat\lambda^\mu_{(\alpha)}(\tau)$, where τ is the proper time along its worldline. For each τ, $\hat\lambda^\mu_{(\alpha)}(\tau)$ coincides with the constant tetrad frame (related to $\lambda^\mu_{(\alpha)}$ by a Lorentz transformation) of the momentarily comoving inertial observer. We note that $d\hat\lambda^\mu_{(\alpha)}/d\tau = \phi_\alpha{}^\beta \hat\lambda^\mu_{(\beta)}$, where $\phi_{\alpha\beta} = -\phi_{\beta\alpha}$ is a tensor such that $\phi_{0i} = (\mathbf{g})_i$ and $\phi_{ij} = \epsilon_{ijk}(\mathbf{\Omega})_k$. Here $\mathbf{g}(\tau)$ is the translational acceleration of the observer and $\mathbf{\Omega}(\tau)$ is the rotational frequency of its spatial frame. Each element of the acceleration tensor $\phi_{\alpha\beta}$ is a scalar under the inhomogeneous Lorentz transformations of the background spacetime. We assume throughout that the acceleration is turned on at $\tau = \tau_0$ and will in general be turned off at $\tau_1 > \tau_0$.

Let $f_{\mu\nu}$ represent an electromagnetic radiation field as measured by the standard set of static inertial observers. According to the hypothesis of locality $\hat f_{\alpha\beta} = f_{\mu\nu} \hat\lambda^\mu_{(\alpha)} \hat\lambda^\nu_{(\beta)}$, i.e. the projection of the field on the instantaneous tetrad frame, would be the field measured by the accelerated observer. On the other hand, let $F_{\alpha\beta}(\tau)$ be the true result of such a measurement. Taking causality into account, the most general linear relationship between $F_{\alpha\beta}(\tau)$

and $\hat{f}_{\alpha\beta}(\tau)$ is

$$F_{\alpha\beta}(\tau) = \hat{f}_{\alpha\beta}(\tau) + \int_{\tau_0}^{\tau} K_{\alpha\beta\gamma\delta}(\tau, \tau') \hat{f}^{\gamma\delta}(\tau') \, d\tau'. \tag{1}$$

This relation refers to quantities that are all scalars under the Poincaré group of spacetime transformations of the underlying inertial coordinate system. We note that the magnitude of the nonlocal part of equation (1) is of the form λ/L if the kernel is proportional to the acceleration of the observer. It follows from Volterra's theorem that in the space of continuous functions the relationship between F and f is unique [10, 11]; this theorem has been extended to the Hilbert space of square-integrable functions by Tricomi [12].

The basic ansatz (1) is consistent with an observation originally put forward by Bohr and Rosenfeld that the electromagnetic field cannot be measured at a spacetime *point*; in fact, an averaging process is necessary over a spacetime neighborhood [13, 14]. In the case of measurements by *inertial* observers envisaged by Bohr and Rosenfeld [13, 14], there is no intrinsic temporal or spatial scale associated with the inertial observers; therefore, one can effectively pass to the limiting case of a point with no difficulty as the dimensions of the spacetime neighborhood can be shrunk to zero without any obstruction. For an accelerated observer, however, the intrinsic acceleration time and length need to be properly taken into account. Hence the nonlocal ansatz (1) may be interpreted in terms of a certain averaging process over the past worldline of the accelerated observer.

To determine the kernel K, let us first mention a basic consequence of the hypothesis of locality for a radiation field. Imagine plane monochromatic electromagnetic waves of frequency ω propagating along the z-axis and an observer rotating uniformly about this axis with frequency Ω_0 in the (x, y)-plane on a circle of radius ρ in the underlying inertial reference frame. We find from $\hat{f}_{\alpha\beta} = f_{\mu\nu}\hat{\lambda}^{\mu}_{(\alpha)}\hat{\lambda}^{\nu}_{(\beta)}$ that according to the rotating observer the frequency of the wave is $\hat{\omega} = \gamma(\omega \mp \Omega_0)$, where γ is the Lorentz factor corresponding to the speed $\rho\Omega_0$ of the observer and the upper (lower) sign refers to incident positive (negative) helicity radiation. This result has a simple intuitive interpretation: In an incident positive (negative) helicity wave the electric and magnetic field vectors rotate with frequency $\omega(-\omega)$ about the direction of propagation of the wave. As seen by the rotating observer, the field vectors rotate with frequency $\omega - \Omega_0 \, (-\omega - \Omega_0)$ with respect to the inertial temporal coordinate t; moreover, the Lorentz factor simply accounts

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for time dilation $dt = \gamma d\tau$. It follows that a positive helicity incident wave can stand completely still with respect to all observers rotating uniformly with frequency $\Omega_0 = \omega$. In terms of energy, we have $\hat{E} = \gamma(E - \boldsymbol{\sigma} \cdot \boldsymbol{\Omega}_0)$, where $\boldsymbol{\sigma}$ is the spin of the incident photon. More generally, for oblique incidence $\hat{E} = \gamma(E - \hbar M \Omega_0)$, where M is the multipole parameter such that $\hbar M$ is the component of the total (orbital plus spin) angular momentum along the z-axis. This is an example of the general phenomenon of spin-rotation coupling; various aspects of this effect and the available observational evidence are discussed in [15, 16, 17, 18, 19, 20]. Again, the incident wave can theoretically stand completely still for all observers rotating with frequency Ω_0 such that $\omega = M\Omega_0$. Let us recall here a fundamental consequence of Lorentz invariance, namely that a radiation field can never stand completely still with respect to an inertial observer. That is, an inertial observer can move along the direction of propagation of a wave so fast that the frequency $\hat{\omega} = \gamma\omega(1 - \beta)$ can approach zero but the mathematical limit of $\hat{\omega} = 0$ is never physically achieved, since the observer's speed cannot reach the speed of light in vacuum ($\beta < 1$). Therefore, for an inertial observer $\hat{\omega} = 0$ implies that $\omega = 0$. On the other hand, while we find that the hypothesis of locality predicts that a circularly polarized wave can stand completely still with respect to a uniformly rotating observer, this possibility can be avoided in the nonlocal theory by an appropriate choice of the kernel.

To implement the requirement that a radiation field can never stand completely still with respect to any observer, we assume that if $F_{\alpha\beta}(\tau)$ turns out to be constant in equation (1), then $f_{\mu\nu}$ must have been originally constant just as in the case of inertial observers in the standard theory of relativity. It is convenient to replace the tensor $f_{\mu\nu}$ by a six-vector f, with electric and magnetic fields as components, and introduce the "Lorentz" matrix Λ such that $\hat{f} = \Lambda f$. Then for constant fields f and F, equation (1) takes the form

$$F = \Lambda(\tau)f + \int_{\tau_0}^{\tau} K(\tau, \tau')\Lambda(\tau')f \, d\tau', \tag{2}$$

where for $\tau = \tau_0$, the matrix $\Lambda_0 := \Lambda(\tau_0)$ is constant and $F = \Lambda_0 f$. Thus in the nonlocal theory the kernel K should be determined from the Volterra integral equation

$$\Lambda_0 = \Lambda(\tau) + \int_{\tau_0}^{\tau} K(\tau, \tau')\Lambda(\tau') \, d\tau'. \tag{3}$$

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It follows from Volterra's theory (see Appendix A) that to every kernel K corresponds a unique *resolvent* kernel $R(\tau, \tau')$ such that

$$\Lambda(\tau) = \Lambda_0 + \int_{\tau_0}^{\tau} R(\tau, \tau')\Lambda_0 \, d\tau'. \tag{4}$$

Therefore, only the integral of the resolvent kernel is completely determined by our physical requirement

$$\int_{\tau_0}^{\tau} R(\tau, \tau') \, d\tau' = \Lambda(\tau)\Lambda_0^{-1} - I, \tag{5}$$

where I is the unit matrix. It is clear at this point that given $\Lambda(\tau)$, relations (3)–(5) are not sufficient to determine the kernel K uniquely. To proceed further, other simplifying restrictions are necessary on K or R,

$$\hat{f}(\tau) = F(\tau) + \int_{\tau_0}^{\tau} R(\tau, \tau')F(\tau') \, d\tau'. \tag{6}$$

This must be done in such a way as to preserve time translation invariance in the underlying inertial coordinate system.

Let us finally remark that for a scalar field, $\Lambda(\tau) = 1$ and equations (3)–(5) simply reduce to the requirement that $K(\tau, \tau')$ must have a vanishing integral over $\tau' : \tau_0 \to \tau$. That is, the connection between the kernel and the acceleration of the observer disappears. This circumstance is further discussed in Section 5.

3   Memory

It is necessary to introduce simplifying assumptions in order to find a unique kernel K. We therefore tentatively postulate that K is a function of a single variable. There are two reasonable possibilities:

$$K(\tau, \tau') = k_0(\tau') \tag{case 1}$$

and

$$K(\tau, \tau') = k(\tau - \tau'); \tag{case 2}$$

in either case, the basic requirement of time translation invariance in the background global inertial frame is satisfied.

