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Holonomy and gravitomagnetism

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Abstract
We analyse parallel transport of a vector field around an equatorial orbit in the Kerr and stationary axisymmetric spacetimes that are reflection symmetric about their equatorial planes. As in the Schwarzschild spacetime, there is a band structure of holonomy invariance. The new feature introduced by rotation is a shift in the timelike component of the vector, which is the holonomic manifestation of the gravitomagnetic clock effect.

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1. Introduction

If a vector is parallel-transported around an equatorial circle in the exterior Schwarzschild spacetime, one may expect that the vector would be unchanged on return to the starting point, given the spherical symmetry and staticity of the spacetime. However, it turns out that, in general, the vector is shifted after a closed loop (i.e. there is a deficit angle), as shown by Rothman *et al* [1]. This result reflects the nonlocal nature of holonomy; parallel transport carries an imprint of the curvature enclosed by the loop. What is also interesting in the results of [1] is that the shift vanishes for n circuits of a closed loop at radius r if n and r satisfy an appropriate condition. In other words, there is a band structure of holonomy invariance in the Schwarzschild spacetime.

In light of these results, a natural question is how this holonomy is modified by rotation. Rotation introduces a gravitomagnetic clock effect [2, 3], whereby co- and counter-rotating orbital periods differ, and one expects that the modified holonomy should reflect the existence of this clock effect. We consider the holonomy around a closed circle in the equatorial plane in the Kerr and stationary axisymmetric spacetimes that, like Kerr, are reflection symmetric about their equatorial planes. As in the Schwarzschild case, there is a band structure of holonomy invariance. Unlike the Schwarzschild case, there is a shift in the timelike component of the transported vector. In Schwarzschild spacetime, parallel transport around a loop leaves the timelike component invariant. Rotation of the source and the associated gravitomagnetism lead to timelike holonomy. For a four-velocity vector, this holonomy is a Lorentz boost, and the corresponding local observer would measure a time dilation.

Thus, in these stationary spacetimes, holonomy is a geometric counterpart of the gravitomagnetic clock effect; these are two related nonlocal signatures of the curvature. (It should be mentioned that a different treatment of holonomy in the Kerr geometry using the Wilson loops is given in [4].)

2. Kerr holonomy

To study the connection between holonomy and gravitomagnetism associated with the rotation of the source, we first consider the exterior Kerr spacetime. In Boyer–Lindquist coordinates [2], the metric is

$$(g_{\mu\nu}) = \begin{pmatrix} -(1-2Mr/\Sigma) & 0 & 0 & -2aMr\sin^2\theta/\Sigma \\ 0 & \Sigma/\Delta & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -2aMr\sin^2\theta/\Sigma & 0 & 0 & \sin^2\theta\left[r^2+a^2+2a^2Mr\sin^2\theta/\Sigma\right] \end{pmatrix} \tag{1}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2 \tag{2}$$

and $x^\mu = (x^0, x^i) = (t, r, \theta, \varphi)$, with $0 < \theta < \pi$ and $0 \leqslant \varphi < 2\pi$. The inverse metric in the equatorial plane, $\theta = \pi/2$, is given by

$$\left(g^{\mu\nu}\big|_{\theta=\pi/2}\right) = \begin{pmatrix} -(r^3+a^2r+2a^2M)/(r\Delta) & 0 & 0 & -2aM/(r\Delta) \\ 0 & \Delta/r^2 & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ -2aM/(r\Delta) & 0 & 0 & (r-2M)/(r\Delta) \end{pmatrix}. \tag{3}$$

Units are chosen so that $G = c = 1$; the mass is M and $a = J/M$ is the specific angular momentum of the Kerr source.

