SIMULTANEOUS LOCAL RESOLUTION ALONG A RATIONAL VALUATION IN TWO DIMENSIONAL POSITIVE CHARACTERISTIC FUNCTION FIELDS

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A dissertation presented to the Faculty of the Graduate School at the University of Missouri–Columbia in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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July of 2022
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David Retzloff, Ph.D.
Dedicated to my mother and father.
ACKNOWLEDGMENTS

I thank my advisor, professor Dale Cutkosky, for his support and guidance throughout my graduate career. I thank him for his patience and understanding as I struggled sometimes to keep up with deadlines. I learned a lot reading his papers and thank him for the vast knowledge he was always happy to share. I thank him for his help in writing this thesis as well.

I also thank the members of the committee, for the fruitful discussions during my comprehensive exams and their full support throughout the program. In particular, I would like to thank professor Ian Aberbach for his classes that were always fun and helped me integrate with life at Mizzou.

I would like to also thank my fellow graduate students at the University of Missouri for the camaraderie and the many stimulating conversations, particularly Nick Cox-Steib, Suprajo Das, Kyle Maddox, Roberto Núñez, and Smita Praharaj.

Finally, I thank my family for their love, encouragement, and support that I can always count on.
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ABSTRACT

We consider the condition that a germ of an algebraic mapping of nonsingular surfaces can be made finite, after sufficient blowing up along a nondiscrete rational rank 1 valuation. This problem has been solved in the affirmative in characteristic zero by Abhyankar. He calls this simultaneous resolution. The essential new case that occurs in positive characteristic p is a chain of Artin-Schreier extensions. It is known that simultaneous local resolution is true (along a nondiscrete rational rank 1 valuation) for a single Artin-Schreier extension. We prove that simultaneous local resolution is true in a tower of two Artin-Schreier extensions.
Chapter 1

Introduction

Let $k$ be an algebraically closed field and $L/K$ a finite separable extension of algebraic function fields of finite transcendence degree over $k$. Let $n$ denote the transcendence degree of $K$ (and $L$) over $k$. Let $\nu$ be a $k$-valuation of $K$, with valuation ring $V_\nu$ and value group $\Gamma_\nu$, and let $\omega$ be an extension of $\nu$ to $L$, with valuation ring $V_\omega$ and value group $\Gamma_\omega$. Suppose that the residue field $V_\nu/m_\nu = k$. Suppose that $R$ and $S$ are algebraic regular local rings with respective quotient fields $K$ and $L$ with $\dim R = \dim S = 2$, such that $\omega$ dominates $S$ and $S$ dominates $R$ (so $\nu$ dominates $R$). Let $m_R$ and $m_S$ denote the maximal ideals of $R$ and $S$ respectively.

Simultaneous local resolution of $R \subseteq S$ along $\omega$ is the statement that there exists a commutative diagram of algebraic regular local rings

$$
\begin{array}{ccc}
R' & \longrightarrow & S' \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$

(1.0.1)

such that the vertical arrows are products of monoidal transforms along $\nu$ and $\omega$ respectively, and $R' \to S'$ is the localization of a finite map.

We observe that, with our assumptions, the statement that $R' \subseteq S'$ is the localization of a finite map is equivalent to having $\sqrt{m_R S'} = m_{S'}$ ($m_{R'} S'$ is $m_{S'}$
primary) (see Lemma 2.0.6).

When \( k \) has characteristic zero, Abhyankar [3, Theorem 2] shows that simultaneous local resolution holds for rational valuations in algebraic function fields of dimension 2. In dimension 2, the only monoidal transform is the quadratic transform, the blow up of the maximal ideal. However, an example of Abhyankar [3, Theorem 12] in dimension 2 and valid in all characteristics shows that simultaneous local resolution is in general false for valuations of rational rank larger than 1.

We have that local monomialization and even strong local monomialization are true in characteristic 0 and arbitrary dimension (see [9, Theorem 1.1] and [8, Theorem 4.8]). A monomialization is a diagram (1.0.1) such that \( R' \to S' \) has the form

\[
x_i = \gamma_i y_1^{a_{i1}} \cdots y_n^{a_{in}},
\]

for \( 1 \leq i \leq n \), where \( x_1, \ldots, x_n \) are regular parameters in \( R' \), \( y_1, \ldots, y_n \) are regular parameters in \( S' \), \( A = (a_{ij}) \) is an \( n \times n \) matrix of natural numbers with non-zero determinant and \( \gamma_1, \ldots, \gamma_n \) are units in \( S' \).

When \( \nu \) is rational, a strong monomialization is a diagram (1.0.1) such that \( R' \to S' \) has the form

\[
x_1 = \gamma y_1^a, \quad x_2 = y_2, \ldots, \quad x_n = y_n
\]

where \( x_1, \ldots, x_n \) are regular parameters in \( R' \), \( y_1, \ldots, y_n \) are regular parameters in \( S' \), \( \gamma \) is a unit in \( S' \) and \( a \) is a positive integer. Thus \( R' \to S' \) is the localization of a finite map. Hence, we have that, simultaneous local resolution holds for rational valuations in algebraic function fields of arbitrary dimension and characteristic zero (see [9, Theorem 4.1] or [8, Theorem 4.8]).
In this thesis, we consider the problem of simultaneous local resolution for rational valuations in dimension 2 and positive characteristic.

The essential difference between characteristic $p > 0$ and characteristic 0 is the possibility of defect in the extension. The defect $\delta(\omega/\nu)$ is defined in Section 7.1 of [8]. If there is no defect (as is always the case in equicharacteristic zero) the extension is called defectless.

If $\nu$ is a rational valuation, the strong monomial form $R' \to S'$ in dimension 2 is a diagram (1.0.1) such that

$$u = \gamma x^a$$  \hspace{1cm} (1.0.2)  

$$v = y$$  \hspace{1cm} (1.0.3)  

where $u, v$ are regular parameters in $R'$, $x, y$ are regular parameters in $S'$, $\gamma$ is a unit in $S'$ and $a$ is a positive integer. In their paper [8], S. D. Cutkosky and O. Piltant, show that strong local monomialization is true in characteristic $p > 0$ dimension 2 defectless extensions (see (2) of [8, Theorem 7.3] for the discrete case and [8, Theorem 7.35] for the rational non-discrete case). Thus simultaneous local resolution holds for rational valuations, in dimension 2 and char $p > 0$, if $L/K$ is defectless.

There are examples where both strong local monomialization ([8, Theorem 7.38]) and even local monomialization ([10]) are false in 2 dimensional defect extensions. In these cases, the local forms of the mapping are necessarily quite complicated. The only type of valuations for which there can be defect in dimension 2 are rational rank 1 non-discrete valuations (see [8, Theorem 7.3]).
From now on, we assume $k$ has characteristic $p > 0$, the transcendence degree of $K$ over $k$ is 2, and that $\nu$ (and thus $\omega$) have rational rank 1 but are not discrete. This is the remaining case in dimension 2 where simultaneous local resolution is not known.

In their paper, “Ramification of Valuations” [8, Corollary 7.30, Theorem 7.33], S. D. Cutkosky and O. Piltant show that one can obtain stable forms of mappings $R' \to S'$, where $S'$ is an iterated quadratic transform of $S$ and $R'$ is an iterated quadratic transform of $R$, along $\nu$ and $\omega$ respectively (constructed by the algorithm of Section 7.4 [8]), such that $R' \to S'$ has the form

\begin{align}
  u &= \gamma x^a \\
  v &= x^b (y^d \tau + x \Sigma)
\end{align}

where $u, v$ are regular parameters in $R'$, $x, y$ are regular parameters in $S'$, $\gamma, \tau \in S'$ are units, $\Sigma \in S'$, $a$ and $d$ are positive integers and $b$ is a non negative integer. These forms continue to hold along canonical sequences of quadratic transforms. Furthermore, we have that the product $ad$ is eventually constant upon further blowing up (see Proposition 3.0.12). However, $a$ and $d$ need not become constant. In fact, they may vary wildly ([6, Theorem 5.4]).

We note that a standard form $R' \to S'$ ((1.0.4) and (1.0.5) above) is the localization of a finite map if and only if $b = 0$ (equivalently $\sqrt{m_{R'}S'} = m_{S'}$).

Therefore, we are interested in the question of whether we can take quadratic transforms of $R$ and $S$, along $\nu$ and $\omega$ respectively, to obtain stable forms $R' \to S'$ with $b = 0$ in (1.0.5).

Now, for $L/K$ a Galois extension, one can obtain from ramification theory, the
following tower of subfields,

\[ K \subseteq K^d \subseteq K^i \subseteq K^r \subseteq L \]

where \( K^d, K^i, K^r \) are the decomposition, inertia and ramification fields respectively for \( \omega/\nu \). Furthermore, we have that \( K \subseteq K^r \) is defectless and hence we observe that simultaneous resolution holds for the sub-extension \( K \subseteq K^r \) ([8, Theorem 7.35]). The remaining part, \( K^r \subseteq L \), is a tower of Artin-Schreier extensions.

This leads us to consider the question of simultaneous local resolution for the special case when \( L/K \) is a tower of Artin-Schreier extensions.

Simultaneous local resolution is known when \( L/K \) is a single Artin-Schreier extension. In this case, an affirmative answer to the question follows from the results in the paper “On the Jung Method in positive characteristic” [11, Lemma 7.3] by O. Piltant.

A proof for the result can also be found in the paper “Erratic Birational Behavior of Mappings in Positive Characteristic” [6, Proposition 3.7] by S. D. Cutkosky.

The stable forms of an Artin-Schreier extension are very simple.

**Theorem 1.0.1.** [11, Lemma 7.3] [6, Proposition 3.7] Let \( k \) be an algebraically closed field of char \( p > 0 \) and \( L/K \) an Artin-Schreier extension of algebraic function fields of transcendence degree 2 over \( k \). Let \( \omega \) be a rational rank 1 nondiscrete \( k \)-valuation of \( L \) with restriction \( \nu = \omega|_K \). Suppose that \( R \) and \( S \) are algebraic regular local rings of \( K \) and \( L \) respectively such that \( \dim R = \dim S = 2 \), \( \omega \) dominates \( S \) and \( S \) dominates \( R \) (so \( \nu \) dominates \( R \)). Then there exists a commutative
such that the vertical arrows are products of quadratic transforms, \( \omega \) dominates \( S' \), \( S' \) dominates \( R' \), \( R' \) has a regular system of parameters \( u, v \) and \( S' \) has a regular system of parameters \( x, y \) such that one of the following three cases hold:

0) \( u = x, v = y \).

1) \( u = x, v = \tau y^p + x\Sigma \) where \( \tau \in S' \) is a unit and \( \Sigma \in S' \).

2) \( u = \gamma x^p, v = y \) where \( \gamma \in S' \) is a unit.

In particular, simultaneous local resolution holds for \( R \rightarrow S \) along \( \omega \).

1.1 Main Result

In this thesis, we consider the case when \( L/K \) is a tower of two Artin-Schreier extensions and prove the following similar result, showing simultaneous local resolution for a tower of two Artin-Schreier extensions.

**Theorem 1.1.1.** Suppose that \( K \rightarrow L \rightarrow M \) is a tower of two Artin-Schreier extensions, where \( K, L, M \) are two dimensional algebraic function fields over an algebraically closed field \( k \) of characteristic \( p > 0 \). Let \( \omega \) be a rational rank 1 nondiscrete \( k \)-valuation of \( M \). Suppose that \( R \) is an algebraic regular local ring of \( K \) and \( T \) is an algebraic regular local ring of \( M \) such that such that \( \dim R = \dim T = 2 \), \( \omega \) dominates \( T \), and \( T \) dominates \( R \). Then there exists a commutative
such that the vertical arrows are products of quadratic transforms, $\omega$ dominates $T'$, $T'$ dominates $R'$, $R'$ has a regular system of parameters $u, v$ and $T'$ has a regular system of parameters $x, y$ such that one of the following holds:

(a) $u = x, v = y$.

