EXTENSION THEOREMS IN VECTOR SPACES OVER FINITE FIELDS

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TABLE OF CONTENTS

LIST OF FIGURES

EXTENSION THEOREMS

IN VECTOR SPACES OVER FINITE FIELDS

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ABSTRACT

We study the $L^p - L^r$ boundedness of the extension operator associated with algebraic varieties such as nondegenerate quadratic surfaces, paraboloids, and cones in vector spaces over finite fields. We obtain the best possible result for the extension theorems related to nondegenerate quadratic curves in two dimensional vector spaces over finite fields. In higher even dimensions, we improve upon the Tomas-Stein exponents which were obtained by Mockenhaupt and Tao ([21]) by studying extension theorems for paraboloids in the finite field setting. We also study extension theorems for cones in vector spaces over finite fields. We give an alternative proof of the best possible result for the extension theorems for cones in three dimensions, which originally is due to Mockenhaupt and Tao ([21]). Moreover, our method enables us to obtain the sharp $L^2 - L^r$ estimate of the extension operator for cones in higher dimensions. In addition, we study the relation between extension theorems for spheres and the Erdös-Falconer distance problems in the finite field setting. Using the sharp extension theorem for circles, we improve upon the best known result, due to A. Iosevich and M. Rudnev ([17]), for the Erdös-Falconer distance problems in two dimensional vector spaces over finite fields. Discrete Fourier analytic machinery, arithmetic considerations, and classical exponential sums play an important role in the proofs.

Chapter 1

Introduction

1.1 Statement of Purpose

Let S be a hypersurface in $\mathbb{R}^d$ and $d\sigma$ a surface measure on S. In the Euclidean space, the classical extension problems ask one to find the set of exponents p and r such that the estimate

$$\|(fd\sigma)^\vee\|_{L^r(\mathbb{R}^d)} \leq C\|f\|_{L^p(S,d\sigma)}, \qquad f \in L^p(S, d\sigma)$$

holds, where the constant $C > 0$ depends only on the exponents p and r, and $(fd\sigma)^\vee$ denotes the inverse Fourier transform of the measure $fd\sigma$. The extension problems have received much attention in the last few decades, because they are related to many interesting problems in harmonic analysis such as the Kakeya problems. For a survey of the development of these ideas and some recent results on the extension problem, see, for example, [29]. See also [8], [33], [24], [2], [23], [4], [28], [9], and [32], and the references contained therein on recent progress related to this problem. In recent years, the extension problems in the finite field setting have been studied, in part because the finite field case serves as a good model for the Euclidean setting. See, for example, [21], [13], [14] and [15].

Mockenhaupt and Tao ([21]) first posed the extension problems in the finite field setting for various algebraic varieties S and obtained good results for the extension problems for cones in three dimensions and for paraboloids in d-dimensional vector spaces over finite fields. The purpose of this dissertation is to investigate and survey the results obtained by the author in the past few years on the extension theorems for paraboloids, nondegenerate quadratic varieties, and cones in d-dimensional vector spaces over finite fields. Moreover, we study the relation between the extension theorems for spheres and the Erdös-Falconer distance problems in the finite field setting. As a result, we shall see that our extension theorem for circles enables us to improve upon the best known result for the Erdös-Falconer distance problems which is due to A. Iosevich and M. Rudnev ([17]).

1.2 A Brief Overview of Main Results

In two dimensional vector spaces over finite fields, the extension problems for the parabola were completely solved by Mockenhaupt and Tao ([21]). Moreover, they obtained the Tomas-Stein exponents for the $L^p - L^r$ boundedness of extension operators related to paraboloids in higher d-dimensional vector spaces over finite fields. In particular, they improved on the Tomas-Stein exponents for the paraboloids in three dimensional vector spaces over finite fields in the case when -1 is not a square in the underlying finite fields. Here and throughout this paper, the Tomas-Stein exponents are defined as the exponents $p \geq 1$ and $r \geq 1$ such that

$$r \geq \frac{2d+2}{d-1} \quad \text{and} \quad r \geq \frac{p(d+1)}{(p-1)(d-1)}, \tag{1.2.1}$$

where d denotes the dimension of the vector space over the finite field. The aforementioned authors used the combinatorial methods to prove the incidence theorems between lines and points in two dimensional vector spaces over finite fields. As a result, they could improve upon the Tomas-Stein exponents for paraboloids in three dimensions. However, if -1 is a square number or the dimension d is greater than three, then combinatorial methods are more difficult to use to establish incidence theorem. We significantly improve the Tomas-Stein exponents for the boundedness of extension operators associated with paraboloids in higher even dimensional vector spaces over finite fields. Moreover, our results in even dimensions hold without the assumption that -1 is not a square in the underlying finite field. We also obtain some improvements in odd dimensions under a variety of assumptions.

We initially study the extension theorems for nondegenerate quadratic varieties in the finite field setting. The underlying Fourier analysis turns out to be considerably more complicated in this case because the Gauss sums, used to study the Fourier transform of the discrete paraboloid are no longer adequate for the task. The estimation of the Fourier transform of the discrete sphere and other quadratic varieties reduces to the consideration of bounds for Kloosterman and Salié sums that arise in analytic number theory and algebraic geometry in the study of the Riemann hypothesis in finite fields and related problems. See, for example, [19] and the references contained therein.

Mockenhaupt and Tao ([21]) obtained the best possible result for extension theorems for the cone in three dimensional vector spaces over finite fields. In order to obtain the best result, they calculated the number of solutions to the following equation: for each $\eta \in \mathbb{F}_q^3$,

$$\xi_1 + \xi_2 = \eta,$$

where ξ_1 and ξ_2 are elements in the cone of three dimensional vector spaces over finite fields $\mathbb{F}_q$. The idea can be also used to study the extension theorems for cones in higher dimensions, but it just gives $L^2 - L^4$ estimate for all dimensions. On the other hand, we shall calculate the Fourier transform of surface measures of cones in all dimensions and apply the usual Tomas-Stein type argument. As a result of the methods, we also give the complete solution to the question about the extension theorems for cones in three dimensions, which Mockenhaupt and Tao ([21]) already proved. Moreover, we obtain the sharp $L^2 - L^r$ estimates for extension theorems in higher dimensional cones.

As an application of our extension theorems for circles in the finite field setting, we will see the improvement of the Erdös-Falconer distance problems in two dimensional vector spaces over finite fields.

Chapter 2

ON THE FINITE FIELD SETTING

2.1 Notation and Discrete Fourier Analytic Machinery

We review some notation and basic properties of the Fourier transform in the finite field setting. We denote by $\mathbb{F}_q$ a finite field with q elements whose characteristic is greater than two. Let $\mathbb{F}_q^d$ be a d-dimensional vector space over the finite field $\mathbb{F}_q$. Let $(\mathbb{F}_q^d, dx)$ denote a d-dimensional vector space, which we endow with the normalized counting measure dx, and let $(\mathbb{F}_q^d, dm)$ denote the dual space, which we endow with the counting measure dm. For any complex-valued function f on $(\mathbb{F}_q^d, dx), d \geq 1$, we define the Fourier transform of f by the formula

$$\widehat{f}(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot m) f(x) \tag{2.1.1}$$

where χ denotes a non-trivial additive character of $\mathbb{F}_q$, and $x \cdot m$ is the usual dot product of x and m. Similarly, we define the inverse Fourier transform of the measure $f d\sigma$ by the relation

$$(f d\sigma)^{\vee}(m) = \frac{1}{|S|} \sum_{x \in S} \chi(x \cdot m) f(x)$$

where $|S|$ denotes the number of elements in an algebraic variety S in $(\mathbb{F}_q^d, dx)$, and $d\sigma$ denotes the normalized surface measure on S. In other words, $\sigma(x) = q^d|S|^{-1}S(x)$. Here and throughout the paper, we identify sets with their characteristic functions. For example, we write $E(x)$ for $\chi_E(x)$. Using the orthogonality relations for non-trivial characters, we obtain the Fourier inversion theorem, that is,

$$f(x) = \sum_{m\in\mathbb{F}_q^d} \chi(x\cdot m)\widehat{f}(m).$$

Given complex-valued functions f, g on $(\mathbb{F}_q^d, dx)$, the Plancherel theorem is given by

$$\sum_{m\in\mathbb{F}_q^d} \widehat{f}(m)\overline{\widehat{g}}(m) = q^{-d}\sum_{x\in\mathbb{F}_q^d} f(x)\overline{g}(x).$$

Note that the Plancherel theorem says in this context that

$$\sum_{m\in\mathbb{F}_q^d} |\widehat{f}(m)|^2 = q^{-d}\sum_{x\in\mathbb{F}_q^d} |f(x)|^2.$$

Since we endow the measure on the "space" variables, dx, with the normalized counting measure given by dividing the counting measure by q^d, and the measure on the "phase" variables, dm, with the usual counting measure, we have the following definitions: for each $1 \le p, r < \infty$,

$$\|f\|^p_{L^p\left(\mathbb{F}_q^d, dx\right)} = q^{-d}\sum_{x\in\mathbb{F}_q^d} |f(x)|^p,$$

$$\|\widehat{f}\|^r_{L^r\left(\mathbb{F}_q^d, dm\right)} = \sum_{m\in\mathbb{F}_q^d} |\widehat{f}(m)|^r$$

and

$$\|f\|^p_{L^p(S,d\sigma)} = \frac{1}{|S|}\sum_{x\in S} |f(x)|^p.$$

Similarly, denote by $\|f\|_{L^\infty}$ the maximum value of f.

2.2 Definition of the Extension Theorem in the Finite Field Setting

Let S be an algebraic variety in $(\mathbb{F}_q^d, dx)$ and $d\sigma$ the surface measure on S. For $1 \le p, r \le \infty$, we define $R^*(p \to r)$ by the smallest constant such that the extension estimate

$$\|(fd\sigma)^\vee\|_{L^r(\mathbb{F}_q^d, dm)} \le R^*(p \to r)\|f\|_{L^p(S,d\sigma)} \tag{2.2.1}$$

holds for all functions f on S. By duality, $R^*(p \to r)$ is also given by the smallest constant such that the restriction estimate

$$\|\widehat{g}\|_{L^{p'}(S,d\sigma)} \le R^*(p \to r)\|g\|_{L^{r'}(\mathbb{F}_q^d, dm)} \tag{2.2.2}$$

holds for all functions g on $(\mathbb{F}_q^d, dm)$. Observe that the value of the left-hand side in (2.2.2) is given by

$$\|\widehat{g}\|_{L^{p'}(S,d\sigma)} = \left(\frac{1}{|S|}\sum_{x\in S}|\widehat{g}(x)|^{p'}\right)^{1/p'},$$

because the normalized surface measure $d\sigma$ is related to the algebraic variety $(S, d\sigma)$. We also note that the $\widehat{g}$ in (2.2.2) takes the following value:

$$\widehat{g}(x) = \sum_{m\in\mathbb{F}_q^d} \chi(-x\cdot m),$$

because the function g is defined on the space $(\mathbb{F}_q^d, dm)$ with a counting measure dm. Compare this with the definition of the Fourier transform given in (2.1.1). Throughout this paper, we always denote the variable x as an element of the space $(\mathbb{F}_q^d, dx)$ related to the normalized counting measure dx. On the other hand, the variable m will be understood as an element of the space $(\mathbb{F}_q^d, dm)$ with the counting measure dm.

We now introduce the definition of the extension theorems related to the algebraic variety S in $(\mathbb{F}_q^d, dx)$. Since $|\mathbb{F}_q| = q$ is finite, the quantity $R^*(p \to r)$ in (2.2.1) or (2.2.2) will be a finite positive real. Moreover, the quantity may depend on "q", the size of the underlying finite field $\mathbb{F}_q$. However, in the finite field setting, the extension theorem for S asks us to determine the set of exponents p and r such that $R^*(p \to r) \leq C_{p,r}$, where the constant $C_{p,r}$ is independent of "q" , the size of the underlying finite field $\mathbb{F}_q$. Using Hölder's inequality and the nesting properties of L^p-norms, we see that

$$R^*(p_1 \to r) \leq R^*(p_2 \to r) \quad \text{for} \quad p_1 \geq p_2 \tag{2.2.3}$$

and

$$R^*(p \to r_1) \leq R^*(p \to r_2) \quad \text{for} \quad r_1 \geq r_2 \tag{2.2.4}$$

which will allow us to reduce the analysis below to certain endpoint estimates.

2.3 General Necessary Conditions for the Boundedness of $R^*(p \to r)$

We introduce the necessary conditions for the boundedness of $R^*(p \to r)$. Let $(S, d\sigma)$ be an algebraic variety in $\mathbb{F}_q^d$ and $d\sigma$ the normalized surface measure on S. We assume that $|S| \sim q^k$, for some $0 < k < d$. In order to obtain the necessary conditions for $R^*(p \to r) \lesssim 1$, we first observe that

$$\sum_{m \in \mathbb{F}_q^d} |(fd\sigma)^\vee(m)|^2 = \frac{q^d}{|S|} \|f\|^2_{L^2(S,d\sigma)}.$$

Using this and Cauchy-Schwarz inequality, we see that if $r \geq 2$, then

$$\frac{q^d}{|S|} \|f\|^2_{L^2(S,d\sigma)} \leq \left(\sum_{m \in \mathbb{F}_q^d} |(fd\sigma)^\vee(m)|^r \right)^{\frac{2}{r}} \cdot q^{\frac{d(r-2)}{r}}.$$

It therefore follows that

$$\begin{aligned}\frac{q^d}{|S|} q^{\frac{-d(r-2)}{r}} \|f\|^2_{L^2(S,d\sigma)} &\leq \|(f d\sigma)^\vee\|^2_{L^r(\mathbb{F}_q^d, dm)} \\ &\leq \left(R^*(p \to r)\right)^2 \|f\|^2_{L^p(S,d\sigma)}.\end{aligned}$$

Choosing $f = 1$ on S and using the fact that $|S| \sim q^k$, we see

$$q^{\frac{-d(r-2)}{r}} q^{d-k} \lesssim \left(R^*(p \to r)\right)^2,$$

because $d\sigma$ is the normalized surface measure on S and it therefore follows that $\|1\|^2_{L^p(S,d\sigma)} = 1$ for $1 \leq p \leq \infty$. Thus one of necessary conditions for $R^*(p \to r) \lesssim 1$ is given by

$$r \geq \frac{2d}{k}. \tag{2.3.1}$$

On the other hand, if we test (2.2.1) with $f = \delta_\alpha$ for some $\alpha \in S$, then we see that

$$|S|^{-1} q^{\frac{d}{r}} \leq R^*(p \to r) |S|^{-\frac{1}{p}},$$

where $\delta_\alpha(x) = 1$ if $x = \alpha$ and 0 otherwise. Since $|S| \sim q^k$, the quantity $R^*(p \to r)$ can be bounded by $O(1)$ only if

$$r \geq \frac{dp}{k(p-1)}. \tag{2.3.2}$$

However, if the algebraic variety S contains an affine subspace $H \subset \mathbb{F}_q^d$ of dimension n $(|H| = q^n)$, then we can improve the necessary condition in (2.3.2). In fact, by testing (2.2.1) with the characteristic function $H(x)$ on the affine subspace H, we see that

$$|H||S|^{-1} q^{\frac{d-n}{r}} \leq R^*(p \to r) (|H||S|^{-1})^{\frac{1}{p}}. \tag{2.3.3}$$

Since $|H| = q^n$ and $|S| \sim q^k$, the inequality (2.3.3) gives the improved necessary condition for $R^*(p \to r) \lesssim 1$, that is,

$$r \geq \frac{p(d-n)}{(p-1)(k-n)}. \tag{2.3.4}$$

From (2.3.1), (2.3.2), and (2.3.4), we have the following theorem related to the necessary conditions for $R^*(p \to r) \lesssim 1$.

Theorem 1. *Let S be an algebraic variety in $\mathbb{F}_q^d$ and $d\sigma$ the surface measure on S. If $|S| \sim q^k$ for some $0 < k < d$, then $R^*(p \to r)$ can be bounded by $O(1)$ only if*

$$r \geq \frac{2d}{k} \quad \textit{and} \quad r \geq \frac{dp}{k(p-1)}. \tag{2.3.5}$$

Moreover, if the algebraic variety S contains an affine subspace $H \subset \mathbb{F}_q^d$ of dimension n, then $R^*(p \to r)$ can be bounded by $O(1)$ only if

$$r \geq \frac{2d}{k} \quad \textit{and} \quad r \geq \frac{p(d-n)}{(p-1)(k-n)}. \tag{2.3.6}$$

2.4 Definition of Some Algebraic Varieties

In this section, we define the paraboloids, nondegenerate quadratic surfaces, and cones in d-dimensional vector spaces over finite fields. Let $\mathbb{F}_q$ denote a finite field with q elements, where q, a power of odd prime, is considered as an asymptotic parameter, and let $\mathbb{F}_q^d$ denote the d-dimensional vector space over the field $\mathbb{F}_q$. As usual, we denote by $\mathbb{F}_q^*$ the multiplicative group of $\mathbb{F}_q$. We define the paraboloid P in $\mathbb{F}_q^d$ by the set

$$P = \{(\underline{x}, x_d) \in \mathbb{F}_q^d : \underline{x} \in \mathbb{F}_q^{d-1}, x_d = \underline{x} \cdot \underline{x} \in \mathbb{F}_q\}, \tag{2.4.1}$$

an analog of the Euclidean paraboloid. We now define the nondegenerate quadratic surfaces in $\mathbb{F}_q^d$ in the usual way. Let $x = (x_1, x_2, \cdots, x_d) \in \mathbb{F}_q^d$. Denote by $Q(x)$ a homogeneous polynomial in $\mathbb{F}_q[x_1, \ldots, x_d]$ of degree 2. Since $char(\mathbb{F}_q) > 2$ throughout this paper, we can express $Q(x)$ in the form

$$Q(x_1, x_2, \ldots, x_d) = \sum_{i,j=1}^{d} a_{ij} x_i x_j \quad \text{with} \quad a_{ij} = a_{ji}.$$

If the $d \times d$ matrix $\{a_{ij}\}$ is invertible, then we say that the polynomial $Q(x)$ is a nondegenerate quadratic form over $\mathbb{F}_q$. For each $j \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, the multiplicative group of $\mathbb{F}_q$, consider a set Q_j in $\mathbb{F}_q^d$ given by

$$Q_j = \{x \in \mathbb{F}_q^d : Q(x_1, \cdots, x_d) = j\}, \tag{2.4.2}$$

where $Q(x)$ is a nondegenerate quadratic form. We call such a set Q_j a nondegenerate quadratic surface in $\mathbb{F}_q^d$. As an example of the nondegenerate quadratic surfaces in $\mathbb{F}_q^d$, we define the sphere as the set

$$S_j = \{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \cdots + x_d^2 = j \neq 0\}. \tag{2.4.3}$$

We define the cone $\mathcal{C}$ in $\mathbb{F}_q^d$ as the following set:

$$\mathcal{C} = \{x \in \mathbb{F}_q^d : x_d^2 = x_1^2 + \ldots + x_{d-1}^2\}. \tag{2.4.4}$$

In (2.4.4), by means of a nonsingular linear substitution, $x_d = u/2 + v/2$ and $x_{d-1} = u/2 - v/2$ and by a change of variable, $u = x_d, v = x_{d-1}$, we can also define the cone $\mathcal{C}$ in $\mathbb{F}_q^d$ as the set

$$\mathcal{C} = \{x \in \mathbb{F}_q^d : x_d x_{d-1} = x_1^2 + \ldots + x_{d-2}^2\}. \tag{2.4.5}$$

Remark 1. *Throughout the paper, we shall use the set $\mathcal{C}$ in (2.4.5) as the definition of the cone in $\mathbb{F}_q^d$.*

Chapter 3

CLASSICAL EXPONENTIAL SUMS

3.1 Gauss Sums

In this section, we shall collect useful lemmas which make an important role in proving our main results in this paper. Such lemmas can be obtained from the well-known facts related to estimates of Gauss sums. In particular, Gauss sums make an important role to estimate the Fourier transform of paraboloids and cones in d-dimensional vector spaces over finite fields. Let χ be a non-trivial additive character of $\mathbb{F}_q$ and η a multiplicative character of $\mathbb{F}_q$ of order two, that is, $\eta(ab) = \eta(a)\eta(b)$ and $\eta^2(a) = 1$ for all $a, b \in \mathbb{F}_q^*$ but $\eta \not\equiv 1$. For each $a \in \mathbb{F}_q$, the Gauss sum $G_a(\eta, \chi)$ is defined by

$$G_a(\eta, \chi) = \sum_{s \in \mathbb{F}_q^*} \eta(s)\chi(as). \tag{3.1.1}$$

The magnitude of the Gauss sum is given by the relation

$$|G_a(\eta, \chi)| = \begin{cases} q^{\frac{1}{2}} & \text{if} \quad a \neq 0 \\ 0 & \text{if} \quad a = 0. \end{cases}$$

Remark 2. *Here, and throughout this paper, we denote by χ and η the canonical additive character and the quadratic character of $\mathbb{F}_q$ respectively.*

The following theorem tells us the explicit value of the Gauss sum $G_1(\eta, \chi)$. For the nice proof, see [19].

Theorem 2. *Let $\mathbb{F}_q$ be a finite field with $q = \underline{p}^l$, where $\underline{p}$ is an odd prime and $l \in \mathbb{N}$. Then we have*

$$G_1(\eta, \chi) = \begin{cases} (-1)^{l-1} q^{\frac{1}{2}} & \text{if} \quad \underline{p} = 1 \ (mod 4) \\ (-1)^{l-1} i^l q^{\frac{1}{2}} & \text{if} \quad \underline{p} = 3 \ (mod 4). \end{cases}$$

In particular, we have

$$\sum_{s \in \mathbb{F}_q} \chi(as^2) = \eta^{-1}(a) G_1(\eta, \chi) = \eta(a) G_1(\eta, \chi) \quad \text{for any} \quad a \neq 0, \tag{3.1.2}$$

because η is the multiplicative character of $\mathbb{F}_q^*$ of order two. For the nice proof for this equality and the magnitude of Gauss sums, see [19] or [12]. As the direct application of the equality in (3.1.2), we have the following estimate.

Lemma 3. *For $\beta \in \mathbb{F}_q^k$ and $t \neq 0$, we have*

$$\sum_{\alpha \in \mathbb{F}_q^k} \chi(t\alpha \cdot \alpha + \beta \cdot \alpha) = \chi\left(\frac{\|\beta\|_2}{-4t}\right) \eta^k(t) \left(G_1(\eta, \chi)\right)^k,$$

where, here and throughout the paper, $\|\beta\|_2 = \beta \cdot \beta$.

