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Getting to the Root of the Problem: An Introduction to Fibonacci's Method of Finding Square Roots of Integers

Introduction

Leonardo of Pisa, famously known as Fibonacci, provided extensive works to the mathematical community in the early 1200's, many of which still influence modern mathematics. He experienced arithmetical studies in northern Africa and the Mediterranean, and later exposed his knowledge to larger portions of the world [3, 336]. Fibonacci provided numerous works on the practical applications of mathematics, many of which are explored in his *De Practica Geometrie*. Included in this book are his methods of finding square roots and cube roots, along with how to perform operations with such, which he demonstrated has useful practical applications in geometrical calculations.

One may find Fibonacci's method of finding the square roots of integers to be interesting, considering most people nowadays depend on a calculator to find such values for them. However, it first must be noted that this mathematician was not the first to explore this topic. The Rhind Papyrus suggests the ancient Egyptians explored this topic earlier than 1650 BCE [1, 30]. Square roots were also studied in ancient India, among many other places. As discussed in the commentary of *De Practica Geometrie* [2, 37], a technique for approximating square roots long before Fibonacci entailed the following: if N is the integer you wish to square root, let $N = a^2 + r$, where a^2 is the largest integer value squared which is less than N, and r is the difference between a^2 and N (for example, 107 would be represented as $10^2 + 7$). A close approximation of the square root of N is $\sqrt{N} = a + r/(2a + 1)$. Traveling to different parts of the world, Fibonacci acquired knowledge such as this and applied it to his method of finding roots.

Precursors for Fibonacci's Calculation of Square Roots of Integers

Fibonacci aspired to find simple and relatively far less time-consuming methods of deriving the square roots of quantities. Most astonishing is the fact that he accomplished what he did without the use of symbols, relying only on explanation via words. However, the lack of symbols, along with the lack of explanation in Fibonacci's works limits the clarity with which a reader can interpret his methods. Modern notation allows us to tackle the task of calculating roots by hand in a much clearer fashion.

To begin root calculations by hand for Fibonacci's method, it was important to know some simple but essential facts about the roots of numbers. Not surprisingly, it was emphasized that the first ten squares be memorized, for aid in simple calculation: that $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, ..., $10^2 = 100$. Furthermore, it was of utmost importance that a certain property about the relation between an integer and its square root be acknowledged: that the number of digits that represent an integer will determine the number of digits that will represent its root. The following table demonstrates this [2, 38]:

Table

# of digits in integer number	1	2	3	4	5	6	7	8	...
# of digits in integer part of root	1	1	2	2	3	3	4	4	...

Also, it is important to understand Fibonacci's notation. When he states that the root of an integer is arr, he means that the original integer must be in the form a^2+r, where a and r are both positive integers (and "r" means "root"). In the example below of 864, the root of 864 is found to be 29r23, meaning $864 = 29^2 + 23$. Considering these facts about numbers and their roots, along with Fibonacci's notation, one can commence the calculating process.

Simple Cases

Fortunately for modern mathematicians, where Fibonacci lacked in clarity he exhausted in demonstration. It is important to consider a few particular cases of calculation Fibonacci provided, along with his means of verbalizing techniques used, to fully grasp the nature of these hand calculations.

As a first example, Fibonacci demonstrated how to find the (square) root of 864 [2, 40]. Below is the image of what the hand calculation would look like as provided in *De Practica Geometrie*: [My comments are in square brackets and smaller font.]

(23 [23 is the remainder r]

1

4 0

8 6 4

2 9 [29 is the integer part of root]

4

The following is the explanation taken directly out his book [2, 40] on how to complete this hand calculation (as so much of Fibonacci's work was written in words rather than symbols), and following that will be a more modern interpretation.

> If you wish to find the root of 864, put 2 under the 6 because 2 is the whole root of 8 [that is, 2 is the largest integer so that its square is less than or equal to 8]. Put the remainder 4 above the 8. Then double 2 to get 4 placing it under the 2. Form 46 from the 4 above the 8 and the 6 in the second place. Now divide the new number 46 by 4 to get 11. From this division we get an idea of the following first digit which must be multiplied by twice the digit you already found. Afterwards, square it. The digit is a little less or exactly as much as what comes from the

