

Measures on Hilbert Spaces and Applications to Hydrodynamics

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by
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Measures on Hilbert Spaces and Applications to Hydrodynamics

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ABSTRACT

Homogeneous and isotropic statistical solutions of the Navier-Stokes equations are produced. These are shown to be approximated by Galerkin statistical solutions on finite dimensional subspaces. Homogeneous and isotropic measures are approximated in the 2nd Wasserstein metric by measures supported on finite dimensional subspaces. The homogeneous measures are then shown to be a subspace of positive curvature of the 2nd Wasserstein space.

0.1 Introduction

The Kolmogorov theory of turbulence describes the behavior of locally homogeneous and isotropic fluid flows, i.e. flows with local statistical properties independent of translations, reflections, and rotations in 3D-space. Kolmogorov [K41] hypothesizes that at high Reynolds numbers all fluid flows are of this form and that their statistical properties depend only on dissipation and viscosity.

In the absence of a rigorous theory for locally homogeneous and isotropic flows, the theory of homogeneous and isotropic flows has been developed, see [VF]. The mathematical equivalent of such flows are translation, reflection, and rotation invariant probability measures P supported by weak Navier-Stokes solutions. Given an appropriate initial measure μ on the space of initial conditions and any finite time interval $[0, T]$, the measure P should yield μ at $t = 0$ in some sense. It should also yield measures μ_t for each time t in $[0, T]$, each invariant under space translations, reflections and rotations, that represent the flow of μ under the Navier-Stokes equations as in [H].

The existence of such measures P that satisfy all the assumptions above for translations in space, called **homogeneous**, was proved by Vishik and Fursikov in [VF1] and is included in detail in [VF]. See also [FT]. The first part of this thesis adopts the construction from [VF1], [VF] to produce measures P that, in addition to being invariant under translations, are also invariant under rotations and reflections. Such measures will be called **homogeneous and isotropic**.

The first main result of this thesis is:

Theorem 0.1.1. *Given $\hat{\mu}$ homogeneous and isotropic measure on a space $\mathcal{H}^0(r)$*

of vector fields on \mathbb{R}^3 with the finite energy density, there exists measure \widehat{P} on $L^2(0, T, \mathcal{H}^0(r))$, homogeneous and isotropic with respect to the space variables, supported by weak solutions of the Navier-Stokes equations, and with finite energy density that satisfies the standard energy inequality. On the support of \widehat{P} right limits with respect to time are well-defined in an appropriate norm and the right limit at $t = 0$ yields $\widehat{\mu}$.

In addition, the measure P is obtained as the weak limit of measures supported on Galerkin approximations of the Navier-Stokes equations.

To prove Theorem 0.1.1 the homogeneous statistical solution P constructed in [VF1], [VF] with initial measure $\widehat{\mu}$ is averaged over all rotations and reflections. The resulting measure \widehat{P} satisfies all conditions of Theorem 0.1.1 above and is therefore the desired homogeneous and isotropic statistical solution.

It has to be emphasized, however, that existence theorems obtained via convergent sequences of “simpler” approximations of the constructed solution are, as a rule, much more useful than pure existence theorems. For example, in numerical simulations of flows it is some appropriate approximation of Navier-Stokes equation that is simulated rather than the equation itself. For this reason part of this thesis is devoted to constructing such approximations of isotropic statistical solutions.

The construction of homogeneous statistical solutions in [VF1], [VF] is based on Galerkin approximations of measures that are supported by divergence free periodic vector fields with trigonometric polynomials as components. The main difficulty in extending this construction to isotropic measures is that the space of such vector fields, whereas invariant under translations, is not invariant under rotations.

The construction of statistical approximation in this thesis is based on the observation that the space of such vector fields AND all their rotations and reflections should suffice for invariance under rotations and reflections. It is then necessary to construct Galerkin approximations of isotropic measures on this class of vector fields. This thesis considers the case of 3D Navier-Stokes equations, although the arguments here are applicable in 2D case as well.

Properties of the correlation functions of homogeneous and isotropic solutions test the validity of the theory when compared to real data.

The construction of homogeneous & isotropic measures supported by weak solutions of the Navier-Stokes equations relies on approximating an initial homogeneous measure μ by explicitly constructed homogeneous measures μ_l supported by trigonometric polynomials of degree l and period $2l$. The approximation is in characteristic: The characteristic functions of the μ_l 's converge to the characteristic function of μ . Convergence in characteristic suffices to show that the restriction of P at time 0 is the initial measure μ . The details of this construction are in [VF], with Appendix II there containing the details of the approximation $\mu_l \rightarrow \mu$.

On the other hand, several properties of the spatial correlation functions R_{ij} of homogeneous and isotropic fluid flows are taken for granted, see [D], for example. Often, arguments for the validity of these properties express the correlations as Fourier transforms. The existence of the Fourier transform of the correlation tensor has not been shown for homogeneous solutions, even when the Fourier transform of the correlation of the initial measure exists. What does hold is that the correlation tensor of any homogeneous measure is the Fourier-Stieltjes transform of a (possibly) non-

differentiable function. This is thoroughly explained in [K].

The μ_l homogeneous measures above have, of course, periodic correlations, and hence Fourier series expansions. One can then try to show that an l -approximation also yields an approximation of the corresponding correlations to the correlation of μ , that this is also true for (almost) all times for statistical solutions, and then use Galerkin correlations to get information for the correlations of the solution itself.

For example, in $\dim=1$, convergence of correlations of an l -approximation of Burgers statistical solutions for almost all times implies immediately that the integral of the correlation function is constant in time, as already anticipated by Burgers, [Bu]. In fact, the main motivation behind this step is to develop tools for examining the spatial decay of correlation functions of Navier-Stokes statistical solutions in dimension 3, cf. [L].

The second main result of this thesis is:

Theorem 0.1.2. *Let μ be a homogeneous measure on the separable Hilbert space $\mathcal{H}^0(r)$ of vector fields on \mathbb{R}^3 and let μ_l be an l -approximation of μ , as in section 5.3.1 below. Then the μ_l 's converge to μ weakly, up to subsequence, and the correlation functions of this subsequence converge to the correlation functions of μ pointwise.*

The method of proof of this theorem is interesting in itself: It first improves the convergence of μ_l to μ from characteristic to weak and then shows that second moments converge.

Recall here a standard result in the theory of optimal transport: Weak convergence and the convergence of second moments is equivalent to convergence in the space $W_2(\mathcal{H}^0(r))$ of probability measures on $\mathcal{H}^0(r)$, with finite second moment, equipped

with the second Wasserstein metric W_2 , see section (1.4.1) for definitions. In this way, this part of the thesis is a first step in revisiting the constructions of homogeneous and isotropic solutions in terms of the geometry of the Wasserstein space.

Kuksin and Shirikian use the first Wasserstein metric W_1 in their study of stochastic (as opposed to statistical) solutions of the 2-dimensional Navier-Stokes equations on periodic domains, see [Ku], [KS].

The above results seem to indicate that it could be advantageous to examine the Kolmogorov theory of turbulence as a flow (in some sense, via Navier-Stokes equations) in the Wasserstein space W_2 of measures with finite second moment. Once the geometry of the subspace of homogeneous and isotropic measures within this larger subspace is understood, it is conceivable that the Kolmogorov hypotheses can be recast as a conjecture on the index of an appropriate operator.

The third part of this thesis makes progress in this direction for the setting of periodic vector fields and for homogeneous measures. In particular, it is shown that

Theorem 0.1.3. *The space of homogeneous measures is geodesically connected in the Wasserstein space W^2 .*

As a corollary, the space of homogeneous measures inherits the metric properties of W_2 and is itself a non-positive curvature space in the sense of Alexandrov.

Chapter 1

Preliminary Definitions and Results

1.1 General definitions

Definition 1.1.1. $\mathcal{P}(\mathcal{H})$ is the space of all probability measures on a Hilbert space \mathcal{H} .

Definition 1.1.2. For M a metric space denote by $\mathcal{B}(M)$ the σ -algebra of Borel sets of M . Let M_1, M_2 be metric spaces, and $\Psi : M_1 \rightarrow M_2$ a measurable map, i.e.

$$\forall B \in \mathcal{B}(M_2) \quad \Psi^{-1}B := \{m \in M_1 : \Psi(m) \in B\} \in \mathcal{B}(M_1). \quad (1.1)$$

For every measure $\nu(A), A \in \mathcal{B}(M_1)$

$$\Psi_{\#}\nu(B) = \nu(\Psi^{-1}B) \quad \forall B \in \mathcal{B}(M_2). \quad (1.2)$$

The measure $\Psi_{\#}\nu$ is called the **push forward of the measure ν under the map Ψ** . Equality (1.2) is equivalent to

$$\int f(u) \Psi_{\#}\nu(du) = \int f(\Psi(v)) \nu(dv), \quad (1.3)$$

see [VF], p. 61.

1.1.1 Gaussian Measures

Definition 1.1.3. A probability measure $\mu \in \mathcal{P}(\mathbb{R})$ is called Gaussian if it has density

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right). \quad (1.4)$$

Here a is the mean, and σ is the variance.

Definition 1.1.4. A probability measure $\mu \in \mathcal{P}(\mathcal{H})$, is Gaussian if, for any v in \mathcal{H} and for $f_v(u) = \langle v, u \rangle_{\mathcal{H}}$, the measure $(f_v)_{\#}\mu$ is Gaussian.

It is standard that if \widehat{K} is a symmetric, positive, and trace class operator on a Hilbert space \mathcal{H} , then

$$\chi(z) = e^{-\langle \widehat{K}z, z \rangle_{\mathcal{H}}} \quad (1.5)$$

is the characteristic functional, according to Definition 1.1.4 of a Gaussian measure of mean 0 and correlation operator \widehat{K} , according to definition Theorem 1, [GS]; p. 350.

Definition 1.1.5. Let \mathcal{H} be a Hilbert space of vector functions on \mathbb{R}^n . A vector function $K(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is positive definite if

$$\int (K(x, y) \cdot v(x)) \cdot v(y) \, dx \, dy \geq 0. \quad (1.6)$$

1.2 Homogeneous Measures

Definition 1.2.1. For u in M , a vector space over \mathbb{R}^n , let T_h be the translation operation defined weakly by

$$T_h u(x) = u(x + h). \quad (1.7)$$

Definition 1.2.2. A measure μ defined on $\mathcal{B}(M)$, is called **homogeneous** if it is translation invariant:

$$(T_h)_\# \mu = \mu \iff \int_M F(u) (T_h)_\# \mu(du) = \int_M F(T_h u) \mu(du) = \int_M F(u) \mu(du), \quad (1.8)$$

for any μ -integrable F on M and for all h in \mathbb{R}^3 .

1.2.1 The spaces $\mathcal{H}^k(r)$

Non-trivial measures invariant under shifts exist on weighted Sobolev spaces of vector fields, but not on L^p spaces, the weight in the norm ensuring that balls in the function space are not be invariant under shifts, [VF], p. 208.

Definition 1.2.3. For k non negative integer and $r < -3/2$, define $\mathcal{H}^k(r)$ to be the space of solenoidal vector fields

$$\begin{aligned} u(x) &= (u_1(x), u_2(x), u_3(x)), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \\ \operatorname{div} u &= \sum_{j=1}^3 \frac{\partial u_j}{\partial x_j} = 0, \end{aligned} \quad (1.9)$$

with finite (k, r) -norm:

$$\|u\|_{k,r}^2 = \int_{\mathbb{R}^3} (1 + |x|^2)^r \sum_{|\alpha| \leq k} \left| \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \right|^2 dx, \quad r < -\frac{3}{2}, \quad (1.10)$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is multi index and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

Here the equality $\operatorname{div} u = 0$ is to be understood weakly, i.e.

$$\int u(x) \cdot \nabla \phi(x) dx = 0 \quad \forall \phi \in C_0^\infty(\mathbb{R}^3) \quad (1.11)$$

Observe that the restriction on r implies that constant and many periodic vector fields are in $\mathcal{H}^k(r)$.

1.2.2 Examples of homogeneous measures

Non-trivial homogeneous measures exist on weighted Sobolev spaces of vector fields defined over \mathbb{R}^3 , p. 208, [VF].

Example 1.2.4. *An example of a homogeneous Gaussian measure.*

Define \widehat{K} on $\mathcal{H} = L^2(\mathbb{T}^n)^n$:

$$\widehat{K} v(x) = \int K(x-y) v(y) dy, \quad (1.12)$$

where $K : \mathbb{T}^n \rightarrow (\mathbb{R}^n)^2$, has the following properties: positive definite, continuous, square integrable, and $K_{ij}(x) = K_{ji}(-x)$. Then \widehat{K} is

1. Continuous:

$$\begin{aligned} |\widehat{K}(v(x) - u(x))| &\leq \int |K(x-y) (v(y) - u(y))| dy \\ &\leq \left(\int K^2(x-y) dy \right)^{1/2} \left(\int (v(y) - u(y))^2 dy \right)^{1/2} \\ &\leq \left(\int K^2(-y) dy \right)^{1/2} \left(\int (v(y) - u(y))^2 dy \right)^{1/2} \\ &= C \|u - v\|_{\mathcal{H}}. \end{aligned} \quad (1.13)$$

2. Positive:

$$\begin{aligned} \langle \widehat{K}v, v \rangle &= \int \int K(x-y) v(y) dy v(x) dx \\ &= \int \int K(x-y) v(y) v(x) dy dx \geq 0 \end{aligned} \quad (1.14)$$

3. Symmetric:

$$\begin{aligned}
\langle \widehat{K}v, v \rangle_{\mathcal{H}} &= \int \int K(x-y) v(y) dy v(x) dx \\
&= \sum_{i=1}^n \int \int \sum_{j=1}^n K_{ij}(x-y) v_j(y) dy v_i(x) dx \\
&= \sum_{i=1, j=1}^n \int \int K_{ij}(x-y) v_j(y) v_i(x) dy dx \\
&= \sum_{i=1, j=1}^n \int \int K_{ji}(y-x) v_j(y) v_i(x) dy dx \tag{1.15} \\
&= \sum_{j=1}^n \int v_j(y) \int \sum_{i=1}^n K_{ji}(y-x) v_i(x) dx dy \\
&= \int v(y) \int K(y-x) v(x) dx dy \\
&= \langle v, \widehat{K}v \rangle_{\mathcal{H}} .
\end{aligned}$$

4. Trace-class: Choose an orthonormal basis, $\{e_i\}$ of the space of functions $L^2(\mathbb{T}^n)$.

Construct an orthonormal basis of \mathcal{H} for arbitrary n by taking n copies of $\{e_i\}$, as follows

$$\bigcup_i \{(e_i, 0, \dots, 0)\} \cup \bigcup_i \{(0, e_i, 0, \dots, 0)\} \dots \cup \bigcup_i \{(0, \dots, 0, e_i)\} \tag{1.16}$$

and let \hat{e}^k be a typical element of this basis. Then

$$\begin{aligned}
\sum_k \langle \widehat{K}\hat{e}^k, \hat{e}^k \rangle_{\mathcal{H}} &= \sum_k \sum_{i=1}^n \int \int \sum_{j=1}^n K_{ij}(x-y) \hat{e}_j^k(y) dy \hat{e}_i^k(x) dx \\
&= \sum_k \int \int \sum_{i=1}^n K_{ii}(x-y) e_k(y) e_k(x) dx dy \tag{1.17}
\end{aligned}$$

shows that \widehat{K} is trace class if and only if

$$\widehat{N}v(x) = \int \sum_{i=1}^n K_{ii}(x-y) v(y) dy \tag{1.18}$$

is trace class. Since K is continuous

$$\int \sum_i K_{ii}(0) dy = |\mathbb{T}^2| \sum_i K_{ii}(0) < \infty, \tag{1.19}$$

which implies that \widehat{N} is trace class, by the Lemma on page 65 in Chapter XI.4 of [RS3].

Therefore, by section 1.1.1, there exists a μ Gaussian such that:

$$e^{-\langle \widehat{K}z, z \rangle_{\mathcal{H}}} = \int e^{i\langle u, z \rangle_{\mathcal{H}}} \mu(du). \quad (1.20)$$

Lemma 1.2.5. *A probability measure μ on $\mathcal{H} = L^2(\mathbb{T}^n)$ is homogeneous if and only if, for each $h \in \mathbb{R}^n$,*

$$\begin{aligned} \int e^{i\langle u, v \rangle_{\mathcal{H}}} \mu(du) &= \int e^{i\langle T_{-h}u, v \rangle_{\mathcal{H}}} \mu(du) \\ &= \int e^{i\langle u, T_h v \rangle_{\mathcal{H}}} \mu(du) \end{aligned} \quad (1.21)$$

for $v \in \mathcal{H}$.

Proof. This is proved in [GT]. □

Proposition 1.2.6. *The measure μ as constructed in subsection 1.2.4 is homogeneous.*

Proof. For $u, v \in \mathcal{H}$, by (1.20)

$$e^{-\langle \widehat{K}v, v \rangle_{\mathcal{H}}} = \int e^{i\langle u, v \rangle_{\mathcal{H}}} \mu(du). \quad (1.22)$$

Since for all $h \in \mathbb{R}^n$

$$\begin{aligned} \int e^{i\langle u, T_h v \rangle_{\mathcal{H}}} \mu(du) &= e^{-\langle \widehat{K}T_h v, T_h v \rangle_{\mathcal{H}}} \\ &= \exp\left(-\int \int K(x-y) v(x+h) v(y+h) dx dy\right) \\ &= \exp\left(-\int \int K(x-h-y+h) v(x) v(y) dx dy\right) \\ &= \exp\left(-\int \int K(x-y) v(x) v(y) dx dy\right) \\ &= e^{-\langle \widehat{K}v, v \rangle_{\mathcal{H}}} \\ &= \int e^{i\langle u, v \rangle_{\mathcal{H}}} \mu(du), \end{aligned} \quad (1.23)$$

it follows from Lemma 1.2.5 that μ is homogeneous. □

Proposition 1.2.7. *Example 1.2.4 yields every homogeneous Gaussian measure on \mathcal{H} .*

Proof. If μ is a homogeneous Gaussian, then

$$\begin{aligned} \int e^{i\langle u, T_h v \rangle_{\mathcal{H}}} \mu(du) &= e^{-\langle \widehat{K} T_h v, T_h v \rangle_{\mathcal{H}}} \\ &= e^{-\langle \widehat{K} v, v \rangle_{\mathcal{H}}} \\ &= \int e^{i\langle u, v \rangle_{\mathcal{H}}} \mu(du). \end{aligned} \tag{1.24}$$

So the operator \widehat{K} must have the property that $\langle \widehat{K} T_h v, T_h v \rangle_2 = \langle \widehat{K} v, v \rangle_2$. The correlation operator of a homogeneous measure has the form

$$\widehat{K}u(x) = \int K(x-y)u(y) dy, \tag{1.25}$$

with \widehat{K} continuous, $K_{ij}(x) = K_{ji}(-x)$. As shown in subsection 5.1.2, the kernel K must have the properties given in Example 1.2.4. \square

1.3 Isotropic Measures

Definition 1.3.1. *Let $O(n)$ of all orthogonal n by n matrices, i.e., if $\omega \in O(n)$, then $\det \omega = \pm 1$, i.e. all rotations and reflections.*

As for isotropic flows, they should have statistical properties invariant under rotations of the coordinate system, [MY], [T]. To find how a vector field $u(x) = (u_1(x), u_2(x), u_3(x))$ is transformed under rotation of the coordinate system it is convenient to write it in the usual manifold notation, cf. [DFN], p. 15:

$$u(x) = u_k(x) \frac{\partial}{\partial x_k} \tag{1.26}$$

(using summation on repeated indices). Let $v(y) = v_j(y) \frac{\partial}{\partial y_j}$ be the description of the vector field (1.26) after the transformation $y = \omega x$ where $\omega = \{\omega_{ij}\}$ is a rotation

matrix (i.e. $\omega^{-1} = \omega^*$). Since $\frac{\partial}{\partial x_k} = \omega_{jk} \frac{\partial}{\partial y_j}$, then

$$v_j(y) \frac{\partial}{\partial y_j} = u_k(\omega^{-1}y) \omega_{jk} \frac{\partial}{\partial y_j}.$$

In other words, returning to the standard notation for vector fields on \mathbb{R}^3 where $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$,

$$v(y) = \omega u(\omega^{-1}y). \quad (1.27)$$

Observe here that since ω is orthogonal the differential form $\sum u_i dx_i$ transforms under ω in the same way, cf. [DFN], page 156.

Then for ω belonging to the group $O(3)$, define its action on vector fields as

$$(R_\omega u)(x) = \omega u(\omega^{-1}x) \quad (1.28)$$

Observe that the standard action identity holds

$$R_{\omega_1}(R_{\omega_2}u)(x) = \omega_1(R_{\omega_2}u)(\omega_1^{-1}x) = \omega_1\omega_2u(\omega_2^{-1}\omega_1^{-1}x) = R_{\omega_1\omega_2}u(x), \quad (1.29)$$

that

$$R_\omega u = v \Leftrightarrow u = (R_{\omega^{-1}})v, \quad (1.30)$$

and that

$$T_h R_\omega u = R_\omega T_{\omega^{-1}h} u. \quad (1.31)$$

Lemma 1.3.2. *For every $\omega \in O(3)$ the operator $R_\omega : \mathcal{H}^0(r) \rightarrow \mathcal{H}^0(r)$ is an isometry, i.e. if $\operatorname{div} u = 0$ then $\operatorname{div} R_\omega u = 0$ and*

$$\|R_\omega u\|_{\mathcal{H}^0(r)} = \|u\|_{\mathcal{H}^0(r)}. \quad (1.32)$$

Proof. The transformation formula for multiple integrals, [A], page 421, gives for the change of variables $y = \omega x$:

$$\int_{\mathbb{R}^3} f(\omega x) dx = |(\det \omega)^{-1}| \int_{\mathbb{R}^3} f(y) dy = \int_{\mathbb{R}^3} f(y) dy, \quad \forall \omega \in O(3), f \in L_1(\mathbb{R}^3), \quad (1.33)$$

since $|\det \omega| = 1$ for any ω in $O(3)$.