3.1 Kinetic memory

In case (1), the kernel k_0 corresponds to a simple weight function that can be determined by differentiating equation (3),

$$k_0(\tau) = -\frac{d\Lambda}{d\tau}\Lambda^{-1}(\tau) = \Lambda(\tau)\frac{d\Lambda^{-1}}{d\tau}. \tag{7}$$

The kernel k_0 is thus directly proportional to the acceleration of the observer. A significant feature of this kernel is that once the acceleration is turned off at $\tau = \tau_1$, then for $\tau > \tau_1$,

$$F(\tau) = \hat{f}(\tau) + \int_{\tau_0}^{\tau_1} k_0(\tau')\hat{f}(\tau')\,d\tau'. \tag{8}$$

There is therefore a *constant* memory of past acceleration and the field F satisfies the standard field equations in the inertial frame. That is, the field equations are linear differential equations and the addition of a constant solution is always permissible but subject to boundary conditions. In terms of actual laboratory devices that have experienced accelerations in the past, such constant fields as in equation (8) would be canceled once the devices are reset. Thus case (1) involves simple "nonpersistent" memory of past acceleration; therefore, we call k_0 the *kinetic memory* kernel.

It is interesting to note that our basic integral equation (2) together with the kinetic memory kernel (7) and an integration by parts takes the form

$$F(\tau) = F(\tau_0) + \int_{[\tau_0,\tau]} \Lambda df,$$

so that $dF = \Lambda df$ along the worldline of the accelerated observer.

3.2 Dynamic memory

The second case involves a convolution type kernel $K = k(\tau - \tau')$. It follows (see Appendix A) that in this case the resolvent kernel is of convolution type as well, $R = r(\tau - \tau')$. Thus equation (5) can be written, after expressing the left side as the area under the graph of the function r from the origin to $\tau - \tau_0 = t$, as

$$r(t) = \frac{d\Lambda(t + \tau_0)}{dt}\Lambda_0^{-1}. \tag{9}$$

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The kernel k is then given by (cf. Appendix A)

$$k(t) = -r(t) + r * r(t) - r * r * r(t) + \cdots , \tag{10}$$

where a star denotes the convolution operation. We note that in this case the *resolvent* kernel is directly proportional to acceleration, so that $r = 0$ and, by equation (10), $k = 0$ for $t < 0$ or $\tau < \tau_0$, i.e. before the acceleration is turned on. However, the character of memory that is indicated by k,

$$\begin{aligned}
F(\tau) &= \hat{f}(\tau) + \int_{\tau_0}^{\tau} k(\tau - \tau') \hat{f}(\tau') \, d\tau' \\
&= \hat{f}(\tau) + \int_0^{\tau - \tau_0} k(t) \hat{f}(\tau - t) \, dt,
\end{aligned} \tag{11}$$

is more complicated than in case (1) due to the intricate relationship between $r(t)$ and $k(t)$ in equation (10). Even if the acceleration is turned off at $\tau = \tau_1$, it turns out that k does not vanish in general for $\tau > \tau_1$ and could even be divergent; in fact, proving the latter point is the main purpose of this paper.

Imagine, for instance, that $k(t)$ is finite everywhere and decays exponentially to zero for $t \to \infty$. Then in equation (11), as $\tau \to \infty$ long after the acceleration has been turned off at $\tau = \tau_1$, the contribution of the nonlocal term in (11) rapidly approaches a constant and we essentially recover the "nonpersistent" kinetic memory familiar from case (1). It turns out, however, that in general case (2) involves situations with persistent or *dynamic* memory such that under certain conditions $k(t)$ could diverge resulting in an asymptotically divergent $F(\tau)$.

The convolution (Faltung) type kernel is generally employed in many branches of physics and mathematics. As in equation (11), to produce the nonlocal part of the output $F(\tau)$, an input signal $\hat{f}(\tau - t)$ is linearly folded, starting from τ and going backwards in proper time until τ_0, with a weight function $k(t)$ that is the impulse response of the system. The use of convolution type kernels is standard practice in phenomenological treatments of the electrodynamics of media [21, 22, 23], feedback control systems [24], etc. We find, however, that for the pure vacuum case the convolution kernel due to *nonuniform acceleration* in general leads to instability and is therefore unacceptable. This proposition is proved in the following section for the translational and rotational accelerations of the observer.

The simplicity of the kinetic memory versus dynamic memory has been particularly stressed by Hehl and Obukhov in their investigations of nonlocal

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electrodynamics [25, 26]; moreover, their work has led to the question of the ultimate physical significance of the convolution type kernel in the nonlocal theory of accelerated systems [25, 26]. This question is settled in the present paper in favor of the kinetic memory kernel.

4 Dynamic memory of accelerated motion

4.1 Linear acceleration

Imagine an observer at rest on the z-axis for $-\infty < \tau < \tau_0$. At $\tau = \tau_0$, the observer accelerates along the positive z-direction with acceleration $g(\tau) > 0$. For $\tau \geq \tau_0$, we set

$$\theta(\tau) = \int_{\tau_0}^{\tau} g(\tau') \, d\tau', \tag{12}$$

$C = \cosh\theta$ and $S = \sinh\theta$. The natural nonrotating orthonormal tetrad frame of the observer along its worldline is given by

$$\begin{aligned}
\hat{\lambda}^{\mu}_{(0)} &= (C, 0, 0, S), \; \hat{\lambda}^{\mu}_{(1)} = (0, 1, 0, 0), \\
\hat{\lambda}^{\mu}_{(2)} &= (0, 0, 1, 0), \quad \hat{\lambda}^{\mu}_{(3)} = (S, 0, 0, C).
\end{aligned} \tag{13}$$

In this case $\Lambda(\tau)$ is given by

$$\Lambda = \begin{bmatrix} U & V \\ -V & U \end{bmatrix}, \quad U = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad V = SI_3, \tag{14}$$

where I_i, $(I_i)_{jk} = -\epsilon_{ijk}$, is a 3×3 matrix proportional to the operator of infinitesimal rotations about the x^i-axis.

Let us first consider case (1), for which the kernel can be easily computed using equation (7),

$$k_0(\tau) = -g(\tau) \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix}, \tag{15}$$

so that when the acceleration is turned off at $\tau = \tau_1$ the kernel k_0 vanishes with the acceleration for $\tau \geq \tau_1$. On the other hand, k_0 is simply constant for uniform acceleration (i.e. hyperbolic motion) with $g(\tau) = g_0$ for $\tau \geq$

τ_0. In the rest of this section, we focus attention on case (2) involving the convolution kernel.

For the convolution kernel, the resolvent kernel is given, via equation (9), by

$$r(\tau - \tau_0) = g(\tau) \begin{bmatrix} SJ_3 & CI_3 \\ -CI_3 & SJ_3 \end{bmatrix}, \tag{16}$$

where $(J_k)_{ij} = \delta_{ij} - \delta_{ik}\delta_{jk}$. In principle, the convolution kernel can be computed via the substitution of equation (16) in equation (10); however, this turns out to be a daunting task in practice. Imagine, for instance, that the acceleration is turned off at $\tau = \tau_1$, so that the resolvent kernel (16) has compact support over a time interval of length $\alpha = \tau_1 - \tau_0$ and vanishes otherwise. It then follows that the r^{*n} term in the expansion (10) has compact support over a time interval of length $n\alpha$. The summation of series (10) turns out to be rather complicated, except for the case of *uniform* acceleration, i.e. $g(\tau) = g_0$ for $\tau \geq \tau_0$, and the result is

$$k = -g_0 \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix}. \tag{17}$$

It is interesting to note that equation (17) is the *same* as the result of case (1), equation (15), for *uniform* acceleration.

In view of the difficulty of summing the series (10) directly, we find it advantageous to use Laplace transforms, which we denote by an overbar, i.e. $\mathcal{L}\{k(t)\} = \bar{k}(s)$, where

$$\bar{k}(s) = \int_0^\infty e^{-st} k(t)\, dt; \tag{18}$$

then, taking the Laplace transform of equation (10) and using the convolution (Faltung) theorem repeatedly, we arrive at

$$\bar{k}(s) = [I + \bar{r}(s)]^{-1} - I, \tag{19}$$

which is consistent with the reciprocity between k and r.

