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On the spectrum of oscillations of a Schwarzschild black hole

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Abstract. The spectral properties of the resonant oscillations of a Schwarzschild black hole are studied. The analogy with the Coulomb wave equation is developed, and it is shown that the high-overtone quasi-normal modes are intimately connected with the long-range nature of the gravitational interaction. This is followed by an explicit construction of the quasi-normal eigenfunctions in analogy with the bound states of the inverted black-hole potentials. The special gravitational quasi-normal mode is discussed and its relationship with the algebraically special solution of the gravitational perturbations is clarified.

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1. Introduction

Astrophysical observations have indicated the possibility of the existence of stellar black holes which might occur as the end result of stellar evolution. Moreover, massive black holes—which might occur as the end result of evolutionary processes in dense stellar systems—are thought to be responsible for the observed activity in active galactic nuclei and quasars. Even the galactic nucleus may harbour a relatively dormant black hole of mass $M \sim 10^6 M_\odot$. However, no definitive evidence has been found thus far for the existence of a completely collapsed configuration [1]. The observed electromagnetic radiation from a system containing a black-hole candidate would be expected to have been emitted mostly by matter that has been accreted by, or in any case surrounds, a black hole. Therefore, it would be difficult to have a unique electromagnetic signature for a black hole. On the other hand, the extreme weakness of the interaction between gravitational waves and matter implies that gravitational wave astronomy might provide a means of uniquely identifying black holes. The effort to detect gravitational radiation from cosmic sources using bar detectors pioneered by Weber has continued in several laboratories; moreover, a programme has been undertaken to detect gravitational waves via large-scale laser interferometry [2].

The exterior black-hole spacetimes represent regular stationary vacuum solutions of the gravitational field equations corresponding to simple configurations of matter in a completely collapsed state of zero volume [3]. The study of the perturbations of black holes was pioneered by Wheeler [4]. When matter collapses to form a black hole, or when a black hole that is already formed is perturbed, it 'rings' in a characteristic way. The ringing modes of a black hole carry information about the mass, charge, and angular momentum of the black hole. In fact, these parameters can be calculated from the frequency and decay

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rate of the gravitational waves that represent the ringing of a black hole. Thus, black-hole oscillations produce characteristic gravitational waves that once detected in the laboratory might lead to an unambiguous detection of black holes due to the unique signature that these waves carry.

The characteristic oscillations of a black hole were first encountered in computer-generated numerical experiments. That is, the equations for the perturbations of a black hole were solved numerically for various incident wavepackets. The response of the black hole at late times was dominated by certain characteristic damped oscillations ('quasi-normal modes'); the frequency and damping rate of a mode were largely independent of the incident perturbations [5]. Theoretical studies of black-hole formation using simplified models have revealed that in the late stages of collapse, the gravitational radiation emitted by the collapsing configuration would contain quasi-normal mode (QNM) oscillations characteristic of the black hole that is being formed [6]. The recognition of the intrinsic significance of QNMs following these studies has led to the development of numerical [7] and analytic [8–23] methods for the determination of QNM frequencies. Let us note, for the sake of concreteness, that the least-damped quadrupolar vibrations of a Schwarzschild black hole of mass M result in gravitational radiation of frequency $\omega_0/2\pi \approx 10^4\ (M_\odot/M)$ Hz and damping constant $\Gamma^{-1} \approx 6\times 10^{-5}(M/M_\odot)$ s.

At a fixed position in space far away from a black hole, a QNM oscillation of the black hole may be expressed as $\mathcal{A}\exp(-\mathrm{i}\omega t)$, where the amplitude $\mathcal{A}$ depends on the strength of the perturbation as well as the nature of the black hole while the complex frequency ω depends solely on the perturbed black-hole parameters. The QNM oscillations are connected with the pole singularities of the scattering amplitude in the background black-hole spacetime; in fact, the location of such a pole singularity in the complex frequency plane is indicated by ω with $\mathrm{Im}(\omega) < 0$, while the residue is reflected in the QNM amplitude $\mathcal{A}$. The spacetime is spherically symmetric in the Schwarzschild field; therefore, the complex frequency of a mode can be characterized by the total angular momentum parameter j as well as the mode number n. The Schwarzschild QNM frequencies are independent of the azimuthal multipole parameter; however, this $(2j+1)$-fold degeneracy is broken for a rotating black hole. It turns out that for a given $j \geq 2$, there is a denumerable infinity of gravitational QNMs for $n = 0, 1, 2, \dots,$ characterizing modes with decreasing relaxation times. Let $\omega = \omega_0 - \mathrm{i}\Gamma$, then the energy of the gravitational perturbation is expected to be dissipated with a time constant $\tau = (2\Gamma)^{-1}$. In fact, τ is the effective relaxation time for the black hole to return to a quiescent state. The frequency of the least-damped vibrations of a black hole increases with j as $(j+\frac{1}{2})$ for $j \gg 1$. On the other hand, we show in the next section that the long-range nature of the gravitational interaction implies that for a given j the high-overtone modes are strongly damped with a damping constant that decreases as $(n+\frac{1}{2})^{-1}$ with increasing $n \gg 1$. The spectral properties of QNMs are discussed in section 3, and in section 4 the status of the algebraically special gravitational perturbation of a Schwarzschild black hole is clarified. The QNMs with $n \sim 1$ have been studied extensively thus far; therefore, we generally focus attention on high-overtone QNMs in this paper.

It is interesting to define—in analogy with a damped harmonic oscillator—a quality factor for a Schwarzschild black hole as a resonator, $Q \approx \omega_0\tau$, using the fundamental (i.e. least-damped) $n = 0$ QNM for a given $j \geq 2$. It turns out that a rough estimate for the Q-factor of the black hole is $Q \approx j$, so that a Schwarzschild black hole is a rather poor oscillator except for modes with extremely high $j \gg 1$. Compared to a Schwarzschild black hole resonating at the basic $j = 2$ mode, an excited atom (nucleus) has a higher quality by a factor of roughly 10^6 (10^{12}). It therefore appears that the observational determination of the properties of a black hole using gravitational wave astronomy might be a difficult task.

In practice, even the theoretical determination of mass (M) and angular momentum (J) of a black hole would encounter difficulties due to several factors. In an astrophysical context, M and J would be continually changing as the black hole accretes matter and radiation. Moreover, many modes would be excited by a realistic perturbation. These modes would be expected to decay rapidly, since a black hole is, in general, a poor resonator. It should be pointed out that the simple analogy between a black hole and a damped harmonic oscillator breaks down for a rapidly rotating black hole. For instance, nearly extreme Kerr black holes ($J \to M^2$) have QNMs with long relaxation times ($\tau \to \infty$); however, for these modes the amplitude of excitation $\mathcal{A} \to 0$ as $\tau \to \infty$. The relaxation time may increase with rotation; however, for the modes that would be mainly excited and therefore would matter most in practice, the effective Q-factor of a rotating black hole is expected to be generally of the same order of magnitude as that of a Schwarzschild black hole with the same mass. This qualitative supposition is borne out by accurate numerical results for the QNMs of rotating black holes [11].

