GLOBAL BIFURCATION AND STABILITY OF SOLITARY WAVES IN TWO-LAYER WATER

A Dissertation
presented to
the Faculty of the Graduate School
University of Missouri

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

by
Daniel Sinambela
Dr. Samuel Walsh, Dissertation Supervisor

JULY 2022
The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

GLOBAL BIFURCATION AND STABILITY OF SOLITARY WAVES IN TWO-LAYER WATER

Daniel Sinambela, a candidate for the degree of Doctor of Philosophy of Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

________________________
Professor Samuel Walsh

________________________
Professor Yuri Latushkin

________________________
Professor Stamatis Dostoglou

________________________
Professor David Retzloff
ACKNOWLEDGEMENTS

First of all, I would like to thank my advisor, Dr. Samuel Walsh, for his guidance, advice, and continuous support during my time in the department. He has been immensely helpful and approachable in the times of need and in all stages of my dissertation writing process. I hope someday I can repay back all of his act of kindness. Thank you, Dr. Walsh!

I would like to also thank all committee members including Dr. Yuri Latushkin, Dr. Stamatis Dostoglou, and Dr. David Retzloff for being helpful reviewers of my research and for their valuable time on fruitful discussion during my study.

Lastly, I would like to thank my family: my parents, brother, and sister for their constant encouragement during my time pursuing my education in the USA. Without their support and prayers, I do not think I can make it this far.
Contents

ACKNOWLEDGEMENTS ii

Abstract vi

1 Introduction 1

1.1 Incompressible Euler equations ......................... 1
1.2 Boundary Conditions .......................................... 6
1.3 Vorticity ......................................................... 8
1.4 Traveling Waves ................................................. 10
1.5 Outline .......................................................... 11

2 Large-amplitude solitary waves in two-layer density stratified water 13

2.1 Introduction ...................................................... 13

2.1.1 Governing equation ........................................... 15
2.1.2 Statement of results ......................................... 18
2.1.3 Outline .......................................................... 22

2.2 Formulation ....................................................... 24

2.2.1 Non-dimensionalization ..................................... 24
2.2.2 Stream function formulation ............................... 27
2.2.3 Height function formulation ............................... 29
2.2.4 Flow force .................................................. 31
2.2.5 Function spaces and the operator equation ............... 32
2.3 Linearized operators .......................................... 34
  2.3.1 Sturm–Liouville type problems ......................... 34
  2.3.2 Fredholm property ....................................... 43
2.4 Qualitative properties ........................................ 45
  2.4.1 Bounds on the Froude number ......................... 45
  2.4.2 Symmetry ................................................. 50
  2.4.3 Asymptotic monotonicity and nodal properties ....... 61
2.5 Small-amplitude existence theory ............................ 68
2.6 Large-amplitude existence theory ............................ 77
  2.6.1 Velocity bound ........................................... 77
  2.6.2 Uniform regularity ...................................... 86
  2.6.3 Proof of the main result ................................. 88

3 Orbital stability/instability ................................. 95
  3.1 Introduction .................................................. 95
    3.1.1 Statement of results .................................. 98
    3.1.2 Idea of the proof ..................................... 103
    3.1.3 Outline ................................................ 105
  3.2 Existence theory ............................................. 105
  3.3 Stability ..................................................... 122
    3.3.1 General theory ....................................... 122
    3.3.2 Notion on stability/instability ....................... 127
3.4 Hamiltonian Formulation .............................................. 128
  3.4.1 Nonlocal operators ............................................. 128
  3.4.2 Function spaces ............................................... 131
  3.4.3 Hamiltonian Structure ......................................... 133
  3.4.4 The symmetry group and momentum .......................... 139
  3.4.5 Bound states .................................................. 140

3.5 Spectral Analysis .................................................... 142
  3.5.1 Rescaled operator ............................................... 150
  3.5.2 Spectrum of the linearized augmented potential ............ 157

3.6 Proof of theorem ................................................... 161

Appendices ................................................................. 166

A Global Bifurcation .................................................. 167

B Stability ................................................................. 170

Vita ......................................................................... 187
ABSTRACT

The present work concerns two mathematical problems on wave in a stratified body of water governed by the incompressible Euler equations. In the first part, we present a large-amplitude existence theory for two-dimensional solitary waves by means of a global bifurcation theoretic approach. That is for any piece-wise smooth upstream density distribution and laminar background current, we construct a global curve of solutions. This curve bifurcates from the background current and, following along the curve, we find waves that are arbitrarily close to having horizontal stagnation points.

The second part of the work tackles the problem on orbital stability of solitary waves under the presence of surface tension and constant vorticities. Using a spatial dynamics technique, we established the existence of such waves. Following that, we proved an orbital stability result. It is accomplished via a variant of Grillakis–Shatah–Strauss method.
Chapter 1

Introduction

Fluids are objects that exist all around us; but the well-known complexity of their motion makes them highly non-trivial to study mathematically. In 1755, Leonhard Euler proposed the first rigorous mathematical model for inviscid fluids. Remarkably, it is still used until today by physicists and engineers, while mathematicians continue to study its rich structure. Viscous fluids, on the other hand, are governed by the famous Navier–Stokes equations. To keep our discussion directed towards the present work, in this thesis we only focus to study the inviscid case.

1.1 Incompressible Euler equations

In this section, we will derive the fundamental governing equations for two-dimensional water waves, both in the bulk and boundaries. At the outset, we would like to emphasize that our derivation has been simplified to capture all the relevant physical phenomena that are directly related to the work in the later part of the dissertation. For more in-depth discussions on the derivation, see [CM93] and [Ach90].

Let $t \geq 0$ denote the time variable. A time-dependent domain where the fluid is contained is described by $\Omega(t) \subset \mathbb{R}^2$. In this regard, we adopt a continuum mechanics point of view: we think of the body of water as a continuum rather than a collec-
tion of discrete molecular structures. This assumption is believed to be valid when considering various macroscopic phenomena in nature.

Imagine a fluid particle in the bulk; as it evolves in time, it traces a curve in the plane at time $t$ given by the function $\mathbf{x}(t) := (x(t), y(t))^T$. Let $\mathbf{u}(t, x, y) := (u, v)^T$ denote the velocity field of the fluid particle. Thus,

$$
\frac{d\mathbf{x}(t)}{dt} = \mathbf{u}(\mathbf{x}).
$$

(1.1)

The *Lagrangian flow map* corresponding to this vector field is defined by

$$
\Xi : \Omega_T \rightarrow \mathbb{R}^2,
$$

(1.2)

$$
(t, \mathbf{x}_0) \mapsto (x(t), y(t)),
$$

where

$$
\Omega_T := \{\{t\} \times \Omega(t) : t \in (0, T)\}, \quad \partial\Omega_T := \{\{t\} \times \partial\Omega_t : t \in (0, T)\}
$$

and $\mathbf{x}(t)$ solves the ODE (1.1) together with an initial condition $\mathbf{x}(0) = \mathbf{x}_0$. This flow map describes the trajectory followed by the fluid particle as time progresses. We will take for granted that it is sufficiently smooth to handle all the desired mathematical operations.

Let us now derive the equations expressing the incompressibility assumption of the fluid. Physically, incompressibility simply means that $\Xi$ is measure preserving. In other words,

$$
|\Omega(t)| := \int_{\Omega(t)} 1 \, d\mathbf{x} = \int_{\Omega(0)} 1 \, d\mathbf{x} =: |\Omega(0)|.
$$

Via change of variables, for all $t$, one can obtain the following relation

$$
|\Omega(t)| := \int_{\Omega(0)} \det D\Xi(t, \mathbf{x}) \, d\mathbf{x},
$$
where $D$ is the Jacobian matrix of the spatial variables.

From the above two equations, we can alternatively say that incompressibility is achieved when

$$ \det D \Xi(t, x) = 1, \quad \text{for all } t \geq 0. \quad (1.3) $$

Differentiating both sides of the equation in (1.3) yields

$$ \partial_t (\det D \Xi(t, x)) = \partial_t ((\partial_x \Xi_1)(\partial_y \Xi_2) - (\partial_x \Xi_2)(\partial_y \Xi_1)) $$

$$ = (u_x(t, \Xi(t, x)) + v_y(t, \Xi(t, x))) ((\partial_x \Xi_1)(\partial_y \Xi_2) - (\partial_x \Xi_2)(\partial_y \Xi_1)) $$

$$ = (\nabla \cdot \mathbf{u})(\det D \Xi(t, x)) $$

$$ = 0. \quad (1.4) $$

Thus, from equation (1.4), the fluid is incompressible if it satisfies the following condition:

$$ \nabla \cdot \mathbf{u} = 0. \quad (1.5) $$

In the remainder of this section, we derive the continuity equation and momentum equation. It is important to note that our derivation relies on the continuum viewpoint mentioned earlier. Under this assumption, the fluid has a mass density $\rho(t, x)$ at fixed time $t$. Let $\Omega'(t)$ be any arbitrary subdomain of $\Omega(t)$. The total mass of the fluid at time $t$ in the subdomain is given by the following integral

$$ m(t, \Omega') = \int_{\Omega'} \rho(t, x) \, d\mathbf{x}. \quad (1.6) $$

Our derivation of the governing equations is based on the following two fundamental principles:

- **Mass Conservation:** Mass is neither created nor destroyed;
• **Balance of Momentum:** The rate of change of momentum of a subdomain in the fluid equals the force applied to it.

The rate of change of the total mass in $\Omega'(t)$ is given by

$$\frac{d}{dt} m(t, \Omega') = \frac{d}{dt} \int_{\Omega'} \rho(t, x) \, dx = \int_{\Omega'} \frac{\partial \rho(t, x)}{\partial t} \, (t, x) \, dx.$$

The principle of mass conservation formally states that the rate of increase in mass density throughout $\Omega'$ is equal to the total flow rate of the mass density crossing the boundary $\partial \Omega'$ in the inward direction. The flow rate of the volume across $\partial \Omega'(t)$ per unit length is given by $\mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward unit normal. Thus, the mass density flow rate per unit length is $\rho \mathbf{u} \cdot \mathbf{n}$. Hence, from the mass conservation principle, we obtain

$$\frac{d}{dt} \int_{\Omega'} \rho(t, x) \, dx = - \int_{\partial \Omega'} \rho \mathbf{u} \cdot \mathbf{n} \, dL$$

(1.7)

where $dL$ denotes the length element of $\partial \Omega'(t)$. Since $\Omega'(t) \subset \Omega(t)$ was arbitrary, via the divergence theorem the integral equation (1.7) gives us

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$  

(1.8)

This equation is a differential form of the mass conservation law. In the literature, it goes by the name of *continuity equation*. Observe that, when the density is taken to be constant, from (1.8), one obtains the incompressibility condition that is

$$\nabla \cdot \mathbf{u} = 0.$$  

(1.9)

Moreover, when the density is non-constant, the incompressibility condition (1.5) and continuity equation (1.8) together yields

$$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0.$$  

(1.10)
Next, we will use the balance of momentum principle to arrive at the governing equation for the momentum. This requires us to understand how body and stress forces act on a continuum. In the context of water, the body force results from the gravitational force \( \mathbf{g} := (0, -\rho g) \), where \( g > 0 \) is the gravitational constant. The total body force exerted on the fluid inside \( \Omega' \) is then given by

\[
\mathbf{B} = \int_{\Omega'} \mathbf{g} \, d\mathbf{x}.
\]  

(1.11)

Meanwhile, the stress force mainly comes from the pressure \( P(t, \mathbf{x}) \). For inviscid fluid, the stress force exerted on the boundary \( \partial \Omega' \) at position \( \mathbf{x} \) and time \( t \) is governed by \( P(t, \mathbf{x}) \mathbf{n} \). Again, for simplicity of derivation, we shall ignore other forces such as viscosity. Note that this force is directed orthogonal to the boundary \( \partial \Omega' \). Meaning, we assume no tangential forces acting on the fluid in \( \Omega' \). The total stress force \( \mathbf{S} \) acting on the the boundary of \( \Omega' \) is given by

\[
\mathbf{S} = -\int_{\partial \Omega'} P \mathbf{n} \, dL,
\]  

(1.12)

where \( \mathbf{n} \) is again the outward normal vector to the boundary. For any arbitrary vector \( \nu \), we have

\[
\nu \cdot \mathbf{S} = -\int_{\partial \Omega'} P \nu \cdot \mathbf{n} \, dL = -\int_{\Omega'} \nabla \cdot (P \nu) \, d\mathbf{x} = -\int_{\Omega'} \nabla P \cdot \nu \, d\mathbf{x}.
\]  

(1.13)

This implies

\[
\mathbf{S} = -\int_{\Omega'} \nabla P \, d\mathbf{x}.
\]  

(1.14)

Combining both equations (1.11) and (1.14), we can say that the total force is

\[
\text{Total Force} = \int_{\Omega'} \mathbf{g} - \nabla P \, d\mathbf{x}.
\]  

(1.15)
Recall the ODE for the velocity field in (1.1). Observe that, differentiating the right hand side with respect to time yields

\[
\frac{du(t, x)}{dt} = \frac{\partial u(t, x)}{\partial t} + \frac{dx}{dt} \cdot \nabla_x u(t, x) = D_t u,
\]

(1.16)

where \(D_t = \frac{\partial}{\partial t} + u \cdot \nabla\) denotes the material derivative. In this context, the material derivative physically describes the rate of change of the velocity field in the direction of the flow. Applying the Newton’s Second Law and the Balance of Momentum, we can conclude that

\[
\int_{\Omega'(t)} D_t(u \rho) \, dx = \int_{\Omega'} (-\nabla P + g) \, dx.
\]

(1.17)

Again, using the continuity equation (1.8) and the fact that \(\Omega'(t)\) is an arbitrary subdomain of \(\Omega(t)\), equation (1.17) reads

\[
\rho D_t u = -\nabla P + g = -\nabla P + (0, -\rho g)^\top.
\]

(1.18)

Combining all three equations in (1.5), (1.8), and (1.18), we arrive at the Euler incompressible equations

\[
\begin{align*}
\nabla \cdot u &= 0, \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0, \\
\rho D_t u &= -\nabla P + (0, -\rho g)^\top.
\end{align*}
\]

(1.19)

1.2 Boundary Conditions

Upon deriving the governing equations above, we are now ready to outline the derivation of motion equations on the boundary of \(\Omega(t)\). To connect directly with the main works of this thesis, we will assume that the fluid is bounded above by a graph of a function

\[
y = \eta(t, x) \quad \text{for all } t \geq 0,
\]

(1.20)
and below by the flat bed \( y = -d \) for some fixed number \( d \geq 0 \). Formally, the domain can be written in the following way:

\[
\Omega(t) = \{(x(t), y(t)) : \quad -d \leq y < \eta(x, t) \text{ and } -\infty < x < \infty\}. \tag{1.21}
\]

In principle, the boundary conditions we are considering here can be divided into two types:

- The kinematic boundary condition: This imposes the condition that all particles that are originally on the boundary stay on the boundary for all time during the evolution;
- The dynamic boundary condition: The normal stress on the boundary must be balanced.

We start by deriving the kinematic condition on the surface \( \eta \). Differentiating (1.20) in \( t \) yields

\[
v = \frac{dy}{dt} = \frac{\partial \eta}{\partial t} + \frac{dx}{dt} \frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x}. \tag{1.22}
\]

After a rearrangement, we obtain

\[
v - u \eta_x = \eta_t \quad \text{on } y = \eta(t, x). \tag{1.23}
\]

The kinematic condition on \( y = -d \) results from the impermeability condition of the bed. Mathematically speaking, the velocity field on the bed has to satisfy

\[
\quad v = 0 \quad \text{on } y = -d. \tag{1.24}
\]

As mentioned earlier, the dynamic condition concerns the normal stress to the boundary. In setting of this work, this translates to requiring

\[
P = \alpha^2 \kappa(\eta) \quad \text{on } y = \eta(t, x), \tag{1.25}
\]
where $P$ denotes the constant pressure, $\alpha^2$ is the coefficient of surface tension, and

$$\kappa(x) = \frac{-\eta''}{(1 + (\eta')^2)^{3/2}}$$

is the mean curvature. The dynamic condition (1.25) can be understood via the Young–Laplace law: the jump in pressure across the surface of the fluid is directly proportional to the mean curvature. In (1.25) we have normalized the pressure above the surface to be zero. On the bed, however, the stress force is negligible. Hence, no dynamic condition can be derived there.

Finally, observe that from the governing equations in the interior (1.19) with kinematic and dynamic boundary conditions (1.23), (1.24), (1.25), we have derived the full water wave problem. In essence, it is a free boundary problem where the function $\eta$ is also one of the unknowns. Mathematically, this does complicate the problem and cause the analysis to be several order of magnitude more difficult.

### 1.3 Vorticity

In studying the mathematical aspects of fluid, another object that is worth discussing is vorticity denoted by $\omega$. In the two-dimensional setting, it is a scalar-valued function defined as follows

$$\omega := \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (1.26)$$

Physically, vorticity measures the local rotation of fluid. In the literature, the large majority of the works on water waves concern the irrotational regime that is when $\omega \equiv 0$ everywhere in the fluid. This regime is proven to be mathematically more tractable than the rotational case ($\omega \neq 0$). It is straightforward to show that the irrotational condition implies the existence of a velocity potential function $\phi$ which is
harmonic. This fact allows the use of tools available in complex analysis, for instance conformal mapping to fix the domain of the fluid.

However, in nature, it is a well-known fact that vorticity can be generated by incoming currents, the presence of wind, temperature or density gradients in the water, or boundary layer effects. In 1809, Gerstner [Ger09] gave the first explicit formula for solutions to the Euler equations where constant vorticity and gravitational force were considered. Dubreil-Jacotin [DJ34] in 1934, via a non-conformal transformation they developed, constructed a family of small-amplitude periodic waves with vorticity. In another rotational regime, Ter-Krikorov [TK63] in 1963 proved the existence of small-amplitude solitary waves. Relatively recently, a number of authors have begun intensively studying steady rotational waves in an effort to better understand these phenomena; see for example the monograph [Con11a]. In the infinite-depth setting, for instance, two-dimensional capillary-gravity waves with compactly supported vorticity were constructed by Shatah, Walsh, and Zeng [SWZ13]. Lastly, we would like to mention that the present work deals with rotational flow of constant vorticity.

For two-dimensional steady incompressible flow, it is trivial from calculus that there exists a stream function $\psi$ such that

$$u = \nabla^\perp \psi, \quad (1.27)$$

where $\nabla^\perp := (-\partial_y, \partial_x)$. Taking the curl of $u$ in (1.27) and combine it with (1.26), we have

$$\Delta \psi = \omega. \quad (1.28)$$
1.4 Traveling Waves

At this point, we have derived formally the equations of motions for two-dimensional, incompressible, and inviscid fluids. We call this the water wave problem. As it is to any other partial differential equations, the water wave problem also supports various types of solutions. One them that is the subject of this work is the traveling or steady waves. They are essentially time-dependent solutions that propagate via translation at a fixed speed $c$ without changing profile. In nature, traveling waves can be found in many different settings ranging from propagation ripples on the surface of a lake or pond, to gas combustion \[1\text{Mv81}\], and even tsunami waves \[1\text{CJ06}\]. Various literature treating these types of waves extend back several centuries ago. The field of nonlinear traveling water waves was born following the observation of Russell \[1\text{Rus44}\] in 1844. It had inspired a myriad of works on such waves in various different regimes.

Mathematically, two-dimensional waves solutions are said to be traveling if the velocity field $\mathbf{u}$ takes the form

$$\mathbf{u}(t, \mathbf{x}) = \tilde{\mathbf{u}}(\mathbf{x} - tc),$$

for some velocity profile $\tilde{\mathbf{u}}$ and $c := (c, 0)$ is the fixed wave speed, similarly for the profile $\eta$, pressure $P$, and density $\rho$. Essentially, we can simply hide the time dependency from the problem by introducing the change of variable $(\mathbf{x} - tc) \mapsto (x, y)$. Under this change of coordinates, the fluid occupies a steady domain; Equation (1.21) now reads

$$\Omega(t) = \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x) \text{ and } -\infty < x < \infty\}. \quad (1.29)$$

In the literature, this is well-known by the name moving frame of reference. Under
this frame of reference, the incompressible Euler equations in (1.19) becomes the Steady incompressible Euler equations

\[
\begin{align*}
\rho(u - c)u_x + \rho vu_y &= -P_x, \\
\rho(u - c)v_x + \rho vv_y &= -P_y - g\rho, \\
(u - c)\rho_x + v\rho_y &= 0.
\end{align*}
\]

(1.30)

1.5 Outline

Next, we provide a brief outline of the thesis. The content of the rest of the thesis comes from two journal articles.

Chapter 2 is based on the author’s published paper [Sin21]. We present a large-amplitude existence theory for 2-D solitary waves moving through a two layer body of water. The fluid is trapped in a domain that is bounded below by an impermeable flat bed and above by a free boundary at constant pressure. Precisely, for any piecewise smooth upstream density distribution and laminar background current, we construct a global curve of solutions. The small-amplitude waves are constructed using a center manifold reduction technique. The large-amplitude theory is obtained through analytical global bifurcation together with refined qualitative properties of the waves.

Chapter 3 of this thesis is based on a recent project that will be submitted for publication. In this work, we investigate orbital stability/instability of two-dimensional capillary-gravity internal water waves. Unlike the domain in Chapter 2, the fluid now is confined in an infinitely long channel bounded above and below by rigid walls. We allow the effects of constant vorticity in each layer. The existence of family of small-amplitude waves is established via a spatial dynamics approach. The orbital
stability/instability is then proved using a variant of the classical Grillakis–Shatah–Strauss approach found in [VWW20].
Chapter 2

Large-amplitude solitary waves in two-layer density stratified water

2.1 Introduction

Internal waves are a common sight in the vicinity of complex coastline structures and narrow passages such as straits or fjords [HM06]. They are a product of the density heterogeneity brought on by variations in temperature and salinity. Together, these effects result in the water being *stratified* into superposed layers, and it is the interfaces between these layers along which internal waves move. While comparatively slow, they can be large in amplitude and carry enormous amounts of energy. In the Lombok Strait, for instance, internal waves have been observed with amplitude exceeding 100 meters and average speed of approximately 1.96 m/s [SMZ05]. They play a pivotal role in general ocean dynamics by transporting and mixing biogenic and non-biogenic components in the water bulk.

This chapter concerns the existence of two-dimensional internal solitary waves in a stratified body of water. Solitary here means that the waves take the form of a spatially localized disturbance moving over a background current. These have been the subject of extensive research, beginning in the 19th century with the famous ob-
servations of Russel [Rus44]. Exact existence results for steady water waves took nearly a century more to prove, but this theory has progressed considerably in recent decades due to advancements in nonlinear functional analysis, harmonic analysis, and PDE theory. In particular, local and global bifurcation techniques have been used to construct small- and large-amplitude traveling water waves in a number of physical settings. Most of these works study the irrotational and homogeneous density case; see, for example, surveys in [Con11b, BGN16]. A major challenge when considering internal waves is that stratification generically creates vorticity, and thus it is necessary to work in the rotational regime.

The first rigorous existence theory in the heterogeneous setting was obtained by Dubreil-Jacotin [DJ34] who constructed small-amplitude periodic waves. Ter-Krikorov [TK60] later showed the existence of infinitesimal solitary water waves as the limit of periodic waves taking period to infinity. For large-amplitude stratified solitary water waves, the first result can be found in the work of Amick [Ami84] and Amick–Turner [AT86]. They considered heterogeneous fluid bounded above and below by infinitely long rigid walls with uniform background current.

Our primary contribution in this chapter is to establish the existence of large-amplitude solitary waves allowing an arbitrary piecewise smooth background current and density distribution. This is done via a global bifurcation theoretic argument that furnishes a locally analytic curve of solutions. We further prove that, following this family to its extreme, one finds waves that come arbitrarily close to horizontal stagnation. This is consistent with the limiting behavior of homogeneous density irrotational solitary waves [AT81b], which are known to terminate at a wave of greatest
Figure 2.1: Configuration of the fluid domain

height with a stagnation point at its crest. A similar result was obtained by Chen, Walsh, and Wheeler [CWW18] for continuously stratified solitary waves. The two-layered stratified case is considerably more complicated, however, and a definitive proof of the stagnation limit requires substantial new analysis.