division. Practice with this procedure will perfect you. So we choose 9 since it is less than 11, and put it under the first digit [Fibonacci used a practice of "guessing" what this number would be, as appropriate for the problem. For this problem, he guessed 9, because only a one digit number can fit in the one digit space under 4 in this case, and 9 is the largest one digit number. Unfortunately, Fibonacci did not demonstrate much of an explanation or justification for procedures such as this, as will be reflected further in later examples]. Multiply 9 by 4 (twice the second term) and subtract the product from 46. The remainder is 10. Put 0 over the 6 and 1 above the 4. Join 10 with 4 in the first place to make 104. Subtract the square of 9 from it to get 23. This is less than 29 the root that has been found [Furthermore, Fibonacci knew that $r \leq 2a$ where N is represented as $N = a^2+r$ for the least misleading representation of the root. For example, the root of 107 could be represented as 9r26, meaning $9^2 + 26 = 107$, but since 26 > 9*2, we know a better representation of the root of 107 must exist, namely 10r7.].

Interpreting this method verbalized in words can be very tricky, especially considering the differences in terminology and notation which we use today. For any number, what we would refer to as the last digit he would refer to as the first, which is based off the right-to-left Arabic writing style [2, 35]. For example, the digit 8 in 864 would be considered the last digit of the number. Additionally, the method itself should seem alien in its nature to most modern readers, as it is "essentially the tremendously tedious technique of the Hindus", as referred to in the commentary of the textbook [2, 35].

In my modern interpretation of the root-finding method, I will provide subscript notation to the numbers to indicate the order in which they are derived in the method, along with keeping the digits of the original number to be rooted in boldface type, in this case **864**, to avoid any confusion. Below is the image of the step-by-step handwritten process:

2_1 is placed below **6** because 2_1 is the greatest number whose square is less than or equal to **8** (because $2_1^2 = 4 <$ **8**). Notice that in this method the 2_1 is being placed below **6**, so that as **864** is a three digit number, its root will be a two digit number (refer to Table p. 54) This follows the aforementioned observation that Fibonacci made in terms of the relation between the number of digits of a number and its root, and this is applied to the technique of finding roots in all of Fibonacci's examples.

8 6 4
2_1

Moving along, $4_2 =$ **8** - 2_1^2, the calculation of the difference between **8** and the greatest square less than it.

4_2
8 6 4
2_1

$4_3 = 2_1$*2.

4_2
8 6 4
2_1
4_3

9_4 is derived in a more complicated manner. On the diagonal in the handwritten calculation, 4_2 and **6** make the value 46. $46/4_3 = 11 + b$ (b being a remainder). Hypothetically, if the value 11 was represented as a one digit number, at this point 11 would simply go in the one's digit place on the right alongside 2_1, which would yield 211 as the integer part of the answer for the root of **864**. However, since this is not the case, further calculation is necessary to find the integer part of the root. Since 9 is the largest single digit number less than 11, 9_4 is placed to the right of 2_1.

4_2

8 6 4

2_1 9_4

4_3

Then 46 - 9_4 * 4_3 = 10 (recall 46 is from the diagonal of 4_2 and **6**), yielding the 1_5 and 0_5 which, with **4**, make 104 on the diagonal.

1_5

4_2 0_5

8 6 4

2_1 9_4

4_3

104 - 9_4^2 = 23, which is the remainder of the root. So the root of **864** is 29r23.

The method functions similarly when working with other numbers. For example, below is my modified version demonstrating the steps to finding the square root of 1234 [2, 41]:

To begin, note that 3_1 is placed in the ten's digit place under **3**, as since **1234** is a four digit number, its root must be a two digit number (again, refer to Table p. 54) One can see how the process is similar: 3_1 is the largest number so that its square is less than or equal to **12**,

1 2 3 4

3_1

3_2 is **12**-3_1^2,

3_2

1 2 3 4

3_1

6_3 is 3_1*2,

3_2

1 2 3 4

3_1

6_3

and 5_4 is derived from 33/6_3 (minus remainder b = 3), where 33 is composed of 3_2 and **3** on the diagonal.

3_2

1 2 3 4

3_1 5_4

6_3

However, since 5_4 is a one digit number (as opposed to the two digit 11 replaced by 9_4 in the case of 864), the calculation following is a little bit simpler than for finding the root of 864 (for 1234, no second diagonal will need to be formed, unlike the 104 for the case of 864; an analogy can be made to adding large digit numbers by hand: when two numbers added yield a one digit number, it can be left as is, but if the sum yields a two digit number, the extra step of carrying the one must be made. This seems worth noting as Fibonacci's method of hand calculation seems to resemble the nature of hand calculations for basic arithmetic. Unfortunately, Fibonacci's demonstration is devoid of an explanation of why this works.). So it is already known that the integer part of the value of the root of 1234 is 35, all that is left is to find the remainder. To do this, recall the remainder of 33/6_3 is b = 3. This 3 joined with **4** makes 34, and 34 - 5_4^2 = 9. Thus the root of **1234** is 35r9.