The fundamental quadrupolar oscillation of a black hole may be hard to observe using gravitational wave detectors under construction; however, it is of basic theoretical significance for the theory of gravitation. General relativity has been tested experimentally in situations where wave phenomena can be treated essentially in the eikonal approximation [24]. In the geometric 'optics' regime, the reduced wavelength of the radiation $\lambda\!\!\!^{-}_0$ is negligible compared to the characteristic gravitational acceleration length $\mathcal{L}_g$ [25]. For a Schwarzschild black hole, this length is comparable to its gravitational radius. It follows that with $\lambda\!\!\!^{-}_0 = {\omega_0}^{-1}$ and $j \sim 1$, $\lambda\!\!\!^{-}_0/\mathcal{L}_g \sim 1$ for the fundamental oscillations of a black hole and hence the detection of these gravitational waves would test general relativity in the wave 'optics' regime.

2. QNMs and the infinite range of the gravitational interaction

The radiative perturbations of a Schwarzschild black hole can be expressed as linear combinations of $\exp(-\mathrm{i}\omega t)\Psi$, where Ψ is a radial wavefunction that satisfies the Regge–Wheeler equation

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}X^2} + \left(\omega^2 - W\right)\Psi = 0\,. \tag{1}$$

Here $X = R + 2M\ln(-1 + R/2M)$, and R is the radial Schwarzschild coordinate such that as R ranges from the horizon at $2M$ to radial infinity, the Regge–Wheeler coordinate X ranges from $-\infty$ to $+\infty$. Throughout this paper, we use units such that Newton's gravitational constant and the speed of light in vacuum are set equal to unity. It turns out that

$$W = \left(1 - \frac{2M}{R}\right)\left(\frac{\lambda}{R^2} + \frac{2M\beta}{R^3}\right), \tag{2}$$

where $\lambda = j(j+1)$ and $\beta = -3, 0, 1$, for the gravitational, electromagnetic or scalar perturbations, respectively. A massless scalar field is included in this analysis for the sake of completeness; in fact, observational data are consistent with the absence of basic scalar fields in nature.

Let $X = 2Mx$, $R = 2Mr$, $2M\omega = \sigma$, and consider the Regge–Wheeler equation in the modified form

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \left(\sigma^2 - kV\right)\psi = 0\,. \tag{3}$$

Here $V = 4M^2W$,

$$V = \left(1 - \frac{1}{r}\right)\left(\frac{\lambda}{r^2} + \frac{\beta}{r^3}\right), \tag{4}$$

and k is a parameter such that for $k = 1$, equation (3) reduces to equation (1). We may regard σ and ψ as functions of k, which is allowed to vary over the real numbers. The quasi-normal modes are solutions of equation (1) with boundary conditions such that the waves are absorbed at the horizon for $r \to 1$ and emitted outward to infinity for $r \to \infty$. Since V vanishes as $x \to \pm\infty$, we find that for $x \to +\infty$, $\psi \to \psi_+(k)\exp(\mathrm{i}\sigma x) + \hat{\psi}_+(k)\exp(-\mathrm{i}\sigma x)$, while for $x \to -\infty$, $\psi \to \psi_-(k)\exp(-\mathrm{i}\sigma x) + \hat{\psi}_-(k)\exp(\mathrm{i}\sigma x)$. Here $\psi_\pm$ and $\hat{\psi}_\pm$ are constants for a given k, and the QNM boundary conditions correspond to $\hat{\psi}_\pm(k) = 0$ for $k > 0$. In this case, $\sigma = \sigma_0 - \mathrm{i}\gamma$ with $\gamma > 0$ and the QNMs are symmetrically distributed about the imaginary axis in the lower half of the complex frequency plane. Thus as $x \to \pm\infty$, the contributions to the wavefunction with amplitudes $\hat{\psi}_\pm$ are, in fact, damped exponentially while the QNM wavefunction diverges exponentially; this is the basic difficulty associated with the determination of QNMs.

The QNM problem under consideration is intimately connected, through the parameter k, with the problem of bound states of the inverted black-hole potential. Let $\mathrm{i}\sigma(k) \equiv -\Omega(k)$ for $k < 0$, then the eigenvalues associated with $\Omega > 0$ correspond to the bound states of the inverted black-hole potential. Once such bound states are determined, the quasi-normal modes may be obtained from the smooth extension of these states to positive values of k. The QNM frequencies would then be given by $\sigma(k) = \mathrm{i}\Omega(k)$ for $k = 1$. It should be remarked here that the exact behaviour of σ as a function of k is not known at present; in particular, Nollert [12] has shown that $\sigma(k)$ is not analytic at $k = 0$, hence the continuation from $k < 0$ to $k > 0$ cannot be simply analytic. The inverted black-hole potential has an infinite number of bound states for $j > 0$; these states are further discussed in appendix A. It has been shown [21] that the number of QNMs is countably infinite as well. The mode number $n = 0, 1, 2, \ldots,$ corresponds to QNMs as well as the bound states of the inverted potential for $j > 0$. It turns out that the ground state ($n = 0$) is related to the fundamental, i.e. least-damped, QNM and the low-lying excited states ($n \sim 1$) correspond to the dominant QNMs with increased damping. For $n \gg 1$, the bound states are reminiscent of the Rydberg states familiar from the Coulomb wave equation and the relevant QNMs are the high-overtone modes that are highly damped. The connection between QNMs and the bound states underlies our work; however, only QNM eigenfunctions will be explicitly treated here. Therefore, we set $k = 1$ in what follows.

The similarity between the Coulomb potential in electrodynamics and Newton's gravitational potential is reflected in the treatment of wave phenomena as well, as can be demonstrated from the following considerations. Just as in the Schrödinger treatment of the hydrogen atom in which the Coulomb potential plays a dominant role for large r, the wave equation in the black-hole field can be transformed such that the Newtonian $1/r$ potential will dominate at large r. To this end, let us note that this transformation can be accomplished using

$$\psi = \left(1 - \frac{1}{r}\right)^{-1/2} \phi, \tag{5}$$

where ϕ satisfies

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \left[\sigma^2 - U(r;\sigma)\right]\phi = 0\,. \tag{6}$$

Here U is given by

$$U(r;\sigma) = -\left[\frac{2\sigma^2 - \lambda - \beta + \frac{1}{2}}{r-1} + \frac{\lambda + \beta - \frac{1}{2}}{r} + \frac{\sigma^2 + \frac{1}{4}}{(r-1)^2} + \frac{\beta - \frac{3}{4}}{r^2}\right]. \tag{7}$$

Let us set $r - 1 = \rho$, such that in terms of the radial coordinate $\rho : 0 \to \infty$, equation (6) becomes analogous to the radial Schrödinger equation. For $\rho \to \infty$,

$$U \sim -\frac{2\sigma^2}{\rho} - \frac{\sigma^2 - \lambda}{\rho^2} + \cdots, \tag{8}$$

where the first term represents the Newtonian potential and the second term corresponds to the centrifugal barrier.