2.1.1 Governing equation

Let us now formulate the problem mathematically. We consider waves propagating through a two-dimensional body of water. They are traveling in the sense that they evolve by translating to the right with a fixed wave speed \( c > 0 \). By adopting a moving reference frame, all time dependence in the system can therefore be eliminated. Suppose the water is organized into two continuously density stratified layers that are separated by a free boundary. The lower layer is bounded from below by an impermeable bed at \( \{ y = -d \} \) for a fixed \( d > 0 \). The upper layer lies below a free boundary above which is vacuum at constant pressure. We, therefore, write the fluid domain as \( \Omega = \Omega_+ \cup \Omega_- \), where \( \Omega_+ \) is the upper layer

\[
\Omega_+ = \{(x, y) \in \mathbb{R}^2 : \zeta(x) < y < \eta(x)\},
\]

and \( \Omega_- \) is the lower layer

\[
\Omega_- = \{(x, y) \in \mathbb{R}^2 : -d < y < \zeta(x)\}.
\]
This assumes that the internal and upper interfaces are the graphs of the a priori unknown functions \( \zeta \) and \( \eta \), respectively. For solitary waves, we must have that \( \zeta \) and \( \eta \) limit to some far-field heights as \( x \to \pm \infty \). We let \( d_\pm \) denotes the asymptotic thickness of the layer \( \Omega_\pm \), so that \( d = d_+ + d_- \); see Figure 3.1.

Denote by \( (u, v): \overline{\Omega_+} \cup \Omega_- \to \mathbb{R}^2 \) the fluid velocity, \( P: \overline{\Omega} \to \mathbb{R} \) the pressure, and let \( \varrho: \overline{\Omega_+} \cup \overline{\Omega_-} \to \mathbb{R} \) be the density. For physical reasons, we require that \( \varrho \) be strictly positive and that \( y \mapsto \varrho(\cdot, y) \) is non-increasing. It is important to note here that the velocity and density will in general not be continuous over the internal interface.

In the moving frame, traveling water waves are governed by the incompressible steady Euler system:

\[
\begin{align*}
  u_x + v_y &= 0 \\
  \varrho(u - c)u_x + \varrho v u_y &= -P_x \quad \text{in } \Omega, \quad (2.1) \\
  \varrho(u - c)v_x + \varrho v v_y &= -P_y - g \varrho 
\end{align*}
\]

where \( g > 0 \) is the gravitational constant. We also assume mass conservation along the flow which is formulated in the following form

\[
(u - c)\varrho_x + v \varrho_y = 0 \quad \text{in } \Omega. \quad (2.2)
\]

On the boundaries, we impose the standard kinematic and dynamic conditions,

\[
\begin{align*}
  v &= 0 \quad \text{on } y = -d, \\
  v &= (u - c)\eta_x \quad \text{on } y = \eta(x), \\
  v &= (u - c)\zeta_x \quad \text{on } y = \zeta(x), \\
  P &= P_{\text{atm}} \quad \text{on } y = \eta(x), \\
  [P] &= 0 \quad \text{on } y = \zeta(x). \quad (2.3)
\end{align*}
\]

Here \( P_{\text{atm}} \) is the (constant) atmospheric pressure. Note also that the third equation in (2.3) implies that \( v/(u - c) \) is continuous over the internal interface. Throughout
Chapter 2, we use the notation $\mathbf{J} \cdot \mathbf{K} := (\cdot)|_{\Omega^+} - (\cdot)|_{\Omega^-}$ to denote the jump operator over the internal interface $y = \zeta(x)$.

We also require that there is no horizontal stagnation:

$$u - c < 0 \quad \text{in } \overline{\Omega}.$$  

(2.4)

This assumption will be crucial for our reformulation of the problem later. Recall that the *streamlines* are the integral curves of the relative velocity field $(u - c, v)$. The first three equations in (2.3) ensure that the bed, internal interface, and upper boundary are streamlines. As a consequence of (2.4), every streamline extends from $-\infty$ to $\infty$ and is given by the graph of a single-valued function of $x$.

To study solitary waves, we must also specify the background current. This takes the form of the asymptotic conditions

$$(u, v) \to (\hat{u}, 0), \quad \rho \to \hat{\rho}, \quad \eta \to 0, \quad \zeta \to -d_+ \quad \text{as } |x| \to \infty,$$  

(2.5)

where the convergence is uniform in $y$. Here, $\hat{u} := \hat{u}(y)$ is the far field horizontal velocity profile and $\hat{\rho} = \hat{\rho}(y)$ is the far field density profile. It is convenient to replace $\hat{u}$ by the scaled asymptotic relative horizontal velocity $u^*$ given by

$$\hat{u} = c - Fu^*.$$  

(2.6)

The parameter $F > 0$ is referred to as the *Froude number* and can be thought of as the dimensionless wave speed. From the literature of traveling waves, it is expected that there exists a critical Froude number $F_{cr}$ that separates the regimes where periodic waves and solitary waves exist. In particular, (nontrivial) solitary waves must be *supercritical* in that $F > F_{cr}$. This fact has recently been proved
for the case of homogeneous density by Kozlov, Lokharu, and Wheeler [KLW20]. A rigorous definition of $F_{cr}$ for the present system is given in Section 2.3.

Lastly, we recall some terminology describing the qualitative properties of water waves. A laminar flow is a wave whose streamlines are all parallel to the bed; these form the class of trivial solutions of the problem. A solitary wave of elevation is a wave where each streamline lies above its limiting height upstream and downstream. In particular, given our coordinates system, this implies $\eta$ is strictly positive. A traveling wave is called symmetric provided that $u$ and $\eta$ are even in $x$ while $v$ is odd. Finally, a symmetric waves is monotone if the slope of the streamlines, $v/(u - c)$, is negative to the left of the crest at $x = 0$ and above the bed.

2.1.2 Statement of results

Our first contribution is a systematic existence theory for large-amplitude stratified solitary waves with arbitrary piecewise smooth density distribution and horizontal velocity profile at infinity.

**Theorem 2.1** (Large-amplitude solitary waves). Fix Hölder exponent $\alpha \in (0, 1)$, wave speed $c > 0$, far-field depths $d_+, d_- > 0$, gravitational constant $g > 0$. For any (strictly positive) asymptotic relative velocity and density profiles

$$u^*, \hat{\varrho} \in \mathcal{C}^{8+\alpha}([-d, -d_+], \mathbb{R}_+) \cap \mathcal{C}^{8+\alpha}([-d_+, 0], \mathbb{R}_+),$$

there exists a continuous global curve

$$\mathcal{C} = \{(u(s), v(s), \eta(s), \zeta(s), F(s)) : s \in (0, \infty)\}$$

of solitary wave solutions to (2.1)–(2.5), exhibiting the regularity

$$u(s), v(s) \in \mathcal{C}^{8+\alpha}(\Omega_+(s)) \cap \mathcal{C}^{8+\alpha}(\Omega_-(s)), \quad \eta(s), \zeta(s) \in \mathcal{C}^{9+\alpha}(\mathbb{R})$$

18
where $\Omega(s) := \Omega_+(s) \cup \Omega_-(s)$ is the corresponding fluid domain. The global solution curve $C$ enjoys the following properties:

(a) (Stagnation limit) Following $C$, we encounter waves that are arbitrarily close to having horizontal stagnation:

$$\lim_{s \to \infty} \inf_{\Omega(s)} |c - u(s)| = 0.$$ 

(b) (Critical laminar flow) The curve begins at the critical laminar flow:

$$\lim_{s \to 0} (u(s), v(s), \eta(s), \zeta(s), F(s)) = (c - F_{cr} u^*, 0, 0, -d_+, F_{cr}).$$

(c) (Symmetry and monotonicity) All solutions on $C$ are symmetric waves of elevation, monotone, and supercritical.

**Remark 2.2.** Let us make a few remarks.

(i) This result assumes a single discontinuity in the far-field density profile. In fact, the theory easily extends to finitely many discontinuities with the only cost being more cumbersome notation. The resulting waves would then be organized into many layers.

(ii) The $C^{8+\alpha}$ regularity asked for here is almost certainly much more than necessary. We impose it in order to satisfy the hypothesis of the center manifold reduction method from [CWW19], which in turn only needs it due to the technical lemma [AT94, Lemma 2.1]. We conjecture that the regularity of $u^*$ and $\dot{\rho}$ in each layer can be relaxed to $C^{2+\alpha}$, which will then give solutions with

$$u, v \in C^{2+\alpha}(\Omega_+) \cap C^{2+\alpha}(\Omega_-), \quad \eta, \zeta \in C^{3+\alpha}(\mathbb{R}).$$

(2.10)
A proof of this fact would require a lengthy digression into the details of those
two papers, and so we do not pursue it here. Following the approach in \[\text{AW19}\],
moreover, one expects that it should be possible to take $u^*$ to be merely Lipschitz
continuous in each layer.

We also establish a number of qualitative properties of stratified solitary waves.
These are of independent interest but also crucially important to the proof of Theo-
rem 2.1. We list here the two most significant, but others can be found in Section 2.4.

The first result states that supercritical solitary waves of elevation are necessarily
symmetric and monotone. This is achieved through a moving planes argument in the
spirit of Li \[\text{Li91}\] and Maia \[\text{Mai97}\].

**Theorem 2.3 (Symmetry).** Let $(u, v, \eta, \zeta, F)$ be a supercritical wave of elevation that
solves (2.1)–(2.4) and enjoys the regularity (2.10) with

$$
\|u\|_{C^2(\Omega_+) \cap C^2(\Omega_-)}, \|v\|_{C^2(\Omega_+) \cap C^2(\Omega_-)}, \|\eta\|_{C^3(\mathbb{R})}, \|\zeta\|_{C^3(\mathbb{R})} < \infty.
$$

Suppose that

$$(u, v) \to (\hat{u}, 0)$$

uniformly as $x \to +\infty$ or as $(x \to -\infty)$, then after an appropriate translation, the
wave is a monotone and symmetric solitary wave.

The second theorem gives a uniform upper bound on the velocity for stratified
solitary waves in terms of a lower bound on the Froude number and a bound away
from horizontal stagnation.

**Theorem 2.4 (Velocity bound).** Let $(u, v, \eta, \zeta, F)$ be a solution to (2.1)–(2.6) that
enjoys the regularity (2.10) and satisfies

\[
F \geq F_0 > 0, \quad \sup_{\Omega} (u - c) \leq -\delta < 0.
\]

Then,

\[
\sup_{\Omega} ((u - c)^2 + v^2) < C,
\]

for a constant \(C = C(F_0, \delta, u^*, \varrho) > 0\).

Note that through Bernoulli’s law, the above theorem can also be used to control the pressure. A result of this type is proved by Chen, Walsh, Wheeler [CWW18] for one-layer stratified waves based on Varvaruca’s [Var09] treatment of the constant density case. That method, however, is not sufficient for the present setting, as the maximum principle argument it relies on struggles with the discontinuity of the velocity across the layers. Our approach combines pressure bounds in the bulk with the “almost monotonicity formula” of Caffarelli–Kenig–Jerison [CJK02] to control the velocity near the internal interface.

Finally, we provide an upper bound on the Froude number in terms of a bound away from stagnation along the crest line; see Theorem 2.14. An analogous result was obtained by Chen, Walsh, and Wheeler [CWW18], which forms the basis for our argument. Without this bound, one must allow for the possibility that the Froude number blows up along the global bifurcation curve even though stagnation is never approached. This alternative was in the original work of Wheeler [Whe13] on solitary homogeneous density waves but eliminated in the paper [CWW18].
2.1.3 Outline

Let us now outline the general structure of the remaining portion of this chapter while explaining the main mathematical difficulties and how we will approach them.

We begin in Section 2.2 by non-dimensionalizing the governing equations. Applying the Dubreil-Jacotin transformation sends the fluid domain $\Omega$ to a slitted rectangular strip. In these variables the incompressible steady Euler system becomes a quasilinear elliptic PDE coupled with nonlinear transmission boundary conditions. Written as an abstract operator equation, it takes the form

$$F(w, F) = 0,$$

where $w$ is a new unknown measuring the deviation of the streamlines relative to the background current. To lay the ground work for the small-amplitude theory, in Section 2.3 we investigate the linearized operator at $w = 0$. Restricting its domain to laminar flows, we arrive at a Sturm–Liouville type problem with a transmission condition. It is shown that there exists a critical value of the Froude number, $F = F_{cr}$, for which 0 is the principal eigenvalue.

As further preparation for the existence theory, in Section 2.4 we prove the qualitative results mentioned above. We also present a result on asymptotic monotonicity and nodal pattern of the solutions. The main tools used here are the maximum principle as well as integral identities.

Section 2.5 is where the small-amplitude existence theory is established. These solutions lie on a local curve, denoted by $\mathcal{C}_{loc}$, that bifurcates from $(w, F) = (0, F_{cr})$. For periodic waves, small solutions are usually found via the classical Lyapunov–Schmidt reduction. However, this method cannot be applied directly here because $F_w(0, F_{cr})$...
is not Fredholm as a consequence of the unboundedness of the domain and the definition of $F_{cr}$. This analytical challenge is intrinsic to the study of (small-amplitude) solitary waves. For constant density rotational waves, Hur [Hur08] constructed solutions using a Nash–Moser technique that generalized Beale’s [Bea77] treatment of the irrotational case. Considering the same problem, Groves and Wahlén in [GW08] used a Hamiltonian spatial dynamics approach. This argument was adapted by Chen, Walsh, and Wheeler [CWW18] to the one-layer continuously stratified regime, and by Wang [Wan17] for two-phase flows with constant density in each layer. In the present work, however, we employ a center manifold reduction “without a phase space” based on the recent paper [CWW19].

In Section 2.6, we continue $C_{loc}$ globally to obtain the curve $C$ that extends into the large-amplitude regime. Again, the unboundedness of the domain presents a significant obstruction to standard bifurcation theoretic techniques. For example, it is not a priori clear that $F^{-1}(0)$ is locally pre-compact or that $F$ is locally proper. This is not just a technical concern. Indeed, it is well-known that in other stratified regimes, solitary waves may broaden into an infinitely long “table top”; see, for example, [TVB88]. Because these waves remain bounded in any Hölder space but do not converge to a localized solution, this scenario implies a lack of compactness for the zero set of $F$.

The classical strategy for constructing large-amplitude solitary waves is to view them as the limit of periodic waves as the period tends to infinity. This is done, for example, by Amick and Toland [AT81a] in their study of the constant density irrotational wave case. They first construct global families of periodic waves, then take
the period to infinity using a uniform estimates and an application of the Whyburn lemma. This results in a global connected set of solutions.

Our approach is based on the analytic global bifurcation theory introduced by Chen, Walsh, and Wheeler [CWW18] which is a variant of the classical work of Dancer [Dan73a, Dan73b] and Buffoni–Toland [BT03]. Essentially, we treat the loss of compactness as an alternative and show that it must manifest as the broadening phenomena mentioned above. In particular, we prove that if a bounded sequence of solutions is not pre-compact, then there is a translated subsequence converging (locally) to a monotone heteroclinic solution whose far-field limits are distinct. In the literature, these types of waves are called (monotone) “fronts” or “bores.” Using the qualitative theory, we can then rule out this possibility leaving only the stagnation limit.

Lastly, for the convenience of the reader, Appendix A contains some results from the literature that are drawn upon throughout the thesis.

2.2 Formulation

In this section, we introduce several reformulations of the problem that will make it more amenable to analysis. We also record a number of notational conventions used throughout Chapter 2.

2.2.1 Non-dimensionalization

Let us denote the density along the free surface as follows:

\[ \rho_0 := \hat{\rho}(0). \]  

(2.11)

Next, we normalize the \( u^* \) to satisfy
\[
\int_{-d}^{0} \sqrt{\bar{\varrho}(y)} u^*(y) \, dy = \sqrt{g \varrho_0 d^3}.
\] (2.12)

In addition, we consider the \textit{(relative) pseudo-volumetric mass} \( m > 0 \):

\[
m := \int_{-d}^{0} \sqrt{\bar{\varrho}(x,y)} (c - u(x,y)) \, dy.
\] (2.13)

One can check that \( m \) is independent of \( x \). Letting \( |x| \to \infty \), we obtain

\[
m = \int_{-d}^{0} \sqrt{\bar{\varrho}(y)} (c - \bar{u}(y)) \, dy = F \int_{-d}^{0} \sqrt{\bar{\varrho}(y)} u^*(y) \, dy.
\]

Using the above equation and (2.12), we see that

\[
\frac{g \varrho_0 d^3}{m^2} = \frac{1}{F^2}.
\] (2.14)

We non-dimensionalize the coordinates using the asymptotic depth \( d \) as the characteristic length scale, which gives us

\[
(\bar{x}, \bar{y}) := \frac{1}{d} (x, y), \quad \bar{\eta}(\bar{x}) := \frac{1}{d} \eta(x), \quad \bar{\zeta}(\bar{x}) := \frac{1}{d} \zeta(x).
\]

Likewise, the density is rescaled using \( \varrho_0 \) in (2.11)

\[
\bar{\varrho}(\bar{x}, \bar{y}) = \frac{1}{\varrho_0} \varrho(x,y), \quad \bar{\varrho}(\bar{y}) := \frac{1}{\varrho_0} \varrho(y),
\]

and the velocity is non-dimensionalized via the Froude number

\[
\bar{u}(\bar{x}, \bar{y}) := \frac{\sqrt{\varrho_0 d}}{m} u(x,y), \quad \bar{v}(\bar{x}, \bar{y}) := \frac{\sqrt{\varrho_0 d}}{m} v(x,y),
\]

\[
\bar{c} := \frac{\sqrt{\varrho_0 d}}{m} c, \quad \bar{\bar{u}}(\bar{y}) := \frac{\sqrt{\varrho_0 d}}{m} \bar{u}(y).
\]

Finally, the pressure is rescaled by taking

\[
\bar{P}(\bar{x}, \bar{y}) := \frac{d^2}{m^2} (P(x,y) - P_{\text{atm}}).
\] (2.15a)
Rewriting (2.1) and (2.2), we finally obtain the non-dimensionalized system

\[
\begin{aligned}
\tilde{u}_\tilde{x} + \tilde{v}_\tilde{y} &= 0 \\
\varrho(\tilde{u} - \tilde{c})\tilde{u}_\tilde{x} + \varrho\tilde{v}\tilde{u}_\tilde{y} &= -\tilde{P}_x \\
\varrho(\tilde{u} - \tilde{c})\tilde{u}_\tilde{x} + \varrho\tilde{v}\tilde{u}_\tilde{y} &= -\tilde{P}_y - \frac{1}{\tilde{F}^2}\varrho \\
(\tilde{u} - \tilde{c})\tilde{v}_\tilde{x} + \tilde{v}\varrho\tilde{y} &= 0
\end{aligned}
\]

in \(\tilde{\Omega}\),

where \(\tilde{\Omega}\) is the rescaled domain:

\[
\tilde{\Omega} := \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : -1 < \tilde{y} < \tilde{\zeta}(\tilde{x}) \cup \tilde{\zeta}(\tilde{x}) < y < \tilde{\eta}(\tilde{x})\}.
\]

The boundary conditions after rescaling read

\[
\begin{aligned}
\tilde{v} &= 0 \quad \text{on } \tilde{y} = -1, \\
\tilde{v} &= (\tilde{u} - \tilde{c})\tilde{\eta}_\tilde{x} \quad \text{on } \tilde{y} = \tilde{\eta}(\tilde{x}), \\
\tilde{v} &= (\tilde{u} - \tilde{c})\tilde{\zeta}_\tilde{x} \quad \text{on } \tilde{y} = \tilde{\zeta}(\tilde{x}), \\
P &= P_{\text{atm}} \quad \text{on } \tilde{y} = \tilde{\eta}(\tilde{x}), \\
\\left[ P \right] &= 0 \quad \text{on } \tilde{y} = \tilde{\zeta}(\tilde{x}).
\end{aligned}
\] (2.16)

Moreover, the asymptotic condition in (2.5) become

\[
(\tilde{u}, \tilde{v}) \to (\tilde{u}, 0), \quad \tilde{v} \to \tilde{\sigma}, \quad \tilde{\eta} \to 0, \quad \tilde{\zeta} \to -\frac{d_+}{d} \quad \text{as } |\tilde{x}| \to \infty. \tag{2.17}
\]

Combining (2.14) and (2.5) gives us

\[
\tilde{u}(\tilde{y}) - \tilde{c} = -\frac{1}{\sqrt{gd}}\tilde{u}^*(\tilde{y}). \tag{2.18}
\]

Note that this means that the asymptotic state in (2.17) is independent of \(F\).

For the sake of cleaner notation, in what follows we will use the dimensionless variables but drop the tildes.
2.2.2 Stream function formulation

Let us introduce the following relative pseudo stream function $\psi$:

$$
\psi_x = -\sqrt{\varrho} v, \quad \psi_y = \sqrt{\varrho}(u - c).
$$

In comparison to the classical definition of the stream function, the pseudo version introduced by Yih [Yih65] takes into account the effects of stratification by including a factor of $\sqrt{\varrho}$. The existence of $\psi$ is guaranteed by the incompressibility of the flow and the fact that density is constant along the streamlines. Indeed, from this definition we see that the streamlines are precisely the level sets of $\psi$. In particular, the kinematic boundary condition tells us that $\psi$ is constant on the surface, internal interface, and bed. Without loss of generality, we may set $\psi = 0$ on $\{y = \eta(x)\}$. Thanks to equation (2.13) together with the rescaling of coordinate, density and velocities, we then have $\psi = 1$ on the floor $\{y = -1\}$. Let us denote its value on $\{y = \zeta(x)\}$ by $-\hat{p}$; the reason for this will become clear in the next subsection. Observe that the assumption of no horizontal stagnation (2.4) becomes:

$$
\psi_y < 0 \quad \text{in } \Omega. \tag{2.19}
$$

Via the mass conservation in (2.2), we know that density is transported, hence constant, along each streamline. That allows us to rewrite the density in terms of $\psi$, otherwise known as streamline density function:

$$
\varrho(x, y) = \rho(-\psi(x, y)).
$$

Naturally, $\rho$ is determined by the limiting density profile $\hat{\varrho}$. It is easily verified that the regularity assumption (2.7) implies $\varrho \in C^{8+\alpha}([-1, \hat{p}]) \cap C^{8+\alpha}([\hat{p}, 0])$. As the
stratification here is assumed to be stable, moreover, we have that $\rho' \leq 0$ in upper and lower domain.

By Bernoulli’s law, we know that the quantity

$$E = \frac{\rho}{2}((u - c)^2 + v^2) + P + \frac{1}{F^2} \varrho y$$

(2.20)

is constant along each streamline. This fact together with the no horizontal stagnation implies that there exists a so-called Bernoulli function $\beta$ such that

$$\frac{dE}{d\psi} = -\beta(\psi) \quad \text{in } \Omega.$$ (2.21)

Since it is constant on streamlines, all of which extend fully upstream and downstream, $\beta$ can be reconstructed from the background current and density profile; see Remark 2.5 below. In particular, for the regularity assumed in (2.7), we find that $\beta \in C^{7+\alpha}([0, -\hat{p}]) \cap C^{7+\alpha}([-\hat{p}, 1])$. Note that the somewhat odd looking choice to view $\rho$ as a function of $-\psi$ while $\beta$ is a function of $\psi$ is done here to be in accordance with previous results in the literature.

Following [CW16, Lemma A.2], the governing equations in (2.1) with the absence stagnation can be reformulated as Yih’s equation:

$$\Delta \psi - \frac{1}{F^2} y \rho'(-\psi) + \beta(\psi) = 0 \quad \text{in } \Omega.$$ (2.22)

Likewise, the boundary conditions become

$$\begin{cases}
\psi = 0 & \text{on } y = \eta(x), \\
|\nabla \psi|^2 + \frac{2}{F^2} \varrho(y + 1) = Q^n & \text{on } y = \eta(x), \\
\|\nabla \psi\|^2 + \frac{2}{F^2} \|\varrho\| (y + 1) = Q^c & \text{on } y = \zeta(x), \\
\psi = 1 & \text{on } y = -1,
\end{cases}$$ (2.23)
where 
\[ Q^n := 2 \left( E + \frac{1}{F^2} \eta \right) \bigg|_{y = \eta(x)} \quad \text{and} \quad Q^\xi := 2 \left( [E] + \frac{1}{F^2} [\varrho] \right) \bigg|_{y = \zeta(x)} \]
are constants. Lastly, the asymptotic conditions in (2.5) now read
\[ \nabla \psi \to \left( 0, \sqrt{\varrho} (\hat{u} - c) \right), \quad \eta \to 0, \quad \zeta \to -d_+, \quad \varrho \to \hat{\varrho} \quad \text{as} \quad |x| \to \infty. \] (2.24)

For later use, we introduce the convention that \( \psi_{\pm} \) denotes the restriction of \( \psi \) to \( \Omega_{\pm} \).