4.2 Stepwise acceleration

Let us specialize to a simple case of stepwise uniform acceleration, namely, we let $g(\tau) = g_0$ for $\tau_0 \leq \tau \leq \tau_1$ and zero otherwise (see Figure 1). In this
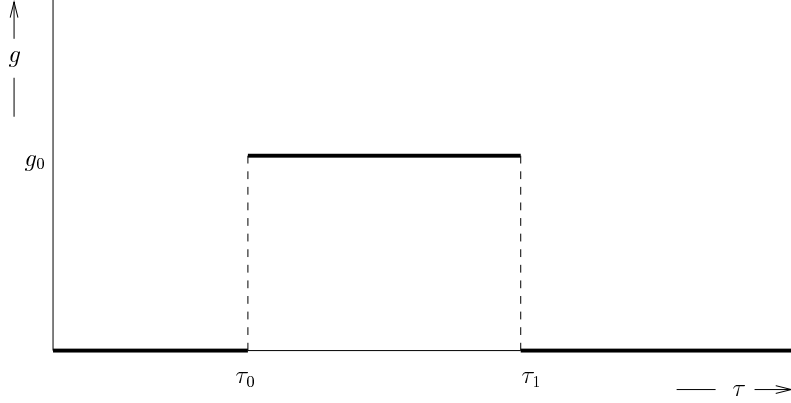
Figure 1: The linear acceleration of an observer that undergoes uniform acceleration g_0 during a period $\alpha = \tau_1 - \tau_2$ of its proper time. If the area under the graph exceeds a critical value given by $\beta_0 \approx 1.2931$, then the convolution kernel leads to divergences.

case,

$$\bar{r}(s) = \begin{bmatrix} \bar{r}_1 & \bar{r}_2 \\ -\bar{r}_2 & \bar{r}_1 \end{bmatrix}, \tag{20}$$

where $\bar{r}_1(s) = q(s)J_3$ and $\bar{r}_2(s) = p(s)I_3$. Here $p(s) = \mathcal{L}\{gC\}$ and $q(s) = \mathcal{L}\{gS\}$. All 6×6 matrices that we consider in this paper have the general form (20), i.e. each is completely determined by two 3×3 matrices just as $\bar{r}_1$ and $\bar{r}_2$ characterize $\bar{r}$ in equation (20); we therefore write $\bar{r} \to [\bar{r}_1; \bar{r}_2]$ to express this decomposition as in equation (20). To find the Laplace transforms of gC and gS, we note that in equation (12), $\theta = g_0(\tau - \tau_0)$ for $\tau \leq \tau_1$ and $\theta = \beta_0 = g_0(\tau_1 - \tau_0)$ for $\tau \geq \tau_1$; therefore,

$$p(s) \pm q(s) = \frac{g_0}{s \mp g_0}[1 - e^{-(s \mp g_0)\alpha}], \tag{21}$$

where $\alpha = \tau_1 - \tau_0 = \beta_0/g_0$ is the acceleration time interval. Using the results of Appendix B, we find from equation (19) that $\bar{k}(s)$ can be expressed as

$$\bar{k}(s) \to [\beta_0 Q(s)J_3; \beta_0 P(s)I_3],$$

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where

$$P(s) = \frac{e^w}{D}[w(-e^w + \cosh \beta_0) + \beta_0 \sinh \beta_0], \qquad (22)$$

$$Q(s) = \frac{1}{D}[e^w(w \sinh \beta_0 - \beta_0 \cosh \beta_0) + \beta_0]. \qquad (23)$$

Here $w := s\alpha$ and the denominator D can be factorized as

$$D = (we^w - \beta_0 e^{\beta_0})(we^w + \beta_0 e^{-\beta_0}). \qquad (24)$$

It is useful to recall that the kernel $k \to [k_1; k_2]$ refers to a system at rest on the z-axis for $\tau \leq \tau_0$ that is uniformly accelerated at $\tau = \tau_0$ with acceleration g_0 until $\tau_0 + \alpha = \tau_1$, and then continues with uniform speed $\tanh \beta_0$ along the positive z-direction for $\tau \geq \tau_1$. Under certain conditions, it is possible to obtain series representations for k_1 and k_2 (see Appendix C); however, to gain insight into the asymptotic behavior of k_1 and k_2 it proves more fruitful to proceed with an investigation of the singularities of $\bar{k}_1(s) = \beta_0 Q(s) J_3$ and $\bar{k}_2(s) = \beta_0 P(s) I_3$ in the complex s-plane. This is due to a simple property of the Laplace transformation in equation (18) extended to the complex s-plane: let us suppose that the convolution kernel $k(t)$ is a bounded function for all $t = \tau - \tau_0 > 0$ as one naturally expects of a function that represents memory; then, for any s in the complex plane with positive real part, i.e. $\text{Re}(s) > 0$, equation (18) implies that the absolute magnitude of $\bar{k}(s)$ should be finite, i.e. $\bar{k}(s)$ cannot be singular. Therefore, if we could show that $\bar{k}(s)$ has in fact pole singularities at complex values of s with $\text{Re}(s) > 0$, then it would simply follow that $k(t)$ cannot be bounded for all $t > 0$ and would thus be unsuitable to represent the memory of finite accelerated motion.

We will prove the following result: If $\beta_0 \exp(\beta_0) > 3\pi/2$, then the corresponding function k is unbounded for $t \geq 0$. It suffices to show that $\bar{k}$ has a pole in the right half of the complex s-plane. In fact, let us suppose that $\bar{k}$ has a pole at $s = s_0$, where $\text{Re}(s_0) > 0$, but $\|k\| := \sup_{t \geq 0} |k(t)| < \infty$. In this case, $\bar{k}$ has a pole in the half-plane $\mathcal{H}$ consisting of all complex numbers s such that $\text{Re}(s) \geq \frac{1}{2} \text{Re}(s_0)$, and therefore $|\bar{k}|$ is not bounded on $\mathcal{H}$. On the other hand, for $s \in \mathcal{H}$, we have that

$$|\bar{k}(s)| \leq \int_0^\infty e^{-\text{Re}(s)t}|k(t)|\, dt \leq \|k\| \int_0^\infty e^{-\text{Re}(s_0)t/2}\, dt < \infty,$$

in contradiction. Thus the rest of this subsection is devoted to the determination of the poles of $\bar{k}(s)$ in the right half-plane.

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The poles of $\bar{k}$ are elements of the zero set of D with $w = s\alpha$ and $\alpha > 0$. Note, however, that the (real) zeros $w = \pm\beta_0$ are removable singularities. Poles in the right half-plane are the zeros of D with nonzero imaginary parts. In view of the definition of D, let us consider the complex roots in the right half-plane of the equation $w \exp(w) = b$, where b is one of the real numbers $\pm\beta_0 \exp(\pm\beta_0)$. Because the zero set of this relation is symmetric with respect to the real axis, it suffices to consider only roots in the first quadrant of the complex w-plane.

We set $w = \xi + i\eta$, where $\xi \geq 0$ and $\eta \geq 0$ are real variables, and note that $w \exp(w) = b$ if and only if

$$\xi e^\xi = b \cos\eta, \qquad \eta e^\xi = -b \sin\eta.$$

If this system of equations has a solution, then, by squaring, adding and rearranging, we have that $\eta^2 = b^2 \exp(-2\xi) - \xi^2$ or, since $\eta \geq 0$, $\eta = \sqrt{b^2 \exp(-2\xi) - \xi^2}$.

There are several cases. For example, for $b > 0$, there is a pole in the right half-plane if the system of equations

$$\eta = \sqrt{b^2 e^{-2\xi} - \xi^2}, \qquad \xi e^\xi = b \cos\eta$$

has a solution with $\xi > 0$ and $\eta \bmod 2\pi \in (3\pi/2, 2\pi)$. Similarly, for $b < 0$, there is a pole in the right half-plane if the system of equations

$$\eta = \sqrt{b^2 e^{-2\xi} - \xi^2}, \qquad \xi e^\xi = b \cos\eta$$

has a solution with $\xi > 0$ and $\eta \bmod 2\pi \in (\pi/2, \pi)$.

A necessary condition for the relation $\eta = \sqrt{b^2 \exp(-2\xi) - \xi^2}$ to have a solution (ξ, η) is that $\xi \exp(\xi) < |b|$. For $b < 0$, we must have $\xi \exp(\xi) < \beta_0 \exp(-\beta_0)$; hence, there is a unique real number ξ_0 such that the necessary condition is met whenever $\xi \leq \xi_0$. On the other hand, for $b > 0$, the necessary condition, $\xi \exp(\xi) < \beta_0 \exp(\beta_0)$, is met if and only if $\xi < \xi_0 = \beta_0$.

Let us view η as a function of ξ and note that $\eta(0) = |b|$, $\eta(\xi_0) = 0$, and

$$\eta \frac{d\eta}{d\xi} = -e^{-2\xi} b^2 - \xi < 0$$

for $\xi \geq 0$. In particular, η decreases monotonically for $0 \leq \xi \leq \xi_0$.