The analogy with the Coulomb problem may be taken a step further by considering the wave equation (6) with the potential (8). The Coulomb aspect of the radiative perturbations of a black hole is in part due to the logarithmic term in the equation connecting the Regge–Wheeler coordinate X with the Schwarzschild radial coordinate R, so that the near field as well as the far field is affected. Imposing QNM boundary conditions for $\rho \to 0$ and $\rho \to \infty$ on the analogue of the Coulomb wave equation,

$$\frac{\mathrm{d}^2\phi_c}{\mathrm{d}\rho^2} + \left(\sigma^2 + \frac{2\sigma^2}{\rho} + \frac{\sigma^2 - \lambda}{\rho^2}\right)\phi_c = 0, \tag{9}$$

results in the asymptotic behaviour of high-overtone QNMs. To show this, let $\phi_c = \rho^\mu \exp(\mathrm{i}\sigma\rho)u(\zeta)$, where $\zeta = -2\mathrm{i}\sigma\rho$ and μ is given by

$$\mu(\mu - 1) = \lambda - \sigma^2\,. \tag{10}$$

Then equation (9) takes the form

$$\zeta\frac{\mathrm{d}^2u}{\mathrm{d}\zeta^2} + (2\mu - \zeta)\frac{\mathrm{d}u}{\mathrm{d}\zeta} - (\mu - \mathrm{i}\sigma)u = 0, \tag{11}$$

which is a confluent hypergeometric equation with two linearly independent solutions given, in general, by the Kummer functions ${}_1F_1(\mu - \mathrm{i}\sigma, 2\mu;\ \zeta)$ and $\zeta^{1-2\mu}\ {}_1F_1(-\mu - \mathrm{i}\sigma + 1, 2 - 2\mu;\ \zeta)$. The confluent hypergeometric functions are essentially the same as the generalized Laguerre functions that are familiar from the treatment of the Coulomb problem in quantum mechanics. Inspection of the general solution reveals that the QNM boundary conditions can be satisfied if either of the confluent hypergeometric functions reduces to a polynomial. In either case (i.e. $\mu - \mathrm{i}\sigma = -n$ or $-\mu - \mathrm{i}\sigma + 1 = -n$ for positive integer n), equation (10) implies that

$$\sigma = -\frac{\mathrm{i}}{2}\left[\left(n + \frac{1}{2}\right) - \frac{(j + \frac{1}{2})^2}{n + \frac{1}{2}}\right], \tag{12}$$

which is consistent with the fact that for a given j, the asymptotic behaviour of QNMs for $n \gg 1$ is such that $\sigma_0/n \sim 0$ and $\gamma/n \sim \frac{1}{2}$. For instance, numerical investigations [11, 12, 20] of the basic $j = 2$ gravitational oscillations of a Schwarzschild black hole have shown that $\sigma_0 \approx 0.087$ and $\gamma \approx (n + \frac{1}{2})/2$ for $n \gg 1$. Moreover, for a given $n \gg 1$ the dependence of γ on j given by equation (12) is a fair analytic approximation to the results of numerical investigations; however, it should be pointed out that better analytic results are available which give

$$\gamma = \frac{1}{2}\left(n + \frac{1}{2}\right) - \gamma_1\frac{j(j+1) - 1}{\sqrt{n + 1/2}} + \cdots, \tag{13}$$

where $\gamma_1 \approx 0.093$ is consistent with available numerical results [23]. It is important to note that for the high-overtone QNMs in the limit $n \to \infty$, ω/n depends only upon the black-hole mass M and is independent of j and β; this is consistent with the interpretation of these modes in terms of the Newtonian gravitational potential.

3. Quasi-normal mode eigenfunctions

The Regge–Wheeler equation (1) is essentially a one-dimensional time-independent Schrödinger equation with a potential barrier W that vanishes as $X \to \pm\infty$. It follows from the intrinsic connection between the QNMs and the bound states of the inverted black-hole potential described in the previous section that the QNM eigenfunctions are intimately related to the bound-state eigenfunctions of the inverted potential. It is well known in quantum mechanics that for one-dimensional bound-state problems the eigenfunction of order $n = 0, 1, 2, \ldots,$ corresponding to the eigenvalue of order n, has exactly n nodes. Thus the eigenfunction has n distinct zeros along the finite x-axis. This property will be employed in the description of QNM eigenfunctions, except that—as a consequence of the continuation from $k < 0$ for bound states to $k > 0$ for QNMs—the nodes of the QNM eigenfunctions generally occur in the complex x-plane.

It proves useful to begin with the wave equation (6), where $U(r;\,\sigma)$ is an analytic potential in the complex r-plane except for the poles at $r = 0$ and $r = 1$. These singular points are regular singularities of equation (6). We assume that the QNM eigenfunction of order n is of the form

$$\phi = f \exp(g), \tag{14}$$

with

$$f = \begin{cases} 1 & \text{for} \quad n = 0\ , \\ \displaystyle\prod_{i=1}^{n}(r - \alpha_i) & \text{for} \quad n = 1, 2, \ldots, \end{cases} \tag{15}$$

and

$$g = ar + b\ln(r-1) + c\ln r - \sum_{s=1}^{\infty}\frac{e_s}{sr^s}, \tag{16}$$

where α_i and a, b, c, e_s are complex constants to be determined. The n zeros of the eigenfunction of order n are explicitly expressed via the function f in equation (15). It is shown in appendix B that equation (6) can be solved with this ansatz, and three kinds of solution can be obtained. In particular, the QNM solution of order n can be determined from a system of $n+1$ algebraic equations for $n+1$ unknowns, namely, c, and $\alpha_1, \alpha_2, \ldots, \alpha_n$. These algebraic equations can be expressed as

$$c^2 + nc + \tfrac{1}{2}\left(\lambda + \beta - \tfrac{1}{2}\right) - b\sum_{s=1}^{\infty} e_s + \sum_{i=1}^{n} D_i = 0, \tag{17}$$

$$D_i + \frac{c}{\alpha_i} + \sum_{s=1}^{\infty}\frac{e_s}{\alpha_i^{s+1}} = 0, \qquad i = 1, 2, \ldots, n, \tag{18}$$

where we have defined n quantities D_i, $i = 1, 2, \ldots, n$, via

$$D_i = a + \frac{b}{\alpha_i - 1} + \sum_{j=1, j\neq i}^{n}\frac{1}{\alpha_i - \alpha_j}\,. \tag{19}$$

We have assumed here that $\alpha_i \neq \alpha_j$, if $i \neq j$; that is, the QNM eigenfunction of order n has n distinct zeros in the complex r-plane in complete correspondence with the n distinct nodes of the bound state of order n. Moreover, the fundamental QNM (i.e. $n = 0$) is included in the formulae given here once we set finite sums from 1 to n, such as in equation (17), equal to zero whenever $n = 0$. Appendix B contains a detailed derivation of these results. The substitution of our ansatz in equation (6) results in a simple expression for the QNM eigenfrequency of order n, namely,

$$\sigma = -\mathrm{i}a = -\tfrac{1}{2}\mathrm{i}\big(n + c + \tfrac{1}{2}\big). \tag{20}$$

