

Spacetime Measurements in Kaluza-Klein Gravity

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Abstract

We extend the classical general relativistic theory of measurement to include the possibility of existence of higher dimensions. The intrusion of these dimensions in the spacetime interval implies that the inertial mass of a particle in general varies along its worldline if the observations are analyzed assuming the existence of only the four spacetime dimensions. The variations of mass and spin are explored in a simple 5D Kaluza-Klein model.

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1 Introduction

The most basic measurements of a physical observer are those of time and space. In principle, such observations may involve an atomic clock for local temporal measurements and three independent spatial axes for the characterization of space in the neighborhood of the observer. In standard general relativity, the observer carries an orthonormal tetrad frame along its worldline and physical observables are scalars that are obtained as the projections of tensors that correspond to various physical variables upon the local tetrad frame of the observer. In general relativity, just as in Newtonian physics, the observer can determine, via local measurements, the acceleration of its local frame. This acceleration could be in the form of the translational acceleration of the observer as well as the rotation of its local spatial frame. Theoretically, a set of three ideal orthogonal torque-free gyroscopes can provide a nonrotating (i.e. Fermi-Walker transported) spatial frame along the path of the observer. Thus in 4D spacetime a free test observer can carry a nonrotating orthonormal frame that is parallel transported along its geodesic worldline. In view of the possibility

of existence of higher dimensions, it is worthwhile to examine how the theory of measurement in general relativity would have to be extended if higher dimensions intrude into the spacetime arena. This intrusion is expected in any realistic higher-dimensional physical theory [1, 2]; nevertheless, the interpretation of observational data currently involves only the standard four spacetime dimensions.

In this paper, we explore the extension of the classical general relativistic theory of measurement to the Kaluza-Klein theory by studying some of the main observational consequences of the dependence of the spacetime metric upon extra dimensions. For instance, we show in section 2 that the mass of a test particle in general varies along its worldline if the motion of matter is not wholly confined to the four spacetime dimensions. This circumstance is expected in realistic higher-dimensional theories [1, 2]. Our physical considerations are motivated by the fact that experimental data are routinely analyzed assuming the existence of only the four spacetime dimensions. In section 3, we show that an initially orthonormal frame does not in general remain orthonormal along the worldline of a test observer. To render these results explicit, we consider a concrete 5D model in section 4 and explore its physical consequences. Section 5 contains a brief discussion of our results.

2 Variation of the Inertial Mass

Imagine that the universe can be described in terms of $4+N$ dimensions with $N \geq 1$ and the 4D spacetime part, which is embedded in the $4+N$ manifold, has a metric of the form

$$ds^2 = g_{\mu\nu}(x, y)dx^\mu dx^\nu. \quad (1)$$

Here x stands for the spacetime coordinates and y stands for the extra dimensions (y^1, \dots, y^N) that are in general reflected in the spacetime metric. Greek indices run from 0 to 3. The path of a test particle in the $4+N$ dimensional manifold involves the 4D velocity $u^\alpha = dx^\alpha/ds$, where s is the proper time along

the path such that

$$g_{\mu\nu}(x, y)u^\mu u^\nu = 1. \quad (2)$$

Differentiating (2) with respect to s , we find that

$$g_{\mu\nu,\alpha}u^\alpha u^\mu u^\nu + \sum_{i=1}^N \frac{\partial g_{\mu\nu}}{\partial y^i} \frac{dy^i}{ds} u^\mu u^\nu + 2g_{\mu\nu}u^\mu \frac{du^\nu}{ds} = 0. \quad (3)$$

This relation may be written in the form

$$(g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} - g_{\nu\alpha,\mu})u^\alpha u^\mu u^\nu + 2g_{\mu\nu}u^\mu \frac{du^\nu}{ds} = - \sum_{i=1}^N \frac{\partial g_{\mu\nu}}{\partial y^i} \frac{dy^i}{ds} u^\mu u^\nu. \quad (4)$$