Consider the relation $\xi \exp(\xi) = b \cos\eta$. At $\xi = 0$, we have $\cos\eta = 0$; therefore, the implicitly defined function η is such that $\eta(0)$ is an odd integer
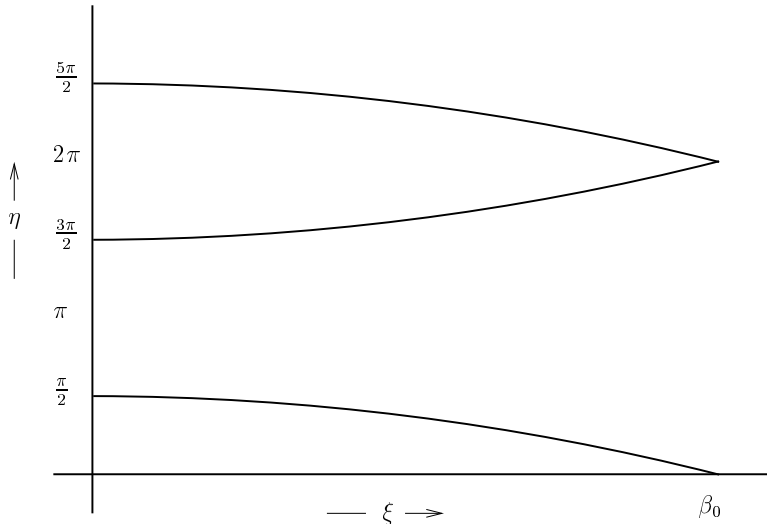
Figure 2: The real branches of $\xi \exp(\xi) = b \cos \eta$ for $b > 0$.

multiple of $\pi/2$. At ξ_0, we have $\cos \eta = \pm 1$ according to the sign of b. In fact, $\eta(\xi_0)$ is an even multiple of π for $b > 0$ and an odd multiple of π for $b < 0$. Also, let us note that

$$(\xi + 1)e^\xi = -b\frac{d\eta}{d\xi} \sin \eta.$$

Suppose that $b > 0$. We will determine the positions of the real branches of the curve defined by $\xi \exp(\xi) = b \cos \eta$. For $0 \leq \eta \leq \pi/2$, we have $\sin \eta > 0$ and $d\eta/d\xi < 0$, so there is a real branch connecting the points $(0, \pi/2)$ and $(\xi_0, 0)$ in the (ξ, η)-plane. For $\pi/2 < \eta < 3\pi/2$, we have $\cos \eta < 0$; thus, there is no real branch in this region. There is a real branch connecting $(0, 3\pi/2)$ and $(\xi_0, 2\pi)$ with $d\eta/d\xi > 0$. This pattern continues as depicted in Figure 2. Note, however, that only the "increasing" branches correspond to poles in the right half-plane. Indeed, for $b > 0$, it is necessary that $\eta \bmod 2\pi$ be in the interval $(3\pi/2, 2\pi)$. In particular, the "lowest" branch corresponding to a pole connects the points $(0, 3\pi/2)$ and $(\xi_0, 2\pi)$. It is now clear that the curve defined by $\eta = \sqrt{b^2 \exp(-2\xi) - \xi^2}$ intersects an increasing branch with $\xi > 0$ if and only if $b > 3\pi/2$. The number of poles in the right half-plane increases by one as b increases past an odd multiple of $\pi/2$.

Suppose that $b < 0$. In this case, the real branches of $\xi \exp(\xi) = b \cos \eta$ exist only if $\cos \eta < 0$ as in Figure 3 and a corresponding pole in the open
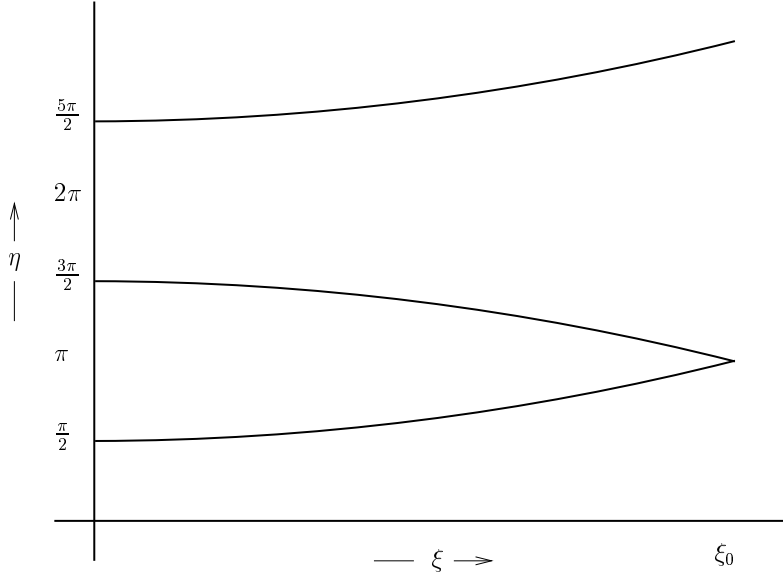
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Figure 3: The real branches of $\xi \exp(\xi) = b\cos\eta$ for $b < 0$.

right half-plane does not exist unless $|b| > \pi/2$. Using the definition of b, this condition is equivalent to the requirement that $\beta_0 \exp(-\beta_0) > \pi/2$. But, the maximum value of $\beta_0 \exp(-\beta_0)$ is $1/e < \pi/2$. Hence, negative values of b do not correspond to poles in the right half-plane.

We conclude that the dynamic memory kernel k for stepwise uniform linear acceleration is unbounded for $\beta_0 = g_0 \alpha > 1.3$.

4.3 Rotation

Imagine next an observer that is initially moving uniformly with speed v in the (x, y)-plane along a line parallel to the y-axis at $x = \rho_0$. At $t = 0$, $x = \rho_0$ and $y = 0$, the observer starts rotating on a circle of radius ρ_0 with uniform frequency $\Omega_0 = v/\rho_0$ in the positive sense around the z-axis. Though the motion is continuous, there is no acceleration for $t < 0$ and uniform circular acceleration for $t > 0$. The natural orthonormal tetrad frame of the uniformly
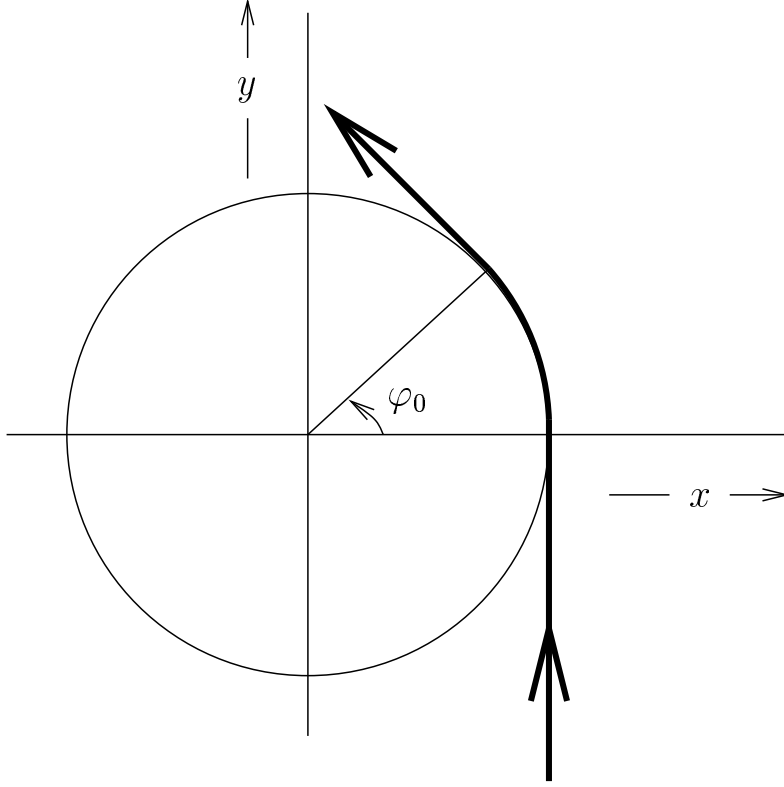
Figure 4: Schematic plot of the motion of the observer that undergoes stepwise uniform rotation of frequency Ω_0 during a period $\alpha = \tau_1 - \tau_2$ of its proper time such that $\varphi_0 = \gamma \Omega_0 \alpha$. If φ_0 exceeds $\pi/2$, then the convolution kernel leads to divergences.

rotating observer is given by

$$
\begin{aligned}
\hat{\lambda}^\mu_{(0)} &= \gamma(1, -v \sin\varphi, v \cos\varphi, 0), \\
\hat{\lambda}^\mu_{(1)} &= (0, \cos\varphi, \sin\varphi, 0), \\
\hat{\lambda}^\mu_{(2)} &= \gamma(v, -\sin\varphi, \cos\varphi, 0), \\
\hat{\lambda}^\mu_{(3)} &= (0, 0, 0, 1),
\end{aligned}
\tag{25}
$$

where $\gamma = (1 - v^2)^{-\frac{1}{2}}$ is the Lorentz factor and $\varphi = \Omega_0 t = \gamma \Omega_0 \tau$, so that we have set $\tau_0 = 0$ in this case. Computing $\phi_{\alpha\beta}$ for the tetrad frame (25), we find as expected that the translational acceleration has only a radial component

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$g_1 = -v\gamma^2\Omega_0$ and the rotational frequency is along the z-direction with magnitude $\Omega_3 = \gamma^2\Omega_0$. Thus $\hat{f} = \Lambda f$, where $\Lambda \to [\Lambda_1; \Lambda_2]$ is given by

$$\Lambda_1 = \begin{bmatrix} \gamma\cos\varphi & \gamma\sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \Lambda_2 = v\gamma \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\cos\varphi & -\sin\varphi & 0 \end{bmatrix}. \tag{26}$$

Let us first consider case (1); the kinetic memory kernel k_0 can be easily computed using the fact that for Λ given by equation (26) we have $\Lambda^{-1} \to [\Lambda_1^T; \Lambda_2^T]$. Then we find that $k_0 \to [\boldsymbol{\Omega}{\cdot}\boldsymbol{I}; -\mathbf{g}\cdot\mathbf{I}]$, where $\boldsymbol{\Omega} = (0, 0, \gamma^2\Omega_0)$ and $\mathbf{g} = (-v\gamma^2\Omega_0, 0, 0)$ with respect to the orthonormal tetrad frame (25). Thus k_0 is a constant kernel so long as the observer rotates uniformly; for instance, if the acceleration is turned off at $\tau_1 = \alpha$ corresponding to $\varphi_0 = \gamma\Omega_0\alpha$, then the observer will have uniform linear motion again with speed v for $\tau > \tau_1$ and the kernel k_0 will vanish (see Figure 4).