Moreover, b is given by

$$b = -\tfrac{1}{2}\big(n + c - \tfrac{1}{2}\big), \tag{21}$$

and e_s, $s = 1, 2, \ldots,$ can be determined from the recurrence relation

$$\begin{aligned} e_{l+2} = \frac{1}{2a}\Bigg[&(l+2)e_{l+1} + (2c - l - 1)e_l + \sum_{s=1}^{l} e_s(e_{l-s} - e_{l+1-s}) \\ &- 2\sum_{i=1}^{n}\alpha_i^{l+1}(\alpha_i - 1)\bigg(D_i + \frac{c}{\alpha_i} + \sum_{s=1}^{l}\frac{e_s}{\alpha_i^{s+1}}\bigg)\Bigg] \qquad l = 1, 2, \ldots \end{aligned} \tag{22}$$

with $e_0 = 0$,

$$e_1 = \frac{1}{2a}\left[c^2 + c + \lambda + \frac{1}{4} - 2\sum_{i=1}^{n}(\alpha_i - 1)D_i\right], \tag{23}$$

and

$$e_2 = \frac{1}{2a}\left[2e_1 + c^2 - c + \beta - \frac{3}{4} - 2\sum_{i=1}^{n}\alpha_i(\alpha_i - 1)\left(D_i + \frac{c}{\alpha_i}\right)\right]. \tag{24}$$

It may be assumed that if the infinite sums involving e_s in equations (16)–(18) converge, then the $n+1$ algebraic equations (17) and (18) have at least one set of solutions for the $n+1$ unknowns α_i and c. Moreover, it follows from our construction that the eigenfunctions given above could satisfy the QNM boundary conditions. Let us denote the QNM eigenfunction of order n by ϕ_n; then, ϕ_n has exactly n zeros in the complex r-plane. Therefore, ϕ_n is linearly independent of ϕ_m if $n \neq m$. Thus *the independent QNM eigenfunctions are enumerably infinite*, as expected. The structure of the zeros of these QNM eigenfunctions is consistent with certain underlying assumptions [22] in the application of the phase-integral method [18] to the QNM problem. Moreover, it is clear from a comparison of the results of the previous section, e.g. equation (12), with the form of equation (20) that in our approach we should choose a solution of equation (17) for the complex constant c such that $c/n \to 0$ as $n \to \infty$.

It can also be shown that if $n \neq m$, then $\sigma_n \neq \sigma_m$; that is, none of the quasi-normal modes is degenerate. Suppose the contrary to be true, and let ϕ_n and ϕ_m be two linearly independent eigenfunctions corresponding to the same eigenvalue. That is, from equation (6), $\phi_n(\phi_m)$ satisfies the wave equation for the eigenvalue $\sigma_n(\sigma_m)$, where $\sigma_n = \sigma_m$ and $n \neq m$. From equation (6), we obtain the Wronskian

$$\phi_n\frac{\mathrm{d}\phi_m}{\mathrm{d}r} - \phi_m\frac{\mathrm{d}\phi_n}{\mathrm{d}r} = W_{nm}, \tag{25}$$

where W_{nm} is a constant. Substituting equations (14)–(16) into (25), we find

$$\phi_n \phi_m F_{nm} = W_{mn}, \tag{26}$$

with

$$F_{nm} = a_n - a_m + \frac{b_n - b_m}{r-1} + \frac{c_n - c_m}{r} + \sum_{s=1}^{\infty} \frac{e_s^{(n)} - e_s^{(m)}}{r^{s+1}} + \sum_{i=1}^{n} \frac{1}{r - \alpha_i^{(n)}} - \sum_{i=1}^{m} \frac{1}{r - \alpha_i^{(m)}}, \tag{27}$$

and the proviso regarding the last two terms in this expression that a finite sum would be zero if $n = 0$ or $m = 0$. We should note here that in this argument $\sigma_n = \sigma_m$ by assumption; therefore, $a_n = a_m$ by equation (20). The eigenfunctions ϕ_n and ϕ_m both diverge *exponentially* as $r \to \infty$ in accordance with the QNM boundary conditions; therefore, equation (26) can be satisfied for all r only when F_{nm} vanishes identically. Thus we must have $F_{nm} = 0$, and hence $W_{nm} = 0$. This means that ϕ_m is proportional to ϕ_n, contradicting our previous assumption. Thus we arrive at the conclusion that *the quasi-normal modes of a Schwarzschild black hole are discrete, non-degenerate, and enumerably infinite*.

It should be pointed out here that these results reflect the consistency of the approach adopted here, namely, the connection between bound states and QNMs. The bound states of the inverted black-hole potential (for $j > 0$) form a countably infinite set of orthonormal eigenfunctions corresponding to discrete non-degenerate bound levels $\Omega_n > 0$, $n = 0, 1, 2, \ldots$.

4. A special QNM solution

Let us now consider a special case of the general solution (17)–(24) in which all the coefficients e_s vanish. For this case, equations (20) and (21) imply that

$$a = \mathrm{i}\sigma, \qquad b = -\mathrm{i}\sigma + \tfrac{1}{2}, \qquad c = 2\mathrm{i}\sigma - n - \tfrac{1}{2}, \tag{28}$$

and equation (24) yields

$$c = \pm\sqrt{1-\beta} + \tfrac{1}{2}, \tag{29}$$

while equation (22) is identically satisfied, and equation (23) is equivalent to equation (17). The remaining equations, (17) and (18), are still not easy to solve. However, we find that equation (28), together with equations (5) and (14)–(16), can reduce the eigenfunction $\tilde{\psi}_n$ for the special mode to the form

$$\tilde{\psi}_n = \mathrm{e}^{\mathrm{i}\sigma r}(r-1)^{-\mathrm{i}\sigma} r^{2\mathrm{i}\sigma} \prod_{i=1}^{n} \left(\frac{r - \alpha_i}{r} \right), \tag{30}$$

which is interesting in view of Leaver's method for the determination of QNMs.

Leaver has suggested that the QNM eigenfunctions can be written as [11]

$$\psi = \mathrm{e}^{\mathrm{i}\sigma r}(r-1)^{-\mathrm{i}\sigma} r^{2\mathrm{i}\sigma} \sum_{m=0}^{\infty} L_m \left(\frac{r-1}{r} \right)^m. \tag{31}$$

It is therefore clear that $\tilde{\psi}_n$ in equation (30) corresponds to a special case of the general QNM eigenfunction (31) in which the infinite series reduces to a polynomial, i.e. the coefficients L_m vanish for $m > n$,

$$\tilde{\psi}_n = \mathrm{e}^{\mathrm{i}\sigma r}(r-1)^{-\mathrm{i}\sigma} r^{2\mathrm{i}\sigma} \sum_{m=0}^{n} \tilde{L}_m \left(\frac{r-1}{r}\right)^m . \tag{32}$$

The substitution of Leaver's ansatz (31) in the Regge–Wheeler equation results in a three-term recurrence relation for Leaver's coefficients given by

$$A_0 L_1 + B_0 L_0 = 0, \tag{33}$$

$$A_m L_{m+1} + B_m L_m + C_m L_{m-1} = 0, \qquad m = 1, 2, \ldots, \tag{34}$$

where A_m, B_m and C_m can be expressed as

$$A_m = (m+1)(m+1-2\mathrm{i}\sigma), \tag{35}$$

$$B_m = -2\left(m - 2\mathrm{i}\sigma + \tfrac{1}{2}\right)^2 - \lambda - \beta + \tfrac{1}{2}, \tag{36}$$

$$C_m = (m - 2\mathrm{i}\sigma)^2 + \beta - 1 . \tag{37}$$

Furthermore, Leaver has used the recurrence relations to derive an infinite continued fraction equation for the QNM eigenfrequencies [11],

$$0 = B_0 - \frac{A_0 C_1}{B_1 -} \frac{A_1 C_2}{B_2 -} \frac{A_2 C_3}{B_3 -} \cdots . \tag{38}$$

