ROTATING CYLINDRICAL SYSTEMS AND GRAVITOMAGNETISM

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Abstract

We discuss gravitomagnetism in connection with rotating cylindrical systems. In particular, the gravitomagnetic clock effect is investigated for the exterior vacuum field of an infinite rotating cylinder. The dependence of the clock effect on the Weyl parameters of the stationary Lewis metric is determined. We illustrate our results by means of the van Stockum spacetime.

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1 Introduction

The purpose of this paper is to discuss the physics of rotating cylindrical systems from the viewpoint of gravitomagnetism. To this end, we focus attention on the gravitomagnetic clock effect that exhibits the special temporal structure around rotating bodies. Briefly, let τ_+ (τ_-) denote the total proper time of a standard clock that completes a direct (retrograde) stable circular geodesic orbit of radius $r \gg 2GM/c^2$ in the equatorial plane of an astrophysical source of mass M and angular momentum J; then,

$$\tau_+ - \tau_- \approx 4\pi \frac{J}{Mc^2}, \tag{1}$$

which is independent of the orbital radius r and the Newtonian gravitational constant G. This *classical* effect is in some ways reminiscent of the Aharonov-Bohm effect that is topological in character [1-3].

It is interesting to observe that for a classical source with uniform density, the clock effect is independent of M. To illustrate this point, consider a spherical body of radius R and constant density rotating rigidly with frequency Ω_0. In this case the moment of inertia I is given by $(2/5)\ MR^2$ and with $J = I\Omega_0$ we obtain from (1)

$$\tau_+ - \tau_- \approx \frac{8\pi}{5} \frac{R^2\Omega_0}{c^2}, \tag{2}$$

which is independent of the total inertial mass of the rotating body. Thus the clock effect under consideration here is a classical gravitational effect that is mainly dependent upon the *rotation* of the source.

Let us now imagine that the shape of the constant density source is changed to a cylindrical configuration that rotates about its axis of cylindrical symmetry with frequency Ω_0. Then equation (1) implies that

$$\tau_+ - \tau_- \approx \frac{2\pi R_0^2 \Omega_0}{c^2}, \tag{3}$$

since the moment of inertia of this configuration is (1/2) MR_0^2, where R_0 is the radius of the circular section of the cylinder. These considerations illustrate the dependence of the clock effect – which is simply proportional to the specific angular momentum of the source – on the shape of the rotating body.

There are no exact solutions of the gravitational field equations for a finite rotating cylinder [4]. On the other hand, equation (3) is independent of the height of the cylinder and this might suggest that the height of the cylinder is not relevant and hence we can study the much simpler case of infinite cylindrical systems; however, the approximation scheme for which (1) is valid (i.e., $r \gg 2GM/c^2$) applies to the exterior field of a finite rotating source and would break down for an infinitely extended cylindrically symmetric configuration of mass-energy since $M \to \infty$. Nevertheless, we consider in this paper the exterior gravitational field of infinite rotating cylindrical systems. Hence we do not expect to recover equation (3) from our exact analysis, since our results here apply only to the "near zone" of the rotating cylinder in contrast to previous work on the gravitomagnetic clock effect that has been generally concerned with the "far zone" of a rotating mass [1-3]. Indeed, the relevance of our results for cylindrically symmetric systems to naturally occurring (astrophysical) sources is quite limited. Instead, it may be possible to study the influence of certain topological aspects of spacetime structure – such as the presence of cosmic strings – on the clock effect, since previous work [1-3] has been mainly concerned with the derivation of the effect in the

Kerr geometry and the possibility of its detection using spaceborne clocks. However, only a beginning is made in this first treatment of the clock effect for cylindrically symmetric fields as we limit our discussion to the Weyl class of exterior Lewis spacetimes for the sake of simplicity.

The general relativistic treatment of cylindrically symmetric systems has been considered by many authors following the early work of Weyl (cf. [5] for a lucid discussion). It is interesting to elucidate certain physical aspects of rotating cylindrical configurations from the standpoint of gravitoelectromagnetism. Our treatment here is by no means exhaustive; instead, we illustrate this point of view by means of an example in this introductory section and devote the rest of this paper to the gravitomagnetic clock effect due to its intrinsic physical significance.