Proof. It follows that

$$\sum_{\alpha \in \mathbb{F}_q^k} \chi(t\alpha \cdot \alpha + \beta \cdot \alpha) = \prod_{j=1}^{k} \sum_{\alpha_j \in \mathbb{F}_q} \chi(t\alpha_j^2 + \beta_j \alpha_j).$$

Completing the square in α_j-variables, applying a change of variable, $\alpha_j + \frac{\beta_j}{2t} \rightarrow \alpha_j$, and using the inequality in (3.1.2), the proof immediately follows. □

We shall introduce the explicit formula of $(d\sigma)^\vee$, the inverse Fourier transform of the surface measure related to the paraboloid.

Lemma 4. *Let $P \subset \mathbb{F}_q^d$ be the paraboloid and $d\sigma$ the surface measure on P. For each $m = (\underline{m}, m_d) \in \mathbb{F}_q^{d-1} \times \mathbb{F}_q$, we have*

$$(d\sigma)^\vee(m) = \begin{cases} q^{-(d-1)}\chi\left(\frac{\|\underline{m}\|_2}{-4m_d}\right)\eta^{d-1}(m_d) \quad (G_1(\eta,\chi))^{d-1} & \text{if} \quad m_d \neq 0 \\ 0 & \text{if} \quad m_d = 0, \underline{m} \neq \underline{0} \\ 1 & \text{if} \quad m = (0,\ldots,0). \end{cases}$$

Proof. For each $m = (\underline{m}, m_d) \in \mathbb{F}_q^d$, we have

$$\begin{aligned} (d\sigma)^\vee(m) &= \frac{1}{|S|}\sum_{x\in S}\chi(m\cdot x) \\ &= q^{-(d-1)}\sum_{\underline{x}\in\mathbb{F}_q^{d-1}}\chi\left(m_d\,\underline{x}\cdot\underline{x} + \underline{m}\cdot\underline{x}\right). \end{aligned}$$

If $m = (0,\ldots,0)$, then it is clear that $(d\sigma)^\vee(m) = 1$. If $m_d = 0, \underline{m} \neq \underline{0}$, then the orthogonality relation of the nontrivial character yields that $(d\sigma)^\vee(m) = 0$. On the other hand, if $m_d \neq 0$, then the proof is complete by Lemma 3. □

The following lemma gives us the explicit formula of $\mathcal{C}^\vee$, the inverse Fourier transform of the characteristic function on $\mathcal{C}$. In the remainder of this paper, we shall write G_a for the Gauss sum $G_a(\eta,\chi)$ defined as in (3.1.1).

Lemma 5. *Let $\mathcal{C} \in \mathbb{F}_q^d$ be the cone defined as before. If $d \geq 3$ is odd, then*

$$\mathcal{C}^\vee(m) = \eta(-1)q^{-d}G_1^{d-1}\eta(m_1^2 + \ldots + m_{d-2}^2 - 4m_dm_{d-1}) \quad \text{if} \quad m \neq (0,\ldots,0),$$

and

$$\mathcal{C}^\vee(0,\ldots,0) = q^{-1}.$$

If $d \geq 2$ is even, then we have

$$\mathcal{C}^\vee(m) = \begin{cases} -q^{-d}G_1^{d-2} & \text{if} \quad m_1^2 + \ldots + m_{d-2}^2 - 4m_dm_{d-1} \neq 0 \\ q^{-1} + q^{-d}(q-1)G_1^{d-2} & \text{if} \quad m = (0,\ldots,0) \\ q^{-d}(q-1)G_1^{d-2} & \text{otherwise.} \end{cases}$$

Proof. Using the definition of inverse Fourier transform and the orthogonality relations of the non-trivial additive character χ of $\mathbb{F}_q$, we see that

$$\begin{aligned}
\mathcal{C}^\vee(m) =& q^{-d}\sum_{x\in\mathcal{C}}\chi(x\cdot m)\\
=& q^{-d-1}\sum_{x\in\mathbb{F}_q^d}\sum_{s\in\mathbb{F}_q}\chi\left(s(x_1^2+\ldots+x_{d-2}^2-x_dx_{d-1})\right)\chi(x\cdot m)\\
=& q^{-1}\delta_0(m)+q^{-d-1}\sum_{x\in\mathbb{F}_q^d}\sum_{s\neq 0}\chi\left(s(x_1^2+\ldots+x_{d-2}^2-x_dx_{d-1})\right)\chi(x\cdot m),
\end{aligned}$$

where $\delta_0(m)=1$ if $m=(0,\ldots,0)$, and $\delta_0(m)=0$ otherwise. Using lemma 3, above expression is written by

$$q^{-1}\delta_0(m)+q^{-d-1}G_1^{d-2}\sum_{s\neq 0}\eta^{d-2}(s)\chi\left(\frac{m_1^2+\ldots+m_{d-2}^2}{-4s}\right)\Omega_m(s),$$

where $\Omega_m(s)=\sum_{x_d,x_{d-1}\in\mathbb{F}_q}\chi\left((-sx_d+m_{d-1})x_{d-1}\right)\chi(m_dx_d)$. By the orthogonality relations of the non-trivial additive character χ of $\mathbb{F}_q$, we see that

$$\Omega_m(s)=q\chi\left(\frac{m_dm_{d-1}}{s}\right)\quad\text{for}\quad s\neq 0.$$

Thus $\mathcal{C}^\vee(m)$ is given by the value

$$q^{-1}\delta_0(m)+q^{-d}G_1^{d-2}\sum_{s\neq 0}\eta^{d-2}(s)\chi\left(\frac{m_1^2+\ldots+m_{d-2}^2-4m_dm_{d-1}}{-4s}\right). \tag{3.1.3}$$

Case I. Suppose that $d\geq 3$ is odd. Then $\eta^{d-2}\equiv\eta$, because η is one of the multiplicative characters of order two. Thus if $m_1^2+\ldots+m_{d-2}^2-4m_dm_{d-1}=0$, then the proof is complete, because $\sum_{s\in\mathbb{F}^*}\eta(s)=0$ and $\eta(0)=0$. In particular, we have $\mathcal{C}^\vee(0,\ldots,0)=q^{-1}$. On the other hand, if $m_1^2+\ldots+m_{d-2}^2-4m_dm_{d-1}\neq 0$, then the result follows by a change of variable, $\frac{m_1^2+\ldots+m_{d-2}^2-4m_dm_{d-1}}{-4s}\to s$, and the facts that $\eta(4)=1$, $\eta(s)=\eta(s^{-1})$ for $s\neq 0$, and $G_1=\sum_{s\neq 0}\eta(s)\chi(s)$.

Case II. Suppose that $d\geq 2$ is even. Then $\eta^{d-2}\equiv 1$. Thus the proof is clear, because $\sum_{s\neq 0}\chi(as)=-1$ for all $a\neq 0$, and $\sum_{s\neq 0}\chi(as)=(q-1)$ if $a=0$. □

Remark 3. *From Lemma 5, we see that*

$$|\mathcal{C}| = \begin{cases} q^{d-1} & \text{if} \quad d \geq 3 \quad \text{is odd} \\ q^{d-1} + G_1^{d-2}(q-1) \sim q^{d-1} & \text{if} \quad d \geq 2 \quad \text{is even,} \end{cases}$$

because $\mathcal{C}^\vee(0, \dots, 0) = \frac{|\mathcal{C}|}{q^d}$.

The following corollary gives us the upper bound of $(d\sigma)^\vee$, the inverse Fourier transform of the surface measure of the cone $\mathcal{C}$.

Corollary 6. *Let* $\mathcal{C} \in \mathbb{F}_q^d$ *be the cone defined as before and* $d\sigma$ *the surface measure on the cone* $\mathcal{C}$. *If* $d \geq 3$ *is odd, then*

$$(d\sigma)^\vee(0, \dots, 0) = 1,$$

and for $m \neq (0, \dots, 0)$, *we have*

$$|(d\sigma)^\vee(m)| = \begin{cases} q^{-\frac{(d-1)}{2}} & \text{if} \quad m_1^2 + \dots + m_{d-2}^2 - 4m_d m_{d-1} \neq 0 \\ 0 & \text{if} \quad m_1^2 + \dots + m_{d-2}^2 - 4m_d m_{d-1} = 0. \end{cases}$$

On the other hand, if $d \geq 2$ *is even, then*

$$(d\sigma)^\vee(0, \dots, 0) = 1,$$

and for $m \neq (0, \dots, 0)$, *we have*

$$|(d\sigma)^\vee(m)| \sim \begin{cases} q^{-\frac{d}{2}} & \text{if} \quad m_1^2 + \dots + m_{d-2}^2 - 4m_d m_{d-1} \neq 0 \\ q^{-\frac{(d-2)}{2}} & \text{if} \quad m_1^2 + \dots + m_{d-2}^2 - 4m_d m_{d-1} = 0. \end{cases}$$

Proof. From Lemma 5, the proof is clear, because we have that $|G_1| = q^{\frac{1}{2}}$ and $(d\sigma)^\vee(m) = \frac{1}{|\mathcal{C}|} \sum_{x \in \mathcal{C}} \chi(x \cdot m) = \frac{q^d}{|\mathcal{C}|} \mathcal{C}^\vee(m) \sim q\mathcal{C}^\vee(m)$. □

3.2 Generalized Kloosterman Sums

The generalized Kloosterman sums make an important role in the proofs of the extension theorems for nondegenerate quadratic surfaces and spheres. The estimate for

Kloosterman sums due to Weil ([30]) is given by

$$\sum_{t\in\mathbb{F}_q^*}\chi(at+bt^{-1})\lesssim q^{\frac{1}{2}} \quad \text{for} \quad a,b\in\mathbb{F}_q^*, \tag{3.2.1}$$

and the estimate for twisted Kloosterman sums due to Salié ([22]) is given by

$$\sum_{t\in\mathbb{F}_q^*}\eta(t)\chi(at+bt^{-1})\lesssim q^{\frac{1}{2}} \quad \text{for} \quad a,b\in\mathbb{F}_q. \tag{3.2.2}$$

We have the following lemma which gives the sharp decay of the inverse Fourier transform of the nondegenerate quadratic surfaces Q_j in $\mathbb{F}_q^d$.

Lemma 7. *Let Q_j be the nondegenerate quadratic surface in $\mathbb{F}_q^d$. Then*

$$|Q_j{}^{\vee}(m)| = \left|q^{-d}\sum_{x\in Q_j}\chi(x\cdot m)\right| \lesssim q^{-\frac{d+1}{2}}$$

if $m\neq(0,\cdots,0)$, and

$$Q_j{}^{\vee}(0,\ldots,0)\sim q^{-1}.$$

Proof. For each $j\in\mathbb{F}_q^*$, the nondegenerate quadratic surface Q_j is given by

$$Q_j=\{x\in\mathbb{F}_q^d : Q(x_1,\cdots,x_d)=j\},$$

where $Q(x)$ is a nondegenerate quadratic form. Using the definition of the inverse Fourier transform and the orthogonality relation of the nontrivial additive character χ of $\mathbb{F}_q$, we see that

$$\begin{aligned}
Q_j{}^{\vee}(m) &= q^{-d}\sum_{x\in Q_j}\chi(x\cdot m)\\
&= q^{-d}\sum_{x\in\mathbb{F}_q^d}\chi(x\cdot m)q^{-1}\sum_{t\in\mathbb{F}_q}\chi(t(Q(x)-j))\\
&= q^{-1}\delta_0(m)+q^{-d-1}\sum_{t\in\mathbb{F}_q^*}\chi(-jt)\sum_{x\in\mathbb{F}_q^d}\chi(tQ(x)+x\cdot m)
\end{aligned}$$

where $\delta_0(m) = 1$ if $m = (0, \cdots, 0)$ and $\delta_0(m) = 0$ otherwise. To complete the proof of Lemma 7, it suffices to show that for $j \neq 0, m \in \mathbb{F}_q^d$,

$$D(j, m) \lesssim q^{\frac{d+1}{2}} \tag{3.2.3}$$

where

$$D(j, m) = \sum_{t \in \mathbb{F}_q^*} \chi(-jt) \sum_{x \in \mathbb{F}_q^d} \chi(tQ(x) + x \cdot m). \tag{3.2.4}$$

Let

$$W_t(m) = \sum_{x \in \mathbb{F}_q^d} \chi(tQ(x) + x \cdot m)$$

for $m \in \mathbb{F}_q^d, t \in \mathbb{F}_q^*$. We shall need the following theorem (see [19]).

Theorem 8. *Every quadratic form* $Q(x) = \sum_{i,k=1}^{d} a_{ik} x_i x_k$ *over* $\mathbb{F}_q, q$ *odd, can be transformed into a diagonal form* $a_1 x_1^2 + \cdots + a_d x_d^2$ *over* $\mathbb{F}_q$ *by means of a nonsingular linear substitution of indeterminates. Moreover if* $Q(x)$ *is a nondegenerate quadratic form, then* $a_i \neq 0$ *for all* $i = 1, 2, \cdots, d$.

Using Theorem 8, we may write that for some $m' = (m_1', \ldots, m_d') \in \mathbb{F}_q^d$, and $a_i \in \mathbb{F}_q^*$ for all $i = 1, 2, \ldots, d$,

$$W_t(m) = \sum_{x \in \mathbb{F}_q^d} \chi(t\|x\|_a + x \cdot m'),$$

where $m' \in \mathbb{F}_q^d$ is determined by $m \in \mathbb{F}_q^d$ and $\|x\|_a$ is given by

$$\|x\|_a = a_1 x_1^2 + \cdots + a_d x_d^2.$$

Since χ is an additive character of $\mathbb{F}_q$, we have

$$
\begin{aligned}
W_t(m) &= \prod_{k=1}^{d} \sum_{x_k \in \mathbb{F}_q} \chi(ta_k x_k^2 + m_k' x_k) \\
&= \prod_{k=1}^{d} \sum_{x_k \in \mathbb{F}_q} \chi\left(ta_k (x_k + (2ta_k)^{-1} m_k')^2 - (4ta_k)^{-1} {m_k'}^2\right) \\
&= \prod_{k=1}^{d} \chi\left(-(4ta_k)^{-1} {m_k'}^2\right) \sum_{x_k \in \mathbb{F}_q} \chi(ta_k x_k^2),
\end{aligned}
$$

where we complete the square in the second equality, and apply a change of variable in the last line. Using (3.1.2), we see that

$$
\sum_{x_k \in \mathbb{F}_q} \chi(ta_k x_k^2) = \eta^{-1}(ta_k) G_1(\eta, \chi),
$$

where η is a multiplicative character of $\mathbb{F}_q^*$ of order two and the Gauss sum $G_1(\eta, \chi)$ is given by $G_1(\eta, \chi) = \sum_{s \in \mathbb{F}_q^*} \chi(s)\eta(s)$. It therefore follows that

$$
\begin{aligned}
W_t(m) &= \eta^{-d}(t)\eta^{-1}(a_1 \cdots a_d)(G_1(\eta, \chi))^d \prod_{k=1}^{d} \chi\left(-(4ta_k)^{-1} {m_k'}^2\right) \\
&= \eta^{-d}(t)\eta^{-1}(a_1 \cdots a_d)(G_1(\eta, \chi))^d \chi\left(t^{-1} \sum_{k=1}^{d} -(4a_k)^{-1} {m_k'}^2\right). \qquad (3.2.5)
\end{aligned}
$$

Combining above fact in (3.2.5) with (3.2.4), we obtain that

$$
D(j, m) = \eta^{-1}(a_1 \cdots a_d)(G(\eta, \chi))^d \sum_{t \in \mathbb{F}_q^*} \chi(-jt + t^{-1} M)\eta^{-d}(t), \qquad (3.2.6)
$$

where M is given by

$$
M = \sum_{k=1}^{d} -(4a_k)^{-1} {m_k'}^2.
$$

Since η is a multiplicative character of order two, we see that $\eta^{-d} = 1$ for d even, and $\eta^{-d} = \eta$ for d odd. Therefore, if the dimension d is odd, then the inequality in (3.2.3) follows from (3.2.2) and (3.2.6) with the Gauss sum estimate. On the other hand, if d is even, then (3.2.1) and (3.2.6) yield the inequality in (3.2.3), because $j \neq 0$. This completes the proof of Lemma 7. □

Corollary 9.

$$|Q_j| \sim q^{d-1}.$$

Proof. Using the second part of Lemma 7, we have

$$q^{-1} \sim Q_j{}^{\vee}(0,\ldots,0) = q^{-d} \sum_{x\in\mathbb{F}_q^d} Q_j(x) = q^{-d}|Q_j|,$$

and the result follows. □

We can obtain the explicit formula for the Fourier transform of spheres S_j in $\mathbb{F}_q^d$.

Lemma 10. *Let S_j be a sphere in $\mathbb{F}_q^d$ defined as in (2.4.3). Then for any $m \in \mathbb{F}_q^d$, we have*

$$\widehat{S_j}(m) = q^{-1}\delta_0(m) + q^{-d-1}\eta^d(-1)G_1^d(\eta,\chi) \sum_{r\in\mathbb{F}_q^*} \eta^d(r)\chi\Big(jr + \frac{\|m\|_2}{4r}\Big),$$

where $\delta_0(m) = 1$ if $m = (0,\ldots,0)$ and $\delta_0(m) = 0$ otherwise.

Proof. Recall that we write $\widehat{S_j}$ for $\widehat{\chi_{S_j}}$. From the definition of the Fourier transform and the orthogonality relations for nontrivial characters, we have

$$\begin{aligned}\widehat{S_j}(m) &= q^{-d}\sum_{x\in\mathbb{F}_q^d} \chi(-x\cdot m) q^{-1} \sum_{r\in\mathbb{F}_q} \chi(-r(\|x\|_2 - j)) \\ &= q^{-1}\delta_0(m) + q^{-d-1}\sum_{r\in\mathbb{F}_q^*}\chi(jr)\sum_{x\in\mathbb{F}_q^d}\chi(-r\|x\|_2 - x\cdot m).\end{aligned}$$

Completing the squares and using the formula in (3.1.2), the proof is complete. □

Remark 4. *From Lemma 10, we see that $\widehat{S_j}(0,\ldots,0) \sim q^{-1}$, and so the number of elements of the sphere S_j is $\sim q^{d-1}$. In other words, $|S_j| \sim q^{d-1}$. Moreover, if $m \neq (0,\ldots,0)$, then $|\widehat{S_j}(m)| \lesssim q^{-\frac{d+1}{2}}$. If d is even, then $\eta^d = 1$, because η is a multiplicative character of $\mathbb{F}_q^*$ of order two. Moreover the exact value of $G_1^d(\eta,\chi)$ is* *given by the equation*

$$G_1^d(\eta,\chi) = Kq^{\frac{d}{2}} \quad \textit{for some} \quad K \in \mathbb{C},$$

where K depends on the additive character χ, the size of $\mathbb{F}_q$, and the dimension of $\mathbb{F}_q^d$. However it is uniformly bounded by 1. For the exact value of K, see Theorem 2. Thus for even d and for each $m \in \mathbb{F}_q^d$, we have

$$\widehat{S}_j(m) = q^{-1}\delta_0(m) + Kq^{-\frac{d+2}{2}} \sum_{r\in\mathbb{F}_q^*} \chi\Big(jr + \frac{\|m\|_2}{4r}\Big), \tag{3.2.7}$$

where the absolute value of K is one.

Remark 5. *Throughout the paper, the constant K may change from a line to another line, but it is uniformly bounded by one.*

Chapter 4

EXTENSION THEOREMS FOR PARABOLOIDS

4.1 Statement of Results

In this section, we state our results for the extension theorems related to the paraboloids in the finite field setting. We have the following $L^p - L^4$ estimate.

Theorem 11. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as in (2.4.1). If $d \geq 4$ is even, then we have*

$$R^*(p \to 4) \lessapprox 1 \quad \textit{for all} \quad p \geq \frac{4d}{3d-2}.$$

Remark 6. *Recall that $X \lesssim Y$ means that there exists $C > 0$, independent of q such that $X \leq CY$, and $X \lessapprox Y$, with the controlling parameter q, means that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $X \leq C_\varepsilon q^\varepsilon Y$.*

In higher even dimensions, we claim that Theorem 11 improves upon the Tomas-Stein exponents which Mockenhaupt and Tao obtained in [21]. To see this, note that the Tomas-Stein exponents in (1.2.1) imply that $R^*((4d-4)/(3d-5) \to 4) \lesssim 1$. Since $4d/(3d-2)$ is less than $(4d-4)/(3d-5)$, the claim immediately follows from the inequality in (2.2.3). Moreover, Theorem 11 is, in general, sharp except for the endpoint in the sense that for every $\varepsilon > 0, R^*(4d/(3d-2) - \varepsilon \to 4) \lesssim 1$ fails to be true. In fact, the sharpness of Theorem 11 follows from the necessary conditions (2.3.6) for $R^*(p \to r) \lesssim 1$ in even dimensions $d \geq 4$. To see this, observe that if $-1 \in \mathbb{F}_q$ is a square number, (say $i^2 = -1$ for some $i \in \mathbb{F}_q$), then the paraboloid P contains the subspace $H \subset \mathbb{F}_q^d$ of dimension $(d-2)/2$ which is given by

$$\left\{(s_1, is_1, \ldots, s_k, is_k, \ldots, s_{\frac{d-2}{2}}, is_{\frac{d-2}{2}}, 0, 0) : s_k \in \mathbb{F}_q,\ k = 1, 2, \ldots, \frac{d-2}{2}\right\}.$$

Since $|H| = q^{\frac{d-2}{2}}$ and $|P| = q^{d-1}$, the sharpness of Theorem 11 immediately follows from the second inequality of (2.3.6)(see Figure 4.2).

Remark 7. *In the case when* $d = 3$ *and* -1 *is not a square number, Mockenhaupt and Tao ([21]) obtained an improvement of the "p" index of the Tomas-Stein exponent* $R^*(2 \to 4)$ *by showing that* $R^*(8/5 \to 4) \lessapprox 1$. *However, if* -1 *is allowed to be a square number , one can not expect the improvement of the "p" index of the Tomas-Stein exponent* $R^*(p \to 4)$ *in the case when the dimension* $d \geq 3$ *is odd. This follows from the fact that if* -1 *is a square number and* d *is odd, then the paraboloid* P *always contains the subspace* $H \subset \mathbb{F}_q^d$ *of dimension* $(d-1)/2$*, defined by*

$$H = \left\{(s_1, is_1, \ldots, s_k, is_k, \ldots, s_{\frac{d-1}{2}}, is_{\frac{d-1}{2}}, 0) : s_k \in \mathbb{F}_q,\ k = 1, 2, \ldots, \frac{d-1}{2}\right\}.$$

Note that $|H| = q^{(d-1)/2}$ *and* $|P| = q^{d-1}$. *From these and the second inequality in (2.3.6), a necessary condition for* $R^*(p \to r) \lesssim 1$ *takes the form*

$$r \geq \frac{p(d+1)}{(p-1)(d-1)}.$$

Note that if $r = 4$*, then this necessary condition exactly matches the Tomas-Stein exponent in (1.2.1) (See Figure 4.1), which justifies the claim above.*

The next theorem is related to our $L^2 - L^r$ estimate for paraboloids P in higher even dimensions.