Via the continuity of the pressure and Bernoulli’s law on the internal interface, we have
\[ \frac{1}{2} (|\nabla \psi_+|^2 - |\nabla \psi_-|^2) = \frac{[\varrho]}{F^2} y - [E]. \] (2.25)

**Remark 2.5.** The Bernoulli function \( \beta \) can be expressed in terms of \( \hat{\varrho} \) and \( \hat{u} \) as follows. Letting \( \hat{y}(p) \) be the asymptotic \( y \)-coordinate of the streamline \( \{ \psi = -p \} \), and defining \( \hat{U}(p) := \hat{u}(\hat{y}(p)) \), by (2.24) we have:
\[ \hat{y}(p) = \int_{-1}^{p} \frac{1}{\sqrt{\rho(s)}} \left( \frac{1}{c - \hat{U}(s)} \right) ds - 1. \] (2.26)

Solving for \( \beta \) in equation (2.22), sending \( x \to \pm \infty \) and applying (2.24), we obtain
\[ \beta(-p) = \left( \frac{1}{F^2} \hat{y} - \frac{1}{2} \left( \hat{U} - c \right)^2 \right) \rho_p + \rho \left( \hat{U} - c \right) \hat{U}_p. \]

### 2.2.3 Height function formulation

The fact that the Yih’s equation is scalar is already a considerable simplification of the system in (2.1). However, due to the free boundary, the domain \( \Omega \) remains a priori unknown which presents a serious difficulty for existence theory. To get around this, we employ the Dubreil-Jacotin transformation to send the domain \( \Omega \) into a fixed slitted rectangular strip \( R \):
\[ (x, y) \mapsto (x, -\psi) =: (q, p), \] (2.27)
\[ \psi = 0 \]
\[ \psi = -\hat{p} \]
\[ \psi = 1 \]

Figure 2.2: The fluid domain with (non-dimensionalized) streamline values labeled. The thick lines represent the upper and internal free boundaries. Also depicted are the height function \( h \) and asymptotic height \( H \).

where

\[ R := R^+ \cup R^- = \{ (q, p) \in \mathbb{R}^2 : p \in (-1, \hat{p}) \} \cup \{ (q, p) \in \mathbb{R}^2 : p \in (\hat{p}, 0) \}. \]

The free boundary \( \{ y = \eta(x) \} \), floor \( \{ y = 0 \} \), and internal interface \( \{ y = \zeta(x) \} \) are mapped to \( T := \{ p = 0 \} \), \( B := \{ p = -1 \} \) and \( I := \{ p = \hat{p} \} \), respectively. The new coordinates \( (q, p) \) are often referred to as *semi-Lagrangian variables*.

Define the *height function* which measures the height above the flat ocean floor,

\[ h(q, p) := y + 1 \geq 0 \quad \text{in} \; \overline{R}. \]

See Figure 2.2 for an illustration. Via elementary computations, we obtain

\[ h_q = \frac{v}{u - c}, \quad h_p = \frac{1}{\sqrt{\rho}(c - u)}, \]

where the left-hand side is evaluated at \( (q, p) \) and the right-hand side is evaluated at \( (x, y) \). As a consequence, the absence of horizontal stagnation now translates to

\[ h_p > 0. \]

Furthermore, the asymptotic conditions in (2.5) become

\[ h(q, p) \to H(p), \quad h_q(q, p) \to 0, \quad h_p(q, p) \to H_p(p) \quad \text{as} \; |q| \to \infty. \]
From equations (2.18), (2.24), and (2.26), we can view the asymptotic height function $H$ (downstream and upstream) as the solution to the following boundary value problem

\[
\begin{cases}
H_p(p) = \frac{1}{\sqrt{\rho(c - \hat{u})}} \bigg|_{y = H(p) - 1} & \text{in } [-1, \hat{p}) \text{ and } [\hat{p}, 0), \\
H(-1) = 0, & H(\hat{p}) = -\frac{d_+}{d} + 1, H(0) = 1.
\end{cases}
\]

Yih’s equation in (2.22) and the boundary conditions in (2.23) can be written as the following quasi-linear PDE with transmission boundary condition:

\[
\begin{cases}
\left( -\frac{1 + h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \right)_p + \left( \frac{h_q}{h_p} \right)_q - \frac{1}{F^2} \rho_p (h - H) = 0 & \text{in } R, \\
\frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2} + \frac{1}{F^2} \rho (h - 1) = 0 & \text{on } T, \\
\left[ \frac{1 + h_q^2}{2h_p^2} \right] - \left[ \frac{1}{2H_p^2} \right] + \frac{1}{F^2} \left[ \rho \right] (h - H) = 0 & \text{on } I, \\
h = 0 & \text{on } B.
\end{cases}
\] (2.28)

The PDE in (2.28) is elliptic as long as $\inf_R h_p > 0$. The boundary condition on $I$ is of transmission type, while that on $T$ is oblique. Observe that for stably stratified flow, $-\rho_p \geq 0$, and hence the maximum principle cannot be applied directly. This is a well-known feature of the problem that we will have to contend with at several stages of the analysis.

### 2.2.4 Flow force

The flow force is defined to be the

\[
\mathcal{J}(x) = \int_{-1}^{n(x)} \left( P + \varrho (u - c)^2 \right) dy.
\] (2.29)

One can check that this quantity is independent of $x$ if evaluated at a solution of the Euler equation. Rewritten in semi-Lagrangian variables, it takes the form

\[
\mathcal{J}(h) = \int_{-1}^0 \left( \frac{1 - h_q^2}{2h_p^2} + \frac{1}{2H_p^2} - \frac{1}{F^2} \rho (h - H) - \frac{1}{F^2} \int_0^p \rho H_p dp' \right) h_p dp,
\] (2.30)
where now we are viewing it as a functional acting on $h$ with $H$. We will make use of the flow force in many ways. For instance, in Section 2.5 of small-amplitude theory, it gives rise to a conserved quantity on the center manifold that is essential to the construction. More generally, the flow force is one of the three conserved quantities that determine the set of conjugate flows for the system; see [Ben71].

### 2.2.5 Function spaces and the operator equation

In this subsection, we introduce the function spaces that we shall be working in. For a generic domain $D \subset \mathbb{R}^2$, non-negative integer $k$, and $\alpha \in [0, 1)$, we define

- $C^{k+\alpha}(D) := \{ f \in C^k(D) : \| \phi f \|_{C^{k+\alpha}} < \infty \text{ for all } \phi \in C_0^\infty \}$,
- $C^{k+\alpha}_b(D) := \{ f \in C^k(D) : \| f \|_{C^{k+\alpha}} < \infty \}$,
- $C^{k+\alpha}_0(D) := \left\{ f \in C^{k+\alpha}_b(D) : \lim_{r \to \infty} \sup_{|x| = r} |\partial^j f(x)| = 0 \text{ for all } 0 \leq j \leq k \right\}$.

In particular, we emphasize that $C^{k+\alpha}$ refers to locally Hölder continuous functions.

The center manifold reduction carried out in Section 2.5 requires us to work with exponentially weighted Hölder space. For $\nu \in \mathbb{R}$, define

$$C^{k+\alpha}_\nu(D) := \left\{ f \in C^{k+\alpha}(D) : \| f \|_{C^{k+\alpha}_\nu(D)} < \infty \right\},$$

(2.31)

where the norm

$$\| f \|_{C^{k+\alpha}_\nu(D)} := \sum_{|\beta| \leq k} \sech(\nu q) \| \partial^\beta f \|_{C^0(D)} + \sum_{|\beta| = k} \sech(\nu q) |\partial^\beta f|_\alpha \|_{C^0(D)},$$

(2.32)

and $| \cdot |_\alpha$ is the usual local Hölder seminorm.

Finally, let $w = w(q, p)$ be

$$w(q, p) := h(q, p) - H(p),$$

32
which measures the deviation of the height function \( h \) in the near-field from its limiting height \( H \) at \( q = \pm \infty \). Note that the decay of \( w \) at infinity implies that the asymptotic conditions are satisfied. One can see that the height equation (2.28) can be formulated in terms of \( w \) as follows:

\[
\begin{cases}
-\frac{1 + w_q^2}{2(H_p + w_p)^2} + \frac{1}{2H_p^2} + \left( \frac{w_q}{H_p + w_p} \right)_q - \frac{1}{F^2 \rho_p} w = 0 & \text{in } R, \\
\frac{1 + w_q^2}{2(H_p + w_p)^2} - \frac{1}{2H_p^2} + \frac{1}{F^2 \rho} w = 0 & \text{on } T, \\
\left[ \frac{1 + w_q^2}{2(H_p + w_p)^2} \right] - \left[ \frac{1}{2H_p^2} \right] + \frac{1}{F^2} \left[ \rho \right] w = 0 & \text{on } I, \\
w = 0 & \text{on } B.
\end{cases}
\] (2.33)

Define the following Banach Spaces,

\[
\begin{align*}
X & := \{ w \in \mathcal{C}_{b,e}^{\alpha}((R^+) \cap C_{b,e}^{\alpha}(R^-) \cap C_0^0(R) \cap C_0^8(R^+) \cap C_0^8(R^-) : w = 0 \text{ on } B \}, \\
Y_1 & := \mathcal{C}_{b,e}^{\alpha}(R^+) \cap C_{b,e}^{\alpha}(R^-) \cap C_0^0(R) \cap C_0^0(R^+) \cap C_0^0(R^-), \\
Y_2 & := \mathcal{C}_{b,e}^{\alpha}(T) \cap C_0^7(R^+) \cap C_0^7(R^-),
\end{align*}
\] (2.34)

and set \( Y := Y_1 \times Y_2 \). Throughout the section, the subscript “e” is used to indicates that the functions are even in \( q \). We write (2.28) as an operator equation acting on functions in the Banach spaces above as

\[
\mathcal{F}(w,F) = 0,
\]

for the mapping

\[
\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : U \subset X \times \mathbb{R} \to Y
\]
given by

\[ F_1(w, F) := \left( -\frac{1 + w_q^2}{2(H_p + w_p)^2} + \frac{1}{2H_p^2} \right)_p + \left( \frac{w_q}{H_p + w_p} \right)_q - \frac{1}{F^2} \rho w, \]

\[ F_2(w, F) := \frac{1 + w_q^2}{2(H_p + w_p)^2} - \frac{1}{2H_p^2} + \frac{1}{F^2} \rho w, \]  \tag{2.35}

\[ F_3(w, F) := \left[ \frac{1 + w_q^2}{2(H_p + w_p)^2} \right] - \left[ \frac{1}{2H_p^2} \right] + \frac{1}{F^2} \left[ \rho \right] w. \]

We are looking for solutions that belong in the open subset

\[ \mathcal{U} := \left\{ (w, F) \in X \times \mathbb{R} : \inf_{R} (w_p + H_p) > 0, F > F_{cr} \right\} \subset X \times \mathbb{R}. \]  \tag{2.36}

Here \( F_{cr} \) is the critical Froude number which will be defined later in Section 2.3.1.

Since \( F \) is a rational function of \( w \) and its derivatives, then it is a real-analytic mapping from \( \mathcal{U} \) to \( Y \).

## 2.3 Linearized operators

This section is devoted to investigating the linearized operator \( F_w(w, F) \). The results presented in Section 2.3.1 concern a Sturm–Liouville-type problem related to the case \( w = 0 \); this will be used to define the critical Froude number \( F_{cr} \). Section 2.3.2 analyzes the linearized operator at an arbitrary \( (w, F) \), which plays a crucial role in proving the local and global existence theory in Sections 2.5 and 2.6.

### 2.3.1 Sturm–Liouville type problems

Let us first consider the spectrum of the transversal linearized operator at the laminar flow \( (w, F) = (0, F) \), by which we mean the restriction of \( F_w(0, F) \) to functions that are independent of \( q \). Thus, we obtain the following Sturm–Liouville-type problem...
\[
\begin{cases}
\left( \frac{\dot{w}_p}{H_p^3} \right)_p - \mu \rho_p \dot{w} = -\nu \frac{\dot{w}}{H_p} & \text{in } (-1, \hat{p}) \text{ and } (\hat{p}, 0), \\
- \frac{\dot{w}_p}{H_p^3} + \mu \rho \dot{w} = 0 & \text{on } p = 0, \\
- \left[ \frac{\dot{w}_p}{H_p^3} \right] + \mu \left[ \rho \right] \dot{w} = 0 & \text{on } p = \hat{p}, \\
\dot{w} = 0 & \text{on } p = -1,
\end{cases}
\] (2.37)

where \( \mu := 1/F^2 \) and \( \nu \) is the eigenvalue.

Heuristically, we expect all spatially localized gravity waves to be supercritical in that their wave speed is faster than that of any linear periodic wave. This idea underlies our approach to constructing small-amplitude solitary waves in Section 2.5. By separating variables, we see that the linearized problem at \( w = 0 \) admits periodic solutions provided (2.37) has a positive eigenvalue. Thus we wish to identify a critical Froude number (which recall is the non-dimensionalized wave speed) at which the transversal linearized problem at the background laminar flow has a 0 as its principal eigenvalue.

With that in mind, take \( \nu = 0 \) in (2.37) above and look for the largest value of \( \mu \) such that

\[
\begin{cases}
\left( \frac{\dot{w}_p}{H_p^3} \right)_p - \mu \rho_p \dot{w} = 0 & \text{in } [-1, \hat{p}) \text{ and } (\hat{p}, 0), \\
- \frac{\dot{w}_p}{H_p^3} + \mu \rho \dot{w} = 0 & \text{on } p = 0, \\
- \left[ \frac{\dot{w}_p}{H_p^3} \right] + \mu \left[ \rho \right] \dot{w} = 0 & \text{on } p = \hat{p}, \\
\dot{w} = 0 & \text{on } p = -1,
\end{cases}
\] (2.38)

has a nontrivial solution. To achieve this, we will consider the solution \( \Phi(p; \mu) \) to the
initial value problem

\[
\begin{cases}
\left(\frac{\Phi_p}{H_p^3}\right)_p = \mu \rho_p \Phi & \text{in } (-1, \hat{p}), \\
\Phi = 0 & \text{on } p = -1, \\
\Phi_p = 1 & \text{on } p = -1.
\end{cases}
\] (2.39a)

Using the jump condition on \(\{p = \hat{p}\}\) in (2.38), we continue this solution into the upper layer corresponding to \(p \in (\hat{p}, 0)\). We denote this extended function by \(\Upsilon\), which is thus determined by

\[
\begin{cases}
\left(\frac{\Upsilon_p}{H_p^3}\right)_p = \mu \rho_p \Upsilon & \text{in } (\hat{p}, 0), \\
\Upsilon = \Phi & \text{on } p = \hat{p}, \\
\Upsilon_p = H_p^3(\hat{p}^+) \frac{\Phi_p(\hat{p}^-)}{H_p^3(\hat{p}^-)} + \mu [\rho] H_p^3(\hat{p}^+ \Phi) & \text{on } p = \hat{p},
\end{cases}
\] (2.39b)

where \(\hat{p}^\pm := \hat{p}|_{\Omega\pm}\). Finally, to satisfy the Bernoulli condition on \(\{p = 0\}\), we introduce the function

\[
A(\mu) = -\frac{\Upsilon_p(0; \mu)}{H_p^3(0)} + \mu \rho(0) \Upsilon(0; \mu).
\] (2.40)

The idea here is that if \(A(\mu) = 0\), then

\[
\Psi(p; \mu) := \begin{cases}
\Upsilon(p; \mu) & \text{for } \hat{p} \leqslant p \leqslant 0 \\
\Phi(p; \mu) & \text{for } -1 \leqslant p < \hat{p}
\end{cases}
\] (2.41)

solves the IVP (2.39), and hence is an eigenfunction for (2.37) corresponding to the eigenvalue \(\nu = 0\).

**Lemma 2.6.** There exists a unique \(\mu_{\text{cr}} > 0\) such that all of the following hold.

(a) For \(\mu = \mu_{\text{cr}}\), the problem (2.38) has a nontrivial solution \(\dot{w} = \Psi(p; \mu_{\text{cr}})\).

(b) For \(0 \leqslant \mu \leqslant \mu_{\text{cr}}\), \(\Psi(p; \mu) > 0\) for \(-1 < p \leqslant 0\) and \(\Psi_p(p, \mu) > 0\) for \(-1 \leqslant p < \hat{p}\) and \(\hat{p} < p \leqslant 0\).
(c) For $0 \leq \mu < \mu_{cr}$, $A(\mu) < 0$.

**Proof.** Note that (2.38) has a non-trivial solution provided $B(\mu) = \mu \rho(0) H^3_p(0)$ where we define

$$
B(\mu) := \frac{\Upsilon_p(0; \mu)}{\Upsilon(0; \mu)}.
$$

(2.42)

Setting $\mu = 0$ and integrating the first equation in (2.39a) and (2.39b), we obtain

$$
\Psi_p(p; 0) = \begin{cases}
\frac{H^3_p(p)}{H^3_p(-1)} & \text{for } p \in [-1, \hat{p}), \\
\frac{\Upsilon_p(\hat{p}^+) H^3_p(\hat{p})}{H^3_p(\hat{p}^+)} & \text{for } p \in (\hat{p}, 0].
\end{cases}
$$

(2.43)

Anti-differentiating (2.43) once more then gives

$$
\Psi(p; 0) = \begin{cases}
\frac{1}{H^3_p(-1)} \int_{-1}^p H^3_p(s) ds & \text{for } p \in [-1, \hat{p}), \\
\Upsilon(\hat{p}^+) + \frac{\Upsilon_p(\hat{p}^+) H^3_p(\hat{p})}{H^3_p(\hat{p}^+)} \int_{\hat{p}}^p H^3_p(s) ds & \text{for } p \in (\hat{p}, 0].
\end{cases}
$$

(2.44)

Inserting this into (2.42) yields the formula

$$
B(0) = \frac{\Upsilon_p(\hat{p}^+) H^3_p(0)}{H^3_p(\hat{p}^+) \Upsilon(\hat{p}^+) + \Upsilon_p(\hat{p}^+) \int_{\hat{p}}^0 H^3_p(p) dp}.
$$

(2.45)

We claim that

$$
B(0) > 0.
$$

(2.46)

Observe that substituting $p = \hat{p}^-$ into the first equation in (2.43) and (2.44) leads to

$$
\Phi_p(\hat{p}^-; 0) = \frac{H^3_p(\hat{p}^-)}{H^3_p(-1)} > 0,
$$

(2.47a)

and

$$
\Phi(\hat{p}^-; 0) = \frac{1}{H^3_p(-1)} \int_{-1}^{\hat{p}^-} H^3_p(p) dp > 0.
$$

(2.47b)

Because $\Phi(\hat{p}) = \Upsilon(\hat{p})$, we have from the inequality in (2.47b) that $\Upsilon(\hat{p}^+; 0) > 0$. Using equation (2.47a) in concert with the last equation in (2.39b) leads to $\Upsilon_p(\hat{p}^+; 0) > 0$. The desired inequality (2.46) now follows.
Next, we claim that
\[ B_\mu < 0. \tag{2.48} \]

Differentiating \( B \) with respect to \( \mu \) gives
\[ B_\mu = \frac{\Upsilon_{\mu p} \Upsilon - \Upsilon_p \Upsilon_\mu}{\Upsilon^2} \bigg|_{p=0}. \tag{2.49} \]

Note, differentiating the last two equations of (2.39b) with respect to \( \mu \) yields
\[
\begin{cases}
\Upsilon_\mu = \Phi_\mu \\
\Upsilon_{\mu p} = H_p^3(\hat{p}^+) \frac{\Phi_{\mu p}(\hat{p}^-)}{H_p^3(\hat{p}^-)} + \left[ \rho \right] H_p^3(\hat{p}^+)\Phi + \mu \left[ \rho \right] H_p^3(\hat{p}^+)\Phi_\mu \\
\end{cases} \quad \text{on } p = \hat{p}. \tag{2.50}
\]

Furthermore, by differentiating (2.39a) with respect to \( \mu \), we obtain the following problem
\[
\begin{cases}
\left( \frac{\Phi_{\mu p}}{H_p^3} \right) = \rho_p \Phi + \mu \rho_p \Phi_\mu & \text{in } (-1, \hat{p}), \\
\Phi_\mu = 0 = \Phi_{\mu p} & \text{on } p = -1. \tag{2.51}
\end{cases}
\]

A computation using integration by parts gives us,
\[
\left( \frac{\Upsilon_{\mu p} \Upsilon - \Upsilon_p \Upsilon_\mu}{H_p^3} \right) \bigg|_{p=0} - \left( \frac{\Upsilon_{\mu p} \Upsilon - \Upsilon_p \Upsilon_\mu}{H_p^3} \right) \bigg|_{p=\hat{p}} = \int_\hat{p}^0 \rho_p \Upsilon^2(p; \mu) \, dp. \tag{2.52}
\]

By rearranging terms we have that
\[
B_\mu = \left( \frac{\Upsilon_{\mu p} \Upsilon - \Upsilon_p \Upsilon_\mu}{\Upsilon^2} \right) \bigg|_{p=0} = H_p^3(0) \left[ \left( \frac{\Upsilon_{\mu p} \Upsilon - \Upsilon_p \Upsilon_\mu}{H_p^3} \right) \bigg|_{p=\hat{p}} + \int_\hat{p}^0 \rho_p \Upsilon^2(p; \mu) \, dp \right].
\]

Therefore, to verify our claim in (2.49), it suffices to show
\[ (\Upsilon_{\mu p} \Upsilon - \Upsilon_p \Upsilon_\mu) \big|_{p=\hat{p}} < 0. \]

Differentiating the first equation in (2.39a) with respect to \( \mu \) and testing against \( \Phi \) yields
\[
\left( \frac{\Phi_{\mu p} \Phi - \Phi_p \Phi_\mu}{H_p^3} \right) \bigg|_{p=-1}^{p=\hat{p}} = \int_{-1}^{\hat{p}} \rho_p \Phi^2(p; \mu) \, dp < 0.
\]
Recall that $\Phi_{\mu p}(-1) = 0 = \Phi_{\mu}(-1)$. Hence, the above inequality simplifies into

$$\frac{\Phi_{\mu p}(\hat{p}^-)}{\Phi_{p}(\hat{p}^-)} < \frac{\Phi_{\mu}(\hat{p}^-)}{\Phi(\hat{p}^-)} = \frac{\Upsilon_{\mu}(\hat{p}^+)}{\Upsilon(\hat{p}^+)}, \quad (2.53)$$

which is equivalent to

$$\Phi_{\mu p} \Upsilon - \Phi_{p} \Upsilon_{\mu} < 0 \quad \text{on } p = \hat{p}. \quad (2.54)$$

Multiplying the second equation in (2.50) by $\Upsilon_{\mu}$ reveals that

$$\Upsilon_{\mu p} \Upsilon = H^3_{p}(\hat{p}^+) \frac{\Phi_{\mu p}(\hat{p}^-) \Upsilon}{H^3_{p}(\hat{p}^-)} + [\rho] H^3_{p}(\hat{p}^+) \Phi \Upsilon + \mu [\rho] H^3_{p}(\hat{p}^+) \Phi_{\mu} \Upsilon \quad \text{on } p = \hat{p}. \quad (2.55)$$

On the other hand, multiplying the third equation in (2.39b) by $\Upsilon_{\mu}$ and evaluating it at $p = \hat{p}$ gives

$$\Upsilon_{p} \Upsilon_{\mu} = H^3_{p}(\hat{p}^+) \frac{\Phi_{p}(\hat{p}^-) \Upsilon_{\mu}}{H^3_{p}(\hat{p}^-)} + \mu [\rho] H^3_{p}(\hat{p}^+) \Phi \Upsilon_{\mu} \quad \text{on } p = \hat{p}. \quad (2.56)$$

Subtracting (2.55) from (2.56) and using the fact in (2.54), we know that $(\Upsilon_{\mu p} \Upsilon - \Upsilon_{p} \Upsilon_{\mu}) < 0$ on $p = \hat{p}$. This proves the claim in (2.48) provided that $\Upsilon(0) \neq 0$.

Now, combining our earlier observations that $B_{\mu} < 0$ and $B(0) > 0$, we can infer that there exists a unique smallest $\mu_{cr}$ such that $B(\mu) = \mu \rho(0) H^3_{p}(0)$. This proves part (a) of the lemma. Observe, by the uniqueness of solution to initial value problem, the numerator and denominator in (2.42) cannot vanish altogether. Thus collectively these facts show that $B(\mu), \Upsilon_{p}(0, \mu), \Upsilon(0, \mu)$ are all strictly positive quantities for $0 \leq \mu \leq \mu_{cr}$. Part (c) of the lemma is then a direct consequence of the fact that $B(\mu) > \mu \rho(0) H^3_{p}(0)$ for all $0 \leq \mu < \mu_{cr}$.

It remains only to prove part (b). We first consider the sign of $\Psi$ through a continuity argument. Define the set

$$\mathcal{E} := \{\mu \in [0, \mu_{cr}] : \Psi(p; \mu) > 0 \text{ for } p \in (-1, 0]\}.$$
Observe that $0 \in \mathcal{E}$ due to (2.43). We claim that $\mathcal{E}$ is closed. Seeking a contradiction, suppose that $\mathcal{E}$ has a limit point $\tilde{\mu}$ and there exists $\tilde{p} \in [-1, 0]$ so that $\Psi(\tilde{p}, \tilde{\mu}) = 0$. By continuity, we can infer that $\Psi(p, \tilde{\mu}) \geq 0$ for all $p \in [-1, 0]$, and hence $\Psi(\cdot; \tilde{\mu})$ attains its minimum at $p = \tilde{p}$. In particular, this implies $\Psi_p(\tilde{p}, \tilde{\mu}) = 0$, where notice this would be true even in the case $\tilde{p} = \hat{p}$. But then $\Psi_p(\tilde{p}, \hat{\mu}) = \Psi(\tilde{p}, \hat{\mu}) = 0$, and so $\Psi$ vanishes identically by uniqueness. Thus we have arrived at a contradiction meaning $\mathcal{E}$ is closed. On the other hand, $\mathcal{E}$ is clearly open because $\Psi$ is continuous in $\mu$ and we have already shown that $\Psi_p > 0$ at $p = 0, -1$. It follows then that $\mathcal{E} = [0, \mu_{cr}]$.