Now (1.10) and (1.33) yield

$$\begin{aligned} \|R_\omega u\|_{0,r}^2 &= \int_{\mathbb{R}^3} (1 + |x|^2)^r |\omega u(\omega^{-1}x)|^2 dx \\ &= \int_{\mathbb{R}^3} (1 + |\omega^{-1}x|^2)^r |u(\omega^{-1}x)|^2 dx \\ &= \int_{\mathbb{R}^3} (1 + |x|^2)^r |u(x)|^2 dx \\ &= \|u\|_{0,r}^2, \end{aligned} \quad (1.34)$$

which proves (1.32).

If $\omega = (\omega_{ij}) \in O(3)$ then $x = \omega y$ is equivalent to $x_i = \omega_{ij}y_j$ and $y = \omega^{-1}x = \omega^*x$ is equivalent to $y_l = \omega_{kl}x_k$ (using summation on repeated indexes). Then

$$\frac{\partial}{\partial x_k} = \omega_{kl} \frac{\partial}{\partial y_l}, \quad \omega_{kj}\omega_{kl} = \delta_{jl}, \quad (1.35)$$

where δ_{kl} is Kroneker symbol. Using these equalities, (1.33), and assuming that u satisfies (1.11), obtain

$$\begin{aligned} \int R_\omega u(x) \cdot \nabla \phi(x) dx &= \int \omega u(\omega^{-1}x) \cdot \nabla \phi(x) dx \\ &= \int \omega_{kj} u_j(y) \omega_{kl} \frac{\partial \phi(\omega y)}{\partial y_l} d\omega y \\ &= \int u_j(y) \frac{\partial \phi(\omega y)}{\partial y_j} dy = 0, \end{aligned} \quad (1.36)$$

where the last equality holds because of (1.11) and the inclusion $\phi \circ \omega \in C_0^\infty(\mathbb{R}^3)$.

Therefore $\operatorname{div} R_\omega u = 0$ if $\operatorname{div} u = 0$. □

Definition 1.3.3. A measure μ on a space of vector fields X is called **isotropic** if it is invariant under rotations: For all ω in $O(n)$,

$$(R_\omega)_\# \mu = \mu \iff \int f(u) \mu(du) = \int f(R_\omega u) \mu(du) = \int f(u) (R_\omega)_\# \mu(du), \quad (1.37)$$

for any μ -integrable f in X .

Remark 1.3.4. The choice of $O(3)$ as space of rotations captures the usual conventions of isotropic flows as flows invariant under “proper” rotations and reflections with respect to coordinate planes, see [Rob], p. 212. I.e. the measure is invariant under the transformations

$$(u_1, u_2, u_3)(x) \mapsto (-u_1, u_2, u_3)(\bar{x}) \quad (1.38)$$

for $\bar{x} = (-x_1, x_2, x_3)$, and similarly for the indices 2 and 3.

It also follows from the definition that the correlation function and all statistical expressions of such a μ have the usual form for isotropic flows, see for example [MY], p. 39.

1.3.1 Examples of isotropic measures

Homogeneous and isotropic measures can easily be constructed from homogeneous measures by standard averaging:

Definition 1.3.5. If μ is homogeneous, define $\hat{\mu}$ on $\mathcal{B}(\mathcal{H}^0(r))$ as

$$\hat{\mu}(A) = \int_{O(3)} (R_\omega)_\# \mu(A) d\omega \quad (1.39)$$

for any $A \in \mathcal{B}(\mathcal{H}^0(r))$ and for $d\omega = H$ the standard Haar measure on $O(3)$ normalized.

By definition (1.3) of the push forward measure and by Fubini's Theorem, equality (1.39) is equivalent to

$$\int f(u) \widehat{\mu}(du) = \int \int_{O(3)} f(R_\omega u) d\omega \mu(du) \quad (1.40)$$

for each μ -integrable function $f(u)$. This definition of $\widehat{\mu}$ is sometimes more convenient than (1.39), as will become clear below.

Proposition 1.3.6. *Let μ be a homogeneous probability measure on $\mathcal{H}^0(r)$. Then $\widehat{\mu}$ is isotropic and still homogeneous.*

Proof. Invariance under rotations follows from

$$\begin{aligned} \int f(u) (R_{\omega_0})_{\#} \widehat{\mu}(du) &= \int \int_{O(3)} f(R_{\omega\omega_0} u) d\omega \mu(du) \\ &= \int \int_{O(3)} f(R_\omega u) d\omega \mu(du) \\ &= \int f(u) \widehat{\mu}(du), \end{aligned} \quad (1.41)$$

with the second equality following from the fact that $O(3)$ is compact, therefore unimodular, therefore the “change of variables” $\omega \rightarrow \omega\omega_0$ has “Jacobian” 1, see [R], p. 498.

Invariance under translations follows from the fact that

$$T_h R_\omega u = R_\omega T_{\omega^{-1}h} u \Leftrightarrow T_{\omega h} R_\omega u = R_\omega T_h u \quad (1.42)$$

and

$$\begin{aligned} \int f(u) (T_h)_{\#} \widehat{\mu}(du) &= \int \int_{O(3)} f(T_h R_\omega u) d\omega \mu(du) \\ &= \int_{O(3)} \int f(R_\omega T_{\omega^{-1}h} u) \mu(du) d\omega \\ &= \int_{O(3)} \int f(R_\omega u) \mu(du) d\omega \\ &= \int \int_{O(3)} f(R_\omega u) d\omega \mu(du) \\ &= \int f(u) \widehat{\mu}(du). \end{aligned} \quad (1.43)$$

□

Proposition 1.3.7. μ as in Example 1.2.4 is isotropic if and only if $K = K(|z|)$.

Proof. As shown in [GT], it suffices to show that

$$\int e^{i\langle u, R_\omega v \rangle_{\mathfrak{H}}} \mu(du) = \int e^{i\langle u, v \rangle_{\mathfrak{H}}} \mu(du). \quad (1.44)$$

$$\begin{aligned} \int e^{i\langle u, R_\omega v \rangle_{\mathfrak{H}}} \mu(du) &= e^{-\langle \widehat{K} R_\omega v, R_\omega v \rangle_{\mathfrak{H}}} \\ &= \exp\left(-\int \int K(x-y) \omega v(\omega^{-1}x) \omega v(\omega^{-1}y) \, dx dy\right) \\ &= \exp\left(-\int \int K(x-y) v(\omega^{-1}x) v(\omega^{-1}y) \, dx dy\right) \\ &= \exp\left(-\int \int K(\omega x - \omega y) v(x) v(y) \, dx dy\right). \end{aligned} \quad (1.45)$$

So for μ to be isotropic,

$$\begin{aligned} \exp\left(-\int \int K(\omega x - \omega y) v(x) v(y) \, dx dy\right) &= \exp\left(-\int \int K(x-y) v(x) v(y) \, dx dy\right) \\ &= e^{-\langle \widehat{K} v, v \rangle_2} \\ &= \int e^{i\langle u, v \rangle_2} \mu(du), \end{aligned} \quad (1.46)$$

which holds for all ω and all v only when $K = K(|z|)$. □

1.4 Optimal Transport and Homogeneous Measures

1.4.1 Wasserstein convergence

On $\mathcal{P}_p(X)$, the space of probability measures μ on a separable Hilbert space X with finite p -moments

$$\int_X \|u\|_X^p \mu(du), \quad (1.47)$$

consider the p -Wasserstein metric:

$$W_p(\mu_1, \mu_2) = \left(\inf_{\pi \in \Gamma(\mu_1, \mu_2)} \int_{X \times X} \|u - v\|_X^p \pi(du, dv) \right)^{1/p}, \quad (1.48)$$

with

$$\Gamma(\mu_1, \mu_2) = \{ \pi \in \mathcal{P}(X \times X) : (pr_1)_\# \pi = \mu_1, (pr_2)_\# \pi = \mu_2 \}. \quad (1.49)$$

(For an equivalent description of Γ in terms of couplings of random variables see [Ku], p. 41.)

The **p-Wasserstein space** is the metric space $W_p(X) = (\mathcal{P}_p(X), W_p)$. It is complete, separable, [AGS], p. 154, not locally compact, [AGS], p. 156, and the following holds as $n \rightarrow \infty$:

$$W_p(\mu_n, \mu) \rightarrow 0 \Leftrightarrow \left\{ \begin{array}{l} \mu_n \rightarrow \mu, \text{ weakly} \\ \int_X \|u\|_X^p \mu_n(du) \rightarrow \int_X \|u\|_X^p \mu(du) \end{array} \right\}, \quad (1.50)$$

see [AGS], p. 154, or [V], p. 212.

1.4.2 Homogeneity in the Wasserstein space

Of concern will be measures on $\mathcal{H}^0(r)$ of finite energy density. Taking ϕ as is defined later in (2.10) to be the integrable weight of the $\mathcal{H}^0(r)$ -norm, such measures satisfy

$$\int_{\mathcal{H}^0(r)} \|u\|_{\mathcal{H}^0(r)}^2 \mu(du) \leq +\infty, \quad (1.51)$$

i.e. they have finite second moment.

For X separable Hilbert, the space $\text{Tan}_\mu W^2(X)$, tangent to $W_2(X)$ at a measure μ , is identified in [AGS] in terms of vector fields on X itself. It consists of vector fields perpendicular to those $v : X \rightarrow X$ satisfying

$$\int \langle \nabla F(u), v(u) \rangle_X \mu(du) = 0, \quad (1.52)$$

for any F cylindrical function on X .

In this way, for $X = \mathcal{H}^0(r)$, the homogeneity of a measure μ with

$$\int_{\mathcal{H}^0(r)} \|u\|_{\mathcal{H}^1(r)}^2 \mu(du) < +\infty \quad (1.53)$$

implies that the vector field

$$u \mapsto \nabla u \cdot h \quad (1.54)$$

is in $\text{Tan}_\mu^\perp W_2(X)$, for all $h \in \mathbb{R}^3$. (Differentiate $\epsilon \mapsto \int_X F(T_{\epsilon h} u) \mu(du)$ at $\epsilon = 0$.)

This restricts the tangent space at a homogeneous μ . That this vector field is in $\text{Tan}_\mu^\perp W_2(X)$ and [AGS], Proposition 8.3.3 also give an approximation of μ other than an l -approximation (via absolutely continuous measures on finite dimensional subspaces), see section 5.3.3.

Chapter 2

Homogeneous and Isotropic Statistical Solutions

Statistical solutions are defined. Then, homogeneous and isotropic probability measures supported by weak solutions of the Navier-Stokes system are produced by averaging over rotations the known homogeneous probability measures, supported by such solutions, of [VF1], [VF].

2.1 Definition of Statistical Solutions

The following preliminary definitions are required to state the properties of statistical solutions of the Navier-Stokes equations:

Definition 2.1.1. *Define \mathcal{G}_{NS} to be the set of all generalized solutions of the Navier-Stokes system, i.e.*

$$\mathcal{G}_{NS} = \left\{ u \in L^2(0, T; \mathcal{H}^0(r)) : \right. \\ \left. L(u, \phi) \equiv \int_0^T \left(\langle u, \frac{\partial \phi}{\partial t} \rangle_2 + \langle u, \Delta \phi \rangle_2 + \sum_{j=1}^3 \langle u_j u, \frac{\partial \phi}{\partial x_j} \rangle_2 \right) dt = 0, \quad (2.1) \right. \\ \left. \text{for all } \phi \in C_0^\infty((0, T) \times \mathbb{R}^3) \cap C((0, T); \mathcal{H}^0(r)) \right\},$$

where $\langle u, v \rangle_2 = \int_{\mathbb{R}^3} u(x) \cdot v(x) dx$.

To define restrictions at any time $t \in [0, T]$ one works with the following norms: For $B_N = \{|x| < N\}$ the ball of radius N in \mathbb{R}^3 , and for $\|\cdot\|_s$ the standard Sobolev norm in $W^{s,2}(\mathbb{R}^3) = L_s^2(\mathbb{R}^3)$, define the dual norm

$$\|v|_{B_N}\|_{-s} = \sup_{w \in C_0^\infty(B_N)} \frac{\langle v, w \rangle_2}{\|w\|_s}. \quad (2.2)$$

Using this and following [VF], p.245, define

$$\|u\|_{BV^{-s}} = \|u\|_{L^2(0,T;\mathcal{H}^0(r))} + \sum_{N=1}^{\infty} \frac{1}{2^{2N}C(N)} |u|_N. \quad (2.3)$$

Here $C(N)$ are constants from (4.3) and (4.4) (defined later) and $|u|_N$ is defined as follows:

$$|u|_N = \text{vrai sup}_{t \in [0,T]} \|u(t, \cdot)|_{B_N}\|_{-s} + \sup_{\{t_j\}} \sum_{j=1}^l \text{vrai sup}_{t, \tau \in [t_{j-1}, t_j]} \|(u(t, \cdot) - u(\tau, \cdot))|_{B_N}\|_{-s}, \quad (2.4)$$

where $\sup_{\{t_j\}}$ is the supremum over all partitions $t_0 < \dots < t_l$, $l \in \mathbb{N}$ of the segment $[0, T]$.

Define

$$BV^{-s} = \{u \in L^2(0, T; \mathcal{H}^0(r)) : \|u\|_{BV^{-s}} < \infty\}. \quad (2.5)$$

The merit of the BV^{-s} norm is that for u in BV^{-s} the limits

$$\gamma_{t_0}(u) := \lim_{t \rightarrow t_0^+} u(t, \cdot) \quad (2.6)$$

exist for all $t_0 \in [0, T]$, if taken with respect to the norm

$$\|u(t, \cdot)\|_{\Phi^{-s}} = \left(\sum_{N=1}^{\infty} \frac{1}{2^{2N}C(N)} \|u(t, \cdot)|_{B_N}\|_{-s}^2 \right)^{1/2}, \quad (2.7)$$

see [VF], Chapter VII, Lemma 8.2.

2.2 Point-wise averages and densities

The homogeneity of a measure μ implies that the functionals on $L^1(\mathbb{R}^3)$

$$\begin{aligned}\phi &\mapsto \int \int |u(x)|^2 \phi(x) \, dx \, \mu(du), \\ \phi &\mapsto \int \int |\nabla u(x)|^2 \phi(x) \, dx \, \mu(du),\end{aligned}\tag{2.8}$$

are invariant under translation of ϕ , therefore the point-wise averages

$$\int |u(x)|^2 \, \mu(du), \quad \int |\nabla u(x)|^2 \, \mu(du),\tag{2.9}$$

can be defined by

$$\begin{aligned}\int \int |u(x)|^2 \phi(x) \, dx \, \mu(du) &= \int |u(x)|^2 \, \mu(du) \int \phi(x) \, dx, \\ \int \int |\nabla u(x)|^2 \phi(x) \, dx \, \mu(du) &= \int |\nabla u(x)|^2 \, \mu(du) \int \phi(x) \, dx,\end{aligned}\tag{2.10}$$

for any $\phi \in L_1(\mathbb{R}^3)$, and they are independent of $x \in \mathbb{R}^3$, see Chapter VII, section 1 of [VF]. The first average in (2.9) will be called **the energy density** and the second one **the density of the energy dissipation**, a terminology justified by the identity $\frac{1}{2} \frac{d}{dt} \overline{u^2} = -\nu \overline{|\nabla u|^2}$, which formally follows after integrating by parts the Navier-Stokes equation of viscosity ν .

Since the translation operator T_h (along x) is well defined on the space $L^2(0, T; \mathcal{H}^0(r))$ of vector fields $u(t, x)$ dependent not only on x but on t as well, the notion of homogeneity in x in Definition 1.2.2 can be extended.

The following definition summarizes the properties of homogeneous statistical solutions of the Navier-Stokes equations as they were produced in Chapter VII of [VF]:

Definition 2.2.1. *Given homogeneous probability measure μ on $\mathcal{B}(\mathcal{H}^0(r))$ possessing finite energy density, a **homogeneous statistical solution of the Navier-Stokes***

equations with initial condition μ is a probability measure P on $\mathcal{B}(L^2(0, T; \mathcal{H}^0(r)))$

such that:

1. P is homogeneous in x .

2. $P(\widehat{W}) = 1$, where $\widehat{W} = L^2(0, T; \mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS}$, $s > \frac{11}{2}$.

3. For all $A \in \mathcal{B}(\mathcal{H}^0(r))$,

$$P(\gamma_0^{-1}A) = \mu(A), \quad \text{where} \quad \gamma_0^{-1}A = \{u \in \widehat{W} : \gamma_0 u \in A\}. \quad (2.11)$$

4. For each t in $[0, T]$,

$$\int \left(|u(t, x)|^2 + \int_0^t |\nabla u|^2(\tau, x) \, d\tau \right) P(du) \leq C \int |u(x)|^2 \mu(du), \quad (2.12)$$

where the expression in the left hand side of (2.12) is defined similarly to (2.9).

The main result of [VF], Chapter VII, then reads as follows:

Theorem 2.2.2. *Given μ homogeneous measure on $\mathcal{H}^0(r)$ with finite energy density,*

$$\int_{\mathcal{H}^0(r)} |u|^2(x) \mu(du) < \infty, \quad (2.13)$$

there exists homogeneous statistical solution of the Navier-Stokes equations P with initial condition μ .

Remark 2.2.3. *The definition above is a rephrasing of Definition 11.1 of [VF], with one minor change: It asks that P is supported by $L^2(0, T; \mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS}$ rather than some subset of it. Since [VF] produces some subset supporting a homogeneous statistical solution, it automatically produces a homogeneous statistical solution according to the definition here.*

Remark 2.2.4. *In addition, the family of homogeneous measures $\mu_t := P \circ \gamma_t^{-1}$ on $\mathcal{H}^0(r)$ satisfies the Hopf equation, [VF], Chapter VIII.*

Define now isotropic and homogeneous statistical solutions. First define **isotropic in x measures** P by Definition 1.3.3 on $\mathcal{B}(L^2(0, T; \mathcal{H}^0(r)))$. (This can be done since for each $\omega \in O(3)$ operator $R_\omega u(t, x) \equiv \omega u(t, \omega^{-1}x)$ is well defined on $L^2(0, T; \mathcal{H}^0(r))$.)

For the definition of homogeneous and isotropic statistical solutions one has only to add in Definition 2.2.1 the property of rotation and reflection invariance:

Definition 2.2.5. *Given homogeneous and isotropic probability measure $\hat{\mu}$ on $\mathcal{B}(\mathcal{H}^0(r))$, a homogeneous and isotropic statistical solution of the Navier-Stokes equations with initial condition $\hat{\mu}$ is a probability measure \hat{P} on $\mathcal{B}(L^2(0, T; \mathcal{H}^0(r)))$ such that:*

1. \hat{P} is homogeneous and isotropic in x .
2. $\hat{P}(\widehat{W}) = 1$, where $\widehat{W} = L^2(0, T; \mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS}$, $s > \frac{11}{2}$.
3. $\hat{P}(\gamma_0^{-1}A) = \hat{\mu}(A)$ for every $A \in \mathcal{B}(\mathcal{H}^0(r))$.
4. For each t in $[0, T]$,

$$\int_{L^2(0, T; \mathcal{H}^0(r))} \left(|u(t, x)|^2 + \int_0^t |\nabla u|^2(\tau, x) d\tau \right) \hat{P}(du) \leq C \int_{\mathcal{H}^0(r)} |u(x)|^2 \hat{\mu}(du). \quad (2.14)$$

2.3 Construction of homogeneous and isotropic statistical solutions

To construct homogeneous and isotropic statistical solutions several preliminary assertions need to be proved first. For these, use the definition of the norm $\|\cdot\|_s$ of

Sobolev space $W^{s,2}(\mathbb{R}^3)$ through Fourier transform:

$$\|\phi\|_s^2 = \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\widehat{\phi}|^2(\xi) d\xi, \quad \text{where} \quad \widehat{\phi}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} \phi(x) dx. \quad (2.15)$$

Lemma 2.3.1. *For any matrix $\omega \in O(3)$ the following equalities hold:*

$$\|R_\omega \phi\|_s = \|\phi\|_s,$$

$$\|R_\omega \phi\|_{BV^{-s}} = \|\phi\|_{BV^{-s}}, \quad (2.16)$$

$$\|R_\omega \phi\|_{\Phi^{-s}} = \|\phi\|_{\Phi^{-s}}.$$

Proof. By the definition of Fourier transform and by virtue of (1.33)

$$\begin{aligned} \widehat{R_\omega \phi}(\xi) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\omega^{-1}x \cdot \omega^{-1}\xi} \omega \phi(\omega^{-1}x) dx \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-iy \cdot \omega^{-1}\xi} \omega \phi(y) dy = R_\omega \widehat{\phi}(\xi). \end{aligned} \quad (2.17)$$

By (2.15), (2.17), and (1.33)

$$\begin{aligned} \|R_\omega \phi\|_s^2 &= \int_{\mathbb{R}^3} (1 + |\omega^{-1}\xi|^2)^s |\omega \widehat{\phi}(\omega^{-1}\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^3} (1 + |\eta|^2)^s |\omega \widehat{\phi}(\eta)|^2 d\eta = \|\phi\|_s^2, \end{aligned} \quad (2.18)$$

which proves the first equality in (2.16).