Using the fact that the 4D connection is given by

$$\Gamma_{\alpha\beta}^\nu(x, y) = \frac{1}{2}g^{\nu\eta}(g_{\eta\alpha,\beta} + g_{\eta\beta,\alpha} - g_{\alpha\beta,\eta}), \quad (5)$$

we can write equation (4) as

$$2g_{\mu\nu}u^\mu \left(\frac{du^\nu}{ds} + \Gamma_{\alpha\beta}^\nu u^\alpha u^\beta \right) = - \sum_{i=1}^N \frac{\partial g_{\mu\nu}}{\partial y^i} \frac{dy^i}{ds} u^\mu u^\nu. \quad (6)$$

The 4D acceleration of the particle is defined by

$$A^\mu = \frac{du^\mu}{ds} + \Gamma_{\alpha\beta}^\mu u^\alpha u^\beta, \quad (7)$$

so that equation (6) implies

$$u_\mu A^\mu = - \frac{1}{2} \sum_{i=1}^N \frac{\partial g_{\mu\nu}}{\partial y^i} \frac{dy^i}{ds} u^\mu u^\nu. \quad (8)$$

If $g_{\mu\nu}$ does not depend on the extra dimensions, then equation (8) becomes $u_\mu A^\mu = 0$, as expected. However, in the higher-dimensional theories $g_{\mu\nu}$ may depend on y and y may vary with respect to s . So the right-hand side of equation (8) may not vanish. Let us note that u^μ is a timelike vector, so $u_\mu A^\mu \neq 0$ indicates that there may be a *timelike* component of acceleration in A^μ in higher-dimensional theories. This is an extraordinary result, since all known basic 4D forces are spacelike and lead to accelerations that are orthogonal to the 4D velocity of the particle [3]. It turns out that the most natural way

to incorporate a *timelike* acceleration into 4D physics is to assume that the “invariant” inertial mass of the test particle varies along its worldline.

Experimental data are reduced and interpreted at present assuming that the most general force law for the motion of a test particle may be written classically as

$$\frac{Dp^\mu}{ds} \equiv \frac{dp^\mu}{ds} + \Gamma_{\alpha\beta}^\mu u^\alpha p^\beta = F^\mu, \quad (9)$$

where $p^\mu \equiv mu^\mu$ is the momentum of the particle and F^μ consists of all forces acting on the particle arising from the known fundamental interactions. In the rest frame of the particle, we expect all forces acting on the particle to be 3D vectors; therefore, $u_\mu F^\mu = 0$. This relation together with the force law (9), $Dp^\mu/ds = (dm/ds)u^\mu + mA^\mu = F^\mu$, implies that

$$\frac{1}{m} \frac{dm}{ds} = \frac{1}{2} \sum_{i=1}^N \frac{\partial g_{\mu\nu}}{\partial y^i} \frac{dy^i}{ds} u^\mu u^\nu, \quad (10)$$

where we have used equation (8). That is, the simplest interpretation of equation (8) in terms of 4D physics is to assume that the invariant “rest” mass of the particle may vary with respect to its proper time s due to the existence of higher dimensions. Conversely, the observation of such a basic variation would indicate the presence of an extra *timelike* acceleration and this could come about precisely because of the intrusion of the extra dimensions into the 4D physics as indicated by equations (1) and (8).

Let us note that equation (10) may be expressed as

$$\delta(m^2) = \left[\sum_{i=1}^N \frac{\partial g_{\mu\nu}(x, y)}{\partial y^i} \delta y^i \right] p^\mu p^\nu, \quad (11)$$

where

$$m^2 = g_{\mu\nu}(x, y) p^\mu p^\nu. \quad (12)$$

Equations (11) and (12) can be used to determine the variable inertial mass in higher-dimensional theories.

The acceleration A^μ and the variation of the extra coordinates of the particle along its worldline are determined by the equation of motion of the theory. In

principle, it is possible that y can vary in just such a way as to render $dm/ds = 0$ in equation (10). This would, of course, require rather special circumstances; therefore, we pursue in this paper the general situation in which $dm/ds \neq 0$.