Finally, we establish the sign of $\Psi_p$ claimed in part (b). Fix $\mu \in [0, \mu_{cr}]$ and consider the function

$$g(p) := \frac{\Psi_p(p; \mu)}{H^3_p(p)}.$$  

Clearly, $g(-1) > 0$ and since $\Psi_p(0; \mu) > 0$, we have $g(0) > 0$. Moreover, the equation satisfied by $\Psi$ gives the identity $g_p = \mu \rho_p \Psi$. From this it is easily seen that $g(\hat{p}^+) > 0$. This shows that $\Psi_p(p; \mu) > 0$ for $\hat{p} < p \leq 0$. Furthermore, from (2.39b), we have

$$\frac{\Upsilon_p(\hat{p}^+; \mu)}{H^3_p(\hat{p}^+)} - \mu \rho \Phi(\hat{p}; \mu) = \frac{\Phi_p(\hat{p}^-; \mu)}{H^3_p(\hat{p}^-)}.$$  

Notice that the left hand side of the above equation is strictly positive. We then conclude that $g(\hat{p}^-) > 0$. Hence, we can conclude that $\Psi_p(p; \mu) > 0$ for all $-1 \leq p < \hat{p}$. This gives the desired inequality in part (b) of the lemma. 

\textbf{Lemma 2.7 (Spectrum).} \textit{Let $\Sigma$ denote the set of eigenvalues for the problem in (2.37) at $\mu = \mu_{cr}$.}

(a) $\Sigma = \{\nu_j\}_{j=0}^\infty$ such that $\nu_j \to \infty$ as $j \to \infty$ and $\{\nu_j\}_{j=0}^\infty$ is a strictly increasing sequence,
(b) \( \nu_0 = 0 \), and

c) each eigenvalue has algebraic and geometric multiplicity 1.

Proof. Fix \( \mu = \mu_{cr} \). Similar in spirit to the proof of Lemma 2.6, we begin by introducing the function \( N(p; \nu) \) which solves the following initial value problem

\[
\begin{cases}
\left( \frac{N_p}{H_p^3} \right)_p - \mu_{cr} \rho_p N = -\nu \frac{N}{H_p} & \text{in } (-1, \hat{p}) \text{ and } (\hat{p}, 0), \\
\left[ \frac{N_p}{H_p^3} \right] = \mu_{cr} \left[ \rho \right] N & \text{on } p = \hat{p}, \\
N = 0 & \text{on } p = -1, \\
N_p = 1 & \text{on } p = -1,
\end{cases}
\quad (2.57)
\]

and the associated function

\[
B(\nu) := \frac{N_p(0; \nu)}{N(0; \nu)}. \quad (2.58)
\]

Observe that \( \dot{w} := N(p; \nu) \) solves (2.37) provided that \( B(\nu) = \mu_{cr} \rho(0)H_p^3(0) \). By construction, \( B \) has singularity at each eigenvalue \( \nu_D \) of the Dirichlet problem

\[
\begin{cases}
- \left( \frac{\dot{w}_p}{H_p^3} \right)_p + \mu_{cr} \rho_p \dot{w} = \nu_D \frac{\dot{w}}{H_p}, & \text{in } (-1, \hat{p}) \text{ and } (\hat{p}, 0), \\
\dot{w} = 0 & \text{on } p = 0, \\
\left[ \frac{\dot{w}_p}{H_p^3} \right] - \mu_{cr} \left[ \rho \right] \dot{w} = 0 & \text{on } p = \hat{p}, \\
\dot{w} = 0 & \text{on } p = -1.
\end{cases}
\quad (2.59)
\]

It is well-known that the set of Dirichlet eigenvalues takes the form \( \Sigma_D = \{ \nu_D^{(j)} \}_{j=1}^\infty \), where each \( \nu_D^{(j)} \) is simple, \( \nu_D^{(j)} \leq \nu_D^{(j+1)} \) for all \( j \in \mathbb{Z}^+ \), and \( \nu_D^{(j)} \to \infty \) as \( j \to \infty \). We claim, moreover, that each Dirichlet eigenvalue is positive. For the sake of contradiction, suppose that there exists a \( \nu_D < 0 \) in \( \Sigma_D \) with eigenfunction \( \dot{w} \). For \( 0 < \delta \ll 1 \), define

\[
\nu^\delta := \frac{\dot{w}}{\left( \Psi_{cr} + \delta \right)}.
\]
Using equations (2.39a) and (2.39b) for $\Psi$ together with the Dirichlet problem (2.59), one can show that

\[- \left( \frac{\Psi_{cr} + \delta}{H_p} \right)_p \left( \frac{\dot{\Psi}}{H_p} + \left( \frac{\mu_{cr} \rho_p}{H_p} - \nu_D \frac{\Psi_{cr} + \delta}{H_p} \right) \right) = 0. \tag{2.60}\]

If $\nu_D < 0$, we can choose small enough $\delta$ such that the coefficient of the zeroth order term in (2.60) is positive. Similarly, for $\nu_D = 0$, we let $\delta \to 0$ so that the coefficient of the zeroth order term in (2.60) vanishes. Thus, in both cases, taking $0 < \delta \ll 1$, we can apply the maximum principle to conclude $\dot{\delta} \equiv 0$ and thus $\dot{w} \equiv 0$. Having arrived at a contradiction, we therefore infer that all the elements of $\Sigma_D$ are strictly positive.

Now, differentiating (2.57) with respect to $\nu$ yields

\[
\begin{cases}
\left( \frac{N_{\nu p}}{H_p^3} \right)_p - \mu_{cr} \rho_p N_\nu = - \frac{N}{H_p} - \frac{\nu N_\nu}{H_p} & \text{in } (\hat{p}, 0), \\
N_\nu(-1; \nu) = 0 = N_{\nu p}(-1; \nu).
\end{cases}
\]

Testing the above equation against $N$ and comparing it to (2.57) tested against $N_\nu$ yields the Green’s identity

\[
\left( \frac{N_{p p} N_\nu}{H_p^3} - \frac{N_{\nu p} N}{H_p^3} \right) \bigg|_{p = \hat{p}}^0 = \int_{\hat{p}}^0 \frac{N^2(p; \nu)}{H_p} \, dp.
\]

Hence we have

\[B'(\nu) = -\frac{H_p^3(0)}{N^2(0)} \int_{\hat{p}}^0 \frac{N^2(p; \nu)}{H_p} \, dp < 0,
\]

as long as $N(0; \nu) \neq 0$. We then conclude that $B$ is strictly decreasing on the complement of the set $\Sigma_D$ denoted by $\Sigma_D^c$. Thus, we must have that $B(\nu) \to \pm \infty$ as $\nu \to \nu_D^{(j)\pm}$ for each $j \in \mathbb{Z}^+$. In particular, on each connected component of $\Sigma_D$, there exists a unique $\nu \in (\nu_D^{(j)}, \nu_D^{(j+1)})$ such that $B(\nu) = \mu_{cr} \rho(0) H_p^3(0)$. Likewise, on $(-\infty, \nu_D^{(1)})$ there can be at most one such value of $\nu$. 42
This analysis shows that the eigenvalues of \((2.37)\) are intertwined with those of the Dirichlet problem \((2.59)\). We have already proved in Lemma 2.6(a) that \(0 \in \Sigma\), and hence it is the unique element of \(\Sigma\) in the interval \((-\infty, \nu_D^{(1)})\). This implies further implies that \(\nu_D^{(2)} > 0\), so parts \([a]\) and \([b]\) now follow. Part \([c]\) is easily verified from classical Sturm–Liouville theory. ■

Finally, let us conclude the subsection by recalling that \(\mu = 1/F^2\). Hence, the critical Froude number is defined by

\[
F_{cr}^2 := \frac{1}{\mu_{cr}},
\]

with \(\mu_{cr}\) given by Lemma 2.6

### 2.3.2 Fredholm property

Now we focus our attention on the full linearized operator at \((w, F) \in \mathcal{U} \). Consider first the problem \(\mathcal{F}_w(0, F)\dot{w} = (f_1, f_2, f_3)\), which reads

\[
\begin{align*}
\left(\frac{\dot{w}_p}{H_p^3}\right)_p + \left(\frac{\dot{w}_q}{H_p}\right)_q - \frac{1}{F^2}\rho p \dot{w} &= f_1 \quad \text{in } R, \\
- \frac{\dot{w}_p}{H_p^3} + \frac{1}{F^2}\rho \dot{w} &= f_2 \quad \text{on } T, \\
- \left[\frac{\dot{w}_p}{H_p^3}\right] + \frac{1}{F^2} \left[\rho\right] \dot{w} &= f_3 \quad \text{on } I, \\
\dot{w} &= 0 \quad \text{on } B.
\end{align*}
\]

Although we know that \(\mathcal{F}_w(0, F)\) is a map from \(X\) to \(Y\), here we shall view it as a map between the larger spaces \(X_b\) to \(Y_b\), where

\[
\begin{align*}
X_b &:= \left\{ w \in C^0_0(\overline{R}) \cap C^{9+\alpha}_b(\overline{R}^+) \cap C^{9+\alpha}_b(\overline{R}^-) : w|_B = 0 \right\}, \\
Y_b &:= \left( C^{7+\alpha}_b(\overline{R}^+) \cap C^{7+\alpha}_b(\overline{R}^-) \right) \times C^{8+\alpha}_b(T) \times C^{8+\alpha}_b(I).
\end{align*}
\]
That is, the requirement that the solution decays at infinity has temporarily been lifted.

Observe that the zeroth-order term in the interior equation in (2.62) has the “bad” sign in that it does not satisfy the the assumptions of the maximum principle Theorem A.1. For supercritical waves, we can fix this by introducing a function \( \tilde{\Psi} \) that is a variant of the function \( \Psi \) found in Lemma 2.6. It is defined as the solution to the same ODE (2.39) but with initial conditions \( \tilde{\Psi}(-1) = \epsilon \) and \( \tilde{\Psi}_p(-1) = 1 \), for some \( 0 < \epsilon \ll 1 \) that depends only on \( F \).

**Lemma 2.8.** Suppose \( F > F_{cr} \), then for \( \epsilon > 0 \) sufficiently small, there exists \( \tilde{\Psi} \) satisfying

\[
\left( \frac{\tilde{\Psi}_p}{H^3_p} \right)_p - \frac{1}{F^2} \rho_p \tilde{\Psi} = 0 \quad \text{in} \quad (-1, \hat{p}) \cup (\hat{p}, 0),
\]

with

\[
\tilde{\Psi} > 0 \quad \text{for} \quad -1 < p \leq 0, \quad \tilde{\Psi}_p > 0 \quad \text{for} \quad (-1, \hat{p}) \quad \text{and} \quad (\hat{p}, 0],
\]

\[
-\frac{\tilde{\Psi}_p}{H^3_p} + \frac{1}{F^2} \rho \tilde{\Psi} < 0 \quad \text{on} \quad p = 0,
\]

and

\[
\left[ \frac{\tilde{\Psi}_p}{H^3_p} \right] - \frac{1}{F^2} \left[ \rho \right] \tilde{\Psi} < 0 \quad \text{on} \quad p = \hat{p}.
\]

**Proof.** By definition, \( \tilde{\Psi} = \Psi \) when \( \epsilon = 0 \). Adapting the proof of Lemma 2.6, it is easy to see that equations (2.63b) and the second inequality in (2.63a) hold for \( 0 < \epsilon \ll 1 \). Further, the first inequality in (2.63a) can be obtained by integrating the second one where \( \epsilon > 0 \) is chosen to be sufficiently small. Lastly, to arrive at (2.63c), we use the transmission equation from the ODE (2.39b) together with the boundary condition \( \tilde{\Psi}(-1, \mu) = \epsilon \). \( \blacksquare \)
Now, letting $\dot{w} =: \tilde{\Psi} v$, we see that (2.62) is equivalent to the following more amenable problem for $v$:

\[
\begin{align*}
\frac{v_p}{H_p^3} + \frac{v_q}{H_p} &= \frac{f_1}{\tilde{\Psi}} & \text{in } R, \\
-\frac{v_p}{H_p^3} + \frac{1}{\tilde{\Psi}} \left( -\tilde{\Psi}_p \right) + \frac{1}{F^2} \tilde{\rho} \tilde{\Psi} v &= \frac{f_2}{\tilde{\Psi}} & \text{on } T, \\
-\left[ \frac{v_p}{H_p^3} \right] &= \frac{f_3}{\tilde{\Psi}} & \text{on } I, \\
v &= 0 & \text{on } B.
\end{align*}
\]

(2.64)

Lemma 2.9 (Invertibility). For $F > F_{cr}$, $F_w(0, F)$ is an invertible map from $X_b$ to $Y_b$ and from $X$ to $Y$.

Proof. Because the maximum principle can be applied to (2.64), one can show following [Whe13 Lemma A.5] and [CWW18 Lemma A.1] that $F_w(0, F)$ is an injective map between $X_b$ and $Y_b$. Surjectivity between these spaces, moreover, follows from the same argument as in [CWW18 Lemma A.3]. Finally, using [Whe13 Corollary A.11], we obtain the invertibility of $F_w(0, F)$ between the spaces $X$ and $Y$.

Lemma 2.10. For all $(w, F) \in \mathcal{U}$, $F_w(w, F)$ is Fredholm index 0 as a map $X \to Y$.

Proof. Fix $(w, F) \in \mathcal{U}$. Since $w \in X$ then the coefficients of the operator $F_w(w, F)$ go to the coefficients of $F_w(0, F)$ as $|q| \to \infty$. By Lemma 2.9 we know that $F_w(0, F)$ is invertible as a map $X$ to $Y$. The proof then follows from an application of [Whe15b Lemma A.12 and A.13].

2.4 Qualitative properties

2.4.1 Bounds on the Froude number

In this section, we derive both upper and lower bounds on the Froude number in our heterogenous regime. The analysis follows closely the arguments from [CWW18]...
Section 4]. These bounds play a crucial role in the global theory discussed in Section 2.6. In particular, they allow us to infer that blowup in norm implies stagnation.

There has been a number of significant applied works devoted to estimating the Froude number of irrotational solitary waves. In this regime, Starr [Sta47] gave a formal proof of Froude number bounds written in terms of an integral of the free surface profile. Numerically, Longuet-Higgins–Fenton [LHF74] obtained the upper and lower bounds on the Froude number, $1 < F < 1.286$. However, for rotational solitary waves, much less is known. In addition to the results in [CWW18], we also mention the work of Wheeler [Whe15a] where a number of bounds on Froude number in the rotational (but constant density) case are obtained.

Lower bound

We begin by showing that every wave with critical Froude number must be trivial. In the global continuation argument, this allows us to conclude that all waves on the solution curve are supercritical. The first step is to establish an integral identity.

**Lemma 2.11.** Let

$$ (w, F) \in C_{b,e}^2(\mathbb{R}^+) \cap C_{b,e}^2(\mathbb{R}^-) \cap C_0^4(\mathbb{R}^+) \cap C_0^4(\mathbb{R}^-) \cap C_0^d(\mathbb{R}) \times \mathbb{R} \quad (2.65) $$

solve equation (2.33). For $M \geq 0$, define

$$ I(M) := \int_{-M}^{M} \int_{-1}^{0} \frac{H_p^3 w_q^2 + (H_p + 2h_p) w_p^2 \Psi_p}{2 h_p^2 H_p^3} dp dq + A \left( \frac{1}{F^2} \right) \int_{-M}^{M} \eta dx, $$

where $A$ is given in (2.40) and $\Psi = \Psi(p; 1/F^2)$ is from (2.41). Then $I \to 0$ as $M \to \infty$.

**Proof.** Multiplying the height equation (2.28) by $\Psi$ then integrating by parts over the finite rectangle $|q| < M$, we obtain
0 = \int_{-M}^{M} \int_{-1}^{0} \left[ \left( \frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) \Psi_p + \frac{\Psi_p}{H_p^3} (h_p - H_p) \right] dp \, dq \\
- \int_{-M}^{M} \left[ \left[ \frac{1 + h_q^2}{2h_p^2} \Phi_p + \frac{1}{2H_p^2} \right] \Psi - \frac{\Psi_p}{H_p^3} (h - H) \right]_I dq \\
+ \int_{-M}^{M} \left[ \left[ \frac{1 + h_q^2}{2h_p^2} \Phi_p + \frac{1}{2H_p^2} \right] \Psi - \frac{\Psi_p}{H_p^3} (h - H) \right]_T dq + \int_{-1}^{0} \frac{h_q}{h_p} \Psi \bigg|_{q=-M}^{q=M} dp.

Here we have used the equation satisfied by $\Psi$ (2.39) to eliminate several terms.

Notice that we can re-write the first integrand above as follows

$$
\left( \frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) \Psi_p + \frac{\Psi_p}{H_p^3} (h_p - H_p) = \frac{H_p^3 w_q^2 + (H_p + 2h_p)w_p^2}{2h_p^2 H_p^3} \Psi_p.
$$

This yields

$$
\int_{-M}^{M} \int_{-1}^{0} \frac{H_p^3 w_q^2 + (H_p + 2h_p)w_p^2}{2h_p^2 H_p^3} \Psi_p \, dp \, dq + \int_{-M}^{M} \left[ \left[ \frac{1 + h_q^2}{2h_p^2} \Phi_p - \frac{1}{2H_p^2} \right] \Psi - \frac{\Psi_p}{H_p^3} (h - H) \right]_I dq \\
+ A \left( \frac{1}{F^2} \right) \int_{-1}^{M} \eta \, dx = - \int_{-1}^{0} \frac{h_q}{h_p} \Psi \bigg|_{q=-M}^{q=M} dp,
$$

(2.66)

where $A(1/F^2)$ is the expression defined in (2.40) and $\eta$ is the free surface. Observe that from (2.28) and (2.39), the second integral on the left hand side of (2.66) vanishes. Now, sending $M \to \infty$ results in $h_q |_{-M}^{M} \to 0$, which forces the right hand side of (2.66) to vanish. Hence, the proof is complete.

**Theorem 2.12** (Critical waves are laminar). Let $(w, F)$ be in the function space defined in (2.65) and solve (2.33).

(a) Suppose $w > 0$ on $T$, and $F$ is chosen such that $\Psi_p \geq 0$ for $[-1, \hat{p})$ and $(\hat{p}, 0]$, then $A(1/F^2) < 0$.

(b) If $F = F_{cr}$, then $w \equiv 0$. 47
Proof. The assumptions on the strict positivity of \( w \) and assuming strict positivity for \( \Psi_p \) force the first integrand in the definition of \( I \) to be positive. Sending \( M \to \infty \) and applying Lemma \( 2.11 \), we therefore prove part (a).

It remains to prove part (b). Thanks to Lemma \( 2.6(a) \), we know that \( A(1/F_{cr}^2) = 0 \). For \( F = F_{cr} \), \( I \) takes the following form

\[
\int_{-M}^{M} \left[ \int_{-1}^{0} \frac{H_p w_q^2 + (H_p + 2h_p)w_p^2}{2h_p^2 H_p^3} (\Psi_{cr})_p dp \right] dq, (2.67)
\]

which still goes to zero as \( M \to \infty \). From Lemma \( 2.6(b) \), it is clear that \( (\Psi_{cr})_p > 0 \) for \( -1 \leq p \leq \hat{p} \) and \( \hat{p} \leq p \leq 0 \). Hence, the integral in (2.67) is non-negative and non-decreasing as a function of \( M \). Hence, the integrand must vanish for all \( M \). This implies that \( w_p \) and \( w_q \) should be equal to zero everywhere. In other words, \( w \equiv 0 \).

Upper bound

Now, we shall derive the upper bound of the Froude number. The argument here is based on [CWW18] which is strongly inspired by the work of Pritchard–Keady [KP74] and Starr [Sta47]. Both earlier results show that for homogeneous irrotational fluid, \( F < \sqrt{2} \). However, due to stratification and vorticity, our estimate here is presented in terms of several quantities associated to the underlying current and a bound away from stagnation along the crest line. The integral identity (2.68) can, in fact, be applied to the homogeneous irrotational regime where it recovers the bound in [Sta47] and [KP74].

Lemma 2.13. Let \( (w, F) \) belong to the space defined in (2.65) and solve (2.33), then

\[
\frac{1}{F^2} \left[ \int_{-\hat{p}}^{\hat{p}} |\rho_p| w(0,p)^2 dp + \int_{-1}^{0} |\rho_p| w(0,p)^2 dp + \rho(0)\eta(0)^2 - [p] w(\hat{p})^2 \right] = \int_{-1}^{0} \frac{w_p^2}{H_p^2 h_p} (0,p) dp. \quad (2.68)
\]
Proof. Recall from the discussion in Section 2.2.4 that the flow force $\mathcal{S}$ is independent of $q$ and in semi-Lagrangian variables can be viewed as the functional \((2.30)\) acting on $h$. Hence, by evaluating it at $q = 0$ and $q = \pm\infty$, we obtain

\[
\mathcal{S}(h) = \int_{-1}^{0} \left( \frac{1}{2H_p^2} + \frac{1}{2H_p^2} - \frac{1}{F^2} \rho(h - H) - \frac{1}{F^2} \int_0^p \rho H_p \ dp' \right) h_p \ dp
\]

Gathering like terms, integrating by parts and simplifying terms lead to

\[
0 = \int_{-1}^{0} \frac{w_p^2}{2H_p^2 h_p} \Bigg|_{q=0} \ dp - \int_{-1}^{0} \frac{1}{F^2} \rho w_p \Bigg|_{q=0} \ dp. \tag{2.69}
\]

Moreover, integrating by parts, we obtain

\[
\int_{-1}^{0} \frac{1}{F^2} \rho w_p \Bigg|_{q=0} \ dp = \int_{-1}^{0} \frac{1}{2F^2} \left[ \rho \right] w(0,p)^2 \ dp + \int_{-1}^{0} \frac{1}{2F^2} \left[ \rho \right] w(0,p)^2 \ dp
\]

Combining the previous two integrals with the one in \((2.69)\), we arrive at equation \((2.68)\).

Theorem 2.14 (Upper bound of $F$). Let $(w, F)$ belong to the space defined in \((2.65)\) and solve \((2.33)\). Then the Froude number $F$ is bounded above:

\[
F^2 \leq \frac{2}{\pi} \left\| \left\| w \right\|_{L^2} \right\|_{L^\infty} \left\| \left\| H_p \right\|_{L^\infty} \right\|_{L^\infty} \left\| h_p(0,\cdot) \right\|_{L^\infty} . \tag{2.70}
\]

Proof. From the Poincaré inequality, it is easy to see that

\[
\int_{-1}^{0} \frac{2}{F^2} \rho w_p \Bigg|_{q=0} \ dp \leq \frac{2}{\pi} \left\| \left\| \rho \right\|_{L^\infty} \right\|_{L^2} \left\| \left\| w_p(0,\cdot) \right\|_{L^2}^2 \right. \]

Moreover,

\[
\int_{-1}^{0} \frac{w_p^2}{2H_p^2 h_p} \Bigg|_{q=0} \ dp \geq \left( \min_p \left( H_p^{-1} \right)^2 \right) \left( \min_p \left( h_p^{-1}(0,p) \right) \right) \left\| w_p(0,\cdot) \right\|_{L^2}^2 .
\]
Combining both estimates with the identity (2.69) and canceling some terms yields

\[
\frac{2}{\pi} \frac{1}{F^2} \|\rho\|_{L^\infty} \geq \left( \min_p (H_p^{-1})^2 \right) \left( \min_p (h_p^{-1})(0, p) \right),
\]

which is equivalent to (2.70).

2.4.2 Symmetry

In this section, we will prove Theorem 2.3 on the symmetry of supercritical solitary waves of elevation. The main machinery used here is the method of moving planes, first introduced by Alexandrov [Ale62] in his study of spheres. Variations of this argument have been used by many authors, for instance, Serrin [Ser71] in dealing with a symmetry problem in potential theory (see also [BN88, BN91]). Due to the full nonlinearity of the problem and the unboundedness of the domain, we adopt the version used in [Li91]. The proof is patterned on that of [CWW18, Theorem 4.13], which in turn is based partially on the work of Maia in [Mai97].

The main result, stated now in semi-Lagrangian variables for convenience, is as follows.