Let us now consider case (2); the dynamic memory kernel is given by the series (10) in terms of the resolvent kernel. This is given by equation (9), $r \to [r_1; r_2]$, where

$$
\begin{aligned}
r_1 &= \gamma\Omega_0 \begin{bmatrix} -\gamma^2\sin\varphi & \gamma\cos\varphi & 0 \\ -\gamma\cos\varphi & -\sin\varphi & 0 \\ 0 & 0 & v^2\gamma^2\sin\varphi \end{bmatrix}, \\
r_2 &= v\gamma^2\Omega_0 \begin{bmatrix} 0 & 0 & \gamma\sin\varphi \\ 0 & 0 & \cos\varphi \\ \gamma\sin\varphi & -\cos\varphi & 0 \end{bmatrix}.
\end{aligned} \tag{27}
$$

The explicit calculation of k using the series (10) for the general case of step-wise uniform rotation from $\tau = 0$ to $\tau_1 = \alpha$ is rather complicated; however, for $\tau_1 \to \infty$ the calculation can be carried through and the result is a constant kernel given by $k \to [\boldsymbol{\Omega}{\cdot}\mathbf{I}; -\mathbf{g}\cdot\mathbf{I}]$. Just as in the case of uniform translational acceleration (cf. Section 4), we have $k_0 = k$ for uniform rotation as well.

To calculate k for the stepwise uniform rotation of duration $\tau_1 - \tau_0 = \alpha > 0$, we use Laplace transforms as in the previous section (see Figure 4). Let $C' = \alpha^{-1}\mathcal{L}\{\cos\varphi\}$ and $S' = \alpha^{-1}\mathcal{L}\{\sin\varphi\}$; then, with $w = s\alpha$ we find

$$C' \pm iS' = \frac{1 - e^{-(w \mp i\varphi_0)}}{w \mp i\varphi_0}, \tag{28}$$

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and hence the Laplace transform of the resolvent kernel is given by $\bar{r} \rightarrow [\bar{r}_1; \bar{r}_2]$, where

$$\bar{r}_1 = \varphi_0 \begin{bmatrix} -\gamma^2 S' & \gamma C' & 0 \\ -\gamma C' & -S' & 0 \\ 0 & 0 & v^2 \gamma^2 S' \end{bmatrix}, \qquad \bar{r}_2 = v\gamma\varphi_0 \begin{bmatrix} 0 & 0 & \gamma S' \\ 0 & 0 & C' \\ \gamma S' & -C' & 0 \end{bmatrix}. \qquad (29)$$

Using methods given in Appendix B, equation (19) leads to

$$\bar{k}(s) \rightarrow [\bar{k}_1(s); \bar{k}_2(s)],$$

where

$$\bar{k}_1(s) = \varphi_0 \begin{bmatrix} \gamma^2 \mathcal{Q} & \gamma \mathcal{P} & 0 \\ -\gamma \mathcal{P} & \mathcal{Q} & 0 \\ 0 & 0 & -v^2 \gamma^2 \mathcal{Q} \end{bmatrix}, \qquad (30)$$

$$\bar{k}_2(s) = v\gamma\varphi_0 \begin{bmatrix} 0 & 0 & -\gamma \mathcal{Q} \\ 0 & 0 & \mathcal{P} \\ -\gamma \mathcal{Q} & -\mathcal{P} & 0 \end{bmatrix}. \qquad (31)$$

Here $\mathcal{P}$ and $\mathcal{Q}$ are given by

$$\mathcal{P} = \frac{e^w}{\mathcal{D}}[w(-e^w + \cos\varphi_0) - \varphi_0 \sin\varphi_0], \qquad (32)$$

$$\mathcal{Q} = \frac{1}{\mathcal{D}}[e^w(-w \sin\varphi_0 + \varphi_0 \cos\varphi_0) - \varphi_0], \qquad (33)$$

and the denominator $\mathcal{D}$ is given by

$$\mathcal{D} = (we^w - i\varphi_0 e^{i\varphi_0})(we^w + i\varphi_0 e^{-i\varphi_0}). \qquad (34)$$

It is interesting to note that if we formally substitute β_0 for $i\varphi_0$ in equations (32)–(34), we obtain results familiar from the previous subsection; specifically, under $i\varphi_0 \rightarrow \beta_0$, $\mathcal{P} \rightarrow P$, $\mathcal{Q} \rightarrow iQ$ and $\mathcal{D} \rightarrow D$, where P, Q and D are given in equations (22)–(24). Therefore, the main results of the previous subsection can also be used in the analysis of stepwise uniform rotation; for instance, with appropriate modifications the explicit expressions given in Appendix C for the convolution kernel in a special case can be employed here as well. However, since $\varphi_0 > 0$, the singularities of $\mathcal{P}$ and $\mathcal{Q}$ are in general different from those in the previous subsection.

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To determine the pole singularities of $\bar{k}(s)$ in the right half-plane in the case of stepwise rotation it suffices to consider the equation

$$we^w = i\varphi_0 e^{i\varphi_0} \tag{35}$$

with $\varphi_0 > 0$. Indeed, note that if w is a solution of this equation, then the complex conjugate of w is a solution of $w \exp(w) = -i\varphi_0 \exp(-i\varphi_0)$.

As before, let us set $w = \xi + i\eta$ and note that equation (35) is equivalent to the system of real equations given by

$$\xi e^\xi = \varphi_0 \sin(\eta - \varphi_0), \qquad \eta e^\xi = \varphi_0 \cos(\eta - \varphi_0), \tag{36}$$

where $\xi \geq 0$ and $\varphi_0 > 0$. We recall here that the solution $w = i\varphi_0$, i.e. $\xi = 0$ and $\eta = \varphi_0$, of equations (35) and (36) corresponds to a removable singularity. A necessary condition for system (36) to have a solution with $\xi > 0$ is that $\sin(\eta - \varphi_0) > 0$ and $\eta \cos(\eta - \varphi_0) > 0$; the latter condition means that $\cos(\eta - \varphi_0)$ and η must have the same sign.

Consider the system

$$(\xi^2 + \eta^2)e^{2\xi} = \varphi_0{}^2, \qquad \xi e^\xi = \varphi_0 \sin(\eta - \varphi_0). \tag{37}$$

If it has a solution (ξ, η), then it follows from display (37) that

$$\eta^2 e^{2\xi} = \varphi_0{}^2 \cos^2(\eta - \varphi_0),$$

and therefore $\eta \exp(\xi) = \pm\varphi_0 \cos(\eta - \varphi_0)$. Comparing this result with system (36), we conclude that we can use system (37) for finding the poles if we keep in mind that η and $\cos(\eta - \varphi_0)$ must have the same sign.

The first equation in display (37) is equivalent to $\eta^2 = \varphi_0{}^2 \exp(-2\xi) - \xi^2$. Its graph in the right half-plane has the form depicted in Figure 5, where ξ_0 is the unique real solution of the equation $\xi \exp(\xi) = \varphi_0$.

The poles we seek correspond to the intersections of the graph in Figure 5 with the real branches of the second curve in display (37). The intercepts of these branches with the η-axis are given by the solutions of the equation $\sin(\eta - \varphi_0) = 0$; that is, η is equal to φ_0 plus an integer multiple of π. Along the line given by $\xi = \xi_0$, the intercepts are given by $\xi_0 \exp(\xi_0) = \varphi_0 \sin(\eta - \varphi_0)$. Because $\xi_0 \exp(\xi_0) = \varphi_0$, these intercepts are the solutions of $\sin(\eta - \varphi_0) = 1$; that is, η is $\varphi_0 + \pi/2$ plus an integer multiple of 2π. The shape of the branches connecting points on the two vertical lines (at $\xi = 0$ and $\xi = \xi_0$) is determined by the sign of $\cos(\eta - \varphi_0)$ along the branch. Indeed, we have
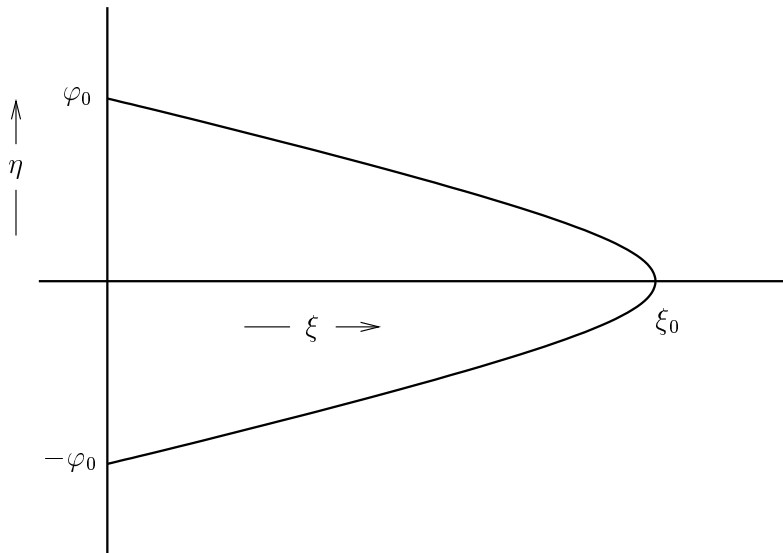
Figure 5: The graph of $\eta^2 = \varphi_0{}^2 \exp(-2\xi) - \xi^2$.

already established that poles occur only at points where η and $\cos(\eta - \varphi_0)$ have the same sign. Note that

$$(\xi + 1)e^\xi = \varphi_0 \frac{d\eta}{d\xi} \cos(\eta - \varphi_0),$$

and therefore the slope of the branch has the same sign as $\cos(\eta - \varphi_0)$. Moreover, only the branches with $\sin(\eta - \varphi_0) > 0$ correspond to poles in the right half-plane.