3 Variation of the Spin

Let us first consider an ideal gyroscope represented by the spin vector σ^μ within the context of classical general relativity. We may imagine in our classical model that we are dealing here with the limiting case of an ideal gyroscope with its magnitude of spin given by $\sigma = I\omega$ in a certain “rest” frame of the system. Here I is the proper moment of inertia of the gyroscope and ω is its angular speed of rotation with respect to the *proper time* of the “pointlike” spinning particle. The general relativistic theory is based on the Mathisson-Papapetrou equations for a “pole-dipole” test particle. It follows from detailed considerations [4, 5] of the classical theory of such ideal spinning “point” particles that $\sigma_\mu u^\mu = 0$. Moreover, σ^μ is nonrotating, i.e.

$$\frac{D\sigma^\mu}{ds} = -(\sigma_\alpha A^\alpha)u^\mu \quad , \quad (13)$$

so that $\sigma_\alpha u^\alpha$ and $\sigma^2 = -g_{\mu\nu}\sigma^\mu\sigma^\nu$ are constants along the worldline of the particle.

Consider next the possibility that the spacetime metric could depend upon higher dimensions. In this case, one can extend the treatment of section 2 to demonstrate that

$$Dg_{\mu\nu} = \sum_{i=1}^N \frac{\partial g_{\mu\nu}}{\partial y^i} dy^i \quad . \quad (14)$$

Using this result, it is straightforward to show that

$$\frac{d}{ds} (\sigma_\mu u^\mu) = T_\mu u^\mu + A_\mu \sigma^\mu + \sum_{i=1}^N \frac{\partial g_{\mu\nu}}{\partial y^i} \frac{dy^i}{ds} \sigma^\mu u^\nu \quad (15)$$

and

$$\frac{d}{ds} (\sigma^2) = -2T_\mu \sigma^\mu - \sum_{i=1}^N \frac{\partial g_{\mu\nu}}{\partial y^i} \frac{dy^i}{ds} \sigma^\mu \sigma^\nu \quad , \quad (16)$$

where $T^\mu = D\sigma^\mu/ds$ is the torque and A^μ is the translational acceleration (7) of the particle. The vectors A^μ and T^μ must be determined from the higher-dimensional theory under consideration. It follows from the extension of these results to the axes of a spatial frame carried by an observer that an initially orthonormal frame will not in general remain orthonormal in the course of time. In particular, the magnitude of spin can change with time. To illustrate how this could come about, we now turn to a simple Kaluza-Klein gravitational model.

4 A 5D Model

We consider a Kaluza-Klein model with one extra noncompactified spacelike dimension y such that the 5D metric is given by

$$dS^2 = \hat{g}_{AB} dx^A dx^B = ds^2 - dy^2 \quad , \quad (17)$$

where $x^A = (x^\mu, y)$ and \hat{g}_{AB} satisfies the 5D vacuum field equations $\hat{R}_{AB} = 0$. It is natural to assume in this theory that a test particle follows the 5D geodesic equation

$$\frac{d^2 x^A}{dS^2} + \hat{\Gamma}_{BC}^A \frac{dx^B}{dS} \frac{dx^C}{dS} = 0 \quad , \quad (18)$$

so that the 5D velocity vector $U^A = dx^A/dS$ is parallel transported along the path. Moreover, we expect that the 5D spin vector Σ^A of an ideal test gyroscope would also be parallel transported along its path

$$\frac{d\Sigma^A}{dS} + \hat{\Gamma}_{BC}^A U^B \Sigma^C = 0 \quad . \quad (19)$$

To connect equations (18)-(19) with our spacetime variables in previous sections, we note that

$$u^\mu = \left(\frac{dS}{ds} \right) U^\mu \quad , \quad \sigma^\mu = \left(\frac{dS}{ds} \right) \Sigma^\mu \quad . \quad (20)$$

The relationship between 4D and 5D velocities follows simply from the definitions, while the corresponding relation for spin is expected by analogy.

The Kaluza-Klein field equations $\hat{R}_{AB} = 0$ can be reduced to certain constraint equations together with the 4D gravitational field equations of standard general relativity with an effective energy-momentum tensor as the source of the gravitational field. This is consistent with recent investigations [6] based on Campbell's theorem that an n -dimensional Riemannian space can be locally embedded in a Ricci-flat $(n + 1)$ -dimensional Riemannian space [7].