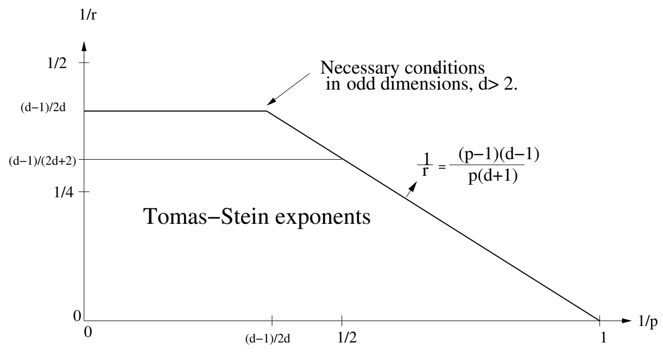

Figure 4.1: In odd dimensions $d \geq 3$, the necessary conditions for $R^*(p \to r)$ bound related to paraboloids.

Theorem 12. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as in (2.4.1). If $d \geq 4$ is even, then we have*

$$R^*(2 \to r) \lessapprox 1 \quad \textit{for all} \quad r \geq \frac{2d^2}{d^2 - 2d + 2}.$$

Note that the Tomas-Stein exponents in (1.2.1) yield that $R^*(2 \to (2d+2)/(d-1)) \lesssim 1$. In higher even dimensions, we therefore see that Theorem 12 gives an improvement of the "r" index of the Tomas-Stein exponent $R^*(2 \to (2d+2)/(d-1))$, because the exponent $2d^2/(d^2 - 2d + 2)$ is less than $(2d+2)/(d-1)$ (See Figure 4.2).

Remark 8. *Interpolating the results of Theorem 11, Theorem 12, and the trivial bound $R^*(1 \to \infty) \lesssim 1$ together with the fact in (2.2.3), our results can be described as in Figure 4.2.*

As mentioned in Remark 7, in general it is impossible to improve the "p" index of the Tomas-Stein exponent $R^*(p \to 4)$ in odd dimensions. However, if we assume that -1 is not a square number in the underlying finite field $\mathbb{F}_q$, then the improvement of the "p" index of the Tomas-Stein exponent $R^*(p \to 4)$ can be obtained in odd dimensions. For example, Mockenhaupt and Tao ([21]) obtained the improvement in three dimensions (see Remark 7). We shall extend their work to higher odd dimensions

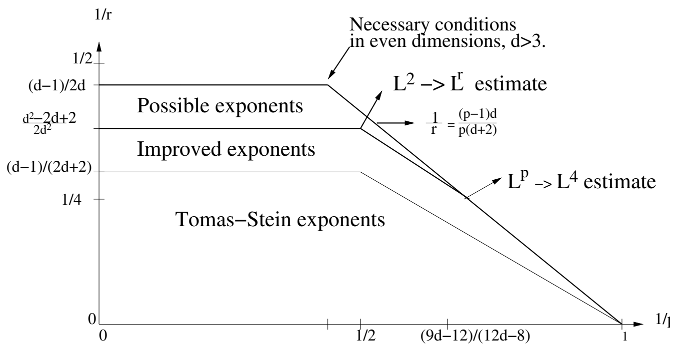

Figure 4.2: In even dimensions $d \geq 4$, exponents for $R^*(p \to r)$ bound related to paraboloids.

$d \geq 7$ in the specific case.

Theorem 13. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as before. Suppose that $d \geq 7$ is odd and $q = \underline{p}^l$ for some odd prime $\underline{p} \equiv 3$ (mod4). If $l(d-1)$ is not a multiple of four, then we have*

$$R^*(p \to 4) \lessapprox 1 \quad \text{for all} \quad p \geq \frac{4d}{3d-2}$$

and

$$R^*(2 \to r) \lessapprox 1 \quad \text{for all} \quad r \geq \frac{2d^2}{d^2-2d+2}.$$

Remark 9. *Observe that the assumptions in Theorem 13 imply that if l is an odd number, $q = \underline{p}^l$ for some odd prime $\underline{p} \equiv 3$ (mod 4), and $d = 4k+3$ for some $k \in \mathbb{N}$, then the conclusions in Theorem 13 hold. However, since any odd power of the prime $\underline{p} \equiv 3$ (mod 4) is $\equiv 3$ (mod 4), we need the condition that $q \equiv 3$ (mod 4) which means -1 is not a square number in $\mathbb{F}_q$. In conclusion, we can say that if -1 is not a square number in $\mathbb{F}_q$ and $d = 4k+3$ for some $k \in \mathbb{N}$, then the conclusions in Theorem 13 hold.*

4.2 Proofs of Results

In this section, we restate and prove the theorems stated in the previous section.

4.2.1 Proof of $L^p - L^4$ Estimate in Even Dimensions

Theorem 11. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as in (2.4.1). If $d \geq 4$ is even, then we have*

$$R^*(p \to 4) \lessapprox 1 \quad \textit{for all} \quad p \geq \frac{4d}{3d-2}.$$

Proof. Let $P \subset \mathbb{F}_q^d$ be the paraboloid defined as in (2.4.1). Using (2.2.3) and the usual dyadic pigeonholing argument (see [10]), it is enough to show that

$$\|(Ed\sigma)^{\vee}\|_{L^4(\mathbb{F}_q^d, dm)} \lesssim \|E\|_{L^{p_0}(P, d\sigma)}, \quad \text{for all} \quad E \subset P, \tag{4.2.1}$$

where $p_0 = 4d/(3d-2)$. Expanding both sizes in (4.2.1), we see that

$$\|E\|_{L^{p_0}(P,d\sigma)} = \left(\frac{|E|}{|P|}\right)^{\frac{1}{p_0}}$$

and

$$\|(Ed\sigma)^{\vee}\|_{L^4(\mathbb{F}_q^d, dm)} = \frac{q^{\frac{d}{4}}}{|P|} \left(\Lambda_4(E)\right)^{\frac{1}{4}}$$

where $\Lambda_4(E) = \sum_{\substack{x,y,z,w \in E \\ : x+y=z+w}} 1$. Since $|P| = q^{d-1}$, it therefore suffices to show that

$$\Lambda_4(E) \lesssim |E|^{\frac{4}{p_0}} q^{3d-4} q^{\frac{-4d+4}{p_0}} \quad \text{for all} \quad E \subset P. \tag{4.2.2}$$

We shall need the following estimate.

Lemma 14. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as before. In addition, we assume that the dimension of $\mathbb{F}_q^d, d \geq 4$, is even. If E is any subset of P, then we have*

$$\Lambda_4(E) \lesssim \min\{|E|^3, q^{-1}|E|^3 + q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} + q^{\frac{d-2}{2}}|E|^2\}.$$

For a moment, we assume that Lemma 14 holds. Note that Lemma 14 implies that if $d \geq 4$ is even and E is any subset of the paraboloid P, then we have

$$\Lambda_4(E) \lesssim \begin{cases} q^{-1}|E|^3 & \text{if } \; q^{\frac{d+2}{2}} \lesssim |E| \lesssim q^{d-1} \\ q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} & \text{if } \; q^{\frac{d-2}{2}} \lesssim |E| \lesssim q^{\frac{d+2}{2}} \\ |E|^3 & \text{if } \; 1 \lesssim |E| \lesssim q^{\frac{d-2}{2}}. \end{cases}$$

Using these upper bounds of $\Lambda_4(E)$ depending on the size of the subset of P, the inequality in (4.2.2) follows by the direct calculation. Thus the proof of Theorem 11 is complete once we establish the proof of Lemma 14. To prove Lemma 14, we first note that $\Lambda_4(E) \leq |E|^3$ for all $E \subset P$, because if we fix $x, y, z \in E$, then there is at most one w with $x + y = z + w$. It therefore suffices to show that

$$\Lambda_4(E) \lesssim q^{-1}|E|^3 + q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} + q^{\frac{d-2}{2}}|E|^2 \quad \text{for all} \quad E \subset P. \tag{4.2.3}$$

Since the set E is a subset of the paraboloid P, we have

$$\Lambda_4(E) = \sum_{\substack{x,y,z,w \in E \\ :x+y=z+w}} 1 \leq \sum_{\substack{x,y,z \in E \\ :x+y-z \in P}} 1.$$

For each $x, y, z \in E$, write $x + y - z = (\underline{x} + \underline{y} - \underline{z},\ x_d + y_d - z_d)$. Then we see that $x + y - z \in P \iff \underline{x} \cdot \underline{y} - \underline{y} \cdot \underline{z} - \underline{x} \cdot \underline{z} + \underline{z} \cdot \underline{z} = 0$, because x, y, z are elements of the paraboloid P. It therefore follows that

$$\Lambda_4(E) \leq \sum_{\underline{x},\underline{y},\underline{z} \in \underline{E}} \delta_0(\underline{x} \cdot \underline{y} - \underline{y} \cdot \underline{z} - \underline{x} \cdot \underline{z} + \underline{z} \cdot \underline{z})$$

where $\underline{E} = \{\underline{x} \in \mathbb{F}_q^{d-1} : (\underline{x}, \underline{x} \cdot \underline{x}) = x \in E\}$, and $\delta_0(t) = 1$ if $t = 0$ and 0 otherwise. It follows that

$$\begin{aligned} \Lambda_4(E) &\leq \sum_{\underline{x},\underline{y},\underline{z} \in \underline{E}} q^{-1} \sum_{s \in \mathbb{F}_q} \chi\left(s(\underline{x} \cdot \underline{y} - \underline{y} \cdot \underline{z} - \underline{x} \cdot \underline{z} + \underline{z} \cdot \underline{z})\right) \\ &= q^{-1}|\underline{E}|^3 + R(\underline{E}) \end{aligned}$$

where $R(\underline{E}) = \sum_{\underline{x},\underline{y},\underline{z}\in\underline{E}} q^{-1}\sum_{s\neq 0}\chi\left(s(\underline{x}\cdot\underline{y}-\underline{y}\cdot\underline{z}-\underline{x}\cdot\underline{z}+\underline{z}\cdot\underline{z})\right)$. In order to prove the inequality in (4.2.3), it therefore suffices to show that

$$|R(\underline{E})|^2 \lesssim q^{\frac{d-2}{2}}|\underline{E}|^5 + q^{d-2}|\underline{E}|^4, \tag{4.2.4}$$

because $|E| = |\underline{E}|$. Let us estimate $|R(\underline{E})|^2$. The Cauchy-Schwarz inequality applied to the sum in the variable $\underline{x}$ yields

$$|R(\underline{E})|^2 \leq q^{-2}|\underline{E}|\sum_{\underline{x}\in\underline{E}}\left|\sum_{\underline{y},\underline{z}\in\underline{E},s\neq 0}\chi\left(s(\underline{x}\cdot\underline{y}-\underline{y}\cdot\underline{z}-\underline{x}\cdot\underline{z}+\underline{z}\cdot\underline{z})\right)\right|^2.$$

Applying the Cauchy-Schwarz inequality to the sum in the variable $\underline{z}$ and then dominating the sum over $\underline{z}\in\underline{E}$ by the sum over $\underline{z}\in\mathbb{F}_q^{d-1}$, we have

$$\begin{aligned}|R(\underline{E})|^2 \leq& q^{-2}|\underline{E}|^2\sum_{\underline{x}\in\underline{E}}\sum_{\underline{z}\in\mathbb{F}_q^{d-1}}\left|\sum_{\underline{y}\in\underline{E},s\neq 0}\chi\left(s(\underline{x}\cdot\underline{y}-\underline{y}\cdot\underline{z}-\underline{x}\cdot\underline{z}+\underline{z}\cdot\underline{z})\right)\right|^2\\ =& q^{-2}|\underline{E}|^2\sum_{\underline{x}\in\underline{E}}M(\underline{x})\end{aligned}$$

where $M(\underline{x}) = \sum_{\underline{z}\in\mathbb{F}_q^{d-1}}\left|\sum_{\underline{y}\in\underline{E},s\neq 0}\chi\left(s(\underline{x}\cdot\underline{y}-\underline{y}\cdot\underline{z}-\underline{x}\cdot\underline{z}+\underline{z}\cdot\underline{z})\right)\right|^2$. To prove the inequality in (4.2.4), it is enough to show that

$$M(\underline{x}) \lesssim q^{\frac{d+2}{2}}|\underline{E}|^2 + q^d|\underline{E}| \quad \text{for all} \quad \underline{x}\in\underline{E}. \tag{4.2.5}$$

Let us estimate the value $M(\underline{x})$ which is written by

$$\begin{aligned}
&\sum_{\substack{\underline{z}\in\mathbb{F}_q^{d-1},\\ \underline{y},\underline{y}'\in\underline{E},\\ s,s'\neq 0}} \chi\left(s(\underline{x}\cdot\underline{y}-\underline{y}\cdot\underline{z}-\underline{x}\cdot\underline{z}+\underline{z}\cdot\underline{z})\right)\chi\left(-s'(\underline{x}\cdot\underline{y}'-\underline{y}'\cdot\underline{z}-\underline{x}\cdot\underline{z}+\underline{z}\cdot\underline{z})\right)\\
=&\sum_{\substack{\underline{z}\in\mathbb{F}_q^{d-1},\\ \underline{y},\underline{y}'\in\underline{E},\\ s,s'\neq 0:s=s'}} \chi\left((s-s')\underline{z}\cdot\underline{z}+(s(-\underline{y}-\underline{x})+s'(\underline{y}'+\underline{x}))\cdot\underline{z}\right)\chi\left((s\underline{y}-s'\underline{y}')\cdot\underline{x}\right)\\
&+\sum_{\substack{\underline{z}\in\mathbb{F}_q^{d-1},\\ \underline{y},\underline{y}'\in\underline{E},\\ s,s'\neq 0:s\neq s'}} \chi\left((s-s')\underline{z}\cdot\underline{z}+(s(-\underline{y}-\underline{x})+s'(\underline{y}'+\underline{x}))\cdot\underline{z}\right)\chi\left((s\underline{y}-s'\underline{y}')\cdot\underline{x}\right)\\
=&I+II.
\end{aligned}$$

Since $s=s'\neq 0$ in the term I, we have

$$\begin{aligned}
I=&\sum_{\substack{\underline{y},\underline{y}'\in\underline{E},\\ s\neq 0}}\sum_{\underline{z}\in\mathbb{F}_q^{d-1}} \chi\left(s(-\underline{y}+\underline{y}')\cdot\underline{z}\right)\chi\left(s(\underline{y}-\underline{y}')\cdot\underline{x}\right)\\
=&q^{d-1}\sum_{\underline{y}\in\underline{E},s\neq 0} 1 = q^{d-1}(q-1)|\underline{E}|\lesssim q^d|\underline{E}|,
\end{aligned}$$

where the second line follows from the orthogonality relations for the non-trivial character χ related to the variable $\underline{z}\in\mathbb{F}_q^d$. To complete the proof of Lemma 14, it remains to show that

$$II\lesssim q^{\frac{d+2}{2}}|\underline{E}|^2. \tag{4.2.6}$$

Setting $a=s, b=\frac{s'}{s}$, we see that

$$II=\sum_{\substack{\underline{z}\in\mathbb{F}_q^{d-1},\\ \underline{y},\underline{y}'\in\underline{E},\\ a\neq 0,b\neq 0,1}} \chi\left(a(1-b)\underline{z}\cdot\underline{z}+a(-\underline{y}-\underline{x}+b(\underline{y}'+\underline{x}))\cdot\underline{z}\right)\chi\left(a(\underline{y}-b\underline{y}')\cdot\underline{x}\right).$$

Calculate the sum over $\underline{z}\in\mathbb{F}_q^{d-1}$ by using Lemma 3 and then the term II is given by $(G_1(\eta,\chi))^{d-1}$ times

$$\sum_{\substack{\underline{y},\underline{y}'\in\underline{E},\\ a\neq 0,b\neq 0,1}} \eta(1-b)\eta(a)\chi\left(\left[\frac{\|(-\underline{y}-\underline{x})+b(\underline{y}'+\underline{x})\|_2}{-4(1-b)}+(\underline{y}-b\underline{y}')\cdot\underline{x}\right]a\right),$$

where we also used the fact that $\eta^{d-1} = \eta$, because d is even and η is a multiplicative character of order two. Since the sum over the variable $a \in \mathbb{F}_q^*$ is the Gauss sum, we conclude that

$$II \lesssim q^{\frac{d-1}{2}} q^{\frac{1}{2}} (q-2)|\underline{E}|^2 \lesssim q^{\frac{d+2}{2}} |\underline{E}|^2.$$

Thus the inequality in (4.2.6) holds and the proof of Lemma 14 is complete. Thus we complete the proof of Theorem 11. □

4.2.2 Proof of $L^2 - L^r$ estimate in Even Dimensions

Theorem 12. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as in (2.4.1). If $d \geq 4$ is even, then we have*

$$R^*(2 \to r) \lessapprox 1 \quad \textit{for all} \quad r \geq \frac{2d^2}{d^2 - 2d + 2}.$$

Proof. In order to prove Theorem 12, we shall use Theorem 11 together with interpolation theorems and some specific properties of the Fourier transform of the paraboloids. By (2.2.4), it suffices to show that

$$R^* \left(2 \to \frac{2d^2}{d^2 - 2d + 2} \right) \lessapprox 1 \quad \text{whenever} \quad d \geq 4 \quad \text{is even.} \tag{4.2.7}$$

Let $R^* : L^p(P, d\sigma) \to L^r(\mathbb{F}_q^d, dm)$ be the extension map $f \to (f d\sigma)^\vee$, and $R : L^{r'}(\mathbb{F}_q^d, dm) \to L^{p'}(P, d\sigma)$ be its dual, the restriction map $g \to \widehat{g}|_P$. Note that $R^* R g = (\widehat{g} d\sigma)^\vee = g * (d\sigma)^\vee$ for all function g on $\mathbb{F}_q^d$. We shall use the following theorem which is known as a TT^* method in the Euclidean setting. For the reader's convenience, we review the proof.

Theorem 15. *Let S be an algebraic variety in $\mathbb{F}_q^d$ and $d\sigma$ the surface measure on S. Suppose that $\|g * (d\sigma)^\vee\|_{L^{p'}(\mathbb{F}_q^d, dm)} \lesssim \|g\|_{L^p(\mathbb{F}_q^d, dm)}$ for all functions g on $\mathbb{F}_q^d$. Then the* *following extension estimate holds:*

$$\|(fd\sigma)^{\vee}\|_{L^{p'}(\mathbb{F}_q^d,dm)} \lesssim \|f\|_{L^2(S,d\sigma)} \textit{ for all } f \textit{ on } S.$$

Namely, $R^*(2 \to p') \lesssim 1$.

Proof of Theorem 15. By duality, it suffices to show that

$$\|\widehat{g}\|_{L^2(S,d\sigma)} \lesssim \|g\|_{L^p(\mathbb{F}_q^d,dm)} \quad \text{for all} \quad g \in L^p(\mathbb{F}_q^d, dm). \tag{4.2.8}$$

By the orthogonality principle, we see that

$$\|\widehat{g}\|^2_{L^2(S,d\sigma)} =< Rg,\ Rg >_{L^2(S,d\sigma)} =< R^*Rg,\ g >_{L^2(\mathbb{F}_q^d,dm)}.$$

By Hölder's inequality and our assumption, (4.2.8) holds and so the proof is complete.

□

We return to the proof of Theorem 12. In order to show the inequality (4.2.7), from Theorem 15 it suffices to show that

$$\|g * (d\sigma)^{\vee}\|_{L^{p_0}(\mathbb{F}_q^d,dm)} \lessapprox \|g\|_{L^{p_0'}(\mathbb{F}_q^d,dm)}$$

where $p_0 = \frac{2d^2}{d^2-2d+2}$ and $p_0' = \frac{2d^2}{d^2+2d-2}$. Note that

$$\begin{aligned}\|g * \delta_0\|_{L^{p_0}(\mathbb{F}_q^d,dm)} &= \|g\|_{L^{p_0}(\mathbb{F}_q^d,dm)} \\ &\leq \|g\|_{L^{p_0'}(\mathbb{F}_q^d,dm)}\end{aligned}$$

where the last line follows from the facts that dm is the counting measure and $p_0 > p_0'$. Thus it is enough to show that

$$\|g * K\|_{L^{p_0}(\mathbb{F}_q^d,dm)} \lessapprox \|g\|_{L^{p_0'}(\mathbb{F}_q^d,dm)}, \tag{4.2.9}$$

where K is the Bochner-Riesz kernel given by $K = (d\sigma)^\vee - \delta_0$. Recall that $K(m) = 0$ if $m = (0, \ldots, 0)$, and $K(m) = (d\sigma)^\vee(m)$ otherwise. We now claim that the following two estimates hold:

$$\|g * K\|_{L^2(\mathbb{F}_q^d, dm)} \lessapprox q\|g\|_{L^2(\mathbb{F}_q^d, dm)} \tag{4.2.10}$$

and

$$\|g * K\|_{L^4(\mathbb{F}_q^d, dm)} \lessapprox q^{\frac{-d+4}{4}} \|g\|_{L^{\frac{4d}{3d-2}}(\mathbb{F}_q^d, dm)}. \tag{4.2.11}$$

The inequality in (4.2.10) follows from elementary properties of Fourier transform. In fact, we have

$$\begin{aligned} \|g * K\|_{L^2(\mathbb{F}_q^d, dm)} &= \|\widehat{g}\widehat{K}\|_{L^2(\mathbb{F}_q^d, dx)} \\ &\le \|\widehat{K}\|_{L^\infty(\mathbb{F}_q^d, dx)} \|g\|_{L^2(\mathbb{F}_q^d, dm)}. \end{aligned}$$

However, $\widehat{K}(x) = d\sigma(x) - \widehat{\delta_0}(x) = qP(x) - 1 \le q$. Therefore the inequality in (4.2.10) holds. For a moment, let us assume that the inequality in (4.2.11) holds. Note that the operator $g \to g * K$ is self-adjoint, because K is the inverse Fourier transform of real-valued function $d\sigma$. Thus the inequality in (4.2.11) implies that

$$\|g * K\|_{L^{\frac{4d}{d+2}}(\mathbb{F}_q^d, dm)} \lessapprox q^{\frac{-d+4}{4}} \|g\|_{L^{\frac{4}{3}}(\mathbb{F}_q^d, dm)}. \tag{4.2.12}$$

Interpolating (4.2.11) and (4.2.12) (with $\theta = \frac{1}{2}$ in the Riesz-Thorin theorem), we also see that

$$\|g * K\|_{L^{\frac{4d}{d+1}}(\mathbb{F}_q^d, dm)} \lessapprox q^{\frac{-d+4}{4}} \|g\|_{L^{\frac{4d}{3d-1}}(\mathbb{F}_q^d, dm)}. \tag{4.2.13}$$