**Theorem 2.15 (Symmetry).** Let

\[
(w, F) \in C^3_b(\mathbb{R}^+ \cap \mathbb{R}^-) \cap C^0(\mathbb{R}) \times \mathbb{R}
\]

be a solution of the height equation (2.33) that is a wave of elevation

\[
w > 0 \quad \text{in } \mathbb{R} \cup I \cup T, \quad (2.71)
\]

supercritical, and satisfies the upstream (or downstream) condition

\[
w, Dw \to 0 \quad \text{uniformly as } q \to -\infty \quad (\text{or } \infty). \quad (2.72)
\]
Then, possibly after translation in $q$, $w$ is a symmetric and monotone solitary wave: there exists $q_0 \in \mathbb{R}$ such that $q \mapsto w(q, \cdot)$ is even about $\{ q = q_0 \}$ and

$$\pm w_q > 0 \text{ for } \pm (q_0 - q) > 0, -1 < p \leq 0. \quad (2.73)$$

Following the usual moving planes approach, we start by considering the reflected height function $h$ about the axis $q = \lambda$,

$$h^\lambda(q,p) := h(2\lambda - q, p).$$

Letting $v^\lambda := h^\lambda - h$, we see that $q = q_0$ is the axis of symmetry if and only if $v^{q_0} \equiv 0$.

For $\lambda \in \mathbb{R}$, we define the sets

$$R^{\pm}_\lambda := R^{\pm} \cap \{ q < \lambda \},$$

and likewise for the boundary components $I_\lambda$, $B_\lambda$, and $T_\lambda$. Throughout the section, we denote the restriction of $v^\lambda$ and $h^\lambda$ to $R^{\pm}_\lambda$ by $v^{\lambda\pm}$ and $h^{\lambda\pm}$, respectively.

Suppose that $h$ is a solution to the height equation (2.28). Then for each $\lambda$,

$$v^\lambda \text{ solves the PDE } \mathcal{L} v^\lambda = 0 \text{ in } R_\lambda, \quad \mathcal{B} v^\lambda = 0 \text{ on } T_\lambda \quad \mathcal{I} v^\lambda = 0 \text{ on } I_\lambda \quad v^\lambda = 0 \text{ on } B_\lambda,$$

where $\mathcal{L}$, $\mathcal{B}$, and $\mathcal{I}$ are given as follows:

$$\mathcal{L} := \frac{1}{h^\lambda_p} \frac{\partial^2}{\partial_q^2} - \frac{2h^\lambda_q}{(h^\lambda_p)^2} \frac{\partial}{\partial_q} \frac{\partial}{\partial_p} + \frac{1 + (h^\lambda_q)^2}{(h^\lambda_p)^2} \frac{\partial^2}{\partial_p^2} + \frac{h^\lambda_{pq}(h^\lambda_q + h^\lambda_p)}{(h^\lambda_p)^3} \frac{\partial}{\partial_q} + \frac{h^\lambda_{qq}(h^\lambda_q + h^\lambda_p) - 2h^\lambda_qh^\lambda_{pq}}{(h^\lambda_p)^3} \frac{\partial}{\partial_p} - \frac{1}{F^2} \rho_p,$$

$$\mathcal{B} := \frac{h^\lambda_q + h^\lambda_p}{2(h^\lambda_p)^2} \frac{\partial}{\partial_q} - \frac{(h^\lambda_q + h^\lambda_p)(1 + (h^\lambda_q)^2)}{2(h^\lambda_p)^2(h^\lambda_p)^2} \frac{\partial}{\partial_p} + \frac{1}{F^2} \rho,$$

$$\mathcal{I} := \left[ \frac{h^\lambda_q + h^\lambda_p}{2(h^\lambda_p)^2} \right] \frac{\partial}{\partial_q} - \left[ \frac{(h^\lambda_q + h^\lambda_p)(1 + (h^\lambda_q)^2)}{2(h^\lambda_p)^2(h^\lambda_p)^2} \right] \frac{\partial}{\partial_p} + \frac{1}{F^2} \mathbb{J}_{\rho}. \quad (2.75)$$
For a detailed derivation of the first two operators above and the ellipticity of $\mathcal{L}$, see $[\text{Wal09a}]$. The expression for the transmission operator $\mathcal{T}$ is new but follows from a similar calculation.

The signs of the zeroth-order coefficients above will not allow us to apply the maximum principle directly. However, for supercritical waves, we can tackle this problem as follows. For $0 < \epsilon \ll 1$, let $\tilde{\Psi} = \tilde{\Psi}(p; F, \epsilon)$ be the solution of the ODE

$$
\begin{align*}
\left( \frac{\tilde{\Psi}_p}{H_p^3(\rho_p - \epsilon)} \right)_{\rho} - \frac{1}{F^2(\rho_p - \epsilon)}\tilde{\Psi} &= 0 \quad \text{on } (-1, 0); \\
\tilde{\Psi}_p(-1) &= 1, \\n\tilde{\Psi}(-1) &= \epsilon.
\end{align*}
$$

(2.76)

in the distributional sense. Note that this implies a transmission condition on $p = \hat{p}$.

A straightforward adaptation of Lemma 2.8 gives the following result.

**Lemma 2.16.** Suppose $F > F_{cr}$, then for $\epsilon > 0$ sufficiently small the solution $\tilde{\Psi}$ to (2.76) satisfies

$$
\tilde{\Psi} > \epsilon \quad \text{for } -1 < p \leq 0, \quad \tilde{\Psi}_p > 0 \quad \text{for } [-1, \hat{p}) \text{ and } (\hat{p}, 0],
$$

and

$$
- \frac{\tilde{\Psi}_p}{H_p^3} + \frac{1}{F^2(\rho_p - \epsilon)}\tilde{\Psi} < 0 \quad \text{on } p = 0 \\
\left[ \frac{\tilde{\Psi}_p H_p^{-3}}{F^2} \right] - \frac{1}{F^2(\rho_p - \epsilon)}\tilde{\Psi} < 0 \quad \text{on } p = \hat{p}.
$$

(2.77)

**Proof.** The proof of this lemma is identical to the one of Lemma 2.8. $\blacksquare$

With this in hand, we can begin the moving planes method. The first step is to show that $v^\lambda$ is sign definite on $R_\lambda$ when $\lambda$ is sufficiently large and negative.

**Lemma 2.17.** Under the hypothesis of Theorem 2.15 there exists $K > 0$ such that

$$
v^\lambda \geq 0 \quad \text{on } R_\lambda \quad \text{for all } \lambda < -K
$$

(2.78)
and
\[ h_q \geq 0 \text{ in } R_\lambda \quad \text{for all } \lambda < -K. \] (2.79)

Proof. Let \( \tilde{\Psi} \) be defined as in equation (2.76). By Lemma 2.16, for \( 0 < \epsilon \ll 1 \), we have that \( \tilde{\Psi} > \epsilon \). This allows us to define \( \tilde{\Psi} u^\lambda := v^\lambda \). One can check that \( u^\lambda \) solves the PDE
\[ \tilde{\mathcal{L}} u^\lambda = 0 \quad \text{in } R_\lambda, \quad \tilde{\mathcal{B}} u^\lambda = 0 \quad \text{on } T_\lambda, \quad \tilde{\mathcal{F}} u^\lambda = 0 \quad \text{on } I_\lambda, \quad u^\lambda = 0 \quad \text{on } B_\lambda, \] (2.80)
where
\[
\tilde{\mathcal{L}} u^\lambda := \tilde{\Psi} (\mathcal{L} u^\lambda + \frac{1}{F^2 \rho_p} u^\lambda) + \left( \frac{2(1 + (h_q^\lambda)^2)}{(h^\lambda)^3} \tilde{\Psi}_p \right) u^\lambda_p - \left( \frac{2h_q^\lambda}{(h_p^\lambda)^2} \tilde{\Psi}_p \right) u^\lambda_q + Zu^\lambda,
\]
\[
\tilde{\mathcal{B}} u^\lambda := \tilde{\Psi} \mathcal{B} u^\lambda + (\mathcal{B}_p \tilde{\Psi}) u^\lambda,
\]
\[
\tilde{\mathcal{F}} u^\lambda := \tilde{\Psi} \mathcal{F} u^\lambda + (\mathcal{F}_p \tilde{\Psi}) u^\lambda.
\]
Here, the zeroth-order coefficient in \( \tilde{\mathcal{L}} \) is given by
\[
Z := \frac{1 + (h_q^\lambda)^2}{(h_p^\lambda)^3} \tilde{\Psi}_{ppp} - \frac{1}{F^2 \rho_p} \tilde{\Psi} + \left( \beta (-p) - \frac{1}{F^2 \rho_p} h \right) \left( (h_p^\lambda)^2 + h_p^\lambda h_p + (h_p^\lambda)^2 \right) \tilde{\Psi}_p \frac{1}{(h_p^\lambda)^3},
\]
and the principal parts of the boundary operators are
\[
\mathcal{B}_p := \frac{h_q^\lambda + h_q}{2h_p} \partial_q - \frac{(h_q^\lambda + h_p)(1 + (h_q^\lambda)^2)}{2h_p^2(h_p^\lambda)^2} \partial_p,
\]
\[
\mathcal{F}_p := \left[ \frac{h_q^\lambda + h_q}{2h_p^2} \right] \partial_q - \left[ \frac{(h_q^\lambda + h_p)(1 + (h_q^\lambda)^2)}{2h_p^2(h_p^\lambda)^2} \right] \partial_p.
\]

We will show that there exists \( K > 0 \) such that \( u^\lambda > 0 \) in \( R_\lambda \) for all \( \lambda \leq -K \) which in turn proves (2.78). For the sake of contradiction, assume that for any \( K \), there exists some \( \lambda_0 \leq -K \) such that \( u^\lambda_0 \) takes a negative value in \( R_{\lambda_0} \). By assumption, we know that \( h \) is a wave of elevation, the same is true for \( h^\lambda \) for any \( \lambda \). Clearly by definition, \( u^\lambda = 0 \) on \( q = \lambda \), and
\[
u^\lambda = \frac{h^\lambda - h}{\tilde{\Psi}} > \frac{H - h}{\tilde{\Psi}}
\]
where we note that the right-hand side of the inequality vanishes in the limit \( q \to -\infty \).

Therefore, if \( u^{\lambda_0} \) is negative in \( R_{\lambda_0} \), then there must be a point \((q_0, p_0) \in R_{\lambda_0} \cup T_{\lambda_0} \cup I_{\lambda_0}\) such that

\[
u^{\lambda_0}(q_0, p_0) = \inf_{R_{\lambda_0}} u^{\lambda_0} < 0.
\]

The cases when \((q_0, p_0) \in R_{\lambda_0}\) and \((q_0, p_0) \in T_{\lambda_0}\) are worked out in [CWW18, Lemma 4.18]. The only new possibility for the two-layer setting is that \((q_0, p_0) \in I_{\lambda_0}\).

Suppose that this is the case. Applying the Hopf lemma to \( R^+_{\lambda_0} \) and \( R^-_{\lambda_0} \), we have the following:

\[
u^{\lambda_0}_+(q_0, \hat{p}) > 0, \quad \nu^{\lambda_0}_-(q_0, \hat{p}) < 0. \tag{2.81}
\]

The minimum point assumption implies that \( u^{\lambda_0}_q(q_0, \hat{p}) = 0 \), which is equivalent to saying

\[
h^{\lambda_0}_q(q_0, \hat{p}) = h_q(q_0, \hat{p}). \tag{2.82}
\]

From the uniform decay stated in (2.72), for any \( \delta > 0 \), we can choose a large enough \( K \) such that we have

\[
|h(q_0, \hat{p}) - H(\hat{p})| < \delta, \quad \left|\left[ h_p(q_0, \hat{p}) - H_p(\hat{p}) \right]\right| < \delta, \tag{2.83}
\]

\[
\left|\left[ h^{\lambda_0}_p(q_0, \hat{p}) - H_p(\hat{p}) \right]\right| < \delta, \quad |h^{\lambda_0}_q(q_0, \hat{p}) - H(\hat{p})| < \delta.
\]

Applying triangle inequalities to (2.83) yields

\[
\left|\left[ h^{\lambda_0}_p(q_0, \hat{p}) - h_p(q, \hat{p}) \right]\right| < C\delta. \tag{2.84}
\]

Using inequalities in (2.83) and (2.84) allows us to write \( \tilde{T} u^{\lambda_0} \) in the following way

\[
\tilde{T} u^{\lambda_0} := \left[-\frac{(h^{\lambda_0}_p + h_p)(1 + (h^{\lambda_0}_q)^2)}{2h^{\lambda_0}_p(h^{\lambda_0}_q)^2} u^{\lambda_0}_p\right] \tilde{\Psi} + \left[ -\frac{1}{H^3_p} \tilde{\Psi}_p \right] + \frac{1}{F^2} \left| h_p \right| \tilde{\Psi} + \mathcal{O}(\delta) \right) u^{\lambda_0} = 0. \tag{2.85}
\]

Observe that via (2.77), we know that the coefficient \( u^{\lambda_0} \) is positive. On the other hand \( u^{\lambda_0}(q_0, \hat{p}) < 0 \). Hence, the second term on the right hand side is negative.
However, (2.81) shows that the first term on the right hand side on (2.85) is negative. These, thereby, lead to a contradiction. We conclude that there exists a large enough $K > 0$ such that (2.78) holds.

To prove (2.79), we first observe that $v^\lambda(\lambda, p) = 0$ by construction. In light of (2.78), this means it attains its global minimum on $R_\lambda$ there, hence $v_q^\lambda(\lambda, p) \leq 0$ for all $\lambda < -K$. But recalling the definition of $v^\lambda$, we then have

$$0 \leq -v_q^\lambda(\lambda, p) = 2h_q(\lambda, p)$$

for all $\lambda < -K$.\]

**Proof of Theorem 2.15** Define

$$\hat{\lambda} := \sup\{\lambda_0 : v^\lambda > 0 \text{ in } R_{\lambda_0} \text{ for all } \lambda < \lambda_0\}. \quad (2.86)$$

Note that $\hat{\lambda}$ is well-defined since the set above is non-empty.

**Case 1.** Let $\hat{\lambda} < \infty$. Under the continuity assumption on $h$, therefore $v^\lambda$, we can say that $v^\hat{\lambda} \geq 0$ in $R_{\hat{\lambda}}$. Since, $v^\hat{\lambda}$ is in the kernel of the elliptic operators (2.74) in $R_{\hat{\lambda}}$, applying maximum principle would guarantee that either $v^\hat{\lambda} > 0$ or $v^\hat{\lambda} \equiv 0$ in $R_{\hat{\lambda}}$. By way of contradiction, suppose that the former holds. Since, $\hat{\lambda}$ is taken to be the supremum of all $\lambda_0$ defined in (2.86), then there exists sequences $\{\lambda_l\}$ and $\{(q_l, p_l)\}$ in which $\lambda_l \searrow \hat{\lambda}$ together with $(q_l, p_l) \in \overline{R_{\lambda_l}}$ such that

$$v^{\lambda_l}(q_l, p_l) = \inf_{R_{\lambda_l}} v^{\lambda_l} < 0.$$

Since $v^{\lambda_l} = 0$ on $B_{\lambda_l}$, via the maximum principle, we know $(q_l, p_l) \in T_{\lambda_l} \cup I_{\lambda_l}$. First, assume that $(q_l, p_l) \in T_{\lambda_l}$ which implies $v^\lambda_p(q_l, 0) \leq 0$ and $v^\lambda_q(q_l, 0) = 0$. Next, we need to show that $q_l$ is bounded below. For the sake of contradiction, assume that for all $l$ large enough, we have $q_l \leq -K$ where $K$ is the positive real number obtained in the previous lemma. Let us look at the function $u^{\lambda_l} := v^{\lambda_l}/\Psi$. By
construction, \( u^{\lambda_l} \) in \( T_{\lambda_l} \) satisfies (2.80). Hence, we run into the same situation as case 2 in [CWW18, Lemma 4.18] where \( \tilde{\mathcal{B}} u^{\lambda_l}(q_l, 0) > 0 \). Hence, a contradiction.

Therefore, \( q_l \) is bounded below by \(-K\) and certainly bounded above by \( \lambda_l \). We can say that we have a convergence up to subsequence such that

\[
(q_l, 0) \to (\hat{q}, 0) \in T_{\hat{\lambda}} \quad \text{as} \ l \to \infty,
\]

for some \( \hat{q} \in [-K, \hat{\lambda}] \). Now, since we assume that \( v^{\hat{\lambda}} > 0 \) in \( R^{\hat{\lambda}} \) then

\[
\lim_{l \to \infty} v^{\lambda_l}(q_l, 0) = v^{\hat{\lambda}}(\hat{q}, 0) = 0.
\]

Suppose that \( \hat{q} < \hat{\lambda} \), then

\[
v^{\hat{\lambda}}(\hat{q}, 0) = v^{\hat{\lambda}}(\hat{q}, 0) = 0. \tag{2.87}
\]

By Hopf, \( v_p^{\hat{\lambda}}(\hat{q}, 0) < 0 \). Examining the operator \( \mathcal{B} \) using these facts, we obtain that \( \mathcal{B} v^{\hat{\lambda}}(\hat{q}, 0) > 0 \). But, since \( v^{\hat{\lambda}} \) is in the kernel of the operators in (2.74), then \( \mathcal{B} v^{\hat{\lambda}} = 0 \). Hence, we have arrived at a contradiction.

On the other hand, suppose that \( \hat{q} = \hat{\lambda} \) i.e. \( (\hat{q}, 0) \) is the corner point of \( R^{\hat{\lambda}} \). From (2.87), it follows that \( h_q^{\hat{\lambda}}(\hat{\lambda}, 0) = 0 \). Moreover, the top boundary operator for \( v^{\hat{\lambda}} \) reads

\[
(h_p^{\hat{\lambda}})^2(h_q^{\hat{\lambda}} + h_q)(h_q^{\hat{\lambda}} + h_q)(1 + (h_q^{\hat{\lambda}})^2)v_p^{\hat{\lambda}} + 2\frac{1}{F^2} \rho h^2_p(h_p^{\hat{\lambda}})^2 v^{\hat{\lambda}} = 0. \tag{2.88}
\]

Letting \( \lambda = \hat{\lambda} \), and taking the derivative of (2.88) with respect to the \( q \)-variable, and computing the result at \( (\hat{\lambda}, 0) \), we arrive at the following equality

\[
2h_p(\hat{\lambda}, 0)v^{\hat{\lambda}}_p(\hat{\lambda}, 0) = 0, \tag{2.89}
\]

where we have used the following facts

\[
h_q^{\hat{\lambda}}(\hat{\lambda}, 0) = -h_q(\hat{\lambda}, 0), \quad h_p^{\hat{\lambda}}(\hat{\lambda}, 0) = h_p(\hat{\lambda}, 0), \quad h_{qp}^{\hat{\lambda}}(\hat{\lambda}, 0) = -h_{qp}(\hat{\lambda}, 0).
\]
Now, since $h_p > 0$, then from (2.89) we conclude that $v_{pq}^\lambda(\hat{\lambda}, 0) = 0$. Furthermore, since $v^\lambda(\hat{\lambda}, \cdot) = 0$, then $v_p^\lambda(\hat{\lambda}, 0) = v_{pp}^\lambda(\hat{\lambda}, 0) = 0$.

Additionally, using the reformulation of the PDE in terms of $v^\lambda$ and solving $v_{qq}^\lambda$ in terms of $v_q^\lambda, v_p^\lambda, v_{pp}^\lambda$, and $v_{qq}^\lambda$, one can show that $v_{qq}^\lambda(\hat{\lambda}, 0) = 0$. Hence, $v^\lambda$ with all its derivatives up to order two vanish at the corner point $(\hat{\lambda}, 0)$. By construction, we know $v^\lambda$ is in the kernel of the elliptic operator in $R^\lambda$. Thereby, it contradicts the Serrin edge point lemma which guarantees the strict signs on the first and second derivatives of $v^\lambda$.

It remains to look at the case when $(q_l, p_l) \in I_{\lambda_l}$. Similar to (2.81), applying the Hopf boundary lemma, we obtain the following inequalities:

$$v_{p}^{\lambda_l+}(q_l, \hat{p}) > 0, \quad v_{p}^{\lambda_l-}(q_l, \hat{p}) < 0.$$  

In other words,

$$\left\| v_{q}^{\lambda_l}(q_l, \hat{p}) \right\| > 0 \quad \text{and} \quad v_{q}^{\lambda_l}(q_l, \hat{p}) = 0.$$  

Next, we need to show that $q_l$ is bounded below. For the sake of contradiction, assume that for all $l$ large enough, we have $q_l \leq -K$ where $K$ is the positive real number obtained in the previous lemma. Let us look at the following function $u^{\lambda_l} := v_{q}^{\lambda_l}/\Psi$. Hence, we run into the same situation as in (2.85) that yields $\mathcal{T} u^{\lambda_l} < 0$. However, by construction, $u^{\lambda_l}$ lies in the kernel of $\mathcal{T}$ in $I_{\lambda_l}$. Hence, we obtain a contradiction. Therefore, $q_l$ is bounded below by $-K$ and certainly bounded above by $\lambda_l$. Therefore, we have a convergence up to subsequence, in particular

$$(q_l, \hat{p}) \to (\hat{q}, \hat{p}) \in \overline{I_{\lambda_l}} \quad \text{as} \ l \to \infty,$$
for some \( \hat{q} \in [-K, \hat{\lambda}] \). Now, since we assume that \( \hat{v}^{\hat{\lambda}} > 0 \) in \( R^{\hat{\lambda}} \) then

\[
\lim_{l \to \infty} v^{\hat{\lambda}}(ql_0, \hat{p}) = \hat{v}^{\hat{\lambda}}(\hat{q}, \hat{p}) = 0.
\]

Suppose that \( \hat{q} < \hat{\lambda} \), then

\[
v^{\hat{\lambda}}(\hat{q}, \hat{p}) = v^{\hat{\lambda}}(q, \hat{p}) = 0.
\]

By Hopf, \( \left[ v^{\hat{\lambda}}_p \right](\hat{q}, \hat{p}) < 0 \). In view of the operator \( T \), we know that \( T v^{\hat{\lambda}}(\hat{q}, \hat{p}) < 0 \). But, since \( v^{\hat{\lambda}} \) solves, then \( T v^{\hat{\lambda}} = 0 \). Hence, we arrive at a contradiction.

Now, it remains to consider the case when \( (\hat{q}, \hat{p}) = (\hat{\lambda}, \hat{p}) \). The argument done here is similar to the one for (2.88). Using the fact that \( v^{\hat{\lambda}}_q(\hat{\lambda}, \hat{p}) = 0 \), \( v^{\hat{\lambda}}_{qq}(\hat{\lambda}, \hat{p}) = 0 \), \( v^{\hat{\lambda}}(\hat{\lambda}, \hat{p}) = 0 \), \( h^{\hat{\lambda}+}(\hat{\lambda}, \hat{p}) = h^{+}(\hat{\lambda}, \hat{p}) \), and \( h^{\hat{\lambda}-}(\hat{\lambda}, \hat{p}) = h^{+}(\hat{\lambda}, \hat{p}) \) along with operator \( T \), we arrive at the following equation

\[
\left( 2(h^{\hat{\lambda}+}_p)^2(h^{\hat{\lambda}-}_p)^2(h^{\hat{\lambda}+}_p + h^{\hat{\lambda}-}_p)(1 + (h^{\hat{\lambda}+}_q)^2) v^{\hat{\lambda}+}_{pq} \right) - \left( 2(h^{\hat{\lambda}+}_p)^2(h^{\hat{\lambda}-}_p)^2(h^{\hat{\lambda}+}_p + h^{\hat{\lambda}-}_p)(1 + (h^{\hat{\lambda}+}_q)^2) v^{\hat{\lambda}-}_{pq} \right) = 0.
\]

Doing some computation on (2.90) gives us

\[
\left[ v^{\hat{\lambda}}_{pq} h^{-3} \right](\hat{\lambda}, \hat{p}) = 0.
\]

Now, if we look at the region \( R^{\hat{\lambda}}_+ \) and \( R^{\hat{\lambda}}_- \) separately, we can apply the Serrin edge point lemma. But, since we know that \( v^{\hat{\lambda}}_q(\hat{q}, \hat{p}) = 0 \) and \( v^{\hat{\lambda}}_p(\hat{q}, \hat{p}) = 0 \), then we can rule out the first conclusion of the Serrin edge point lemma that states the strict sign on the derivative of \( v^{\hat{\lambda}} \) in outward direction. Hence, the second derivative of \( v^{\hat{\lambda}} \) in outward direction has a strict sign (see Theorem A.1(c)). Consider the following outward vectors associated to \( R^{\hat{\lambda}}_+ \) and \( R^{\hat{\lambda}}_- \) respectively:

\[
t := \frac{1}{(h^{\hat{\lambda}+}_p)^{3/2}}(1, -1), \quad \text{and} \quad s := \frac{1}{(h^{\hat{\lambda}-}_p)^{3/2}}(1, 1).
\]

58
Evaluating $\partial^2_s v^\lambda$ and $\partial^2_t v^\lambda$ at $(\hat{\lambda}, \hat{p})$, the Serrin edge point lemma gives the inequalities

$$\frac{1}{(h_p^+)^3} v^\lambda_{pq} \bigg|_{(\hat{\lambda}, \hat{p})} > 0 \quad \text{and} \quad \frac{1}{(h_p^-)^3} v^\lambda_{pq} \bigg|_{(\hat{\lambda}, \hat{p})} < 0.$$ 

This implies that $\left[v^\lambda_{pq} h_p^{-3}\right](\hat{\lambda}, \hat{p}) < 0$, which contradicts (2.91). Therefore, it must be that $v^\lambda \equiv 0$ in $R_{\hat{\lambda}}$, and hence $h$ is symmetric with respect to the axis $q = \hat{\lambda}$.