To proceed further, we need an explicit solution of the field equations. It is possible to show that with

$$g_{\mu\nu}(x, y) = \frac{y^2}{L^2} \tilde{g}_{\mu\nu}(x) \quad (21)$$

in the spacetime interval (1), \hat{g}_{AB} in equation (17) is a solution of $\hat{R}_{AB} = 0$, provided that $\tilde{g}_{\mu\nu}(x)$ is *any* source-free solution of general relativity with a cosmological constant $\tilde{\Lambda} = 3/L^2$. Here L is a constant length. If $\tilde{g}_{\mu\nu}(x)$ is the de Sitter solution, then $\hat{R}_{ABCD} = 0$ and the 5D metric is flat. However, for more complicated Einstein spaces such as the Kerr-de Sitter solution the 5D manifold is curved. Let us note that for $g_{\mu\nu}(x, y)$, the 4D Ricci tensor is then given by $R_{\alpha\beta} = -3y^{-2}g_{\alpha\beta}$. Thus the spacetime metric can be interpreted as a source-free solution of general relativity with a cosmological "constant" $\Lambda = 3/y^2$. Various aspects of the gravitational model under consideration here have been explored in a number of publications [8]. A recent discussion of variable cosmological "constant" is contained in [9].

With a spacetime metric of the form (21), equation (18) reduces to $Du^\alpha/ds = A^\alpha$ with

$$A^\alpha = -\frac{1}{y} \frac{dy}{ds} u^\alpha \quad (22)$$

and

$$y \frac{d^2y}{ds^2} - \left(\frac{dy}{ds} \right)^2 + 1 = 0 \quad (23)$$

Equation (10) — or, equivalently, equation (22) — implies that $m = \lambda y$, where λ is a constant. Moreover, equation (23) can be solved by studying the variation of dy/ds with respect to y . The result is

$$\left(\frac{dy}{ds}\right)^2 + \frac{K}{L_0^2} y^2 = 1 \quad , \quad (24)$$

where $K = 0, \pm 1$ and L_0 is a constant length. It follows that

$$\pm y = \begin{cases} L_0 \sin\left(\frac{s-s_0}{L_0}\right) & \text{for } K = +1, \\ s - s_0 & \text{for } K = 0, \\ L_0 \sinh\left(\frac{s-s_0}{L_0}\right) & \text{for } K = -1. \end{cases} \quad (25)$$

If we now substitute for y from equation (25) in $dS^2 = ds^2 - dy^2$, we find that dS^2 is greater than, equal to, or less than zero for $K = +1, 0, -1$, respectively. In general relativity, we know that the worldline of a massive (massless) particle should have $ds^2 > 0$ ($ds^2 = 0$). Extending this requirement from 4D to 5D, we choose $dS^2 > 0$ ($dS^2 = 0$) for the motion of a massive (massless) particle. This implies that $K = +1$ and $K = 0$ in equation (25) hold for massive and massless particles, respectively. It follows from this interpretation that for $K = 0$ we have $ds = 0$, so that $s = s_0$ and hence $y = 0$. Let us note that this interpretation is consistent with equation (25), since for $s - s_0 \rightarrow 0$ the timelike and spacelike cases $K = \pm 1$ reduce to the null case $K = 0$ regardless of the value of L_0 . Moreover, for $s - s_0 \rightarrow 0, y \rightarrow 0$, and hence $m = \lambda y \rightarrow 0$, so that lightlike propagation occurs only for a massless particle. Thus null rays propagate only in the 4D spacetime part of the 5D manifold.

The case of a massless particle is important for the treatment of light propagation; therefore, it is necessary to show in detail how the null geodesic case comes about as a limiting case of the motion of a massive particle for $m \rightarrow 0$. This involves a standard limiting procedure in general relativity that will be adapted here to the situation at hand. Let us note that for a massive particle the momentum $p^\alpha = mu^\alpha$ is covariantly constant along its 4D worldline. Here $m = \lambda y$ and y is given by equation (25). Thus we choose a new variable z defined by $s - s_0 = \varepsilon z$, where $\varepsilon, 0 < \varepsilon \ll 1$, is a constant such that $\varepsilon \rightarrow 0$ in the

massless limit. If we now let $q = \pm\lambda^{-1} \ln z$ be an affine parameter along the worldline, then it is simple to show that for $\varepsilon \rightarrow 0$, $p^\alpha \rightarrow k^\alpha = dx^\alpha/dq$, which is tangent to a null ray, and that the equation of motion reduces in this limit to the null geodesic equation $Dk^\alpha/dq = 0$.