Once again interpolate (4.2.10) and (4.2.13) (with $\theta = \frac{4}{d}$ in the Riesz-Thorin theorem), and then we obtain the inequality in (4.2.9). Thus the proof of Theorem 12 is complete if we can prove the inequality (4.2.11) . The inequality (4.2.11) is a routine modification of the proof of Theorem 6.2 in [21]. We now introduce the proof of the inequality (4.2.11). For $d \geq 4$ even, we must show that

$$\|g * K\|_{L^4(\mathbb{F}_q^d, dm)} \lessapprox q^{\frac{-d+4}{4}} \|g\|_{L^{\frac{4d}{3d-2}}(\mathbb{F}_q^d, dm)}$$

where g is the arbitrary function on $\mathbb{F}_q^d$ and $K = (d\sigma)^\vee - \delta_0$. Let $m = (m_1, \ldots, m_{d-1}, m_d) \in \mathbb{F}_q^d$. For each $a \in \mathbb{F}_q$ and the function g on $\mathbb{F}_q^d$, we define the function g_a as the restriction of g to the hyperplane $\{m \in \mathbb{F}_q^d : m_d = a\}$. Then it is enough to show that for each $a \in \mathbb{F}_q$, the estimate

$$\|g_a * K\|_{L^4(\mathbb{F}_q^d, dm)} \lessapprox q^{\frac{-d^2+3d-2}{4d}} \|g_a\|_{L^{\frac{4d}{3d-2}}(\mathbb{F}_q^d, dm)} \tag{4.2.14}$$

holds, because the estimate (4.2.14) yields the following estimates:

$$\begin{aligned}
\|g * K\|_{L^4(\mathbb{F}_q^d, dm)} &\leq \sum_{a \in \mathbb{F}_q} \|g_a * K\|_{L^4(\mathbb{F}_q^d, dm)} \\
&\lessapprox q^{\frac{-d^2+3d-2}{4d}} \sum_{a \in \mathbb{F}_q} \|g_a\|_{L^{\frac{4d}{3d-2}}(\mathbb{F}_q^d, dm)} \\
&\leq q^{\frac{-d^2+3d-2}{4d}} q^{\frac{d+2}{4d}} \left(\sum_{a \in \mathbb{F}_q} \|g_a\|^{\frac{4d}{3d-2}}_{L^{\frac{4d}{3d-2}}(\mathbb{F}_q^d, dm)} \right)^{\frac{3d-2}{4d}} \\
&= q^{\frac{-d+4}{4}} \|g\|_{L^{\frac{4d}{3d-2}}(\mathbb{F}_q^d, dm)},
\end{aligned}$$

where (4.2.14) is used in the second inequality, Hölder's inequality is applied in the third line, and the last equality follows from the fact that the supports of g_a and g_b are disjoint if $a \neq b$. Without loss of generality, we may assume that $a = 0$, because of translation invariance. Thus it suffices to show that

$$\|g_0 * K\|_{L^4(\mathbb{F}_q^d, dm)} \lessapprox q^{\frac{-d^2+3d-2}{4d}} \|g_0\|_{L^{\frac{4d}{3d-2}}(\mathbb{F}_q^d, dm)}. \tag{4.2.15}$$

By Lemma 4 and the definition of K, note that

$$K(\underline{m}, m_d) = q^{-(d-1)} \chi\left(\frac{\|\underline{m}\|_2}{-4m_d}\right) \eta^{d-1}(m_d) \left(G_1(\eta, \chi)\right)^{d-1}$$

when $m_d \neq 0$, and $K(\underline{m}, 0) = 0$. Using this and the definition of the function g_0, the left-hand side of (4.2.15) is given by

$$q^{\frac{-d+1}{2}} \left(\sum_{\underline{m} \in \mathbb{F}_q^{d-1}} \sum_{m_d \neq 0} \left| \sum_{\underline{m}' \in \mathbb{F}_q^{d-1}} g(\underline{m}', 0) \chi\left(\frac{\|\underline{m} - \underline{m}'\|_2}{4m_d}\right) \right|^4 \right)^{\frac{1}{4}}, \tag{4.2.16}$$

where we used the facts that $|G_1(\eta, \chi)| = q^{\frac{1}{2}}$ and $|\eta| = 1$ and then a change of variable, $-m_d \to m_d$. Letting $\underline{u} = \frac{-m}{2m_d}$ and $s = \frac{1}{4m_d}$, we see that

$$\frac{\|\underline{m} - \underline{m}'\|_2}{4m_d} = \frac{1}{4s}\, \underline{u} \cdot \underline{u} + \underline{u} \cdot \underline{m}' + s\, \underline{m}' \cdot \underline{m}'.$$

Thus the term (4.2.16) becomes

$$\begin{aligned} & q^{\frac{-d+1}{2}} \left(\sum_{\underline{u} \in \mathbb{F}_q^{d-1}} \sum_{s \neq 0} \left| \chi(\frac{1}{4s}\, \underline{u} \cdot \underline{u}) \sum_{\underline{m}' \in \mathbb{F}_q^{d-1}} g(\underline{m}', 0) \chi\left((\underline{u}, s) \cdot (\underline{m}', \underline{m}' \cdot \underline{m}')\right) \right|^4 \right)^{\frac{1}{4}} \\ \leq & q^{\frac{-d+1}{2}} \left(\sum_{(\underline{u}, s) \in \mathbb{F}_q^d} |(Gd\sigma)^{\vee}(\underline{u}, s)|^4 \right)^{\frac{1}{4}} = q^{\frac{-d+1}{2}} \|(Gd\sigma)^{\vee}\|_{L^4(\mathbb{F}_q^d, dm)}, \end{aligned}$$

where the function G on the paraboloid P is defined by

$$G(\underline{x}, \underline{x} \cdot \underline{x}) = |P| g(\underline{x}, 0) = q^{d-1} g(\underline{x}, 0).$$

Using Theorem 11, above expression can be bounded by

$$\lessapprox q^{\frac{-d+1}{2}} \|G\|_{L^{\frac{4d}{3d-2}}(P, d\sigma)} = q^{\frac{-d^2+3d-2}{4d}} \|g_0\|_{L^{\frac{4d}{3d-2}}(\mathbb{F}_q, dm)},$$

where the equality follows from the observation that

$$\begin{aligned}\|G\|_{L^{\frac{4d}{3d-2}}(P,d\sigma)} &= \left(\frac{1}{|P|}\sum_{x\in P}|G(x)|^{\frac{4d}{3d-2}}\right)^{\frac{3d-2}{4d}}\\ &= q^{\frac{-(d-1)(3d-2)}{4d}}\left(\sum_{\underline{m}\in\mathbb{F}_q^{d-1}}\left|q^{d-1}g(\underline{m},0)\right|^{\frac{4d}{3d-2}}\right)^{\frac{3d-2}{4d}}\\ &= q^{\frac{(d+2)(d-1)}{4d}}\|g_0\|_{L^{\frac{4d}{3d-2}}(\mathbb{F}_q,dm)}.\end{aligned}$$

Thus the estimate in (4.2.15) holds and so the proof of Theorem 12 is complete. □

4.2.3 Proof of $L^p - L^r$ Estimate in Odd Dimensions

Theorem 13. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as before. Suppose that $d \geq 7$ is odd and $q = \underline{p}^l$ for some odd prime $\underline{p} \equiv 3 \ (mod 4)$. If $l(d-1)$ is not a multiple of four, then we have*

$$R^*(p \to 4) \lessapprox 1 \quad \textit{for all} \quad p \geq \frac{4d}{3d-2} \tag{4.2.17}$$

and

$$R^*(2 \to r) \lessapprox 1 \quad \textit{for all} \quad r \geq \frac{2d^2}{d^2-2d+2}. \tag{4.2.18}$$

Proof. Since the proof of the second part (4.2.18) of Theorem 13 follows from the same argument as that of Theorem 12, we shall only prove the first part (4.2.17) of Theorem 13. The proof of the first part (4.2.17) of Theorem 13 is almost similar to that of Theorem 11, but we shall use the explicit formula for Gauss sums in Theorem 2. Let us prove the first part (4.2.17) of Theorem 13. Using the arguments for the proof of Theorem 11, it suffices to show that

$$\Lambda_4(E) \lesssim |E|^{\frac{4}{p_0}} q^{3d-4} q^{\frac{-4d+4}{p_0}} \quad \text{for all} \quad E \subset P, \tag{4.2.19}$$

where $p_0=\frac{4d}{3d-2}$ and $\Lambda_4(E)=\sum\limits_{\substack{x,y,z,w\in E\\ :x+y=z+w}} 1$. We shall use the following lemma.

Lemma 16. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as before. Suppose that d is odd and $q=\underline{p}^l$ for some prime $\underline{p}=3$ (mod4). If $l(d-1)$ is not a multiple of four and E is any subset of P, then we have*

$$\Lambda_4(E)\lesssim \min\{|E|^3, q^{-1}|E|^3+q^{\frac{d-3}{4}}|E|^{\frac{5}{2}}+q^{\frac{d-2}{2}}|E|^2\}.$$

We assume this lemma for a moment. By Lemma 16, we see that

$$\Lambda_4(E)\lesssim \begin{cases} q^{-1}|E|^3 & \text{if} \quad q^{\frac{d+1}{2}}\lesssim |E|\lesssim q^{d-1} \\ q^{\frac{d-3}{4}}|E|^{\frac{5}{2}} & \text{if} \quad q^{\frac{d-1}{2}}\lesssim |E|\lesssim q^{\frac{d+1}{2}} \\ q^{\frac{d-2}{2}}|E|^2 & \text{if} \quad q^{\frac{d-2}{2}}\lesssim |E|\lesssim q^{\frac{d-1}{2}} \\ |E|^3 & \text{if} \quad 1\lesssim |E|\lesssim q^{\frac{d-2}{2}}. \end{cases}$$

Note that this estimate yields the inequality in (4.2.19). Thus it is enough to show that Lemma 16 holds. Let us prove Lemma 16. Let E be a subset of the paraboloid P. Since $\Lambda_4(E)\le |E|^3$ for each $E\subset P$, it suffice to show

$$\Lambda_4(E)\lesssim q^{-1}|E|^3+q^{\frac{d-3}{4}}|E|^{\frac{5}{2}}+q^{\frac{d-2}{2}}|E|^2 \quad \text{for} \quad E\subset P. \tag{4.2.20}$$

Define the set $\underline{E}\subset\mathbb{F}_q^{d-1}$ by $\underline{E}=\{\underline{x}\in\mathbb{F}_q^{d-1}:(\underline{x},\underline{x}\cdot\underline{x})=x\in E\}$. Repeating the same arguments as in the proof of Lemma 14, we see that

$$\Lambda_4(E)\le q^{-1}|\underline{E}|^3+R(\underline{E}),$$

where $R(\underline{E})=\sum\limits_{\underline{x},\underline{y},\underline{z}\in\underline{E}} q^{-1}\sum\limits_{s\neq 0}\chi\left(s(\underline{x}\cdot\underline{y}-\underline{y}\cdot\underline{z}-\underline{x}\cdot\underline{z}+\underline{z}\cdot\underline{z})\right)$. From this and the fact that $|E|=|\underline{E}|$, it is enough to show that

$$|R(\underline{E})|^2\lesssim q^{\frac{d-3}{2}}|\underline{E}|^5+q^{d-2}|\underline{E}|^4.$$

As in the proof of Lemma 14, we have

$$|R(\underline{E})|^2\le q^{-2}|\underline{E}|^2\sum_{\underline{x}\in\underline{E}}M(\underline{x}),$$

where $M(\underline{x}) = \sum_{\underline{z}\in\mathbb{F}_q^{d-1}} \left| \sum_{\underline{y}\in\underline{E}, s\neq 0} \chi\left(s(\underline{x}\cdot\underline{y} - \underline{y}\cdot\underline{z} - \underline{x}\cdot\underline{z} + \underline{z}\cdot\underline{z})\right)\right|^2$. It therefore suffices to show that

$$M(\underline{x}) \lesssim q^{\frac{d+1}{2}}|\underline{E}|^2 + q^d|\underline{E}| \quad \text{for all} \quad \underline{x}\in\underline{E}. \tag{4.2.21}$$

From the proof of Lemma 14, recall that $M(\underline{x})$ is given by

$$M(\underline{x}) = q^{d-1}(q-1)|\underline{E}| + II, \tag{4.2.22}$$

where

$$II = \sum_{\substack{\underline{z}\in\mathbb{F}_q^{d-1},\\ \underline{y},\underline{y'}\in\underline{E},\\ a\neq 0, b\neq 0,1}} \chi\left(a(1-b)\underline{z}\cdot\underline{z} + a(-\underline{y}-\underline{x}+b(\underline{y'}+\underline{x}))\cdot\underline{z}\right)\chi\left(a(\underline{y}-b\underline{y'})\cdot\underline{x}\right).$$

Here, the main point is to observe that since $M(\underline{x})$ is a nonnegative real number, the term II must be a real number. Thus if II is a negative real number, then we see from (4.2.22) that

$$0 \le M(\underline{x}) < q^{d-1}(q-1)|\underline{E}| \sim q^d|\underline{E}|,$$

which implies that the inequality in (4.2.21) holds. Without loss of generality, we therefore assume that the term II is a nonnegative real number. To complete the proof of the first part (4.2.17) of Theorem 13, it therefore suffices to show that

$$II \le q^{\frac{d+1}{2}}|\underline{E}|^2, \tag{4.2.23}$$

where we assume that II is a nonnegative. To estimate II, calculate the sum over $\underline{z}\in\mathbb{F}_q^{d-1}$ by using Lemma 3 and then we see that the term II is given by

$$(G_1(\eta,\chi))^{d-1} \sum_{\substack{\underline{y},\underline{y'}\in\underline{E},\\ a\neq 0, b\neq 0,1}} \chi\left(\left[\frac{\|(-\underline{y}-\underline{x})+b(\underline{y'}+\underline{x})\|_2}{-4(1-b)} + (\underline{y}-b\underline{y'})\cdot\underline{x}\right] a\right),$$

where we also used the fact that $\eta^{d-1} = 1$, because d is odd and η is a multiplicative character of order two. Using Theorem 2 together with the assumptions of Lemma 16 , we see that $(G_1(\eta,\chi))^{d-1} = -q^{\frac{d-1}{2}}$. Letting $\Gamma_{\underline{x}}(\underline{y},\underline{y}',b) = \left[\frac{\|(-\underline{y}-\underline{x})+b(\underline{y}'+\underline{x})\|_2}{-4(1-b)} + (\underline{y}-b\underline{y}')\cdot\underline{x}\right]$, it therefore follows that

$$\begin{aligned}
0 \leq II = &- q^{\frac{d-1}{2}} \sum_{\substack{\underline{y},\underline{y}'\in\underline{E},\\ a\neq 0, b\neq 0,1\\ :\Gamma_{\underline{x}}(\underline{y},\underline{y}',b)\neq 0}} \chi\left(\Gamma_{\underline{x}}(\underline{y},\underline{y}',b)\cdot a\right) - q^{\frac{d-1}{2}} \sum_{\substack{\underline{y},\underline{y}'\in\underline{E},\\ a\neq 0, b\neq 0,1\\ :\Gamma_{\underline{x}}(\underline{y},\underline{y}',b)=0}} 1\\
\leq &- q^{\frac{d-1}{2}} \sum_{\substack{\underline{y},\underline{y}'\in\underline{E},\\ a\neq 0, b\neq 0,1\\ :\Gamma_{\underline{x}}(\underline{y},\underline{y}',b)\neq 0}} \chi\left(\Gamma_{\underline{x}}(\underline{y},\underline{y}',b)\cdot a\right)\\
=&q^{\frac{d-1}{2}} \sum_{\substack{\underline{y},\underline{y}'\in\underline{E}, b\neq 0,1\\ :\Gamma_{\underline{x}}(\underline{y},\underline{y}',b)\neq 0}} 1 \leq q^{\frac{d+1}{2}}|\underline{E}|^2,
\end{aligned}$$

where the third line follows from the fact that $\sum_{a\in\mathbb{F}_q^*} \chi(t\cdot a) = -1$ for $t\neq 0$. Thus the inequality in (4.2.23) holds and we complete the proof of the first part (4.2.17) of Theorem 13. □

Chapter 5

EXTENSION THEOREMS FOR QUADRATIC SURFACES

5.1 Statement of Results

In this section, we collect our results related to the extension theorems for nondegenerate quadratic surfaces in the finite field setting. The following theorem follows from our sharp decay estimate for the Fourier Transform of the nondegenerate quadratic surfaces.

Theorem 17. *Let Q_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ defined as in (2.4.2). If $d \geq 2$ and $r \geq \frac{2d+2}{d-1}$, then*

$$R^*(2 \to r) \lesssim 1.$$

Theorem 17 gives the Tomas-Stein exponents, which can be obtained by calculating the sharp estimate of the Fourier transform of the nondegenerate quadratic surfaces. However, if the dimension d is two, then we can obtain the improved extension theorem which is the best possible result for extension theorems for the nondegenerate quadratic surfaces in $\mathbb{F}_q^d$.

Theorem 18. *Let $d \geq 2$. Let Q_j be the nondegenerate quadratic surface in $\mathbb{F}_q^d$ defined* *as in (2.4.2). Then we have*

$$R^*(2 \to 4) \lesssim 1.$$

Observe that Theorem 18 is stronger in two dimensions. Theorem 17 and Theorem 18 are the same in three dimensions, and Theorem 17 is stronger in dimension four and higher. In the case when $d = 2$, Theorem 18 shows that the necessary conditions for the boundedness of $R^*(p \to r)$ in (2.3.5) are also sufficient when the surfaces are the nondegenerate quadratic curves. To see this, observe from Corollary 9 that $|Q_j| \sim q$ for $d = 2$. Thus the necessary conditions in (2.3.5) take the form

$$r \geq 4 \quad \text{and} \quad r \geq \frac{2p}{p-1}. \tag{5.1.1}$$

Combining (2.2.3) with Theorem 18, we see that

$$R^*(p \to 4) \lesssim 1 \quad \text{for} \quad 2 \leq p \leq \infty. \tag{5.1.2}$$

By direct estimation, we have

$$R^*(p \to \infty) \lesssim 1 \quad \text{for} \quad 1 \leq p \leq \infty. \tag{5.1.3}$$

Interpolating (5.1.2) and (5.1.3), we see that the necessary conditions given by (2.3.5) are in fact sufficient.

5.2 Proofs of Results

In this section, we provide the proofs of our results for the extension theorems for the nondegenerate quadratic surfaces.

5.2.1 Proof for the Tomas-Stein Exponents

Theorem 17. *Let Q_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ defined as in (2.4.2). If $d \geq 2$ and $r \geq \frac{2d+2}{d-1}$, then*

$$R^*(2 \to r) \lesssim 1.$$

Proof. We want to prove that $R^*\left(2 \to \frac{2d+2}{d-1}\right) \lesssim 1$. By Theorem 15, it is enough to show that the estimate

$$\|g * (d\sigma)^\vee\|_{L^{\frac{2d+2}{d-1}}(\mathbb{F}_q^d, dm)} \lesssim \|g\|_{L^{\frac{2d+2}{d+3}}(\mathbb{F}_q^d, dm)}, \tag{5.2.1}$$

holds for all functions g defined on $\mathbb{F}_q^d$. Consider the Bochner-Riesz kernel K given by the relation, $K = (d\sigma)^\vee - \delta_0$. Since $(d\sigma)^\vee(0, \ldots, 0) = 1$, we see that $K(m) = 0$ if $m = (0, \ldots, 0)$, and $K(m) = (d\sigma)^\vee(m)$ otherwise. In addition, we see that

$$\begin{aligned}\|g * \delta_0\|_{L^{\frac{2d+2}{d-1}}(\mathbb{F}_q^d, dm)} =& \|g\|_{L^{\frac{2d+2}{d-1}}(\mathbb{F}_q^d, dm)} \\ \leq& \|g\|_{L^{\frac{2d+2}{d+3}}(\mathbb{F}_q^d, dm)},\end{aligned}$$

where above inequality follows from the facts that dm is the counting measure, and $\frac{2d+2}{d-1} \geq \frac{2d+2}{d+3}$. In order to show that (5.2.1) holds, it therefore suffices to show that

$$\|g * K\|_{L^{\frac{2d+2}{d-1}}(\mathbb{F}_q^d, dm)} \lesssim \|g\|_{L^{\frac{2d+2}{d+3}}(\mathbb{F}_q^d, dm)} \quad \text{for all } g \text{ on } \mathbb{F}_q^d. \tag{5.2.2}$$

We now claim that the following two estimates hold.

$$\|g * K\|_{L^2(\mathbb{F}_q^d, dm)} \lesssim q\|g\|_{L^2(\mathbb{F}_q^d, dm)} \quad \text{for all } g \text{ on } \mathbb{F}_q^d, \tag{5.2.3}$$

and

$$\|g * K\|_{L^\infty(\mathbb{F}_q^d,dm)} \lesssim q^{-\frac{d-1}{2}} \|g\|_{L^1(\mathbb{F}_q^d,dm)} \quad \text{for all } g \text{ on } \mathbb{F}_q^d. \tag{5.2.4}$$

Note that the estimate in (5.2.2) follows by interpolating (5.2.3) and (5.2.4) with $\theta = \frac{2}{d+1}$ in the Riesz-Thorin theorem. Thus it remains to show that both (5.2.3) and (5.2.4) hold. The inequality in (5.2.3) follows from the following observation.

$$\begin{aligned} \|g * K\|_{L^2(\mathbb{F}_q^d,dm)} =& \|\widehat{g}\widehat{K}\|_{L^2(\mathbb{F}_q^d,dx)} \\ \leq& \|\widehat{K}\|_{L^\infty(\mathbb{F}_q^d,dx)} \|\widehat{g}\|_{L^2(\mathbb{F}_q^d,dx)} \\ \lesssim& q\|g\|_{L^2(\mathbb{F}_q^d,dm)}, \end{aligned}$$

where we used the fact, in the last inequality, that for each $x \in \mathbb{F}_q^d$, $\widehat{K}(x) = d\sigma(x) - \widehat{\delta_0}(x) = \frac{q^d}{|Q_j|} Q_j(x) - 1 \lesssim q$. In order to show that the estimate in (5.2.4) holds, we observe that

$$\|g * K\|_{L^\infty(\mathbb{F}_q^d,dm)} \leq \|K\|_{L^\infty(\mathbb{F}_q^d,dm)} \|g\|_{L^1(\mathbb{F}_q^d,dm)}. \tag{5.2.5}$$

Since $(d\sigma)^\vee(m) = q^d|Q_j|^{-1}Q_j^\vee(m)$, using the definition of K and the first part of Lemma 7, we also see that

$$\|K\|_{L^\infty(\mathbb{F}_q^d,dm)} \lesssim q^{-\frac{(d-1)}{2}}. \tag{5.2.6}$$

Combining (5.2.5) with (5.2.6), the inequality in (5.2.4) follows and so the proof of Theorem 17 is complete. □

5.2.2 Proof of our Sharp Result in Dimension Two

Theorem 18. *Let $d \geq 2$. Let Q_j be the nondegenerate quadratic surface in $\mathbb{F}_q^d$ defined as in (2.4.2). Then we have*

$$R^*(2 \to 4) \lesssim 1.$$

Proof. To prove Theorem 18, we make the following reduction which Mockenhaupt and Tao ([21]) used to prove the sharp extension theorem for the parabola.