It is left to show the strict monotonicity of $h$. We know that for a fixed $\lambda < \hat{\lambda}$, $v^\lambda > 0$ in $R_\lambda$. Since $v^\lambda$ vanishes on $q = \lambda$ (right boundary of $R_\lambda$), then $v^\lambda$ attains its minimum there. Applying the Hopf boundary lemma yields

$$v^\lambda_q(\lambda, p) < 0.$$ 

But, $v^\lambda_q(\lambda, p) = -2h_q(\lambda, p)$. Hence, we have

$$h_q(\lambda, p) > 0, \quad \text{for all } \lambda < \hat{\lambda} \text{ in } [-1, 0].$$

Next, we consider the case when $p = \hat{p}$. By continuity of $h_q$, for fixed $\lambda < \hat{\lambda}$, we have that $h_q(\lambda, \hat{p}) \geq 0$. Suppose $h_q(\lambda, \hat{p}) = 0$. This implies that $v_q(\lambda, \hat{p}) = 0$. We also know the following facts:

$$v^\lambda_{qq} = 0, \quad v^\lambda = 0, \quad h^\lambda_+ = h^+_p, \quad h^\lambda_- = h^+_p \quad \text{at } (\lambda, \hat{p}). \quad (2.93)$$

We now look at the operator $\mathcal{B}$ in (2.75). Differentiating $\mathcal{B}v^\lambda$ with respect to $q$, evaluating it at $(\lambda, \hat{p})$ and using the facts mentioned previously in (2.93), we obtain a similar equation to the one in (2.91):

$$\left[v^\lambda_{pq} h_p^{-3}\right](\lambda, \hat{p}) = 0. \quad (2.94)$$

Again, let $t$ and $s$ denote the outward normal vectors to $R^+_{\lambda}$ and $R^-_{\lambda}$, respectively, as given by (2.92). Appealing to the Serrin edge point once more, we see immediately
that (2.94) cannot hold. Hence, \( h_q(\lambda, \hat{p}) > 0 \). Thus, we proved the strict monotonicity of \( h \) at the internal interface.

On the top boundary \( T^\hat{\lambda} \), we have \( h_q \geq 0 \) by continuity. We aim to show that \( h_q \) has a strict sign there. By way of contradiction, suppose that there exists \( \lambda < \hat{\lambda} \) such that \( h_q(\lambda, 0) = 0 \) as well. Differentiating the equation in (2.88) with respect to \( q \) and evaluating it at \((\lambda, 0)\) utilizing the following identities:

\[
\begin{align*}
  h_q &= -h_q^\lambda = 0, \\
  h_p &= h_p^\lambda, \\
  h_{qp} &= -h_{qp}^\lambda, \\
  v^\lambda &= v_q^\lambda = 0 \\
\end{align*}
\]

we arrive at the equation

\[
2h_p^\lambda(\lambda, 0)v_{qp}^\lambda(\lambda, 0) = 0.
\]

Thus, the above equation and the no horizontal stagnation condition implies

\[
v_{qp}^\lambda(\lambda, 0) = 0.
\]

As before, we also have that \( v_p^\lambda(\lambda, 0) = v_{pp}^\lambda(\lambda, 0) = v_{qq}^\lambda(\lambda, 0) = 0 \). Therefore, via the Serrin edge point lemma, we arrive at a contradiction. We then conclude that \( h_q > 0 \) on \( T^\hat{\lambda} \). Hence, \( h \) is monotone on \( R_\lambda \cup T_\lambda \cup I_\lambda \).

**Case 2.** Let \( \hat{\lambda} = \infty \). This just means that \( v^\lambda \geq 0 \) in \( R_\lambda \) for all \( \lambda \). Since \( v^\lambda \) is in the kernel of operators \( \mathcal{L} \) in (2.75), therefore we can apply the maximum principle which guarantees that \( v^\lambda > 0 \) in \( R_\lambda \) for all \( \lambda \). Applying the same argument as above, we see that \( h_q > 0 \) in \( R_\lambda \cup T_\lambda \cup I_\lambda \). In other words, we now have that \( h_q > 0 \) in \( R \). Thus, this implies that \( h \) is a monotone front. But this violates the nonexistence of monotone fronts (see Theorem 2.36). Hence, we can exclude case 2. Thus, the proof is complete. ■
2.4.3 Asymptotic monotonicity and nodal properties

The monotonicity property (2.73) will eventually be crucial for the large-amplitude theory where it is used to prove pre-compactness of $F^{-1}(0)$. The small-amplitude waves that will be constructed in Section 2.5 are waves of elevation, and hence monotone simply as a consequence of Theorem 2.15. We will then need to show that this property holds along the global bifurcation curve.

However, the set of monotone functions is neither open nor closed in the topology we are working with. This is a common issue in global bifurcation theoretic studies of elliptic PDE. To remedy it, we introduce additional sign conditions on the derivative of the solutions that are collectively called nodal properties. These conditions will in particular imply monotonicity, but are also open and closed in a relevant topology. Once we confirm that they are exhibited by the small-amplitude solutions, it immediately follows that they hold along a connected set extending the local curve. The main tool here is the maximum principle. As in the previous subsection, we will take advantage of the translation invariant structure of the equation.

Let $q = 0$ be the axis of even symmetry. We start by dividing the region to the right of the crest into four sub-domains: the upper and lower rectangles are each split into a finite sub-rectangle and semi-infinite rectangle. Proving the nodal properties on the finite rectangle is done essentially as in the periodic case in [Wal09b, Section 5] and so can be omitted. We focus instead on the semi-infinite rectangles for which we adopt the idea of [Whe15b, Section 2.2].
To fix notation, we define the following regions:

\[ R_{\pm} := R_{\pm} \cap \{ q > 0 \}, \quad L_0^+ := \{ (0, p) : \hat{p} < p \leq 0 \}, \quad L_0^- := \{ (0, p) : -1 \leq p < \hat{p} \}, \]

(2.95)

with the boundary components \( T_>, I_>, B_> \) given accordingly. We will use \( R_> \) to denote \( R_+ \cup R_- \). In a similar way, \( L_0 := L_0^+ \cup L_0^- \).

We begin by showing that small solutions \( w \) in the half-strip, for which \( w_q \) has a sign along the left boundary, must be monotone throughout the half-strip. This fact will allow us to infer monotonicity on the semi-infinite tail regions once monotonicity on the finite extent rectangles is known.

**Proposition 2.18** (Asymptotic monotonicity). There exists \( \delta > 0 \) such that, if

\[(w, F) \in C^3_b(R_+) \cap C^3_b(R_-) \cap C^0_0(R >) \cap C^1_0(R_+) \cap C^1_0(R_-) \times \mathbb{R}\]

is a solution of the height equation (2.33) in \( R_+ \) for \( F > F_{cr} \), with \( \|w\|_{C^2(R_+)} < \delta \) and \( \pm w_q \leq 0 \) on \( L_0 \), then

\[ \pm w_q < 0 \text{ in } R_+ \setminus (B_+ \cup L_0). \quad (2.96) \]

**Proof.** Due to the translation invariance of the height equation (2.33), we know that \( v := h_q \) is in the kernel of the linearized operator:

\[
\left\{ \begin{array}{l}
\left( -\frac{h_q v_q}{h_p^2} + \frac{(1 + h_q^2) v_p}{h_p^3} \right)_p + \left( \frac{v_q}{h_p} - \frac{h_q v_p}{h_p^2} \right)_q - \frac{1}{F^2} \rho_p v = 0 \quad \text{in } R_+, \\
\frac{h_q v_q}{h_p^2} - \frac{(1 + h_q^2) v_p}{h_p^3} + \frac{1}{F^2} \rho v = 0 \quad \text{on } T_>, \\
\left[ \frac{h_q v_q}{h_p^2} \right] - (1 + h_q^2) \left[ \frac{v_q}{h_p^3} \right] + \frac{1}{F^2} \left[ \rho \right] v = 0 \quad \text{on } I_>, \\
v = 0 \quad \text{on } B_>.
\end{array} \right. \quad (2.97)
\]

Set \( v := \tilde{\Psi} u \) for \( \tilde{\Psi} \) as in Lemma 2.16. Thus, we can rewrite each equation in (2.97)
in terms of $u$. In particular, the interior equation becomes
\[
\left(\frac{\tilde{\Psi}}{h_p^3} + \frac{\tilde{\Psi} h_q^2}{h_p^3} \right) u_p - \left(\frac{\tilde{\Psi}}{h_p^3} - \frac{\tilde{\Psi} h_q}{h_p^2} \right) u_q + \left(\frac{\tilde{\Psi}}{h_q^3} u_q - \left(1 + h_q^2\right) \tilde{\Psi} p u_p - \frac{h_q}{h_p^2} \tilde{\Psi}_p u_q \right)
+ \left(\frac{\tilde{\Psi} h_q}{h_p^2} u_p - \left(1 + h_q^2\right) \tilde{\Psi} p u_q \right) - \left(\frac{h_q}{h_p^2} \tilde{\Psi} u_q \right) = 0.
\]
Taking $\|w\|_{C^2(R)} = \|h - H\|_{C^2(R)}$ to be small enough ensures the equation above represents a uniformly elliptic operator acting on $u$ on $R$. From the equation (2.76) satisfied by $\tilde{\Psi}$, we see that the coefficient of $u$ on the last line is strictly negative for $0 < \delta \ll 1$, and therefore the maximum principle can be applied to infer that $u$ attains no non-negative maximum in the interior.

We can do the same thing to the equation on the top boundary:
\[
- \frac{\left(1 + h_q^2\right) \tilde{\Psi}}{h_p^2} u_p + \frac{h_q}{h_p^2} \tilde{\Psi} u_q + \left(\frac{\tilde{\Psi}}{h_q^3} + \frac{1}{F^2} \rho \tilde{\Psi} + \frac{h_q^2}{h_p^3} \tilde{\Psi} p u_p - \frac{1}{h_p^3} \tilde{\Psi} p u_q \right) u = 0.
\]
From (2.77) and the smallness of $w$ in $C^2$, we can conclude that the coefficient of the zeroth-order term above is negative. Suppose then that $u$ attains its non-negative maximum on $T$. At that point, $u_q = 0$, while by the Hopf lemma $u_p > 0$, giving a contradiction.

Now, consider the equation on the internal interface $I$. By the same type of computation, we arrive at:
\[
\left[\frac{h_q}{h_p^2}\right] \tilde{\Psi} u_q - \left(1 + h_q^2\right) \left[\frac{u_p}{h_q^3 h_p^3}\right] \tilde{\Psi} + \left(-(1 + h_q^2) \left[\frac{\tilde{\Psi}}{h_q^3 h_p^3}\right] + \frac{1}{F^2} \left[\rho\right] \tilde{\Psi}\right) u = 0. \tag{2.98}
\]
Suppose that $u \geq 0$ and attains its maximum on the $I$. Thus $u_q$ vanishes there and by Hopf we know that $u_p^+ > 0$ and $u_p^- < 0$. This implies $\left[ u_p h_q^{-3} \right] > 0$. On the other hand, in view of (2.77), the coefficient of the zeroth-order term in (2.98) is also negative for $\delta$ sufficiently small. Thus we have shown that in (2.98), the first
term vanishes while the second and third are strictly negative. This clearly leads to a contradiction as sum must be 0.

In total, this reasoning shows that $u$, and hence $w_q$, is strictly negative in $R_\succ \setminus (B_\succ \cup L_0)$. The proof of the proposition is therefore complete. □

Consider the following nodal properties:

$$w_q < 0 \text{ in } R_\succ^+ \cup R_\succ^- \cup I_\succ \cup T_\succ, \quad (2.99a)$$

$$w_{qq} < 0 \text{ on } L_0 \setminus \{(0, -1)\}, \quad (2.99b)$$

$$w_{qp} < 0 \text{ on } B_\succ, \quad (2.99c)$$

$$w_{qq} < 0 \text{ at } (0, -1). \quad (2.99d)$$

Shortly, we will prove that these define open and closed sets in an appropriate topology. First, however, we present the following result that show one can deduce the full set of nodal properties from just (2.99a).

**Lemma 2.19.** Let $(w, F)$ be a solution of the height equation (2.33) in $R_\succ$, where $w \in C^3_{b,e}(R_\succ^+) \cap C^3_{b,e}(R_\succ^-) \cap C^1_0(R_\succ^+) \cap C^1_0(R_\succ^-) \cap C^0_0(R_\succ)$ satisfies (2.99a). Then $w$ satisfies all of the nodal properties (2.99).

**Proof.** Observe that the statement of the lemma does not require $w$ to be “small” nor that $F$ is supercritical. We shall prove each of the sign conditions in (2.99) consecutively. To begin, (2.99a) is true by hypothesis.

Consider next (2.99b). By (2.99a), $w_q$ is non-positive in $R_\succ$ and by symmetry it vanishes on $L_0$. Applying the Hopf lemma, we conclude that $w_{qq} < 0$ on $L_0$. It remains to show that $w_{qq} < 0$ holds at the points $(0, 0)$ and $(0, \hat{p})$. The argument for the first point is the same as in [CWW18, Lemma 4.20], so we focus on the second
one which is new. Via continuity of \( w_{qq} \), we know that \( w_{qq} \leq 0 \) at \((0, \hat{p})\). Seeking a contradiction, suppose that \( w_{qq} = 0 \). By evenness,\[ w_q = 0 \text{ and } w_{qp} \equiv 0 \text{ on } L_0. \tag{2.100} \]

Consider the internal equation:
\[
\left[ \frac{1 + h_q^2}{2h_p^2} \right] - \left[ \frac{1}{2H_p^2} \right] + \frac{1}{F^2} \left[ \rho \right] (h - H) = 0 \quad \text{on } I. \tag{2.101} \]

We differentiate (2.101) in the \( q \)-variable twice and evaluate it at \((0, \hat{p})\). Further, using the fact that \( h_{qq} = 0 \) combined with (2.100), we then obtain
\[
\left[ h_{qq}h_p^{-3} \right] (0, \hat{p}) = 0.
\]

Now, if we look at the region \( R^+ \) and \( R^- \) separately, we can apply the Serrin edge point lemma. But, since we know that \( h_{qq}(0, \hat{p}) = 0 \) and \( h_{qp}(0, \hat{p}) = 0 \), then we rule out the first possibility of the conclusion of the Serrin edge point lemma for each of the two regions. Hence, the later conclusion of the Serrin edge point lemma should hold. Consider the following two outward vectors associated to each \( R^+ \) and \( R^- \) respectively:
\[
t := \frac{1}{(h_p^+)^3} (-1, -1) \quad \text{and} \quad s := \frac{1}{(h_p^-)^3} (-1, 1). \tag{2.102}
\]

Evaluating \( \partial_t^2 h_q \) and \( \partial_s^2 h_q \) at \((0, \hat{p})\), the Serrin lemma gives the inequalities
\[
\frac{1}{(h_p^+)^3} h_{qqp}^+ > 0 \quad \text{and} \quad \frac{1}{(h_p^-)^3} h_{qqp}^- < 0,
\]
whence
\[
\left[ h_{qq}h_p^{-3} \right] (0, \hat{p}) > 0.
\]

Thus we have produced a contradiction. Therefore, (2.99b) holds.
Next, (2.99c) follows from the Hopf boundary lemma since $w_q$ vanishes identically along $B_\succ$. It remains only to prove the inequality in (2.99d). Recall, on $L_0^+$, $L_0^-$, and $B_\succ$ we know that $w_q \equiv 0$. Therefore,

$$w_q = w_{qp} = w_{qq} = w_{qqp} = w_{qqq} = 0 \quad \text{at} \quad (0, -1).$$

By the Serrin Edge point lemma, we conclude that $w_{qqp} < 0$ at $(0, -1)$ which then proves (2.99d). \[\blacksquare\]

**Lemma 2.20** (Open property). Let $(w, F)$ and $(\tilde{w}, \tilde{F})$ be supercritical solutions of the height equation (2.33) on $R_\succ$ with

$$w, \tilde{w} \in C^3_{b,e}(R_\succ^+) \cap C^3_{b,e}(R_\succ^-) \cap C^1_0(\overline{R_\succ}) \cap C^1_0(\overline{R_\succ^-}).$$

If $w$ satisfies the nodal properties (2.99), then there exists $\epsilon = \epsilon(w) > 0$ such that

$$\|w - \tilde{w}\|_{C^3(R_\succ)} + |F - \tilde{F}| < \epsilon$$

implies $\tilde{w}$ also satisfies the nodal properties (2.99).

**Proof.** Recall, to prove that $\tilde{w}$ satisfies (2.99), it is enough to show that it exhibits the monotonicity (2.99a). We start by dividing $R_\succ^+$ into two overlapping regions namely:

$$R_{1,\succ}^+ := \{(q, p) \in R_\succ^+ : q < 2K\} \quad R_{2,\succ}^+ := \{(q, p) \in R_\succ^+ : q > K\}.$$  

Likewise, the lower region $R_\succ^-$ is divided into $R_{1,\succ}^-$ and $R_{2,\succ}^-$. The top, bottom, internal and vertical boundaries of these rectangles are denoted $T_{i,\succ}$, $B_{i,\succ}$, $I_{i,\succ}$, $L_{i,0}^+$, and $L_{i,0}^-$ for $i \in \{1, 2\}$.

Let us first look at $R_{j,\succ}^+$, for $j \in \{+, -\}$. These two finite rectangles behave in the same way as in periodic case. Therefore, the proof could be done in the same way as
in [CS04, Lemma 5.1]. The basic idea is that, for any $K > 0$, there exists $\epsilon_K$ such that taking $0 < \epsilon < \epsilon_K$ ensures $\tilde{w}_q < 0$ in the interior. One then uses a Taylor expansion of $w$ and the nodal properties to conclude the same holds up to the boundary.

On the other hand, since $w \in C_0^2(R)$, one can choose large enough $K$ such that

$$\|w\|_{C^2(R^+_2)} < \delta/2,$$

where $\delta$ is given as in Proposition 2.18. Hence, setting $\epsilon := \min\{\delta/2, \epsilon_K\}$, we have $\tilde{w}_q < 0$ in $\overline{R^+_1} \setminus (B_{1,+} \cup \overline{L_{1,0}})$ which then implies $\tilde{w}_q \leq 0$ in $L^+_2$. Again, applying Proposition 2.18 we infer that $\tilde{w}_q < 0$ in $\overline{R^+_2} \setminus (B_{2,+} \cup \overline{L_{2,0}})$. Together with the previous paragraph, this shows $\tilde{w}$ satisfies (2.99a).

Lemma 2.21 (Closed property). Let $\{(w_n, F_n)\} \subset \mathcal{U}$ be a sequence of solutions to the height equation (2.33) on $R_\succ$. Suppose that there exists $(w, F) \in \mathcal{U}$ such that $(w_n, F_n) \to (w, F)$ in $C^2_0(R^+_2) \cap C^1_0(R^-_2) \cap C^0_0(\overline{R^+_2}) \cap C^1_0(\overline{R^-_2}) \times \mathbb{R}$. If each $w_n$ satisfies the nodal properties (2.99), then $w$ also satisfies the nodal properties (2.99) unless $w \equiv 0$.

Proof. Let $v := w_q$. Because each $w_n$ satisfies the nodal properties, we may infer that $w_q \leq 0$ in $\overline{R_\succ}$. Moreover, $v$ solves of the uniformly elliptic PDE (2.97) in $R_\succ$. Furthermore, since $v = 0$ on $L_{0}$ and $B_\succ$, we can apply the maximum principle and conclude the following three possibilities; (i) $v < 0$ in $\overline{R^-_2} \setminus (B_\succ \cup L_0)$; or (ii) there exists some point $(q^*, 0) \in T_\succ$; or (iii) there exists some point $(q^*, \hat{p}) \in I_\succ$ such that $v(q^*, 0) = 0$ or $v(q^*, \hat{p}) = 0$.

The proof of the lemma if either the first and second possibilities occur follows from [CWW18, Lemma 4.22]. Therefore, we only consider the third possibility here.
Assume that there exists \((q^*, \hat{p}) \in I_\succ\) such that \(v(q^*, \hat{p}) = 0\). By the transmission boundary condition in (2.97), we obtain \(\left\| \frac{v_p}{h_p^3} \right\| (q^*, \hat{p}) = 0\). But via Hopf boundary lemma, we know that \(v_p(q^*, \hat{p}^+) < 0\) and \(v_p(q^*, \hat{p}^-) > 0\). Hence, we arrive at a contradiction unless \(v \equiv 0\) in \(\overline{R_\succ}\) which is equivalent to saying \(w \equiv 0\).

2.5 Small-amplitude existence theory

In this section, we will construct a curve of small-amplitude solutions to the height equation (2.33) that bifurcates from the trivial solution \(w = 0\) at the critical Froude number \(F_{cr}\) defined in (2.61). With that in mind, we introduce the non-negative parameter \(\epsilon^2 := \mu_{cr} - \mu\), which will be positive for supercritical waves. The corresponding Froude number is thus \(F^\epsilon := (1/F_{cr}^2 - \epsilon^2)^{-1/2}\). We will frequently abuse notation by writing \((w, \epsilon)\) rather than \((w, F)\).

**Theorem 2.22** (Small-amplitude waves). There exists \(\epsilon_* > 0\) and a continuous local curve

\[ C_{loc} := \{(w^\epsilon, F^\epsilon) : 0 < \epsilon < \epsilon_*\} \subset X \times \mathbb{R} \quad (2.103) \]

of solutions to \(\mathcal{F}(w, F) = 0\) with the following properties

(a) (Continuity) The mapping \(\epsilon \mapsto w^\epsilon\) is continuous from \((0, \epsilon_*)\) to \(X\), with \(\|w^\epsilon\|_X \to 0\) as \(\epsilon \to 0\).

(b) (Invertibility) The linearized operator \(\mathcal{F}_w(w^\epsilon, F^\epsilon)\) is invertible \(X \to Y\) for all \(\epsilon \in (0, \epsilon_*)\)

(c) (Uniqueness) If \(w \in X\) satisfies \(w > 0\) on \(T\) and \(\|w\|_X\) is small enough, then for any \(\epsilon \in (0, \epsilon_*)\), \(\mathcal{F}(w, F^\epsilon) = 0\) implies \(w = w^\epsilon\).
(d) (Elevation) \((w^\varepsilon, F^\varepsilon)\) is a wave of elevation: \(w^\varepsilon > 0\) on \(R \cup I \cup T\).

In proving Theorem 2.22 we will use the center manifold reduction technique introduced in [CWW19], which is a variation of the classical theory due to Kirchgässner [Kir82] and Mielke [Mie86, Mie88a]. This newer version is well-suited to the present work as it is conducted entirely in spaces of Hölder class functions and the computation of the reduced equation on the center manifold is done through a power series expansion that is comparatively straightforward. Moreover, the resulting ODE directly governs the internal interface, which allows us to prove that \(w^\varepsilon\) is a wave of elevation rather easily.

Recall that from Lemma 2.7 the spectrum of the transversal linearized operator at the trivial solution \((w, \varepsilon) = (0, 0)\) and the critical Froude number consists of a simple 0 eigenvalue with the remainder being strictly negative. For convenience, in this section, we write the linearized operator around the trivial flow as \(\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)\) with

\[
\begin{aligned}
\mathcal{L}_1 w &:= \left( \frac{w_p}{H_p^3} \right)_p + \left( \frac{w_q}{H_p} \right)_q - \mu_{cr} \rho_p w, \\
\mathcal{L}_2 w &:= \left( - \frac{w_p}{H_p^3} + \mu_{cr} \rho w \right)_T, \\
\mathcal{L}_3 w &:= \left( - \left[ \frac{w_p}{H_p^3} \right] + \mu_{cr} \left[ \rho \right] w \right)_I.
\end{aligned}
\]

(2.104)

Here we have reintroduced the shorthand \(\mu_{cr} = 1/F_{cr}^2\).