However, the derivation of the continued fraction equation (38) breaks down whenever the infinite series in equation (31) reduces to a polynomial. To see this, let us define $\mathcal{F}_0 = 0$ and $\mathcal{F}_m = -C_m(L_{m-1}/L_m)$ for $m = 1, 2, \ldots,$ so that equation (34) can be written as $\mathcal{F}_m = B_m - A_m C_{m+1}/\mathcal{F}_{m+1}$ for $m = 0, 1, 2, \ldots$; the repeated application of this equation would result in the infinite continued fraction equation (38) as well as its inversions. This method is thus successful if all $\mathcal{F}_m$, $m = 0, 1, 2, \ldots,$ are well defined; however, it would break down once $\mathcal{F}_m$, for $m = n+1, \ldots,$ are undefined. Thus equation (38) does not hold for the special eigenfunction $\tilde{\psi}_n$ given by equation (32). Therefore, to find the eigenfrequency $\tilde{\sigma}_n$ corresponding to $\tilde{\psi}_n$, we need to start with the recurrence relations (33) and (34) directly. Since $\tilde{L}_m = 0$ for $m > n$, the recurrence relation (34) reduces to a finite number of equations of which the last one requires $C_{n+1} = 0$. Using equation (37), we must then have

$$2i\tilde{\sigma} = n + 1 \pm \sqrt{1-\beta} . \tag{39}$$

Let us now consider gravitational perturbations with $\beta = -3$, and choose the minus sign in equation (39), so that

$$2\mathrm{i}\tilde{\sigma} = n - 1, \tag{40}$$

and hence, by equations (35) and (37),

$$A_{n-2} = C_{n-3} = 0 . \tag{41}$$

The recurrence relations can then be reduced to

$$A_0\tilde{L}_1 + B_0\tilde{L}_0 = 0, \tag{42}$$
$$A_m\tilde{L}_{m+1} + B_m\tilde{L}_m + C_m\tilde{L}_{m-1} = 0, \qquad m = 1, 2, \ldots, n-4, \tag{43}$$
$$A_{n-3}\tilde{L}_{n-2} + B_{n-3}\tilde{L}_{n-3} = 0, \tag{44}$$
$$B_{n-2}\tilde{L}_{n-2} + C_{n-2}\tilde{L}_{n-3} = 0, \tag{45}$$
$$A_{n-1}\tilde{L}_n + B_{n-1}\tilde{L}_{n-1} + C_{n-1}\tilde{L}_{n-2} = 0, \tag{46}$$
$$B_n\tilde{L}_n + C_n\tilde{L}_{n-1} = 0, \tag{47}$$

of which equations (44) and (45) yield a characteristic relation

$$A_{n-3}C_{n-2} = B_{n-2}B_{n-3}\,. \tag{48}$$

This relation—which could serve as the condition for the existence of special modes—implies

$$n = \frac{(\lambda+1)(\lambda-3)}{3} + 2, \tag{49}$$

so that from equation (40),

$$\tilde{\sigma} = -\frac{\mathrm{i}}{6}\lambda(\lambda-2)\,. \tag{50}$$

It is interesting to note that $n = 1 + (j-1)j(j+1)(j+2)/3$; therefore, n is an integer for $j = 2, 3, 4, \ldots$. It turns out that the other possibilities do not lead to any consistent result, so that this gravitational mode is the only known special mode; a similar derivation of the special mode is already contained in the work of Nollert [12]. Moreover, it can be checked that the bound state corresponding to the special QNM does not have a simple form analogous to equation (32); that is, a similar treatment of the bound-state problem reveals the absence of a special solution in that case. The coefficients $\tilde{L}_m$ can now be calculated without difficulty. As an illustration, we give the solution for $j = 2$ (i.e. $\lambda = 6$): $\tilde{\sigma} = -4\mathrm{i}$, and

$$\tilde{\psi}_9 = \mathrm{e}^{4r}(r-1)^{-4}r^8\sum_{m=0}^{9}\tilde{L}_m\left(\frac{r-1}{r}\right)^m, \tag{51}$$

with

$$\begin{aligned}
&\tilde{L}_0 = 945 && \tilde{L}_1 = -15\,525\\
&\tilde{L}_2 = 116\,100 && \tilde{L}_3 = -520\,740\\
&\tilde{L}_4 = 1\,551\,870 && \tilde{L}_5 = -3\,209\,958\\
&\tilde{L}_6 = 4\,659\,060 && \tilde{L}_7 = -4\,659\,060\\
&\tilde{L}_8 = 2\,717\,785 && \tilde{L}_9 = -1\,164\,765,
\end{aligned} \tag{52}$$

where we have chosen $\tilde{L}_0 = 945$ for the sake of simplicity. One can check that the polynomial in equation (51) leads to an algebraic equation of degree nine in r with nine solutions in the complex r-plane (excluding $r = 0, 1$). It should be pointed out here that this assignment of $n = 9$ to the special mode differs from that of Leaver [11]; whereas in Leaver's work [11] the ninth mode is special, this happens to be the tenth mode here. The difference may be due to the breakdown of the infinite fraction method in this special case. On the other hand, we have assumed that all the modes corresponding to bound states actually exist, i.e. the series involving e_s, etc, do indeed converge. It may be possible to test this assumption by subjecting the approach of section 3 to numerical investigation; in fact, this may result in a new method of numerically evaluating QNM frequencies [26].

The eigenfrequency of the special mode (50) is the same as that of the algebraically special gravitational perturbation [27]. This relationship is brought out in appendix B.

5. Discussion

The problem under consideration in this paper is a special case of the general problem of resonant states in scattering theory. The QNMs correspond to the poles of the amplitude for scattering of waves from black-hole barrier potentials. It has not been possible, in general, to find explicit analytic expressions for the QNM eigenfrequencies in closed form; therefore, various methods have been developed [8–23] for the approximate determination of these eigenfrequencies. These methods are generally based on the imposition of QNM boundary conditions on the solution of the wave equation such that the waves are outgoing at infinity and ingoing at the horizon; in this way, results have been obtained that are in general agreement with each other. In particular, these methods have resulted in highly accurate numerical algorithms that are due to Leaver [11], Nollert [12] and Andersson [20]. On the other hand, the special QNM discussed in section 4 depends on the *exact* form of the black-hole potential for gravitational perturbations; therefore, it is not expected to appear as a QNM in any method that, in effect, replaces the black-hole potential by an approximate one. This explains the absence of the special mode in some approaches to the QNM problem [13].

The fundamental (i.e. least-damped) Schwarzschild QNM and the first few overtones are basically determined by the height and curvature of the barrier potential at its maximum, while the strongly damped high-overtone QNMs correspond to the long-range nature of the gravitational interaction. As the mode number n increases from zero to infinity, the QNMs of a black hole provide a characterization of the near field out to the far field. Far away from the source, the gravitational potential must vary with radial distance just like the Coulomb potential in correspondence with the Newtonian theory of gravitation; therefore, the high-overtone modes are expected to be essentially common to black holes of mass M regardless of their angular momenta, while the dominant ($n \sim 1$) modes are expected to depend on the near-field characteristics of a rotating black hole.