For a massive particle we have from the first equation in (25)

$$m = m_{\max} \sin \frac{s-s_0}{L_0} \quad , \quad \frac{1}{m} \frac{dm}{ds} = \frac{1}{L_0} \cot \frac{s-s_0}{L_0} \quad , \quad (26)$$

where $m_{\max} = \pm\lambda L_0$. The present upper limit on $|m^{-1}(dm/ds)|$ is of order $10^{-12}/\text{yr}$, so that L_0 must be a sufficiently large cosmic length to render equation (26) compatible with observation [10].

Let us now explore the implications of equation (19) for the motion of the gyro axis. We find that $D\sigma^\mu/ds = T^\mu$, where the torque is given by

$$T^\mu = -\frac{1}{y} \left(\frac{dS}{ds} \right) \Sigma^4 u^\mu \quad (27)$$

and the variation of Σ^4 is governed by

$$\frac{d\Sigma^4}{ds} + \frac{1}{y} \left(\frac{ds}{dS} \right) \sigma_\alpha u^\alpha = 0 \quad . \quad (28)$$

These equations are consistent with the fact that $\Sigma_A U^A$ and $\Sigma_A \Sigma^A$ are constants along the path. For instance,

$$\left(\frac{ds}{dS} \right)^2 \sigma^2 + (\Sigma^4)^2 = \Sigma_0^2 \quad , \quad (29)$$

where $\Sigma_0 = (-\Sigma_A \Sigma^A)^{1/2}$ is the constant magnitude of the 5D spin vector. One can now compute the variation of $\sigma_\alpha u^\alpha$ and σ^2 along the path using equations (15) and (16). In particular, we find that $\sigma_\alpha u^\alpha = C_0 + C_1 \cos \theta$ and

$$\Sigma^4 = \xi(C_0 \cot \theta + C_1 \csc \theta) \quad , \quad (30)$$

where $\theta = (s-s_0)/L_0$ and $\xi^2 = 1$ (i.e. ξ is either +1 or -1). Here C_0 and C_1 are constants of integration. To simplify matters, let us assume that $C_0 = \Sigma_A U^A$

vanishes along the path, generalizing the standard 4D constraint for a “pointlike” gyroscope. Moreover, we set $C_1 = 0$; then, $T^\mu = 0$ and $\sigma_\alpha u^\alpha = 0$. It follows that one can choose initial conditions such that σ^μ is nonrotating along the path, since $\sigma_\mu A^\mu = 0$ follows from our assumptions. However, the magnitude of spin would still vary in accordance with equation (29), i.e. $\sigma = \lambda' y$, where $\lambda' \neq 0$ is a constant. Hence the “pole-dipole” particle’s mass and spin both vary along its trajectory in such a way that σ/m remains constant.

5 Discussion

A preliminary analysis of the basic spacetime measurements in higher-dimensional gravity theory reveals that the inertial mass and spin of an ideal classical “pointlike” gyroscope may vary along its worldline. These results could be significant in Kaluza-Klein cosmology as well as the search for extra dimensions.

Our discussion has been primarily concerned with classical general relativity; however, the results are expected to be of more general validity. In fact, the intrusion of the extra dimensions in the spacetime domain implies that it is not possible in general to reduce the spacetime metric to the Minkowski form at an arbitrary event in spacetime. This would indicate a breakdown of some of the basic concepts of standard relativistic physics; for instance, the inertial mass and the magnitude of spin are invariant constants that characterize the irreducible unitary representations of the Poincaré group, but could now become variables. We have explored this circumstance in this paper within the eikonal approximation of the classical theory of gravitation; nevertheless, our treatment could be extended to other physical quantities such as the variation of the phase of a wave.

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