Now consider parallel transport around a circular orbit (r and t constant) in the equatorial plane to define a vector field $u^\mu(\varphi)$. The field satisfies

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\varphi} + \Gamma^\mu_{3\nu}u^\nu = 0 \tag{4}$$

on the orbit. Since $\Gamma^2_{3\mu} = 0$ in the equatorial plane, we have $\mathrm{d}u^2/\mathrm{d}\varphi = 0$; moreover, since $\Gamma^\mu_{23} = 0$ in the equatorial plane as well, we can consistently set $u^2 = 0$ in what follows. For the remaining components,

$$\frac{\mathrm{d}u^0}{\mathrm{d}\varphi} = \frac{aM}{\Delta}\left(3 + \frac{a^2}{r^2}\right)u^1 \tag{5}$$

$$\frac{\mathrm{d}u^1}{\mathrm{d}\varphi} = \frac{\Delta}{r^4}[aM\,u^0 + (r^3 - a^2M)u^3] \tag{6}$$

$$\frac{\mathrm{d}u^3}{\mathrm{d}\varphi} = \frac{1}{r^2\Delta}(a^2M + 2Mr^2 - r^3)u^1\,. \tag{7}$$

The first important conclusion following from these equations is that in the Schwarzschild spacetime, $\mathrm{d}u^0/\mathrm{d}\varphi = 0$. Thus, *there is no holonomy effect on the timelike component in the absence of rotation*. This is consistent with the absence of a gravitomagnetic clock effect in the Schwarzschild spacetime.

In the general case with $a \neq 0$, the system in equations (5)–(7) gives a third-order equation for u^0:

$$\frac{\mathrm{d}^3u^0}{\mathrm{d}\varphi^3} + \frac{F(r)}{r^4}\frac{\mathrm{d}u^0}{\mathrm{d}\varphi} = 0 \tag{8}$$

where

$$F(r) = r^4 - 2Mr^3 - 2a^2Mr - a^2M^2. \tag{9}$$

The equation $F(r) = 0$ has only one positive root, r_*, by Descartes' rule of signs. To investigate it, we write the equation in the following forms:

$$F(r_*) = r_*^2\Delta(r_*) - a^2(r_* + M)^2 = 0 \qquad r_*^3(r_* - 2M) = a^2M(2r_* + M). \tag{10}$$

It follows that $F(r)$ does not vanish on the horizons $r = r_\pm$ (with $r_+ > r_-$) at $\Delta = 0$ for $a^2 > M^2$, and that $r_* > 2M$ in general. Specifically,

$$M^2 > a^2 > 0 \quad \Rightarrow \quad r_+ < 2M < r_* < \tfrac{5}{2}M \tag{11}$$

$$M^2 = a^2 \quad \Rightarrow \quad r_+ < r_* = (1 + \sqrt{2})M \tag{12}$$

$$M^2 < a^2 \quad \Rightarrow \quad r_* > 2M. \tag{13}$$

Thus r_* lies beyond the ergosphere, and beyond the horizon in each case except the last, where there is no horizon. In the limit of vanishing a/M, however, r_* approaches the Schwarzschild horizon, since, for $a^2/M^2 \ll 1$,

$$r_* = 2M\left[1 + \epsilon - \tfrac{11}{5}\epsilon^2 + O(\epsilon^3)\right] \qquad \epsilon = \frac{5}{16}\left(\frac{a}{M}\right)^2. \tag{14}$$

The root r_* has a further significance, since it is a critical value for 'geodesic meeting point' (gmp) observers [5, 6], for which

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2_{\mathrm{gmp}} = \frac{r^2F(r)}{(r^3 - Ma^2)^2} \tag{15}$$

where τ is the proper time for gmp observers. Thus gmp observers exist only for $r > r_*$ [6].

It is useful to divide the solution of equation (8) into three parts.

(a) For $r > r_*$, i.e. $F(r) > 0$, the general solution is

$$u^0(\varphi) = C + A\sin[f(r)\varphi] + B\cos[f(r)\varphi] \tag{16}$$

$$u^1(\varphi) = \frac{f(r)\Delta(r)}{\Gamma(r)}\{A\cos[f(r)\varphi] - B\sin[f(r)\varphi]\} \tag{17}$$

$$u^3(\varphi) = \left[\frac{1}{a} - \frac{r + M}{\Gamma(r)}\right]u^0(\varphi) + \frac{CF(r)}{(r^3 - a^2M)\Gamma(r)} \tag{18}$$

where A, B and C are constants determined by initial conditions and

$$f(r) = \frac{\sqrt{F(r)}}{r^2} \qquad \Gamma(r) = aM\left(3 + \frac{a^2}{r^2}\right). \tag{19}$$