A new formalism has been developed in this paper to determine the QNM eigenfunctions. Using this method, it is possible, in principle, to construct the infinite sequence of non-degenerate eigenfunctions as well as an analytic formula for the QNM eigenfrequencies. For the important case of gravitational perturbations, the special mode of frequency $-\mathrm{i}(j-1)j(j+1)(j+2)/6$ has been investigated and its relationship with the algebraically special solution of the gravitational perturbations has been clarified. Furthermore, Leaver's general method has been discussed briefly and it has been shown that the infinite fraction approach breaks down for the special mode; this could presumably be the source of the discrepancy in the assignment of mode number to the special QNM solution. This issue as well as a detailed comparison of our results with those of other authors [12–22] would require further investigation.

Acknowledgment

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Appendix A

The purpose of this appendix is to discuss the relationship between QNMs and the bound states of inverted black-hole potentials.

Consider the scattering of radiation of helicity h from a neutral spherical configuration of mass M. If the flux of the incident radiation is sufficiently small and the radiation is scattered purely by the gravitational field of the source, then it is possible to regard the radiation as a small perturbation on the static background of a Schwarzschild black hole of mass M. The scattering may be described in terms of partial waves with spherical harmonic parameter $j \geq h$; the partial wave amplitude then satisfies equation (1) with $\beta = 1 - h^2$.

The standard treatment of scattering of radiation from a black hole is based on the scattering cross section which usually involves an incident plane wave. This approach is interesting for the discussion of many wave phenomena such as the glory effect in black-hole scattering; for instance, it can be shown that for electromagnetic as well as gravitational radiation the amplitude for exact backward scattering ('glory') is precisely zero. This follows from the fact that the backward scattered waves must have a helicity opposite to the helicity of the incident waves by the law of conservation of angular momentum along the direction of incidence; however, this helicity flip is forbidden by the conservation of helicity in black-hole scattering. The waves scattered in the backward direction thus interfere destructively for $h > 0$ [28]. It is clear that the argument presented here for the absence of glory would fail in the case of scalar radiation [29].

The existence of quasi-normal modes is not evident from the scattering cross section, since QNMs occur for incident wavepackets. The response of a black hole to an incident wavepacket is dominated at late times by the singularities of the reflection amplitude, as illustrated schematically in figure A1. The analytic approach to black-hole QNMs had its origin in the recognition that a black hole would present a potential barrier to incident waves with a reflection coefficient given approximately by

$$\mathcal{R} = \frac{1}{1 + \mathrm{e}^{2\pi\epsilon}}, \tag{A1}$$

with

$$\epsilon = \frac{\omega^2 - W_m}{(2K)^{1/2}}, \tag{A2}$$

where W_m and K are the height and curvature of the barrier at its maximum, respectively. This result is valid, above and below the barrier maximum, to the extent that the barrier can be approximated by an inverted harmonic oscillator potential [30]. The QNMs occur when $\mathcal{R}$ diverges, i.e. $\epsilon = \pm\mathrm{i}(n + 1/2), n = 0, 1, 2, \ldots$. It follows that the QNMs of the inverted harmonic oscillator potential are given by $\pm\omega_0 - \mathrm{i}\Gamma$, $\omega_0 > \Gamma > 0$, where

$$2\omega_0^2 = \left[{W_m}^2 + 2K\left(n + \tfrac{1}{2}\right)^2\right]^{1/2} + W_m, \tag{A3}$$

and

$$2\Gamma^2 = \left[{W_m}^2 + 2K\left(n + \tfrac{1}{2}\right)^2\right]^{1/2} - W_m, \tag{A4}$$

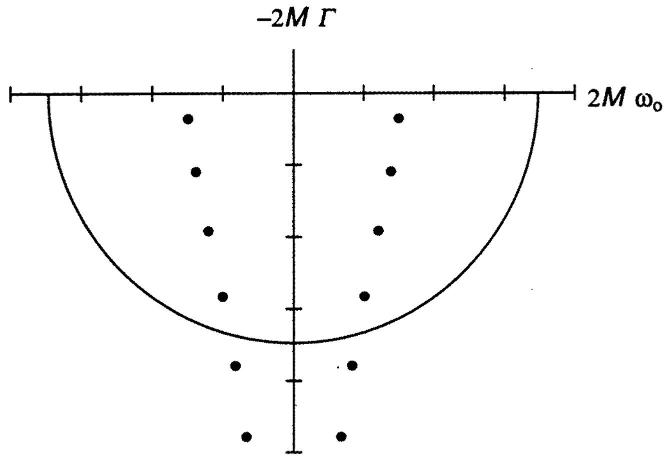

Figure A1. The complex frequencies ($\sigma = \sigma_0 - \mathrm{i}\gamma$) associated with the quadrupolar oscillations of a Schwarzschild black hole. For a given j, the high-overtone modes approach lines parallel to the imaginary axis as $n \to \infty$. Imagine an incident wavepacket of amplitude $\int_{-\infty}^{\infty} \mathcal{I}(\omega) \exp[-\mathrm{i}\omega(t + X)]\,\mathrm{d}\omega$; for $X \to \infty$, the scattered radiation has an amplitude given by $\int_{-\infty}^{\infty} \mathcal{S}(\omega)\mathcal{I}(\omega) \exp[-\mathrm{i}\omega(t - X)]\,\mathrm{d}\omega$, where $\mathcal{S}(\omega)$ is the reflection amplitude. At late times $t > X$ far from the black hole, the scattered amplitude might be written as an integral over the semicircular contour with its radius approaching infinity. It follows from the application of Jordan's lemma that the late-time behaviour of the radiation amplitude is dominated by the QNMs (i.e. the poles of $\mathcal{S}$) as well as the contribution of any singularities from $\mathcal{I}$. It is important to note that a nonpropagating mode (such as the special mode) would also contribute to the scattered radiation.

for $n = 0, 1, 2, \ldots$ (see figure A2). These results provided the first analytic formulae for the fundamental QNMs of the black-hole barrier potentials [8]; in this way, theoretical explanations could be provided for the main numerical results that had been available for a decade [8–10]. The correspondence between these results, i.e. equations (A3), (A4), with the Bohr–Sommerfeld quantization rule and the JWKB approximation was followed up by a number of authors [13–14, 18–20]. A better approximation to a black-hole barrier potential, such as the Eckart potential, can lead to the determination of overtone modes as well [9]; in fact, the distribution of QNMs of the Eckart potential in the complex frequency plane is qualitatively the same as for Schwarzschild black holes. The number of overtone modes that can be determined in this way depends on the accuracy of the fit. That is, the explicit determination of overtone modes in this approach depends on the precision with which the corresponding bound states of the inverted black-hole potential can be calculated. Using an approximate potential, all the modes corresponding to bound levels that occur up to the point in which the model potential well deviates considerably from the inverted black-hole potential can be approximately determined; clearly, the accuracy of this determination can be estimated from the residual error in the method of matching potentials.