Consider the gravitational field of a "nonrelativistic" rotating astronomical source. In the linear approximation, the exterior spacetime metric may be expressed as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. Here Greek indices run from 0 to 3 and Latin indices run from 1 to 3, the quasi-Cartesian coordinates are $x^\mu = (ct, \vec{x})$ and the signature of the metric is +2. Let $h = \mathrm{tr}(h_{\mu\nu})$ and define $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$; then, the gravitational field equations take the form

$$\Box\bar{h}_{\mu\nu} = -16\frac{\pi G}{c^4}T_{\mu\nu}, \tag{4}$$

once the Lorentz gauge condition $\bar{h}^{\mu\nu}{}_{,\nu} = 0$ is imposed. We are interested in the particular solution

$$\bar{h}_{\mu\nu} = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(ct - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} \; d^3x'. \tag{5}$$

For an axisymmetric time-independent source, we find that the stationary field is given by $\bar{h}_{00} = 4\Phi/c^2$, $\bar{h}_{0i} = -2A_i/c^2$ and $\bar{h}_{ij} = O(c^{-4})$, where $\Phi(\vec{x})$ is the gravitoelectric potential, $\vec{A}(\vec{x})$ is the gravitomagnetic vector potential ($\nabla \cdot \vec{A} = 0$) and we neglect the tensor potential $\bar{h}_{ij}$. Thus the exterior metric is of the form

$$ds^2 = -\left(1 - \frac{2\Phi}{c^2}\right)(dx^0)^2 - \frac{4}{c^4}(\vec{A} \cdot d\vec{x})dx^0 + \left(1 + \frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j, \qquad (6)$$

and we define the gravitoelectric and gravitomagnetic fields by $\vec{E} = -\nabla\Phi$ and $\vec{B} = \nabla \times \vec{A}$, respectively, in complete analogy with electromagnetism.

Imagine now the curvature of spacetime generated by such a source

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}). \qquad (7)$$

The components of the curvature tensor as measured by a standard set of exterior observers may be expressed as $R_{\mu\nu\rho\sigma}\lambda^{\mu}_{(\alpha)}\lambda^{\nu}_{(\beta)}\lambda^{\rho}_{(\gamma)}\lambda^{\sigma}_{(\delta)}$ and at the linear order of approximation under consideration here we may set $\lambda^{\mu}_{(\alpha)} = \delta^{\mu}{}_{\alpha}$ either for static Killing observers or for the free (geodesic) test observers. The components of the Riemann tensor may be represented in terms of symmetric and traceless 3×3 matrices $\mathcal{E}$ and $\mathcal{B}$ as

$$\mathcal{R} = \begin{pmatrix} \mathcal{E} & \mathcal{B} \\ \mathcal{B} & -\mathcal{E} \end{pmatrix}. \qquad (8)$$

Here $\mathcal{E}$ represents the *electric* components of the curvature tensor,

$$\mathcal{E}_{ij} = -E_{j,i} \quad , \qquad (9)$$

and $\mathcal{B}$ represents the *magnetic* components of the curvature tensor,

$$\mathcal{B}_{ij} = -B_{j,i} \quad . \qquad (10)$$

The stationary exterior "Maxwell" equations for $\vec{E}$ and $\vec{B}$ are equivalent to the equations that result from the fact that $\mathcal{E}$ and $\mathcal{B}$ as given by (9) and (10) are indeed symmetric and traceless. If the gravitoelectromagnetic field is uniform, then (9) and (10) imply that $\mathcal{R} = 0$ and the spacetime is flat at the level of approximation under consideration here.

Let us now consider the gravitational field of a slowly rotating long cylindrical shell using our approximate treatment. The analogy with electrodynamics implies that the gravitoelectric field vanishes inside the shell. On the other hand, the gravitomagnetic field is uniform inside the shell and vanishes outside. Hence the spacetime is *flat* inside the shell ($\mathcal{R} = 0$) and the exterior field is *static* ($\vec{B} = 0$). These conclusions agree with the results of the exact theory [6-8], though the general situation is more complicated [6].