Equalities (2.18), (1.33), and (2.2) give:

$$\begin{aligned} \|R_\omega v|_{B_N}\|_{-s} &= \sup_{\phi \in C_0^\infty(B_N)} \frac{\langle R_\omega v, \phi \rangle_2}{\|\phi\|_s} \\ &= \sup_{\phi \in C_0^\infty(B_N)} \frac{\langle v, (R_\omega)^{-1} \phi \rangle_2}{\|(R_\omega)^{-1} \phi\|_s} = \|v|_{B_N}\|_{-s}, \end{aligned} \quad (2.19)$$

since $R_\omega : C_0^\infty(B_N) \rightarrow C_0^\infty(B_N)$ is an isomorphism.

Equality (2.19) and definition (2.4) of $|\cdot|_N$ imply the equality

$$|R_\omega \phi|_N = |\phi|_N. \quad (2.20)$$

This identity, (2.19), and the definitions (2.3), (2.7) of the norms $\|\cdot\|_{BV^{-s}}, \|\cdot\|_{\Phi^{-s}}$,

imply directly the second and third equalities in (2.16). \square

Lemma 2.3.2. For every $\omega \in O(3)$ and for each homogeneous measure $\mu(A)$, $A \in \mathcal{B}(\mathcal{H}^0(r))$

$$\begin{aligned} \int |u(x)|^2 (R_\omega)_\# \mu(du) &= \int |u(x)|^2 \mu(du), \\ \int |\nabla u(x)|^2 (R_\omega)_\# \mu(du) &= \int |\nabla u(x)|^2 \mu(du). \end{aligned} \quad (2.21)$$

Proof. By (1.2), (1.3), (2.10), and (1.33), for each $\omega \in O(3)$

$$\begin{aligned} \int |u(x)|^2 (R_\omega)_\# \mu(du) \int \phi(x) dx &= \int \int_{\mathbb{R}^3} |\omega u(\omega^{-1}x)|^2 \phi(\omega \omega^{-1}x) dx d\mu(u) \\ &= \int \int_{\mathbb{R}^3} |u(y)|^2 \phi(\omega y) dy d\mu(u) \\ &= \int |u(y)|^2 d\mu(u) \int_{\mathbb{R}^3} \phi(\omega y) dy \\ &= \int |u(x)|^2 d\mu(u) \int_{\mathbb{R}^3} \phi(x) dx, \end{aligned} \quad (2.22)$$

which proves the first equality of (2.21). If $y = \omega^{-1}x$, i.e. $y_l = \omega_{kl}x_k$, then by (1.33),

(1.35), obtain

$$\begin{aligned} \int \sum_j \left| \omega \frac{\partial u(\omega^{-1}x)}{\partial x_j} \right|^2 \phi(x) dx &= \int \sum_j \left| \frac{\partial u(\omega^{-1}x)}{\partial x_j} \right|^2 \phi(\omega \omega^{-1}x) dx \\ &= \int \sum_{j,p} \omega_{jl} \frac{\partial u_p(y)}{\partial y_l} \omega_{jm} \frac{\partial u_p(y)}{\partial y_m} \phi(\omega y) dy \\ &= \int |\nabla_y u(y)|^2 \phi(\omega y) dy. \end{aligned} \quad (2.23)$$

Using these identities, the second equality of (2.21) can be proved similarly to (2.22). □

Recall now that the set \mathcal{G}_{NS} has been introduced in Definition 2.1.1.

Lemma 2.3.3. For each $\omega \in O(3)$ the equality $R_\omega \mathcal{G}_{NS} = \mathcal{G}_{NS}$ holds.

Proof. Prove first that if u satisfies (2.1) then $R_\omega u$ satisfies (2.1) as well, for every $\omega \in O(3)$. For this note that, by Lemma 1.3.2,

$$u \in L^2(0, T; \mathcal{H}^0(r)) \Rightarrow R_\omega u \in L^2(0, T; \mathcal{H}^0(r)),$$

$$\phi \in C((0, T); \mathcal{H}^0(r)) \cap C_0^\infty((0, T) \times \mathbb{R}^3) \Rightarrow R_\omega \phi \in C((0, T); \mathcal{H}^0(r)) \cap C_0^\infty((0, T) \times \mathbb{R}^3). \quad (2.24)$$

So let u satisfy (2.1). Then (1.33) and the well-known fact that Laplace operator is invariant under orthogonal change of variables yield:

$$\langle R_\omega u, \frac{\partial \phi}{\partial t} + \Delta \phi \rangle = \langle u, \frac{\partial (R_{\omega^{-1}} \phi)}{\partial t} + \Delta (R_{\omega^{-1}} \phi) \rangle. \quad (2.25)$$

If $y = \omega^{-1}x = \omega^*x$, i.e. $y_l = \omega_{kl}x_k$, then taking into account (1.35) and (1.33) calculate:

$$\begin{aligned} \int (R_\omega u)_j R_\omega u \cdot \frac{\partial \phi}{\partial x_j} dx &= \int \omega_{jk} u_k(\omega^{-1}x) \omega_{lm} u_m(\omega^{-1}x) \frac{\partial \phi_l(x)}{\partial x_j} dx \\ &= \int \omega_{jk} u_k(y) \omega_{lm} u_m(y) \omega_{jp} \frac{\partial \phi_l(\omega y)}{\partial y_p} dy \\ &= \int u_k(y) u_m(y) \frac{\partial \omega_{lm} \phi_l(\omega y)}{\partial y_k} dy. \end{aligned} \quad (2.26)$$

Then

$$\sum_{j=1}^3 \langle (R_\omega u)_j R_\omega u, \frac{\partial \phi}{\partial x_j} \rangle_2 = \sum_{j=1}^3 \langle u_j u, \frac{\partial (R_{\omega^{-1}} \phi)}{\partial x_j} \rangle_2. \quad (2.27)$$

Adding (2.25) and (2.27), integrating the resulting equality with respect to t over $[0, T]$, and taking into account that u satisfies (2.1), shows that $R_\omega u$ satisfies (2.1) as well. \square

Lemma 2.3.4. γ_0 commutes with R_ω for any rotation ω .

Proof. By Lemma 2.3.1, $\|u(t, \cdot)\|_{\Phi^{-s}} = \|R_\omega u(t, \cdot)\|_{\Phi^{-s}}$. Therefore if $\lim_{t \rightarrow 0^+} u(t, \cdot) = \gamma_0(u)$, then $\lim_{t \rightarrow 0^+} R_\omega u(t, \cdot) = R_\omega \gamma_0(u)$, and $\lim_{t \rightarrow 0^+} R_\omega u(t, \cdot) = \gamma_0(R_\omega u)$, i.e. $\gamma_0(R_\omega u) = R_\omega \gamma_0(u)$. \square

Recall that for each $B \in \mathcal{B}(\mathcal{H}^0(r))$

$$\gamma_0^{-1}B = \{u(t, x) \in \widehat{W} : \gamma_0 u \in B\}, \quad (2.28)$$

where \widehat{W} is the set of Definition 2.2.1 or (equivalently) of Definition 2.2.5.

Lemma 2.3.5. For $B \in \mathcal{B}(\mathcal{H}^0(r))$,

$$R_\omega \gamma_0^{-1}(B) = \gamma_0^{-1}(R_\omega B), \quad \forall \omega \in O(3). \quad (2.29)$$

Proof. Using (1.35), one can prove similarly to Lemma 1.3.2 that

$$R_\omega \mathcal{H}^1(r) = \mathcal{H}^1(r) \quad \forall \omega \in O(3), \quad (2.30)$$

for $\mathcal{H}^1(r)$ as in Definition 1.2.3. This, together with lemmas 2.3.1 and 2.3.3, imply that

$$R_\omega \widehat{W} = \widehat{W} \quad \forall \omega \in O(3). \quad (2.31)$$

Therefore

$$\begin{aligned} u \in R_\omega \gamma_0^{-1}(B) &\Rightarrow u = R_\omega v, v \in \gamma_0^{-1}(B) \\ &\Rightarrow \gamma_0 u = \gamma_0(R_\omega v), v \in \gamma_0^{-1}(B) \\ &\Rightarrow \gamma_0 u = R_\omega \gamma_0(v), v \in \gamma_0^{-1}(B), \text{ by Lemma 2.3.4,} \\ &\Rightarrow \gamma_0 u = R_\omega b, b \in B \\ &\Rightarrow u = \gamma_0^{-1} R_\omega b, b \in B \\ &\Rightarrow u \in \gamma_0^{-1}(R_\omega B). \end{aligned} \quad (2.32)$$

Conversely,

$$\begin{aligned} u \in \gamma_0^{-1}(R_\omega B) &\Rightarrow \gamma_0(u) = R_\omega b, b \in B \\ &\Rightarrow R_{\omega^{-1}} \gamma_0(u) = b, b \in B \\ &\Rightarrow \gamma_0(R_{\omega^{-1}} u) = b, b \in B, \text{ by Lemma 2.3.4,} \\ &\Rightarrow R_{\omega^{-1}} u = \gamma_0^{-1}(b), b \in B \\ &\Rightarrow R_{\omega^{-1}} u \in \gamma_0^{-1}(B), \\ &\Rightarrow u \in R_\omega \gamma_0^{-1}(B). \end{aligned} \quad (2.33)$$

□

Theorem 2.3.6. *Given $\widehat{\mu}$ homogeneous and isotropic measure on $\mathcal{H}^0(r)$ with finite energy density,*

$$\int_{\mathcal{H}^0(r)} |u|^2(x) \widehat{\mu}(du) < \infty, \quad (2.34)$$

there exists homogeneous and isotropic statistical solution \widehat{P} of the Navier-Stokes equations with initial condition $\widehat{\mu}$.

Proof. Ignoring for the moment that $\widehat{\mu}$ is also isotropic, let P be the homogeneous statistical solution with initial condition the homogeneous $\widehat{\mu}$ guaranteed by Theorem 2.2.2. The set $\widehat{W} = L^2(0, T; \mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS}$ is invariant under rotations by (2.31). Applying the analogue of operation (1.39) on the homogeneous measure P obtain:

$$\widehat{P}(A) = \int_{O(3)} (R_\omega)_\# P(A) d\omega = \int_{O(3)} P(R_{\omega^{-1}}A) d\omega, \quad (2.35)$$

for any $A \in (B)(\widehat{W})$. Repeating the proof of Proposition 1.3.6 for the measure \widehat{P} shows that \widehat{P} is homogeneous and isotropic in x . Since $P(\widehat{W}) = 1$ by Theorem 2.2.2, equality (2.31) implies that $(R_\omega)_\# P(\widehat{W}) = P(R_{\omega^{-1}}\widehat{W}) = P(\widehat{W}) = 1$ for each $\omega \in O(3)$. Hence, $\widehat{P}(\widehat{W}) = 1$ by definition (2.35).

That \widehat{P} has initial condition $\widehat{\mu}$ follows from

$$\begin{aligned} \widehat{P}(\gamma_0^{-1}B) &= \int_{O(3)} P(R_{\omega^{-1}}\gamma_0^{-1}(B)) d\omega \\ &= \int_{O(3)} P(\gamma_0^{-1}(R_{\omega^{-1}}B)) d\omega, \text{ by Lemma 2.3.5} \\ &= \int_{O(3)} \widehat{\mu}(R_{\omega^{-1}}B) d\omega, \text{ by (2.11)} \\ &= \widehat{\mu}(B), \text{ since } \widehat{\mu} \text{ is also isotropic,} \end{aligned} \quad (2.36)$$

for any $B \in \mathcal{B}(\mathcal{H}^0(r))$.

For the energy inequality, use (2.35), Lemma 2.3.2, and (2.12) to get:

$$\begin{aligned}
& \int_{L^2(0,T;\mathcal{H}^0(r))} \left(|u(t,x)|^2 + \int_0^t |\nabla u|^2(\tau,x) d\tau \right) \widehat{P}(du) \\
&= \int_{O(3)} \int_{L^2(0,T;\mathcal{H}^0(r))} \left(|u(t,x)|^2 + \int_0^t |\nabla u|^2(\tau,x) d\tau \right) R_{\omega\#} P(du) d\omega \\
&= \int_{L^2(0,T;\mathcal{H}^0(r))} \left(|u(t,x)|^2 + \int_0^t |\nabla u|^2(\tau,x) d\tau \right) P(du) \\
&\leq C \int_{\mathcal{H}^0(r)} |u(x)|^2 \widehat{\mu}(du),
\end{aligned} \tag{2.37}$$

which proves (2.14). □

Chapter 3

Galerkin Statistical Solutions

It is shown how to approximate (in the sense of convergence of characteristic functionals) any isotropic measure on a certain space of vector fields by isotropic measures supported by periodic vector fields and their rotations. This is achieved without loss of uniqueness for the Galerkin system, allowing for the Galerkin approximations of homogeneous statistical Navier-Stokes solutions to be adopted to isotropic approximations.

3.1 Isotropic measures on periodic vector fields

Let \mathcal{M}_l be as in [VF]:

$$\mathcal{M}_l = \left\{ \sum_{\substack{k \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k| \leq l}} a_k e^{ik \cdot x} : a_k \cdot k = 0, a_k = \bar{a}_{-k} \forall k \right\}, \quad (3.1)$$

the finite-dimensional space of divergence-free, $3D$, real, vector valued trigonometric polynomials of degree l and period $2l$. Then the inclusion

$$\mathcal{M}_l \subset \mathcal{H}^0(r) \quad (3.2)$$

holds for all l . [VF], Appendix II, shows explicitly how, starting from any homogeneous probability measure on $\mathcal{H}^0(r)$, one can construct homogeneous probability

measures μ_l on $\mathcal{H}^0(r)$, supported solely by \mathcal{M}_l for each l , and approximating μ in the sense of characteristic functionals. The trouble, of course, is that \mathcal{M}_l is not invariant under rotations. The following definitions address this point.

Definition 3.1.1. Let $\widehat{\mathcal{M}}_l$ be the union of all rotations of elements of \mathcal{M}_l :

$$\widehat{\mathcal{M}}_l = \bigcup_{\omega \in O(3)} R_\omega M(l) \quad (3.3)$$

in $\mathcal{H}^0(r)$.

Consider on $\widehat{\mathcal{M}}_l$ the topology τ generated by sets of the form

$$\{R_\omega m : \omega \in \rho, m \in \sigma, \text{ where } \rho \subset O(3), \sigma \subset \mathcal{M}_l \text{ are open sets}\}. \quad (3.4)$$

Since $O(3)$ and \mathcal{M}_l are finite-dimensional sets, the topology τ coincides with the topology generated by the enveloping space $\mathcal{H}^0(r)$. Therefore, the Borel σ -algebra $\mathcal{B}(\widehat{\mathcal{M}}_l)$ is generated by sets of the form (3.4). Moreover, it is clear that

$$\mathcal{B}(\widehat{\mathcal{M}}_l) = \mathcal{B}(\mathcal{H}^0(r)) \cap \widehat{\mathcal{M}}_l \equiv \{A \cap \widehat{\mathcal{M}}_l : A \in \mathcal{B}(\mathcal{H}^0(r))\} \quad (3.5)$$

Note that for each fixed ω the elements of $R_\omega \mathcal{M}_l$ are of the form

$$\sum_{\substack{k \in \frac{\pi}{l} \mathbb{Z}^3, \\ |k| \leq l}} b_k e^{i\omega k \cdot x}, \quad b_k \cdot \omega k = 0 \text{ for all } k. \quad (3.6)$$

Using definition (1.2) of the push forward measure, proceed to:

Definition 3.1.2. Let $\widehat{\mu}_l(A)$, $A \in \mathcal{B}(\widehat{\mathcal{M}}_l)$ be the push-forward of the product of the Haar measure on $O(3)$ and the measure μ_l on \mathcal{M}_l via the map $(\omega, u) \mapsto R_\omega u$:

$$\widehat{\mu}_l(A) = (H \times \mu_l) \{(\omega, u) \in O(3) \times \mathcal{M}_l : R_\omega u \in A\}. \quad (3.7)$$

As for any push-forward measure, by (1.3),

$$\int_{\widehat{\mathcal{M}}_l} f(v) \widehat{\mu}_l(dv) = \int_{\mathcal{M}_l} \int_{O(3)} f(R_\omega u) d\omega \mu_l(du), \quad (3.8)$$

for any $\widehat{\mu}_l$ -integrable f .

Since μ_l is supported on $\mathcal{M}_l \subset \mathcal{H}^0(r)$ and $\widehat{\mu}_l$ is supported on $\widehat{\mathcal{M}}_l \subset \mathcal{H}^0(r)$, the domains of integration $\widehat{\mathcal{M}}_l, \mathcal{M}_l$ in (3.8) can change to $\mathcal{H}^0(r)$. Comparing then (3.7), (3.8) to the definitions of averaging (1.39), (1.40) it follows that the measure $\widehat{\mu}_l$ is the averaging of μ_l over $O(3)$:

$$\widehat{\mu}_l(A) = \int_{O(3)} (R_\omega)_\# \mu_l(A) d\omega \quad \forall A \in \mathcal{B}(\mathcal{H}^0(r)) \quad (3.9)$$

Proposition 3.1.3. *$\widehat{\mu}_l$ is homogeneous and isotropic.*

Proof. Using the equality $R_{\omega^{-1}} \widehat{\mathcal{M}}_l = \widehat{\mathcal{M}}_l$ and the invariance of the Haar measure, obtain for each $\widehat{\mu}_l$ -integrable f :

$$\begin{aligned} \int_{\widehat{\mathcal{M}}_l} f(w) (R_{\omega_0})_\# \widehat{\mu}_l(dw) &= \int_{\widehat{\mathcal{M}}_l} f(R_{\omega_0} v) \widehat{\mu}_l(dv) \\ &= \int_{\mathcal{M}_l} \int_{O(3)} f(R_{\omega_0} R_\omega u) d\omega \mu_l(du) \\ &= \int_{\mathcal{M}_l} \int_{O(3)} f(R_\omega u) d\omega \mu_l(du) \\ &= \int_{\widehat{\mathcal{M}}_l} f(v) \widehat{\mu}_l(dv), \end{aligned} \quad (3.10)$$

i.e. $\widehat{\mu}$ is invariant with respect of rotations: $(R_{\omega_0})_\# \widehat{\mu} = \widehat{\mu}$ for each $\omega_0 \in O(3)$.

Similarly, the equalities $T_h^{-1}\widehat{\mathcal{M}}_l = \widehat{\mathcal{M}}_l$, $T_{\omega h}^{-1}\mathcal{M}_l = \mathcal{M}_l$ imply:

$$\begin{aligned}
\int_{\widehat{\mathcal{M}}_l} f(w)(T_h)_\# \widehat{\mu}_l(dw) &= \int_{\widehat{\mathcal{M}}_l} f(T_h v) \widehat{\mu}_l(dv) \\
&= \int_{\mathcal{M}_l} \int_{O(3)} f(T_h R_\omega u) d\omega \mu_l(du) \\
&= \int_{O(3)} \int_{\mathcal{M}_l} f(R_\omega T_{\omega^{-1}h} u) \mu_l(du) d\omega \\
&= \int_{O(3)} \int_{\mathcal{M}_l} f(R_\omega u) \mu_l(du) d\omega, \text{ by the homogeneity of } \mu_l, \\
&= \int_{\mathcal{M}_l} \int_{O(3)} f(R_\omega u) d\omega \mu_l(du) \\
&= \int_{\widehat{\mathcal{M}}_l} f(v) \widehat{\mu}_l(dv).
\end{aligned} \tag{3.11}$$

□

3.2 Galerkin equations for Fourier coefficients on $R_\omega \mathcal{M}_l$

Let $H^s(\Pi_l)$ be the space of periodic vector fields

$$H^s(\Pi_l) = \left\{ u(x) = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3} a_k e^{ik \cdot x}, a_k = (a_{k1}, a_{k2}, a_{k3}), \|u\|_s^2 = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3} (1+|k|^2)^s |a_k|^2 < \infty \right\}. \tag{3.12}$$

Here

$$\Pi_l = \{x = (x_1, x_2, x_3) : |x_j| \leq l, j = 1, 2, 3\} \tag{3.13}$$

is the cube of periods for these vector fields.