There are several cases depending on the size of φ_0. For $0 < \varphi_0 < \pi/2$, it is easy to see that the important branches are as depicted in Figure 6. These would not intersect the graph in Figure 5; hence, there are no poles in the right half-plane.

We will next show that if $\varphi_0 > \pi/2$, then there is at least one pole in the right half-plane. For φ_0 in this range, there is an integer $j \geq 1$ such that $j\pi/2 \leq \varphi_0 < (j+1)\pi/2$. In particular, we have that $\varphi_0 - j\pi/2 \geq 0$ and $\varphi_0 - (j+1)\pi/2 < 0$. There are four cases. (1) Suppose that j is even and $\cos(j\pi/2) = 1$. The branch of the curve $\xi \exp(\xi) = \varphi_0 \sin(\eta - \varphi_0)$ with η-intercept $\varphi_0 - j\pi/2 \geq 0$ has positive slope (like the upper branch in Figure 6). Because $\varphi_0 - j\pi/2 < \varphi_0$, this branch intersects the curve depicted in Figure 5 in the upper half-plane. This point corresponds to a
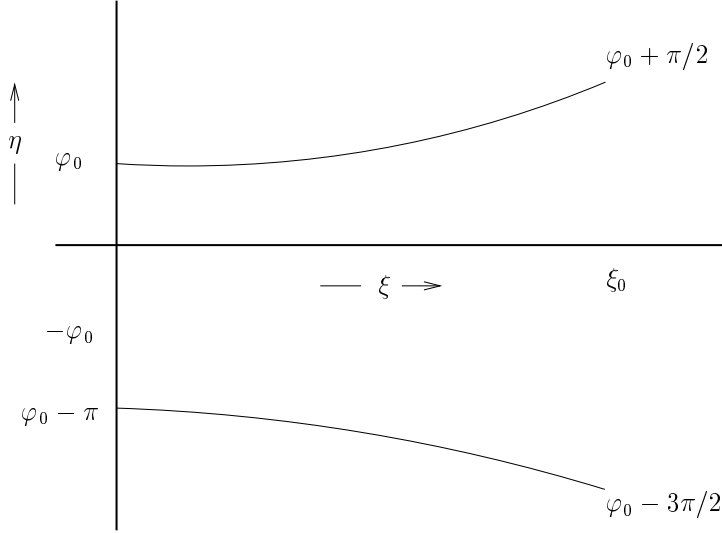
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Figure 6: The graph of $\xi \exp(\xi) = \varphi_0 \sin(\eta - \varphi_0)$ for $0 < \varphi_0 < \pi/2$.

pole. Indeed, at the point of intersection $\sin(\eta - \varphi_0) > 0$ and $\eta \cos(\eta - \varphi_0) > 0$. (2) Suppose that j is even and $\cos(j\pi/2) = -1$. The branch with η-intercept $\varphi_0 - j\pi/2 \geq 0$ has negative slope and meets the line $\xi = \xi_0$ with ordinate $\varphi_0 - (j+1)\pi/2 < 0$. Hence, this branch intersects the curve depicted in Figure 5 in the lower half-plane. This point corresponds to a pole. (3) Suppose that j is odd and $\cos((j+1)\pi/2) = 1$. The branch of the curve with η-intercept $\varphi_0 - (j+1)\pi/2$ has positive slope and it meets the curve depicted in Figure 5 in the upper half-plane where the intersection point corresponds to a pole. For the subcase where $j = 3$ and $\varphi_0 = 3\pi/2$, it is interesting to note that $\eta = 0$ and ξ_0, such that $\xi_0 \exp(\xi_0) = 3\pi/2$, is the pole. (4) Suppose that j is odd and $\cos((j+1)\pi/2) = -1$. The curve with η-intercept $\varphi_0 - (j+1)\pi/2$ has negative slope and $-\varphi_0 \leq \varphi_0 - (j+1)\pi/2$. Hence, this branch meets the curve depicted in Figure 5 in the lower half-plane where the intersection corresponds to a pole.

We conclude that the dynamic memory kernel k for stepwise uniform rotation is unbounded for $\varphi_0 = \gamma \Omega_0 \alpha > \pi/2$.

4.4 Smooth acceleration

We have demonstrated that the convolution kernel k is unbounded for certain stepwise translational and rotational accelerations. Could this result be due to the discontinuities of these accelerations at τ_0 and τ_1? To prove that this is *not* the case, we are interested here instead in smooth accelerations that closely approximate the stepwise ones already studied. The translational and rotational cases are in fact closely related as we have demonstrated; therefore in this subsection we show the same result for the simpler case of *smooth* translational acceleration.

Let us consider an acceleration g with compact support in the interval $[\tau_0, \tau_1]$. By the definition of Λ and the choice of g, the matrix $\Lambda(\tau_0) = \Lambda_0$ is the 6×6 identity matrix. Using this fact and equations (9) and (14), we find that $r(t) = [g(\tau)S(\tau)J_3; g(\tau)C(\tau)I_3]$, where $t = \tau - \tau_0$, $S(\tau) = \sinh\theta$, $C(\tau) = \cosh\theta$ and $\theta(\tau)$ is given by equation (12). It follows from equation (18) that $\bar{r}(s) = [\mathcal{S}(s)J_3; \mathcal{C}(s)I_3]$, where

$$\mathcal{C}(s) \pm \mathcal{S}(s) = \int_0^\infty e^{-st}g(t + \tau_0)e^{\pm\theta(t+\tau_0)}\,dt.$$

Using equation (19) and the results of Appendix B, we find that the Laplace transform of the convolution kernel k is given by $\bar{k}(s) = [\mathcal{H}_1(s)J_3; \mathcal{H}_2(s)I_3]$, where

$$\mathcal{H}_1(s) = \frac{1 + \mathcal{S}}{(1 + \mathcal{S})^2 - \mathcal{C}^2} - 1, \qquad \mathcal{H}_2(s) = -\frac{\mathcal{C}}{(1 + \mathcal{S})^2 - \mathcal{C}^2}.$$

We are interested in the zeros of the denominator

$$(1 + \mathcal{S})^2 - \mathcal{C}^2 = (1 + \mathcal{S} + \mathcal{C})(1 + \mathcal{S} - \mathcal{C}).$$

It suffices to demonstrate that $1 + \mathcal{S} + \mathcal{C}$ has a zero in the right half of the complex s-plane. Because g has compact support in the interval $[\tau_0, \tau_1]$, the function $1 + \mathcal{S} + \mathcal{C}$ is given by

$$s \mapsto 1 + \int_0^\alpha e^{-st}e^{\int_0^t g(\sigma+\tau_0)\,d\sigma}g(t + \tau_0)\,dt,$$

where $\alpha = \tau_1 - \tau_0$. If g is the stepwise uniform acceleration considered previously, then this function reduces to

$$s \mapsto \frac{e^{-\alpha s}}{s - g_0}(se^{\alpha s} - g_0 e^{\alpha g_0});$$

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and, by the results in Subsection 4.2 for equation (24), if $\beta_0 \exp(\beta_0) > 3\pi/2$, it has a zero in the right half of the complex s-plane corresponding to a pole of $\bar{k}$. We will show that such a pole persists for a smooth acceleration that is sufficiently close to the stepwise acceleration.

For an arbitrary acceleration g with support in the interval $[\tau_0, \tau_1]$, we define the associated real-valued function ζ on the interval $[0, \alpha]$ given by $\zeta(t) = g(t + \tau_0)$. Also, recall that the L^1-norm of a real-valued function v defined on the interval $[0, \alpha]$ is given by

$$\|v\|_1 := \int_0^\alpha |v(t)|\, dt.$$

Suppose that ζ and v are real-valued functions defined on the interval $[0, \alpha]$ such that $\|\zeta\| < \infty$ and $\|v\|_1 < \infty$, and consider the complex-valued analytic functions Z and Υ of the complex variable s given by

$$Z(s) \;=\; 1 + \int_0^\alpha e^{-st} e^{\int_0^t \zeta(\sigma)\, d\sigma} \zeta(t)\, dt,$$

$$\Upsilon(s) \;=\; 1 + \int_0^\alpha e^{-st} e^{\int_0^t v(\sigma)\, d\sigma} v(t)\, dt.$$

We will prove the following proposition. *If Z has a zero in the open right-half of the complex s-plane and $\|v - \zeta\|_1$ is sufficiently small, then Υ has a zero in the open right-half of the complex s-plane.* By a standard result from mathematical analysis (see [27]), if ζ is an L^1 function (for example, if $\zeta(t) = g(t + \tau_0)$ for the stepwise acceleration g), then $\|\zeta - v\|_1$ can be made as small as desired for a C^∞ function v. Hence, by the proposition, there is a smooth acceleration with compact support such that its associated convolution kernel is unbounded.