(b) $u = x, v = \tau y + x\Sigma$ where $\tau$ is a unit in $T'$ and $\Sigma \in T'$.

(c) $u = \gamma x, v = y$ where $\gamma$ is a unit in $T'$.

(d) $u = \gamma x, v = \tau y + x\Sigma$ where $\gamma, \tau$ are units in $T'$ and $\Sigma \in T'$.

(e) $u = x, v = \tau y^2 + x\Sigma$ where $\tau$ is a unit in $T'$ and $\Sigma \in T'$.

(f) $u = \gamma x^2, v = y$ where $\gamma$ is a unit in $T'$.

In particular, simultaneous local resolution holds for $R \to T$ along $\omega$. 

7
Chapter 2

Notations and Terminology

Definition 2.0.1. Let \((R, m_R)\) and \((S, m_S)\) be local rings such that we have an inclusion \(R \subset S\). We say \(S\) dominates \(R\) if \(m_S \cap R = m_R\).

Definition 2.0.2. Let \(k\) be a field and \(K\) a field containing \(k\). \(K\) is said to be an algebraic function field over the base field \(k\) if \(K\) is a finitely generated field extension of \(k\), \(K = k(x_1, ..., x_n)\) for some \(x_1, ..., x_n \in K\). A local ring \(k \subset A \subset K\) is said to be an algebraic local ring of \(K\) if \(A\) is the localization of a finitely generated \(k\)-algebra and \(QF(A) = K\).

Definition 2.0.3. [14, Section 8, Chapter VI, p. 32] Let \(K\) be an algebraic function field over a base field \(k\). A \(k\)-valuation \(\mu\) of \(K\) is a mapping from \(K^\times\), the multiplicative group of non-zero elements of \(K\), onto a totally ordered Abelian group \(\Gamma_\mu\)

\[ \mu : K^\times \to \Gamma_\mu \]

such that the following conditions are satisfied:

(a) \(\mu(fg) = \mu(f) + \mu(g)\) for \(f, g \in K^\times\).

(b) \(\mu(f + g) \geq \min\{\mu(f), \mu(g)\}\) for \(f, g \in K^\times\).

(c) \(\mu(\alpha) = 0\) for \(\alpha \neq 0 \in k\).
We extend \( \mu \) to \( K \) by defining \( \mu(0) = \infty \) to be larger than all elements of \( \Gamma_{\mu} \). The collection of elements

\[
V_{\mu} = \{ f \in K \mid \mu(f) \geq 0 \}
\]

forms a local ring with maximal ideal \( m_{\mu} = \{ f \in K \mid \mu(f) > 0 \} \) and is called the valuation ring of \( \mu \).

**Definition 2.0.4.** Let \( K \) be an algebraic function field over a base field \( k \) and let \( \mu \) be a \( k \)-valuation of \( K \). Let \( (A, m_A) \) be a local ring of \( K \). We say \( \mu \) dominates \( A \) if the valuation ring \( V_{\mu} \) dominates \( A \) (\( A \subseteq V_{\mu} \) and \( m_{\mu} \cap A = m_A \)).

**Lemma 2.0.5.** Let \( k \) be a field and let \( K \) be an algebraic function field over \( k \) of finite transcendence degree. Let \( (A, m) \) be an algebraic local ring of \( K \). Then,

\[
\text{trdeg}_k A/m + \text{dim } A = \text{trdeg}_k K.
\]  

(2.0.1)

In particular, suppose \( K \) is a two dimensional algebraic function field over \( k \) and \( (A, m) \) is an algebraic local ring of \( K \) such that \( \text{dim } A = 2 \). We have then, \( \text{trdeg}_k A/m = 0 \) and hence \( A/m = k \) in the case when \( k \) is algebraically closed.

**Proof.** Since \( A \) is an algebraic local ring of \( K \), we can write \( (A, m) \) as \( (A, m) = (B_P, PB_P) \) for some finitely generated \( k \)-algebra \( B \) and a prime ideal \( P \) of \( B \). Since \( B \) is a finitely generated \( k \)-algebra, and is a domain, we have,

\[
\text{dim } B/P + \text{height } P = \text{dim } B
\]  

(2.0.2)

([5, Chapter 1, Theorem 1.64, p. 19]). And since \( B \) and \( B/P \) are finitely generated \( k \)-algebras, that are domains, we have \( \text{dim } B = \text{trdeg}_k \text{QF}(B) = \text{trdeg}_k \text{QF}(A) = \text{trdeg}_k K \) and \( \text{dim } B/P = \text{trdeg}_k \text{QF}(B/P) = \text{trdeg}_k A/m \) (see [5, Chapter 1, Theorem 1.63, p. 19]). Furthermore, we note, \( \text{dim } A = \text{height } P \). Hence, the above
equation (2.0.2), becomes

\[ \text{trdeg}_k A/m + \text{dim} \ A = \text{trdeg}_k K. \]

And, therefore, if \( \text{dim} \ A = \text{trdeg}_k K \) then we get \( \text{trdeg}_k A/m = 0 \) and hence \( A/m = k \), when \( k \) is algebraically closed.

\[ \square \]

**Lemma 2.0.6.** Let \( L/K \) be a separable extension of two dimensional algebraic function fields over a base field \( k \). Let \( R \) and \( S \) be algebraic normal local rings of \( K \) and \( L \) respectively such that \( \text{dim} \ S = \text{dim} \ R \) and \( S \) dominates \( R \). Then the following are equivalent:

(i) \( R \rightarrow S \) is the localization of a finite map.

(ii) \( S = \overline{R}_{m_S \cap \overline{R}} \) where \( \overline{R} \) is the integral closure of \( R \) in \( L \).

(iii) \( \sqrt{m_R S} = m_S \).

**Proof.** We show (iii) \( \implies \) (ii) \( \implies \) (i) \( \implies \) (iii).

We start by showing (iii) \( \implies \) (ii).

Let \( \overline{R} \) be the integral closure of \( R \) in \( L \). Since \( S \) is normal we have \( \overline{R} \subseteq S \). Furthermore, since \( R \subseteq \overline{R} \) is integral and \( (m_S \cap \overline{R}) \cap R = m_S \cap R = m_R \), we obtain that \( m_S \cap \overline{R} \) is a maximal ideal of \( \overline{R} \). Additionally, since \( S \) is a local ring with maximal ideal \( m_S \), we obtain that \( S \) dominates \( \overline{R}_{m_S \cap \overline{R}} \). We use the local version of Zariski’s main theorem (see [12, Theorem 14] or see [5, Theorem 9.2, Chapter 9, p. 148]) to show equality. We show that the conditions required to apply local version of Zariski’s main theorem holds.
We start by noticing that, since \( L/K \) is separable, we have \( R \to \overline{R} \) is a finite map (see [13, Corollary 1, p. 265]). Therefore, since \( R \) is an algebraic local ring of \( K \), we obtain, \( \overline{R}_{mS \cap R} \) is an algebraic local ring of \( L \). Since \( \overline{R} \) is the integral closure of \( R \) in \( L \), we have \( \overline{R}_{mS \cap R} \) is normal as well. Furthermore, in our assumptions, we are assuming \( S \) is an algebraic local ring of \( L \).

Next, we show \( \dim \overline{R}_{mS \cap R} = \dim S \).

Since \( R \subseteq \overline{R} \) is integral with \((m_S \cap \overline{R}) \cap R = m_R \), and additionally \( R \) is normal, we have that \( \dim \overline{R}_{mS \cap R} = \text{ht}(m_S \cap \overline{R}) = \text{ht}(m_R) = \dim R \) (see going up theorem [13, Corollary, p. 259] and going down theorem [13, Theorem 6, p. 262]). Since we are assuming \( \dim R = \dim S \), we hence have the desired equality of dimensions of the two rings.

Next, we show that the extension of residue fields is a finite extension.

We have \( k \subseteq \overline{R}/(m_S \cap \overline{R}) \subseteq S/m_S \). Now, since we have \( \overline{R}_{mS \cap R} \) and \( S \) are algebraic local rings of \( L \) of equal dimension, from Lemma 2.0.5, we obtain that \( S/m_S \) and \( \overline{R}/(m_S \cap \overline{R}) \) have the same transcendence degree over \( k \). Hence, \( \overline{R}/(m_S \cap \overline{R}) \subseteq S/m_S \) is an algebraic extension. Furthermore, we have, \( S/m_S \) is a finitely generated extension of \( \overline{R}/(m_S \cap \overline{R}) \) (This is because, \( S \) being an algebraic local ring, \( S \) can be written as \( S = B_P \) for some finitely generated \( k \) algebra \( B \) and a prime ideal \( P \) of \( B \). Hence, \( S/m_S = QF(B/P) \) is a finitely generated field extension of \( k \) and hence a finitely generated field extension of \( \overline{R}/(m_S \cap \overline{R}) \)). Hence, we conclude that the extension of residue fields is a finite extension.

Furthermore, our assumption for (iii), that \( \sqrt{m_RS} = m_S \), ensures \( \sqrt{(m_S \cap \overline{R})S} = m_S \) since \( m_R \subseteq m_S \cap \overline{R} \).

Using the local version of Zariski’s main theorem, we conclude that \( S = \overline{R}_{mS \cap R} \).

Next, we show (ii) \( \implies \) (i).

Since \( L/K \) is separable, we have \( R \to \overline{R} \) is a finite extension (see [13, Corollary 1, p. 265]). Hence (ii) \( \implies \) (i) follows.
Next, we show \((i) \implies (iii)\).

Suppose we have \(S = R'_P\) where \(R \to R'\) is a finite map and \(P\) is a prime ideal of \(R'\). Since \(S\) dominates \(R\), we have that \(P \cap R = m_R\). Now, to find \(\sqrt{m_R R'_P}\), we note, if a prime ideal \(Q \subseteq P\) of \(R'\) contains \(m_R\), then \(Q \cap R = m_R\) too, and hence \(Q = P\) since \(R \to R'\) is integral (see [13, Remark (1), Section 2, Chapter V, pg 259]). Hence, we conclude that \(\sqrt{m_R S} = m_S\).

\(\square\)

Let \(K\) be an algebraic function field over a base field \(k\) and let \(\nu : K^\times \to \Gamma_\nu\) be a \(k\)-valuation of \(K\). Assume \(\nu\) is non trivial (\(\Gamma_\nu \neq \{0\}\)).

**Definition 2.0.7.** The totally ordered Abelian group \(\Gamma_\nu = \nu(K^\times)\) is called the **value group** of \(\nu\).

**Definition 2.0.8.** The **rational rank** of \(\nu\) is defined to be the vector space dimension of the vector space \(\Gamma_\nu \otimes_\mathbb{Z} \mathbb{Q}\) over \(\mathbb{Q}\):

\[
\text{rational rank } \nu = \dim_\mathbb{Q} \Gamma_\nu \otimes_\mathbb{Z} \mathbb{Q}.
\]

**Definition 2.0.9.** ([14, Section 10, Chapter VI, p. 40])

Let \(\Gamma\) be a totally ordered Abelian group. A proper subgroup \(\Delta\) of \(\Gamma\) is said to be an **isolated subgroup** if it satisfies the property that if \(\alpha \in \Delta\) then \(\{\beta \in \Gamma \mid -\alpha \leq \beta \leq \alpha\} \subseteq \Delta\). The isolated subgroups of \(\Gamma\) form a chain and the rank of \(\Gamma\) is defined to be the length of this chain (length of the chain of isolated subgroups of \(\Gamma\)).

The **rank** of a valuation \(\nu\) is defined to be the rank of its value group \(\Gamma_\nu\). The isolated subgroups of \(\Gamma_\nu\) correspond to the prime ideals of the valuation ring \(V_\nu\).
and hence rank of $\nu$ can also be seen as the length of the chain of prime ideals of $V_\nu$ (the dimension of the ring $V_\nu$).