Lemma 19. *Let Q_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ defined as in (2.4.2). Suppose that for any $x \in (\mathbb{F}_q^d)^* = \mathbb{F}_q^d \setminus (0, \ldots, 0)$, we have*

$$\sum_{\{(\alpha,\beta)\in Q_j\times Q_j:\alpha+\beta=x\}} 1 \lesssim q^{d-2}.$$

Then for $d \geq 2$,

$$R^*(2 \to 4) \lesssim 1.$$

Proof of Lemma 19. We have to show that

$$\|\widehat{fd\sigma}\|_{L^4(\mathbb{F}_q^d,dm)} \lesssim \|f\|_{L^2(Q_j,d\sigma)}$$

for all functions f on Q_j. Using Plancherel, we have

$$\begin{aligned}\|\widehat{fd\sigma}\|_{L^4(\mathbb{F}_q^d,dm)} &= \|\widehat{fd\sigma}\widehat{fd\sigma}\|^{\frac{1}{2}}_{L^2(\mathbb{F}_q^d,dm)}\\ &= \|fd\sigma * fd\sigma\|^{\frac{1}{2}}_{L^2(\mathbb{F}_q^d,dx)}\end{aligned}$$

and so it suffices to show that

$$\|fd\sigma * fd\sigma\|^2_{L^2(\mathbb{F}_q^d,dx)} \lesssim \|f\|^4_{L^2(Q_j,d\sigma)}.$$

It follows that

$$\begin{aligned}&\|fd\sigma * fd\sigma\|^2_{L^2(\mathbb{F}_q^d,dx)}\\ &= q^{-d}|fd\sigma * fd\sigma(0,\cdots,0)|^2 + \|fd\sigma * fd\sigma\|^2_{L^2((\mathbb{F}_q^d)^*,dx)}\end{aligned}$$

Thus it will suffice to show that

$$q^{-d}|fd\sigma * fd\sigma(0,\cdots,0)|^2 \lesssim \|f\|^4_{L^2(Q_j,d\sigma)} \tag{5.2.7}$$

and

$$\|fd\sigma * fd\sigma\|^2_{L^2((\mathbb{F}_q^d)^*,dx)} \lesssim \|f\|^4_{L^2(Q_j,d\sigma)} \tag{5.2.8}$$

We first show that the inequality in (5.2.7) holds. We have

$$\begin{aligned}|fd\sigma * fd\sigma(0,\dots,0)| &\le \sum_{m\in\mathbb{F}_q^d} |\widehat{fd\sigma}(m)|^2 \\ &= |Q_j|^{-2} q^d \sum_{x\in Q_j} |f(x)|^2 \\ &= |Q_j|^{-1} q^d \|f\|^2_{L^2(Q_j,d\sigma)} \sim q\|f\|^2_{L^2(Q_j,d\sigma)}.\end{aligned}$$

Thus the inequality in (5.2.7) holds because $d \ge 2$. It remains to show that the inequality in (5.2.8) holds. Without loss of generality, we may assume that f is positive. Using the Cauchy Schwartz inequality, we see that

$$\begin{aligned}& fd\sigma * fd\sigma(x) \\ &= |Q_j|^{-2} q^d \sum_{\{(\alpha,\beta)\in Q_j\times Q_j:\alpha+\beta=x\}} f(\alpha)f(\beta) \\ &\le |Q_j|^{-2} q^d \Big(\sum_{\{(\alpha,\beta)\in Q_j\times Q_j:\alpha+\beta=x\}} 1\Big)^{\frac{1}{2}} \Big(\sum_{\{(\alpha,\beta)\in Q_j\times Q_j:\alpha+\beta=x\}} f^2(\alpha)f^2(\beta)\Big)^{\frac{1}{2}} \\ &= (d\sigma * d\sigma)^{\frac{1}{2}}(x)(f^2 d\sigma * f^2 d\sigma)^{\frac{1}{2}}(x).\end{aligned} \tag{5.2.9}$$

From our hypothesis and the fact that $|Q_j| \sim q^{d-1}$, we obtain that for $x \ne (0,\dots,0)$,

$$d\sigma * d\sigma(x) \sim q^{-d+2} \sum_{\{(\alpha,\beta)\in Q_j\times Q_j:\alpha+\beta=x\}} 1 \lesssim 1. \tag{5.2.10}$$

From Fubini's theorem, we also have

$$\|f^2 d\sigma * f^2 d\sigma\|_{L^1(\mathbb{F}_q^d,dx)} = \|f\|^4_{L^2(Q_j,d\sigma)}. \tag{5.2.11}$$

Using Hölder's inequality and estimates (5.2.9), (5.2.10) , and (5.2.11), we obtain that

$$\begin{aligned}
\|fd\sigma * fd\sigma\|^2_{L^2((\mathbb{F}_q^d)^*,dx)} &= \|(fd\sigma * fd\sigma)^2\|_{L^1((\mathbb{F}_q^d)^*,dx)} \\
&\le \|(d\sigma * d\sigma)\cdot(f^2d\sigma * f^2d\sigma)\|_{L^1((\mathbb{F}_q^d)^*,dx)} \\
&\le \|d\sigma * d\sigma\|_{L^\infty((\mathbb{F}_q^d)^*,dx)}\|f^2d\sigma * f^2d\sigma\|_{L^1((\mathbb{F}_q^d)^*,dx)} \\
&\lesssim \|f\|^4_{L^2(Q_j,d\sigma)}.
\end{aligned}$$

Thus the inequality in (5.2.8) holds and so the proof of Lemma 19 is complete. □

We now return to the proof of Theorem 18. By Lemma 19, it is enough to show that for any $x \in (\mathbb{F}_q^d)^*, d \ge 2$,

$$\sum_{\{(\alpha,\beta)\in Q_j\times Q_j:\alpha+\beta=x\}} 1 \lesssim q^{d-2} \tag{5.2.12}$$

where Q_j is the nondegenerate quadratic surface in $\mathbb{F}_q^d$. Using Theorem 8, we may assume that the nondegenerate quadratic surface in $\mathbb{F}_q^d$ is given by

$$Q_j = \{y \in \mathbb{F}_q^d : a_1y_1^2 + \cdots + a_dy_d^2 = j \ne 0\}$$

for all $a_k \ne 0,\ \ k = 1, 2, \ldots, d$. Therefore the left hand side of the equation in (5.2.12) can be estimated by the number of common solutions $\alpha = (\alpha_1, \cdots, \alpha_d)$ in $\mathbb{F}_q^d$ of the equations

$$\begin{aligned}
a_1\alpha_1^2 + \cdots + a_d\alpha_d^2 &= j \\
2a_1x_1\alpha_1 + \cdots + 2a_dx_d\alpha_d &= \sum_{k=1}^{d} a_kx_k^2
\end{aligned} \tag{5.2.13}$$

for $x = (x_1, \ldots, x_d) \ne (0, \ldots, 0)$ and $a_k \ne 0$ for all $k = 1, 2, \ldots, d$. Note that $2a_kx_k \ne 0$ for some $k = 1, 2, \ldots, d$ because $x \ne (0, \ldots, 0)$ and $a_k \ne 0$. Thus a routine algebraic computation shows that the number of common solutions of equations in (5.2.13) is less than equal to $2q^{d-2}$. This means that the inequality in (5.2.12) holds and so we complete the proof of Theorem 18. $\square$

Chapter 6

SPHERICAL EXTENSION THEOREMS

6.1 Conjecture for the Boundedness of $R^*(p \to r)$

In this section, we shall investigate the necessary conditions for the boundedness of the extension operators for spheres S_j defined as in (2.4.3). We need the following theorem.

Theorem 20. *Let $H \subset \mathbb{F}_q^d$ be an affine subspace of dimension k. Then we have*

$$|H \cap S_j| \lesssim q^{k-1} + q^{\frac{d-1}{2}}.$$

Proof. Using the Plancherel theorem, we see that

$$\begin{aligned} |H \cap S_j| =& \sum_{x \in \mathbb{F}_q^d} H(x) S_j(x) = q^d \sum_{m \in \mathbb{F}_q^d} \widehat{H}(m) \widehat{S_j}(m) \\ =& q^d \widehat{H}(0, \ldots, 0) \widehat{S_j}(0, \ldots, 0) + q^d \sum_{m \neq (0,\ldots,0)} \widehat{H}(m) \widehat{S_j}(m) \\ =& I + II. \end{aligned}$$

Since $|H| = q^k$ and $|S_j| \sim q^{d-1}$, we obtain that

$$I = q^d \frac{|H|}{q^d} \frac{|S_j|}{q^d} \sim q^{k-1}. \tag{6.1.1}$$

On the other hand, we observe that

$$\begin{aligned}|II| \leq & q^d \max_{\theta \neq (0,\ldots,0)} |\widehat{S_j}(\theta)| \sum_{m \in \mathbb{F}_q^d} |q^{-d} \sum_{x \in H} \chi(-x \cdot m)| \\ \lesssim & q^d q^{-\frac{d+1}{2}} q^{-d} q^{d-k} |H| = q^{\frac{d-1}{2}},\end{aligned}$$

where we used the facts that H is an affine subspace of dimension k and $|\widehat{S_j}(m)| \lesssim q^{-\frac{d+1}{2}}$ if $m \neq (0, \ldots, 0)$ (see Remark 4). Combining this with (6.1.1), the proof immediately follows. □

From Theorem 20, we obtain the following corollary.

Corollary 21. *Let $H \subset \mathbb{F}_q^d$ be an affine subspace of dimension k. Moreover, we assume that $H \subset S_j$. Then we have*

$$|H| \lesssim q^{\frac{d-1}{2}}.$$

In addition, if $d \geq 2$, the dimension of $\mathbb{F}_q^d$, is even, then we have

$$|H| \lesssim q^{\frac{d-2}{2}}.$$

Proof. The first part of Corollary 21 clearly follows from Theorem 20 and the second part of Corollary 21 follows from the fact that the dimensions of the affine subspaces are non-negative integers. □

If -1 is a square number in $\mathbb{F}_q$ and d is odd, then there exists a $(d-1)/2$-dimensional affine subspace H contained in the sphere S_j in $\mathbb{F}_q^d$ (see, e.g., Example 4.4 in [17]). Thus if d is odd, then the necessary conditions in (2.3.6) take the form

$$r \geq \frac{2d}{d-1} \quad \text{and} \quad r \geq \frac{p(d+1)}{(p-1)(d-1)},$$

because $|S_j| \sim q^{d-1}$ (see Corollary 9) and $|H| = q^{(d-1)/2}$. Recall that the Tomas-Stein exponents with $R^*(p \to r) \lesssim 1$ take the form

$$r \geq \frac{2d+2}{d-1} \quad \text{and} \quad r \geq \frac{p(d+1)}{(p-1)(d-1)}.$$

Thus if $d \geq 3$ is odd and $r \geq (2d+2)/(d-1)$, then the Tomas-Stein exponents give sharp "p" values such that $R^*(p \to r) \lesssim 1$. For example, if $r = 4$, then we can not improve the Tomas-Stein exponents $R^*((4d-4)/(3d-5) \to 4)$ (see Figure 6.1). However if $d \geq 2$ is even, then we may improve the Tomas-Stein exponents, because the sphere S_j contains at most a $(d-2)/2$-dimensional affine subspace H, which is a result from Corollary 21. From this and (2.3.6), we may conjecture, in even dimensions $d \geq 2$, that $R^*(p \to r) \lesssim 1$ iff

$$r \geq \frac{2d}{d-1} \quad \text{and} \quad r \geq \frac{p(d+2)}{(p-1)d}.$$

Theorem 22 below partially supports above conjecture (see Figure 6.2).

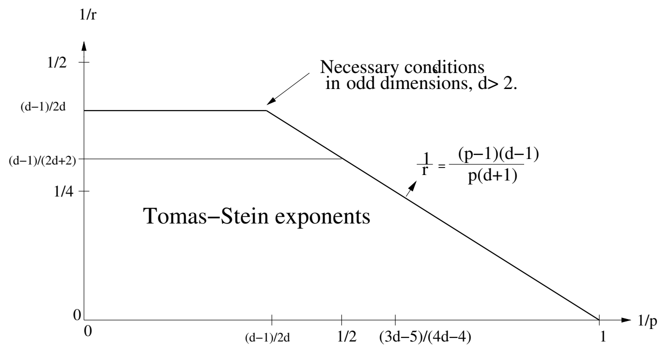

Figure 6.1: Necessary conditions for $R^*(p \to r) \lesssim 1$ related to spheres in odd dimensions.

6.2 Improved $L^p - L^4$ Estimate

Our next result shows that in higher even dimensions, $d \geq 4$, we have the "p" index improvement of the Tomas-Stein exponents (Theorem 17) on which $R^*(p \to 4) \lesssim 1$

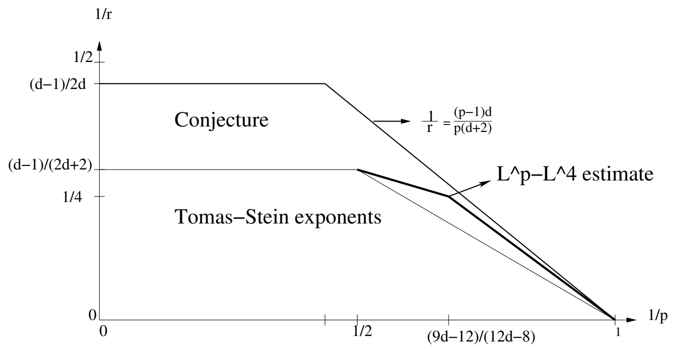

Figure 6.2: In even dimensions $d \geq 4$, exponents for $R^*(p \to r)$ bound related to spheres.

in the specific case when the nondegenerate quadratic surfaces are spheres defined as in (2.4.3).

Theorem 22. *Let S_j be a sphere in $\mathbb{F}_q^d$ defined as in (2.4.3). Suppose $d \geq 4$ is even and $p \geq \frac{12d-8}{9d-12}$. Then we have*

$$R^*(p \to 4) \lessapprox 1.$$

Remark 10. *Note that Theorem 22 holds without any restriction on the underlying finite field $\mathbb{F}_q$. In particular, (2.2.3) implies that our result $R^*((12d-8)/(9d-12) \to 4) \lessapprox 1$ in Theorem 22 is much better than the result, $R^*((4d-4)/(3d-5) \to 4) \lesssim 1$, given by the Tomas-Stein exponents, because the number $(12d-8)/(9d-12)$ is less than $(4d-4)/(3d-5)$ (see Figure 6.2).*

6.2.1 Dot Product Sets

The idea for the proof of Theorem 22 is similar to that of Theorem 11, but many estimates are needed to complete the proof of Theorem 22, in part because Kloosterman sums are related to the extension problems for the spheres and they are very challenging to understand. In order to prove Theorem 22, we first need to estimate the number of pair $(x, z) \in E \times E \subset S_j \times S_j$ such that the dot product $x \cdot z$ is exactly the radius of sphere S_j.

Theorem 23. *Let S_j be a sphere in $\mathbb{F}_q^d$ with $d \geq 4$ even, defined as in (2.4.3). If E is any subset of the sphere S_j, then we have*

$$\sum_{(x,y)\in E^2: x\cdot y = j} 1 \lesssim q^{-1}|E|^2 + q^{\frac{d-2}{2}}|E|.$$

Proof. We begin by noting that

$$\begin{aligned}\sum_{(x,y)\in E^2: x\cdot y = j} 1 &= \sum_{x,y\in E} \delta_0(x\cdot y - j) \\ &= \sum_{x,y\in E} q^{-1} \sum_{s\in\mathbb{F}_q} \chi\left(-s(x\cdot y - j)\right) \\ &= q^{-1}|E|^2 + I(j),\end{aligned} \tag{6.2.1}$$

where

$$I(j) = \sum_{x,y\in E} q^{-1} \sum_{s\in\mathbb{F}_q^*} \chi\left(-s(x\cdot y - j)\right).$$

Viewing $I(j)$ as a sum in $x \in E$, and applying the triangle inequality and the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}|I(j)|^2 \leq & q^{-2}\left(\sum_{x\in E} \Big| \sum_{y\in E, s\in\mathbb{F}_q^*} \chi\left(-s(x\cdot y - j)\right)\Big|\right)^2 \\ \leq & q^{-2}|E| \sum_{x\in E} \Big| \sum_{y\in E, s\in\mathbb{F}_q^*} \chi\left(-s(x\cdot y - j\right)\Big|^2.\end{aligned}$$

Since $E \subset S_j$, we see that

$$\begin{aligned}
|I(j)|^2 \leq & q^{-2}|E| \sum_{x \in S_j} \sum_{\substack{y,y' \in E \\ s,s' \in \mathbb{F}_q^*}} \chi\left(-s(x \cdot y - j)\right) \chi\left(s'(x \cdot y' - j)\right) \\
= & q^{d-2}|E| \sum_{\substack{y,y' \in E \\ s,s' \in \mathbb{F}_q^*}} \widehat{S}_j(sy - s'y') \chi\left((s-s')j\right) \\
= & \sum_{\substack{y,y' \in E \\ s,s' \in \mathbb{F}_q^* \\ : y=y', s=s'}} \bar{G}(y,y',s,s') + \sum_{\substack{y,y' \in E \\ s,s' \in \mathbb{F}_q^* \\ : y=y', s \neq s'}} \bar{G}(y,y',s,s') \\
& + \sum_{\substack{y,y' \in E \\ s,s' \in \mathbb{F}_q^* \\ : y \neq y'}} \bar{G}(y,y',s,s') = \bar{A} + \bar{B} + \bar{C},
\end{aligned}$$

where

$$\bar{G}(y,y',s,s') = q^{d-2}|E|\widehat{S}_j(sy - s'y')\chi\left((s-s')j\right).$$

From this and (6.2.1), it suffices to show that

$$\sqrt{\bar{A} + \bar{B} + \bar{C}} \lesssim q^{-1}|E|^2 + q^{\frac{d-2}{2}}|E| \qquad \text{for all} \qquad E \subset S_j.$$

In order to complete the proof, we will actually show that $\bar{A}, \bar{B}$, and $\bar{C}$ are dominated by

$$\sim q^{-2}|E|^4 + q^{d-2}|E|^2. \tag{6.2.2}$$

Since $y = y', s = s'$, the term $\bar{A}$ is easily estimated by

$$\bar{A} = \sum_{y \in E, s \in \mathbb{F}_q^*} q^{d-2}|E|\widehat{S}_j(0, \ldots, 0) \lesssim q^{d-2}|E|^2,$$

which is dominated by the number in (6.2.2) as wanted. Let us estimate the term $\bar{B}$. Since $y = y'$, we have

$$\bar{B} = \sum_{\substack{y \in E, s, s' \in \mathbb{F}_q^* \\ : s \neq s'}} q^{d-2}|E|\widehat{S}_j\left((s-s')y\right)\chi\left((s-s')j\right).$$

Since $s \neq s'$ and $s, s' \neq 0, (s - s')$ runs through the element of $\mathbb{F}_q^*$ exactly $(q-2)$ times. Thus we see that

$$\bar{B} = q^{d-2}(q-2)|E| \sum_{y \in E, s \in \mathbb{F}_q^*} \widehat{S}_j(sy)\chi(sj).$$

Note that $sy \neq (0, \ldots, 0)$, because $s \neq 0$ and the subset E of sphere S_j doesn't contain the origin. Therefore, using the formula in (3.2.7), we have

$$\bar{B} = Kq^{\frac{d-6}{2}}(q-2)|E| \sum_{y \in E, s, r \in \mathbb{F}_q^*} \chi\left(jr + \frac{js^2}{4r}\right)\chi(sj),$$

where we used the fact that $\|sy\|_2 = s^2 j$ for all $y \in S_j$. After completing the square in s-variable, we have

$$\begin{aligned}
\bar{B} =& Kq^{\frac{d-6}{2}}(q-2)|E| \sum_{y \in E, s, r \in \mathbb{F}_q^*} \chi\left(\frac{j}{4r}(s+2r)^2\right) \\
=& Kq^{\frac{d-6}{2}}(q-2)|E| \sum_{y \in E, s \in \mathbb{F}_q, r \in \mathbb{F}_q^*} \chi\left(\frac{j}{4r}(s+2r)^2\right) \\
&- Kq^{\frac{d-6}{2}}(q-2)|E| \sum_{y \in E, r \in \mathbb{F}_q^*} \chi(jr) \\
=& \bar{B}_1 + \bar{B}_2.
\end{aligned}$$

Recall that we have assumed that j is not zero, because it is the radius of sphere S_j. Therefore, we see that $\sum_{r \in \mathbb{F}_q^*} \chi(jr) = -1$, and so $\bar{B}_2$ is estimated as

$$\bar{B}_2 \lesssim q^{\frac{d-4}{2}}|E|^2. \tag{6.2.3}$$

Let us estimate the term $\bar{B}_1$. Applying Change of variable, $s + 2r \to s$, and using the formula in (3.1.2), we have

$$\bar{B}_1 = K^2 q^{\frac{d-5}{2}}(q-1)|E| \sum_{y \in E, r \in \mathbb{F}_q^*} \eta(\frac{j}{4r}).$$

Since $\sum_{r\in\mathbb{F}_q^*} \eta(\frac{1}{r}) = 0$, $\bar{B}_1$ is exactly 0. From this and (6.2.3), we conclude that $\bar{B} \lesssim q^{\frac{d-4}{2}}|E|^2$, which is dominated by the number in (6.2.2) as wanted. Finally, it remains to estimate the term $\bar{C}$. Recall that

$$\bar{C} = q^{d-2}|E| \sum_{\substack{y,y'\in E, s,s'\in\mathbb{F}_q^* \\ : y\neq y'}} \widehat{S_j}(sy - s'y')\chi\left((s-s')j\right).$$

Write the term $\bar{C}$ as two parts

$$\begin{aligned}
\bar{C} =& q^{d-2}|E| \sum_{\substack{y,y'\in E, s,s'\in\mathbb{F}_q^* \\ : y\neq y', sy-s'y'=(0,\ldots,0)}} \widehat{S_j}(0,\ldots,0)\chi\left((s-s')j\right) \\
&+ q^{d-2}|E| \sum_{\substack{y,y'\in E, s,s'\in\mathbb{F}_q^* \\ : y\neq y', sy-s'y'\neq(0,\ldots,0)}} \widehat{S_j}(sy - s'y')\chi\left((s-s')j\right) \\
&= \bar{C}_1 + \bar{C}_2.
\end{aligned}$$