Our proof begins with two estimates. For notational convenience, let us define

$$\hat\zeta(t) = e^{\int_0^t \zeta(\sigma)\, d\sigma} \zeta(t), \qquad \hat v(t) = e^{\int_0^t v(\sigma)\, d\sigma} v(t).$$

The first estimate is

$$|\Upsilon(s) - Z(s)| \leq \|\hat v - \hat\zeta\|_1 \tag{38}$$

for all s such that $\mathrm{Re}(s) \geq 0$. To prove it, note that

$$|\Upsilon(s) - Z(s)| \leq \int_0^\alpha |e^{-st}||\hat v(t) - \hat\zeta(t)|\, dt.$$

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Because $|\exp(-st)| \leq 1$ for $\operatorname{Re}(s) \geq 0$, we have the inequality

$$|\Upsilon(s) - Z(s)| \leq \|\hat{v} - \hat{\zeta}\|_1$$

for all s in the closed right half-plane. The second estimate is

$$\|\hat{v} - \hat{\zeta}\|_1 \leq e^{\|\zeta\|_1}(1 + \alpha\|\zeta\|)e^{\|v-\zeta\|_1}\|v - \zeta\|_1. \tag{39}$$

To prove it, we have the triangle law estimate

$$
\begin{aligned}
|\hat{v}(t) - \hat{\zeta}(t)| &\leq |e^{\int_0^t v(\sigma)\,d\sigma}v(t) - e^{\int_0^t v(\sigma)\,d\sigma}\zeta(t)| \\
&\quad + |e^{\int_0^t v(\sigma)\,d\sigma}\zeta(t) - e^{\int_0^t \zeta(\sigma)\,d\sigma}\zeta(t)| \\
&\leq e^{\int_0^t |v(\sigma)|\,d\sigma}|v - \zeta| + |\zeta||e^{\int_0^t v(\sigma)\,d\sigma} - e^{\int_0^t \zeta(\sigma)\,d\sigma}| \tag{40}
\end{aligned}
$$

and, by the mean value theorem (applied to the exponential function), the inequality

$$\left|e^{\int_0^t v(\sigma)\,d\sigma} - e^{\int_0^t \zeta(\sigma)\,d\sigma}\right| \leq e^{\varsigma}\left|\int_0^t v(\sigma)\,d\sigma - \int_0^t \zeta(\sigma)\,d\sigma\right|,$$

where ς is some number between $\int_0^t v(\sigma)\,d\sigma$ and $\int_0^t \zeta(\sigma)\,d\sigma$. If $\varsigma \leq 0$, then $\exp(\varsigma) < 1$; and if $\varsigma > 0$, then $\varsigma < \max\{\|v\|_1, \|\zeta\|_1\}$. Hence,

$$e^{\varsigma} \leq e^{\max\{\|v\|_1, \|\zeta\|_1\}}$$

and, because $\|v\|_1 \leq \|v - \zeta\|_1 + \|\zeta\|_1$, we have that

$$e^{\varsigma} \leq e^{\|\zeta\|_1}e^{\|v-\zeta\|_1}.$$

Using this result and the estimate (40), it follows that

$$
\begin{aligned}
|\hat{v}(t) - \hat{\zeta}(t)| &\leq e^{\|v\|_1}|v(t) - \zeta(t)| + \|\zeta\|e^{\|\zeta\|_1}e^{\|v-\zeta\|_1}\int_0^{\alpha}|v(\sigma) - \zeta(\sigma)|\,d\sigma \\
&\leq e^{\|\zeta\|_1}e^{\|v-\zeta\|_1}(|v(t) - \zeta(t)| + \|\zeta\|\|v - \zeta\|_1).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\hat{v} - \hat{\zeta}\|_1 &= \int_0^{\alpha}|\hat{v} - \hat{\zeta}|\,dt \\
&\leq e^{\|\zeta\|_1}e^{\|v-\zeta\|_1}(\|v - \zeta\|_1 + \alpha\|\zeta\|\|v - \zeta\|_1)
\end{aligned}
$$

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and a rearrangement of the right-hand side of the last inequality gives the desired result.

In the rest of this section, we let ζ represent the stepwise uniform linear acceleration and υ the smooth linear acceleration that approximates it sufficiently closely. We then choose a circle centered at a zero of Z in the open right half of the complex s-plane such that the circle does not pass through a zero of Z and such that the circle is contained in the open right half-plane. Let κ, a complex-valued function defined on the interval $[0, 2\pi]$, be a continuous parametrization of this circle and define two new functions κ_Z and κ_Υ on this interval by

$$\kappa_Z(\vartheta) = Z(\kappa(\vartheta)), \qquad \kappa_\Upsilon(\vartheta) = \Upsilon(\kappa(\vartheta)).$$

The images of these functions are closed curves in the complex s-plane. In complex analysis, the principle of the argument theorem [28] for an analytic function Δ relates the winding number of the image of κ_Δ with respect to the origin to the number of zeros of the function Δ inside the circle, provided that the circle does not pass through any zero of Δ. If we show that κ_Z and κ_Υ are homotopic and therefore have the same winding number with respect to the origin and that the circle does not pass through a zero of Υ, then Z and Υ must have the same number of zeros inside the circle.

We claim that if $\|\upsilon - \zeta\|_1$ is sufficiently small, then the image of κ does not pass through a zero of Υ. To prove the claim, note that

$$m := \min\{|\kappa_Z(\vartheta)| : 0 \leq \vartheta \leq 2\pi\} > 0$$

(because κ does not pass through a zero of Z) and using the triangle inequality

$$0 < m \leq |\kappa_\Upsilon(\vartheta)| + \|\kappa_\Upsilon - \kappa_Z\|,$$

where $\|\kappa_\Upsilon - \kappa_Z\|$ is the supremum of $|\kappa_\Upsilon(\vartheta) - \kappa_Z(\vartheta)|$ for $0 \leq \vartheta \leq 2\pi$. Using the estimates (38) and (39), we have that

$$|\kappa_\Upsilon(\vartheta) - \kappa_Z(\vartheta)| \leq e^{\|\zeta\|_1} e^{\|\upsilon - \zeta\|_1} (1 + \alpha\|\zeta\|) \|\upsilon - \zeta\|_1. \tag{41}$$

By the estimate (41), $\|\kappa_\Upsilon - \kappa_Z\|$ can be made small, say less than $m/2$, by taking $\|\upsilon - \zeta\|_1$ sufficiently small. For all υ satisfying this requirement, which we impose for the remainder of the proof, we have that $|\kappa_\Upsilon(\vartheta)| > 0$; that is, κ does not pass through a zero of Υ.

It remains to show that κ_Υ is homotopic to κ_Z. Assuming this homotopy relation, the image curves of κ_Υ and κ_Z would have the same winding number with respect to the origin. By the choice of κ and the argument principle (see [28]), the curve κ_Z has a nonzero winding number. Hence, κ_Υ would have the same nonzero winding number. Again, by the argument principle, Υ must then have a zero in the disk bounded by the circle parametrized by κ, which is the desired result.

To complete the proof we need to show that κ_Υ and κ_Z are indeed homotopic. Let $\mathbb{C}$ denote the complex numbers. We will show that $H : [0,1] \times [0, 2\pi] \to \mathbb{C} \setminus \{0\}$ given by

$$H(\sigma, \vartheta) = \kappa_Z(\vartheta) - \sigma(\kappa_Z(\vartheta) - \kappa_\Upsilon(\vartheta))$$

is the required homotopy. By inspection, H is continuous, $H(0, \vartheta) = \kappa_Z(\vartheta)$ and $H(1, \vartheta) = \kappa_\Upsilon(\vartheta)$. Hence, it suffices to show that $H(\sigma, \vartheta) \neq 0$ for all $(\sigma, \vartheta) \in [0,1] \times [0, 2\pi]$. By our choice of υ, we have that $\|\kappa_\Upsilon - \kappa_Z\| < m/2$; therefore

$$|H(\sigma, \vartheta)| \geq |\kappa_Z(\vartheta)| - |\sigma||\kappa_\Upsilon(\vartheta) - \kappa_Z(\vartheta)| \geq m - \|\kappa_\Upsilon - \kappa_Z\| > m/2,$$

as required.

We conclude that the dynamic memory kernel k for the smooth linear acceleration that closely approximates the stepwise acceleration is unbounded if the area under the graph of $g(\tau)$ exceeds a critical value ~ 1.

5   Discussion

We have investigated the properties of the nonlocal kernel that is induced by accelerated motion in Minkowski spacetime. The physical principles outlined in this paper do not completely determine the kernel; therefore, simplifying mathematical assumptions need to be introduced in order to identify a unique kernel. Two possibilities have been explored in this work corresponding to kinetic memory (k_0) and dynamic memory (k). We show that for accelerated motion that is uniform (linear or circular), the two kernels give the same constant result $k_0 = k$. They differ, however, if the acceleration is turned off at a certain moment. We have therefore studied piecewise uniform acceleration (linear and circular) and have demonstrated that the dynamic memory (convolution) kernel could be divergent and is therefore ruled out. Furthermore,

this conclusion is shown to be independent of the stepwise character of the linear acceleration considered.