We have the relations

$$\text{rank } \nu \leq \text{rational rank } \nu \leq \text{trdeg}_k K$$  \hspace{1cm} (2.0.3)

(see Corollary and Note on p. 50 of [14, Section 10, Chapter VI]).

Furthermore, we have, a valuation $\nu$ has rank 1 if and only if there is an order-preserving isomorphism from the value group $\Gamma_\nu$ to a subgroup of the ordered additive group of real numbers ([14, Section 10, Chapter VI, p. 45]). Valuations of rank 1 are also referred to as real valuations.

**Definition 2.0.10.** A valuation $\nu$ with rational rank 1 is called a **rational valuation**. From (2.0.3), we note $\nu$ is then necessarily a real valuation. We have $\nu$ is a rational valuation if and only if $\Gamma_\nu$ is order isomorphic to a subgroup of the ordered additive group of rational numbers.

**Definition 2.0.11.** [14, Chapter VI, Section 10, Remark A, p. 48]

Let $\Gamma$ be a totally ordered Abelian group of finite rank $n$. Let $\Gamma_0 = (0), \Gamma_1, ..., \Gamma_{n-1}$ be the isolated subgroups of $\Gamma$: $\Gamma_0 = (0) < \Gamma_1 < \cdots < \Gamma_{n-1} < \Gamma$. We note, then, for $i = 0, ..., n-1$, $\Gamma_{i+1}/\Gamma_i$ have rank 1. If each of these quotient groups is isomorphic to the additive group of integers, we say the ordered group $\Gamma$ is discrete. A valuation $\nu$ is said to be a **discrete valuation** if its value group $\Gamma_\nu$ is a discrete group. When $\nu$ has rank 1, $0 < \Gamma_\nu$ is the chain of isolated subgroups of $\Gamma_\nu$, and we note, $\nu$ is discrete if the value group $\Gamma_\nu$ is isomorphic to the additive group of integers.
We have $\nu$ is rank 1 discrete if and only if the valuation ring $V_\nu$ is Noetherian ([14, Theorem 16, Section 10, Chapter VI, p. 41]).

Since $K$ is the quotient field of $V_\nu$, we have, $\text{trdeg}_k V_\nu/m_\nu \leq \text{trdeg}_k K$. We have equality only for the trivial valuation ($V_\nu = K$). When $\text{trdeg}_k V_\nu/m_\nu = \text{trdeg}_k K - 1$, we say the valuation $\nu$ is a divisorial valuation. We have $\nu$ is divisorial if and only if the valuation ring $V_\nu$ is an algebraic regular local ring of $K$ (see Theorem 31 in p. 89 and Corollary in p. 92 of [14, Section 14, Chapter VI]). Therefore, $\nu$ is, in particular, a rank 1 discrete valuation.

When $\text{trdeg}_k K = 2$ and $\nu$ is not divisorial, we have $\text{trdeg}_k V_\nu/m_\nu = 0$. Therefore, in particular, when $K$ is a two dimensional algebraic function field over an algebraically closed field $k$ and $\nu$ is a rational non-discrete valuation, we have, the residue field $V_\nu/m_\nu = k$.

Now, let $L$ be a finite extension of $K$. Let $\omega$ be an extension of $\nu$ to $L$, with valuation ring $V_\omega$ and value group $\Gamma_\omega$ (see [14, Corollary 2, Section 7, Chapter VI, p. 27]).

We have then, $\nu$ and $\omega$ have the same rational rank and rank (see [14, Lemma 1, Section 11, Chapter VI, p. 51] and [14, Lemma 2, Section 11, Chapter VI, p. 51]).

Furthermore, we have, $\nu$ is discrete if and only if $\omega$ is discrete (see [14, Corollary, p. 52, Section 11, Chapter VI]).

We have $|\Gamma_\omega/\Gamma_\nu| \leq [L : K]$ is finite ([14, Corollary, p. 52, Section 11, Chapter VI]) and is defined to be the reduced ramification index $e(\omega/\nu)$ of $\omega$ with respect
to $\nu$:

$$e(\omega/\nu) = |\Gamma_\omega/\Gamma_\nu|.$$  

Additionally, the degree of extension of the residue fields, $[V_\omega/m_\omega : V_\nu/m_\nu] \leq [L : K]$ is finite ([14, Corollary 2, p. 26, Section 6, Chapter VI]) and is called the relative degree $f(\omega/\nu)$ of $\omega$ with respect to $\nu$:

$$f(\omega/\nu) = [V_\omega/m_\omega : V_\nu/m_\nu].$$

Furthermore, the total number of extensions of $\nu$ to $L$ is bounded by $[L : K]$ ([14, Corollary 4, Section 7, Chapter VI, p. 29]) and is denoted by the letter $g$:

$$g = \text{Number of extensions of } \nu \text{ to } L.$$

We observe, in the case when $\nu$ and $\omega$ are rational rank 1 and non discrete and $k$ is algebraically closed, the inclusions $k \subseteq V_\nu/m_\nu \subseteq V_\omega/m_\omega$ are all equalities and we get

$$f = [V_\omega/m_\omega : V_\nu/m_\nu] = 1. \quad (2.0.4)$$

The defect $\delta(\omega/\nu)$ is defined in Section 7.1 of [8]. It is always equal to 1 in equicharacteristic zero. When $L/K$ is a Galois extension, we have the equality

$$e(\omega/\nu) \cdot f(\omega/\nu) \cdot g \cdot \delta(\omega/\nu) = [L : K] \quad (2.0.5)$$

([14, Section 12, Chapter VI, Corollary to Theorem 25, p. 78]).

For $K$ an algebraic function field of transcendence degree 2 over a base field $k$ and $\nu$ a rational non-discrete valuation, we have the following lemma regarding change of variables.

**Lemma 2.0.12.** Let $K$ be an algebraic function field of transcendence degree 2 over a base field $k$. Let $A \subseteq K$ be an algebraic regular local ring over $k$ with regular
system of parameters $x, y$. Let $\nu$ be a $k$-valuation of $K$ such that $\nu$ has rational rank 1 and is non discrete with residue field $k$ and $\nu$ dominates $A$. Suppose $\nu(y) \in \nu(x)Z$. Then there exits $p(x) \in k[x]$ such that for $\overline{y} = y - p(x)$, $\nu(\overline{y}) \notin \nu(x)Z$.

Proof. Suppose $\nu(y) = n_1\nu(x)$ for some positive integer $n_1$.

Therefore, there exists $\lambda_1 \neq 0 \in k$, such that $y_1 := \frac{y}{x^{n_1}} - \lambda_1 = (y - \lambda_1 x^{n_1})/x^{n_1}$ has value $> 0$. We obtain $\nu(y - \lambda_1 x^{n_1}) > \nu(x^{n_1})$. Let $p_1(x) = \lambda_1 x^{n_1}$. Now, if $\nu(y_1) \notin \nu(x)Z$, we have the desired result as then $\nu(y - p_1(x)) = \nu(y - \lambda_1 x^{n_1}) \notin \nu(x)Z$.

If not, we repeat the above process with $y_1$. Inductively, suppose we have constructed $y_i = (y - p_i(x))/x^{n_1 + \cdots + n_i}$ such that $\nu(y - p_i(x)) > (n_1 + \cdots + n_i)\nu(x)$.

If $\nu(y_i) \notin \nu(x)Z$, $p(x) = p_i(x)$ gives the desired result. If not, suppose $\nu(y_i) = n_{i+1}\nu(x)$ for some positive integer $n_{i+1}$. Therefore, we obtain, there exists $\lambda_{i+1} \neq 0 \in k$, such that $y_{i+1} := \frac{y}{x^{n_{i+1}}} - \lambda_{i+1} = (y - p_i(x) - \lambda_{i+1} x^{n_{i+1}})/x^{n_1 + \cdots + n_i + n_{i+1}}$ has value $> 0$, giving us $\nu(y - p_i(x) - \lambda_{i+1} x^{n_{i+1}}) > \nu(x^{n_1 + \cdots + n_i + n_{i+1}})$. Let $p_{i+1}(x) = p_i(x) + \lambda_{i+1} x^{n_1 + \cdots + n_i + n_{i+1}}$. If $\nu(y_{i+1}) \notin \nu(x)Z$, we have the desired result as then $\nu(y - p_{i+1}(x)) \notin \nu(x)Z$.

We claim that this process must end after some finite number of steps.

Suppose not. Note that $n_1, \ldots, n_i$ are positive integers and $\nu$ is rank 1 with $\nu(x) > 0$ (since $\nu$ dominates $A$). Therefore, we obtain a series $\Sigma_i a_i x^i$ such that $\nu(y - \Sigma_i a_i x^i) \to \infty$ as $n \to \infty$.

We show that this is a contradiction to $\nu$ being non discrete. To see this, given $f(x, y) \in k[x, y]$, we choose $\Sigma_i a_i x^i$ with $n$ large enough such that $y - \Sigma_i a_i x^i$ has value strictly bigger than that of $f(x, y)$. Then replacing $y$ with $(y - \Sigma_i a_i x^i) + (\Sigma_i a_i x^i)$ in $f(x, y)$, we see that the value of $f(x, y)$ is same as the value of a polynomial in $x$, and therefore, giving us that the value of $f(x, y) \in Z$. And going to the quotient field $k(x, y)$, we get that $\nu$ is discrete rank 1 on $k(x, y)$.

Furthermore, we have $K$ is a finite extension of $k(x, y)$ since $K$ is an algebraic function field of transcendence degree 2 over $k$. Hence, $\nu$ being discrete rank 1 on $k(x, y)$ implies $\nu$ is discrete rank 1 on $K$ as well, which is a contradiction to our
assumption that $\nu$ is a rank 1 non-discrete valuation of $K$. Therefore, the above process must end after some finite number of steps and we hence conclude that there exists $p(x) \in k[x]$ such that for $\overline{y} = y - p(x)$, $\nu(\overline{y}) \notin \nu(x)\mathbb{Z}$. □
Chapter 3

Extensions of two dimensional regular local rings

In this chapter, we state the results and definitions from Section 3 of [6] for quick reference.

Let $K'$ be an algebraic function field of transcendence degree 2 over an algebraically closed base field $k$ of char $p > 0$ and let $\mu$ be a non-discrete rational rank 1 $k$-valuation of $K'$ with valuation ring $(V_\mu, m_\mu)$. Let $A$ be a dimension 2 algebraic regular local ring of $K'$ such that $\mu$ dominates $A$.

A quadratic transform of $A$ is an extension $A \to A_1$ where $A_1$ is a local ring of the blowup of the maximal ideal of $A$ such that $A_1$ dominates $A$ and $A_1$ has dimension 2. A quadratic transform $A \to A_1$ is said to be along the valuation $\mu$ if $\mu$ dominates $A_1$.

Let $x_1, x_2$ be a regular system of parameters of $A$ and suppose that we have arranged $x_1, x_2$ such that $\mu(x_1) \leq \mu(x_2)$. Then, the quadratic transform of $A$ along $\mu$, is given by, $A_1 = A[x_2/x_1]_{m_\mu \cap A[x_2/x_1]}$. The quadratic transform $A_1$ is an algebraic regular local ring of $K'$ (see [4, Lemma 3.20] and [4, Definition 3.21]).
$A_1$ has dimension 2 since $\mu$ is not divisorial (see Lemma 2.0.5).

Now let $A_0 = A$ and inductively define $A_n$ by letting $A_n$ be the quadratic transform of $A_{n-1}$ along $\mu$. $A_n$ is called “the n-th quadratic transform of $R$ along $\mu$”. If $A'$ is the n-th quadratic transform of $A$ along $\mu$ for some $n$, we say that “$A'$ is a product of quadratic transforms of $A$ along $\mu$”.

Let

\[ A = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \]

be the sequence of all quadratic transforms of $A$ along $\mu$. Since trdeg$_{k'K'} K' = 2$, we have that $\bigcup A_i = V_\mu$, where $V_\mu$ is the valuation ring of $\mu$ (see [1, Lemma 12]).