Complications in Subtleties in Calculations

It is important to note the differences in calculation when the first diagonal divided by some one digit value x_3 equals a two digit number versus when it equals a one digit number. Furthermore, there are other small differences in calculation steps when other similar situations arise. Unfortunately, Fibonacci's methods are exceptionally vague for some specific cases.

Such an example is for finding the root of 153 [2, 39]. Unlike the previously discussed root finding methods for 864 and 1234, for 153 Fibonacci finds both digits of the integer part of the root in a single step: "You will find the root of 1 in the third place to be 1. Place it under the 5 and put 2 before it under the 3." Fibonacci spends the rest of the demonstration calculating the remainder. He does not explain how he was able to find the one's place digit of the root, 2, in the same step as finding the ten's place digit, 1. Perhaps his reasoning lay in the fact that the square root of 1 is 1 as well, so the 2 in the one's place of the root can be found in the simple method that all first digits of roots can be found: 2 being the largest integer in which its square is less than or equal to the middle digit 5 of the number 153 ($2^2 = 4 \leq 5$).

1 (9
1 5 3
1 2

Although Fibonacci's choice to skip the steps in demonstration which he did not ignore for 864 and 1234 may be misleading, one can find the root of 153 by the same method as demonstrated with these two already explored numbers. Again, the boldface and subscript notation will be used in my demonstration: 1_1 is the largest number whose square is less than or equal to **1**.

1 5 3
1_1

0_2 is $\mathbf{1}^2 - 1_1^2$.

0_2
1 5 3
1_1

2_3 is $1_1 * 2$.

0_2
1 5 3
1_1
2_3

2_4 is derived from $5/2_3$ (minus remainder $b = 1_5$), where 5 is composed of 0_2 and **5** on the diagonal.

0_2
1 5 3
1_1 2_4
2_3

0_2 1_5
1 5 3
1_1 2_4
2_3

To find the remainder, as demonstrated in the method before, 1_5 composed with **3** makes 13. $13 - 2_4^2 = 9$. So the root of 153 is 12r9. This result matches the result Fibonacci achieved while "skipping steps" in solving for this root. From this, it can be concluded that at least in the specific case of 153, and perhaps in the cases of all the numbers whose largest digit's place is 1, the "skipping step" method of finding more than one digit of the root in a single step is possible. However, for the modern student of mathematics, how to make calculations only in very specific cases is not very useful; it is better to have an overall general algorithm which can be applied to all cases.

In finding the root of 960 [2, 40], it can be found that attempting to formulate a general algorithm for Fibonacci's method will reach complications. As with 153, steps that would aid in clarification for demonstration are omitted by Fibonacci for 960.

(60

9 6 0

3 0

6

As with the case of 153, one may attempt to calculate the root of 960 without skipping steps, to assure that the method continues to work for this case:

Following the usual process, 3_1 is the largest integer so that its square is less than or equal to **9**.

$\mathbf{9}\ \mathbf{6}\ \mathbf{0}$

3_1

0_2 is $\mathbf{9}^2 - 3_1^2$.

0_2

$\mathbf{9}\ \mathbf{6}\ \mathbf{0}$

3_1

$6_3 = 3_1 * 2$.

0_2

$\mathbf{9}\ \mathbf{6}\ \mathbf{0}$

3_1

6_3

To find the one's place digit of the root, 0_2 is composed with **6**, yielding 6. This 6 is divided by 6_3, with the quotient equaling 1_4 (with remainder $b = 0_5$).

0_2

$\mathbf{9}\ \mathbf{6}\ \mathbf{0}$

$3_1\ 1_4$

6_3

$0_2\ 0_5$

$\mathbf{9}\ \mathbf{6}\ \mathbf{0}$

$3_1\ 1_4$

6_3

However, this would mean that the integer part of the root of 960 would be 31, not 30 as Fibonacci showed. Additionally, the process would yield the remainder part of the root $r = -1$ (where 0_5 is composed with **0**, then $0 - 1_4^2 = -1$), so that the result would be 31r(-1). Although putting this in the form $N = a^2 + r$ works, as $31^2 + (-1) = 960$, this does not follow Fibonacci's simple format for representing square roots, in which negative values were avoided. This comes to show that the method of finding and representing roots as discussed so far in this paper is not general. There are many algebraic subtleties which Fibonacci did not bother to explain in his demonstrations of these calculations. It may lead one to question what can be learned from Fibonacci's demonstrations for finding roots overall, and what precautions must be taken.