As always, the computation of the center manifold reduction requires (temporarily) expanding our function spaces to allow small exponential growth in \(q\). In particular, we will view \(\mathcal{L}\) as mapping \(X_\nu \rightarrow Y_\nu\) for some \(0 < \nu \ll 1\), where \(X_\nu\) and \(Y_\nu\) correspond to \(X\) and \(Y\) with the standard Hölder norm replaced by the exponentially weighted version defined in (2.31). Then it is easily confirmed that the kernel of this operator
is two dimensional and takes the form

$$\ker L : X_\nu \to Y_\nu = \left\{ (A + Bq)\Phi_0(p) \in X_\nu : (A, B) \in \mathbb{R}^2 \right\}, \quad (2.105)$$

where $\Phi_0$ generates the null space of the transversal linearized operator and is normalized so that $\Phi_0(\hat{p}) = 1$. It will be convenient to introduce a projection $Q : X_\nu \to X_\nu$ onto this kernel given by

$$Qw = (w(0, \hat{p}) + w_q(0, \hat{p}) q) \Phi_0(p).$$

Finally, note that the height equation (2.33) can be written as a quasilinear transmission problem for the elliptic PDE operator

$$\nabla \cdot A(p, w, \nabla w) + B(p, w, \epsilon),$$

where $A$ and $B$ are $C^8$ in their arguments due to the regularity assumptions in (2.7). Of course, $A$ and $B$ are actually analytic with respect to $w$ and $\nabla w$, but they have finite smoothness in $p$ because the coefficients involve $\rho$ and $H_p$.

Together these facts ensure that the hypotheses of the center manifold reduction result \cite[Theorem 1.1]{CWW19} are satisfied (specifically, we use the extension of that theorem to transmission problems given in \cite[Section 2.7]{CWW19}). As a direct consequence, we obtain the following.

**Lemma 2.23** (Center Manifold). There exists $0 < \nu \ll 1$, neighborhoods $U \subset X \times \mathbb{R}$ and $V \subset \mathbb{R}^3$, and a $C^5$ coordinate map $\Lambda = \Lambda(A, B, \epsilon) : \mathbb{R}^3 \to X_\nu$ satisfying

$$\Lambda(0, 0, \epsilon) = \Lambda_A(0, 0, \epsilon) = \Lambda_B(0, 0, \epsilon) = 0 \text{ for all } \epsilon,$$

such that the following hold.
(a) Suppose that \((w, \epsilon) \in \mathcal{U}\) solves \((2.33)\). Then \(v(q) := w(q, \hat{p})\) solves the second-order ODE
\[
v'' = f(v, v', \epsilon),
\]
where \(f : \mathbb{R}^3 \mapsto \mathbb{R}\) which is defined as follows
\[
f(A, B, \epsilon) := \lim_{q \to 0} \frac{d^2}{dq^2} \Lambda(A, B, \epsilon)(q, \hat{p}).
\]

(b) Conversely, if \(v : \mathbb{R} \to \mathbb{R}\) solves the ODE \((2.106)\) and \((v(q), v'(q), \epsilon) \in \mathcal{V}\) for all \(q\), then \(v := w(\cdot, \hat{p})\) for solution \((w, \epsilon) \in \mathcal{U}\) of the PDE \((2.33)\). Moreover, we write it as
\[
w(q + \tau, p) = v(q)\Phi_0(p) + v'(q)\tau\Phi_0(p) + \Lambda(v(q), v'(q), \epsilon)(\tau, p),
\]
for all \(\tau \in \mathbb{R}\).

\textbf{Remark 2.24.} By inspection, it is easy to verify that the height equation \((2.33)\) is invariant under the reversal transformation \(w \mapsto w(-\cdot, \cdot)\). One can show that this gives rise to a symmetry for the coordinate map:
\[
\Lambda(A, B, \epsilon)(q, p) = \Lambda(A, -B, \epsilon)(-q, p),
\]
and hence \(f\) is even in \(B\).

The next step is to derive the reduced ODE \((2.106)\) on the center manifold. In \cite[Theorem 1.3]{CWW19}, it is proved that the coordinate map \(\Lambda\) admits the Taylor expansion
\[
\Lambda(A, B, \epsilon) := \sum_J \Lambda_{ijk} A^i B^j \epsilon^k + O\left((|A| + |B|)(|A| + |B| + |\epsilon|)^4 \right),
\]

71
where

\[ J = \{(i, j, k) \in \mathbb{N}^3 : 2i + 3j + k \leq 4, i + j + k \geq 2, i + j \geq 1\}. \]

Each of the coefficient functions \( \Lambda_{ijk} \in X_\nu \) lies in the kernel of \( Q \) and satisfies

\[
\partial_i A \partial_j B \partial_k \lambda(A, B, \lambda) = (0, 0, 0)
\]

\[
F((A + Bq)\Phi_0(p) + \Lambda(A, B, \lambda), \lambda) = 0 \quad \text{for all } 2i + 3j + k \leq 4,
\]

with the above derivatives being of the formal Gâteaux type. By [CWW19, Lemma 2.3], this determines the \( \Lambda_{ijk} \) uniquely. Note that our need to expand to fourth order, and hence for \( \Lambda \) to be \( C^5 \), is precisely the reason behind the regularity of the background flow assumed in (2.9).

Explicitly, the index set \( J \) only contains the following 3-tuples:

\[ J = \{(2, 0, 0), (0, 1, 1), (1, 0, 1), (1, 0, 2)\}. \]

Following the procedure outlined in [CWW19, Section 2.6], computing the coefficients in the expansion (2.109) requires us to solve a hierarchy of equations taking the general form

\[
\begin{cases}
L \Lambda_{ijk} = R_{ijk}, \\
Q \Lambda_{ijk} = 0,
\end{cases}
\]

where each \( R_{ijk} \in X_\nu \) depends on previously computed terms. This calculation is largely elementary but quite onerous. For that reason, we use a computer algebra package to verify the results.

Computing the above Gâteaux derivatives, we see that \( L \Lambda_{101} = 0 \), and hence \( \Lambda_{101} = 0 \) by uniqueness. The same type of calculation will also show that \( \Lambda_{011} = 0 \). The remaining coefficients, however, are nontrivial. Indeed, we find that

\[
L \Lambda_{102} = \begin{pmatrix}
-\rho_p \Phi_0 \\
\rho(0)\Phi_0(0) \\
[\rho] \Phi_0(\hat{p})
\end{pmatrix}
= \begin{pmatrix}
R_1 \\
R_2 \\
R_3
\end{pmatrix},
\]

\( Q \Lambda_{102} = 0 \). (2.110)
Since, $\mathcal{L}\Lambda_{102}$ is independent of $q$, we infer that $\mathcal{L}(\partial_q\Lambda_{102}) = 0$. Therefore $\partial_q\Lambda_{102}$ is in the kernel of $\mathcal{L}$, and so by (2.105) it must take the form

$$\Lambda_{102}(q, p) = (A_1q + \frac{1}{2}B_1q^2)\Phi_0(p) + K_1(p), \quad (2.111)$$

for some constants $A_1, B_1$, and function $K_1$ to be determined. Applying the operator $\mathcal{L}$ to this ansatz and recalling (2.110), we obtain

$$\mathcal{L}'K_1 = \begin{pmatrix} \mathcal{R}_1 - \frac{B_1\Phi_0}{H_p} \\ \mathcal{R}_2 \\ \mathcal{R}_3 \end{pmatrix}. \quad (2.112)$$

Here, $\mathcal{L}'$ is the transversal linearized operator found by restricting $\mathcal{L}$ to $q$-independent functions. Note that $\Phi_0$ generates the kernel of $\mathcal{L}'$ by definition.

Now, multiplying the first component of the equation in (2.112) by $\Phi_0$ and integrating by parts, we find that

$$B_1\int_{-1}^{0} \frac{\Phi_0^2}{H_p} dp + \left[ (\Phi_0)p_K_1 - \Phi_0(K_1)_p \right] \Bigg|_{p=0} = \int_{-1}^{0} \mathcal{R}_1 \Phi_0 dp.$$

Using the fact that $\mathcal{L}'\Phi_0 = 0$, the above identity simplifies to

$$B_1\int_{-1}^{0} \frac{\Phi_0^2}{H_p} dp + \mathcal{R}_3\Phi_0(\hat{p}) - \mathcal{R}_2\Phi_0(0) = \int_{-1}^{0} \mathcal{R}_1 \Phi_0 dp.$$

Hence, $B_1$ takes form

$$B_1 = \frac{\int_{-1}^{0} \mathcal{R}_1 \Phi_0 dp - \mathcal{R}_3\Phi_0(\hat{p}) + \mathcal{R}_2\Phi_0(0)}{\int_{-1}^{0} \frac{\Phi_0^2}{H_p} dp} > 0.$$

Notice that $A_1q\Phi_0$ is in the kernel of $\mathcal{Q}$, and hence will not enter into the reduced equation. Thus we have computed the relevant part of $\Lambda_{102}$.

Following the same strategy for $\Lambda_{200}$ yields

$$\Lambda_{200} = (A_2q + \frac{1}{2}B_2q^2)\Phi_0(p) + K_2(p). \quad (2.113)$$
where the coefficient
\[
B_2 = -\frac{3}{2} \frac{\int_{-1}^{0} \frac{(\Phi_0)^3}{H_p^4} \, dp}{\int_{-1}^{0} \frac{\Phi_0^2}{H_p} \, dp} < 0.
\]

At this stage, we have all the information needed to find the reduced equation (2.107). First, we consider the truncated ODE where only the leading order terms of \( \Lambda \) are retained:

\[
v_0^{\prime\prime} = B_1 \epsilon^2 v^0 - B_2 (v^0)^2,
\]

where \( v^0(q) := w(q, \hat{p}) \). Note that the functions \( K_1 \) and \( K_2 \) play no role as they are independent of \( q \). One can verify directly that

\[
v^0(q) = \frac{3B_1 \epsilon^2}{2B_2} \text{sech}^2 \left( \frac{\epsilon \sqrt{B_1}}{2} q \right),
\]

is an explicit solution to the truncated reduced equation that is homoclinic to 0. It is left to show that this orbit persists for the full reduced equation.

**Proof of Theorem 2.22.** Let us introduce the scaled variables

\[
q = \epsilon^{-1} Q, \quad v(q) = \epsilon^2 V(Q).
\]

We can then rewrite the reduced ODE (2.114) as the following planar system:

\[
\begin{cases}
V_Q = W, \\
W_Q = B_1 V - B_2 V^2 + R(V, W, \epsilon).
\end{cases}
\]

The error term \( R(A, B, \epsilon) = \mathcal{O}(\epsilon^2 |A| + |\epsilon||B|) \) by (2.109) and is even in \( B \) (see Remark 2.24). Taking \( \epsilon = 0 \), we get back a rescaled version of the truncated reduced ODE in (2.114). Moreover, the explicit solution \( v^0 \) (2.115) in the rescaled variables becomes

\[
V^0(Q) = \frac{3B_1}{2B_2} \text{sech}^2 \left( \frac{\sqrt{B_1}}{2} Q \right), \quad W^0(Q) = -\frac{3B_1^{3/2}}{2B_2} \tanh \left( \frac{\sqrt{B_1}}{2} Q \right) \text{sech}^2 \left( \frac{\sqrt{B_1}}{2} Q \right),
\]
which is an explicit solution to (2.117) when $\epsilon = 0$. Moreover, this orbit is homoclinic to the origin and intersects the $V$-axis transversally. The symmetry property exhibited by $\Lambda$ (2.108) implies that the ODE (2.117) is reversible in the sense that it is invariant with respect to $(V(Q), W(Q)) \mapsto (V(-Q), -W(-Q))$. A standard planar systems argument then implies that the homoclinic orbit $(V^0, W^0)$ persists for sufficiently small $\epsilon$, giving a continuous one-parameter family of homoclinic solutions. Undoing the scaling, we obtain the local curve $C_{\text{loc}}$, proving (2.103). Part (a) is a consequence of the continuity of the reduction function.

Next, we will show that $C_{\text{loc}}$ consists of waves of elevation as claimed in (d). It is easy to see from the equation (2.117) satisfied by $V^\epsilon := w^\epsilon (\cdot / \epsilon, \hat{p}) / \epsilon^2$ that $V^\epsilon > 0$ and exponentially localized for $0 < \epsilon \ll 1$. Moreover from the phase portrait as $Q \to \pm \infty$ we can infer that

$$\lim_{Q \to \pm \infty} \frac{W(Q)}{V(Q)} = \pm \sqrt{B_1} + \mathcal{O}(\epsilon).$$

Taking $\epsilon$ small enough and undoing the scaling, this yields

$$|v'(q)| \lesssim C\epsilon |v(q)|,$$

which holds for some $C > 0$ and all $0 < \epsilon \ll 1$.

On the other hand, combining the solution ansatz given by Lemma (2.23)[b] with the expansion of the coordinate map $\Lambda$ in (2.109), we have

$$w^\epsilon(q, p) = v(q)\Phi_0(p) + \Lambda_{200}(0, p)v(q)^2 + \Lambda_{102}(0, p)v(q)\epsilon^2 + r(v(q), v'(q), \epsilon)(0, p),$$

where the remainder term satisfies

$$r(A, B, \epsilon)(0, \cdot) = \mathcal{O}\left((|A| + |B|)(|A| + |B| + |\epsilon|)^4\right) \quad \text{in } C^2([-1, \hat{p}]) \cap C^2(\hat{p}, 0].$$
In concert with (2.119) and (2.118), this shows that for $\epsilon$ sufficiently small we have $w_p > 0$ in $R^- \cup R^+$. Since $w = 0$ on $p = -1$, this gives that $w > 0$ in $R$, proving part (d).

Lastly, we will show that the linearized operator $\mathcal{F}_w(w, F)$ on the $C_{\text{loc}}$ is invertible. Recall also that from Lemma 2.10 we know that $\mathcal{F}_w(w, F)$ is Fredholm of index 0. As a consequence, for all $(w, F) \in C_{\text{loc}}$, $\mathcal{F}_w(w, F)$ is invertible if and only if it has a trivial kernel. As we have seen many times, the translation invariance in $q$ means that $\mathcal{F}_w(w^\epsilon, F^\epsilon)w_q^\epsilon = 0$.

To identify other potential solutions of the linearized problem, we make use of [CWW19, Theorem 1.6]. This result states that $\dot{w}$ satisfies $\mathcal{F}_w(w^\epsilon, F^\epsilon)\dot{w} = 0$ if and only if $\dot{v} := \dot{w}(\cdot, \hat{p})$ solves the linearized reduced ODE

$$\frac{d^2}{dq^2} \dot{v} = f(A,B)(v^\epsilon, v_q^\epsilon, \epsilon) \cdot (\dot{v}, \dot{v}_q),$$

where $v^\epsilon = w^\epsilon(\cdot, \hat{p})$. Performing the same rescaling as before, we see that the corresponding (linear) planar system is given by

$$\begin{pmatrix} \dot{V}_Q \\ \dot{W}_Q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ f_A & f_B \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix},$$

where $f_A$ and $f_B$ are evaluated at $(\epsilon^2 V^\epsilon_Q(Q), \epsilon V^\epsilon(Q), \epsilon)$. Sending $|Q| \to \infty$, we therefore obtain

$$\lim_{Q \to \pm \infty} \begin{pmatrix} 0 & 1 \\ f_V & f_W \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B_1 + \mathcal{O}(\epsilon^2) & 0 \end{pmatrix}.$$ 

Clearly, the eigenvalues of the above matrix are both real, with one strictly positive and the other strictly negative. By standard dynamical system theory, there cannot exist two linearly independent bounded solutions to the reduced ODE. Hence, the only nontrivial solution of $\mathcal{F}_w(w^\epsilon, F^\epsilon)\dot{w} = 0$ is $\dot{w} = w_q^\epsilon$. But we have already established
that \( w^\epsilon \) is even, hence \( w_q^\epsilon \) is not in \( X \). Thus the kernel of \( \mathcal{F}_w(w^\epsilon, F^\epsilon) : X \rightarrow Y \) is trivial, completing the proof. ■

2.6 Large-amplitude existence theory

In this final section, we complete the argument for the existence of the curve \( \mathcal{C} \) of large-amplitude solutions and show that it exhibits the properties asserted in Theorem 2.1. The global curve is constructed through continuation of the local curve \( \mathcal{C}_{\text{loc}} \) obtained in Section 2.5. Our strategy is in the spirit of Wheeler’s [Whe13, Whe15a] work on homogeneous rotational waves and that of Chen, Walsh, and Wheeler’s [CWW18] study of continuously stratified fluids. In particular, the latter of these papers develops a general analytic global bifurcation theory that is adapted to monotone solutions on unbounded domains; see Appendix A. Using that machinery allows us to prove the existence \( \mathcal{C} \). To verify the extreme wave limit requires the bounds on the velocity field given by Theorem 2.4, which we prove in the next subsection, and uniform regularity estimates that are tackled in Section 2.6.2.

2.6.1 Velocity bound

Here, we derive some uniform \( L^\infty \) bounds on the velocity for solitary stratified waves. Throughout the analysis, the far-field state (as described by \( H \) and \( \rho \) or equivalently \( \hat{u} \) and \( \varrho^* \)) is fixed. To simplify the presentation, we do not track how the constants depend on these quantities. Recall that the relative velocity in terms of the stream function is given by \( \nabla^\perp \psi \). Having uniform \( L^\infty \) control on the velocity will ensures the uniformly ellipticity of the height equation \( (2.28) \) along \( \mathcal{C} \).

In earlier studies of rotational waves in constant density water [Var09] and contin-
uous stratified fluids \cite{CW18}, velocity bounds were obtained by first establishing a lower bound of the pressure via the maximum principle. Bernoulli’s equation then allows one to uniformly control the magnitude of the relative velocity. However, in the present work, it is not obvious that we can apply the same strategy due to the transmission boundary condition. Adopting instead the approach of \cite{AT86}, we start by deriving a “local” $L^2$ bound of $\nabla \psi$ which is recorded in the lemma below.

**Lemma 2.25 (Local velocity bound).** There exists $C = C(F_0, K) > 0$ such that, for any solution $(\psi, \eta)$ to (2.22)–(2.23) with $\psi_y < -1/K$, $F \geq F_0 > 0$, and any $m \in \mathbb{R}$,

$$\int_{m-1}^{m+1} \int_{-1}^{\eta(x)} |\nabla \psi|^2 \, dy \, dx < C.$$  

**Proof.** Working in semi-Lagrangian variables, this is equivalent to

$$\int_{m-1}^{m+1} \int_{-1}^{0} \frac{1 + h_q^2}{h_p} \, dp \, dq \leq C.$$  

Let $\xi = \xi(q)$ be a bump function supported in $[m-2, m+2]$, with $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $[m-1, m+1]$. If we multiply the interior height equation (2.28) by $\xi^2 w$, and integrate over the domain, we obtain

$$\int \int_R \left( \frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) w_p - \left( \frac{h_q}{h_p} \right) w_q \xi^2 \, dp \, dq$$

$$= \int \int_R \frac{2h_q^2}{h_p} \xi \xi_q + \frac{1}{F^2} \rho_p w^2 \xi^2 \, dp \, dq + \int_{T} \left( \frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) w \xi^2 \, dq$$

$$+ \int_{I} \left[ - \frac{1 + h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \right] w \xi^2 \, dq. \quad (2.121)$$

Observe that the factor of $\xi^2$ in the integrand on the left hand side of the above equation can be rewritten as follows:

$$\left( \frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) w_p - \left( \frac{h_q}{h_p} \right) w_q = \frac{2H_p + w_p}{2h_p^2} \left( - \frac{w_p^2}{H_p^2} - w_q^2 \right).$$
Therefore, via (2.121), the above equality along with Young’s inequality, and integration by parts we have

\[
\iint_R \frac{2H_p + w_p}{2h_p^2} \left(\frac{w_p^2}{H_p^2} + w_q^2\right) \xi^2 \, dp \, dq = -\iint_R \frac{2h_q}{h_p} w\xi \xi_q \, dp \, dq \\
+ 2\iint_R \frac{1}{F^2} \rho w_p w\xi^2 \, dp \, dq \\
\lesssim \epsilon_1 \iint_R \frac{h_q^2}{h_p^2} w^2 \xi^2 \, dp \, dq + \frac{1}{\epsilon_1} \int_R \xi_q^2 \, dq \\
+ \epsilon_2 \iint_R \rho^2 w_p^2 w^2 \xi^2 \, dp \, dq + \frac{1}{F^2\epsilon_2} \int_R \xi^2 \, dq,
\]

(2.122)

where the constants \(\epsilon_1\) and \(\epsilon_2\) are defined to be

\[
\epsilon_1 := \inf H_p \frac{1}{2K^2}, \quad \epsilon_2 := \frac{1}{2\|\rho\|^2_{L^\infty} K^2 \|H_p\|_{L^\infty}}.
\]

Upon grouping and simplifying (2.122), we obtain

\[
\iint_R \frac{1}{h_p} \xi^2 \, dp \, dq + \iint_R \frac{h_q^2}{h_p} \xi^2 \, dp \, dq \leq C(F_0, K).
\]

This then proves the estimate in (2.120). \(\blacksquare\)

Next, we show that along the internal interface, the velocity is uniformly controlled in \(L^\infty\). Our approach is based on that of Amick and Turner [AT86] and [CWW20]. In both those papers, however, the stream function is harmonic in each layer, which permits them to use the classical monotonicity formula of Alt–Caffarelli–Friedman [ACF84, Lemma 5.1]. Because we allow for general stratification, we must instead use the slightly weaker “almost monotonicity formula” due to Caffarelli–Jerison–Kenig [CJK02]. The precise application is presented in the next lemma. However, prior to using the formula, an intermediate step is done to make sure that the upper and internal layers are uniformly separated. This is the content of the next corollary.
which follows from the local estimate \((2.120)\) and the fact that the relative pseudo-volumetric mass flux is fixed.

**Corollary 2.26** (Interface separation bound). Under the hypothesis of Lemma 2.25, we have

\[ \inf_x (\eta(x) - \zeta(x)) > C|\hat{p}|^2, \]  

(2.123)

where the constant \(C = C(F_0, K) > 0\).

**Proof.** Via the definition of the relative pseudo-volumetric mass flux in the upper layer, we have for all \(x\) and \(m \in \mathbb{R}\)

\[
|\hat{p}| = \int_{\zeta(x)}^{\eta(x)} \sqrt{\rho(c-u)} \, dy = \frac{1}{2} \int_{m-1}^{m+1} \int_{\zeta(x)}^{\eta(x)} \sqrt{\rho(c-u)} \, dy \, dx \\
\leq \frac{1}{2} \left( \int_{m-1}^{m+1} \int_{\zeta(x)}^{\eta(x)} 1 \, dy \, dx \right)^{1/2} \left( \int_{m-1}^{m+1} \int_{\zeta(x)}^{\eta(x)} \rho(c-u)^2 \, dy \, dx \right)^{1/2} \\
< C\sqrt{\eta(x) - \zeta(x)},
\]

where we have used the fact that \(\rho(c-u)^2 \leq |\nabla \psi|^2\) and the local velocity bound (2.25). This then leads to the inequality in (2.123). \(\blacksquare\)

**Lemma 2.27** (Interfacial velocity bounds). Any solution \((\psi, \eta, \zeta)\) to the water wave problem \((2.22)-(2.23)\) with \(\sup \psi_y < -1/K\) and \(F \geq F_0 > 0\) satisfies the bound

\[ |\nabla \psi_+|^2 + |\nabla \psi_-|^2 < C \quad \text{on } \mathcal{I}, \]  

(2.124)

where the constant \(C = C(F_0, K) > 0\).

**Proof.** Again, we will use \(C\) to denote a generic positive constant depending only on the quantities listed in the statement.

Fix \((x_0, y_0) \in \mathcal{I}\), where recall \(\mathcal{I}\) denotes the internal interface in the original coordinate system. Let \(a := \text{dist}((x_0, y_0), \partial \Omega \setminus \mathcal{I})\). Observe that, in view of Corollary 2.26, \(a\) is uniformly positive.
We will work with the rescaled coordinates \((\tilde{x}, \tilde{y}) = ((x - x_0)/a, (y - y_0)/a)\). Likewise, we introduce the modified and rescaled stream functions \(\tilde{\psi}_-\) and \(\tilde{\psi}_+\) defined by

\[
\tilde{\psi}_-(\tilde{x}, \tilde{y}) := \max \left\{ 0, \frac{\psi(x, y) + \hat{p}}{M_-} \right\}, \quad \tilde{\psi}_+(\tilde{x}, \tilde{y}) := \max \left\{ 0, -\frac{\psi(x, y) + \hat{p}}{M_+} \right\}
\]

for \((\tilde{x}, \tilde{y}) \in B_1\), the unit ball centered at \((0, 0)\) in the \((\tilde{x}, \tilde{y})\)-plane. Here,

\[
M_- := 2a^2 \max \left\{ \|\beta_+\|_{L^\infty}, \frac{1}{F^2} K \|\rho_p\|_{L^\infty} \right\}, \quad M_+ := 2a^2 \max \left\{ \|\beta_-\|_{L^\infty}, \frac{1}{F^2} \|\rho_p\|_{L^\infty} \right\}.
\]

Notice that \(\tilde{\psi}_\pm\) is non-negative and \(C^{0+\alpha}(B_1)\) but the product \(\tilde{\psi}_-\tilde{\psi}_+\) vanishes identically. Moreover, from the definitions of \(M_\pm\) and Yih’s equation (2.22), we find that

\[
\Delta_{(\tilde{x}, \tilde{y})} \tilde{\psi}_\pm \geq -1 \quad \text{on } B_1,
\]

with the inequality holding in the sense of distributions.