On the space $C^1(0, T; H^2(\Pi_l))$ the Navier-Stokes system can be written in the form:

$$\partial_t u - \Delta u + \pi(u, \nabla)u = 0, \operatorname{div} u = 0, \tag{3.14}$$

where $\pi : L^2(\Pi_l) \rightarrow \{u \in L^2(\Pi_l) : \operatorname{div} u = 0\}$ is the projection on solenoidal vector fields. It is standard that substitution of the Fourier series $u(x) = \sum_k a_k e^{ik \cdot x}$ into

(3.14) yields the following system for the Fourier coefficients $a_k(t)$:

$$\begin{aligned} \partial_t a_k + |k|^2 a_k + \sum_{\substack{k'+k''=k, \\ k', k'' \in \frac{\pi}{l}\mathbb{Z}^3}} i((a_{k'} \cdot k'') a_{k''} - \frac{(a_{k'} \cdot k'')(a_{k''} \cdot k)}{|k|^2} k) = 0, \quad a_k \cdot k = 0, \\ k \in \frac{\pi}{l}\mathbb{Z}^3. \end{aligned} \quad (3.15)$$

Let $p_l : H^2(\Pi_l) \rightarrow \mathcal{M}_l$ be projection on trigonometric polynomials:

$$H^2(\Pi_l) \ni u(x) = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3} a_k e^{ikx} \mapsto p_l u(x) = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3, |k| \leq l} a_k e^{ikx} \quad (3.16)$$

As is well-known, to get Galerkin approximations of Navier-Stokes system one restricts (3.14) to $C^1(0, T; \mathcal{M}_l)$ and applies the operator p_l to (3.14) to obtain:

$$\partial_t u - \Delta u + p_l \pi(u, \nabla) u = 0, \quad \text{div } u = 0, \quad (3.17)$$

where $u \in C^1(0, T; \mathcal{M}_l)$. In terms of the Fourier coefficients of the Galerkin approximations this will have the form:

$$\begin{aligned} \partial_t a_k + |k|^2 a_k + \sum_{\substack{k'+k''=k, \\ k', k'' \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k'| \leq l, |k''| \leq l}} i((a_{k'} \cdot k'') a_{k''} - \frac{(a_{k'} \cdot k'')(a_{k''} \cdot k)}{|k|^2} k) = 0, \quad a_k \cdot k = 0, \\ k \in \frac{\pi}{l}\mathbb{Z}^3, \quad |k| \leq l. \end{aligned} \quad (3.18)$$

Proposition 3.2.1. *For each $\omega \in O(3)$ the following holds:*

$$R_\omega \mathcal{M}_l = \left\{ v(x) = \sum_{\substack{m \in \frac{\pi}{l}\omega\mathbb{Z}^3, \\ |m| \leq l}} b_m e^{imx} \right\}. \quad (3.19)$$

Moreover,

$$\left. \begin{aligned} u(x) &= \sum_{\substack{k \in \frac{\pi}{l}\mathbb{Z}^3 \\ |k| \leq l}} a_k e^{ikx} \\ R_\omega u(x) &= \sum_{\substack{m \in \frac{\pi}{l}\omega\mathbb{Z}^3 \\ |m| \leq l}} b_m e^{imx} \end{aligned} \right\} \Rightarrow b_m = \omega a_{\omega^{-1}m}. \quad (3.20)$$

Proof. Let $u(x) = \sum_{|k| \leq l} a_k e^{ikx} \in \mathcal{M}_l$. Then using the definition $R_\omega u(x) = \omega u(\omega^{-1}x)$ and applying the change of variables $\omega k = m$ get:

$$R_\omega u(x) = \sum_{\substack{k \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k| \leq l}} \omega a_k e^{i\omega k x} = \sum_{\substack{m \in \frac{\pi}{l}\omega\mathbb{Z}^3, \\ |m| \leq l}} \omega a_{\omega^{-1}m} e^{imx} \quad (3.21)$$

This proves (3.19) and (3.20). \square

Proposition 3.2.2. *For each $\omega \in O(3)$ the Galerkin approximations for the Navier-Stokes equations on the space $C^1(0, T; R_\omega \mathcal{M}_l)$ are of the following form:*

$$\begin{aligned} \partial_t b_m(t) + |m|^2 b_m + \sum_{\substack{m' + m'' = m, \\ m', m'' \in \frac{\pi}{l}\omega\mathbb{Z}^3, \\ |m'| \leq l, |m''| \leq l}} i \left((b_{m'} \cdot m'') b_{m''} - \frac{(b_{m'} \cdot m'')(b_{m''} \cdot m)}{|m|^2} m \right) = 0, \quad b_m \cdot m = 0, \\ m \in \frac{\pi}{l}\omega\mathbb{Z}^3, \quad |m| \leq l. \end{aligned} \quad (3.22)$$

Proof. To obtain the Galerkin approximations on $C^1(0, T; R_\omega \mathcal{M}_l)$ for the Navier-Stokes equations, repeat the procedure above that leads to the Galerkin approximations (3.18): Re-write (3.14) on the space of periodic fields $C^1(0, T; R_\omega H^2(\Pi_l))$ in terms of Fourier coefficients to get the following analog of (3.15):

$$\begin{aligned} \partial_t b_m + |m|^2 b_m + \sum_{\substack{m' + m'' = m, \\ m', m'' \in \frac{\pi}{l}\omega\mathbb{Z}^3}} i \left((b_{m'} \cdot m'') b_{m''} - \frac{(b_{m'} \cdot m'')(b_{m''} \cdot m)}{|m|^2} m \right) = 0, \quad b_m \cdot m = 0, \\ m \in \frac{\pi}{l}\omega\mathbb{Z}^3, \end{aligned} \quad (3.23)$$

and then repeat the derivation of (3.17), (3.18) from (3.14), (3.15) to finally get (3.22) from (3.23). \square

Now supplement (3.17) and (3.18) with the initial condition

$$u(t, x)|_{t=0} = \sum_{\substack{k \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k| \leq l}} a_k(t) e^{ikx}|_{t=0} = u_0(x) = \sum_{\substack{k \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k| \leq l}} a_{k0} e^{ikx} \quad (3.24)$$

and

$$a_k(t)|_{t=0} = a_{k0}, \quad k \in \frac{\pi}{l}\mathbb{Z}^3, |k| \leq l \quad (3.25)$$

Moreover, we supplement (3.22) with initial condition

$$b_m(t)|_{t=0} = b_{m0}, \quad m \in \frac{\pi}{l}\omega\mathbb{Z}^3, |k| \leq l \quad (3.26)$$

Then, as is well-known, the Cauchy problem (3.17), (3.24), (equivalently, the Cauchy problem for the ordinary differential equations (3.18), (3.25)) has a unique solution in $C^1(0, T; \mathcal{M}_l)$. Call this solution $S_l(u_0)$. Analogously, the Cauchy Problem (3.22), (3.26) possesses a unique solution. Write this solution as the Fourier polynomial

$$S_l(v_0) = \sum_{\substack{m \in \frac{\pi}{l}\omega\mathbb{Z}^3, \\ |k| \leq l}} b_m(t)e^{imx}, \quad \text{where } v_0 = \sum_{\substack{m \in \frac{\pi}{l}\omega\mathbb{Z}^3, \\ |k| \leq l}} b_{m0}e^{imx} \quad (3.27)$$

Lemma 3.2.3. *Let $u_0 \in \mathcal{M}_l$. Then $R_\omega S_l(u_0)$ solves (3.22) with initial condition $v_0 = R_\omega u_0$. Moreover, if $S_l(u_0)$ admits the Fourier decomposition*

$$S_l(u_0) = \sum_{\substack{k \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k| \leq l}} a_k(t)e^{ikx}, \quad (3.28)$$

then $\{\omega a_k\}$ satisfies

$$\begin{aligned} \partial_t \omega a_k + |k|^2 \omega a_k + \sum_{\substack{k'+k''=k, \\ k', k'' \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k'| \leq l, |k''| \leq l}} i \left((\omega a_{k'} \cdot \omega k'') \omega a_{k''} - \frac{(\omega a_{k'} \cdot \omega k'') (\omega a_{k''} \cdot \omega k)}{|k|^2} \omega k \right) = 0, \\ \omega a_k \cdot \omega k = 0, \quad k \in \frac{\pi}{l}\mathbb{Z}^3, |k| \leq l. \end{aligned} \quad (3.29)$$

Proof. Let

$$R_\omega S_l(u_0) = \sum_{\substack{m \in \frac{\pi}{l}\omega\mathbb{Z}^3, \\ |m| \leq l}} b_m(t)e^{imx} \quad (3.30)$$

where, by (3.20),

$$b_m = a_{\omega^{-1}m}. \quad (3.31)$$

The assertion of the Lemma will be proved once it shown that the $\{b_m(t)\}$ satisfy (3.22). Substitution of (3.31) into the left hand side of (3.22), the change of variables $m = \omega k$, and the fact that $\omega \in O(3)$ yield:

$$\begin{aligned} & \forall k \in \frac{\pi}{l}\mathbb{Z}^3, |k| \leq l; \\ & \partial_t \omega a_k + |\omega k|^2 \omega a_k + \sum_{\substack{k'+k''=k, \\ k',k'' \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k'| \leq l, |k''| \leq l}} i \left((\omega a_{k'} \cdot \omega k'') \omega a_{k''} - \frac{(\omega a_{k'} \cdot \omega k'') (\omega a_{k''} \cdot \omega k)}{|\omega k|^2} \omega k \right) \\ & = \omega \left\{ \partial_t a_k + |k|^2 a_k + \sum_{\substack{k'+k''=k, \\ k',k'' \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k'| \leq l, |k''| \leq l}} i \left((a_{k'} \cdot k'') a_{k''} - \frac{(a_{k'} \cdot k'') (a_{k''} \cdot k)}{|k|^2} k \right) \right\} = 0, \end{aligned} \quad (3.32)$$

since $\{a_k(t)\}$ satisfies (3.18). \square

Remark 3.2.4. *As will be shown, more is true: The Galerkin PDE has a unique solution for any initial condition v in $\widehat{\mathcal{M}}_l$, see (3.47) below.*

The task now is to show that inclusions $u_1 \in \mathcal{M}_l$ and $R_\omega u_1 = u_2 \in \mathcal{M}_l$, for some ω , implies that for the same ω , $R_\omega S_l u_1$ stays in \mathcal{M}_l for all t , still solving the Galerkin system on \mathcal{M}_l . This will then show that $R_\omega S_l u_1 = S_l u_2$.

Definition 3.2.5. *Given an isometry ω of \mathbb{R}^3 , let \mathcal{K}_ω be the set of all elements k in the lattice $\frac{\pi}{l}\mathbb{Z}^3$ such that ωk also belongs to the lattice.*

Lemma 3.2.6. *Let u_1, u_2 be vector fields of the form*

$$u_1(x) = \sum_{|k| \leq l} a_k e^{ik \cdot x}, \quad u_2(x) = \sum_{|k| \leq l} b_k e^{ik \cdot x}, \quad \text{where all } k \in \frac{\pi}{l}\mathbb{Z}^3, \quad (3.33)$$

not necessarily divergence free. Then for some ω isometry of \mathbb{R}^3 , $R_\omega u_1 = u_2$ if and only if all k 's in the representation of u_1 are in \mathcal{K}_ω .

Proof. Since $R_\omega u_1 = u_2$, $R_\omega u_1$ is periodic of period $2l$. Therefore

$$\begin{aligned} \sum_{|k| \leq l} \omega a_k e^{i\omega k \cdot x} &= \sum_{|k| \leq l} \omega a_k e^{i\omega k \cdot (x + (2l)_j)} \\ &= \sum_{|k| \leq l} \omega a_k e^{i2l(\omega k)_j} e^{i\omega k \cdot x}, \quad j = 1, 2, 3, \end{aligned} \quad (3.34)$$

for $(2l)_j$ denoting the vector with $2l$ as j coordinate and zeroes on the rest. Now since $e^{i\omega k \cdot x}$ are orthonormal on the image of the cube $\Pi_l = [-l, l]^3$ under ω , this implies that

$$1 = e^{i2l(\omega k)_j}, \quad (3.35)$$

therefore

$$2l(\omega k)_j = 2\pi N_{k_j}, \quad N_{k_j} \in \mathbb{Z}. \quad (3.36)$$

Therefore $\omega k \in \frac{\pi}{l} \mathbb{Z}^3$. The converse is clear. \square

Lemma 3.2.7. *Let u_1 in \mathcal{M}_l of the form ,*

$$u_1 = \sum_{|k| \leq l} a_k e^{ik \cdot x}, \quad k \in \mathcal{K}_\omega. \quad (3.37)$$

Then the Galerkin solution in \mathcal{M}_l with initial condition u_1 is of the form

$$u(t) = \sum_{|k| \leq l} a_k(t) e^{ik \cdot x}, \quad a_k(t) = 0 \text{ for all } t \text{ for } k \notin \mathcal{K}_\omega. \quad (3.38)$$

Proof. In \mathcal{M}_l , first solve the system

$$\begin{aligned} \partial_t a_k &= -|k|^2 a_k, \quad k \notin \mathcal{K}_\omega, \quad |k| \leq l \\ \partial_t a_k + \sum_j \sum_{\substack{k'+k''=k \\ |k'| \leq l \\ |k''| \leq l}} \left((a_{k'})_j i k''_j a_{k''} - \frac{(a_{k'})_j i k''_j a_{k''} \cdot k}{|k|^2} k \right) &= -|k|^2 a_k, \quad k \in \mathcal{K}_\omega, \quad |k| \leq l \end{aligned} \quad (3.39)$$

with initial conditions

$$a_k(0) = 0, \quad k \notin \mathcal{K}_\omega \quad (3.40)$$

$$a_k(0) = a_k, \quad k \in \mathcal{K}_\omega,$$

using the a_k 's from (3.37).

In particular

$$a_k(t) = 0, \quad \text{for all } t, \text{ for } k \text{ not in } \mathcal{K}_\omega. \quad (3.41)$$

Now if $k \notin \mathcal{K}_\omega$, and $k' + k'' = k$, then either $k' \notin \mathcal{K}_\omega$ or $k'' \notin \mathcal{K}_\omega$. Then the unique solution of (3.39) (3.40) also solves the system

$$\begin{aligned} \partial_t a_k + \sum_j \sum_{\substack{k'+k''=k \\ |k'| \leq l \\ |k''| \leq l}} \left((a_{k'})_j i k''_j a_{k''} - \frac{(a_{k'})_j i k''_j a_{k''} \cdot k}{|k|^2} k \right) &= -|k|^2 a_k, \quad k \notin \mathcal{K}_\omega, \quad |k| \leq l \\ \partial_t a_k + \sum_j \sum_{\substack{k'+k''=k \\ |k'| \leq l \\ |k''| \leq l}} \left((a_{k'})_j i k''_j a_{k''} - \frac{(a_{k'})_j i k''_j a_{k''} \cdot k}{|k|^2} k \right) &= -|k|^2 a_k, \quad k \in \mathcal{K}_\omega, \quad |k| \leq l, \end{aligned} \quad (3.42)$$

i.e. solves the Galerkin system with initial condition u_1 . \square

Proposition 3.2.8. *Let u_1, u_2 be in \mathcal{M}_l of the form*

$$u_1(x) = \sum_{|k| \leq l} a_k e^{ik \cdot x}, \quad u_2(x) = \sum_{|\kappa| \leq l} b_\kappa e^{i\kappa \cdot x}, \quad (3.43)$$

Then $u_2 = R_\omega u_1$ implies $S_l u_2 = R_\omega S_l u_1$.

Proof. Let

$$S_l u_1 = \sum_{|\kappa| \leq l} a_\kappa(t) e^{i\kappa \cdot x}, \quad a_\kappa(0) = a_\kappa, \quad S_l u_2 = \sum_{|\kappa| \leq l} b_\kappa(t) e^{i\kappa \cdot x}, \quad b_\kappa(0) = b_\kappa. \quad (3.44)$$

By Lemma 3.2.3, $R_\omega S_l u_1$ is the unique solution of the Galerkin system on $R_\omega \mathcal{M}_l$ with initial condition $R_\omega u_1$, i.e. by (3.29) it solves

$$\partial_t \omega a_k + \sum_j \sum_{\substack{k'+k''=k \\ |k'| \leq l \\ |k''| \leq l}} \left((\omega a_{k'})_j i (\omega k'')_j \omega a_{k''} - \frac{(\omega a_{k'})_j i (\omega k'')_j \omega a_{k''} \cdot \omega k}{|\omega k|^2} \omega k \right) = -|\omega k|^2 \omega a_k. \quad (3.45)$$

Observe that in this system if $\omega k \neq \kappa$ for some κ in the lattice $\frac{\pi}{l}\mathbb{Z}^3$ then $\omega a_k(t) = 0$ for all t , by Lemma 3.2.7. Now rename $\omega k = \kappa$, $\omega a_k(t) = c_\kappa(t)$, $\omega a_k(0) = c_\kappa(0) = b_\kappa$, $\omega k' = \kappa'$, $\omega k'' = \kappa''$ to get

$$\partial_t c_\kappa + \sum_j \sum_{\substack{\kappa'+\kappa''=\kappa \\ |\kappa'| \leq l \\ |\kappa''| \leq l}} \left((c_{\kappa'})_j i(\kappa'')_j c_{\kappa''} - \frac{(c_{\kappa'})_j i(\kappa'')_j c_{\kappa''} \cdot \kappa}{|\kappa|^2} \kappa \right) = -|\kappa|^2 c_\kappa, \quad (3.46)$$

with initial conditions Ra_k . This gives a permutation of the Galerkin system on \mathcal{M}_l with initial condition u_2 . Therefore $c_\kappa(t) = b_\kappa(t)$ for all $\kappa \in \frac{\pi}{l}\mathbb{Z}^3$. \square

3.3 Isotropic Galerkin approximations of statistical solution

Now given v in $\widehat{\mathcal{M}}_l$, there are ω in $O(3)$ and u in \mathcal{M}_l such that $v = R_\omega u$. Extend S_l from \mathcal{M}_l to $\widehat{\mathcal{M}}_l$ as

$$\widehat{S}_l v = R_\omega S_l u. \quad (3.47)$$

This is well defined by Proposition 3.2.8: If $R_{\omega_1} u_1 = R_{\omega_2} u_2$ then a straightforward calculation shows that $R_{\omega_2^{-1}\omega_1} u_1 = u_2$, therefore $R_{\omega_2^{-1}\omega_1} S_l u_1 = S_l u_2$, or $R_{\omega_1} S_l u_1 = R_{\omega_2} S_l u_2$. In particular, \widehat{S}_l satisfies

$$\widehat{S}_l R_\omega u = R_\omega S_l u \quad \forall u \in \mathcal{M}_l. \quad (3.48)$$

Let $v = R_\omega u$ and $w = R_{\omega_1} v = R_{\omega\omega_1} u$ where $u \in \mathcal{M}_l$. Applying R_{ω_1} to both parts of (3.47) gives $R_{\omega_1} \widehat{S}_l v = R_{\omega\omega_1} S_l u$. On the other hand, (3.47) for w can be written as $\widehat{S}_l w = R_{\omega\omega_1} S_l u$. Comparing these two equalities gives

$$R_{\omega_1} \widehat{S}_l v = \widehat{S}_l R_{\omega_1} v, \quad \forall v \in \widehat{\mathcal{M}}_l. \quad (3.49)$$

[VF], p. 219 shows that

$$T_h S_l = S_l T_h. \quad (3.50)$$

Applying to both parts of (3.47) the translation operator T_h and using (1.42), (3.50) gives

$$T_h \widehat{S}_l v = R_\omega T_{\omega^{-1}h} S_l u = R_\omega S_l T_{\omega^{-1}h} u. \quad (3.51)$$

On the other hand, applying \widehat{S}_l as defined in (3.47) to $T_h v = T_h R_\omega u = R_\omega T_{\omega^{-1}h} u$, obtain $\widehat{S}_l T_h v = R_\omega S_l T_{\omega^{-1}h} u$. Comparing this equality with (3.51) obtain

$$T_h \widehat{S}_l v = \widehat{S}_l T_h v \quad \forall v \in \widehat{\mathcal{M}}_l. \quad (3.52)$$

Define

$$\widehat{P}_l(A) = \widehat{\mu}_l(\widehat{S}_l^{-1} A), \quad (3.53)$$

for any Borel subset A of $L^2(0, T, \mathcal{H}^0(r))$ where, recall,

$$\widehat{S}_l^{-1} A = \gamma_0(A \cap \widehat{S}_l \widehat{\mathcal{M}}_l). \quad (3.54)$$

Since the measure $\widehat{\mu}_l$ is supported on $\widehat{\mathcal{M}}_l$ it is enough to consider Borel sets A satisfying $A \cap \widehat{S}_l \widehat{\mathcal{M}}_l \neq \emptyset$.

Definition (3.53) is the isotropic version of the measure P_l defined in [VF]:

$$P_l(A) = \mu_l(S_l^{-1} A) \quad (3.55)$$

for any Borel subset A of $L^2(0, T, \mathcal{H}^0(r))$.

Lemma 3.3.1. *\widehat{P}_l is homogeneous and isotropic.*

Proof. It suffices to show that for all $A \in \mathcal{B}(L^2(0, T; \mathcal{H}^0(r)))$:

$$\widehat{S}_l^{-1} R_\omega A = R_\omega \widehat{S}_l^{-1} A \quad (3.56)$$

and

$$\widehat{S}_l^{-1} T_h A = T_h \widehat{S}_l^{-1} A, \quad (3.57)$$

since $\widehat{P}_l(A) = \widehat{\mu}_l(\widehat{S}_l^{-1}A)$ and $\widehat{\mu}_l$ is homogeneous and isotropic.

Using (3.54), (3.49),

$$\begin{aligned}
\widehat{S}_l^{-1}R_\omega A &= \gamma_0(R_\omega A \cap R_\omega R_{\omega^{-1}}\widehat{S}_l\widehat{\mathcal{M}}_l) \\
&= \gamma_0 R_\omega(A \cap \widehat{S}_l R_{\omega^{-1}}\widehat{\mathcal{M}}_l) \\
&= R_\omega \gamma_0(A \cap \widehat{S}_l\widehat{\mathcal{M}}_l) \\
&= R_\omega \widehat{S}_l^{-1}A.
\end{aligned} \tag{3.58}$$

Also, by (3.54), (3.52)

$$\begin{aligned}
\widehat{S}_l^{-1}T_h A &= \gamma_0(T_h A \cap T_h T_{-h}\widehat{S}_l\widehat{\mathcal{M}}_l) \\
&= \gamma_0 T_h(A \cap \widehat{S}_l T_{-h}\widehat{\mathcal{M}}_l) \\
&= T_h \gamma_0(A \cap \widehat{S}_l\widehat{\mathcal{M}}_l) \\
&= T_h \widehat{S}_l^{-1}A. \quad \square
\end{aligned}$$

The following relation between P_l and \widehat{P}_l allows the known estimates on for P_l to be carried over to \widehat{P}_l . Once again, let a to be the action map of rotations on vector fields:

$$a(\omega, u) = R_\omega u \tag{3.59}$$

Lemma 3.3.2. *The equality holds:*

$$\widehat{P}_l(A) = (P_l \times H)(a^{-1}A).$$

where H is the Haar measure on $O(3)$, normalized.