In order to find what general observations may be made about Fibonacci's methods for finding roots, first, it seems wise to consider all other concerned examples in *De Practica Geometrie* which demonstrate unique properties and methods worth noting. To begin, consider the finding of the root of 8172 [2, 42].

(72

8 1 7 2

9 0

1 8

As with the case of 1234, the largest digit of the root of 8172 will be in the ten's place, as any four digit number has a two digit root (see Table p. 54) and from the usual method, this digit in the ten's place will be 9. As usual, the number to be placed under the 9 should be simply 9 * 2. However, this is a two digit number, 18. A new special case has arisen. Fibonacci describes:

"…put the 8 under the 9 and the 1 after it to the left. Now the 1 and 8 must be multiplied by the first digit, one at a time. Then square the first digit. And thus there are three products to be subtracted gradually from 72, the remainder from the 81 after finding of the root of 81. Whence, as we obviously know, nothing comes after it except 0. Since a step is lacking, it is the first product that can be subtracted . Because if the first product is subtracted from 7, the second needs to be subtracted from 2. But then there is no place from where to subtract the third product. Or in another way: because the first place is a factor with any step, that step arises from the multiplication. Since the product of the digit in the first place and the digit in the third place, namely by 1, fills the third place, there is no place for 72. Therefore the root of 8172 is 90 and the remainder is 72."

Similarly, in finding the root of 6142 [2, 42], 7*2 is a two digit number, 14, and as in the case of 8172 special subtractions must be made for this circumstance.

				(58
1	2	1	2	
6	1	4	2	
		7	8	
	1	4		

However, in this case, yet another unique circumstance appears, as **61** - 7_2^2 = 12, a two digit number. Recall that if such a number is only one digit, it is placed over the lower digit of the number in which it is derived (see the case for 1234, in which 3_2 is derived from **12** and then the 3_2 is placed above the **2**). Since in the case of 6142 the number derived from **61** is 12 which is two digits, the digit 2 from 12 is placed above **1** and the digit 1 from 12 is placed above **6**. Also, recall that to find the one's place digit of the root as discussed in previous examples in this paper, the value placed above the original number to be rooted will be composed with the digit of the original number which is on the diagonal down and to the right, and then more steps follow. For the example of 1234, 3_3 is composed with **3**, and then steps follow which yield the value of 5 as the digit in the one's place for the root. Similarly, for the case of 6142, 12 is composed with **42** to make 1242. Altogether, since the values extracted 14 and 12 are each two digit numbers, the techniques involved in finding the root of 6142 is all the more complicated. In maintaining the focus on the techniques used in special cases, however, no more is necessary to be discussed about this specific case.

As a final set of examples to demonstrate Fibonacci's methods, consider this: "If you wish to find the root of any number of 5 digits, [first] find the root by the foregoing instructions for the last three digits," [2, 42] and "If you wish to find the root of a number with six digits, first find the root of the last four digits and join the remainder with the following two digits, and continue as before. For example, if we want to find the root of 123456, first find the root of 1234…" [2, 44], etc. As a general rule, it must be understood that to find the roots of larger numbers, first find the root of the number which consists of the digits of the largest place values. Essentially, this means that sometimes to find the root of a number using Fibonacci's method, another calculation must be completed first. This makes sense, as Fibonacci emphasized memorizing the first ten squares: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, …, $10^2 = 100$; but memorization of values further out would be arduous.