Fix a height one prime ideal $P$ in $A$ such that $A/P$ is a regular local ring.

We use the following definitions from [8] and [6]:

**Definition 3.0.1.** ([8, Definition 7.5] and [6, Section 3]) $A_i$ is said to be free if the radical of $PA_i$ is a height one prime ideal (so that $A_i/\sqrt{PA_i}$ is a regular local ring). In particular, $A_0 = A$ is free.

**Definition 3.0.2.** ([8, Definition 7.11] and [6, Section 3]) Let $A = A_0 = A_{r_1}$. For all $i \geq 1$, let $(r'_{i+1}, \tau_i)$ be the pair of integers with the following properties:

1. $\tau_i$ is the largest integer $r \geq r'_i$ such that $A_r$ is free for all $r'$ with $r'_i \leq r' \leq r$.

2. $r'_{i+1}$ is the smallest integer $r > \tau_i$ such that $A_r$ is free.

Now suppose that $A_j$ is free. Therefore, for notations as in Definition 3.0.2 above, there exists $i$ such that $A_{r'_i} \subseteq A_j \subseteq A_r_i$. 

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Definition 3.0.3. Suppose that $A_j$ is free. Let $u, v$ be regular parameters in $A_j$ such that $u = 0$ is a local equation of $Z(\mathcal{P}A_j)$; that is, $(u) = \sqrt{\mathcal{P}A_j}$. We will say that $u, v$ are allowable parameters in $A_j$.

Let $u, v$ be allowable parameters in $A_j$, where $A_j$ is free with $A_{r_i} \subseteq A_j \subseteq A_{r_i}$ for some $i$. Since $\mu$ is rational, there exist positive integers $m, q$ such that $\frac{\mu(u)}{\mu(v)} = \frac{m}{q}$ with $\gcd(m, q) = 1$. Let $a', b'$ be such that $mb' - qa' = 1$. We have $\mu\left(\frac{v^m}{u^q}\right) = 0$ and let $0 \neq \alpha \in k$ be such that $\nu\left(\frac{v^m}{u^q} - \alpha\right) > 0$. Let $u_1, v_1$ be defined by

$$u = u_1^m(v_1 + \alpha)^{a'}, v = u_1^q(v_1 + \alpha)^{b'}$$

and let $A_j \to A_k$ be the sequence of quadratic transforms such that $u_1, v_1$ are regular parameters in $A_k$. We observe that then $A_k$ is free, and $u_1 = 0$ is a local equation of $Z(\mathcal{P}A_k)$, so that $u_1, v_1$ are allowable parameters in $A_k$. If $\mu(v) \notin \mu(u)Z$, then $m > 1$ and $A_k = A_{r_i+1}$. If $\mu(v) \in \mu(u)Z$, then $m = 1$ and $A_{r_i} \subseteq A_j \subseteq A_k \subseteq A_{r_i}$.

We note that since $\mu$ is non-discrete, given an allowable regular system of parameters $u, v$, by possibly replacing $v$ with the difference of $v$ and a suitable polynomial $p(u) \in k[u]$ (which necessarily has no constant term), $\overline{v} = v - p(u)$, we may obtain an allowable system of parameters $u, \overline{v}$ such that $\mu(\overline{v}) \notin \mu(u)Z$ (see Lemma 2.0.12).

We will call the sequence $A = A_0 = A_{r_1} \to A_{r_2} \to A_{r_3} \to \cdots$ the sequence of standard sequences of quadratic transforms along $\mu$. Observe that this sequence depends on the choice of $P$ in $A$.

Now let $L/K$ be a separable extension of algebraic function fields of transcendence degree 2 over an algebraically closed base field $k$ of characteristic $p > 0$, $\nu$
a $k$-valuation of $K$ and $\omega$ an extension of $\nu$ to $L$ such that $\nu$ (and thus $\omega$) have rational rank 1 but are not discrete. Let $R$ and $S$ be algebraic regular local rings, respectively of $K$ and $L$, such that $\dim R = \dim S = 2$, $\omega$ dominates $S$ and $S$ dominates $R$ (so $\nu$ dominates $R$).

**Definition 3.0.4.** Let $u, v$ and $x, y$ be regular parameters in $R$ and $S$ respectively. Then, the **Jacobian ideal** $J(S/R)$ is defined as

$$J(S/R) := \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right).$$

The Jacobian ideal $J(S/R)$ is independent of the choice of regular parameters and we define the critical locus of $\text{Spec}(S) \to \text{Spec}(R)$ to be $Z(J(S/R))$.

Let

$$R \to R_1 \to \cdots \to R_i \to \cdots$$

be the sequence of quadratic transforms of $R$ along $\nu$ and let

$$S \to S_1 \to \cdots \to S_j \to \cdots$$

be the sequence of quadratic transforms of $S$ along $\omega$. Let $P$ be a height one prime ideal in $R$ such that $R/P$ is a regular local ring and let $Q$ be a height one prime ideal in $S$ such that $S/Q$ is a regular local ring. We do not insist in this definition that the good condition that $Q \cap R = P$ holds. Let $E_i$ be the divisor $Z(PR_i)$ and $F_j$ be the divisor $Z(QS_j)$. Let $r$ and $s$ be such that $S_s$ dominates $R_r$.

**Definition 3.0.5.** ([8, Definition 7.5]) Suppose that $S_s$ dominates $R_r$. The map $R_r \to S_s$ is said to be **prepared** if both $R_r$ and $S_s$ are free, the critical locus of $\text{Spec}(S_s) \to \text{Spec}(R_r)$ is contained in $F_s$ and we have an expression $u = \gamma x^n$, where $u$ is part of a regular system of parameters of $R_r$ such that $u = 0$ is a local
equation of $E_r$, $x$ is part of a regular system of parameters of $S_s$ such that $x = 0$ is a local equation of $F_s$ and $\gamma$ is a unit in $S_s$.

**Definition 3.0.6.** Suppose that $R_r \to S_s$ is prepared. Let $u,v$ and $x,y$ be allowable parameters for $R_r$ and $S_s$ respectively; that is, $u = 0$ is a local equation of $E_r$, $x = 0$ is a local equation of $F_s$. Since $R_r \to S_s$ is prepared we have that $u = \gamma x^a$ where $\gamma$ is a unit in $S_s$. We say $(u,v)$ and $(x,y)$ are **admissible parameters** in $R_r$ and $S_s$ respectively. We have the expression

$$u = \gamma x^a, v = x^b f$$

where $\gamma \in S_s$ is a unit, $a > 0$ and $b \geq 0$ are integers, $f \in S_s$ and $x$ does not divide $f$ in $S_s$.

**Definition 3.0.7.** Suppose that $R_r \to S_s$ is prepared. Let $u,v$ and $x,y$ be admissible parameters in $R_r$ and $S_s$ respectively. For notations as in Definition 3.0.6, we say $R_r \to S_s$ is well prepared if $f$ is not a unit in $S_s$.

The following remark follows from [1, Theorem 2] (also see proof of [6, Proposition 3.7] (Proposition 3.0.13 below) ).

**Remark 3.0.8.** Given $r_0 > 0$ and $s_0 > 0$ there exist $r \geq r_0$ and $s \geq s_0$ such that $S_s$ dominates $R_r$ and $R_r \to S_s$ is well prepared. This result is true for any initial choice of $P$ in $R$ and $Q$ in $S$.

Suppose that $R_r \to S_s$ is well prepared. Since $x \nmid f$ in $S_s$, we may write $f$ as $f = \tau y^d + x \Sigma$ where $\tau \in S_s$ is a unit, $\Sigma \in S_s$, and $d$ is the order of the residue of $f$ in the one dimensional regular local ring $S_s/(x)$.
Definition 3.0.9. [8, Definition 7.9] Suppose $S_s$ dominates $R_r$ and $R_r \to S_s$ is well prepared with admissible regular system of parameters $(u, v)$ and $(x, y)$ of $R_r$ and $S_s$ respectively such that we have

$$u = \gamma x^a$$

$$v = x^b(\tau y^d + x^\Sigma)$$

where $a > 0$ and $b \geq 0$ are integers, $\gamma, \tau \in S$ are units and $\Sigma \in S$. Then the product $ad$ depends only on the extension $R_r \to S_s$ and is called the **complexity of the extension** $R_r \to S_s$ (see [8, Proposition 7.2] below).

Proposition 3.0.10. ([8, Proposition 7.2]) Suppose that $S_s$ dominates $R_r$, $R_r \to S_s$ is well prepared and $R_r$ and $S_s$ have admissible parameters $(u, v)$ and $(x, y)$ satisfying the equation (3.0.1). Let $S^*$ be the unique two dimensional algebraic normal local ring over $k$ lying above $R$ with $QF(S^*) = L$ and $S^* \subset S$. Then there exists a commutative diagram

$$
\begin{array}{ccc}
S^* & \to & S_s \\
\uparrow & & \uparrow \\
R_r & \to & R^*
\end{array}
$$

where all arrows are dominating maps such that $R^*$ is a two dimensional algebraic normal local ring of $K$ such that $S_s$ lies over $R^*$. We have

$$[QF(\tilde{S}_s) : QF(\tilde{R}^*)] = ad$$

where $d$ is the order of the residue of $f$ in the one dimensional regular local ring $S_s/(x)$.

Remark 3.0.11. ([6, Remark 3.4]) Suppose that assumptions are as in Propo-
position 3.0.10 with the additional assumption that $K \rightarrow L$ is Galois. Then the complexity $ad$ of $R_r \rightarrow S_s$ divides the degree $[L : K]$.

We further have that the complexity stabilizes eventually for well prepared extensions along a rational non-discrete valuation (also see Remark 3.0.8).

**Proposition 3.0.12.** ([6, Proposition 3.6], [7, Proposition 3.4], [8, Section 7.9])

Let $K \rightarrow L$ be a finite separable extension of two dimensional algebraic function fields over an algebraically closed field $k$ of characteristic $p > 0$, $\omega$ is a rational rank 1 nondiscrete valuation of $L$ (with residue field $k$) and $\nu$ is the restriction of $\omega$ to $K$.

Suppose that $A$ is an algebraic local ring of $K$ which is dominated by $\nu$. Then there exists an algebraic regular local ring $R'$ of $K$ which is dominated by $\nu$ and dominates $A$ such that if $R$ is a regular algebraic local ring of $K$ which dominates $R'$ and $S$ is a regular algebraic local ring of $L$ which is dominated by $\omega$ and dominates $R$ such that there are regular parameters $x, y$ in $S$ and $u, v$ in $R$ such that $u = \gamma x^a$ and $v = x^b f$ with $\gamma$ a unit in $S$ and $f \in S$ such that $f$ is not a unit in $S$, $x$ does not divide $f$ and the critical locus of $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is contained in $Z(xS)$. Then letting $d = \dim_k S/(f, x)$, we have that the complexity $ad = e(\omega/\nu)\delta(\omega/\nu)$.

Let $L/K$ be a separable extension of algebraic function fields of transcendence degree 2 over an algebraically closed base field $k$ of characteristic $p > 0$, $\nu$ a $k$-valuation of $K$ and $\omega$ an extension of $\nu$ to $L$ such that $\nu$ (and thus $\omega$) have rational rank 1 but are not discrete. Let $R$ and $S$ be algebraic regular local rings of $K$ and $L$ respectively such that $\dim R = \dim S = 2$, $\omega$ dominates $S$ and $S$ dominates $R$ (so $\nu$ dominates $R$).