Proof.

$$\begin{aligned}
\widehat{P}_l(A) &= \widehat{\mu}_l(\widehat{S}_l^{-1}A) \\
&= (\mu_l \times H)(a^{-1}\widehat{S}_l^{-1}A) \\
&= (\mu_l \times H)\{(u_0, \omega) \in \mathcal{M}_l \times O(3) : R_\omega u_0 \in \widehat{S}_l^{-1}A\} \\
&= (\mu_l \times H)\{(u_0, \omega) \in \mathcal{M}_l \times O(3) : \widehat{S}_l R_\omega u_0 \in A\} \tag{3.60} \\
&= (\mu_l \times H)\{(u_0, \omega) \in \mathcal{M}_l \times O(3) : R_\omega S_l u_0 \in A\}, \text{ by (3.48)} \\
&= (P_l \times H)\{(S_l u_0, \omega) \in C^1(0, T; \mathcal{M}_l) \times O(3) : R_\omega S_l u_0 \in A\} \\
&= (P_l \times H)(a^{-1}A).
\end{aligned}$$

□

Chapter 4

Homogeneous and Isotropic Statistical Solution of the Navier-Stokes Equations via Galerkin Approximations

The construction of homogeneous measures in [VF1], [VF] applies to show that the Galerkin statistical solutions of the previous chapter approximate homogeneous and isotropic probability measures supported by weak solutions of the Navier-Stokes equations. The restriction of the measures at $t = 0$ is well defined and coincides with the initial measure. It is then shown that the Navier-Stokes statistical solution obtained this way coincides with the statistical solution of Chapter 2.

4.1 Convergence of isotropic Galerkin approximations

4.1.1 Galerkin Approximation of Homogeneous Statistical Solutions

This section summarizes the Galerkin approximation in [VF]. Recall that the following are shown in Chapter VII there:

Given any μ homogeneous probability measure on $\mathcal{H}^0(r)$, there exist for each l

homogeneous probability measures μ_l on $\mathcal{H}^0(r)$, supported on \mathcal{M}_l , converging to μ in characteristic i.e.:

$$\int_{\mathcal{H}^0(r)} e^{i\langle u, \nu \rangle} \mu_l(du) \rightarrow \int_{\mathcal{H}^0(r)} e^{i\langle u, \nu \rangle} \mu(du), \quad l \rightarrow \infty, \quad (4.1)$$

for any test function ν .

Convergence.

The probability measures P_l , defined by (3.55) are homogeneous in x if the initial μ is homogeneous, and converge weakly to a homogeneous probability measure P on $L^2(0, T; \mathcal{H}^0(r))$. Weak convergence relies on the following three uniform estimates on P_l : There are constants $C, C(N)$ independent of l such that for each t in $[0, T]$,

$$\int \left(|u(t, x)|^2 + \int_0^t |\nabla u|^2(\tau, x) d\tau \right) P_l(du) \leq C \int |u_0(x)|^2 \mu_l(du_0), \quad (4.2)$$

$$\int \|u|_{B_N}\|_{-s} P_l(du) \leq C(N), \quad (4.3)$$

$$\int \|\partial_t u|_{B_N}\|_{-s} P_l(du) \leq C(N) \quad (4.4)$$

for $\|v|_{B_N}\|_{-s}$ the dual norm (2.2), and with $s > 11/2$.

P is a homogeneous statistical solution of the Navier-Stokes system.

As already remarked, the weak limit P is supported on $L^2(0, T; \mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS}$ so that each $u(., .)$ in the support of P satisfies the weak form (2.1) of Navier-Stokes equations, and the right limit in time $\gamma_t u$ exists for each t with respect to the Φ^{-s} norm. This extra regularity of P relies on the estimate

$$\int \|u\|_{BV^{-s}} P(du) \leq \infty, \quad s > 11/2 \quad (4.5)$$

for $\|u\|_{BV^{-s}}$ the norm (2.3). In addition, P satisfies energy estimate (2.12).

P solves the initial value problem

The measure defined by $P(\gamma_0^{-1}A)$, for A Borel of $\mathcal{H}^0(r)$, is the initial measure μ . The proof of this relies on the convergence (4.1).

4.1.2 Homogeneous and Isotropic Statistical Solutions via isotropic Galerkin Approximations

In the setting of isotropic measures, the construction of the previous subsection can be repeated as follows:

Approximation of initial measure.

Given initial homogeneous and isotropic $\widehat{\mu}$ on $\mathcal{H}^0(r)$, construct its (merely) homogeneous approximation μ_l , with $\mu_l \rightarrow \widehat{\mu}$ in characteristic as in the previous section. Then use Definitions 3.1.1 and 3.1.2 to obtain from the μ_l 's probability measures $\widehat{\mu}_l$ that are now homogeneous and isotropic and supported by $\widehat{\mathcal{M}}_l$. Convergence in characteristic still holds, thanks to the following:

Lemma 4.1.1. $\widehat{\mu}_l$ converges to $\widehat{\mu}$ in characteristic as $l \rightarrow \infty$.

Proof. For $\widehat{\mu}_l$ defined as above,

$$\begin{aligned}
\chi_{\widehat{\mu}_l}(\nu) &= \int_{\mathcal{H}^0(r)} e^{iu \cdot \nu} \widehat{\mu}_l(du) \\
&= \int_{\mathcal{H}^0(r)} \int_{O(3)} e^{iR_\omega u \cdot \nu} d\omega \mu_l(du) \\
&= \int_{O(3)} \int_{\mathcal{H}^0(r)} e^{iu \cdot R_{\omega^{-1}} \nu} \mu_l(du) d\omega \\
&= \int_{O(3)} \chi_{\mu_l}(R_{\omega^{-1}} \nu) d\omega.
\end{aligned} \tag{4.6}$$

Since $|\chi_{\mu_l}(R_{\omega^{-1}} \nu)| \leq 1$ and $\int_{O(3)} 1 d\omega = 1$, (4.1) implies, via the Lebesgue Dominated Convergence Theorem,

$$\chi_{\widehat{\mu}_l}(\nu) = \int_{O(3)} \chi_{\mu_l}(R_{\omega^{-1}} \nu) d\omega \rightarrow \int_{O(3)} \chi_{\widehat{\mu}}(R_{\omega^{-1}} \nu) d\omega \tag{4.7}$$

as $l \rightarrow \infty$. Therefore, since

$$\int_{O(3)} \chi_{\widehat{\mu}}(R_{\omega^{-1}\nu}) d\omega = \int_{O(3)} \int_{\mathcal{H}^0(r)} e^{iR_{\omega}u \cdot \nu} \widehat{\mu}(du) d\omega = \chi_{\widehat{\mu}}(\nu), \quad (4.8)$$

$\chi_{\widehat{\mu}_l}(\nu) \rightarrow \chi_{\widehat{\mu}}(\nu)$ as $l \rightarrow \infty$. □

Convergence to a homogeneous and isotropic measure

Now construct the homogeneous and isotropic measures \widehat{P}_l according to (3.53). To prove weak convergence the following is needed

Lemma 4.1.2. *The following estimates hold:*

$$\int \|u|_{B_N}\|_{-s} \widehat{P}_l(du) \leq C(N), \quad (4.9)$$

$$\int \|\partial_t u|_{B_N}\|_{-s} \widehat{P}_l(du) \leq C(N). \quad (4.10)$$

Proof. By Lemma 3.3.2 and relations (2.19),

$$\begin{aligned} \int \|\partial_t u|_{B_N}\|_{-s} \widehat{P}_l(du) &= \int \int_{O(3)} \|R_{\omega} \partial_t u|_{B_N}\|_{-s} d\omega P_l(du) \\ &= \int \|\partial_t u|_{B_N}\|_{-s} P_l(du). \end{aligned} \quad (4.11)$$

[VF] Chapter VII, Theorem 3.1 proves that the right hand side of (4.11) is bounded by a constant $C(N)$ that depends on N but does not depend on l . This proves (4.10).

The bound (4.9) is proved similarly. □

The following is also needed to repeat the proof of the weak convergence of the \widehat{P}_l 's.

Lemma 4.1.3. *With pointwise averages defined as in (2.9), (2.10), for any t in $[0, T]$,*

$$\begin{aligned} \int \left(|u|^2(t, x) + \int_0^t (|u|^2(\tau, x) + |\nabla u|^2(\tau, x)) d\tau \right) \widehat{P}_l(du) \\ \leq C \int |u|^2(x) \widehat{\mu}_l(du). \end{aligned} \quad (4.12)$$

Proof. Using an equality similar to (2.10) one can show that

$$\int |u|^2(t, x) \widehat{\mu}_l(du) = \int_{O(3)} \int |u|^2(t, x) (R_\omega)_\# \mu_l(du) d\omega, \quad (4.13)$$

$$\begin{aligned} & \int \left(|u|^2(t, x) + \int_0^t (|u|^2(t, x) + |\nabla u|^2(t, x)) d\tau \right) \widehat{P}_l(du) \\ &= \int_{O(3)} \int \left(|u|^2(t, x) + \int_0^t (|u|^2(t, x) + |\nabla u|^2(t, x)) d\tau \right) (R_\omega)_\# P_l(du) d\omega. \end{aligned} \quad (4.14)$$

Applying to (4.13), (4.14) Lemma 2.3.2 and taking into account the inequality for Galerkin approximations of the homogeneous statistical solution

$$\begin{aligned} & \int \left(|u|^2(t, x) + \int_0^t (|u|^2(t, x) + |\nabla u|^2(t, x)) d\tau \right) P_l(du) \\ & \leq C \int |u|^2(x) \mu_l(du) \end{aligned} \quad (4.15)$$

that was proved in [VF] Chapter VII, Lemma 2.4, yields (4.12). \square

Lemma 4.1.4. *The measures $\widehat{\mu}_l$ satisfy*

$$\int |u|^2(x) \widehat{\mu}_l(du) \leq \int |u|^2(x) \mu_l(du) \quad (4.16)$$

Proof. This follows from (4.13) and Lemma 2.3.2. \square

With Lemmas 4.1.2, 4.1.3, 4.1.4 established, there are no further obstacles in repeating the arguments in [VF] to show that the family \widehat{P}_l is weakly compact. The argument for this in [VF] is that the measures P_l are supported on the space Ω of elements in $L^2(0, T; \mathcal{H}^0(r))$ with the following norm finite:

$$\begin{aligned} & \sum_{N=1}^{\infty} \frac{1}{2^N C(N)} (\|u\|_{L^2(0, T; \mathcal{H}^{-s}(B_N))} + \|\frac{\partial u}{\partial t}\|_{L^1(0, T; \mathcal{H}^{-s}(B_N))} + \|u\|_{L^2(0, T; \mathcal{H}^1(r_1))}) < \infty, \\ & s > 11/2, \quad r < r_1 < -\frac{3}{2}. \end{aligned} \quad (4.17)$$

Then Ω is compactly embedded in $L^2(0, T; \mathcal{H}^0(r))$, [VF], Chapter VII, Lemma 5.2, and

$$\sup_l \int \|u\|_\Omega P_l(du) < \infty, \quad (4.18)$$

[VF], proof of Theorem 6.1 in Chapter VII. From the estimates of Lemmas 4.1.2, 4.1.3 for the measures \widehat{P}_l , the energy conservation (4.12), and the uniform estimate (4.1.4), it becomes clear that the all measures \widehat{P}_l are supported on Ω and the uniform estimate (4.18), with P_l changed to \widehat{P}_l , still holds.

Let \widehat{Q} be the limit of some weakly convergent subsequence of \widehat{P}_l 's, as $l \rightarrow \infty$.

Lemma 4.1.5. *If \widehat{P}_l are homogeneous and isotropic and $\widehat{P}_l \Rightarrow \widehat{Q}$ weakly on $L^2(0, T; \mathcal{H}^0(r))$, then \widehat{Q} is homogeneous and isotropic.*

Proof. If $\widehat{P}_l \Rightarrow \widehat{Q}$ weakly then by definition

$$\int f(u) \widehat{P}_l(du) \rightarrow \int f(u) \widehat{Q}(du). \quad (4.19)$$

for any f continuous and bounded on $L^2(0, T; \mathcal{H}^0(r))$. Since the measures \widehat{P}_l are homogeneous and isotropic, by definitions 1.2.2, 1.3.3

$$\begin{aligned} \int f(u) \widehat{P}_l(du) &= \int f(T_h u) \widehat{P}_l(du) \\ &= \int f(R_\omega u) \widehat{P}_l(du), \end{aligned} \quad (4.20)$$

for any $h \in \mathbb{R}^3$ and $\omega \in O(3)$. These equalities and

$$\begin{aligned} \int f(T_h u) \widehat{P}_l(du) &\rightarrow \int f(T_h u) \widehat{Q}(du), \\ \int f(R_\omega u) \widehat{P}_l(du) &\rightarrow \int f(R_\omega u) \widehat{Q}(du), \end{aligned} \quad (4.21)$$

imply

$$\begin{aligned} \int f(u) \widehat{Q}(du) &= \int f(T_h u) \widehat{Q}(du) \\ &= \int f(R_\omega u) \widehat{Q}(du). \end{aligned} \quad (4.22)$$

□

Extra regularity for right t -limits and the initial condition

That the support of \widehat{Q} is in addition in $L^2(0, T; \mathcal{H}^1(r)) \cap BV^{-s}$ (where right limits with respect to time are well defined by [VF], Chapter VII, Lemma 8.2) uses only the estimate

$$\int \|u\|_{BV^{-s}} \widehat{Q}(du) \leq \infty \quad (4.23)$$

for $\|u\|_{BV^{-s}}$ the norm (2.3), which follows as in the proof of Lemma 4.1.2. Define therefore γ_0 by (2.6) and think of $(\gamma_0)_\# \widehat{Q}$ as the initial value of \widehat{Q} .

That the initial value $(\gamma_0)_\# P$ of the the homogeneous statistical solution P of Theorem 2.2.2 is the initial measure μ is shown in [VF] as Theorem 10.1, Lemma 10.1, Lemma 10.2, and Theorem 10.2 of Chapter VII there. Of these, Lemma 10.1, Lemma 10.2, and Theorem 10.2 of Chapter VII are valid verbatim for \widehat{Q} . Theorem 10.1 uses only the convergence in characteristic of the μ_l 's to μ . The analogous convergence of the $\widehat{\mu}_l$'s to $\widehat{\mu}$ was established here as Lemma 4.1.1.

4.2 The support of the measure \widehat{Q} .

This section uses the approach of [VF] Chapter VII, Section 7, to show that the homogeneous and isotropic measure \widehat{Q} is supported by generalized solutions of the Navier-Stokes equations. To realize this approach, some subtle points regarding the definition of the equations for isotropic Galerkin approximations in the x -representation need to be addressed first.

4.2.1 Equations for isotropic Galerkin approximations in the x -representation

Chapter 3 defined isotropic Galerkin approximations by introducing and investigating the Galerkin equations in terms of Fourier coefficients. Here, a complete description of the Galerkin equations in the x -representation is given, beginning with a more precise determination of the domain of their definition.

In addition to the sets \mathcal{M}_l and $\widehat{\mathcal{M}}_l$ defined by (3.1), (3.3), introduce the set of periodic vector fields \mathcal{N}_l with the cube of periods Π_l defined by (3.13):

$$\mathcal{N}_l = \{u(x) = (u_1, u_2, u_3) \in L^2(\Pi_l) : \operatorname{div} u(x) = 0\}, \quad (4.24)$$

for $\operatorname{div} u$ understood in the weak sense, see (1.11). Also define the space

$$\widehat{\mathcal{N}}_l = \cup_{\omega \in O(3)} R_\omega \mathcal{N}_l \quad (4.25)$$

which, of course, is not linear. Since $\widehat{\mathcal{N}}_l \subset \mathcal{H}^0(r)$ for $r < -3/2$, $\widehat{\mathcal{N}}_l$ is a metric space with the metric generated by the norm of $\mathcal{H}^0(r)$. Now use the set $C^1(0, T; \widehat{\mathcal{N}}_l)$ to define the Galerkin equation, recalling that this equation was defined in (3.17) only for $u \in C^1(0, T; \mathcal{M}_l) \subset C^1(0, T; \mathcal{N}_l)$. To extend this definition from $C^1(0, T; \mathcal{M}_l)$ to $C^1(0, T; \widehat{\mathcal{M}}_l)$, first extend the operator p_l to \mathcal{N}_l as in (3.16):

$$p_l : \mathcal{N}_l \rightarrow \mathcal{M}_l; \quad \mathcal{N}_l \ni u(x) = \sum_{k \in \frac{\pi}{l} \mathbb{Z}^3} a_k e^{ikx} \mapsto p_l u(x) = \sum_{k \in \frac{\pi}{l} \mathbb{Z}^3, |k| \leq l} a_k e^{ikx} \in \mathcal{M}_l. \quad (4.26)$$

For each $\omega \in O(3)$ the operator p_l induces operator

$$p_{l,\omega} = R_\omega p_l R_\omega^{-1} : R_\omega \mathcal{N}_l \rightarrow R_\omega \mathcal{M}_l. \quad (4.27)$$

The family of operators $p_{l,\omega}$, $\omega \in O(3)$ defines the operator

$$\widehat{p}_l : \widehat{\mathcal{N}}_l \rightarrow \widehat{\mathcal{M}}_l \quad (4.28)$$

as follows: Since for each $u \in \widehat{\mathcal{N}}_l$ there exist $\omega \in O(3)$ and $v \in \mathcal{N}_l$ such that $u = R_\omega v$, define $\widehat{p}_l u = p_{l,\omega} u = R_\omega p_l v$. This is well defined by the obvious extension of Proposition 3.2.1 to infinite series.

Similarly, the projection operator $\pi_l : L^2(\Pi_l) \rightarrow \mathcal{N}_l$ of periodic vector fields onto solenoidal periodic vector fields yields an operator on $R_\omega L^2(\Pi_l)$ by

$$\pi_{l,\omega} = R_\omega \pi_l R_\omega^{-1} : R_\omega L^2(\Pi_l) \rightarrow R_\omega \mathcal{N}_l, \quad (4.29)$$

and the family $\pi_{l,\omega}$, $\omega \in O(3)$ defines the operator

$$\widehat{\pi}_l : \cup_{\omega \in O(3)} R_\omega L^2(\Pi_l) \rightarrow \widehat{\mathcal{N}}_l \quad (4.30)$$

by

$$\cup_{\omega \in O(3)} R_\omega L^2(\Pi) \ni u = R_{\omega_0} v \rightarrow \widehat{\pi}_l u = \pi_{l,\omega_0} u = R_{\omega_0} \pi_l v. \quad (4.31)$$

The remaining operators Δ and ∇ in equation (3.17), are already well defined on the larger space $C^1(0, T; \mathcal{H}^2(r)) \supset C^1(0, T; \widehat{\mathcal{M}}_l)$ and therefore need not be redefined specifically for $C^1(0, T; \widehat{\mathcal{M}}_l)$.

Thus, the Galerkin equation on the set $C^1(0, T; \widehat{\mathcal{M}}_l)$ is now defined as follows:

$$\partial_t u - \Delta u + \widehat{p}_l \widehat{\pi}_l [(u, \nabla) u] = 0, \text{ where } u = u(t, x) \in C^1(0, T; \widehat{\mathcal{M}}_l) \quad (4.32)$$

Equation (4.32) is the x -representation of the Galerkin equation that was written in terms of Fourier coefficients and was studied in Chapter 3. In particular, the resolving operator $\widehat{S}_l v$ of the Cauchy problem for this equation was defined in (3.47).

4.2.2 Definition and estimates on a functional

Let

$$v(t, x) \in G^\infty \equiv C_0^\infty((0, T) \times \mathbb{R}^3) \cap C(0, T; \mathcal{H}^0(r)) \quad (4.33)$$

be a vector field and $B \subset \mathbb{R}^3$ be a ball with center at the origin, satisfying

$$\text{supp } v(t, x) \subset B \quad \forall t \in [0, T] \quad . \quad (4.34)$$

Then there exists $l_0 > 0$ such that for each $l \geq l_0$

$$B \subset \bigcap_{\omega \in O(3)} \omega \Pi_l. \quad (4.35)$$

From now on consider only $l \geq l_0$. For every such l and for each $\omega \in O(3)$ extend $v(t, x)|_{(0, T) \times \omega \Pi_l}$ from $(0, T) \times \omega \Pi_l$ into $(0, T) \times \mathbb{R}^3$ as a periodic in x vector field $v_{l, \omega}(t, x)$ with cube of periods $\omega \Pi_l$. Denote the family of functions $v_{l, \omega}(t, x)$, $\omega \in O(3)$ by $\widehat{v}_l(t, x)$.

Note that (4.24) implies

$$R_\omega \mathcal{N}_l = \{v(x) = (v_1, v_2, v_3) \in L^2(\omega \Pi_l) : \text{div } v(x) = 0\}. \quad (4.36)$$

For $u, v \in L^2(0, T; R_\omega \mathcal{N}_l)$, define

$$[u, v]_{l, \omega} = \int_0^T \int_{\omega \Pi_l} u(t, x) \cdot v(t, x) \, dx \, dt. \quad (4.37)$$

Finally, for $v \in G^\infty$ as above define, for each $j = 1, 2, 3$, the functional $F_{j, v}$ on $C^1(0, T; \widehat{\mathcal{M}}_l)$ as follows: If $u \in C^1(0, T; \widehat{\mathcal{M}}_l)$, with $u = (u_1, u_2, u_3) = \omega \tilde{u}$ for $\omega \in O(3)$, $\tilde{u} \in C^1(0, T; \mathcal{M}_l)$, then

$$F_{j, v}(u) = [(I - p_{l, \omega}) \pi_{l, \omega}(u_j u), v_{l, \omega}]_{l, \omega}, \quad (4.38)$$

for I the identity operator.