The basic notions of gravitoelectromagnetism can thus be used to interpret physically some of the results of the study of cylindrical systems. In the rest of this paper, however, we consider standard clocks on stable circular geodesic orbits around infinite rotating cylindrical systems and we shall set $G = 1$ and $c = 1$ for the sake of simplicity. The general exterior vacuum solution in this case is discussed in section 2. We confine our discussion throughout to the Weyl class of exterior stationary spacetimes. The stable circular geodesic orbits are studied in section 3. The clock effect is examined in section 4, and it is explicitly worked out in the special case of the exterior van Stockum spacetime in the appendix. Finally, section 5 contains a discussion of our main result (49) and a brief examination of its dependence on the parameters of the Lewis metric.

2 Exterior Lewis metric

The exterior spacetime of a rotating infinitely long cylindrical source has been obtained by Lewis [9] and is given by

$$ds^2 = -f dt^2 + 2k dt\, d\phi + e^{\mu}(dr^2 + dz^2) + l d\phi^2. \tag{11}$$

Here $f(r)$, $k(r)$, $\mu(r)$ and $l(r)$ are given by

$$f = ar^{1-n} - \frac{c_0^2}{n^2 a} r^{1+n}, \tag{12}$$

$$k = -abr^{1-n} - \frac{c_0}{na}\left(1 - \frac{bc_0}{n}\right) r^{1+n}, \tag{13}$$

$$e^{\mu} = r^{(n^2-1)/2}, \tag{14}$$

$$l = -ab^2 r^{1-n} + \frac{1}{a}\left(1 - \frac{bc_0}{n}\right)^2 r^{1+n}, \tag{15}$$

where n, a, b, and c_0 are constant parameters. We limit our study here to the Weyl class [10], where all these parameters are real. If these parameters are complex, then the solution reduces to the Lewis class [11] that is beyond the scope of this work. In this paper, we are concerned with the behavior of certain measurable quantities ("readings of clocks") in the Lewis spacetime. It is therefore important to discuss the physical significance of the coordinates that appear in (11). To this end, let us choose a definite temporal or spatial scale λ and express all spatial and temporal quantities in units of λ; then, the spacetime *interval* expressed in terms of such dimensionless quantities is given by (11)-(15). In this way, we always deal with dimensionless quantities insofar as the Lewis metric is concerned. Moreover, the nature of the circular cylindrical coordinates in (11) is intimately connected with the rotating source, since the general form of the Lewis metric (11) is appropriate for matching to the interior solution. We expect that in the complete absence of matter and radiation, the spacetime would become Minkowskian; hence, the coordinates in (11) derive their particular significance in relation to the source. The exterior van Stockum solution [12] discussed in theappendix provides a particularly clear illustration of this circumstance.

The Lewis metric (11) contains a set of dimensionless parameters (n, a, b, c_0). The physical and geometrical interpretations of these parameters have been the subject of the pioneering work of van Stockum [12] followed by various authors (see, e.g., [4-8], [10-11], [13-16] and the references cited therein). It follows from these studies that the parameter n is associated with the Newtonian mass per unit length of a uniform line-mass source in the low-density approximation. The parameter a is connected with the constant arbitrary potential that exists in the corresponding Newtonian solution. In the static and locally flat limit of the Weyl class, $a > 1$ produces a linear energy density along a string. The parameter b is associated, in the locally flat limit, with the angular momentum of the spinning string. The parameter c_0 is produced by the vorticity of matter when it is represented by a general stationary completely anisotropic source. This parameter c_0 together with b is responsible for the stationarity of the Weyl-class metric.

It can be proved [10], using the Cartan curvature scalars, that only n helps to curve spacetime locally and that the three parameters a, b and c_0 only influence the global structure of spacetime. As a consequence, the stationary Lewis metric for the Weyl class is indistinguishable locally from the static Levi-Civita metric [17] as far as the curvature of spacetime is concerned.