Let us estimate the term $\bar{C}_1$. Note that when $y \neq y'$, $sy - s'y' = (0,\ldots,0)$ happens only if $s = -s$ and $y = -y'$. Combining this with the fact that $\widehat{S_j}(0,\ldots,0) \sim q^{-1}$, we see that $\bar{C}_1 \lesssim q^{d-2}|E|^2$, which is dominated by the term in (6.2.2). It remains to estimate the term $\bar{C}_2$. Since $sy - s'y' \neq (0,\ldots,0)$, using the formula for $\widehat{S_j}$ in (3.2.7), we have

$$\begin{aligned}
\bar{C_2} =& Kq^{\frac{d-6}{2}}|E| \sum_{\substack{y,y'\in E, s,s',r\in\mathbb{F}_q^* \\ : y\neq y', sy-s'y'\neq(0,\ldots,0)}} \chi\left(jr + \frac{\|sy-s'y'\|_2}{4r}\right)\chi\left((s-s')j\right) \\
=& Kq^{\frac{d-6}{2}}|E| \sum_{\substack{y,y'\in E, s,s',r\in\mathbb{F}_q^* \\ : y\neq y'}} \chi\left(jr + \frac{\|sy-s'y'\|_2}{4r}\right)\chi\left((s-s')j\right) \\
&- Kq^{\frac{d-6}{2}}|E| \sum_{\substack{y,y'\in E, s,s',r\in\mathbb{F}_q^* \\ : y\neq y', sy-s'y'=(0,\ldots,0)}} \chi(jr)\chi\left((s-s')j\right) \\
=& \bar{C_{21}} + \bar{C_{22}}.
\end{aligned}$$

To estimate $\bar{C_{22}}$, note that the sum over r-variable is exactly -1 and the condition, $y \neq y', sy - s'y' = (0,\ldots,0)$ happens only if $s = -s$ and $y = -y'$. Thus we have

$\bar{C_{22}} \lesssim q^{\frac{d-4}{2}}|E|^2$ which is dominated by the term in (6.2.2). To estimate the term $\bar{C_{21}}$, note that $\|sy - s'y'\|_2 = js^2 + js'^2 - 2ss'y \cdot y'$. Write

$$\sum_{\substack{y,y' \in E \\ s,s',r \in \mathbb{F}_q^* \\ : y \neq y'}} = \sum_{\substack{y,y' \in E \\ s,s' \in \mathbb{F}_q, r \in \mathbb{F}_q^* \\ : y \neq y'}} - \sum_{\substack{y,y' \in E \\ s,r \in \mathbb{F}_q^*, s'=0 \\ : y \neq y'}} - \sum_{\substack{y,y' \in E \\ s=0, s', r \in \mathbb{F}_q^*, \\ : y \neq y'}} - \sum_{\substack{y,y' \in E \\ s=0=s', r \in \mathbb{F}_q^*, \\ : y \neq y'}}$$

and define $\bar{\Gamma}(y, y', s, s', r)$ as the value

$$Kq^{\frac{d-6}{2}}|E|\chi\left(jr + \frac{js^2 + js'^2 - 2ss'y \cdot y'}{4r}\right)\chi\left((s - s')j\right).$$

Then we have

$$\begin{aligned}
\bar{C_{21}} = & \sum_{\substack{y,y' \in E \\ s,s' \in \mathbb{F}_q, r \in \mathbb{F}_q^* \\ : y \neq y'}} \bar{\Gamma}(y, y', s, s', r) - \sum_{\substack{y,y' \in E \\ s,r \in \mathbb{F}_q^*, s'=0 \\ : y \neq y'}} \bar{\Gamma}(y, y', s, s', r) \\
& - \sum_{\substack{y,y' \in E \\ s=0, s', r \in \mathbb{F}_q^*, \\ : y \neq y'}} \bar{\Gamma}(y, y', s, s', r) - \sum_{\substack{y,y' \in E \\ s=0=s', r \in \mathbb{F}_q^*, \\ : y \neq y'}} \bar{\Gamma}(y, y', s, s', r) \\
= & I + II + III + IV.
\end{aligned}$$

Let us first estimate the term IV. Note that IV is estimated as

$$\begin{aligned}
IV = & -Kq^{\frac{d-6}{2}}|E| \sum_{\substack{y,y' \in E, r \in \mathbb{F}_q^* \\ : y \neq y'}} \chi(jr) \\
= & Kq^{\frac{d-6}{2}}|E| \sum_{y,y' \in E : y \neq y'} 1 \lesssim q^{\frac{d-6}{2}}|E|^3,
\end{aligned}$$

which is dominated by the term in (6.2.2), because the term in (6.2.2) dominates $\sqrt{(q^{-2}|E|^4)(q^{d-2}|E|^2)} \gtrsim q^{\frac{d-6}{2}}|E|^3$ using the fact that $a + b \geq 2\sqrt{ab}$ for $a, b \geq 0$. Let us estimate the term III. It follows that

$$\begin{aligned}
III =& - Kq^{\frac{d-6}{2}}|E| \sum_{\substack{y,y'\in E, s', r\in\mathbb{F}_q^* \\ :y\neq y'}} \chi\left(jr + \frac{js'^2}{4r}\right)\chi(-s'j) \\
=& - Kq^{\frac{d-6}{2}}|E| \sum_{\substack{y,y'\in E, s'\in\mathbb{F}_q, r\in\mathbb{F}_q^* \\ :y\neq y'}} \chi\left(jr + \frac{js'^2}{4r}\right)\chi(-s'j) \\
&+ Kq^{\frac{d-6}{2}}|E| \sum_{\substack{y,y'\in E, r\in\mathbb{F}_q^* \\ :y\neq y'}} \chi(jr) \\
=& III_1 + III_2.
\end{aligned}$$

Note that the term III_2 is dominated by $\sim q^{\frac{d-6}{2}}|E|^3$, because the sum over r-variable is exactly -1. To estimate the term III_1, completing the square in s'-variable and a change of variable, $(s'-2r) \to s'$, and using the formula in (3.1.2), we see that

$$III_1 = -K^2 q^{\frac{d-5}{2}}|E| \sum_{\substack{y,y'\in E, r\in\mathbb{F}_q^* \\ :y\neq y'}} \eta(\frac{j}{4r}) = 0,$$

where the fact, $\sum_{r\in\mathbb{F}_q^*} \eta(\frac{1}{r}) = 0$, was used to get the last equality. From this and the estimate for the term III_2, we have $III \lesssim q^{\frac{d-6}{2}}|E|^3$, which is also bounded by the term in (6.2.2) as before. Note that the estimate of the term II is exactly same as that of the term III. Thus the term II is also bounded by the term in (6.2.2). To complete the proof, we need to show that the term I is bounded by the term in (6.2.2). Recall that the term I is given by the value

$$Kq^{\frac{d-6}{2}}|E| \sum_{\substack{y,y'\in E, s,s'\in\mathbb{F}_q, r\in\mathbb{F}_q^* \\ :y\neq y'}} \chi\left(jr + \frac{js^2 + js'^2 - 2ss'y\cdot y'}{4r}\right)\chi\left((s-s')j\right).$$

If we complete the square in s-variable above, apply a change of variable, $s+(-j^{-1}s'y\cdot$

$y' + 2r) \to s$, and use the formula in (3.1.2), then we see that

$$\begin{aligned}
I =& Kq^{\frac{d-5}{2}}|E| \sum_{\substack{y,y'\in E\\ s'\in\mathbb{F}_q, r\in\mathbb{F}_q^*\\ :y\neq y'}} \eta(\frac{j}{4r})\chi\left(\frac{1}{4r}\left(j - j^{-1}(y\cdot y')^2\right)s'^2 + (y\cdot y' - j)s'\right)\\
=& Kq^{\frac{d-5}{2}}|E| \sum_{\substack{y,y'\in E\\ s'\in\mathbb{F}_q, r\in\mathbb{F}_q^*\\ :y\neq y', y\cdot y'=\pm j}} \eta(\frac{j}{4r})\chi\left((y\cdot y' - j)s'\right)\\
&+ Kq^{\frac{d-5}{2}}|E| \sum_{\substack{y,y'\in E\\ s'\in\mathbb{F}_q, r\in\mathbb{F}_q^*\\ :y\neq y', y\cdot y'\neq\pm j}} \eta(\frac{j}{4r})\chi\left(\frac{1}{4r}\left(j - j^{-1}(y\cdot y')^2\right)s'^2 + (y\cdot y' - j)s'\right)\\
=& I_1 + I_2.
\end{aligned}$$

The term I_1 is exactly 0. To see this, observe that the sum over r-variable is zero, because $\sum_{r\in\mathbb{F}_q^*} \eta(\frac{1}{r}) = 0$. To estimate the term I_2, completing the square in s'-variable, applying a change of variable, $s' + \frac{-2rj}{y\cdot y'+j} \to s'$, and using the formula in (3.1.2), we see that the value I_2 is given by

$$Kq^{\frac{d-4}{2}}|E|\eta(-1) \sum_{\substack{y,y'\in E\\ r\in\mathbb{F}_q^*\\ :y\neq y', y\cdot y'\neq\pm j}} \eta\left((y\cdot y')^2 - j^2\right)\chi\left(\frac{rj(y\cdot y' - j)}{y\cdot y' + j}\right),$$

where we used the fact that η is a multiplicative character of $\mathbb{F}_q^*$ of order two. Note that the sum over r-variable above is exactly -1. Thus we conclude that $I_2 \lesssim q^{\frac{d-4}{2}}|E|^3$, which is also dominated by the value in (6.2.2), because the first term in (6.2.2) dominates $q^{\frac{d-4}{2}}|E|^3$ if $|E| \gtrsim q^{\frac{d}{2}}$ and the second term in (6.2.2) dominates $q^{\frac{d-4}{2}}|E|^3$ if $|E| \lesssim q^{\frac{d}{2}}$. Thus the proof is complete. □

As the direct application of Theorem 23, we obtain the following corollary which shall make an important role in the proof of Theorem 22.

Corollary 24. *With the same assumptions as in Theorem 23, we have*

$$\sum_{\substack{(x,z,z',s,s')\in E^3\times(\mathbb{F}_q^*)^2\\ :z\neq z'\\ s(-x+z)+s'(x-z')=(0,\dots,0)}} 1 \lesssim q|E|^2+q^{\frac{d+2}{2}}|E|.$$

Proof. Let E be a subset of the sphere S_j defined as in (2.4.3). Suppose $x,z,z'\in E, s,s'\in\mathbb{F}_q^*$. Then we first observe that if $s(-x+z)+s'(x-z')=(0,\dots,0)$, then x,z,and z' must be on a line, because $s,s'\neq 0$. Moreover if $z\neq z'$, then $s(-x+z)+s'(x-z')=(0,\dots,0)$ never happens if $x=z$ or $x=z'$. Thus if $z\neq z'$ and $s(-x+z)+s'(x-z')=(0,\dots,0)$, then x,z,z' are three different points on a line. This implies that the line passing through two points $x,z\in E\subset S_j$ should contain one point on S_j which is different from x and z. In other words, it satisfies that $\|x+\alpha(-x+z)\|_2=j$ for some $\alpha\in\mathbb{F}_q\setminus\{0,1\}$. Since $x,z\in S_j$, simple calculation of $\|x+\alpha(-x+z)\|_2=j$ yields that $\alpha(\alpha-1)(j-x\cdot z)=0$ for some $\alpha\in\mathbb{F}_q\setminus\{0,1\}$. Thus we conclude that if $z\neq z'$and $s(-x+z)+s'(x-z')=(0,\dots,0)$, then we have $j-x\cdot z=0$. Using this fact, we see that

$$\begin{aligned}
&\sum_{\substack{(x,z,z',s,s')\in E^3\times(\mathbb{F}_q^*)^2\\ :z\neq z'\\ s(-x+z)+s'(x-z')=(0,\dots,0)}} 1\\
\leq &\sum_{x,z\in E:x\cdot z=j}\ \sum_{\substack{z'\in E,s,s'\in\mathbb{F}_q^*\\ :s(-x+z)+s'(x-z')=(0,\dots,0)}} 1\\
\leq &\sum_{x,z\in E:x\cdot z=j} q^2,
\end{aligned}$$

where the last line can be obtained by using the facts that a line has q elements, and if we fix x,z, then the maximum number of choices of $z'\in E$ is at most q, because x,z,z' are exactly on one line , and so if x,z,z' are determined, then the number of choices of the pair $(s,s')\in(\mathbb{F}_q^*)^2$ satisfying $s(-x+z)+s'(x-z')=(0,\dots,0)$ is at most q. From Theorem 23, the proof is complete. $\square$

6.2.2 Proof of Theorem 22

We now prove Theorem 22. Using the inequality in (2.2.3) and the usual dyadic pigeonholing argument, it suffices to show that for $p = (12d-8)/(9d-12)$, we have

$$\|(Ed\sigma)^\vee\|_{L^4(\mathbb{F}_q^d,dm)} \lesssim \|E\|_{L^p(S_j,d\sigma)} \quad \text{for all} \quad E \subset S_j. \tag{6.2.4}$$

Expanding both terms in (6.2.4) and using the fact that $|S_j| \sim q^{d-1}$, it is enough to prove that

$$\Lambda_4(E) \lesssim |E|^{\frac{4}{p}} q^{3d-4} q^{\frac{-4d+4}{p}} \quad \text{for} \quad p = \frac{12d-8}{9d-12}, \tag{6.2.5}$$

where $\Lambda_4(E) = \sum_{\substack{(x,y,z,k)\in E^4\\ :x+y=z+k}} 1$. We need the following theorem.

Theorem 25. *Let S_j be a sphere in $\mathbb{F}_q^d$ defined as before. In addition, we assume that the dimension of $\mathbb{F}_q^d$, $d \geq 4$, is even. If E is any subset of S_j then we have*

$$\begin{aligned}\Lambda_4(E) = &\sum_{\substack{(x,y,z,k)\in E^4\\ :x+y=z+k}} 1\\ &\lesssim \min\{|E|^3, \quad q^{-1}|E|^3 + q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} + q^{\frac{3d-4}{4}}|E|^{\frac{3}{2}}\}.\end{aligned}$$

We assume that Theorem 25 holds and continue proving Theorem 22. Note that Theorem 25 implies that if $d \geq 4$ is even and E is any subset of the sphere S_j, then

$$\Lambda_4(E) \lesssim \begin{cases} q^{-1}|E|^3 & \text{if} \quad q^{\frac{d+2}{2}} \lesssim |E| \lesssim q^{d-1} \\ q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} & \text{if} \quad q^{\frac{d-1}{2}} \lesssim |E| \lesssim q^{\frac{d+2}{2}} \\ q^{\frac{3d-4}{4}}|E|^{\frac{3}{2}} & \text{if} \quad q^{\frac{3d-4}{6}} \lesssim |E| \lesssim q^{\frac{d-1}{2}} \\ |E|^3 & \text{if} \quad 1 \lesssim |E| \lesssim q^{\frac{3d-4}{6}}. \end{cases}$$

Using these upper bounds of $\Lambda_4(E)$ depending on the size of the subset E of S_j, the inequality in (6.2.5) follows by the direct calculation, and the proof of Theorem

22 is complete if we can prove Theorem 25. We now introduce the proof of Theorem 25. We first note that it is trivial that $\Lambda_4(E) \lesssim |E|^3$, because if we fix $x, y, z \in E$ then there is at most one k such that $x + y = z + k$. Thus it suffices to show that

$$\Lambda_4(E) \lesssim q^{-1}|E|^3 + q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} + q^{\frac{3d-4}{4}}|E|^{\frac{3}{2}}.$$

Since the set E is a subset of the sphere S_j, we see that

$$\Lambda_4(E) \leq \sum_{\substack{(x,y,z)\in E^3 \\ :x+y-z\in S_j}} 1 = \sum_{\substack{(x,y,z)\in E^3 \\ :\|x+y-z\|_2=j}} 1.$$

Therefore we need to estimate the number of elements of the following set:

$$\begin{aligned} &\{(x, y, z) \in E^3 : \|x + y - z\|_2 = j\} \\ =&\{(x, y, z) \in E^3 : x \cdot y - x \cdot z - y \cdot z = -j\}, \end{aligned}$$

where we used the fact that x, y, and z are elements of the sphere S_j. It therefore follows that

$$\begin{aligned} \Lambda_4(E) \leq &\sum_{(x,y,z)\in E^3} \delta_0(x \cdot y - x \cdot z - y \cdot z + j) \\ = &\sum_{(x,y,z)\in E^3} q^{-1} \sum_{s\in\mathbb{F}_q} \chi\left(s(x \cdot y - x \cdot z - y \cdot z + j)\right) \\ =&q^{-1}|E|^3 + R(j), \end{aligned} \tag{6.2.6}$$

where

$$R(j) = \sum_{(x,y,z)\in E^3} q^{-1} \sum_{s\in\mathbb{F}_q^*} \chi\left(s(x \cdot y - x \cdot z - y \cdot z + j)\right).$$

Thus our work is to find the upper bound of $|R(j)|$. Viewing $R(j)$ as a sum in $x \in E$, and applying the triangle inequality and the Cauchy-Schwarz inequality in x-variable, we have

$$|R(j)| \leq q^{-1}|E|^{\frac{1}{2}} \left(\sum_{x\in E} M_j(x)\right)^{\frac{1}{2}}, \tag{6.2.7}$$

where

$$M_j(x) = \left| \sum_{(y,z,s)\in E^2\times\mathbb{F}_q^*} \chi\left(s(x\cdot y - x\cdot z - y\cdot z + j)\right)\right|^2.$$

Let us estimate $M_j(x)$ for $x\in E$. Viewing $M_j(x)^{\frac{1}{2}}$ as a sum in $y\in E$, applying the triangle inequality and the Cauchy-Schwarz inequality in y-variable, and dominating the sum over $y\in E$ by the sum over $y\in S_j$, we see that $M_j(x)$ is dominated by the value

$$\begin{aligned}&|E|\sum_{y\in S_j}\sum_{\substack{z,z'\in E\\ s,s'\in\mathbb{F}_q^*}} \chi\left(s(x\cdot y - x\cdot z - y\cdot z + j)\right)\chi\left(-s'(x\cdot y - x\cdot z' - y\cdot z' + j)\right)\\ =&q^d|E|\sum_{\substack{z,z'\in E\\ s,s'\in\mathbb{F}_q^*}} \widehat{S}_j\left(s(-x+z)+s'(x-z')\right)\chi\left(j(s-s')+s(-x\cdot z)+s'(x\cdot z')\right).\end{aligned}$$

To estimate $M_j(x)$ for each $x\in E$, we write above sum into three parts

$$\sum_{\substack{z,z'\in E\\ s,s'\in\mathbb{F}_q^*}} = \sum_{\substack{(z,z',s,s')\in E^2\times(\mathbb{F}_q^*)^2\\ :z=z',s=s'}} + \sum_{\substack{(z,z',s,s')\in E^2\times(\mathbb{F}_q^*)^2\\ :z=z',s\neq s'}} + \sum_{\substack{(z,z',s,s')\in E^2\times(\mathbb{F}_q^*)^2\\ :z\neq z'}}.$$

For $x,z,z'\in E$ and $s,s'\in\mathbb{F}_q^*$, we denote by $G(x,z,z',s,s')$ the value

$$q^d|E|\widehat{S}_j\left(s(-x+z)+s'(x-z')\right)\chi\left(j(s-s')+s(-x\cdot z)+s'(x\cdot z')\right).$$

Then we have the following upper bound of $\sum_{x\in E} M_j(x)$ given as in (6.2.7):

$$\begin{aligned}\sum_{x\in E} M_j(x) \le & \sum_{\substack{(x,z,z',s,s')\in E^3\times(\mathbb{F}_q^*)^2\\ :z=z',s=s'}} G(x,z,z',s,s')\\ &+ \sum_{\substack{(x,z,z',s,s')\in E^3\times(\mathbb{F}_q^*)^2\\ :z=z',s\neq s'}} G(x,z,z',s,s')\\ &+ \sum_{\substack{(x,z,z',s,s')\in E^3\times(\mathbb{F}_q^*)^2\\ :z\neq z'}} G(x,z,z',s,s')\\ =&A_j+B_j+C_j. \qquad (6.2.8)\end{aligned}$$

From (6.2.6), (6.2.7), and (6.2.8), we see that for each $E \subset S_j$,

$$\Lambda_4(E) \le q^{-1}|E|^3 + q^{-1}|E|^{\frac{1}{2}} \left(|A_j|^{\frac{1}{2}} + |B_j|^{\frac{1}{2}} + |C_j|^{\frac{1}{2}}\right). \tag{6.2.9}$$

In order to complete the proof of Theorem 25, we shall carefully estimate the three terms, A_j, B_j, C_j. Since $\widehat{S_j}(0,\ldots,0) \sim q^{-1}, z = z'$, and $s = s'$, A_j is easily estimated as

$$A_j = q^d|E| \sum_{(x,z,s)\in E^2\times\mathbb{F}_q^*} \widehat{S_j}(0,\ldots,0) \lesssim q^d|E|^3.$$

From this and (6.2.9), we obtain that

$$\Lambda_4(E) \lesssim q^{-1}|E|^3 + q^{\frac{d-2}{2}}|E|^2 + q^{-1}|E|^{\frac{1}{2}} \left(|B_j|^{\frac{1}{2}} + |C_j|^{\frac{1}{2}}\right).$$

Using the fact that $a + b \ge 2\sqrt{ab}$ if $a, b \ge 0$, we see that the term $q^{\frac{d-2}{2}}|E|^2$ is $\lesssim q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} + q^{\frac{3d-4}{4}}|E|^{\frac{3}{2}}$. Therefore, in order to complete the proof, it suffices to show that $q^{-1}|E|^{\frac{1}{2}}\left(|B_j|^{\frac{1}{2}} + |C_j|^{\frac{1}{2}}\right) \lesssim q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} + q^{\frac{3d-4}{4}}|E|^{\frac{3}{2}}$. To show this, we shall prove the following two estimates:

$$|B_j| \lesssim q^{\frac{d+2}{2}}|E|^4 + q^{\frac{3d}{2}}|E|^2, \tag{6.2.10}$$

and

$$|C_j| \lesssim q^{\frac{d+2}{2}}|E|^4 + q^{\frac{3d}{2}}|E|^2. \tag{6.2.11}$$

We shall estimate the terms, B_j and C_j to prove above two inequalities.