Let

$$R \rightarrow R_1 \rightarrow R_2 \rightarrow \cdots \quad \text{and} \quad S \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \quad (3.0.2)$$
be the sequences of quadratic transforms along \( \nu \) and \( \omega \) respectively. Let \( P \) be a height one prime ideal in \( R \) such that \( R/P \) is a regular local ring and let \( Q \) be a height one prime ideal in \( S \) such that \( S/Q \) is a regular local ring. From Proposition \ref{prop:3.0.12}, we observe that there exists a positive integer \( r_0 \) such that whenever \( r \geq r_0 \) and \( R_s \rightarrow S_s \) is well prepared, we have that the complexity \( ad \) of the extension is a constant which depends only on the extension of valuations. We will call this the stable complexity of the sequences \( (3.0.2) \).

The following proposition is established in \cite{8} and also proved in \cite[Proposition 3.7]{6}.

**Proposition 3.0.13.** \cite[Proposition 3.7]{6} Suppose that \( A \) is an algebraic local ring of \( K \) and \( B \) is an algebraic local ring of \( L \) which is dominated by a rational rank \( 1 \) nondiscrete valuation \( \omega \) of \( L \) such that \( B \) dominates \( A \). Then there exists a commutative diagram of homomorphisms

\[
\begin{array}{ccc}
R & \rightarrow & S \\
\uparrow & & \uparrow \\
A & \rightarrow & B \\
\end{array}
\]

such that \( R \) is a regular algebraic local ring of \( K \) with regular parameters \( u, v \), \( S \) is a regular algebraic local ring of \( L \) with regular parameters \( x, y \) such that \( S \) is dominated by \( \omega \), \( S \) dominates \( R \), \( J(S/R) = (x^\tau) \) for some non negative integer \( \tau \) and there is an expression

\[
u = \gamma x^a, v = x^b(y^d \tau + x\Omega)
\]  \hspace{1cm} (3.0.3)

where \( \tau, \gamma \) are units in \( S \), \( \Omega \in S \) and \( d > 0, 0 \leq b < a \). Thus the quadratic transform of \( R \) along \( \nu \) is not dominated by \( S \).
Now let \( L/K \) be an Artin-Schreier extension. Therefore, we have \( L/K \) is a Galois extension and \([L : K] = p\). In the following remark, we note what are the possibilities for the stable complexity of the sequences (3.0.2) and the cases in which they occur.

**Remark 3.0.14.** [6, Remark 3.8] Let \( K \rightarrow L \) be an Artin-Schreier extension of two dimensional algebraic function fields over an algebraically closed field \( k \) of characteristic \( p > 0 \). Let \( \omega \) be a rational rank 1 nondiscrete valuation of \( L \) (with residue field \( k \)) and \( \nu \) be the restriction of \( \omega \) to \( K \). Since \( L \) is Galois over \( K \), we have that \( g(\omega/\nu)e(\omega/\nu)\delta(\omega/\nu) = p \) where \( g = g(\omega/\nu) \) is the number of extensions of \( \nu \) to \( L \). So we either have that \( g = 1 \) or \( g = p \). If \( g = 1 \), then \( \omega \) is the unique extension of \( \nu \) to \( L \) and either \( e(\omega/\nu) = p \) and \( \delta(\omega/\nu) = 1 \) or \( e(\omega/\nu) = 1 \) and \( \delta(\omega/\nu) = p \). If \( g = 1 \), we have by Proposition 3.0.12 that the stable complexity of the sequences (3.0.2) is \( ad = p \). If \( g = p \), then \( e(\omega/\nu) = 1 \) and \( \delta(\omega/\nu) = 1 \) and the stable complexity of the sequences (3.0.2) is \( ad = 1 \).

Additionally, for \( L/K \) an Artin-Schreier extension, taking further quadratic transforms if necessary, we can obtain \( b = 0 \) in Proposition 3.0.13. The following proposition is established in [11]. A proof can also be found in [6].

**Proposition 3.0.15.** ([11, Lemma 7.3]) ([6, Proposition 3.9]) Suppose that \( K \rightarrow L \) is an Artin-Schreier extension of two dimensional algebraic function fields over an algebraically closed field \( k \) of characteristic \( p > 0 \), \( \omega \) is a rational rank 1 nondiscrete valuation of \( L \) with restriction \( \nu = \omega|K \). Further, suppose that \( A \) is an algebraic local ring of \( K \) and \( B \) is an algebraic local ring of \( L \) which is dominated by \( \omega \) such that \( B \) dominates \( A \). Then there exists a commutative diagram of homomorphisms

\[
\begin{array}{ccc}
R & \rightarrow & S \\
\uparrow & & \uparrow \\
A & \rightarrow & B
\end{array}
\]
such that $R$ is a regular algebraic local ring of $K$ with regular parameters $u, v$, $S$ is a regular algebraic local ring of $L$ with regular parameters $x, y$ such that $S$ is dominated by $\omega$, $S$ dominates $R$, $R \to S$ is well prepared with admissible parameters $u, v$ in $R$ and $x, y$ in $S$ (with respect to the prime ideals $P = uR$ in $R$ and $Q = xS$ in $S$). We further have that $S$ is a localization of a finite extension of $R$, $J(S/R) = (x^\bar{c})$ for some non-negative integer $\bar{c}$ and one of the following three cases holds:

0) $u = x, v = y$.

1) $u = x, v = yp\gamma + x\Sigma$ where $\gamma$ is a unit in $S$ and $\Sigma \in S$.

2) $u = \gamma x^p, v = y$ where $\gamma$ is a unit in $S$.

Since, in particular, we have $\sqrt{m_R S} = m_S$, we have that $R \to S$ is the localization of a finite map and hence simultaneous local resolution holds along a rational non-discrete valuation when $L/K$ is an Artin-Schreier extension of two dimensional algebraic function fields over an algebraically closed base field of char $p > 0$. 

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Chapter 4

Some calculations in two dimensional Artin-Schreier extensions

In this chapter, we state the results and definitions from Section 4 of [6] for quick reference.

Let $L/K$ be an Artin-Schreier extension of two dimensional algebraic function fields over an algebraically closed base field $k$ of characteristic $p > 0$. Let $\nu$ be a $k$-valuation of $K$ and $\omega$ an extension of $\nu$ to $L$ such that $\nu$ (and thus $\omega$) have rational rank 1 but are not discrete. Let $R$ and $S$ be algebraic regular local rings with respective quotient fields $K$ and $L$ such that $\dim R = \dim S = 2$, $\omega$ dominates $S$ and $S$ dominates $R$ (so $\nu$ dominates $R$).

Definition 4.0.1. Let $u,v$ be regular parameters in $R$ and $x,y$ be regular parameters in $S$.

0) We say $R \to S$ is of type 0 with respect to the parameters $u,v$ and $x,y$ if we have the relations

$$u = \gamma x, v = \tau y + x\Sigma$$
where $\gamma, \tau \in S$ are units and $\Sigma \in S$.

1) We say $R \to S$ is of type 1 with respect to the parameters $u, v$ and $x, y$ if we have the relations

$$u = \gamma x, \quad v = \tau y^p + x\Sigma$$

where $\gamma, \tau \in S$ are units and $\Sigma \in S$.

2) We say $R \to S$ is of type 2 with respect to the parameters $u, v$ and $x, y$ if we have the relations

$$u = \gamma x^p, \quad v = \tau y + x\Sigma$$

where $\gamma, \tau \in S$ are units and $\Sigma \in S$.

Let $R \to S$ along with regular parameters $u, v$ and $x, y$ be as above such that one of the three types above holds. Now let $\overline{u}, \overline{v}$ and $\overline{x}, \overline{y}$ be regular parameters of $R$ and $S$ respectively such that $\overline{u}$ is a unit in $R$ times $u$ and $\overline{x}$ is a unit in $S$ times $x$. We have $\overline{u} = \tau_1 v + u\Sigma_1$ where $\tau_1$ is a unit in $R$, $\Sigma_1 \in R$ and $\overline{y} = \tau_2 y + x\Sigma_2$ where $\tau_2$ is a unit in $S$, $\Sigma_2 \in S$. Plugging into the expressions above, we observe that $R \to S$ is of the same type with respect to the new parameters $\overline{u}, \overline{v}$ and $\overline{x}, \overline{y}$ as well.

If we replace $R \to S$ with a well prepared map $R_i \to S_j$ in the sequences (3.0.2), we will insist that the above parameters be admissible. And our discussion above shows that the three types are preserved by allowable changes of variables. In particular, we note that up to an allowable change of variables the types above correspond to the respective cases 0), 1) and 2) of Proposition 3.0.15.

Now suppose that $R \to S$ is of type 1 with respect to regular parameters $x, y$ in $S$ and $u, v$ in $R$. Theorem [6, Theorem 4.1] and Remark [6, Remark 4.2] (Theorem
4.0.2 and Remark 4.0.3 below) show how we can take quadratic transforms of $R$ and $S$ to obtain a commutative diagram

$$
\begin{array}{c}
R_1 \rightarrow S_1 \\
\uparrow \quad \uparrow \\
R \rightarrow S
\end{array}
$$

where $R \subsetneq R_1$, $S \subsetneq S_1$, $R \rightarrow R_1$ and $S \rightarrow S_1$ are products of quadratic transforms, $\omega$ dominates $S_1$, $S_1$ dominates $R_1$ and $R_1 \rightarrow S_1$ is of type 0, 1 or 2. Theorem [6, Theorem 4.1] and Remark [6, Remark 4.2] also provide conditions that determine whether $R_1 \rightarrow S_1$ has type 0, 1 or 2.

**Theorem 4.0.2.** [6, Theorem 4.1] Suppose that $R \rightarrow S$ is of type 1 with respect to regular parameters $x, y$ in $S$ and $u, v$ in $R$ and that $J(S/R) = (x^p)$. Let $\overline{x} = u$, $\overline{y} = y - g(\overline{x})$ where $g(\overline{x}) \in k[\overline{x}]$ is a polynomial with zero constant term, so that $\overline{x}, \overline{y}$ are regular parameters in $S$. Computing the Jacobian determinate $J(S/R)$, we see that

$$u = \overline{x}, v = \overline{y}^p \gamma + \overline{x}^p \overline{y}^q + f(\overline{x})$$

where $\gamma, \tau$ are unit series in $\overline{S}$ and $f(\overline{x}) = \sum e_i \overline{x}^i \in k[\overline{x}]$. Make the change of variables $\overline{v} = v - \sum e_i u^i$ where the sum is over $i$ such that $i \leq \frac{pq}{m}$ so that $u, \overline{v}$ are regular parameters in $R$.

Suppose that $m, q$ are positive integers with $m > 1$ and gcd$(m, q) = 1$. Let $\alpha$ be a nonzero element of $k$. Consider the sequence of quadratic transforms $S \rightarrow S_1$ so that $S_1$ has regular parameters $x_1, y_1$ defined by

$$\overline{x} = x_1^m (y_1 + \alpha)^{a'}, \overline{y} = x_1^q (y_1 + \alpha)^{b'}$$

where $a', b' \in \mathbb{N}$ are such that $mb' - qa' = 1$.

We have that $R \rightarrow S$ is of type 1 with respect to the regular parameters $\overline{x}, \overline{y}$ and $u, v$. Let $\sigma = \text{gcd}(m, pq)$ which is 1 or $p$. 

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There exists a unique sequence of quadratic transforms $R \rightarrow R_1$ such that $R_1$ has regular parameters $u_1,v_1$ defined by

$$u = u_1^{m_1}(v_1 + \beta)^{c'}_1, \quad v = u_1^{q_1}(v_1 + \beta)^{d'}_1$$

with $0 \neq \beta \in k$ giving a commutative diagram of homomorphisms

$$
\begin{array}{ccc}
R_1 & \rightarrow & S_1 \\
\uparrow & & \uparrow \\
R & \rightarrow & S
\end{array}
$$

such that $S_1$ is a localization of a finite extension of $R_1$. We have that $J(S_1/R_1) = (x_1^{c_1})$ for some positive integer $c_1$. Further:

0) If $\frac{a}{m} \geq \frac{c}{p-1}$ then $R_1 \rightarrow S_1$ is of type 0.