3 Circular geodesics

To study the geodesics of the Lewis metric (11), we start with the Lagrangian

$$\mathcal{L} = \frac{1}{2}f\dot{t}^2 - k\dot{t}\dot{\phi} - \frac{1}{2}e^{\mu}(\dot{r}^2 + \dot{z}^2) - \frac{1}{2}l\dot{\phi}^2, \tag{16}$$

where a dot stands for $d/d\tau$, τ being the proper time. From the Euler-Lagrange equations for (16) we obtain

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{\partial \mathcal{L}}{\partial t} = 0, \tag{17}$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi} = 0, \tag{18}$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = \frac{\partial \mathcal{L}}{\partial z} = 0, \tag{19}$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r}. \tag{20}$$

Some aspects of the geodesics of the general Lewis metric have been studied in [15]. Equations (17)-(19) reduce to

$$f\dot{t} - k\dot{\phi} = E, \tag{21}$$

$$k\dot{t} + l\dot{\phi} = L, \tag{22}$$

$$e^{\mu}\dot{z} = P, \tag{23}$$

where E, L and P are constants and represent, respectively, the total energy, orbital angular momentum and linear momentum along the z-axis of the test particle divided by its inertial mass. We assume no motion along the z-axis, hence $P = 0$; therefore, we choose $z = 0$ and confine our considerations to the (x, y)-plane. As is well known, the equation of radial motion (20) has a first integral that is equivalent to $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ for the timelike path under consideration. This implies that

$$-f\dot{t}^2 + 2k\dot{t}\dot{\phi} + e^\mu\dot{r}^2 + l\dot{\phi}^2 + 1 = 0, \tag{24}$$

so that $\mathcal{L}$ in (16) is equal to $\frac{1}{2}$ along the path. From (21) and (22) we have

$$\dot{t} = \frac{El + Lk}{fl + k^2}, \tag{25}$$

$$\dot{\phi} = \frac{-Ek + Lf}{fl + k^2}. \tag{26}$$

Thus equation (24) can be written as

$$e^\mu\dot{r}^2 + V(r) = 0, \tag{27}$$

where

$$V(r) \equiv 1 - \frac{1}{fl + k^2}(E^2l + 2ELk - L^2f). \tag{28}$$

Defining new parameters p and q,

$$p \equiv \frac{1}{a}\left[E - \frac{c_0}{n}(Eb + L)\right]^2, \tag{29}$$

$$q \equiv a(Eb + L)^2, \tag{30}$$

and using the metric functions, we find that $fl + k^2 = r^2$ and equation (28) can be represented as

$$V(r) = 1 - \frac{p}{r^{1-n}} + \frac{q}{r^{1+n}}. \tag{31}$$

The conditions for circular motion of the test particle at radius $r = r_0$ are

$$V(r_0) = 0, \tag{32}$$

$$\left(\frac{dV}{dr}\right)_{r=r_0} = 0. \tag{33}$$

Furthermore, the circular motion is stable if,

$$\left(\frac{d^2V}{dr^2}\right)_{r=r_0} > 0. \tag{34}$$

The condition (33) gives

$$r_0 = \left[\frac{(1+n)q}{(1-n)p}\right]^{1/(2n)} . \tag{35}$$

Only positive roots are to be taken throughout this paper. Moreover, equations (32) and (35) imply that

$$r_0 = \left(\frac{2np}{1+n}\right)^{1/(1-n)} . \tag{36}$$

Hence, E and L can be obtained in principle from (35) and (36) in terms of r_0. From (31) and (35)-(36), we have

$$\left(\frac{d^2V}{dr^2}\right)_{r=r_0} = \frac{4n^2pq}{r_0^4}, \tag{37}$$

which is manifestly positive once r_0 can be properly defined. To satisfy this latter condition, we find from (35) and (36) that $-1 < n < 1$ and $na > 0$ must hold. Then circular geodesic orbits exist and are stable.