We first estimate the term B_j. Since $z = z'$ in the sum of the term B_j ,we first see that the term B_j is given by the value

$$q^d|E| \sum_{\substack{(x,z,s,s')\in E^2\times(\mathbb{F}_q^*)^2 \\ :s\neq s'}} \widehat{S_j}\left((s-s')(-x+z)\right)\chi\left((s-s')(j-x\cdot z)\right).$$

Now observe that $s - s'$ runs through each value in $\mathbb{F}_q^*$ exactly $(q-2)$ times, because $s, s' \neq 0, s \neq s'$. Therefore we have

$$
\begin{aligned}
B_j =& q^d (q-2)|E| \sum_{(x,z,s) \in E^2 \times \mathbb{F}_q^*} \widehat{S_j}\left(s(-x+z)\right) \chi\left(s(j - x \cdot z)\right) \\
=& q^d (q-2)|E| \sum_{\substack{(x,z,s) \in E^2 \times \mathbb{F}_q^* \\ : x = z}} \widehat{S_j}(0, \ldots, 0) \\
&+ q^d (q-2)|E| \sum_{\substack{(x,z,s) \in E^2 \times \mathbb{F}_q^* \\ : x \neq z}} \widehat{S_j}\left(s(-x+z)\right) \chi\left(s(j - x \cdot z)\right) \\
=& B_{j,1} + B_{j,2}.
\end{aligned}
$$

Since $\widehat{S_j}(0, \ldots, 0) \sim q^{-1}$ and $x = z \in E$, we have $B_{j,1} \lesssim q^{d+1}|E|^2$ which is dominated by the value in (6.2.10) for $d \geq 2$ as wanted. To estimate the term $B_{j,2}$, observe that $s(-x+z) \neq (0, \ldots, 0)$, because $x \neq z$ and $s \neq 0$. Noting that $\|s(-x+z)\|_2 = (2j - 2x \cdot z)s^2$ for $x, z \in E \subset S_j$ and using the explicit form for $\widehat{S_j}$ in (3.2.7), we see that

$$
\begin{aligned}
B_{j,2} =& K q^{\frac{d-2}{2}} (q-2)|E| \sum_{\substack{(x,z,s,t) \in E^2 \times (\mathbb{F}_q^*)^2 \\ : x \neq z}} \chi\left(jt + \frac{(j - x \cdot z)s^2}{2t}\right) \chi\left((j - x \cdot z)s\right) \\
=& K q^{\frac{d-2}{2}} (q-2)|E| \sum_{\substack{(x,z,s,t) \in E^2 \times (\mathbb{F}_q^*)^2 \\ : x \neq z, j - x \cdot z = 0}} \chi(jt) \\
&+ K q^{\frac{d-2}{2}} (q-2)|E| \sum_{\substack{(x,z,s,t) \in E^2 \times (\mathbb{F}_q^*)^2 \\ : x \neq z, j - x \cdot z \neq 0}} \chi\left(jt + \frac{(j - x \cdot z)s^2}{2t}\right) \chi\left((j - x \cdot z)s\right) \\
=& B_{j,21} + B_{j,22},
\end{aligned}
$$

where K is the complex value which is bounded by 1. Recall that "j" is the radius of the sphere S_j, and not zero. Therefore, $\sum_{t \in \mathbb{F}_q^*} \chi(jt)$ is exactly -1 and so the term $B_{j,21}$ is written by the value

$$
B_{j,21} = -K q^{\frac{d-2}{2}} (q-1)(q-2)|E| \sum_{\substack{(x,z) \in E^2 \\ : x \neq z, j - x \cdot z = 0}} 1.
$$

Using Theorem 23, we obtain that

$$B_{j,21} \lesssim q^{\frac{d}{2}}|E|^3 + q^d|E|^2.$$

Thus $B_{j,21}$ is also dominated by the value in (6.2.10) as wanted. To estimate the term $B_{j,22}$, write the term $B_{j,22}$ as follows:

$$\begin{aligned} &Kq^{\frac{d-2}{2}}(q-2)|E| \sum_{\substack{(x,z,s,t)\in E^2\times\mathbb{F}_q\times\mathbb{F}_q^* \\ :x\neq z, j-x\cdot z\neq 0}} \chi\left(jt + \frac{(j-x\cdot z)s^2}{2t}\right)\chi\left((j-x\cdot z)s\right) \\ &- Kq^{\frac{d-2}{2}}(q-2)|E| \sum_{\substack{(x,z,t)\in E^2\times\mathbb{F}_q^* \\ :x\neq z, j-x\cdot z\neq 0}} \chi(jt) = B_{j,221} + B_{j,222}. \end{aligned}$$

Then the term $B_{j,222}$ is clearly dominated by $\sim q^{\frac{d}{2}}|E|^3$, because the sum over $t \in \mathbb{F}_q^*$ above is exactly -1. Thus the term $B_{j,222}$ is clearly dominated by the value in (6.2.10). To justify the inequality in (6.2.10), it remains to show that the term $B_{j,221}$ is dominated by the value in (6.2.10). Let us estimate the term $B_{j,221}$. After completing the square in s-variable and applying a change of variable, $s + t \to s$, we see that

$$B_{j,221} = kq^{\frac{d-2}{2}}(q-2)|E| \sum_{\substack{(x,z,s,t)\in E^2\times\mathbb{F}_q\times\mathbb{F}_q^* \\ :x\neq z, j-x\cdot z\neq 0}} \chi\left(\frac{(j-x\cdot z)s^2}{2t}\right)\chi\left(\frac{(j+x\cdot z)t}{2}\right),$$

Since $j - x\cdot z \neq 0$ and $t \neq 0$, we can apply the formula in (3.1.2) to get the Gauss sum from the sum over s-variable. As a consequence, we obtain that

$$B_{j,221} = k^2 q^{\frac{d-1}{2}}(q-2)|E| \sum_{\substack{(x,z,t)\in E^2\times\mathbb{F}_q^* \\ :x\neq z, j-x\cdot z\neq 0}} \eta\left(\frac{j-x\cdot z}{2t}\right)\chi\left(\frac{(j+x\cdot z)t}{2}\right).$$

where η is the multiplicative character of $\mathbb{F}_q^*$ of order two. Note that $\eta(\frac{1}{t}) = \eta(t)$ for all $t \in \mathbb{F}_q^*$, because the order of the character η is two. Then we see that the sum over t-variable above is just one of the twisted Kloosterman sums introduced in (3.2.2).

Thus we obtain that

$$B_{j,221} \lesssim q^{\frac{d+2}{2}}|E|^3.$$

This clearly implies that $B_{j,221}$ is dominated by the value in (6.2.10) and so the inequality in (6.2.10) holds. In order to complete the proof of Theorem 25, we must estimate the term C_j given by

$$\sum_{\substack{(x,z,z',s,s')\in E^3\times(\mathbb{F}_q^*)^2\\ :z\neq z'}} G(x,z,z',s,s'),$$

where $G(x,z,z',s,s')$ is defined by the value

$$q^d|E|\widehat{S_j}\left(s(-x+z)+s'(x-z')\right)\chi\left(j(s-s')+s(-x\cdot z)+s'(x\cdot z')\right).$$

Recall that we need to prove that the inequality in (6.2.11) holds to complete the proof of Theorem 25. Let us begin by writing the term C_j as two parts

$$\sum_{\substack{(x,z,z',s,s')\in E^3\times(\mathbb{F}_q^*)^2\\ :z\neq z',\\ s(-x+z)+s'(x-z')=(0,\ldots,0)}} q^d|E|\widehat{S_j}(0,\ldots,0)\chi\left(j(s-s')+s(-x\cdot z)+s'(x\cdot z')\right)$$
$$+\sum_{\substack{(x,z,z',s,s')\in E^3\times(\mathbb{F}_q^*)^2\\ :z\neq z'\\ s(-x+z)+s'(x-z')\neq(0,\ldots,0)}} G(x,z,z',s,s') = C_{j,1}+C_{j,2}.$$

We shall first estimate the term $C_{j,1}$. The condition $s(-x+z)+s'(x-z')=(0,\ldots,0)$ clearly implies that $x\cdot(s(-x+z)+s'(x-z'))=0$. By the direct calculation of the dot product, we see that $j(s-s')+s(-x\cdot z)+s'(x\cdot z')=0$, because $x\in E\subset S_j$. Using this and the fact that $\widehat{S_j}(0,\ldots,0)\sim q^{-1}$, we obtain that

$$C_{j,1}\sim q^{d-1}|E|\sum_{\substack{(x,z,z',s,s')\in E^3\times(\mathbb{F}_q^*)^2\\ :z\neq z',\\ s(-x+z)+s'(x-z')=(0,\ldots,0)}} 1.$$

Combining this with Corollary 24 , we conclude that

$$C_{j,1} \lesssim q^d|E|^3 + q^{\frac{3d}{2}}|E|^2.$$

Note that the term $C_{j,1}$ is dominated by the value in (6.2.11). To see this, use the fact that $a + b \geq 2\sqrt{ab}$ for $a, b \geq 0$ so that the term $q^d|E|^3$ is dominated by $\sim q^{\frac{d+2}{2}}|E|^4 + q^{\frac{3d}{2}}|E|^2$. We shall estimate the term $C_{j,2}$. Since $s(-x+z) + s'(x - z') \neq (0, \dots, 0)$, using the explicit form for $\widehat{S_j}$ in (3.2.7) yields that

$$C_{j,2} = Kq^{\frac{d-2}{2}}|E| \sum_{\substack{(x,z,z',s,s',t)\in E^3\times(\mathbb{F}_q^*)^3 \\ :z\neq z' \\ s(-x+z)+s'(x-z')\neq(0,\dots,0)}} \Omega_j(x, z, z', s, s', t),$$

where $\Omega_j(x, z, z', s, s', t)$ is given by

$$\chi\left(jt + \frac{\|s(-x+z) + s'(x - z')\|_2}{4t}\right)\chi\left(j(s - s') + s(-x \cdot z) + s'(x \cdot z')\right).$$

To eliminate the condition $s(-x + z) + s'(x - z') \neq (0, \dots, 0)$, we rewrite the term $C_{j,2}$ as following:

$$\begin{aligned} C_{j,2} =& Kq^{\frac{d-2}{2}}|E| \sum_{\substack{(x,z,z',s,s',t)\in E^3\times(\mathbb{F}_q^*)^3 \\ :z\neq z'}} \Omega_j(x, z, z', s, s', t) \\ &- Kq^{\frac{d-2}{2}}|E| \sum_{\substack{(x,z,z',s,s',t)\in E^3\times(\mathbb{F}_q^*)^3 \\ :z\neq z' \\ s(-x+z)+s'(x-z')=(0,\dots,0)}} \Omega_j(x, z, z', s, s', t) \\ &= C_{j,21} + C_{j,22}. \end{aligned}$$

Using the arguments for the estimate of $C_{j,1}$ as before, we can easily obtain

$$C_{j,22} \lesssim q^{\frac{d}{2}}|E|^3 + q^d|E|^2,$$

where we also used the fact that $\sum_{t\in\mathbb{F}_q^*} \chi(jt) = -1$. Thus the term $C_{j,22}$ is clearly dominated by the value in (6.2.11). Let us estimate the term $C_{j,21}$. For the simple notation, let $P = j - x \cdot z, Q = j - x \cdot z'$, and $U = z \cdot z' - j$. Then direct calculation shows that

$$j(s - s') + s(-x \cdot z) + s'(x \cdot z') = Ps - Qs',$$

and

$$\frac{\|s(-x+z) + s'(x - z')\|_2}{4t} = \frac{Ps^2 - (P+Q+U)ss' + Qs'^2}{2t}.$$

Thus the term $C_{j,21}$ is given by the value

$$Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E \\ s,s',t \in \mathbb{F}_q^* \\ : z \neq z'}} \chi(jt)\chi\left(\frac{Ps^2 - (P+Q+U)ss' + 2Pts + Qs'^2 - 2Qts'}{2t}\right).$$

Define $\Gamma_j(x, z, z', s, s', t)$ as the following value:

$$Kq^{\frac{d-2}{2}}|E|\chi(jt)\chi\left(\frac{Ps^2 - (P+Q+U)ss' + 2Pts + Qs'^2 - 2Qts'}{2t}\right),$$

and write $C_{j,21}$ as four terms:

$$\begin{aligned} C_{j,21} = & \sum_{\substack{x,z,z' \in E \\ s,s',t \in \mathbb{F}_q^* \\ : z \neq z' \\ P=Q=0}} \Gamma_j(x, z, z', s, s', t) + \sum_{\substack{x,z,z' \in E \\ s,s',t \in \mathbb{F}_q^* \\ : z \neq z' \\ P \neq 0 = Q}} \Gamma_j(x, z, z', s, s', t) \\ & + \sum_{\substack{x,z,z' \in E \\ s,s',t \in \mathbb{F}_q^* \\ : z \neq z' \\ P = 0 \neq Q}} \Gamma_j(x, z, z', s, s', t) + \sum_{\substack{x,z,z' \in E \\ s,s',t \in \mathbb{F}_q^* \\ : z \neq z' \\ P \neq 0 \neq Q}} \Gamma_j(x, z, z', s, s', t) \\ = & C_{j,211} + C_{j,212} + C_{j,213} + C_{j,214}. \end{aligned}$$

We shall estimate the term $C_{j,211}$. From the condition $P = Q = 0$, we see that the value $C_{j,211}$ is given by

$$C_{j,211} = Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E \\ s,s',t \in \mathbb{F}_q^* \\ : z \neq z' \\ P=Q=0}} \chi(jt)\chi\left(\frac{-Uss'}{2t}\right).$$

We remark that the values P, Q, and U are independent of the variables $s, s', t \in \mathbb{F}_q^*$. In case $U = 0$, we claim that the contribution to the bound of the term $C_{j,211}$ is given by

$$q^{\frac{d}{2}}|E|^4 + q^d|E|^3. \tag{6.2.12}$$

To justify the claim, note that the sum over $t \in \mathbb{F}_q^*$ is exactly -1 in case $U = 0$. It therefore follows that

$$\begin{aligned} Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E, s,s',t \in \mathbb{F}_q^* \\ :z \neq z', P=Q=U=0}} \chi(jt) &\lesssim q^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E, s,s' \in \mathbb{F}_q^* \\ :z \neq z', P=Q=U=0}} 1 \\ &\lesssim q^{\frac{d+2}{2}}|E|^2 \sum_{\substack{x,z \in E \\ :x \cdot z = j}} 1 \\ &\lesssim q^{\frac{d}{2}}|E|^4 + q^d|E|^3, \end{aligned}$$

where we used the fact that $P = j - x \cdot z = 0$ in the second inequality, and Theorem 23 in the last inequality. Thus the claim in (6.2.12) is complete. On the other hand, if $U \neq 0$, the contribution to the bound of the term $C_{j,211}$ is given by the value

$$q^{\frac{d-2}{2}}|E|^4 + q^{d-1}|E|^3. \tag{6.2.13}$$

To see this, note that if $U \neq 0$ then after applying a change of variable, $\frac{-Uss'}{2t} \to s'$, we see that

$$\begin{aligned} &Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E, s,s',t \in \mathbb{F}_q^* \\ :z \neq z', P=Q=0, U \neq 0}} \chi(jt)\chi\left(\frac{-Uss'}{2t}\right) \\ \sim &q^{\frac{d}{2}}|E| \sum_{\substack{x,z,z' \in E \\ :z \neq z', P=Q=0, U \neq 0}} 1 \lesssim q^{\frac{d}{2}}|E|^2 \sum_{\substack{x,z \in E \\ :x \cdot z = j}} 1 \\ \lesssim &q^{\frac{d-2}{2}}|E|^4 + q^{d-1}|E|^3, \end{aligned}$$

where we also used Theorem 23. Combining (6.2.12) with (6.2.13), we obtain that

$$C_{j,211} \lesssim q^{\frac{d}{2}}|E|^4 + q^d|E|^3.$$

Thus the term $C_{j,211}$ is also dominated by the term in (6.2.11), because the term $q^d|E|^3$ is dominated by $\sim q^{\frac{d+2}{2}}|E|^4 + q^{\frac{3d}{2}}|E|^2$ using the fact that $a+b \geq 2\sqrt{ab}$ if $a, b \geq 0$ as before. Let us estimate the term $C_{j,212}$. Since $P \neq 0$ and $Q = 0$, the term $C_{j,212}$ is given by

$$Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E, s,s',t \in \mathbb{F}_q^* \\ : z \neq z', P \neq 0, Q=0}} \chi(jt)\chi\left(\frac{Ps^2 - (P+U)ss' + 2Pts}{2t}\right),$$

where we recall that $P = j - x \cdot z, Q = j - x \cdot z'$ and $U = z \cdot z' - j$. We now write the term $C_{j,212}$ as following:

$$\begin{aligned} &Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E, s \in \mathbb{F}_q, s',t \in \mathbb{F}_q^* \\ : z \neq z', P \neq 0, Q=0}} \chi(jt)\chi\left(\frac{Ps^2 - (P+U)ss' + 2Pts}{2t}\right) \\ &-Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E, s',t \in \mathbb{F}_q^* \\ : z \neq z', P \neq 0, Q=0}} \chi(jt) = I + II \end{aligned}$$

As before, the term II is easily estimated as

$$\begin{aligned} II \lesssim\ & q^{\frac{d}{2}}|E|^2 \sum_{x,z' \in E : x \cdot z' = j} 1 \\ \lesssim\ & q^{\frac{d-2}{2}}|E|^4 + q^{d-1}|E|^3, \end{aligned}$$

where Theorem 23 was used in the last line. As before, we therefore see that the term II is also dominated by the term in (6.2.11). We shall estimate the term I. Rewrite the term I as following:

$$\begin{aligned} I =&Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z'\in E, s\in\mathbb{F}_q, s',t\in\mathbb{F}_q^* \\ :z\neq z', P\neq 0, Q=0=P+U}} \chi(jt)\chi\left(\frac{Ps^2+2Pts}{2t}\right) \\ &+ Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z'\in E, s\in\mathbb{F}_q, s',t\in\mathbb{F}_q^* \\ :z\neq z', P\neq 0, Q=0, P+U\neq 0}} \chi(jt)\chi\left(\frac{Ps^2-(P+U)ss'+2Pts}{2t}\right) \\ =&I_1+I_2. \end{aligned}$$

Let us estimate the term I_1. Completing the square in s-variable and applying a change of variable, $s+t\rightarrow s$, we see that

$$I_1 = Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z'\in E, s\in\mathbb{F}_q, s',t\in\mathbb{F}_q^* \\ :z\neq z', P\neq 0, Q=0=P+U}} \chi(jt)\chi(\frac{Ps^2}{2t})\chi(\frac{-Pt}{2}).$$

Since $P\neq 0$ and $t\neq 0$, we can apply the formula in (3.1.2) to obtain the Gauss sum from the exponential sum in s-variable. As a result, we have

$$I_1 = K^2q^{\frac{d-1}{2}}|E| \sum_{\substack{x,z,z'\in E, s',t\in\mathbb{F}_q^* \\ :z\neq z', P\neq 0, Q=0=P+U}} \eta(\frac{P}{2t})\chi(jt+\frac{-Pt}{2}),$$

where we used the fact that the Gauss sum is exactly given by $Kq^{\frac{1}{2}}$ for some $K\in\mathbb{C}$. Notice that the sum over t-variable above is a twisted Kloosterman sum introduced as in (3.2.2). From this and Theorem 23, I_1 is easily estimated as

$$\begin{aligned} I_1 \lesssim& q^{\frac{d+2}{2}}|E| \sum_{\substack{x,z,z'\in E \\ j-x\cdot z'=0}} 1 \\ \lesssim& q^{\frac{d}{2}}|E|^4 + q^d|E|^3, \end{aligned}$$

which is clearly dominated by the value in (6.2.11). We shall estimate the term I_2. Since $P\neq 0, P+U\neq 0, s\neq 0\neq t$, after using a change of variable, $\frac{-(P+U)ss'}{2t}\rightarrow s'$, and then applying a change of variable, $s+t \to s$, we see that

$$I_2 = Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E, s\in \mathbb{F}_q, s',t\in \mathbb{F}_q^* \\ :z\neq z', P\neq 0, Q=0, P+U\neq 0}} \chi(\frac{Ps^2}{2t})\chi(s')\chi\left((j-\frac{P}{2})t\right).$$

Observe that the sum over $s' \in \mathbb{F}_q^*$ is exactly -1, and use the formula in (3.1.2) to obtain the Gauss sum. Then we see that

$$I_2 = -K^2 q^{\frac{d-1}{2}}|E| \sum_{\substack{x,z,z' \in E, t\in \mathbb{F}_q^* \\ :z\neq z', P\neq 0, Q=0, P+U\neq 0}} \eta(\frac{P}{2t})\chi\left((j-\frac{P}{2})t\right).$$

Observe that $\eta(\frac{1}{t}) = \eta(t)$ for $t \in \mathbb{F}_q^*$, because η is multiplicative character of $\mathbb{F}_q^*$ of order two. Then the sum over t-variable above is just a twisted Kloosterman sum and so we obtain that

$$\begin{aligned} I_2 \lesssim & q^{\frac{d}{2}}|E| \sum_{\substack{x,z,z' \in E \\ :x\cdot z'=j}} 1 \\ \lesssim & q^{\frac{d-2}{2}}|E|^4 + q^{d-1}|E|^3, \end{aligned}$$

where we used Theorem 23 in the last inequality. Note that the term I_2 is also dominated by the value in (6.2.11). It remains to estimate the terms $C_{j,213}$ and $C_{j,214}$.To get the upper bound of $C_{j,213}$, modify the processes used to obtain the upper bound of $C_{j,212}$ after switching the role of P and Q. Then we can show that the term $C_{j,213}$ has the same upper bound as $C_{j,212}$. Thus we also see that the term $C_{j,213}$ is dominated by the term in (6.2.11). Finally, we shall estimate the term $C_{j,214}$. Recall that the term $C_{j,214}$ is given by the value

$$Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E \\ s,s',t\in \mathbb{F}_q^* \\ :z\neq z' \\ P\neq 0\neq Q}} \chi(jt)\chi\left(\frac{Ps^2 - (P+Q+U)ss' + 2Pts + Qs'^2 - 2Qts'}{2t}\right).$$

We now write the term $C_{j,214}$ by the two terms

$$
\begin{aligned}
&Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E\\ s\in\mathbb{F}_q\\ s',t\in\mathbb{F}_q^*\\ :z\neq z'\\ P\neq 0\neq Q}} \chi(jt)\chi\left(\frac{Ps^2-(P+Q+U)ss'+2Pts+Qs'^2-2Qts'}{2t}\right)\\
-&Kq^{\frac{d-2}{2}}|E| \sum_{\substack{x,z,z' \in E\\ s',t\in\mathbb{F}_q^*\\ :z\neq z'\\ P\neq 0\neq Q}} \chi(jt)\chi\left(\frac{Qs'^2-2Qts'}{2t}\right) = G+H.
\end{aligned}
$$

Let us estimate the term G. Letting $T=P+Q+U$ and completing the square over s-variable, we note that

$$
\begin{aligned}
&Ps^2-(P+Q+U)ss'+2Pts+Qs'^2-2Qts'\\
=&P\left(s+\frac{-Ts'+2tP}{2P}\right)^2+\frac{(4PQ-T^2)s'^2+(-8PQt+4PTt)s'-4P^2t^2}{4P}.
\end{aligned}
$$

After applying a change of variable, $s+\frac{-Ts'+2tP}{2P}\to s$, and using the formula in (3.1.2), we have $G=K^2q^{\frac{d-1}{2}}|E|$

$$
\times \sum_{\substack{x,z,z'\in E, s',t\in\mathbb{F}_q^*\\ :z\neq z'\\ P\neq 0\neq Q}} \eta(\frac{P}{2t})\chi\left(\frac{(-2Q+T)s'}{2}\right)\chi\left((j-\frac{P}{2})t+\frac{(4PQ-T^2)s'^2}{8Pt}\right).
$$

Observe that the sum over t-variable is a twisted Kloosterman sum which is bounded by $\sim q^{\frac{1}{2}}$. Thus, we obtain that

$$
G \lesssim q^{\frac{d+2}{2}}|E|^4. \tag{6.2.14}
$$

To get the upper bound of the term H, we use the trivial estimate so that we can easily obtain that

$$
H \lesssim q^{\frac{d+2}{2}}|E|^4.
$$

From this estimate and (6.2.14), we have

$$C_{j,214} \lesssim q^{\frac{d+2}{2}} |E|^4.$$

Thus the term $C_{j,214}$ is dominated by the value in (6.2.11) and the proof of Theorem 25 is complete. We therefore establish the proof of Theorem 22.