The QNMs of the inverted harmonic oscillator potential are clearly connected with the bound states of the harmonic oscillator potential, i.e. the bound-state energy is essentially given by

$$- W_m + (2K)^{1/2}\left(n + \tfrac{1}{2}\right); \tag{A5}$$

in fact, this connection first indicated the general relationship between QNMs and the bound states of the inverted potential. The QNMs are defined by the singularities of the scattering

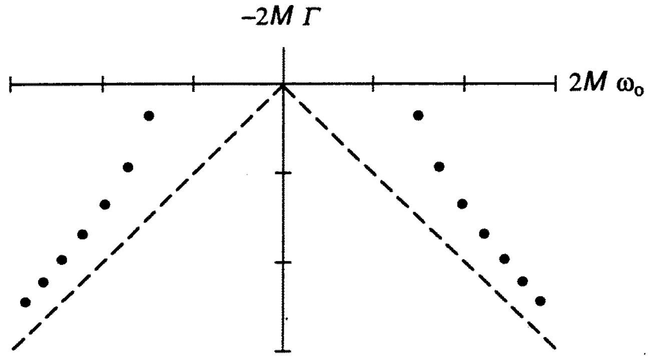

Figure A2. The resonant modes of the inverted harmonic oscillator potential. This potential fits the black-hole barrier potential (with $\lambda = 6$ and $\beta = -3$) at its maximum; therefore, the fundamental mode ($n = 0$) corresponds to the least-damped $j = 2$ gravitational mode of the Schwarzschild black hole. The graph indicates the asymptotic behaviour of the high-overtone modes ($n \to \infty$). See [8, 13, 17] for detailed discussions.

amplitude; therefore, the approach adopted in this work is consistent with the well known correspondence in the quantum theory between the singularities of the scattering matrix and the bound states.

The black-hole potentials are everywhere positive barriers which decay exponentially for $x \to -\infty$ and for $x \to +\infty$ fall off as x^{-2} (x^{-3}) for $j > 0$ ($j = 0$); in each case, the barrier maximum occurs at $x > 0$. In particular, the barrier potential for $j = 0$, $V_0(x)$, is simply given in terms of r by $r^{-3} - r^{-4}$.

The number of bound states of the inverted black-hole potential is the number of zeros of the wavefunction for $\Omega = 0$. If the potential goes as px^{-2} for $x \to \infty$, then it is a well known result from quantum mechanics [31] that the number of bound states is finite for $p < \frac{1}{4}$ and infinite for $p > \frac{1}{4}$. For $x \to \infty$ and $j > 0$, the black-hole potential falls off as $j(j+1)/x^2$; therefore, the number of bound states of the inverted potential is infinite in this case. This result can also be seen from the quasi-classical approximation ('Bohr–Sommerfeld quantization'), since $\pi^{-1}\int_{-\infty}^{\infty} V^{1/2}\,\mathrm{d}x$ is the number of bound states according to this scheme; that is, a region of area 2π in the two-dimensional phase space can accommodate only one eigenstate.

Let us now consider the scalar case with $j = 0$, where the potential falls off as x^{-3} for $x \to \infty$. The number of bound states of $-V_0$ is finite, since the wave equation for $\Omega = 0$ can be reduced in this case to a hypergeometric equation with solutions that have a finite number of zeros. To determine the number of bound states $N \geq 1$, it proves useful to reduce the problem to one involving the radial Schrödinger equation. To this end, let us note that the number of bound states cannot decrease if the potential is made more negative; therefore, we consider a one-dimensional symmetric potential $\tilde{V}_0(x)$ such that $\tilde{V}_0(x) = V_0(x)$ for $x \geq 0$. Let $\tilde{N}$ be the number of bound states of $-\tilde{V}_0(x)$, then $N \leq \tilde{N}$. The bound states of $-\tilde{V}_0(x)$ are either even or odd functions of x; in fact, the ground state is even, the first excited state is odd, and so on. Now consider the odd excited states for $x \geq 0$; these satisfy the boundary conditions for the radial Schrödinger eigenfunction with 'radial' potential $-V_0(x)$ for $x \geq 0$. According to a well known result [32], the number of such bound states for the radial Schrödinger equation with $j = 0$ is less than

$$I = \int_0^{\infty} x\, V_0(x)\,\mathrm{d}x\,. \tag{A6}$$

If $I < 1$, then there is no odd excited state and hence $\tilde{N} = N = 1$. On the other hand, if $I > 1$, then the number of odd excited states is less than I and hence

$$N \leq \tilde{N} < 2I + 1 . \tag{A7}$$

In our case, it turns out that

$$I = \frac{1}{2} \ln\left(1 + \frac{1}{\rho_0}\right), \tag{A8}$$

where ρ_0 is a solution of $\rho_0 + \ln \rho_0 = -1$. It is easy to see that $3 < \rho_0^{-1} < 4$ and hence $I < 1$, so that the inverted scalar potential with $j = 0$ has only a single bound state.

The bound states of the inverted black-hole potential for $j > 0$ are infinite in number; therefore, it was suggested [8] that the number of QNMs should be infinite as well. This was demonstrated numerically by Leaver [11], and proved by Bachelot and Motet-Bachelot [21]. It should be pointed out, however, that the equality of the number of QNMs with the number of bound states certainly does not hold in general; in particular, for $j = 0$ there is just one bound state, while the QNMs are infinite in number. Another example is provided by the Pöschl–Teller potential, which has a finite number of bound states but an infinite number of QNMs [10]. To see that the reverse holds true as well, let us consider for instance the potential $W_* = j(j+1)/(d + |x|)^2$, where $d > 0$ and $j \geq 1$. In this case, the bound states are infinite in number, while there are only $j + 1$ singularities of the scattering amplitude $\mathcal{S}(\omega)$. If ω is a QNM, then in general so is $-\omega^*$; therefore, in this case there are $(j+1)/2$ QNMs with $\omega_0 \geq 0$ for odd j, while there is at least one QNM with $\omega_0 = 0$ for even j. Finally, let us mention that the Dirac δ-function potential $W_D = 2\Delta\delta(X)$, $\Delta > 0$, has a non-propagating QNM at $\omega = -\mathrm{i}\Delta$ with residue $-\mathrm{i}\Delta$, i.e. the scattering amplitude is given by $\mathcal{S}_D(\omega) = -\mathrm{i}\Delta/(\omega + \mathrm{i}\Delta)$. This QNM corresponds exactly to a single bound state of the inverted potential.

Appendix B

The purpose of this appendix is to discuss the derivation of equations (17)–(24).