Starting from $\varphi = 0$ the shift δu^μ after n closed loops of parallel transport is

$$\delta u^0 = A\sin[2\pi nf(r)] - 2B\sin^2[\pi nf(r)] \tag{20}$$

$$\delta u^1 = -\frac{f(r)\Delta(r)}{\Gamma(r)}\{A\sin^2[\pi nf(r)] + B\sin[2\pi nf(r)]\} \tag{21}$$

$$\delta u^3 = \left[\frac{1}{a} - \frac{r + M}{\Gamma(r)}\right]\delta u^0. \tag{22}$$

Just as in the Schwarzschild geometry [1], there exists a band structure of holonomy invariance in the Kerr geometry for $r > r_*$, i.e. there are radius values r for which n circuits lead to a net zero shift in the vector. The condition for holonomy invariance in this case is

$$nf(r) = m \quad \Rightarrow \quad F(r) = \frac{m^2}{n^2} r^4 \tag{23}$$

where m is a positive integer. For $r < r_*$, equation (23) implies that $m^2 < n^2$, just as in the Schwarzschild case. It follows from Descartes' rule of signs that equation (23) has only one positive root r^* for $m^2 < n^2$. Furthermore, since F is positive and monotonically increasing for $r > r_*$, this root is such that $r^* > r_*$. For fixed m, there is a minimum n that results in holonomy invariance; in particular, no such invariance exists for finite r if $n = m$. For instance, the holonomy around a constant-time circle of radius $r^* = 3M$ vanishes for $n = 9$ and $m = 5$ if the Kerr black hole has $a^2/M^2 = \frac{2}{7}$. Another example of holonomy invariance for this circle is provided by $n = 9$, $m = 4$ and $a^2/M^2 = \frac{11}{7}$.

We note that as $a^2/M^2 \to 0$, the band structure of holonomy invariance reduces to that of the Schwarzschild geometry studied in [1]. In fact, for $a^2/M^2 \ll 1$ and fixed integers n and m, the solution of equation (23) is given by

$$r^* = 2M\left[\left(1 - \frac{m^2}{n^2}\right)^{-1} + \epsilon\left(1 - \frac{m^2}{n^2}\right)\left(1 - \frac{m^2}{5n^2}\right) + O(\epsilon^2)\right] \tag{24}$$

where ϵ is given in equation (14).
(b) For $r = r_*$, i.e. $F(r) = 0$, the solution of equation (8) is simply

$$u^0 = \tilde{C} + \tilde{A}\varphi + \tilde{B}\varphi^2 \tag{25}$$

$$u^1 = \frac{\Delta(r_*)}{\Gamma(r_*)}\left[\tilde{A} + 2\tilde{B}\varphi\right] \tag{26}$$

$$u^3 = \left[\frac{1}{a} - \frac{r_* + M}{\Gamma(r_*)}\right]u^0 + \frac{2r_*^4\tilde{B}}{(r_*^3 - a^2M)\Gamma(r_*)} \tag{27}$$

where $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are constants. The shift in u^μ after n closed loops is

$$\delta u^0 = 2\pi n\tilde{A} + 4\pi^2 n^2\tilde{B} \tag{28}$$

$$\delta u^1 = 4\pi n\frac{\tilde{B}\Delta(r_*)}{\Gamma(r_*)} \tag{29}$$

$$\delta u^3 = \left[\frac{1}{a} - \frac{r_* + M}{\Gamma(r_*)}\right]\delta u^0. \tag{30}$$

(c) For $r < r_*$, i.e. $F(r) < 0$, the solution of equation (8) is

$$u^0(\varphi) = \hat{C} + \hat{A}\sinh[h(r)\varphi] + \hat{B}\cosh[h(r)\varphi] \tag{31}$$

where $\hat{A}$, $\hat{B}$ and $\hat{C}$ are constants, and

$$h(r) = \frac{\sqrt{-F(r)}}{r^2}. \tag{32}$$

The other components can be obtained in a similar way from equations (16)–(18) by letting $f \to ih$. The shift in u^0 after n closed loops is

$$\delta u^0 = \hat{A}\sinh[2\pi n h(r)] + 2\hat{B}\sinh^2[\pi n h(r)] \tag{33}$$

and the other components of the shift can be determined using equations (21) and (22).