It is interesting to discuss the Newtonian limit of such orbits; to this end, we set f =exp(-2Φ) and ignore relativistic rotation parameters b and c_0. Thus to make contact with Newtonian gravitation via the gravitoelectric potential Φ, we must take $a > 0$ and then $0 < n < 1$ in this case. Then, equation (12) implies that

$$\Phi = -\frac{1}{2}\ln a - \frac{1}{2}(1-n)\ln r, \tag{38}$$

which should be compared with the corresponding Newtonian potential for a line-mass, namely, $-2\sigma \ln\, r+$ constant. It follows from this comparison that $4\sigma = 1 - n$, where σ is the mass per unit length of the cylindrical configuration and the Newtonian limit would then correspond to $0 < \sigma \ll 1$. We can express V in terms of Φ as

$$V(r) = 1 - ape^{2\Phi} + \frac{q}{ar^2}e^{-2\Phi} \tag{39}$$

for $b = c_0 = 0$. Expanding this expression to first order in σ, we recover the correspondence with the Newtonian theory of gravitation.

Finally, a remark is in order regarding *null* circular geodesics of the spacetime under consideration here. It is simple to show by the methods of this section that no such geodesics exist.

4 Gravitomagnetic clock effect

The aim of this section is to calculate the proper time difference between two free test particles, one corotating and the other counterrotating with respect to the rotating cylindrical source, on an exterior circular orbit of radius r. We do this by using the results of the previous section on the circular geodesics for the Weyl class of the Lewis metric and provide a physical explanation for our main formula describing the clock effect in this case.

In principle, we could integrate around the circle the expression for $d\tau/d\phi$ that can be obtained from the inverse of $\dot{\phi}$ given in (26); however, it proves more straightforward to start from the geodesic equation for radial motion (20) restricted to the circular orbit. This implies that

$$f'\dot{t}^2 - 2k'\dot{t}\dot{\phi} - l'\dot{\phi}^2 = 0, \tag{40}$$

where a prime stands for d/dr. Defining the "angular velocity" for a test particle as $\omega = \dot{\phi}/\dot{t} = d\phi/dt$, we obtain from (40) that $\omega = [-k' \pm (k'^2 + f'l')^{1/2}]/l'$. From equations (12), (13), and (15), we find that $k'^2 + f'l' = 1 - n^2$. Let us now define N and D as

$$N = (1+n)\frac{c_0}{na}\left(1 - \frac{bc_0}{n}\right) r^n + (1-n)abr^{-n} \pm (1-n^2)^{1/2}, \tag{41}$$

$$D = \frac{1+n}{a}\left(1 - \frac{bc_0}{n}\right)^2 r^n - (1-n)ab^2 r^{-n}. \tag{42}$$

Then, the angular velocity of the particle is given by

$$\omega = \frac{N}{D}. \tag{43}$$

If $b = c_0 = 0$, the Weyl-class Lewis metric (12) becomes the static Levi-Civita metric and (43) reduces to $\omega = \pm\omega_0$, where $\omega_0 > 0$ is given by

$$\omega_0^2 = \left(\frac{1-n}{1+n}\right)\frac{a^2}{r^{2n}}. \tag{44}$$

With the help of (44), we can rewrite (43) as

$$\omega = \frac{\pm\omega_0 + \frac{c_0}{n}}{1 - b\left(\pm\omega_0 + \frac{c_0}{n}\right)}. \tag{45}$$

It is important to note here that ω vanishes for $\omega_0 = \mp c_0/n$; therefore, the free particle is in this case simply static in the spatial coordinates under consideration. This could come about if the "centrifugal repulsion" balances the gravitational attraction (cf. the appendix). From (11) we have for the proper time of circular orbits

$$d\tau^2 = f dt^2 - 2k dt d\phi - l d\phi^2, \tag{46}$$

which becomes

$$\left(\frac{d\tau}{d\phi}\right)^2 = f\left(\frac{1}{\omega} + \frac{c_0 r^{1+n}}{naf} + b\right)^2 - \frac{r^2}{f}. \tag{47}$$

We note that $b + 1/\omega = (\pm\omega_0 + c_0/n)^{-1}$ from (45), hence

$$\left|\frac{d\tau}{d\phi}\right| = \left(\frac{2n}{1+n}\, ar^{1-n}\right)^{1/2} \left|\o mega_0 \pm \frac{c_0}{n}\right|^{-1}. \tag{48}$$