The substitution of equation (14) in the Regge–Wheeler equation (6) yields

$$g'' + g'^2 + (f'' + 2f'g')f^{-1} + \sigma^2 - U(r;\sigma) = 0 . \tag{B1}$$

Using equations (15) and (16), we find

$$\begin{aligned} g'' + g'^2 = {} & a^2 + 2b\left(a + c + \sum_{s=1}^{\infty} e_s\right)(r-1)^{-1} + \left[2c(a-b) - 2b\sum_{s=1}^{\infty} e_s\right]r^{-1} \\ & + (b^2 - b)(r-1)^{-2} + \left[c^2 - c + 2ae_1 - 2b\sum_{s=1}^{\infty} e_s\right]r^{-2} \\ & + \sum_{l=1}^{\infty}\left[2ae_{l+1} + (2c - l - 1)e_l - 2b\sum_{s=1}^{\infty} e_{s+l} + \sum_{s=1}^{\infty} e_{l-s}e_s\right]r^{-(l+2)}, \end{aligned} \tag{B2}$$

and

$$\frac{f'' + 2f'g'}{2f} = \sum_{i=1}^{n} \frac{1}{r - \alpha_i}(S_i + g'), \tag{B3}$$

where $e_s \equiv 0$ for $s \leq 0$ and

$$S_i \equiv \sum_{j=1, j\neq i}^{n} \frac{1}{\alpha_i - \alpha_j} . \tag{B4}$$

It is assumed here that S_i is well defined, and for $n = 1$, $S_1 = 0$. Furthermore, any finite sum from 1 to n, as in equation (B3), is set equal to zero when $n = 0$. It is easy to prove the following useful relations:

$$\frac{1}{r-\alpha_i}\frac{1}{r-1} = \frac{1}{\alpha_i - 1}\left(\frac{1}{r-\alpha_i} - \frac{1}{r-1}\right), \tag{B5}$$

$$\frac{1}{r-\alpha_i}\frac{1}{r^l} = \frac{1}{\alpha_i^l}\left(\frac{1}{r-\alpha_i} - \frac{1}{r} - \frac{\alpha_i}{r^2} - \cdots - \frac{\alpha_i^{l-1}}{r^l}\right), \qquad l = 1, 2, \ldots, \tag{B6}$$

which can put equation (B3) in the form

$$\begin{aligned}\frac{f'' + 2f'g'}{2f} = {} & \sum_{i=1}^{n} \frac{1}{r-\alpha_i}\left(S_i + a + \frac{b}{\alpha_i - 1} + \frac{c}{\alpha_i} + \sum_{s=1}^{\infty}\frac{e_s}{\alpha_i^{s+1}}\right) - \frac{1}{r-1}\sum_{i=1}^{n}\frac{b}{\alpha_i - 1} \\ & - \frac{1}{r}\sum_{i=1}^{n}\left[\frac{c}{\alpha_i} + \sum_{s=1}^{\infty}\frac{e_s}{\alpha_i^{s+1}}\right] - \frac{1}{r^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}\frac{e_s}{\alpha_i^s} - \sum_{l=1}^{\infty}\frac{1}{r^{l+2}}\sum_{i=1}^{n}\sum_{s=1}^{\infty}\frac{e_{l+s}}{\alpha_i^s} .\end{aligned} \tag{B7}$$

In order to satisfy the wave equation (6), it is necessary to choose α_i such that

$$S_i + a + \frac{b}{\alpha_i - 1} + \frac{c}{\alpha_i} + \sum_{s=1}^{\infty}\frac{e_s}{\alpha_i^{s+1}} = 0, \qquad i = 1, 2, \ldots, n; \tag{B8}$$

then, the substitution of equations (B2), (B7) and (7) into (B1) results in an algebraic equation which can be expressed as a series in inverse powers of $r - 1$ and r. This equation must hold for any r; therefore, each term in the equation must vanish. Thus we obtain

$$a^2 + \sigma^2 = 0, \tag{B9}$$

$$b^2 - b + \sigma^2 + \tfrac{1}{4} = 0, \tag{B10}$$

$$2b\left(a + c + \sum_{s=1}^{\infty} e_s\right) - 2\sum_{i=1}^{n}\frac{b}{\alpha_i - 1} + 2\sigma^2 - \lambda - \beta + \frac{1}{2} = 0, \tag{B11}$$

$$2c(a-b) - 2b\sum_{s=1}^{\infty} e_s - 2\sum_{i=1}^{n}\left(\frac{c}{\alpha_i} + \sum_{s=1}^{\infty}\frac{e_s}{\alpha_i^{s+1}}\right) + \lambda + \beta - \frac{1}{2} = 0, \tag{B12}$$

$$c^2 - c + 2ae_1 - 2b\sum_{s=1}^{\infty} e_s - 2\sum_{i=1}^{n}\sum_{s=1}^{\infty}\frac{e_s}{\alpha_i^s} + \beta - \frac{3}{4} = 0, \tag{B13}$$

$$2ae_{l+1} + (2c - l - 1)e_l - 2b\sum_{s=1}^{\infty} e_{s+l} + \sum_{s=1}^{\infty} e_{l-s}e_s - 2\sum_{i=1}^{n}\sum_{s=1}^{\infty}\frac{e_{l+s}}{\alpha_i^s} = 0, \qquad l = 1, 2, \ldots . \tag{B14}$$

To solve these equations, we find that from the definition of S_i in equation (B4) one can derive some very useful relations such as

$$\sum_{i=1}^{n} S_i = 0, \qquad \sum_{i=1}^{n} S_i\alpha_i = \frac{n(n-1)}{2}, \ldots . \tag{B15}$$

Adding equations (B11) and (B12) and using equations (B8) and (B15), we obtain

$$a(b + c + n - a) = 0\,. \tag{B16}$$

On the other hand, from equations (B9) and (B10) we find

$$b = \pm a + \tfrac{1}{2}\,. \tag{B17}$$

It follows from equations (B16) and (B17) that we can obtain three kinds of solution, of which the first kind (i.e. $a = b + c + n$, $b = -a + \frac{1}{2}$) is given in equations (20) and (21) and would, in general, correspond to modes that are outgoing at infinity and ingoing at the horizon.

The second kind of solution is

$$\sigma = \mathrm{i}a, \qquad b = a + \tfrac{1}{2}, \qquad c = -(n + \tfrac{1}{2}), \tag{B18}$$

which would correspond in general to purely ingoing modes at the horizon. Let us note that the algebraically special solution [27] is contained here as a special case in which $\beta = -3$, $e_s = 0$ for all s, and $n = 1$. This solution vanishes for $r \to \infty$. The eigenfunction and eigenvalue in this case are

$$\psi_s = \mathrm{e}^{ar}(r-1)^a r^{-1}(r - \alpha), \qquad \sigma = \mathrm{i}a = -\tfrac{1}{6}\mathrm{i}\lambda(\lambda - 2), \tag{B19}$$

where $\alpha = -3/(\lambda-2)$. The second independent solution of the wave equation corresponding to this eigenvalue, $\bar{\psi}_s$, can be obtained explicitly; we find that $\bar{\psi}_s$ can be chosen to be purely outgoing at infinity such that for $r \to 1$, $\bar{\psi}_s \to 0$. It is then a simple matter to show explicitly that $\tilde{\psi}_n$ given by equation (32) can be expressed as a linear combination of ψ_s and $\bar{\psi}_s$ with constant coefficients.

The third kind of solution is given by

$$a = 0, \qquad \sigma = 0, \qquad b = \tfrac{1}{2}, \tag{B20}$$

which would represent zero-frequency modes corresponding to static perturbations.

For the QNMs discussed in section 3, the substitution of equations (20) and (21) into equation (B12) yields equation (17). We then subtract equation (B12) from (B13); this, together with equation (B15), would yield equation (23). Similarly, the subtraction of equation (B13) from equation (B14) for $l = 1$ yields equation (24). Finally, we change the index l in equation (B14) to $l + 1$, and then subtract equation (B14) from this new equation to obtain the recurrence relation (22). Thus we arrive at the simplified form of our results for the QNMs given in equations (17)–(24).

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