The use of convolution kernels is standard practice in the nonlocal electrodynamics of continuous media, where it is assumed phenomenologically that memory always fades. In our treatment of acceleration-induced nonlocality in vacuum, however, the behavior of memory must be determined from first principles. In this connection, the possible advantage of kinetic memory in terms of its simplicity was first emphasized by Hehl and Obukhov [25, 26].

The theory developed here is applicable to any basic field; however, for the sake of concreteness and in view of possible observational consequences, we employ electromagnetic radiation fields throughout. A basic consequence of the nonlocal theory of accelerated systems is that it is incompatible with the existence of a basic scalar field; that is, in this case $\Lambda(\tau) = 1$, $k_0 = 0$ and the nonlocality disappears so that a basic scalar radiation field can stay completely at rest with respect to a rotating observer in contradiction with our fundamental physical assumption. This prediction of the nonlocal theory is in agreement with present experimental data. Further confrontation of the nonlocal theory with observation is urgently needed.

Appendix A

Consider an integral equation of the form

$$\phi(x) = \psi(x) + \epsilon \int_a^x K(x, y)\phi(y)\, dy, \qquad (A1)$$

where ψ is a continuous function, the kernel K is continuous and ϵ is a constant parameter. There is a unique continuous resolvent kernel R such that

$$\psi(x) = \phi(x) + \epsilon \int_a^x R(x, y)\psi(y)\, dy. \qquad (A2)$$

In turn, K can be thought of as the resolvent kernel for R; this follows from the complete reciprocity between K and R.

The proof of the existence and uniqueness of the resolvent kernel is by successive approximation. In fact, the solution ϕ can be obtained as the uniform limit of the sequence of continuous functions $\{\phi_n\}_{n=0}^\infty$ defined as

follows: $\phi_0(x) = \psi(x)$ and

$$\phi_{n+1}(x) = \psi(x) + \epsilon \int_a^x K(x,y)\phi_n(y)\,dy. \tag{A3}$$

Thus

$$\phi_1(x) = \psi(x) + \epsilon \int_a^x K(x,y)\psi(y)\,dy, \tag{A4}$$

$$\phi_2(x) = \psi(x) + \epsilon \int_a^x K(x,y)[\psi(y) + \epsilon \int_a^y K(y,z)\psi(z)\,dz]\,dy$$

$$= \phi_1(x) + \epsilon^2 \int_a^x K(x,y) \int_a^y K(y,z)\psi(z)\,dzdy. \tag{A5}$$

The integration in (A5) is over a triangular domain in the (y,z)-plane defined by the vertices (a,a), (x,a) and (x,x). Changing the order of the integration in (A5) results in the equality

$$\int_a^x K(x,y) \left[\int_a^y K(y,z)\psi(z)dz \right] dy$$

$$= \int_a^x \left[\int_z^x K(x,y)K(y,z)\,dy \right] \psi(z)\,dz. \tag{A6}$$

Let us define the successive iterated kernels of K by $K_1(x,z) = K(x,z)$ and

$$K_{n+1}(x,z) = \int_z^x K(x,y)K_n(y,z)\,dy. \tag{A7}$$

Then we can write (A5) as

$$\phi_2(x) = \phi_1(x) + \epsilon^2 \int_a^x K_2(x,z)\psi(z)\,dz, \tag{A8}$$

and similarly

$$\phi_3(x) = \phi_2(x) + \epsilon^3 \int_a^x K_3(x,z)\psi(z)\,dz, \tag{A9}$$

etc., such that in general

$$\phi_m(x) = \phi_{m-1}(x) + \epsilon^m \int_a^x K_m(x,z)\psi(z)\,dz. \tag{A10}$$

Iterating (A10) for $m = 1, 2, 3, \ldots, n$ and summing the equations results in

$$\phi_n(x) = \psi(x) + \int_a^x \left[\sum_{m=1}^{n} \epsilon^m K_m(x, z) \right] \psi(z)\, dz, \qquad (A11)$$

which can be rewritten as

$$\psi(x) = \phi_n(x) + \epsilon \int_a^x \left[-\sum_{m=1}^{n} \epsilon^{m-1} K_m(x, y) \right] \psi(y)\, dy. \qquad (A12)$$

It can be shown that the uniform limit as $n \to \infty$ exists (see [10, 11, 12]). Thus, we obtain equation (A2) with

$$R(x, y) = -\sum_{n=1}^{\infty} \epsilon^{n-1} K_n(x, y). \qquad (A13)$$

In case (1), $K(x, y) = k_0(y)$, the iterated kernels K_n for $n > 1$ and the resolvent kernel R are in general functions of both x and y.

In case (2), $K(x, y) = k(x - y)$, i.e. the kernel is of the convolution (Faltung) type, it follows from (A7) that

$$k_{n+1}(t) = \int_0^t k(u)k_n(t - u)\, du, \qquad (A14)$$

where $x - z = t$ and $x - y = u$; therefore, all of the iterated kernels are of the convolution type and can be obtained by successive convolutions of k with itself. More precisely, let a star denote the Faltung operation,

$$\phi * \chi(t) = \int_0^t \phi(t)\chi(t - u)\, du = \chi * \phi(t), \qquad (A15)$$

and write $\phi^{*2} = \phi * \phi$, etc. Then, the resolvent kernel (A13) can be expressed as $R(x, y) = r(x - y)$, where

$$r(t) = -\sum_{n=1}^{\infty} \epsilon^{n-1} k^{*n}(t). \qquad (A16)$$

Appendix B

In this paper, we deal with 6×6 matrices of the form

$$\mathcal{M} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}, \tag{B1}$$

where $\det A \neq 0$ and $\det B = 0$. The inverse of the matrix $\mathcal{M}$ is given by

$$\mathcal{M}^{-1} = \begin{bmatrix} G & H \\ -H & G \end{bmatrix}, \tag{B2}$$

where

$$G = (A + BA^{-1}B)^{-1}, \quad H = -GBA^{-1} = -A^{-1}BG. \tag{B3}$$

Appendix C

Let us rewrite $P(s)$ and $Q(s)$ given by equations (22) and (23) in the form

$$2\beta_0 \, P(s) = \frac{1 - \frac{\beta_0}{w}}{1 - \zeta_+} - \frac{1 + \frac{\beta_0}{w}}{1 + \zeta_-}, \tag{C1}$$

$$2\beta_0 \, Q(s) = -2 + \frac{1 - \frac{\beta_0}{w}}{1 - \zeta_+} + \frac{1 + \frac{\beta_0}{w}}{1 + \zeta_-}, \tag{C2}$$

where $w = s\alpha$, $\beta_0 = g_0\alpha$ and $\zeta_\pm$ are given by

$$\zeta_\pm = \frac{\beta_0}{w} \exp(-w \pm \beta_0). \tag{C3}$$

If we assume that $\mathrm{Re}(s) > g_0$, then $|\zeta_\pm| < 1$. We can therefore expand $(1 \mp \zeta_\pm)^{-1}$ in powers of $\zeta_\pm$ and use the relation

$$\mathcal{L}\left\{ u_{n\alpha}(t) \frac{(t - n\alpha)^{\ell-1}}{(\ell - 1)!} \right\} = \frac{e^{-n\alpha s}}{s^\ell} \tag{C4}$$

for integers $n \geq 0$ and $\ell > 1$ to find $k(t) \rightarrow [k_1(t); k_2(t)]$. Here we use unit step functions such that $u_{n\alpha}(t) = u_0(t - n\alpha)$ and $u_0(t)$ is the standard unit step function, i.e. $u_0(t) = 1$ for $t \geq 0$ and $u_0(t) = 0$ for $t < 0$.

We find that $k_1 = \tilde{k}_1 J_3$, $k_2 = \tilde{k}_2 I_3$ and

$$g_0^{-1}\tilde{k}_1(t) = \mathcal{S}_1 u_\alpha(t) + \mathcal{C}_2 u_{2\alpha}(t) + \mathcal{S}_3 u_{3\alpha}(t) + \mathcal{C}_4 u_{4\alpha}(t) + \cdots, \quad \text{(C5)}$$

$$g_0^{-1}\tilde{k}_2(t) + u_0(t) = \mathcal{C}_1 u_\alpha(t) + \mathcal{S}_2 u_{2\alpha}(t) + \mathcal{C}_3 u_{3\alpha}(t) + \mathcal{S}_4 u_{4\alpha}(t) + \cdots, \quad \text{(C6)}$$

where

$$\mathcal{C}_n \pm \mathcal{S}_n = e^{\pm n\beta_0} \left[\frac{(g_0 t - n\beta_0)^{n-1}}{(n-1)!} \mp \frac{(g_0 t - n\beta_0)^n}{n!} \right]. \quad \text{(C7)}$$

Note that for any fixed value of t, only a finite number of terms contribute to the kernel $k(t)$.

Acknowledgements

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