Integrating this expression, we obtain

$$\tau_\pm = 2\pi\left(\frac{2n}{1+n} ar^{1-n}\right)^{1/2} \left|\omega_0 \pm \frac{c_0}{n}\right|^{-1}, \tag{49}$$

where $\tau_+(\tau_-)$ represents the proper time period registered by a standard clock on a corotating (counterrotating) circular orbit around the axis of symmetry. More generally, the two clocks could both be corotating or counterrotating as viewed from a rotating system of coordinates (cf. the appendix). In any case, in the discussion of the clock effect we are interested in $\tau_+ - \tau_-$. Therefore, we assume in the following that $n\omega_0 > |c_0|$ for the sake of concreteness; moreover, the case $n\omega_0 \le |c_0|$ will be discussed in the appendix for the exterior van Stockum spacetime. The proper time difference between the two periods can be obtained from (49) giving

$$\tau_+ - \tau_- = 4\pi\left(\frac{2n}{1+n} ar^{1-n}\right)^{1/2} \frac{\frac{c_0}{n}}{\left(\frac{c_0}{n}\right)^2 - \omega_0^2}. \tag{50}$$

It follows from this result that the gravitomagnetic clock effect is directly proportional to c_0. To understand our main result physically, it is important to develop an intuitive interpretation of formula (50). Let us first note that this result in independent of b; therefore, to simplify matters let us just consider $b = 0$ for the rest of this argument. Hence the physical argument below is actually valid only for $b = 0$; however, it can be generalized to include $b \neq 0$ as discussed in section 5. In [10], it has been demonstrated that for $b = 0$ the Lewis metric in the Weyl class can be transformed into the static Levi-Civita metric by making the coordinate transformation,

$$\bar{\phi} = \phi - (c_0/n)\, t. \tag{51}$$

Therefore, starting from this static metric we can recover the original stationary metric by a rotation of frequency

$$\Omega = -\frac{c_0}{n}. \tag{52}$$

Following the analysis given in [3], the result (52) means that for $b = 0$ the gravitomagnetic field in the Weyl class is *constant* and just as in the Larmor theorem [18] can be replaced by a simple Larmor rotation of frequency Ω.

In the rotating system, consider two clocks moving along the same circular path with physical radius $\rho = \rho(r)$ but one going prograde and the other retrograde. The time t_+ that takes the clock moving in the prograde sense to make a complete revolution in the rotating frame is given by

$$(v - \rho\Omega)t_+ = 2\pi\rho, \tag{53}$$

while the corresponding period t_- for the retrograde orbit is

$$(v + \rho\Omega)t_- = 2\pi\rho. \tag{54}$$

Here v is the speed of circular motion in the *static* metric,

$$v = \rho\frac{d\bar{\phi}}{dt} = \rho\,\omega_0, \tag{55}$$

since in this explanation the rotation corresponds to gravitomagnetism by the gravitational Larmor theorem. Therefore, from (53) and (54), we have

$$t_+ - t_- = \frac{4\pi\rho^2\Omega}{v^2 - \rho^2\Omega^2}, \tag{56}$$

and with (52) and (55)

$$t_+ - t_- = 4\pi\frac{\frac{c_0}{n}}{\left(\frac{c_0}{n}\right)^2 - \omega_0^2}. \tag{57}$$

We note that in (57), $\rho(r)$ drops out. In the static case, the relationship between t and τ for a circular orbit is given by

$$d\tau^2 = ar^{1-n}dt^2 - \frac{r^{1+n}}{a}d\bar{\phi}^2, \tag{58}$$

and considering (44) we have

$$\frac{d\tau}{dt} = \left(\frac{2n}{1+n}ar^{1-n}\right)^{1/2}, \tag{59}$$

if we assume $d\tau/dt > 0$. Since (59) is constant for r =constant, we get

$$\tau_+ - \tau_- = \left(\frac{2n}{1+n}ar^{1-n}\right)^{1/2}(t_+ - t_-). \tag{60}$$

Putting this expression together with (57), we recover equation (50). In this way, the connection between the main result (50) for the Weyl class of the Lewis metric and the analogue of the Sagnac effect [19] is established.