1) If $\frac{a}{m} < \frac{c}{p-1}$ and $\sigma = 1$ then $R_1 \rightarrow S_1$ is of type 1 and

$$\left( \frac{c_1}{p-1} \right) = \left( \frac{c}{p-1} \right) m - q.$$

2) If $\frac{a}{m} < \frac{c}{p-1}$ and $\sigma = p$ then $R_1 \rightarrow S_1$ is of type 2 and

$$\left( \frac{c_1}{p-1} \right) = \left( \frac{c}{p-1} \right) m - q + 1.$$

In cases 1) and 2), $m = \sigma m, pq = \sigma q$ and $m c' - q d' = 1$.

**Remark 4.0.3.** [6, Remark 4.2] Suppose that $\omega$ is a rational rank 1 nondiscrete valuation of $L$ dominating $S$ and $R \rightarrow S$ is of type 1. Let $\nu$ be the restriction of $\omega$ to $K$. Let $\bar{\omega} = u$ and $\overline{Y}$ be the difference of $y$ and a nonzero polynomial in $\bar{\omega}$ so that $\omega(\overline{Y}) \not\in \omega(\bar{\omega})\mathbb{Z}$. Let $v$ be the change of variables in Theorem 4.0.2.

Define $m$ and $q$ to be the unique relatively prime positive integers such that $m \omega(\overline{Y}) = q \omega(\bar{\omega})$. We have that $m > 0$. There exist $0 \neq \alpha \in k$ and $a', b' \in \mathbb{N}$ such
that \( mb' - qa' = 1 \) and if \( S \rightarrow S_1 \) is the sequence of quadratic transforms defined by

\[
\overline{x} = x_1^m(y_1 + \alpha)^n', \quad \overline{y} = x_1^q(y_1 + \alpha)^n'
\]

then \( \omega \) dominates \( S_1 \).

Let \( \nu \) be the restriction of \( \omega \) to \( K \). The formulas of Cases 0), 1) and 2) of Theorem 4.0.2 can then be stated in terms of the valuation \( \omega \). They are:

0) If \( \frac{a}{m} \geq \frac{\sigma}{p-1} \) then \( R_1 \rightarrow S_1 \) is of type 0.

1) If \( \frac{a}{m} < \frac{\sigma}{p-1} \) and \( \sigma = 1 \) then \( R_1 \rightarrow S_1 \) is of type 1 and

\[
\left( \frac{c_1}{p-1} \right) \omega(x_1) = \left( \frac{\sigma}{p-1} \right) \omega(x) - \omega(\overline{y}).
\]

2) If \( \frac{a}{m} < \frac{\sigma}{p-1} \) and \( \sigma = p \) then \( R_1 \rightarrow S_1 \) is of type 2 and

\[
\left( \frac{c_1}{p-1} \right) \omega(x_1) = \left( \frac{\sigma}{p-1} \right) \omega(x) - \omega(\overline{y}) + \omega(x_1).
\]

In the conclusions of the theorem, suppose that \( R_1 \rightarrow S_1 \) is of type 1. Then we necessarily have that \( \nu(\overline{v}) \notin \nu(u)\mathbb{Z} \) since \( \sigma = 1 \) and thus \( \overline{m} = m > 1 \).

Next, suppose that \( R \rightarrow S \) is of type 2 with respect to regular parameters \( x, y \) in \( S \) and \( u, v \) in \( R \). Theorem [6, Theorem 4.3] and Remark [6, Remark 4.4] (Theorem 4.0.4 and Remark 4.0.5 below) show how we can take quadratic transforms of \( R \) and \( S \) to obtain a commutative diagram

\[
\begin{array}{ccc}
R_1 & \rightarrow & S_1 \\
\uparrow & & \uparrow \\
R & \rightarrow & S
\end{array}
\]

where \( R \subsetneq R_1, S \subsetneq S_1, R \rightarrow R_1 \) and \( S \rightarrow S_1 \) are products of quadratic transforms, \( \omega \) dominates \( S_1 \), \( S_1 \) dominates \( R_1 \) and \( R_1 \rightarrow S_1 \) is of type 1 or 2. Theorem [6,
Theorem 4.3] and Remark [6, Remark 4.4] also provide conditions that determine whether $R_1 \to S_1$ has type 1 or 2.

**Theorem 4.0.4.** [6, Theorem 4.3] Suppose that $R \to S$ is of type 2 with respect to regular parameters $x, y$ in $S$ and $u, v$ in $R$ and that $J(S/R) = (x^c)$. Let $g(u) \in k[u]$ be a polynomial with no constant term. Make the change of variables, letting $\overline{v} = v - g(u)$ and $\overline{y} = \overline{v}$, so that $x, \overline{y}$ are regular parameters in $S$ and $u, \overline{v}$ are regular parameters in $R$.

Suppose that $m, q$ are positive integers with $\gcd(m, q) = 1$. Let $\alpha$ be a nonzero element of $k$. Consider the sequence of quadratic transforms $S \to S_1$ so that $S_1$ has regular parameters $x_1, y_1$ defined by

$$x = x^m(y_1 + \alpha)^{a'}, \overline{y} = x^q(y_1 + \alpha)^{b'}$$

where $a', b' \in \mathbb{N}$ are such that $mb' = qa' = 1$.

Let $\sigma = \gcd(pm, q)$ which is 1 or $p$. There exists a unique sequence of quadratic transforms $R \to R_1$ such that $R_1$ has regular parameters $u_1, v_1$ defined by

$$u = u^m(v_1 + \beta)^{c'}, \overline{v} = u^q(v_1 + \beta)^{d'}$$

where $pm = \sigma m$, $q = \sigma q$, $md' = c'q = 1$ and $0 \neq \beta \in k$, giving a commutative diagram of homomorphisms

$$\begin{array}{ccc}
R_1 & \to & S_1 \\
\uparrow & & \uparrow \\
R & \to & S
\end{array}$$

such that $S_1$ is a localization of a finite extension of $R_1$. We have that $J(S_1/R_1) = (x_1^{c_1})$ for some positive integer $c_1$. Further:
1) If $\sigma = 1$ then $R_1 \rightarrow S_1$ is of type 1 and
\[
\left( \frac{c_1}{p-1} \right) = \left( \frac{\tau}{p-1} \right) m - m.
\]

2) If $\sigma = p$ then $R_1 \rightarrow S_1$ is of type 2 and
\[
\left( \frac{c_1}{p-1} \right) = \left( \frac{\tau}{p-1} \right) m - m + 1.
\]

Remark 4.0.5. [6, Remark 4.4] Suppose that $\omega$ is a nondiscrete rational rank 1 valuation of $L$ dominating $S$ and $R \rightarrow S$ is of type 2. Let $\nu$ be the restriction of $\omega$ to $K$. Make the change of variables, letting $\overline{v}$ be the difference of $v$ and a polynomial in $u$ so that $\omega(\overline{v}) \not\in \omega(u)\mathbb{Z}$ and letting $\overline{y} = \overline{v}$.

Define $m$ and $q$ to be the unique relatively prime positive integers such that $m\omega(\overline{y}) = q\omega(x)$. There exist $0 \neq \alpha \in k$ and $a', b' \in \mathbb{N}$ such that $mb' - qa' = 1$ and if $S \rightarrow S_1$ is the sequence of quadratic transforms defined by
\[
x = x_1^m(y_1 + \alpha)^{a'}, \quad \overline{y} = x_1^q(y_1 + \alpha)^{b'}
\]
then $\omega$ dominates $S_1$.

The formulas of Cases 1) and 2) of Theorem 4.0.4 can then be stated in terms of the valuation $\omega$. They are:

1) If $\sigma = 1$ then $R_1 \rightarrow S_1$ is of type 1 and
\[
\left( \frac{c_1}{p-1} \right) \omega(x_1) = \left( \frac{\tau}{p-1} \right) \omega(x) - \omega(x).
\]

2) If $\sigma = p$ then $R_1 \rightarrow S_1$ is of type 2 and
\[
\left( \frac{c_1}{p-1} \right) \omega(x_1) = \left( \frac{\tau}{p-1} \right) \omega(x) - \omega(x) + \omega(x_1).
\]
If $R_1 \to S_1$ is of type 2, then we have that $\nu(\overline{y}) \notin \nu(x)\mathbb{Z}$, since $\gcd(pm, q) = p$. 
Chapter 5

Simultaneous resolution in a tower of two Artin-Schreier extensions

In this chapter, we prove our main result Theorem 1.1.1.

Lemma 5.0.1. Let $K$ be a two dimensional algebraic function field over an algebraically closed field $k$. Suppose that $\nu$ is a nondiscrete rational rank 1 valuation of $K$. Let $R$ be an algebraic regular local ring of $K$ which is dominated by $\nu$. Let $P$ be a height one prime ideal of $R$ such that $R/P$ is a regular local ring and let

$$R = R_{r_1} \to R_{r_2} \to \cdots$$

be the sequences of quadratic transforms above $R$ which are dominated by $\nu$ which are determined by $P$.

Let $\tilde{R}$ also be an algebraic regular local ring of $K$ which is dominated by $\nu$. Let $\tilde{P}$ be a height one prime ideal of $\tilde{R}$ such that $\tilde{R}/\tilde{P}$ is a regular local ring and let

$$\tilde{R} = \tilde{R}_{a_1} \to \tilde{R}_{a_2} \to \cdots$$
be the sequences of quadratic transforms above \( \tilde{R} \) which are dominated by \( \nu \) which are determined by \( \tilde{P} \).

Then there exists \( i, j \in \mathbb{Z}_{>0} \) such that \( R_{r_{i+l}'} = \tilde{R}_{a'_{j+l}} \) and \( R_{\pi_{i+l}} = \tilde{R}_{\pi_{j+l}} \) for all \( l \in \mathbb{N} \).

Proof. There exists a regular algebraic local ring \( A \) of \( K \) which is dominated by \( \nu \) and dominates both \( R \) and \( \tilde{R} \), such that \( R \to A \) and \( \tilde{R} \to A \) are products of quadratic transforms along \( \nu \). Let \( B \) be the quadratic transform of \( A \) along \( \nu \).

We can choose \( A \) so that \( \sqrt{PB} = \sqrt{\tilde{P}B} \) is the ideal of the exceptional divisor of \( \text{Spec}(B) \to \text{Spec}(A) \) by [1, Theorem 2]. Thus \( B \) is free over both \( R \) and \( \tilde{R} \). There exists \( i \) such that \( R_{r_{i}} \subset B \subset R_{\pi_{i}} \) and there exists \( j \) such that \( \tilde{R}_{a'_{j}} \subset B \subset \tilde{R}_{\pi_{j}} \).

Thus \( R_{\pi_{i+l}} = \tilde{R}_{\pi_{j+l}} \) for all \( l \in \mathbb{N} \) and \( R_{r_{i+l}'} = \tilde{R}_{a'_{j+l}} \) for all \( l \in \mathbb{Z}_{>0} \). \qed

Suppose that \( K \) and \( L \) are two dimensional algebraic function fields over an algebraically closed field \( k \) of characteristic \( p > 0 \) and \( K \to L \) is an Artin-Schreier extension. Suppose that \( \omega \) is a nondiscrete rational rank 1 valuation of \( L \). Let \( \nu = \omega|K \). Suppose that \( R_0 \) is a regular algebraic local ring of \( K \) and \( S_0 \) is a regular algebraic local ring of \( L \) such that \( \omega \) dominates \( S_0 \), \( S_0 \) dominates \( R_0 \) and \( R_0 \to S_0 \) is of type 0, 1 or 2 as in Definition 4.0.1, with \( \sqrt{J(S_0/R_0)} = xS_0 \). Letting \( P = uR_0 \) and \( Q = xS_0 \), these prime ideals make \( R_0 \to S_0 \) well prepared, with \( u, v \) admissible parameters in \( R_0 \) and \( x, y \) admissible parameters in \( S_0 \).