3. Stationary axisymmetric holonomy

It is possible to extend the main results of our analysis in section 2 to stationary axisymmetric spacetimes that are symmetric under a reflection about their equatorial planes. Let us therefore consider a general stationary axisymmetric spacetime such that in symmetry-adapted coordinates the metric can be written in the form

$$(g_{\mu\nu}) = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\varphi} \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{t\varphi} & 0 & 0 & g_{\varphi\varphi} \end{pmatrix} \tag{34}$$

and the inverse metric is given by

$$(g^{\mu\nu}) = \begin{pmatrix} -\Psi^{-1}g_{\varphi\varphi} & 0 & 0 & \Psi^{-1}g_{t\varphi} \\ 0 & g_{rr}^{-1} & 0 & 0 \\ 0 & 0 & g_{\theta\theta}^{-1} & 0 \\ \Psi^{-1}g_{t\varphi} & 0 & 0 & -\Psi^{-1}g_{tt} \end{pmatrix} \tag{35}$$

where $\Psi \equiv -\left(g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2\right)$. Note that Ψ reduces, in the case of the Kerr metric, to $\Delta\sin^2\theta$, where Δ is defined in equation (2).

The spacetime is stationary, so that the vector normal to the hypersurfaces of constant time t can be expressed as $\partial_t - G\partial_\varphi$, where $G = -g^{t\varphi}/g^{tt} = g_{t\varphi}/g_{\varphi\varphi}$. In general $G \neq 0$ and so this vector is different from the timelike Killing vector ∂_t.

The requirement that the spacetime be reflection symmetric about the equatorial plane implies that $g_{\mu\nu}(t, r, \theta, \varphi) = g_{\mu\nu}(t, r, \pi - \theta, \varphi)$. Therefore, $g_{\mu\nu,\theta} = 0$ for $\theta = \pi/2$. In this case, one can prove that in general there are circular geodesic orbits in the equatorial plane. In fact, the radial component of the geodesic equation for circular orbits with $\theta = \pi/2$ reduces to

$$g_{\varphi\varphi,r}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + 2g_{t\varphi,r}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) + g_{tt,r} = 0 \tag{36}$$

since $\Gamma^r_{\alpha\beta} = -\frac{1}{2}g^{rr}g_{\alpha\beta,r}$ for $\alpha, \beta = t$ or φ. Thus there are, in general, two angular frequencies in this case corresponding to co- and counter-rotating orbits. The clock effect arises since in general $g_{t\varphi,r} \neq 0$. (For further details see [6].)

For the spacetime considered here, equations (5)–(7) describing the parallel transport of an arbitrary vector field around a circle of fixed t and r in the equatorial plane take the form

$$\frac{\mathrm{d}u^0}{\mathrm{d}\varphi} = \left[\frac{g_{\varphi\varphi}^2}{2\Psi}G_{,r}\right]_{\theta=\pi/2} u^1 \tag{37}$$

$$\frac{\mathrm{d}u^1}{\mathrm{d}\varphi} = \left[\frac{g_{t\varphi,r}}{2g_{rr}}\right]_{\theta=\pi/2} u^0 + \left[\frac{g_{\varphi\varphi,r}}{2g_{rr}}\right]_{\theta=\pi/2} u^3 \tag{38}$$

$$\frac{\mathrm{d}u^3}{\mathrm{d}\varphi} = \left[\frac{g_{tt}g_{\varphi\varphi,r} - g_{t\varphi}g_{t\varphi,r}}{2\Psi}\right]_{\theta=\pi/2} u^1. \tag{39}$$