Equation (50) and its physical interpretation via the gravitational Larmor theorem have been presented here explicitly for $n\omega_0 > |c_0|$; however, our approach is general and can be used to show that similar results hold in *appropriately modified form* for $n\omega_0 < |c_0|$ as well.

5 Discussion

The calculation of the gravitomagnetic clock effect for the exterior field of a rotating cylindrical configuration of matter has been restricted to the Weyl case, where the parameters of the general Lewis metric are real; in fact, a separate analysis is required for the Lewis case, where the parameters are complex. The spacetime metric employed here can be joined to any slowly rotating cylindrical source. The physical meaning of the parameters (n, a, b, c_0) emerges from matching the exterior solution to particular interior solutions. It follows from such studies that in general a, b, and c_0 are global parameters that could possibly have topological significance. The clock effect is meaningful if the static parameters (n, a) are such that $0 < n < 1$ and $a > 0$. For instance, the source could be a cosmic string for $n = 1$ with a flat exterior spacetime that would then rule out circular geodesic orbits for clocks. Of the stationary parameters (b, c_0), only c_0 appears explicitly in the formula for the clock effect. The prograde period τ_+ is longer than the retrograde period τ_-, just as for the exterior equatorial plane of a rotating mass, if $c_0 < 0$. The clock effect loses its physical significance in the interesting situation where $c_0 = \pm n\omega_0$ (cf. the appendix).

To gain a physical understanding of the independence of the clock effect from the stationary parameter b, let us introduce a new *periodic* temporal coordinate $\hat{t}$ given by $\hat{t} = t + b\phi$. Consider next a rotation of frequency c_0/n with respect to the periodic time $\hat{t}$ about the z-axis, i.e. $\phi \to \hat{\phi} = \phi - (c_0/n)\,\hat{t}$. Under the transformation $(t, \phi) \to \left(\hat{t}, \hat{\phi}\right)$, whose determinant is unity just like a regular rotation, the Lewis metric (11) takes the form

$$ds^2 = -ar^{1-n}d\hat{t}^2 + e^{\mu}\left(dr^2 + dz^2\right) + \frac{1}{a}r^{1+n}d\hat{\phi}^2. \tag{61}$$

It follows from this result that the physical connection between the gravitomagnetic clock effect and the analogue of the Sagnac effect established in section 4 for $b = 0$ is valid for $b \neq 0$ as well insofar as time t can be replaced by *periodic* time $\hat{t}$ (with period $2\pi b$). In fact, equation (45) implies that $d\hat{\phi}/d\hat{t} = \pm\omega_0$ as would be expected for a circular orbit in the static spacetime (61). Indeed, the metric form (61) appears to be static with respect to $\hat{t}$; however, since the periodic temporal coordinate $\hat{t}$ does not monotonically increase along future-directed causal curves, Bonnor [6] has described the spacetime represented by (61) as only *locally* static. It is important to note that this global stationarity due to $b \neq 0$ does not affect standard clocks under consideration in this work. This is ultimately based on the fact that time is itself measured via a simple periodic motion. In our case, for instance, imagine measuring time by means of a timepiece based on circular motion with period $2\pi b$. It should therefore be clear that Bonnor's sound logical distinction aside, there is no difference between periodic time and regular time that would be directly measurable by a clock. This operational viewpoint is based on the fact that in timekeeping – from the simple mechanical timepieces to modern atomic clocks – one basically counts a fundamental period. Upon further elementary transformations $\tilde{t} = \alpha\hat{t}$, $\tilde{r} = \alpha^{-1}r$, $\tilde{\phi} = \hat{\phi}$ and $\tilde{z} = \alpha^{-1}z$ with $\alpha^{1+n} = a$, the spacetime metric (61) becomes locally equivalent to Levi-Civita's metric for a static line-mass with mass per unit length given approximately by $(1-n)/4$ [6,20].