Generalities Drawn from Fibonacci's Methods

Overall, although Fibonacci's methods of finding square roots has many twists and turns, depending on algebraic subtleties and special cases, much can be learned from his studies. If nothing more, one can grasp the general methods present in all cases, and some simple cases should be handled with ease. Here composed is a short list of traits found uniform in Fibonacci's methods which can be considerably useful:

1. Obtaining the "last" (that is, of largest place value) digits of a root is always simple.
2. Obtaining the "first" digits of a root can be a difficult task, depending on the subtleties of the specific case. This is where Fibonacci's demonstrations were significantly obscure. However, in general, the method for finding the "first" digits requires a sort of "guessing" nonetheless, and if all else fails, one can easily check a calculation by simply squaring it (Fibonacci actually includes a demonstration for checking a calculation, but such will be omitted here).
3. For larger numbers (five digits or more), it is necessary to find the root of the "last" digits first in a separate calculation.
4. Obtaining the remainder value r also can be a difficult task, depending on the specific case. However, as a criticism against Fibonacci's method, it seems that actually calculating the remainder in the way that he did was not necessary. Simply, a root $\sqrt{N} = arr$ with remainder equaling r means $N = a^2 + r$, so once the integer part of the root is found, no more calculation is necessary.
5. Where N is represented as $N = a^2 + r$, we need $r \leq 2a$ for the best representation of the root.

Relief! A Modern Method; and Conclusion

Quite fortunately, in the modern world humanity is blessed with a square root finding algorithm which is general enough to work in all cases. Here is a useful example [4, 1]:

Find √645 to one decimal place. First group the numbers under the root in pairs from right to left, leaving either one or two digits on the left (6 in this case). For each pair of numbers you will get one digit in the square root. To start, find a number whose square is less than or

equal to the first pair or first number, and write it above the square root line (2).

2
√6.45

2 √6.45 - 4 245	2 √6.45 - 4 (4 _) 245	2 √6.45 - 4 (45) 245
Square the 2, giving 4, write that underneath the 6, and subtract. Bring down the next pair of digits.	Then double the number above the square root symbol line (highlighted), and write it down in parenthesis with an empty line next to it as shown.	Next think what single digit number *something* could go on the empty line so that forty-*something* times *something* would be less than or equal to 245. 45 x 5 = 225 46 x 6 = 276, so 5 works.

```
   2 5
  √6.45.00
  - 4
(45)  245
     - 225
       20 00
```

Write 5 on top of line.
Calculate 5 x 45, write that below 245, subtract, bring down the next pair of digits (in this case the decimal digits 00).

```
   2 5
  √6.45.00
  - 4
(45)  245
     - 225
(50_)20 00
```

Then double the number above the line (25), and write the doubled number (50) in parenthesis with an empoty line next to it as indicated.

```
   2 5 . 3
  √6.45.00
  - 4
(45)  245
     - 225
 (503)20 00
```

Think what single digit number *something* could go on the empty line so that five hundred-*something* times *something* would be less than or equal to 2000.
503 x 3 = 1509
504 x 4 = 2016, so 3 works.

```
   2 5 . 3
  √6.45.00.00
  - 4
(45)  245
     - 225
 (503)20 00
      - 15 09
        491  00
```

Calculate 3 x 503, write that below 2000, subtract, bring down the next digits.

```
   2 5 . 3
  √6.45.00.00
  - 4
(45)  245
     - 225
 (503)20 00
      - 15 09
(506_)  491  00
```

Then double the 'number' 253 which is above the line (ignoring the decimal point), and write the doubled number 506 in parenthesis with an empty line next to it as indicated:

```
   2 5 . 3   9
  √6.45.00.00
  - 4
(45)  245
     - 225
 (503)20 00
      - 15 09
(506_)  491  00
```

5068 x 8 = 40544
5069 x 9 = 45621, which is less than 49100, so 9 works.

Thus to one decimal place, √645 = 25.4.

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From this, one can greatly appreciate the centuries of experience today's mathematical community has over the one in which Fibonacci was exposed. However, notice the similarities in some of the techniques between the modern method and Fibonacci's, and the similarities in mathematical properties considered. Indeed, Fibonacci could have very well been on the way to creating a more general method for finding roots, and if nothing more, his methods were at least accurate and usable enough for the community in which he prospered as a mathematician.

For those curious about the more in-depth mechanisms of Fibonacci's methods of finding the square roots of integers, along with finding the square roots of irrationals, finding cube roots, and performing the operations addition, subtraction, multiplication, and division with roots, it is suggested that one refers to *De Practica Geometrie* for further study. One may refer to Euclid's *Elements* (*II.4*) to understand the old technique of the Hindus [2, 35]. Portions of Fibonacci's *Liber Abaci* can also be found useful. So much can be learned from the old techniques Fibonacci exhibited, and further study can reveal even more intriguing characteristics in all of which the mathematician studied involving roots, including all the interesting and unique practical applications he demonstrated.

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