We have used the fact that, as before, $\Gamma^2_{3\mu} = 0 = \Gamma^{\mu}_{23}$ in the equatorial plane, and we can consistently set $u^2 = 0$. This system leads to

$$\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}\varphi^2} + \mathcal{F}^2\mathbf{X} = 0 \tag{40}$$

where $\mathbf{X} = (u^1, \mathrm{d}u^0/\mathrm{d}\varphi, \mathrm{d}u^3/\mathrm{d}\varphi)$ and

$$\mathcal{F}^2(r) = -\left\{\frac{1}{4g_{rr}\Psi}\left[g_{tt}(g_{\varphi\varphi,r})^2 - 2g_{t\varphi}g_{\varphi\varphi,r}g_{t\varphi,r} + g_{\varphi\varphi}(g_{t\varphi,r})^2\right]\right\}_{\theta=\pi/2}. \tag{41}$$

In the Kerr case, $\mathcal{F}$ reduces to f, defined in equation (19). For $\mathcal{F}^2 > 0$ the band structure of holonomy invariance exists in general, as in the Kerr case.

4. Conclusion

For the Kerr spacetime it follows from cases (a)–(c) considered in section 2 that the shift in the timelike component (i.e. temporal holonomy) is nonzero in general. For instance, if u^μ is given initially at $\varphi = 0$ by (0, 1, 0, 0,), then after n loops, u^0 has grown by virtue of the angular momentum of the Kerr source. For $r > r_*$,

$$u^0\big|_{\varphi=0} = 0 \longrightarrow u^0\big|_{\varphi=2\pi n} = \frac{aM(a^2+3r^2)}{\Delta(r)\sqrt{F(r)}}\sin[2\pi n f(r)]. \tag{42}$$

The band structure of holonomy invariance extends from the Schwarzschild case to the Kerr case, but it does not exist for $r \leqslant r_*$ in the Kerr case.

These results refer to the holonomy around a constant-time circle in the equatorial plane of the Kerr spacetime for an arbitrary vector field. On the other hand, the corresponding gravitomagnetic clock effect [2, 3] refers to the motion of clocks on *timelike* circular geodesic orbits. Let $t_+(t_-)$ be the period of a clock on a co-rotating (counter-rotating) circular equatorial orbit as measured by asymptotically static inertial observers at infinity; then,

$$t_+ - t_- = 4\pi a. \tag{43}$$

Moreover, the proper periods $\tau_\pm$ accumulated by the clocks in their complete revolutions around the Kerr source [2, 3] are such that

$$\tau_+ - \tau_- \approx 4\pi a\left[1 + \frac{3}{2}\frac{M}{r} + \frac{27}{8}\frac{M^2}{r^2} + \left(\frac{135}{16} + \frac{1}{2}\frac{a^2}{M^2}\right)\frac{M^3}{r^3} + O\left(\frac{M^4}{r^4}\right)\right]. \tag{44}$$

Hence, $\tau_+ - \tau_- \approx 4\pi a$ for $r \gg 2M$. It follows from these results that there exists a special temporal structure, characterized by the specific angular momentum a, around a rotating mass. This temporal structure is responsible for the fact that the temporal holonomy involving the *timelike* component of the vector field in the Kerr spacetime does not vanish in general. For a four-velocity vector, the temporal holonomy is a Lorentz boost, and the associated time dilation signals the existence of the clock the effect. The clock effect and the holonomic Lorentz boost are purely rotational; both vanish in the Schwarzschild spacetime.

This is the key point of the paper, established first for the Kerr metric and then extended to the stationary axisymmetric case; circular equatorial holonomy produces a Lorentz boost, and an equivalent time dilation, which is a signature of the clock effect. If $g_{t\varphi,r} \neq 0$ in the equatorial plane, then in general there is a clock effect by equation (36). In addition $u^0(\varphi)$ is not constant in general, since $G_{,r} \neq 0$ in equation (37), so that there is timelike holonomy.

We have also shown that the band structure of holonomy invariance recently demonstrated in the Schwarzschild spacetime survives in appropriately modified form if the source rotates.

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