Appendix: Clock effect for the van Stockum spacetime

The purpose of this appendix is to study equation (49) explicitly for the rigidly rotating dust cylinder of van Stockum [12,6].

Imagine a background inertial frame and an infinite cylindrical configuration of radius R_0 in which free dust particles rotate with constant frequency Ω_0 about the axis of cylindrical symmetry. Let $\beta = R_0\Omega_0$ be the speed of the dust at the rim of the cylinder. Let us now imagine that we corotate with the cylinder; that is, we choose a comoving reference frame in which the particles of the dust cylinder are all at rest. Moreover, we choose our basic spacetime scale λ to be $\lambda = R_0/e^{1/2}$. Thus the radius of the van Stockum cylinder in our units is simply $r = e^{1/2}$ and the frequency of its rotation (relative to the background inertial frame) is $\alpha = \lambda\Omega_0 = \beta/e^{1/2}$. In the comoving system of coordinates, the interior $\left(0 \leq r \leq e^{1/2}\right)$ van Stockum metric is given by

$$ds^2 = -\left(dt - \alpha r^2 d\phi\right)^2 + e^{-\alpha^2 r^2}\left(dr^2 + dz^2\right) + r^2 d\phi^2, \qquad \text{(A1)}$$

which is a solution of the gravitational field equations for dust of density $\alpha^2 \exp(\alpha^2 r^2)/2\pi$.

The exterior van Stockum spacetime is a member of the Weyl class only for $0 \leq \beta < 1/2$. For $\beta \geq 1/2$, the exterior solution belongs to the Lewis class. Thus we limit our discussion here to $\beta < 1/2$; then, it is straightforward to show that for the exterior solution $\left(r \geq e^{1/2}\right)$ in (11)-(15),

$$n = \left(1 - 4\beta^2\right)^{1/2}, \qquad \text{(A2)}$$

$$a = \frac{1}{2}\left(1+\frac{1}{n}\right)\exp\left[-2\beta^2/(1+n)\right], \tag{A3}$$

$$b = 4e^{1/2}\frac{\beta^3}{(1+n)^2}, \tag{A4}$$

$$c_0 = -\alpha. \tag{A5}$$

Let us first note that from (44), (A2) and (A3) we have $\omega_0 = \alpha\chi^{-n}/n$, where $\chi = r/e^{1/2}$. Thus

$$\pm\omega_0 + \frac{c_0}{n} = -\frac{\alpha}{n}\left(1 \mp \chi^{-n}\right), \tag{A6}$$

which vanishes only for the upper sign at the boundary surface $\chi = 1$ and is negative throughout the exterior region. Hence no free particle ("clock") would stay at rest in the spatial Weyl coordinates except when it coincides with a dust particle at the boundary and then $\tau_+ = \infty$. Moreover, it is clear from (45) and (A6) that $\omega < 0$; therefore, both clocks at a given r are counterrotating as viewed from the comoving frame. It then follows from equation (49) that we always have $\tau_+ > \tau_-$ in the exterior region just as in the equatorial plane of a rotating mass [1-3].

The clock effect for the exterior van Stockum spacetime ($\beta < 1/2$) is given – on the basis of equation (49) – by

$$\tau_+ - \tau_- = 4\pi\frac{n}{\alpha}T_n(\chi), \tag{A7}$$

where

$$T_n(\chi) = \frac{\chi^{(n+1)/2}}{\chi^{2n}-1}. \tag{A8}$$

The function $T_n(\chi)$ diverges at the boundary $\chi = 1$ and decreases monotonically for $1 < \chi < \tilde{\chi}$, where $\tilde{\chi}^{2n} = (1+n)/(1-n)$. In fact, $\tilde{r} = e^{1/2}\tilde{\chi}$ is the outer boundary of the region of interest in this case since the coordinate system is no longer admissible beyond this point. It is interesting to note that for $\beta \ll 1$, $\tilde{r} \simeq \alpha^{-1}$ as would be expected for the radius of the light cylinder in a frame rotating with frequency α.

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