Inductively applying Theorems [6, Theorem 4.1 and Remark 4.2] and [6, Theorem 4.3 and Remark 4.4] (see Theorem 4.0.2, Remark 4.0.3, Theorem 4.0.4 and Remark 4.0.5), and making choices for the construction of \( S_i \to S_{i+1} \) and \( R_i \to R_{i+1} \) consistent with the assumptions of Theorem 4.0.2 and Remark 4.0.3, Theorem 4.0.4 and Remark 4.0.5, we construct a diagram where the horizontal sequences
are birational extensions of regular local rings (sequences of quadratic transforms)

\[ S_0 = S^0 \to S^1 \to S^2 \to \cdots \]
\[ R_0 = R^0 \to R^1 \to R^2 \to \cdots \]  \hspace{1cm} \text{(5.0.1)}

such that \( \omega \) dominates each \( S^i \) (and thus \( \nu \) dominates each \( R^i \)). Each extension \( R^i \to S^i \) is well prepared and of one of the three types 0, 1 or 2.

If \( R^i \to S^i \) is of type 0, then \( R^i \to R^{i+1} \) is the quadratic transform along \( \nu \) and \( S^i \to S^{i+1} \) is the quadratic transform along \( \omega \). \( R^{i+1} \to S^{i+1} \) is then of type 0.

In particular, if \( R^i \to S^i \) is of type 0 for some \( i \) in the sequence, then \( R^j \to S^j \) is of type 0 for all \( j \geq i \).

Each \( R^i \) has allowable regular parameters \( (u_i, v_i) \) and \( (\overline{u}_i, \overline{v}_i) \) and each \( S^i \) has allowable regular parameters \( (x_i, y_i) \) and \( (\overline{x}_i, \overline{y}_i) \). If \( R^i \to S^i \) is of type 1 or 2, the map \( S^i \to S^{i+1} \) is defined by

\[ \overline{x}_i = x_i^{m_i+1} (y_i + \alpha_i) \beta_i, \quad \overline{y}_i = x_i^{q_i+1} (y_i + \alpha_i) \beta_i \]  \hspace{1cm} \text{(5.0.2)}

and the map \( R^i \to R^{i+1} \) is defined by

\[ u_i = u_{i+1}^{m_i+1} (v_i + \beta_i) \beta_i, \quad \overline{v}_i = u_{i+1}^{q_i+1} (v_i + \beta_i) \beta_i \]  \hspace{1cm} \text{(5.0.3)}

where \( 0 \neq \alpha_{i+1}, 0 \neq \beta_{i+1} \in k \). Each \( R^i \to S^i \) is of type 0, 1 or 2. Further, \( \sqrt{J(S^i/R^i)} = x_i S^i \).

If \( R^i \to S^i \) is of type 1 or of type 2 then \( \overline{x}_i, \overline{y}_i \) and \( \overline{v}_i \) are defined by our changes of variables in Theorem 4.0.2 or Theorem 4.0.4. If \( R^i \to S^i \) is of type 0, then we take \( \overline{x}_i = u_i \) and \( \overline{y}_i = \overline{v}_i = v_i \). If \( R^i \to S^i \) is of type 2, we will impose the extra condition that

\[ m_{i+1} = \frac{pm_{i+1}}{\gcd(pm_{i+1}, q_{i+1})} > 1. \]  \hspace{1cm} \text{(5.0.4)}

Equation (5.0.4) is just the statement that \( \nu(\overline{v}_i) \notin \nu(u_i)Z \). The condition that
all \( m_{i+1} > 1 \) in Theorem 4.0.2 is just the statement that \( \omega(\overline{y}_i) \not\in \omega(\overline{x}_i)\mathbb{Z} \).

We now explain the relationship between the sequence (5.0.1), and the sequence of all quadratic transforms above \( R_0 \) and \( S_0 \) which are dominated by \( \omega \). To avoid excess notation, we will work this out explicitly for the first iteration of the algorithm, starting with \( R_0 \rightarrow S_0 \). The conclusions starting with \( R^i \rightarrow S^i \) are the same. Let

\[
R_0 \rightarrow R_1 \rightarrow R_2 \rightarrow \cdots
\]

be the sequence of all quadratic transforms along \( \nu \) above \( R_0 \) and let

\[
S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots
\]

be the sequence of all quadratic transforms along \( \omega \) above \( S_0 \).

Following the notations of Definition 3.0.1 and Definition 3.0.2, the first part of the sequence of all quadratic transforms along \( \nu \) above \( R_0 \) can be factored as

\[
R_0 = R_{r'_1} \rightarrow \cdots \rightarrow R_{r_1} \rightarrow \cdots \rightarrow R_{r'_2}.
\]

The first part of the quadratic transform sequence along \( \omega \) above \( S_0 \) can be factored as

\[
S_0 = S_{s'_1} \rightarrow \cdots \rightarrow S_{s_1} \rightarrow \cdots \rightarrow S_{s'_2}.
\]

First suppose that \( R_0 \rightarrow S_0 \) is of type 0. Then \( R^1 = R_{r'_1+1} \) is the quadratic transform of \( R_0 = R_{r'_1} \) along \( \nu \), \( S^1 = S_{s'_1+1} \) is the quadratic transform of \( S_0 = S_{s'_1} \) along \( \omega \) and \( R^1 \rightarrow S^1 \) is of type 0.

Suppose that \( R_0 \rightarrow S_0 \) is of type 1, so that we have an expression

\[
u = \gamma x, v = y \tau + x \Omega
\]

where \((u = u_0, v = v_0)\) are allowable regular parameters in \( R_0 \) and \((x = x_0, y = y_0)\) are allowable regular parameters in \( S_0 \). Then by Theorem 4.0.2, Remark 4.0.3 and the algorithm as explained above, make the changes of variables to \( u_0, v_0 \) and \( x_0, y_0 \)}
as specified in Theorem 4.0.2. We have that $S^1 = S'_{s_2}$ since $\omega(y_0) \notin \omega(x_0)\mathbb{Z}$. If $\nu(y_0) \notin \nu(u_0)\mathbb{Z}$ then we have that $R^1 = R_{r_2}$ and $R^1 \rightarrow S^1$ can be of type 0, 1 or 2. If $\nu(y_0) \in \nu(u_0)\mathbb{Z}$, then we have that $R_0 \subset R^1 \subset R_{r_1}$. In this case we must have that $R^1 \rightarrow S^1$ is of type 0 or 2. Therefore, for $R_0 \rightarrow S_0$ of type 1 and our construction of $R^1 \rightarrow S^1$ as above, we have, either $R^1 = R_{r_2}$ and $S^1 = S'_{s_2}$ with $R^1 \rightarrow S^1$ of type 0, 1 or 2, or $R \subset R^1 \subset R_{r_1}$ and $S^1 = S'_{s_2}$ with $R^1 \rightarrow S^1$ necessarily of type 0 or 2.

Now suppose that $R_0 \rightarrow S_0$ is of type 2, so that we have an expression

$$u = \gamma x^p, \; v = y\tau + x\Omega$$

where $(u = u_0, v = v_0)$ are allowable regular parameters in $R_0$ and $(x = x_0, y = y_0)$ are allowable regular parameters in $S_0$. Then by Theorem 4.0.4, Remark 4.0.5 and the algorithm as explained above, make the changes of variables to $u_0, v_0$ and $x_0, y_0$ as specified in Theorem 4.0.4 and (5.0.4). We have that $R^1 = R_{r_2}$ since $\nu(y_0) \notin \nu(u_0)\mathbb{Z}$. If $\omega(y_0) \notin \omega(x_0)\mathbb{Z}$ then $S^1 = S'_{s_2}$ and $R^1 \rightarrow S^1$ can be of type 1 or 2. If $\omega(y_0) \in \omega(x_0)\mathbb{Z}$ then $S \subset S^1 \subset S_{r_1}$. In this case we must have that $R^1 \rightarrow S^1$ is of type 1. Therefore, for $R_0 \rightarrow S_0$ of type 2 and our construction of $R^1 \rightarrow S^1$ as above, we have, either $R^1 = R_{r_2}$ and $S^1 = S'_{s_2}$ with $R^1 \rightarrow S^1$ of type 1 or 2, or $R^1 = R_{r_2}$ and $S_0 \subset S^1 \subset S_{r_1}$ with $R^1 \rightarrow S^1$ necessarily of type 1.

**Proposition 5.0.2.** Suppose that $i \geq 1$. Then there exists $j$ such that $R^j = R_{r_i}$ and there exists $k$ such that $S^k = S'_{s_i}$. That is, every $R_{r_i}$ appears in the sequence

$$R_0 \rightarrow R^1 \rightarrow R^2 \rightarrow \cdots$$

and every $S'_{s_i}$ appears in the sequence

$$S_0 \rightarrow S^1 \rightarrow S^2 \rightarrow \cdots .$$

If $R^l \rightarrow S^l$ is of type 0 for some $l$, then $R^l \rightarrow S'^{l'}$ is of type 0 for all $l' \geq l$ and
$R^{l+1}$ is the quadratic transform of $R^l$ along $\nu$, $S^{l+1}$ is the quadratic transform of $S^l$ along $\omega$.

Proof. We start by observing that if $R^l \to S^l$ is of type 0 for some $l$, then $R^{l+1}$ is the quadratic transform of $R^l$ along $\nu$, $S^{l+1}$ is the quadratic transform of $S^l$ along $\omega$ and $R^{l'} \to S^{l'}$ is of type 0 for all $l' \geq l$.

We use induction to prove the proposition.

For the base case, $i = 1$, we note that, we have $R_0 = R_0^{r_i'1} = R^{r_i}$ and $S_0 = S_0^{s_i'1} = S^{s_i}$. Now let $i \geq 1$ and assume that there exists $j \geq 0$ and $k \geq 0$ such that $R^j = R^{r_i}_i$ and $S^k = S^{s_i}_i$. We show that then there exists $j'$ and $k'$ such that $R^{j'} = R^{r_{i+1}}_{r_i'}$ and $S^{k'} = S^{s_{i+1}}_{s_i'}$.

We first show the existence of $j'$. We split the proof into three cases.

For $j$ as above, first suppose that $R^{r_i}_i = R^j \to S^j$ has type 0. From our observation above, we have that then, for $n \geq 0$, $R^{j+n} = R^{r_i+n}_{r_i'}$ (the $n$th quadratic transform of $R^j = R^{r_i}_i$). In particular, $R^{j+r_{i+1}+1-r_i'} = R^{r_{i+1}}_{r_i'}$. Hence, $j' = j + r_{i+1} - r_i'$ gives the desired outcome in this case.

Next suppose that $R^{r_i}_i = R^j \to S^j$ has type 2. From our discussion above for the type 2 case, now applied to $R^j \to S^j$, we note that in either possibilities for $R^{j+1} \to S^{j+1}$, we have, $R^{j+1} = R^{r_{i+1}}_{r_i'}$. Hence, $j' = j + 1$ gives the desired outcome in this case.

Next suppose that $R^{r_i}_i = R^j \to S^j$ has type 1. From our discussion above for the type 1 case, now applied to $R^j \to S^j$, we note that we have two possibilities.
We have either $R^{j+1} = R_{r_i}^j + 1$, in which case $j' = j + 1$ gives the desired outcome, or we have $R^j = R_{r_i}^j \subset R^{j+1} \subset R_{r_i}$. If $R_{r_i}^j \subset R^{j+1} \subset R_{r_i}$, then we additionally have that $R^{j+1} \to S^{j+1}$ is necessarily of type 0 or 2. If $R^{j+1} \to S^{j+1}$ is of type 2, our discussion for the type 2 case now applied to $R^j \to S^j$ gives that then $R^{j+2} = R_{r_i}^j + 1$ since $R_{r_i}^j \subset R^{j+1} \subset R_{r_i}$. Hence, $j' = j + 2$ gives the desired outcome in this subcase. If $R^{j+1} \to S^{j+1}$ is of type 0, from our observation above for the type 0 case and since $R_{r_i}^j \subset R^{j+1} \subset R_{r_i}$, we get that then $R^{j'} = R_{r_i}^j + 1$ for some $j' \leq j + 1 + r_{i+1}' - r_i'$.