Lemma 4.2.1. *For each $u \in C^1(0, T; \widehat{\mathcal{M}}_l)$ the functional (4.38) satisfies, for all j , the estimate :*

$$|F_{j, v}(u)| \leq C(1 + 3l^2)^{-1} \|u\|_{L^2(0, T; \mathcal{H}^0(r))} \|v\|_{C(0, T; H^{s-r+1}(B))}, \quad (4.39)$$

where $s > 3/2$, $r < -3/2$, and $C > 0$ does not depend on l , u , or v .

Proof. Using the decomposition in Fourier series one sees that the operators $p_{l,\omega}$ and $\pi_{l,\omega}$ commute: $p_{l,\omega}\pi_{l,\omega} = \pi_{l,\omega}p_{l,\omega}$. In addition, the operators $p_{l,\omega}$ and $\pi_{l,\omega}$ are symmetric. Therefore, with $\operatorname{div} v_{l,\omega} = 0$, obtain:

$$\begin{aligned} F_{j,v}(u) &= [(I - p_{l,\omega})(u_j u), \pi_{l,\omega} v_{l,\omega}]_{l,\omega} \\ &= [u_j u, (I - p_{l,\omega})v_{l,\omega}]_{l,\omega}. \end{aligned} \quad (4.40)$$

Using the Sobolev Embedding Theorem: $H^s(\omega\Pi_l) \subset C(\omega\Pi_l)$ for $s > 3/2$, and (4.40),

$$\begin{aligned} |F_{j,v}(u)| &\leq C_1 \|u\|_{L^2(0,T;H^0(\omega\Pi_l))}^2 \sup_{[0,T] \times \omega\Pi_l} |(I - p_{l,\omega})v_{l,\omega}| \\ &\leq C \|u\|_{L^2(0,T;H^0(\omega\Pi_l))}^2 \|(I - p_{l,\omega})v_{l,\omega}\|_{C(0,T;H^s(\omega\Pi_l))}, \end{aligned} \quad (4.41)$$

with $s > 3/2$. Clearly, for periodic vector fields with cube of periods $\omega\Pi_l$

$$\|u\|_{L^2(0,T;H^0(\omega\Pi_l))}^2 \leq (1 + 3l^2)^{-r} \|u\|_{L^2(0,T;\mathcal{H}^0(r))}^2, \quad \text{for } r < 0. \quad (4.42)$$

Decomposition in Fourier series yields:

$$\begin{aligned} \|(I - p_{l,\omega})v_{l,\omega}\|_{C(0,T;H^s(\omega\Pi_l))} &= \sup_{t \in [0,T]} \sum_{m \in \frac{\pi}{l}\omega\mathbb{Z}^3, m > l} |\hat{v}(m)|^2 (1 + |m|^2)^s \\ &\leq (1 + 3l^2)^{r-1} \sup_{t \in [0,T]} \|v_{l,\omega}(t, \cdot)\|_{H^{s-r+1}(\omega\Pi_l)}^2 \\ &\leq (1 + 3l^2)^{r-1} \|v\|_{C(0,T;H^{s-r+1}(B))}, \end{aligned} \quad (4.43)$$

where the last inequality holds by (4.35). Now (4.39) follows from (4.41), (4.42), (4.43). \square

4.2.3 The main result

The goal of this subsection is to prove

Theorem 4.2.2. *Let \widehat{Q} be a weak limit of isotropic Galerkin approximations \widehat{P}_l of statistical solution as $l \rightarrow \infty$. Then \widehat{Q} is supported on weak solutions of Navier-Stokes equations, i.e. on the set \mathcal{G}_{NS} of Definition 2.1.1*

Proof. Let the function $\varphi_R(\lambda) \in C^\infty(\mathbb{R}_+)$ satisfy

$$\varphi_R(\lambda) = \begin{cases} 1, & \lambda \leq R \\ 0, & \lambda \geq R + 1. \end{cases} \quad (4.44)$$

For v, ψ in G^∞ (see (4.33)), construct the families $\widehat{v}_{l,\cdot} = v_{l\omega}(t, x)$, $\widehat{\psi}_{l,\cdot} = \psi_{l\omega}(t, x)$ as explained immediately after relation (4.35). Analogously to (4.38), define the following functionals on $C^1(0, T; \widehat{\mathcal{M}}_l)$: If $u = \omega \tilde{u}$ for $\omega \in O(3)$ and $\tilde{u} \in C^1(0, T; \mathcal{M}_l)$, then

$$u \rightarrow [u, \widehat{\psi}_{l,\cdot}]_{l,\cdot} = [u, \psi_{l,\omega}]_{l,\omega}, \quad (4.45)$$

and

$$u \rightarrow L_l(u, \widehat{v}_l) = [u, \partial_t v_{l,\omega} + \Delta v_{l,\omega}]_{l,\omega} + \sum_{j=1}^3 [p_{l,\omega} \pi_{l,\omega}(u_j u), \frac{\partial v_{l,\omega}}{\partial x_j}]_{l,\omega}. \quad (4.46)$$

By the definition (3.53) of the isotropic Galerkin approximations \widehat{P}_l of a statistical solution and taking into account that $\widehat{S}_l v$ is the solution operator of the Galerkin equations (4.32), the measure \widehat{P}_l is supported by solutions of (4.32) belonging to $C^1(0, T; \widehat{\mathcal{M}}_l)$. Therefore, comparing (4.32) to (4.46),

$$\int_{C^1(0, T; \widehat{\mathcal{M}}_l)} \varphi_R(\|u\|_{L^2(0, T; \mathcal{H}^0(r))}) L_l(u, \widehat{v}_l) e^{i[u, \widehat{\psi}_l]_{l,\cdot}} \widehat{P}_l(du) = 0 \quad (4.47)$$

for each $v, \psi \in G^\infty$ and $l \geq l_0$, with l_0 defined by (4.35), and with the ball B now containing both the support of v and ψ .

Let

$$[u, v] = \int_0^T \int_{\mathbb{R}^3} u(t, x) \cdot v(t, x) \, dx dt. \quad (4.48)$$

Since v and ψ satisfy (4.34), then

$$[u, \widehat{v}_{l,\cdot}]_{l,\cdot} = [u, v], \quad [u, \widehat{\psi}_{l,\cdot}]_{l,\cdot} = [u, \psi] \quad \forall u \in C^1(0, T; \widehat{\mathcal{M}}_l), \quad (4.49)$$

where $[u, \widehat{\psi}_l, \cdot]_{l, \cdot}, [u, \psi]$ are defined by (4.45), (4.48) respectively. Therefore,

$$L_l(u, \widehat{v}_l, \cdot) = L(u, v) - \sum_{j=1}^3 F_{j, \frac{\partial v}{\partial x_j}}(u) \quad \forall u \in C^1(0, T; \widehat{\mathcal{M}}_l), \quad (4.50)$$

where $L_l(u, \widehat{v}_l, \cdot)$, $L(u, v)$, and $F_{j, v}(u)$ are defined by (4.46), (2.1), and (4.38) respectively.

Substitution of (4.49), (4.50) into (4.47) yields:

$$\begin{aligned} & \int \varphi_R(\|u\|_{L^2(0, T; \mathcal{H}^0(r))}) L(u, v) e^{i[u, \psi]} \widehat{P}_l(du) \\ & \quad - \sum_{j=1}^3 \int \varphi_R(\|u\|_{L^2(0, T; \mathcal{H}^0(r))}) F_{j, \frac{\partial v}{\partial x_j}}(u) e^{i[u, \psi]} \widehat{P}_l(du) = 0. \end{aligned} \quad (4.51)$$

Since the integrand of the first integral in (4.51) can be extended to a bounded continuous functional on $L^2(0, T; \mathcal{H}^0(r))$, the of weak convergence $\widehat{P}_l \rightarrow \widehat{Q}$ on this space gives:

$$\int \varphi_R(\|u\|_{L^2(0, T; \mathcal{H}^0(r))}) L(u, v) e^{i[u, \psi]} \widehat{P}_l(du) \rightarrow \int \varphi_R(\|u\|_{L^2(0, T; \mathcal{H}^0(r))}) L(u, v) e^{i[u, \psi]} \widehat{Q}(du), \quad (4.52)$$

as $l \rightarrow \infty$. Moreover, by Lemma 4.2.1, for $j = 1, 2, 3$,

$$\begin{aligned} & \left| \int \varphi_R(\|u\|_{L^2(0, T; \mathcal{H}^0(r))}) F_{j, \frac{\partial v}{\partial x_j}}(u) e^{i[u, \psi]} \widehat{P}_l(du) \right| \\ & \leq C \frac{\|\nabla v\|_{L^2(0, T; H^{s-r+1}(B))}}{(1 + 3l^2)} \int \varphi_R(\|u\|_{L^2(0, T; \mathcal{H}^0(r))}) \|u\|_{L^2(0, T; \mathcal{H}^0(r))} \widehat{P}_l(du), \end{aligned} \quad (4.53)$$

therefore

$$\left| \sum_{j=1}^3 \int \varphi_R(\|u\|_{L^2(0, T; \mathcal{H}^0(r))}) F_{j, \frac{\partial v}{\partial x_j}}(u) e^{i[u, \psi]} \widehat{P}_l(du) \right| \rightarrow 0, \quad l \rightarrow \infty. \quad (4.54)$$

Equations (4.51), (4.52), (4.54) imply

$$\int \varphi_R(\|u\|_{L^2(0, T; \mathcal{H}^0(r))}) L(u, v) e^{i[u, \psi]} \widehat{Q}(du) = 0 \quad \forall v, \psi \in G^\infty. \quad (4.55)$$

With this, repeat the arguments from [VF], p. 243-244, to derive from (4.55) the assertion of the theorem. \square

As a final result obtain the following theorem:

Theorem 4.2.3. \widehat{Q} is a homogeneous and isotropic statistical solution of the Navier-Stokes equations with initial condition $\widehat{\mu}$.

4.3 Comparison of \widehat{P} and \widehat{Q}

To conclude, observe that the isotropic solutions of Chapter 2 and the isotropic solutions constructed in Chapter 3, sections 4.1, and 4.2 coincide in the following sense:

Let P be a homogeneous statistical solution with initial condition $\widehat{\mu}$ according to Theorem 2.2.2, and let \widehat{P} its isotropic average according to (2.35). (Whether $\widehat{\mu}$ is only homogeneous or not is irrelevant to the point about to be made.) The construction of P is via Galerkin approximations, therefore P is the weak limit of a sequence of homogeneous P_l 's. Construct the corresponding \widehat{P}_l for each l according to (3.53).

Probability measures on metric spaces are determined by their integrals on bounded and continuous functions, see [B], page 8. For the probability measures \widehat{P} and \widehat{Q} , calculate for each f continuous and bounded on $L^2(0, T; \mathcal{H}^0(r))$:

$$\begin{aligned}
\int_{L^2(0, T; \mathcal{H}^0(r))} f(u) \widehat{P}(du) &= \int_{O(3)} \int_{L^2(0, T; \mathcal{H}^0(r))} f(R_\omega u) P(du) d\omega \quad (\text{by (2.35)}) \\
&= \lim_{l \rightarrow \infty} \int_{O(3)} \int_{L^2(0, T; \mathcal{H}^0(r))} f(R_\omega u) P_l(du) d\omega \\
&\quad (\text{by weak and dominated convergence}) \\
&= \lim_{l \rightarrow \infty} \int_{L^2(0, T; \mathcal{H}^0(r))} f(u) \widehat{P}_l(du) \quad (\text{by Lemma 3.3.2}) \\
&= \int f(u) \widehat{Q}(du).
\end{aligned} \tag{4.56}$$

Chapter 5

Approximation of Homogeneous and Isotropic Measures in the 2-Wasserstein Metric

It is shown that, on certain weighted spaces of vector fields on \mathbb{R}^3 , any homogeneous and isotropic measure of finite energy density and dissipation can be approximated in the second Wasserstein distance by homogeneous and isotropic measures supported by finite trigonometric polynomials of increasing period and degree. In particular, the periodic correlation functions of the approximation converge uniformly on compact sets of \mathbb{R}^3 to the correlation function of the given measure.

5.1 Spaces and measures

5.1.1 Compact embeddings of $\mathcal{H}^k(r)$ spaces

Recall the spaces $\mathcal{H}^k(r)$ defined in Definition 1.2.3.

Lemma 5.1.1. *$\mathcal{H}^1(r')$ compactly embeds into $\mathcal{H}^0(r)$ for $r < r'$.*

Proof. For any u in the ball of radius M in $\mathcal{H}^1(r')$

$$\int_{\mathbb{R}^3} (1 + |x|^2)^{r'} |u(x)|^2 dx < M. \tag{5.1}$$

Then for any $\epsilon > 0$ there exists an R_ϵ such that for all $R > R_\epsilon$,

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus B_R} (1 + |x|^2)^r |u_n(x)|^2 dx &= \int_{\mathbb{R}^3 \setminus B_R} (1 + |x|^2)^{r-r'} (1 + |x|^2)^{r'} |u_n(x)|^2 dx \\ &\leq (1 + R^2)^{r-r'} M < \frac{\epsilon}{2}, \end{aligned} \quad (5.2)$$

for all such u .

At the same time, the restrictions $u|_{B_R}$ form a bounded set in $W^{1,2}(B_R)$, and therefore a precompact set in $L^2(B_R)$, by standard Sobolev embedding. In particular, [B], p. 239, there exist w_i in $L^2(B_R)$, $i = 1, 2, \dots, N(\epsilon)$, such that for any u as above there exists an i with

$$\int_{B_R} (1 + |x|^2)^r |u(x) - w_i(x)|^2 dx \leq \|u - w_i\|_{L^2(B_R)} < \frac{\epsilon}{2}. \quad (5.3)$$

Now extend each w_i trivially by setting it zero outside of B_R . Then

$$\begin{aligned} &\int_{\mathbb{R}^n} (1 + |x|^2)^r |u(x) - w_i(x)|^2 dx \\ &= \int_{B_R} (1 + |x|^2)^r |u(x) - w_i(x)|^2 dx + \int_{B'_R} (1 + |x|^2)^r |u(x)|^2 dx < \epsilon. \end{aligned} \quad (5.4)$$

Therefore the u 's form a precompact set in $\mathcal{H}^0(r)$. \square

5.1.2 Correlations

Use the homogeneity of the measure as in (2.10) for any $h \in \mathbb{R}^3$ and any $\phi \in L^1(\mathbb{R}^3)$

to see that there is $R_{ij}(h)$ such that

$$\int_{\mathcal{H}^0(r)} \int_{\mathbb{R}^3} u_i(x) u_j(x+h) \phi(x) dx \mu(du) = R_{ij}(h) \int_{\mathbb{R}^3} \phi(x) dx. \quad (5.5)$$

Call the function $h \mapsto R_{ij}(h)$ on \mathbb{R}^3 the (i, j) -th **correlation function** of μ .

Correlation functions are often defined as $\overline{u_i(x) u_j(x+h)}$, with the overline indicating some average. This corresponds here to the bilinear form

$$\int_{\mathcal{H}^0(r)} \langle u_i, \phi \rangle \langle u_j, \psi \rangle \mu(du), \quad (5.6)$$

for ϕ and ψ smooth, with compact supports concentrated around x and $x + h$ respectively, and for \langle, \rangle the L^2 -inner product. By Hölder, this is continuous on $\mathcal{H}^0(-r) \times \mathcal{H}^0(-r)$, for r still smaller than $-3/2$, as in Definition 1.2.3.

The following relates correlations as defined in (5.5) to the bilinear form (5.6) and will be used later. It is an elementary instance of the Kernel Theorem, cf. [GV], pp 167-169:

Lemma 5.1.2. *For μ homogeneous on $\mathcal{H}^0(r)$ with finite second moment, the following holds:*

$$\int_{\mathcal{H}^0(r)} \langle u_i, \phi \rangle \langle u_j, \psi \rangle \mu(du) = \int_{\mathbb{R}^3} R_{ij}(h) \int_{\mathbb{R}^3} \phi(x) \psi(x+h) dx dh \quad (5.7)$$

for any $\phi, \psi \in C_0^\infty(\mathbb{R}^3)$.

Proof. A simple change of variables, Fubini's Theorem (valid by the second moment assumption), and the definition of the correlation functions give:

$$\begin{aligned} & \int_{\mathcal{H}^0(r)} \langle u_i, \phi \rangle \langle u_j, \psi \rangle \mu(du) \\ &= \int_{\mathcal{H}^0(r)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u_i(x) \phi(x) u_j(y) \psi(y) dx dy \mu(du) \\ &= \int_{\mathcal{H}^0(r)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u_i(x) \phi(x) u_j(x+h) \psi(x+h) dx dh \mu(du) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathcal{H}^0(r)} u_i(x) u_j(x+h) \mu(du) \phi(x) \psi(x+h) dx dh, \\ &= \int_{\mathbb{R}^3} R_{ij}(h) \int_{\mathbb{R}^3} \phi(x) \psi(x+h) dx dh. \quad \square \end{aligned}$$

Remark 5.1.3. (*Regular Kernel.*) *For μ homogeneous with finite energy density and finite dissipation rate (assumptions that will be used below), the correlation functions R_{ij} are bounded and in C^2 with bounded derivatives, see [VF], Lemma VIII.7.3. When μ is the evaluation of a Navier-Stokes homogeneous statistical solution at some positive*

time decay at spatial infinity for the R_{ij} 's is expected, but not rigorously shown. The hydrodynamic pressure and the decay rate of the correlation of the initial measure are expected to determine the rate of decay for $t > 0$, cf. [BP], [L], [S].

5.1.3 Convergence in characteristic.

Recall the characteristic function of a measure μ on X

$$\chi_\mu(\phi) = \int_X e^{i\langle u, \phi \rangle} \mu(du), \quad (5.8)$$

for ϕ test function. Also recall that $\mu_n \rightarrow \mu$ **in characteristic** if

$$\chi_{\mu_n}(\phi) \rightarrow \chi_\mu(\phi), \quad (5.9)$$

for any ϕ . Finally recall that given μ and ν probability measures of finite first moments such that $\chi_\mu(\phi) = \chi_\nu(\phi)$ for all ϕ in a dense set in X , then $\chi_\mu(u) = \chi_\nu(u)$ for all u in X (and hence $\mu = \nu$): Indeed, given $u_0 \in X, \phi_n \rightarrow u$ in X ,

$$\int (e^{i\langle \phi_n, u \rangle} - e^{i\langle u_0, u \rangle}) \mu(du) \leq \|\phi_n - u_0\| \int \|u\| \mu(du). \quad (5.10)$$

5.2 General results

Theorem 5.2.1. *Let $\{\mu_l\}_{l>0}$ be a family of homogeneous measures on $\mathcal{H}^0(r), r < -3/2$, with*

$$\int (|u(x)|^2 + |\nabla u(x)|^2) \mu_l(du) \leq C \quad (5.11)$$

for all l and C independent of l . Then there is subsequence $\{\mu_{l(i)}\}_{i \in \mathbb{N}}$ converging weakly to some (necessarily homogeneous) measure μ on $\mathcal{H}^0(r)$, and if

$$\int |u(x)|^2 \mu_{l(i)}(du) \leq \int |u(x)|^2 \mu(du) < \infty, \quad i \in \mathbb{N}, \quad (5.12)$$

then $\mu_{l(i)} \rightarrow \mu$ in $W_2(\mathcal{H}^0(r))$.

Proof. Step 1: For some subsequence $l(i), i \in \mathbb{N}$, $\mu_{l(i)} \rightarrow \mu$ weakly as measures on $\mathcal{H}^0(r)$: For the given r , pick any r' satisfying $r < r' < -\frac{3}{2}$. By (5.11) and the definition of pointwise averages (2.10),

$$\int \|u\|_{\mathcal{H}^1(r')}^2 \mu_l(du) < +\infty, \quad (5.13)$$

for all l . Therefore all Borel subsets of $\mathcal{H}^0(r)$ with infinite $\mathcal{H}^1(r')$ norm have μ_l -measure zero, for any l , i.e. all μ_l 's are supported on $\mathcal{H}^1(r')$, for any such r' . Note that (5.11) implies that the homogeneous μ_l 's are supported in $\mathcal{H}^1(r)$, for any $r < -3/2$.

Given the compactness of the embedding $\mathcal{H}^1(r')$ into $\mathcal{H}^0(r)$ from Lemma 5.1.1, it suffices to show that for all l

$$\int \|u\|_{\mathcal{H}^1(r')} \mu_l < C, \quad (5.14)$$

for C independent of l , cf. Lemma II.3.1 in [VF], or Remark 5.1.5 in [AGS]. This follows from (5.11) and Hölder.

Now rename $l(i)$ to l .