Chapter 7

CONICAL EXTENSION THEOREMS

7.1 Necessary Conditions

We observe the necessary conditions for the boundedness of $R^*(p \to r)$ associated with the cones in d-dimensional vector spaces over finite fields. The following theorem gives us the range of exponents "r" such that $R^*(p \to r) \lesssim 1$. In the dimension three, the authors in [21] proved the following theorem using the standard estimate for Gauss sums. Using the similar idea, we obtain one of necessary conditions for the boundedness of $R^*(p \to r)$ for all dimensions.

Theorem 26. *If $r < \frac{2d-2}{d-2}$, then $R^*(p \to r)$ is unbounded for all $1 \le p \le \infty$.*

Proof. Let $\mathbb{X} = \{s \in \mathbb{F}_q^* : s \text{ is not a square number}\}$. Note that $\eta(s) = -1$ for all $s \in \mathbb{X}$, and $|\mathbb{X}| = \frac{q-1}{2}$. Consider a subset M of $\mathbb{F}_q^d$ given by

$$M = \left\{ (m_1, \ldots, m_{d-1}, m_d) \in \mathbb{F}_q^{d-1} \times \mathbb{X} : m_{d-1} = \frac{m_1^2 + \ldots + m_{d-2}^2}{4m_d} \right\}.$$

It is clear that $|M| = q^{d-2}|\mathbb{X}| \sim q^{d-1}$. To complete the proof, we test (2.2.2) with $g = \chi_M$. Since the measure "dm" is the counting measure, we see that

$$\|M\|_{L^{r'}(\mathbb{F}_q^d, dm)} = |M|^{\frac{1}{r'}} \sim q^{\frac{d-1}{r'}}. \tag{7.1.1}$$

In order to complete the proof, it therefore suffices to show that for all $1 \le p \le \infty$,

$$q^{\frac{d}{2}} \lesssim \|\widehat{M}\|_{L^{p'}(\mathcal{C},d\sigma)}. \tag{7.1.2}$$

However, the inequality in (7.1.2) follows from the following claim. For each $x \in \mathcal{C}$ with $x_{d-1} \neq 0$, we have

$$|\widehat{M}(x)| = q^{\frac{d-2}{2}}|\mathbb{X}| \sim q^{\frac{d}{2}}. \tag{7.1.3}$$

To justify the claim in (7.1.3), we note that if $x \in \mathcal{C}$ with $x_{d-1} \neq 0$ then

$$\begin{aligned}\widehat{M}(x) &= \sum_{m\in M} \chi(-x\cdot m)\\ &= \sum_{m_d\in\mathbb{X}} \sum_{m_1,\ldots,m_{d-2}\in\mathbb{F}_q} \chi\left(\mathbb{P}_x(m_1,\ldots,m_{d-2},m_d)\right),\end{aligned}$$

where $\mathbb{P}_x(m_1,\ldots,m_{d-2},m_d)$ is given by the value

$$-x_1m_1 - \ldots - x_{d-2}m_{d-2} - x_{d-1}\frac{m_1^2+\ldots+m_{d-2}^2}{4m_d} - x_dm_d.$$

Using Lemma 3, we see that

$$\widehat{M}(x) = G_1^{d-2}\eta^{d-2}(-x_{d-1}/4)\sum_{m_d\in\mathbb{X}} \eta^{d-2}(m_d^{-1}),$$

where we also used the fact that $x_d - \frac{x_1^2+\ldots+x_{d-2}^2}{x_{d-1}} = 0$, because $x \in \mathcal{C}$ with $x_{d-1} \neq 0$. Recall that $\eta(s) = \eta(s^{-1}) = -1$ for $s \in \mathbb{X}$. Thus we conclude that

$$|\widehat{M}(x)| = q^{\frac{d-2}{2}}|\mathbb{X}| \sim q^{\frac{d}{2}},$$

and the proof is complete. $\square$

We now investigate another necessary condition for the boundedness of $R^*(p \to r)$. Suppose that the cone $\mathcal{C} \in \mathbb{F}_q^d$ contains a subspace H of dimension k ($|H| = q^k$). We test (2.2.1) with f equal to the characteristic function on the subspace H. Then we see that

$$\|H\|_{L^p(\mathcal{C},d\sigma)} = \left(|H||\mathcal{C}|^{-1}\right)^{\frac{1}{p}}, \tag{7.1.4}$$

and

$$\|(Hd\sigma)^\vee\|_{L^r(\mathbb{F}_q^d,dm)} = |H||\mathcal{C}|^{-1}q^{\frac{d-k}{r}}. \tag{7.1.5}$$

By (7.1.4), and (7.1.5) together with (2.2.1), we must have

$$|H||\mathcal{C}|^{-1}q^{\frac{d-k}{r}} \lesssim \left(|H||\mathcal{C}|^{-1}\right)^{\frac{1}{p}}. \tag{7.1.6}$$

Since $|H| = q^k$ and $|\mathcal{C}| \sim q^{d-1}$, the inequality in (7.1.6) yields the necessary condition for $R^*(p \to r) \lesssim 1$, that is,

$$r \geq \frac{p(d-k)}{(p-1)(d-k-1)}. \tag{7.1.7}$$

However, if $d \geq 2$ is even and $-1 \in \mathbb{F}_q$ is a square number (say $i^2 = -1$ for some $i \in \mathbb{F}_q$), then the cone $\mathcal{C}$ always contains the $\frac{d}{2}-$dimensional subspace H given by

$$H = \left\{(t_1, it_1, \ldots, t_{\frac{d-2}{2}}, it_{\frac{d-2}{2}}, t_{\frac{d}{2}}, 0) : t_k \in \mathbb{F}_q,\ k = 1, 2, \ldots, d/2\right\}.$$

Thus if $d \geq 2$ is even, then the necessary condition in (7.1.7) takes the form

$$r \geq \frac{pd}{(p-1)(d-2)}, \tag{7.1.8}$$

because $|H| = q^{\frac{d}{2}}$ and so $k = \frac{d}{2}$. On the other hand, if $d \geq 3$ is odd and $-1 \in \mathbb{F}_q$ is a square number, then the cone $\mathcal{C}$ contains the subspace H of dimension $(d-1)/2$, given by the set

$$\left\{(t_1, it_1, \ldots, t_{\frac{d-3}{2}}, it_{\frac{d-3}{2}}, t_{\frac{d-1}{2}}, it_{\frac{d-1}{2}}, -it_{\frac{d-1}{2}}) : t_k \in \mathbb{F}_q,\ k = 1, 2, \ldots, \frac{d-1}{2}\right\}.$$

From this and (7.1.7), we obtain, in odd dimensions, the necessary condition for the boundedness of $R^*(p \to r)$, that is,

$$r \geq \frac{p(d+1)}{(p-1)(d-1)}. \tag{7.1.9}$$

In conclusion, using Theorem 26, (7.1.8), and (7.1.9), we have the following theorem related to necessary conditions for $R^*(p \to r) \lesssim 1$ (see Figure 7.1 and 7.2).

Theorem 27. *If $d \geq 2$ is even, then $R^*(p \to r) \lesssim 1$ only if*

$$r \geq \frac{2d-2}{d-2} \quad \textit{and} \quad r \geq \frac{pd}{(p-1)(d-2)}.$$

On the other hand, if $d \geq 3$ is odd, then $R^*(p \to r) \lesssim 1$ only if

$$r \geq \frac{2d-2}{d-2} \quad \textit{and} \quad r \geq \frac{p(d+1)}{(p-1)(d-1)}.$$

7.2 Main Results for the Conical Extension Theorems

We state our main results of extension theorems for the cone $\mathbb{C}$ in $\mathbb{F}_q^d$. The authors in [21] obtained the complete solution to the question about the extension theorems for the cone in three dimensions by showing that $R^*(2 \to 4) \lesssim 1$. Our main results below can be viewed as the extension of their work to higher dimensions.

Theorem 28. *Let $\mathcal{C}$ be the cone in $\mathbb{F}_q^d$ defined as in (2.4.5). Suppose $d \geq 3$ is odd. Then we have*

$$R^*\left(2 \to \frac{2d+2}{d-1}\right) \lesssim 1.$$

Theorem 28 says that if the dimension d is odd, then the Tomas-Stein exponents guarantee that $R^*(p \to r) \lesssim 1$. From the necessary conditions in Theorem 27, we see that Theorem 28 yields the sharp $L^2 - L^r$ estimate (see Figure 7.1). Moreover, if the dimension d is three, then Theorem 28 gives the complete solution to the question about the conical extension theorems in three dimensions. This was already proved by Mockenhaupt and Tao ([21]), but their proof can only yield the $L^2 - L^4$ estimate for all dimensions (see [21]). Our calculation of the sharp decay of the Fourier transform of the cones enables us to investigate the boundedness of the conical extension operators in all dimensions.

The proof of Theorem 28 is exactly same as the proof of Theorem 17, because the decay of the Fourier transform of the cone in odd dimensions and the decay of the Fourier transform of the nondegenerate quadratic surfaces are almost same (see Lemma 5 and Lemma 7).

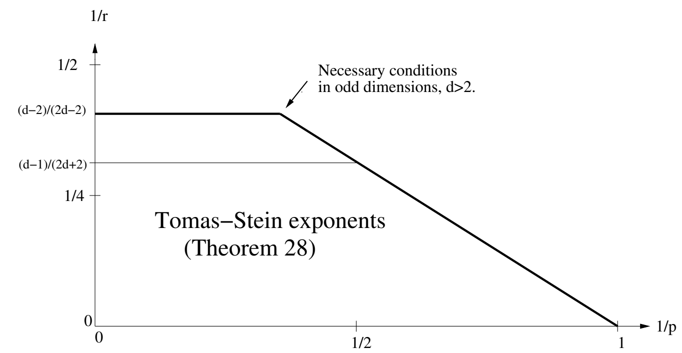

Figure 7.1: Theorem 28 and necessary conditions for boundedness of $R^*(p \to r)$ related to cones in odd dimensions, $d \geq 3$.

From Lemma 5, we also see that the decay of the Fourier transform of the cones in even dimensions is not as good as in odd dimensions. However, the decay is enough to obtain the sharp $L^2 - L^r$ estimate for the cones in even dimensions. In fact, we have the following theorem.

Theorem 29. *Let $\mathcal{C}$ be the cone in $\mathbb{F}_q^d$ defined as in (2.4.5). Suppose $d \geq 2$ is even.*

Then we have

$$R^*\left(2 \to \frac{2d}{d-2}\right) \lesssim 1.$$

From the necessary conditions for the boundedness of $R^*(p \to r)$ in Theorem 27, it is clear that Theorem 29 gives the sharp $L^2 - L^r$ estimate in even dimensions (see Figure 7.2). The proof of Theorem 29 is based on the decay of the Fourier transform of the cones in even dimensions, and it is just a routine modification of the proof of Theorem 17.

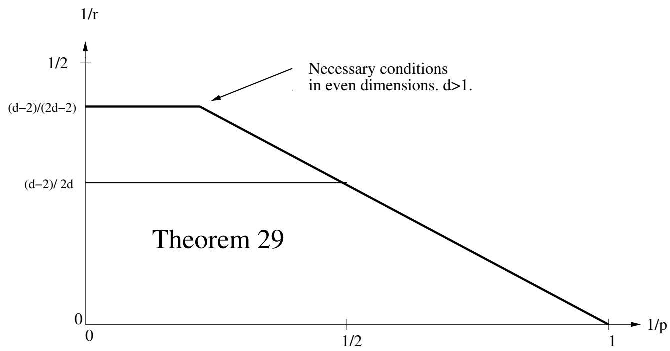

Figure 7.2: Theorem 29 and necessary conditions for boundedness of $R^*(p \to r)$ related to cones in even dimensions, $d \geq 2$.

Chapter 8

APPLICATION OF EXTENSION THEOREMS

8.1 Euclidean Distance Problems

The Erdös distance conjecture in the Euclidean space says that if E is a finite subset of $\mathbb{R}^d$, $d \geq 2$, then

$$|\Delta(E)| \gtrapprox |E|^{\frac{2}{d}}, \tag{8.1.1}$$

where

$$\Delta(E) = \{|x-y| : x, y \in E\},$$

with $|x-y|^2 = (x_1-y_1)^2 + \cdot + (x_d-y_d)^2$ and here, and throughout the paper, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$, and $X \lessapprox Y$, with the controlling parameter N, means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon N^\epsilon Y$.

Taking $E = \mathbb{Z}^d \cap [0, N^{\frac{1}{d}}]^d$ shows that (8.1.1) cannot in general be improved. The conjecture has not been solved in any dimension. See, for example, [20], [1], and the references contained therein for the description of the conjecture, background material, and a survey of recent results. In the Euclidean plane, the best known results due to Katz and Tardos ([18]) are based on a previous breakthrough by Solymosi and Toth ([25]). For the best known results in higher dimensions see [26] and [27]. These results are a culmination of efforts going back to the 1945 paper by Erdös ([5]).

On the other hand, the Falconer distance conjecture in the Euclidean setting says that if the Hausdorff dimension of a set in $\mathbb{R}^d$ exceeds $\frac{d}{2}$, then the Lebesgue measure of the distance set is positive. The Falconer distance problem is considered as the continuous analog of the Erdös distance problem. The best known result on the Falconer distance problems is due to Wolff ([31]), in two dimensions, and Erdoğan ([6]), in higher dimensions. These are a culmination of efforts going back to Falconer in 1986 ([7]) and Bourgain ([3]) a few years later.

8.2 Distance Problems in the Finite Field Setting

In the finite field setting, A. Iosevich and M. Rudnev ([17]) introduced and developed the distance problems which turn out to have features of both the Erdös and Falconer distance problems. Let $\mathbb{F}_q$ denote the finite field with q elements, and let $\mathbb{F}_q^d$ denote the d-dimensional vector space over this field. Let $E \subset \mathbb{F}_q^d$, $d \geq 2$. Then a possible analog of the classical Erdös-Falconer distance problem is to determine the smallest possible cardinality of the set

$$\Delta_n(E) = \{||x-y||_n = (x_1 - y_1)^n + \cdots + (x_d - y_d)^n : x, y \in E\},$$

with n a positive integer ≥ 2, viewed as a subset of $\mathbb{F}_q$.

In the finite field setting, the estimate (8.1.1) cannot hold without further restrictions. To see this, let $E = \mathbb{F}_q^d$. Then $|E| = q^d$ and $|\Delta(E)| = q$. Furthermore, an interesting feature of the Erdös distance problem in the finite field setting with $n = 2$ is the existence of non-trivial spheres of zero radius. These are sets of the form $\{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \cdots + x_d^2 = 0\}$ and several assumptions in the statements of results in [16] are there precisely to deal with issues created by the presence of this object. For example, suppose -1 is a square in $\mathbb{F}_q$. Using spheres of zero radius one can show, in even dimensions, that there exists a set of cardinality precisely $q^{\frac{d}{2}}$ such that all the distances, $(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2$ are zero. What's more, suppose $\mathbb{F}_q$ is a finite field, such that $q = p^2$, where p is a prime. Then $E = \mathbb{F}_p^d$ is naturally embedded in $\mathbb{F}_q^d$, has cardinality $q^{\frac{d}{2}}$, and determines only $\sqrt{q}$ distances. If $n > 2$, the situation is equally fascinating. For example, if $n = 3$ and $d = 2$, the equation $x_1^3 + x_2^3 = 0$ always has at least q solutions, since cube root of -1 is -1. This equation may have as many as $3q$ solutions if the primitive cube root of -1 is in the field.

With these examples as guide, the author along with A. Iosevich ([16]) generalized the conjecture originally stated in [17] in the case $n = 2$ as follows.

Conjecture 30. *Let $E \subset \mathbb{F}_q^d$ of cardinality $\geq Cq^{\frac{d}{2}}$, with C sufficiently large. Then*

$$|\Delta_n(E)| \gtrapprox q.$$

However, it turned out that Conjecture 30 is not true in the case when $n = 2$ and the dimension d is odd. In fact, arithmetic examples constructed by the author, D. Hart, A. Iosevich, and M. Rudnev in [11] show that the exponent $(d+1)/2$ is sharp. However, the author strongly believes that Conjecture 30 holds if the dimension d is even. As a partial evidence of the belief, we have the following theorem.

Theorem 31. *Let $E \subset \mathbb{F}_q^2$ of cardinality $\geq Cq^{4/3}$, with C sufficiently large. Then we have*

$$|\Delta_2(E)| \gtrsim q.$$

Proof. Before we prove Theorem 31, we note that if $d = 2$, then the exponent $4/3$ in Theorem 31 is better than the exponent $(d+1)/2$ which gives a sharp exponent in odd dimensions. The proof of Theorem 31 is based on both the our sharp result of extension theorem for circles and the following theorem due to A. Iosevich and M. Rudnev ([17]).

Theorem 32. *Let $E \subset \mathbb{F}_q^d, d \geq 2$. Suppose that $|E| \geq Cq^{\frac{d}{2}}$ with C sufficiently large. Then*

$$|\Delta(E)| \gtrsim \min\left\{q, \frac{q}{\mathbb{M}_E(q)}\right\},$$

where $\mathbb{M}_E(q)$ is given by the formula

$$\mathbb{M}_E(q) = \frac{q^{3d+1}}{|E|^4} \sum_{t \in \mathbb{F}_q^*} \sigma_E^2(t) \quad with \quad \sigma_E(t) = \sum_{m \in S_t} |\widehat{E}(m)|^2.$$

Let us prove Theorem 31. Using Theorem 32, it suffices to show that for each $E \in \mathbb{F}_q^2$ with $|E| \gtrsim q^{\frac{4}{3}}$,

$$\frac{q^7}{|E|^4}\frac{|E|}{q^2} \max_{t \in \mathbb{F}_q^*} \sigma_E(t) \lesssim 1,$$

because $\sum_{t \in \mathbb{F}_q} \sigma_E(t) = \frac{|E|}{q^2}$. Thus it is enough to show that for each $E \subset \mathbb{F}_q^2$ with $|E| \gtrsim q^{\frac{4}{3}}$, we have

$$\sigma_E(t) = \sum_{m \in S_t} |\widehat{E}(m)|^2 \lesssim \frac{|E|^3}{q^5} \quad \text{for all} \quad t \in \mathbb{F}_q^*. \tag{8.2.1}$$

Using Theorem 18, we see that

$$\|(fd\sigma)^\vee\|_{L^4(\mathbb{F}_q^2, dm)} \lesssim \|f\|_{L^2(d\sigma, S_t)}, \tag{8.2.2}$$

because the spheres S_t are nondegenerate quadratic surfaces. By duality, the inequality (8.2.2) yields the following restriction estimate:

$$\|\widehat{g}\|_{L^2(S_t, d\sigma)} \lesssim \|g\|_{L^{\frac{4}{3}}(\mathbb{F}_q^d, dm)} \quad \text{for all functions} \quad g \quad \text{on} \quad (\mathbb{F}_q^d, dm). \tag{8.2.3}$$

We now return to the proof of the inequality (8.2.1). We have

$$\begin{aligned}\sigma_E(t) &= \sum_{m\in S_t} |\widehat{E}(m)|^2 \\ &= q^{-4}\sum_{m\in S_t}\sum_{y,z\in\mathbb{F}_q^2} \chi\left(-m\cdot(x-y)\right)E(x)E(y). \end{aligned} \tag{8.2.4}$$

Since the space $(\mathbb{F}_q^2, dm)$ is isomorphic to the dual space $(\mathbb{F}_q^2, dx)$, the value $\sigma_E(t)$ can be written by

$$\sigma_E(t) = q^{-4}\sum_{x\in S_t}\sum_{m_1,m_2\in\mathbb{F}_q^2} \chi\left(-x\cdot(m_1-m_2)\right)E(m_1)E(m_2), \tag{8.2.5}$$

where the set E is a subset of $(\mathbb{F}_q^2, dm)$ with $|E| \gtrsim q^{\frac{4}{3}}$, and S_t denotes the sphere with the radius $t \neq 0$ in $(\mathbb{F}_q^2, dx)$. If we take the function g in (8.2.3) as $E(m)$, the characteristic function on $E \subset (\mathbb{F}_q^2, dm)$, then we have

$$\|\widehat{E}\|_{L^2(S_t,d\sigma)} \lesssim \|E\|_{L^{\frac{4}{3}}(\mathbb{F}_q^d,dm)}. \tag{8.2.6}$$

Since dm is the counting measure and $d\sigma$ is the normalized counting measure on the sphere S_t, the inequality (8.2.6) can be written by

$$\left(q^{-1}\sum_{x\in S_t}|\widehat{E}(x)|^2\right)^{\frac{1}{2}} \lesssim |E|^{\frac{3}{4}}, \tag{8.2.7}$$

where we recall that $\sigma(x) \sim qS_t(x)$ and dx is the normalized counting measure. Note that the Fourier transform $\widehat{E}(x)$ in (8.2.7) is defined by the formula

$$\widehat{E}(x) = \sum_{m\in\mathbb{F}_q^2} \chi(-x\cdot m)E(m),$$

because the function $E(m)$ is defined on the space $(\mathbb{F}_q^2, dm)$ with the counting measure dm. Thus from (8.2.7) and (8.2.5), we see that

$$\sigma_E(t) \lesssim q^{-3}|E|^{\frac{3}{2}}.$$

Since $|E| \geq Cq^{\frac{4}{3}}$ with sufficiently large $C > 0$, we have

$$\sigma_E(t) \lesssim q^{-3}|E|^{\frac{3}{2}} \leq q^{-5}|E|^3.$$

Thus the inequality (8.2.1) holds and the proof is complete. □

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