Next, we show the existence of $k'$. We split the proof into three cases.

For $k$ as above, first suppose that $R^k \to S^k = S_{s_i}^j$ has type 0. We observe that then, for $n \geq 0$, $S^{k+n} = S_{s_i+n}$ (the $n$th quadratic transform of $S^k = S_{s_i}$). In particular, $S^{k+s_i'-s_i} = S_{s_i'+1}$. Hence, $k' = k + s_i' - s_i$ gives the desired outcome in this case.

Next suppose that $R^k \to S^k = S_{s_i}^j$ has type 1. From our discussion above for the type 1 case, now applied to $R^k \to S^k$, we note that in either possibilities for $R^{k+1} \to S^{k+1}$, we have $S^{k+1} = S_{s_i'+1}$. Hence, $k' = k + 1$ gives the desired outcome in this case.

Next suppose that $R^k \to S^k = S_{s_i}^j$ has type 2. From our discussion above for the type 2 case, now applied to $R^k \to S^k$, we note that we have two possibilities. We have either $S^{k+1} = S_{s_i'+1}$, in which case $k' = k + 1$ gives the desired outcome, or we have $S^k = S_{s_i'} \subset S^{k+1} \subset S_{r_i}$. If $S_{s_i'} \subset S^{k+1} \subset S_{r_i}$, then we additionally have that $R^{k+1} \to S^{k+1}$ is necessarily of type 1. Therefore, from our discussion for the type 1 case, now applied to $R^{k+1} \to S^{k+1}$, with $S_{s_i'} \subset S^{k+1} \subset S_{r_i}$, we note that then $S^{k+2} = S_{s_i'+1}$. Hence, $k' = k + 2$ gives the desired outcome in this subcase.
Hence, we have shown the existence of $j'$ and $k'$ and this concludes our proof by induction. \hfill \Box

**Theorem 5.0.3.** Suppose that $K \to L \to M$ is a tower of two Artin-Schreier extensions, where $K, L, M$ are two dimensional algebraic function fields over an algebraically closed field $k$ of characteristic $p > 0$. Let $\omega$ be a rational rank 1 nondiscrete valuation of $M$. Suppose that $R$ is a regular algebraic local ring of $K$ and $T$ is a regular algebraic local ring of $M$ such that $\omega$ dominates $T$ and $T$ dominates $R$. Then there exists a commutative diagram

\[
\begin{array}{ccc}
R' & \to & T' \\
\uparrow & & \uparrow \\
R & \to & T
\end{array}
\]

such that the vertical arrows are products of quadratic transforms, $\omega$ dominates $T'$, $T'$ dominates $R'$ and $T'$ is a localization of a finite extension of $R'$.

**Proof.** Let $S'$ denote the localization at the center of $\omega|_L$ of the integral closure of $R$ in $L$. Since $R$ is a regular local ring, $R$ is, in particular, a Noetherian normal ring. Therefore, we have $S'$ is an algebraic local ring of $L$ since $L/K$ is separable (see [13, Chapter V, Section 4, Theorem 7, Corollary 1, p. 265]). Now, since $\omega|_L$ dominates $S'$ and $S'$ is algebraic, using two dimensional local uniformization, [2], we may find an algebraic regular local ring $S$ of $L$ such that $\omega|_L$ dominates $S$ and $S$ dominates $S'$. Applying [6, Proposition 3.7] or [11, Lemma 7.3] (Proposition 3.0.15), for the inclusion $R \to S$, we obtain a commutative diagram

\[
\begin{array}{ccc}
R_0 & \to & S_0 \\
\uparrow & & \uparrow \\
S & \\
\uparrow \\
R & \to & S'
\end{array}
\]
such that $R_0$, $S_0$ are algebraic regular local rings, $\omega|_L$ dominates $S_0$, $S' \to S$ is birational, $R \to R_0$ and $S \to S_0$ are products of quadratic transforms and $R_0 \to S_0$ is of type 0, 1 or 2.

Now, note that, we have $S' \subseteq T$ since $R \subseteq T$, $T$ is a regular (hence normal) local ring dominated by $\omega$ and $L \subseteq M$. Similarly, since $S_0$ is algebraic and dominated by $\omega$, we may find a quadratic transform $T''$ of $T$ such that $T''$ dominates $S_0$. Applying [6, Proposition 3.7] (or [11, Lemma 7.3]) (Proposition 3.0.15), for the inclusion $S_0 \to T''$, we obtain a commutative diagram of algebraic local rings

\[
\begin{array}{ccc}
0S & \longrightarrow & 0T \\
\uparrow & & \uparrow \\
S_0 & \longrightarrow & T'' \\
\uparrow & & \uparrow \\
S & \longrightarrow & T'' \\
\uparrow & & \uparrow \\
S' & \longrightarrow & T \\
\end{array}
\]

such that $\omega$ dominates $0T$, $S' \to S$ is birational, $S \to S_0 \to 0S$ and $T \to T'' \to 0T$ are products of quadratic transforms, and $0S \to 0T$ is of type 0, 1 or 2.

Hence, we have obtained the following commutative diagram of algebraic local rings

\[
\begin{array}{ccc}
0S & \longrightarrow & 0T \\
\uparrow & & \uparrow \\
R_0 & \longrightarrow & S_0 \\
\uparrow & & \uparrow \\
R & \longrightarrow & S' \\
\uparrow & & \uparrow \\
S' & \longrightarrow & T \\
\end{array}
\]

where $\omega$ dominates $0T$, $S'$ is the localization at the center of $\omega|_L$ of the integral closure of $R$ in $L$, $S$ is an algebraic regular local ring of $L$ dominated by $\omega|_L$ and dominating $S'$, $S' \to S$ is birational, $R \to R_0$, $S \to 0S$ and $T \to 0T$ are products of
quadratic transforms, $R_0 \to S_0$ is well prepared with respect to height one prime ideals $P_0$ of $R_0$ and $Q_0$ of $S_0$ and is of type 0, 1 or 2 and $S_0 \to T_0$ is well prepared with respect to height one prime ideals $Q_0$ of $S_0$ and $I_0$ of $T_0$ and is of type 0, 1 or 2.

Applying the algorithm explained at the beginning of the section, with $R^0 = R_0$ and $S^0 = S_0$, we construct the following commutative diagram of sequences of quadratic transforms

\[
\begin{array}{ccc}
\vdots & \vdots & \\
\uparrow & \uparrow & \\
R^k & \longrightarrow & S^k \\
\uparrow & \uparrow & \\
S^k & \longrightarrow & R^k \\
\uparrow & \uparrow & \\
R_0 & \longrightarrow & S_0
\end{array}
\]

such that $\omega$ dominates $S^k$, $S^k$ dominates $R^k$, and $R^k \to S^k$ is of type 0, 1 or 2 for all $k$.

Similarly, applying the algorithm starting with $S^0 = S_0$ and $T^0 = T_0$, we construct the following commutative diagram of sequences of quadratic transforms

\[
\begin{array}{ccc}
\vdots & \vdots & \\
\uparrow & \uparrow & \\
^1S & \longrightarrow & ^1T \\
\uparrow & \uparrow & \\
^1T & \longrightarrow & ^1S \\
\uparrow & \uparrow & \\
S_0 & \longrightarrow & T_0
\end{array}
\]
such that \( \omega \) dominates \( T \), \( T \) dominates \( S \), and \( S \to T \) is of type 0, 1 or 2 for all \( l \).

For notations as in Definition 3.0.1 and Definition 3.0.2, we have the sequence of all quadratic transforms of \( S_0 \) along \( \omega|_L \) as determined by \( Q_0 \), can be factored as

\[
S_0 = S_{s_1} \to \cdots \to S_{s_2} \to \cdots \to S_{s_1} \to \cdots .
\]

Further, the sequence of all quadratic transforms of \( 0 \cdot S \) along \( \omega|_L \), as determined by \( 0 \cdot Q \), can be factored as

\[
0 \cdot S = a_1' \cdot S \to \cdots \to \pi_1 \cdot S \to \cdots \to a_2' \cdot S \to \cdots \to \pi_2 \cdot S \to \cdots .
\]

By Lemma 5.0.1, there exist \( i, j \in \mathbb{Z}_{>0} \) such that \( S_{s_{i+1}} = a_{j+i}' \cdot S \) for all \( l \in \mathbb{N} \).

By Proposition 5.0.2, there exist \( \alpha, \beta \in \mathbb{Z}_{>0} \) such that \( S^\alpha = S_{s_i}' = a_j' \cdot S = \beta \cdot S \).

Let \( R' := R^\alpha \), \( S' := S^\alpha = \beta \cdot S \) and \( T' := \beta \cdot T \). Let \( m_{R'}, m_{S'} \) and \( m_{T'} \) denote their corresponding maximal ideals. Since \( R \to S' \) and \( S' \to T' \) are of type 0, 1 or 2, we have that \( \sqrt{m_{R'} \cdot S'} = m_{S'} \) and \( \sqrt{m_{S'} \cdot T'} = m_{T'} \). Therefore, we have \( \sqrt{m_{R'} \cdot T'} = m_{T'} \) and hence \( T' \) is a localization of a finite extension of \( R' \) (see Lemma 2.0.6).

\[
\square
\]

Let \( K \to L \to M \) be a tower of two Artin-Schreier extensions, where \( K, L, M \) are two dimensional algebraic function fields over an algebraically closed field \( k \) of characteristic \( p > 0 \). Let \( \omega \) be a rational rank 1 nondiscrete valuation of \( M \).

Let \( R \) and \( T \) be algebraic regular local rings of \( K \) and \( M \) respectively such that \( \omega \) dominates \( T \) and \( T \) dominates \( R \). We now prove Theorem 1.1.1.

From Theorem 5.0.3, we have that there exists algebraic regular local rings \( R' \) and \( T' \) of \( K \) and \( M \) respectively such that \( R \to R' \), \( T \to T' \) are products of quadratic transforms, \( \omega \) dominates \( T' \), \( T' \) dominates \( R' \) and \( T' \) is a localization of
a finite extension of $R'$. Furthermore, from the proof of Theorem 5.0.3, we note that $R' \to T'$ can be written as a composition $R' \to S' \to T'$ of maps of types 0, 1 or 2, where $R' \to S'$ is well prepared with respect to divisors $E'$ on Spec($R'$) and $F'$ on Spec($S'$) and $S' \to T'$ is well prepared with respect to the divisor $F'$ on Spec($S'$) and a divisor $G'$ on Spec($T'$). Let $u, v$ and $z, w$ be respective admissible parameters for $R' \to S'$ and $z', w'$ and $x, y$ be respective admissible parameters for $S' \to T'$. Then $z = 0$ and $z' = 0$ are both local equations of $F'$ in $S'$ so the change of variables from $z, w$ to $z', w'$ has the form

$$z = \alpha z', w = \beta w' + z' \Lambda \quad (5.0.5)$$

where $\alpha, \beta$ are units in $S'$ and $\Lambda \in S'$. Composing the expressions of types 0, 1 or 2 of $u, v$ in terms of $z, w$, the change of variables (5.0.5) from $z, w$ to $z', w'$ and the expressions of $z', w'$ in terms of $x, y$, we obtain the forms of the conclusions of Theorem 1.1.1.

Furthermore, since $\sqrt{m_{R'} T'} = m_{T'}$, $R' \to T'$ is the localization of a finite map and hence we have simultaneous local resolution holds for $R \to T$ along $\omega$.

This concludes our proof of Theorem 1.1.1, showing simultaneous local resolution holds along a rational non-discrete valuation when $K \to M$ is a tower of two Artin-Schreier extensions of two dimensional algebraic function fields over an algebraically closed base field of char $p > 0$. 

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BIBLIOGRAPHY


VITA

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