Step 2: The following holds:

$$\int \|u\|_{\mathcal{H}^0(r)}^2 \mu_l(du) \rightarrow \int \|u\|_{\mathcal{H}^0(r)}^2 \mu(du), \quad l \rightarrow \infty. \quad (5.15)$$

Indeed, it is standard that the weak convergence of μ_l to μ as measures on $\mathcal{H}^0(r)$ implies that

$$\liminf_l \int \|u\|_{\mathcal{H}^0(r)}^2 \mu_l(du) \geq \int \|u\|_{\mathcal{H}^0(r)}^2 \mu(du). \quad (5.16)$$

Then (5.12) implies that in addition,

$$\limsup_l \int \|u\|_{\mathcal{H}^0(r)}^2 \mu_l(du) \leq \int \|u\|_{\mathcal{H}^0(r)}^2 \mu(du). \quad (5.17)$$

This in turn implies that the second moments

$$\int \|u\|_{\mathcal{H}^0(r)}^2 \mu_l(du) \quad (5.18)$$

are uniformly integrable in l , i.e.

$$\lim_{R \rightarrow \infty} \int_{\{\|u\| > R\}} \|u\|_{\mathcal{H}^0(r)}^2 \mu_l(du) \rightarrow 0, \quad (5.19)$$

uniformly in l , see Lemma 5.1.7, [AGS]. It is also standard that this uniform integrability implies (5.15), see same Lemma in [AGS]. \square

Remark 5.2.2. *By Hölder and (1.48), $W_2(\mathcal{H}^0(r))$ convergence implies $W_1(\mathcal{H}^0(r))$ convergence, hence convergence of expectations.*

Now convergence in $W_2(X)$ implies convergence of integrals of continuous functions of 2-growth, i.e.

$$\int f(u) \mu_l(du) \rightarrow \int f(u) \mu(du) \quad (5.20)$$

for any f satisfying $|f(u)| \leq C(\|u\|_X^2 + 1)$, see Proposition 7.1.5 and Lemma 5.1.7 of [AGS], or Theorem 7.12 of [V]. Since for each fixed test ϕ the function

$$u \rightarrow \langle u, \phi \rangle^2 \quad (5.21)$$

is a continuous function of 2-growth,

$$\begin{aligned} \int \langle u, \phi + \psi \rangle^2 \mu_l(du) &\rightarrow \int \langle u, \phi + \psi \rangle^2 \mu(du), \\ \int \langle u, \phi \rangle^2 \mu_l(du) &\rightarrow \int \langle u, \phi \rangle^2 \mu(du), \\ \int \langle u, \psi \rangle^2 \mu_l(du) &\rightarrow \int \langle u, \psi \rangle^2 \mu(du), \end{aligned} \quad (5.22)$$

hence

$$\int \langle u, \phi \rangle \langle u, \psi \rangle \mu_l(du) \rightarrow \int \langle u, \phi \rangle \langle u, \psi \rangle \mu(du), \quad (5.23)$$

for any ϕ, ψ test functions.

Then for the particular case of homogeneous measures, where correlation functions are defined, Wasserstein convergence implies the following:

Theorem 5.2.3. *Let μ_l, μ be as in Theorem 5.2.1. Then for each (i, j) there exists subsequence of the correlation functions R_{ij}^l which converges to the correlation function R_{ij} pointwise, uniformly on compact subsets of \mathbb{R}^3 .*

The proof of Theorem 5.2.3 uses the following:

Lemma 5.2.4. *Under the assumptions of Theorem 5.2.1, the correlation functions R_{ij}^l corresponding to the measures μ_l and their first derivatives are uniformly bounded in l .*

Proof. First recall that, as for any homogeneous measure,

$$\frac{\partial}{\partial h_k} R_{ij}^l(h) = \int \frac{\partial u_i}{\partial x_k}(x+h) u_j(x) \mu_l(du), \quad (5.24)$$

see Lemma VIII.7.2 of [VF]. Following the definition of pointwise averages, easily calculate

$$\begin{aligned} \left| \frac{\partial}{\partial h_k} R_{ij}^l(h) \right| &= \left| \int \frac{\partial u_i}{\partial x_k}(x+h) u_j(x) \mu_l(du) \right| \\ &\leq \left[\int \left| \frac{\partial u_i}{\partial x_k}(x+h) \right|^2 \mu_l(du) \right]^{\frac{1}{2}} \left[\int |u_j(x)|^2 \mu_l(du) \right]^{\frac{1}{2}} \\ &= \left[\int \left| \frac{\partial u_i}{\partial x_k}(x) \right|^2 \mu_l(du) \right]^{\frac{1}{2}} \left[\int |u_j(x)|^2 \mu_l(du) \right]^{\frac{1}{2}}, \end{aligned} \quad (5.25)$$

which by (5.11) are bounded above uniformly in l by C . Similarly,

$$|R_{ij}^l(h)| \leq C. \quad (5.26)$$

□

Proof of theorem 5.2.3. Since the ∇R_{ij}^l 's are uniformly bounded by the previous lemma, the R_{ij}^l 's are uniformly equicontinuous. Also by the previous lemma, the sequence is equibounded, therefore by Arzela-Ascoli there exists Q_{ij} on \mathbb{R}^3 such that

$$R_{ij}^l \rightarrow Q_{ij} \quad (5.27)$$

pointwise, uniformly on compact subsets of \mathbb{R}^3 , up to subsequence. In particular, for this subsequence,

$$\int \int_B R_{ij}^l(y-x) \Phi(x,y) dx dy \rightarrow \int \int_B Q_{ij}(y-x) \Phi(x,y) dx dy \quad (5.28)$$

on any B compact in \mathbb{R}^6 and Φ smooth with compact support in B . On the other hand, since

$$\int \langle u, \phi \rangle \langle u, \psi \rangle \mu_l(du) \rightarrow \int \langle u, \phi \rangle \langle u, \psi \rangle \mu(du) \quad (5.29)$$

for ϕ and ψ in $C_0^\infty(\mathbb{R}^3)$ by (5.23),

$$\int \int R_{ij}^l(y-x) \phi(x) \psi(y) dx dy \rightarrow \int \int R_{ij}(y-x) \phi(x) \psi(y) dx dy, \quad (5.30)$$

by (5.7). Now linear combinations of products $\phi(x)\psi(y)$ are dense in $C_0^\infty(\mathbb{R}^6)$, (see for example [F], Theorem 4.3.1), therefore, as the R_{ij}^l 's are bounded uniformly in l by Lemma 5.2.4,

$$\int \int R_{ij}^l(y-x) \Phi(x,y) dx dy \rightarrow \int \int R_{ij}(y-x) \Phi(x,y) dx dy, \quad (5.31)$$

on any B . Therefore $Q_{ij} = R_{ij}$. In particular, R_{ij}^l converge pointwise to R_{ij} , and uniformly so on compacts. \square

5.3 Application: Homogeneous measures on trigonometric polynomials

5.3.1 Overview of l -approximations

Recall that the construction of homogeneous and isotropic statistical solutions of the Navier-Stokes equations is based on approximating ANY homogeneous μ on $\mathcal{H}^0(r)$ by homogeneous μ_l 's supported on:

$$\mathcal{M}_l = \left\{ \sum_{\substack{k \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k| \leq l}} a_k e^{ik \cdot x} : a_k \cdot k = 0, a_k = \bar{a}_{-k} \forall k \right\}. \quad (5.32)$$

A concise description of the μ_l 's follows, with full details available at Appendix II of [VF]. (The construction is not straightforward as one must obtain **divergence free** periodic vector fields.)

- Given l , fix cut-off function ψ_l with support well within $\mathbb{T}_l = [-l, l]^3$. This is used to cut in x -space.
- Also fix for the given l a cut-off ζ_l with support in a ball of radius decreasing in l . This is used to cut in frequency space.
- Given $u \in \mathcal{H}^0(r)$, define

$$w_l(x) = u(x) - \int u(y) \int e^{i(x-y)\xi} \zeta_l(\xi) d\xi dy. \quad (5.33)$$

- Define u_l^s to be the divergence free part of the projection on \mathcal{M}_l of the periodization

$$u_l^T(x) = \sum_{j \in \mathbb{Z}^3} (\psi_l w_l)(x + 2lj) + C_u, \quad (5.34)$$

for C_u a constant that, as only derivatives will be of concern, does not need to be specified here. Then define

$$\begin{aligned} U_l^s : \mathcal{H}^0(r) &\rightarrow \mathcal{M}_l \\ u &\mapsto u_l^s. \end{aligned} \tag{5.35}$$

- Finally, define

$$\mu_l = (\alpha \circ (U_l^s \times Id))_{\#}(\mu \times \tau_l), \tag{5.36}$$

where τ_l is the normalized Lebesgue measure on \mathbb{T}_l , Id the identity on \mathbb{T}_l , and $\alpha(u, h) = T_h u$.

Given μ homogeneous, an approximation μ_l of μ constructed according to (5.33)–(5.36) will be referred to as **an l -approximation of μ** .

Having averaged push forwards via l -periodics over \mathbb{T}_l , μ_l is homogeneous with respect to all shifts in \mathbb{R}^3 , therefore μ_l -pointwise averages can be defined. The main result of Appendix II in [VF] then reads:

Theorem 5.3.1. *$\mu_l \rightarrow \mu$ in characteristic as $l \rightarrow \infty$, and*

$$\int |u(x)|^2 \mu_l(du) \leq \int |u(x)|^2 \mu(du). \tag{5.37}$$

Remark 5.3.2. *Note that the correlations of the μ_l 's are also $2l$ -periodic, as for any test ϕ*

$$\int_{\mathbb{R}^3} u_i(x + 2l + h)u_j(x)\phi(x) dx = \int_{\mathbb{R}^3} u_i(x + h)u_j(x)\phi(x) dx, \tag{5.38}$$

for any u in the support of μ_l .

5.3.2 An improved energy estimate

The following extends part 3, Proposition 2.1, Appendix II, in [VF]:

Lemma 5.3.3. *For any homogeneous measure μ on $\mathcal{H}^0(r)$ with*

$$\int_{\mathcal{H}^0(r)} \|u\|_{\mathcal{H}^1(r)}^2 \mu(du) < \infty, \quad (5.39)$$

there exist finite complex measures $\mathfrak{M}_{ij}, \mathfrak{N}_{ij}$ on \mathbb{R}^3 such that the following hold for averages of the distributional Fourier transforms of u 's and any ψ of rapid decay:

$$\begin{aligned} \int_{\mathcal{H}^0(r)} \langle \widetilde{u}_i, \psi \rangle \overline{\langle \widetilde{u}_j, \psi \rangle} \mu(du) &= \int_{\mathbb{R}^3} |\psi(x)|^2 \mathfrak{M}_{ij}(dx), \\ \int_{\mathcal{H}^0(r)} \langle \widetilde{\nabla u}_i, \psi \rangle \overline{\langle \widetilde{\nabla u}_j, \psi \rangle} \mu(du) &= \int_{\mathbb{R}^3} |\psi(x)|^2 \mathfrak{N}_{ij}(dx), \end{aligned} \quad (5.40)$$

with Hermitian inner products in \mathbb{C} and \mathbb{C}^3 used in the integrands of the left hand sides. In particular,

$$\begin{aligned} \sum_{i=1}^3 \int \mathfrak{M}_{ii}(dx) &= \int |u(x)|^2 \mu(du), \\ \sum_{i=1}^3 \int \mathfrak{N}_{ii}(dx) &= \int |\nabla u(x)|^2 \mu(du). \end{aligned} \quad (5.41)$$

Proof. For the second equality in (5.40):

$$\begin{aligned} &\int_{\mathcal{H}^0(r)} \langle \widetilde{\nabla u}_i, \psi \rangle \overline{\langle \widetilde{\nabla u}_j, \psi \rangle} \mu(du) \\ &= \int_{\mathbb{R}^3} \int_{\mathcal{H}^0(r)} \int_{\mathbb{R}^3} \nabla u_i(x+h) \nabla u_j(x) \tilde{\psi}(x+h) \tilde{\psi}(-x) dx \mu(du) dh, \end{aligned} \quad (5.42)$$

where Fubini is justified by (5.39). Then the definition of the correlation function and the identity

$$\begin{aligned} (\partial_n \partial_m R_{ij})(h) &= \int \partial_n u_i(x) \partial_m u_j(x+h) \mu(du) \\ &= \int \partial_m u_i(x) \partial_n u_j(x+h) \mu(du), \end{aligned} \quad (5.43)$$

(see Lemma VII.7.2 of [VF] for this), imply that

$$\begin{aligned}
& \int_{\mathcal{H}^0(r)} \langle \widetilde{\nabla u_i}, \psi \rangle \langle \widetilde{\nabla u_j}, \psi \rangle \mu(du) \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \tilde{\psi}(x+h) \tilde{\psi}(-x) dx (\nabla^2 R_{ij})(h) dh \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(y') \psi(y) e^{-ih \cdot y'} e^{-ix \cdot (y' - y)} dy' dy dx (\nabla^2 R_{ij})(h) dh \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(y') \psi(y) \delta(y' - y) dy e^{-ih \cdot y'} dy' (\nabla^2 R_{ij})(h) dh \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\psi(y')|^2 e^{-ih \cdot y'} dy' (\nabla^2 R_{ij})(h) dh \\
&= \langle \widetilde{\nabla^2 R_{ij}}, |\psi|^2 \rangle .
\end{aligned} \tag{5.44}$$

The first equality of (5.40) is proved by exactly the same argument, cf. [VF], p. 539. \square

Lemma 5.3.4. *For μ homogeneous measure on $\mathcal{H}^0(r)$ supported on $\mathcal{H}^1(r)$ and μ_l an l -approximation of μ as a measure on $\mathcal{H}^0(r)$, the following holds:*

$$\int_{\mathcal{H}^0(r)} |\nabla u(x)|^2 \mu_l(du) \leq C \int_{\mathcal{H}^0(r)} (|\nabla u(x)|^2 + |u(x)|^2) \mu(du). \tag{5.45}$$

Proof. First note that by the definition of μ_l

$$\begin{aligned}
\int_{\mathcal{H}^0(r)} |\nabla u(x)|^2 \mu_l(du) &= \int_{\mathcal{H}^0(r)} \int_{\mathbb{T}_l} |\nabla u_l^s(x+h)|^2 \tau_l(dh) \mu(du) \\
&= \int_{\mathcal{H}^0(r)} \left(\sum_{k \in \Gamma_l} |\widehat{\nabla u_l^s}(k)|^2 \right) \mu(du) \\
&= C \sum_{k \in \Gamma_l} |k|^2 \int_{\mathcal{H}^0(r)} |\widehat{u_l^s}(k)|^2 \mu(du) \\
&\leq C \sum_{k \in \Gamma_l, |k| \leq l} |k|^2 \int_{\mathcal{H}^0(r)} |\widehat{u_l^T}(k)|^2 \mu(du),
\end{aligned} \tag{5.46}$$

for u_l^T as in (5.34). Using (2.28') of Appendix II of [VF], rewrite this as

$$\begin{aligned}
& \frac{C}{|\mathbb{T}_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} |k|^2 \int_{\mathcal{H}^0(r)} \left| \int_{\mathbb{R}^3} (1 - \zeta_l(\xi)) \tilde{u}(\xi) \tilde{\psi}_l(\xi - k) d\xi \right|^2 \mu(du) \\
&= \frac{C}{|\mathbb{T}_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \sum_i \int_{\mathcal{H}^0(r)} \left| \int_{\mathbb{R}^3} (1 - \zeta_l(\xi)) \tilde{u}_i(\xi) k \tilde{\psi}_l(\xi - k) d\xi \right|^2 \mu(du) \\
&= \frac{C}{|\mathbb{T}_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \sum_i \int_{\mathcal{H}^0(r)} \left| \int_{\mathbb{R}^3} \left\{ (1 - \zeta_l(\xi)) \tilde{u}_i(\xi) (k - \xi) \tilde{\psi}_l(\xi - k) \right. \right. \\
&\quad \left. \left. + (1 - \zeta_l(\xi)) \xi \tilde{u}_i(\xi) \tilde{\psi}_l(\xi - k) \right\} d\xi \right|^2 \mu(du) \quad (5.47) \\
&= \frac{C}{|\mathbb{T}_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \sum_i \int_{\mathcal{H}^0(r)} \left| \int_{\mathbb{R}^3} \left\{ \tilde{u}_i(\xi) (1 - \zeta_l(\xi)) \widetilde{\nabla} \tilde{\psi}_l(\xi - k) \right. \right. \\
&\quad \left. \left. + \widetilde{\nabla} \tilde{u}_i(\xi) (1 - \zeta_l(\xi)) \tilde{\psi}_l(\xi - k) \right\} d\xi \right|^2 \mu(du) \\
&\leq \frac{C}{|\mathbb{T}_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \sum_i \int_{\mathcal{H}^0(r)} \left\{ \left| \langle \tilde{u}_i, (1 - \zeta_l) \widetilde{\nabla} \tilde{\psi}_l(\cdot - k) \rangle \right|^2 \right. \\
&\quad \left. + \left| \langle \widetilde{\nabla} \tilde{u}_i, (1 - \zeta_l) \tilde{\psi}_l(\cdot - k) \rangle \right|^2 \right\} \mu(du).
\end{aligned}$$

Now use $\mathfrak{M} = \sum_i \mathfrak{M}_{ii}$ and $\mathfrak{N} = \sum_i \mathfrak{N}_{ii}$ to rewrite this as:

$$\begin{aligned}
& C \left\{ \int_{\mathbb{R}^3} (1 - \zeta_l(\xi))^2 \sum_{k \in \Gamma_l, |k| \leq l} \frac{|\widetilde{\nabla} \tilde{\psi}_l(\xi - k)|^2}{|\mathbb{T}_l|^2} \mathfrak{M}(d\xi) \right. \\
&\quad \left. + \int_{\mathbb{R}^3} (1 - \zeta_l(\xi))^2 \sum_{k \in \Gamma_l, |k| \leq l} \frac{|\tilde{\psi}_l(\xi - k)|^2}{|\mathbb{T}_l|^2} \mathfrak{N}(d\xi) \right\} \quad (5.48) \\
&\leq C \left\{ \int_{\mathbb{R}^3} \sum_{k \in \Gamma_l, |k| \leq l} \frac{|\widetilde{\nabla} \tilde{\psi}_l(\xi - k)|^2}{|\mathbb{T}_l|^2} \mathfrak{M}(d\xi) + \int_{\mathbb{R}^3} \sum_{k \in \Gamma_l, |k| \leq l} \frac{|\tilde{\psi}_l(\xi - k)|^2}{|\mathbb{T}_l|^2} \mathfrak{N}(d\xi) \right\}.
\end{aligned}$$

Observe next that there is l_0 such that for $l \geq l_0$, using Parseval,

$$\begin{aligned}
& \frac{1}{|\mathbb{T}_l|^2} \sum_k |\widetilde{\nabla} \tilde{\psi}_l(\xi - k)|^2 = \frac{1}{|\mathbb{T}_l|} \int_{\mathbb{T}_l} |\nabla \psi_l|^2 dx, \\
&\leq \frac{C}{|\mathbb{T}_l| l^{2\kappa}} \int_{\mathbb{T}_l} 1 dx, \quad \text{by (2.12) in Appendix II of [VF],} \quad (5.49) \\
&= \frac{C}{l^{2\kappa}} \leq 1.
\end{aligned}$$

With this and the original estimate (2.44) from [VF], p. 547, the right hand side of

(5.48) is smaller than

$$C \left\{ \int_{\mathbb{R}^3} \mathfrak{M}(d\xi) + \int_{\mathbb{R}^3} \mathfrak{N}(d\xi) \right\}. \quad (5.50)$$

Using (5.41), obtain

$$\int_{\mathcal{H}^0(r)} |\nabla u(x)|^2 \mu_l(du) \leq C \int_{\mathcal{H}^0(r)} (|u(x)|^2 + |\nabla u(x)|^2) \mu(du). \quad (5.51)$$

□

Theorem 5.3.5. *Given μ homogeneous on $H^0(r)$, let μ_l be an l -approximation defined by (5.33)–(5.36). Then, up to subsequence, $W_2(\mu_l, \mu) \rightarrow 0$ as $l \rightarrow \infty$ and the correlation functions R_{ij}^l of μ_l 's converge to the correlation functions R_{ij} of μ uniformly on compact subsets of \mathbb{R}^3 .*

Proof. Lemma 5.3.4 shows that (5.11) holds. Hence, the μ_l 's converge weakly to a homogeneous measure μ on $\mathcal{H}^0(r)$ by Theorem 5.2.1. At the same time, Theorem 5.3.1 gives μ as limit of the μ_l 's in characteristic. It is standard that these two limits must be equal, cf. [GS], p.370.

It follows from (5.37) that (5.12) is also satisfied. Therefore, from Theorem 5.2.1 it follows that $\mu_l \rightarrow \mu$ in $W_2(\mathcal{H}^0(r))$. And from Theorem 5.2.3 finally follows that the correlation functions R_{ij}^l converge to the correlation functions R_{ij} uniformly on compact subsets of \mathbb{R}^3 . □

5.3.3 An alternative approximation in $W_2(\mathcal{H}^0(r))$

It follows from (1.54) above and [AGS], Proposition 8.3.3, that a homogeneous μ with finite energy density and finite density of energy dissipation can be approximated, also in $W_2(\mathcal{H}^0(r))$, by measures μ_n , $n \in \mathbb{N}$, such that for each n : μ_n is supported on some n -dimensional subspace of $\mathcal{H}^0(r)$, is absolutely continuous with respect to the

n -Lebesgue measure, and satisfies

$$\int \langle \nabla \Phi(u), u_n \rangle_{\mathcal{H}^0(r)} \mu_n(du) = 0, \quad (5.52)$$

for u_n smoothings of

$$\int_{\{\text{pr}_n(v)=u\}} \text{pr}_n(\nabla v \cdot h) \mu_u(dv), \quad (5.53)$$

for μ_u the disintegration of μ with respect to $(\text{pr}_n)_\# \mu$.

5.4 Homogeneous and isotropic solutions

5.4.1 Isotropic construction

The following corollary of Proposition 5.3.4 is needed to prove a corollary of Theorem 5.3.5.

Corollary 5.4.1. *For μ homogeneous and isotropic measure on $\mathcal{H}^0(r)$, supported on $\mathcal{H}^1(r)$ and $\hat{\mu}_l$ the approximation by homogeneous and isotropic measures of μ as a measure on $\mathcal{H}^0(r)$, as in Definition 4.2 of [DFK], the following holds:*

$$\int_X |\nabla u|^2(x) \hat{\mu}_l(du) \leq C \int_X (|\nabla u|^2(x) + |u|^2(x)) \mu(du). \quad (5.54)$$

Proof. First note that by the definition of $\hat{\mu}_l$:

$$\begin{aligned} \int_X |\nabla u|^2(x) \hat{\mu}_l(du) &= \int_{\mathcal{H}^0(r)} \int_{T_l} \int_{O(3)} |\nabla (R_\omega u)_l^s|^2(x+h) d\omega \tau_l(dh) \mu(du) \\ &= E \left(\sum_{k \in \Gamma_l} |\nabla (\widehat{R_\omega u})_l^s|^2(k) \right) \\ &= C \sum_{k \in \Gamma_l} |k|^2 E |\widehat{(R_\omega u)}_l^s|^2(k) \\ &\leq C \sum_{k \in \Gamma_l, |k| \leq l} |k|^2 E |\widehat{(R_\omega u)}_l^T|^2(k), \end{aligned} \quad (5.55)$$

for u_l^T as in (5.34), with E denoting the expectation with respect to $\mu(du)d\omega$. Using

(2.28') of Appendix II of [VF], rewrite this as

$$\begin{aligned}
& \frac{C}{|T_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} |k|^2 E \left(\int_{\mathbb{R}^3} (1 - \zeta_l(\xi)) (\widetilde{R_\omega u})(\xi) \widetilde{\psi}_l(\xi - k) d\xi \right)^2 \\
&= \frac{C}{|T_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \sum_i E \left(\int_{\mathbb{R}^3} (1 - \zeta_l(\xi)) (\widetilde{R_\omega u})_i(\xi) k \widetilde{\psi}_l(\xi - k) d\xi \right)^2 \\
&= \frac{C}{|T_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \sum_i \\
&\quad E \left(\int_{\mathbb{R}^3} (1 - \zeta_l(\xi)) \left\{ (\widetilde{R_\omega u})_i(\xi) (k - \xi) \widetilde{\psi}_l(\xi - k) + \xi (\widetilde{R_\omega u})_i(\xi) \widetilde{\psi}_l(\xi - k) \right\} d\xi \right)^2 \\
&= \frac{C}{|T_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \sum_i \\
&\quad E \left(\int_{\mathbb{R}^3} (1 - \zeta_l(\xi)) \left\{ (\widetilde{R_\omega u})_i(\xi) \widetilde{\nabla} \widetilde{\psi}_l(\xi - k) + \nabla (\widetilde{R_\omega u})_i(\xi) \widetilde{\psi}_l(\xi - k) \right\} d\xi \right)^2 \\
&\leq \frac{C}{|T_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \sum_i E \left\{ \left(\int_{\mathbb{R}^3} (1 - \zeta_l(\xi)) (\widetilde{R_\omega u})_i(\xi) \widetilde{\nabla} \widetilde{\psi}_l(\xi - k) d\xi \right)^2 \right. \\
&\quad \left. + \left(\int_{\mathbb{R}^3} (1 - \zeta_l(\xi)) \nabla (\widetilde{R_\omega u})_i(\xi) \widetilde{\psi}_l(\xi - k) d\xi \right)^2 \right\} \\
&\leq \frac{C}{|T_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \sum_i E \left\{ \langle (\widetilde{R_\omega u})_i(\xi), \widetilde{\nabla} \widetilde{\psi}_l(\xi - k) \rangle^2 + \langle \nabla (\widetilde{R_\omega u})_i(\xi), \widetilde{\psi}_l(\xi - k) \rangle^2 \right\}
\end{aligned} \tag{5.56}$$

Now use $\mathfrak{M}, \mathfrak{N}$ from (5.40) to rewrite (noting that the assumptions still hold):

$$\begin{aligned}
& \frac{C}{|T_l|^2} \sum_{k \in \Gamma_l, |k| \leq l} \left\{ \int_{\mathbb{R}^3} |\widetilde{\nabla} \widetilde{\psi}_l(\xi - k)|^2 \mathfrak{M}(d\xi) + \int_{\mathbb{R}^3} |\widetilde{\psi}_l(\xi - k)|^2 \mathfrak{N}(d\xi) \right\} \\
&= C \left\{ \int_{\mathbb{R}^3} \sum_{k \in \Gamma_l, |k| \leq l} \frac{|\widetilde{\nabla} \widetilde{\psi}_l(\xi - k)|^2}{|T_l|^2} \mathfrak{M}(d\xi) + \int_{\mathbb{R}^3} \sum_{k \in \Gamma_l, |k| \leq l} \frac{|\widetilde{\psi}_l(\xi - k)|^2}{|T_l|^2} \mathfrak{N}(d\xi) \right\} \tag{5.57} \\
&\leq C \left\{ \int_{\mathbb{R}^3} 1 \mathfrak{M}(d\xi) + \int_{\mathbb{R}^3} 1 \mathfrak{N}(d\xi) \right\}.
\end{aligned}$$

Now observe that there is l_0 such that for $l \geq l_0$, using Parseval,

$$\begin{aligned}
\frac{1}{|T_l|^2} \sum_k |\widetilde{\nabla} \widetilde{\psi}_l(\xi - k)|^2 &= \frac{1}{|T_l|} \int_{T_l} |\nabla \psi_l|^2 dx, \\
&\leq \frac{C}{|T_l| l^{2\kappa}} \int_{T_l} 1 dx, \quad \text{by in Appendix II of [VF],} \tag{5.58} \\
&= \frac{C}{l^{2\kappa}} \leq 1,
\end{aligned}$$

cf. equation (2.10) and Proposition 4.1 of Appendix II of [VF]. Using (5.58) get that

$$\int_X |\nabla u|^2(x) \mu_l(du) \leq C \int_X (|u|^2(x) + |\nabla u|^2(x)) \mu(du). \quad (5.59)$$

□

The following is a corollary of Theorem 5.3.5

Theorem 5.4.2. *Given μ homogeneous and isotropic on $H^0(r)$, let $\hat{\mu}_l$ be the approximation defined by Definition 3.1.2. Then $W_2(\hat{\mu}_l, \mu) \rightarrow 0$ as $l \rightarrow \infty$ and the correlation functions R_{ij}^l of $\hat{\mu}_l$'s converge to the correlation functions R_{ij} of μ uniformly on compact subsets of \mathbb{R}^3 .*

Proof. Corollary 5.4.1 shows that (5.11) holds. Hence, these $\hat{\mu}_l$'s converge weakly to a homogeneous measure μ on $\mathcal{H}^0(r)$ by Theorem 5.2.1. At the same time, Lemma 4.1.1 gives a limit of the μ_l 's in characteristic. It is standard that these two limits must be equal, cf. [GS], p.370.

It follows from Lemma 4.1.4 that (5.12) is also satisfied. Therefore, from Theorem 5.2.1 it follows that $\hat{\mu}_l \rightarrow \mu$ in $W_2(\mathcal{H}^0(r))$. And from Theorem 5.2.3 it finally follows that the correlation functions R_{ij}^l converge to the correlation functions R_{ij} uniformly on compact subsets of \mathbb{R}^3 . □

Chapter 6

Homogeneous Geodesics in W_2

One way to interpret the Kolmogorov theory is as a flow problem in the space of homogeneous & isotropic measures. The question, “How does the space of all homogeneous & isotropic measures lie within the space of all measures?” then naturally arises. The following makes a preliminary step towards determining the geometry of such measures in the second Wasserstein space. In particular, it will be shown that they form subspace of the 2-Wasserstein space, and are therefore a space of non-positive curvature, inheriting the topology from the 2-Wasserstein space.

6.1 Homogeneous geodesics for $\mathcal{H} = L^2(\mathbb{T}^n)$

That a Dirac measure can be approximated by Gaussian measures is standard. The following shows that such Gaussians can be taken to be homogeneous:

Lemma 6.1.1. *There exists a sequence μ_n of homogeneous and Gaussian measures in $\mathcal{P}(\mathcal{H})$, with mean 0 such that μ_n converges weakly to δ_0 , the Dirac at 0.*

Proof. For μ the measure of Example 1.2.4 with covariance operator \widehat{K} , let μ_n be the Gaussian measure with mean 0 and covariance operator $\widehat{K}_n = \frac{1}{n}\widehat{K}$. Then

$$\sup_n \operatorname{tr} \widehat{K}_n = \sup_n \frac{1}{n} \operatorname{tr} \widehat{K} < \infty. \quad (6.1)$$

Let $\{e_i\}$ be an orthonormal basis of \mathcal{H} . Then, since

$$\sum_i (\widehat{K}_n e_i, e_i)_{\mathcal{H}} = \frac{1}{n} \sum_i (\widehat{K} e_i, e_i)_{\mathcal{H}}, \quad (6.2)$$

the sum converges uniformly in n . Then μ_n is a family of relatively weakly compact measures, see Example 3.8.13 in [Bo]. Therefore there exists a subsequence, also called μ_n , converging weakly to some measure, say μ_∞ . On the other hand, the μ_n converge in characteristic to δ_0 by construction:

$$\int e^{i\langle u, z \rangle_{\mathcal{H}}} \mu_n(du) = e^{-\langle \widehat{K}_n z, z \rangle_{\mathcal{H}}} = e^{-\frac{1}{n} \langle \widehat{K} z, z \rangle_{\mathcal{H}}} \rightarrow 1. \quad (6.3)$$

Therefore $\mu_\infty = \delta_0$. □

Definition 6.1.2. μ is regular if $\mu(B) = 0$ for any B Gaussian null, i.e. any B that is given measure zero by all Gaussians.

Definition 6.1.3. Given measures μ and ν , define their convolution $\mu * \nu$ as the push-forward of the product measure $\mu \times \nu$ by $(u, v) \mapsto u + v$.

Lemma 6.1.4. Let $\mu \in W_2$ be homogeneous and regular, and let $\nu \in W_2$ be homogeneous. Then $\mu * \nu$ is homogeneous and regular.

Proof. Let A be a Gaussian null set. For μ_v defined by $\mu_v(B) = \mu(B - v)$ for any v in \mathcal{H} , note that μ_v is regular, and that $A - v$ is Gaussian null for any v in \mathcal{H} , i.e. $\int \chi_{A-v}(u) \mu(du) = 0$. Therefore

$$\mu * \nu(A) = \int \int \chi_A(u + v) \mu(du) \nu(dv) = \int \int \chi_{A-v}(u) \mu(du) \nu(dv) = 0. \quad (6.4)$$

Therefore $\mu * \nu$ is regular. Furthermore,

$$\begin{aligned}
\int f(T_h u) \mu * \nu(du) &= \int \int f(T_h(u+v)) \mu(du) \nu(dv) \\
&= \int \int f(T_h u + T_h v) \mu(du) \nu(dv) \\
&= \int \int f(u + T_h v) \mu(du) \nu(dv) \\
&= \int \int f(u+v) \mu(du) \nu(dv) \\
&= \int f(u) \mu * \nu(du),
\end{aligned} \tag{6.5}$$

for all f such that $f(u+v)$ is integrable on $\mathcal{H} \times \mathcal{H}$. Therefore $\mu * \nu$ is homogeneous. \square

Lemma 6.1.5. *Let $\mu_n \in W_2$ converge weakly to δ_0 , and let $\nu \in W_2$. Then $\mu_n * \nu$ converges weakly to ν .*

Proof. Let $f \in C_b$, and let $F(u) = \int f(u+v) \nu(dv)$, also a function in C_b . Then

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \left| \int f(v) \mu_n * \nu(dv) - \int f(v) \delta_0 * \nu(dv) \right| \\
&= \lim_{n \rightarrow \infty} \left| \int \int f(u+v) \mu_n(du) \nu(dv) - \int \int f(u+v) \delta_0(du) \nu(dv) \right| \\
&= \lim_{n \rightarrow \infty} \left| \int \int f(u+v) \nu(dv) \mu_n(du) - \int f(v) \nu(dv) \right| \\
&= \lim_{n \rightarrow \infty} \left| \int F(u) \mu_n(du) - F(0) \right|
\end{aligned} \tag{6.6}$$

and

$$\int F(u) \mu_n(du) \rightarrow F(0), \quad n \rightarrow \infty, \tag{6.7}$$

give

$$\int f(v) \mu_n * \nu(dv) \rightarrow F(0) = \int f(v) \nu(dv). \tag{6.8}$$

\square

Lemma 6.1.6. *Let \mathcal{H} be a Hilbert space of vector fields on \mathbb{R}^n with translation invariant norm, i.e., for each $h \in \mathbb{R}^n$ and $u \in \mathcal{H}$, $\|T_h(u)\| = \|u\|$.*

Let $\mu \in W_2(\mathcal{H})$ be homogeneous and regular, and let $\nu \in W_2(\mathcal{H})$ be homogeneous.

There is a unique geodesic from μ to ν which passes only through homogeneous measures. In other words, if $\underline{\mu} \in \Gamma_0(\mu, \nu)$ then $\mu_t = [(1-t)\pi_1 + t\pi_2]_{\#}\underline{\mu}$ is homogeneous.

Proof. By theorem 6.2.10 in [AGS], if $\underline{\mu} \in \Gamma_0(\mu, \nu)$, the set of all geodesics from μ to ν , then $\underline{\mu}$ is unique and there exists a map $r \in L^2(\mathcal{H}, \mu; \mathcal{H})$ such that $\nu = r_{\#}\mu$ and $\underline{\mu} = (i \times r)_{\#}\mu$.

Fix $h \in \mathbb{R}^3$.

Since μ and ν are homogeneous, for any integrable f it follows that:

$$\begin{aligned} \int f(u) (T_h \circ r \circ T_{-h})_{\#}\mu(du) &= \int f \circ T_h \circ r \circ T_{-h}(u) \mu(du) \\ &= \int f \circ T_h \circ r(u) \mu(du) \\ &= \int f \circ T_h(u) \nu(du) \\ &= \int f(u) \nu(du). \end{aligned} \tag{6.9}$$

Therefore $\nu = (T_h \circ r \circ T_{-h})_{\#}\mu$ and $(i \times T_h \circ r \circ T_{-h})_{\#}\mu \in \Gamma(\mu, \nu)$.

Since $\underline{\mu} \in \Gamma_0(\mu, \nu)$, it minimizes $\int \|u - v\|^2 \underline{\mu}(du, dv)$. Then

$$\begin{aligned} \int \|u - v\|^2 \underline{\mu}(du, dv) &= \int \|u - v\|^2 (i \times r)_{\#}\mu(du, dv) \\ &= \int \|u - r(u)\|^2 \mu(du) \\ &= \int \|T_h(u) - T_h \circ r(u)\|^2 \mu(du) \\ &= \int \|T_h \circ T_{-h}(u) - T_h \circ r \circ T_{-h}(u)\|^2 \mu(du) \\ &= \int \|u - T_h \circ r \circ T_{-h}(u)\|^2 \mu(du) \\ &= \int \|u - v\|^2 (i \times T_h \circ r \circ T_{-h})_{\#}\mu(du, dv) \end{aligned} \tag{6.10}$$

implies that $(i \times T_h \circ r \circ T_{-h})_{\#}\mu \in \Gamma_0(\mu, \nu)$. But, since $\underline{\mu}$ was the unique member of $\Gamma_0(\mu, \nu)$,

$$(i \times T_h \circ r \circ T_{-h})_{\#}\mu = (i \times r)_{\#}\mu. \tag{6.11}$$

That μ_t is homogeneous follows from:

$$\begin{aligned}
\int f(T_h u) \mu_t(du) &= \int f \circ T_h(u) [(1-t)\pi_1 + t\pi_2]_{\#} \underline{\mu}(du) \\
&= \int f \circ T_h((1-t)u + tv) \underline{\mu}(du, dv) \\
&= \int f \circ T_h((1-t)u + tv) (i \times r)_{\#} \mu(du, dv) \\
&= \int f \circ T_h((1-t)u + tr(u)) \mu(du) \\
&= \int f((1-t)T_h u + tT_h \circ r(u)) \mu(du) \\
&= \int f((1-t)u + tT_h \circ r \circ T_{-h}(u)) \mu(du) \\
&= \int f((1-t)u + tv) (i \times T_h \circ r \circ T_{-h})_{\#} \mu(du, dv) \\
&= \int f((1-t)u + tv) \underline{\mu}(du, dv) \\
&= \int f(u) [(1-t)\pi_1 + t\pi_2]_{\#} \underline{\mu}(du) \\
&= \int f(u) \mu_t(du). \quad \square
\end{aligned}$$

Theorem 6.1.7. *The space of homogeneous measures is geodesically connected in the Wasserstein space $W^2(\mathcal{H})$. That is, given two homogeneous measures $\mu, \nu \in W^2(\mathcal{H})$, there exists an optimal plan, $\underline{\mu} \in \Gamma_0(\mu, \nu)$, such that, for each $t \in [0, 1]$,*

$$\mu_t = [(1-t)\pi_1 + t\pi_2]_{\#} \underline{\mu} \tag{6.12}$$

is homogeneous.

Proof. Let μ_n be as in Lemma 6.1.1. Define $\hat{\mu}_n = \mu * \mu_n$. $\hat{\mu}_n$ is homogeneous and regular by Lemma 6.1.4, and converges weakly to μ by Lemma 6.1.5. Let $\underline{\mu}_n \in \Gamma_0(\hat{\mu}_n, \nu)$. By Proposition 7.1.3 in [AGS], $\underline{\mu}_n$ has a weakly converging subsequence, also called $\underline{\mu}_n$, which converges to some measure, $\underline{\mu} \in \Gamma_0(\mu, \nu)$. Define $\mu_{n,t} = [(1-t)$

$t)\pi_1 + t\pi_2]_{\#}\underline{\mu}_n$ and $\mu_t = [(1-t)\pi_1 + t\pi_2]_{\#}\underline{\mu}$. Then for every $f \in C_0^\infty$,

$$\begin{aligned} \int f(u) \mu_{n,t}(du) &= \int \int f((1-t)u + tv) \underline{\mu}_n(du, dv) \\ &\rightarrow \int \int f((1-t)u + tv) \underline{\mu}(du, dv) \\ &= \int f(u) \mu_t(du) \end{aligned} \quad (6.13)$$

and

$$\begin{aligned} \int f \circ T_h(u) \mu_{n,t}(du) &= \int \int f \circ T_h((1-t)u + tv) \underline{\mu}_n(du, dv) \\ &\rightarrow \int \int f \circ T_h((1-t)u + tv) \underline{\mu}(du, dv) \\ &= \int f \circ T_h(u) \mu_t(du). \end{aligned} \quad (6.14)$$

However, since $\mu_{n,t} = [(1-t)\pi_1 + t\pi_2]_{\#}\underline{\mu}_n$ is homogeneous by Lemma 6.1.6,

$$\int f(u) \mu_{n,t}(du) = \int f \circ T_h(u) \mu_{n,t}(du). \quad (6.15)$$

Therefore

$$\int f(u) \mu_t(du) = \int f \circ T_h(u) \mu_t(du). \quad (6.16)$$

□

6.2 Homogeneous measures and positive curvature

It is standard, see Appendix of [AGS], that W^2 is a metric space of positive curvature in the A.D. Aleksandrov sense, i.e., the Wasserstein distance from a fixed measure μ^3 along the constant speed geodesic $\mu_t^{1 \rightarrow 2}$ connecting any measures μ^1 to μ^2 satisfies:

$$W_2^2(\mu_t^{1 \rightarrow 2}, \mu^3) \geq (1-t)W_2^2(\mu^1, \mu^3) + tW_2^2(\mu^2, \mu^3) - t(1-t)W_2^2(\mu^1, \mu^2). \quad (6.17)$$

Corollary 6.2.1. *The space of homogeneous measures is a subspace of W^2 , and is therefore a PC space, a metric space of positive curvature in the sense of Aleksandrov.*

Proof. Restrict the 2-Wasserstein metric as in (1.48),

$$W_2(\mu_1, \mu_2) = \left(\inf_{\pi \in \Gamma_{hom}(\mu_1, \mu_2)} \int_{X \times X} \|u - v\|_X^2 \pi(du, dv) \right)^{1/2}, \quad (6.18)$$

with, for μ_1, μ_2 homogeneous,

$$\Gamma_{hom}(\mu_1, \mu_2) = \{\pi \in \mathcal{P}(X \times X) : (pr_1)_\# \pi = \mu_1, (pr_2)_\# \pi = \mu_2\}. \quad (6.19)$$

Theorem 6.1.7 gives that this metric agrees with the metric defined in (1.48) when the measures are homogeneous, since geodesics realize the metric and the homogeneous measures are connected by geodesics. Therefore the space of homogeneous measures form a subspace of W^2 , which itself is PC, and inherit this property. \square

Direction for the Future

Now that homogeneous and isotropic statistical solutions of the Navier-Stokes equations have been shown to exist, the question remains, “how do they lie in the 2nd Wasserstein space?” Results from Chapter 5 show that initial measures can be approximated in W_2 , and results from Chapter 6 show that periodic homogeneous measures form a PC space. The next goal would be to show that statistical solutions can be produced in a PC space, and to use the geometry of that space to analyze various conjectures regarding turbulent flow. To achieve this goal, it would be useful to show that the homogeneous and isotropic measures form a PC space for the non-periodic case and that these statistical solutions can be approximated in the 2nd Wasserstein space for positive times.

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Vita

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