Consider the function

\[
\phi(r) := \left( \frac{1}{r^2} \iint_{B_r} |\nabla \tilde{\psi}_+|^2 \, dx \, dy \right) \left( \frac{1}{r^2} \iint_{B_r} |\nabla \tilde{\psi}_-|^2 \, dx \, dy \right) \quad \text{for } 0 < r < 1,
\]

where \(B_r\) denotes the ball of radius \(r\) centered at the origin in the \((\tilde{x}, \tilde{y})\) variables. Applying \[\text{CJK02}\] Theorem 1.3], we can infer that for any \(0 < r < 1\),

\[
\phi(r) \leq C_0 \left( 1 + \iint_{B_1} |\nabla \tilde{\psi}_+|^2 \, dx \, dy + \iint_{B_1} |\nabla \tilde{\psi}_-|^2 \, dx \, dy \right)^2, \quad (2.125)
\]

for a universal constant \(C_0 > 0\). Because of the Hölder regularity of \(\tilde{\psi}_\pm\), from \[\text{CJK02}\] Theorem 1.6], we know that \(\phi(r)\) has a limiting value as \(r \to 0\), which must then coincide with

\[
\phi(0) := \frac{\pi^2}{4} |\nabla \tilde{\psi}_+(0, 0)|^2 |\nabla \tilde{\psi}_-(0, 0)|^2.
\]
Combining this with (2.125) and undoing the scaling, we can say that on internal interface

\[ |\nabla \psi_+(x_0, y_0)|^2 |\nabla \psi_-(x_0, y_0)|^2 < C. \] (2.126)

Finally, from equation (2.126) and the Bernoulli condition (2.25), we arrive at the desired bound (2.124).

Having derived the velocity bounds on the interface, we are now prepared to prove Theorem 2.4. We state this result in the Dubreil-Jacotin variables as this is most convenient for applying it to the global bifurcation theory.

**Theorem 2.28** (Global velocity bounds). Let \((h, F) \in X \times \mathbb{R}\) be a solution to the height equation (2.28) with \(\|h_p\|_{L^\infty} < K\) and \(F \geq F_0 > 0\). Then

\[
\left\| \frac{1}{h_p} \right\|_{L^\infty(R)} + \left\| \frac{h_q}{h_p} \right\|_{L^\infty(R)} \leq C,
\]

where the constant \(C = C(F_0, K) > 0\).

**Proof.** Throughout the proof, we use \(C\) to denote a generic positive constant depending on the quantities in the statement of the theorem. Observe that, when converted to Eulerian variables, these correspond to the same quantities appearing in the statements of Lemma 2.25, Corollary 2.26, and Lemma 2.124. In particular, the previous lemma shows that \(|\nabla \psi| < C\) on the internal interface.

It remains, to control \(|\nabla \psi|\) away from \(\mathcal{J}\). For this, we use a maximum principle argument based on [CWW18, Proposition 4.1]. Define

\[ f := P + M\psi, \]

for \(M > 0\) to be determined. Using the boundedness of \(|\nabla \psi|\) and Bernoulli’s law, one can then show that \(|P| < C\) on \(\mathcal{J}\). From Yih’s equation (2.22) and an elementary
calculation, we see that \( f \) satisfies the elliptic PDE
\[
\Delta f - b_1 f_x - b_2 f_y = \frac{2F^2 - \rho(2M + \Delta \psi)\psi_y - 2F^{-4}\rho^2}{|\nabla \psi|^2} - (2M + \Delta \psi)M + \frac{1}{F^2} \rho \psi_y, \quad (2.127)
\]
where \( b_1 \) and \( b_2 \) are given as follows:
\[
b_1 := 2\frac{\psi_x(2M + \Delta \psi)}{|\nabla \psi|^2}, \quad b_2 := 2\frac{\psi_y(2M + \Delta \psi) - 2F^{-4}\rho}{|\nabla \psi|^2}.
\]
The point here is that there are no zeroth-order terms on the left-hand side of (2.127).

By proving the right-hand side is non-negative, we will be able to apply the maximum principle to \( f \).

With that in mind, observe that from Bernoulli’s law we have
\[
\frac{\eta}{F^2} < \frac{(\bar{u}(0) - c)^2}{2} = \frac{u^*(0)^2}{2gd}.
\]

Via Yih’s equation, we know that
\[
\Delta \psi = -\beta + \frac{1}{F^2} y \rho \rho_p \geq -\|\beta_+\|_{L^\infty} - \frac{u^*(0)^2}{2gd} \|\rho_p\|_{L^\infty}.
\]

Hence, choosing
\[
M > M_1 := \|\beta_+\|_{L^\infty} + \frac{u^*(0)^2}{2gd} \|\rho_p\|_{L^\infty},
\]
yields \( \Delta \psi + M \geq 0 \). In conjunction with (2.127), this leads to the inequality
\[
\Delta f - b_1 f_x - b_2 f_y \leq -\left[M^2 - \frac{1}{F^2} \rho \psi_y\right].
\]

In view of the last inequality, we define
\[
M_2 := \frac{1}{F_0} \|\psi_y\|_{L^\infty}^{1/2} \|\rho_p\|_{L^\infty}^{1/2}.
\]
Provided \( M \geq M_1, M_2 \), we then have
\[
\Delta f - b_1 f_x - b_2 f_y \leq 0.
\]
Furthermore, as $x \to \pm \infty$, the pressure approaches hydrostatic, which implies that $f$ is non-negative in the upstream and downstream limits.

First, we shall show that $\inf f \geq 0$ in $\overline{\Omega}_+$. On the upper surface this holds by definition. Furthermore, we have already established that $|P| < C$ on $\mathcal{S}$. Thus, there exists some $M_3$ depending on the same constants, so that $f = P + M_3|\hat{p}| \geq 0$ on $\mathcal{S}$.

Finally, setting $M := \max\{M_1, M_2, M_3\}$, we have that $f \geq 0$ in $\overline{\Omega}_+$ via the maximum principle.

Similarly, we can show that $f \geq 0$ in $\overline{\Omega}_-$. On the bed,

$$f_y = P_y + M\psi_y = -\frac{1}{F^2}\rho + M\psi_y < 0.$$ 

By the Hopf lemma, this implies that $f$ cannot attain a minimum there. The claim follows easily via the maximum principle.

Thus, $f \geq 0$ throughout the domain. Recalling its definition, this gives a lower bound on the pressure. Using Bernoulli’s law, we can then control the velocity:

$$-\frac{1}{2}|\nabla \psi|^2 - \frac{1}{F^2}\rho y + E = P \geq -M\psi \geq -M.$$ 

After rearrangement and using the fact that $y > -1$, $F > F_0$, this becomes

$$|\nabla \psi|^2 \leq 2M - \frac{2}{F^2}\rho y + 2E \leq 2M + \frac{2}{F_0^2} + 2E. \tag{2.128}$$

Suppose that $M_2 \geq M_1, M_3$, that is $M = M_2$. Taking the supremum of the left hand side of (2.128) and dropping the $\psi_x$ term followed by applying Young’s inequality yields

$$\|\psi_y\|_{L^\infty}^2 \leq \frac{3}{F_0^{4/3}} \|\rho_p\|_{L^\infty}^{2/3} + \frac{4}{F_0} + 4\|E\|_{L^\infty}.$$
Using the definition of $M_2$ and the $L^\infty$ bound of $\psi_y$ above, we can infer

$$M_2 \leq \frac{1}{F_0} \|\rho_p\|_{L^\infty}^{1/2} \left( \frac{3}{F_0^{4/3}} \|\rho_p\|_{L^\infty}^{2/3} + \frac{4}{F_0} + 4 \|E\|_{L^\infty} \right)^{1/4}.$$ 

Plugging $M_2$ back into (2.128) gives

$$\|\nabla \psi\|_{L^\infty}^2 \leq \frac{2}{F_0} \|\rho_p\|_{L^\infty}^{1/2} \left( \frac{3}{F_0^{4/3}} \|\rho_p\|_{L^\infty}^{2/3} + \frac{4}{F_0} + 4 \|E\|_{L^\infty} \right)^{1/4} + \frac{2}{F_0} + 2 \|E\|_{L^\infty}.$$ 

Now, let us look into the case when $M_1 \geq M_2, M_3$, that is $M = M_1$. Plugging $M_1$ into (2.128), this directly gives us

$$\|\nabla \psi\|_{L^\infty}^2 \leq 2M_1 + \frac{2}{F_0} + 2 \|E\|_{L^\infty} \leq C.$$ 

Lastly, if $M_3 \geq M_1, M_2$, then setting $M = M_3$ leads to

$$\|\nabla \psi\|_{L^\infty}^2 \leq 2M_3 + \frac{2}{F_0} + 2 \|E\|_{L^\infty} \leq C.$$ 

Rewriting the above inequality in terms of semi-Lagrangian variables, we arrive at the desired inequality stated in Theorem 2.28.

**Remark 2.29.** As a result of the previous theorem, we can infer that $\|h_q\|_{L^\infty}$ is controlled by $\|h_p\|_{L^\infty}$. Moreover, because

$$w(q, p) = \int_{-1}^p w_p(q, p') \, dp',$$

we have

$$\|w\|_{L^\infty(R)} \lesssim \|w_p\|_{L^\infty(R)}.$$  \hfill (2.129)

**Corollary 2.30** (Bounds on $w$ and $\nabla w$). There exist constants $C = C(K)$ and $\delta = \delta(K)$ such that every supercritical solitary wave with $\|h_p\|_{C^0} < K$ satisfies

$$\inf_{R} (w_p + H_p) \geq \delta,$$
and

\[ \|w\|_{C^1(R)} \leq C. \]

**Proof.** As before, we will use \( C \) to denote a generic positive depending on \( K \). Taking \( F_0 = F_{\text{cr}} \) and applying Theorem 2.28 gives the bound

\[ \frac{1}{h_q^2} + \frac{h_q^2}{h_p^2} < C. \]  \tag{2.130}

Dropping the second term on the left hand side of (2.130) gives

\[ \inf_R h_p \geq \frac{1}{C} : = \delta. \]  \tag{2.131}

Similarly, dropping the first term on the left hand side of (2.130) gives

\[ \sup_R |h_q| \leq \sqrt{C} \sup_R h_p \leq C \left( 1 + \|w_p\|_{C^0(R)} \right). \]

Combining this with (2.129), we obtain the desired \( C^1 \) bound for \( w \). \( \blacksquare \)

### 2.6.2 Uniform regularity

The purpose of this subsection is to establish that the full \( X \) norm of \( w \) can be controlled in terms of \( \|w_p\|_{L^\infty} \). This will be used later to prove that, following the global bifurcation curve \( \mathcal{C} \), blow-up in norm corresponds to the onset of horizontal stagnation. Specifically, the main result is as follows.

**Theorem 2.31** (Uniform Regularity). For all \( \delta > 0 \), there exists \( C : = C(\delta) > 0 \) such that, if \( (w, F) \in X \times \mathbb{R} \) is a solution of the height equation (2.33) satisfying

\[ \inf_R (H_p + w_p) \geq \delta, \quad \|w_p\|_{C^0} + \|H_p\|_{C^0} < \frac{1}{\delta}, \]  \tag{2.132}

then it obeys the bound:

\[ \|w\|_{C^{3+\alpha}(R)} \leq C. \]
For steady waves in constant density or continuously stratified density water, this type of result is very well-known (see e.g., [CS04, Wal09b, Whe13, Wal14]). It is a direct consequence of the ellipticity of the height equation and obliqueness of the Bernoulli condition on the upper boundary — which are consequences of (2.132) — along with the translation invariance of the system. However, the two-fluid problem considered in the present work requires a significantly different approach. To derive estimates near the internal interface, one can adopt the idea of [AT86] and work with a weak formulation of the height equation (2.33). The proof of Theorem 2.31 is then straightforward to obtain following the general argument in [CWW20, Section 5.6]. For that reason, we will only sketch the details.

Specifically, the height equation (2.33) is recast as the distributional equation

\[
\begin{cases}
\nabla \cdot \left( G(\nabla w, H_p) - \frac{1}{F^2} \begin{pmatrix} 0 \\ \rho w \end{pmatrix} \right) + \frac{1}{F^2} \rho w_p = 0 & \text{in } R \cup I, \\
G_2(\nabla w, H_p) - \frac{1}{F^2} \rho w = 0 & \text{on } T, \\
w = 0 & \text{on } B,
\end{cases}
\] (2.133)

where \( G = (G_1, G_2) = (\partial_{\xi_1} f, \partial_{\xi_2} f) \), for \( f : \mathbb{R}^3 \to \mathbb{R} \) defined by,

\[
f(\xi_1, \xi_2, a) := \frac{a^2 \xi_1^2 + \xi_2^2}{2(a + \xi_2)a^2}.
\] (2.134)

Notice that the transmission condition on the internal interface is enforced by the fact that the first equation above holds on \( R \cup I \).

The main tool for proving regularity near the internal interface is the following theorem of Meyers, stated here in a simplified form appropriate to our setting.

**Theorem 2.32** (Meyers, [Mey63]). Let \( \mathcal{D} \subset \mathbb{R}^2 \) be a smooth domain. Consider

\[
\nabla \cdot (A \nabla u) = \nabla \cdot G + g \quad \text{in} \quad \mathcal{D},
\] (2.135)
and
\[ u = 0 \text{ on } \partial \mathcal{D}, \]
where \( A = A(x) \) is a matrix with measurable coefficients and enjoys \( c_1 I \leq A \leq c_1^{-1} I \) for some \( c_1 > 0 \), where \( I \) is the \( 2 \times 2 \) identity matrix. Then there exists \( r = r(c_1) \geq 2 \) such that for all \( G \in L^r(\mathcal{D}) \) and \( g \in L^2(\mathcal{D}) \), (2.135) admits a unique solution \( u \in W^{1,r}_0(\mathcal{D}) \), and satisfies the following inequality
\[ \| \nabla u \|_{L^r(\mathcal{D})} \leq C \left( \| G \|_{L^r(\mathcal{D})} + \| g \|_{L^2(\mathcal{D})} \right). \]

To use this result, we differentiate the equation (2.133) in the \( q \)-variable \( k \) times, say, regroup terms as in (2.135) treating \( u := \partial^k_q w \) as the unknown. Iterating this process furnishes \( W^{1,r} \) estimates for successively higher-order \( q \) derivative of \( w \). Eventually, by Morrey’s inequality, this will lead to a sufficient Hölder regularity of the trace of \( w \) on the boundary. Applying a simple Schauder estimates for the Dirichlet problem yields the desired regularity.

### 2.6.3 Proof of the main result

Now we are at last prepared to give the proof of Theorem 2.1. Recall from Section 2.2 that the height equation is expressed in terms of the nonlinear operator \( F : \mathcal{U} \subset X \times \mathbb{R} \to Y \), where the open set \( \mathcal{U} \) was defined in (2.36). Theorem 2.22 states that there exists a continuous local curve of solutions to the height equation (2.33) denoted by
\[ \mathcal{C}_{\text{loc}} := \{(w^\epsilon, F^\epsilon) : 0 < \epsilon < \epsilon_*\}, \]
which contains nontrivial waves of elevation that are symmetric, monotone and slightly supercritical. First, we show in the next theorem that \( \mathcal{C}_{\text{loc}} \) can be extended to a global
curve of solutions.

**Theorem 2.33** (Global continuation). The continuous local curve of solutions \( \mathcal{C}_{\text{loc}} \) of the nonlinear operator \( \mathcal{F}(w, F) = 0 \) is contained in a global \( C^0 \) curve \( \mathcal{C} \) parameterized as

\[
\mathcal{C} := \{(w(s), F(s)) : 0 < s < \infty\} \subset \mathcal{U},
\]

and exhibiting the following properties

(a) One of the following alternatives must hold:

(i) (Blowup) as \( s \to \infty \),

\[
N(s) := \|w(s)\|_X + \frac{1}{\inf_R (w_p(s) + H_p)} + F(s) + \frac{1}{F(s) - F_{cr}} \to \infty.
\]

(ii) (Loss of compactness) There exists a sequence of \( s_n \to \infty \) as \( n \to \infty \) such that \( \sup_n N(s_n) < \infty \) but \( \{w(s_n)\} \) does not have subsequences converging in \( X \).

(b) Fix parameter \( s^* \in (0, \infty) \), around the neighborhood \( (w(s^*), F(s^*)) \in \mathcal{C} \), we can reparametrize \( \mathcal{C} \) so that \( s \mapsto (w(s), F(s)) \) is real-analytic.

(c) For all \( s \gg 1 \), \( (w(s), F(s)) \notin \mathcal{C}_{\text{loc}} \).

**Proof.** Clearly, \( \mathcal{F} \) is a real analytic as a mapping \( \mathcal{U} \subset X \times \mathbb{R} \to Y \). Moreover, from Lemma 2.10 the linearized operator \( \mathcal{F}_w(w, F) \) is Fredholm of index zero for all \( (w, F) \in \mathcal{U} \). In Theorem 2.22 we proved that \( \mathcal{F}_w(w^\epsilon, F^\epsilon) \) has a trivial kernel for all \( (w^\epsilon, F^\epsilon) \in \mathcal{C}_{\text{loc}} \). Together these facts import invertibility of \( \mathcal{F}_w \) along the local curve \( \mathcal{C}_{\text{loc}} \). The statements of the theorem now follows directly from an application of the abstract global bifurcation result Theorem A.2. \( \blacksquare \)
The above theorem establishes the existence of a global curve, but we have yet to show that the solutions along it limit to stagnation as claimed in Theorem 2.1(a).

For that, we prove a series of lemmas bounding the various quantities occurring in the definition of \( N(s) \).

**Lemma 2.34.** The nodal properties (2.99) hold along the global curve \( \mathcal{C} \).

**Proof.** We start by showing that the nodal properties hold along the local curve \( \mathcal{C}_{\text{loc}} \).

Let \((w, F) \in \mathcal{C}_{\text{loc}}\). By Theorem 2.22(d), \((w, F)\) is a wave of elevation which means that \( w > 0 \) in \( \overline{\mathbb{R}} \setminus B \). Then, Theorem 2.15 tells us that \( w_q < 0 \) in \( \overline{\mathbb{R}} \setminus (B \cup L_0) \). Hence, by Lemma 2.19, we know that the nodal properties hold along the local solutions curve \( \mathcal{C}_{\text{loc}} \). It remains to show that these properties get passed on to solutions along the global curve.

Let \( \mathcal{S} \subset \mathcal{C} \) contains the solutions \((w, F) \in \mathcal{C}\) that satisfy the nodal properties (2.99). From Theorem 2.33, the curve \( \mathcal{C} \) is continuous, therefore connected as a subset of \( X \times \mathbb{R} \). Further, via Lemmas 2.20 and 2.21, we know that \( \mathcal{S} \) is a relatively open and closed subset of \( \mathcal{C} \). Also, the argument in the previous paragraph guarantees that \( \mathcal{C}_{\text{loc}} \subset \mathcal{S} \), hence \( \mathcal{S} \neq \emptyset \). Thus, we conclude that \( \mathcal{S} = \mathcal{C} \) which completes the proof of the lemma.

**Lemma 2.35.** For every \( K > 0 \), there exists a constant \( C = C(K) > 0 \) such that every \((w, F) \in \mathcal{U} \cap F^{-1}(0)\) with \( \|w\|_{C^3(\mathbb{R})} \leq K \) obeys the bound

\[
\frac{1}{\inf_{\mathbb{R}}(w_p + H_p)} + F < C.
\]

**Proof.** The bounds are a direct consequence of Theorem 2.14 and Corollary 2.30.

Next, we will rule out the loss of compactness alternative in Theorem 2.33(a).
key component of the argument is the nonexistence of (nontrivial) monotone front-type solutions of the height equation. This fact is proved in [CWW18] (see the remark between Corollary 4.11 and 4.12) and recalled below.

**Theorem 2.36** (Nonexistence of monotone fronts). Suppose that \( h \in C^2_b(\mathbb{R}^+) \cap C^2_b(\mathbb{R}^-) \cap C^0(\mathbb{R}) \) is a front solution to the height equation (2.28) in the sense that
\[
h(q,p) \to H_{\pm}(p) \quad \text{as} \quad q \to \pm \infty.
\]
If \( \inf_R h_p > 0 \) and
\[
H_+ \geq H_- = H \text{ on } [-1,0] \quad \text{or} \quad H_+ \leq H_- = H \text{ on } [-1,0],
\]
then \( H_+ = H_- = H \).

**Lemma 2.37** (Local compactness). Suppose that \( \{(w_n,F_n)\} \subset \mathcal{W} \) is a sequence of monotone solutions to the height equation (2.33) that is uniformly bounded in \( X \times \mathbb{R} \). Then we can extract a subsequence converging in \( X \times \mathbb{R} \) to some \( (w,F) \in X \times [F_{cr}, \infty) \).

**Proof.** Let a sequence \( \{(w_n,F_n)\} \) be given as above. By Lemma 2.35 and boundedness, it follows that these solutions lie in a subset of \( X \times \mathbb{R} \) on which the height equation (2.33) is uniformly elliptic with a uniformly oblique boundary condition on the top and a co-normal transmission condition on the interior. By a straightforward adaptation of [CWW18] Lemma 6.3] to transmission problems, we may then conclude that either

(i) \( \{(w_n,F_n)\} \) is pre-compact in \( X \times \mathbb{R} \); or

(ii) we can extract a subsequence and find \( q_n \to \infty \) so that the translated sequence \( \{\tilde{w}_n\} \) defined by \( \tilde{w}_n := w_n(\cdot + q_n, \cdot) \) converges in \( C^0_{\text{loc}} \) to some \( \tilde{w} \in X_b \) which solves (2.33) and has \( \tilde{w} \not\equiv 0 \) and \( \partial_q \tilde{w} \leq 0 \).
However, the second of these alternatives is impossible in light of Theorem 2.36. To see this, observe that were it to occur, then $\tilde{h} := \tilde{w} + H$ would be a front-type solution of the height equation (2.28) in the sense that

$$\tilde{h} \to \tilde{H}_- \text{ as } q \to -\infty \quad \tilde{h} \to H \text{ as } q \to \infty,$$

for some $q$-independent solution $\tilde{H}_-$ to (2.28). Since $\tilde{w}_q \leq 0$, it must be that $\tilde{H}_- \geq H$. Theorem 2.36 then ensures that $\tilde{H}_- = H$, meaning $\tilde{h} \equiv H$ or equivalently $\tilde{w} \equiv 0$, a contradiction. The proof is therefore complete.

The next lemma applies the above result to conclude that the extreme of $\mathcal{C}$ does not limit to a critical flow.

**Lemma 2.38** (Asymptotic supercriticality). If $\|w(s)\|_X$ is bounded uniformly along the bifurcation curve $\mathcal{C}$, then

$$\liminf_{s \to \infty} F(s) > F_{cr}.$$

**Proof.** We follow closely the argument in [CWW18, Lemma 6.9]. By way of contradiction, suppose that there exists a sequence $s_n \to \infty$ such that

$$\limsup_{n \to \infty} \|w(s_n)\|_X < \infty \quad \text{and} \quad \lim_{n \to \infty} F(s_n) = F_{cr}.$$

We have already proved that each $w(s_n)$ is a monotone and even solution to the quasilinear elliptic PDE (2.33). Lemma 2.37 therefore tells us that the sequence is pre-compact and so passing to a subsequence, we may assume that $\{(w(s_n), F(s_n))\}$ converges in $X \times \mathbb{R}$ to some $(w^*, F^*)$ with $\mathcal{F}(w^*, F^*) = 0$ where $F^* = F_{cr}$. However, by Theorem 2.12(b), this implies that $w^* \equiv 0$ which is equivalent to saying $\|w(s_n)\|_X \to 0$ as $n \to \infty$. Lemma 2.34 ensures that that each $w(s_n)$ is a wave of elevation, therefore
by the uniqueness of small-amplitude solutions, \((w(s_n), F(s_n)) \in C_{loc}\) for \(n \gg 1\). But, this contradicts the statement in Theorem 2.33(c) that the curve does not reconnect to the trivial solution. The proof of the lemma is therefore complete.

Finally, we are ready to complete the proof of Theorem 2.1. It only remains to assemble all the information obtained earlier.

**Proof of Theorem 2.1.** Let \(C\) be the global curve given by Theorem 2.33. The statement in part (b) follows by construction of the local curve \(C_{loc}\), specifically Theorem 2.22(a). To prove part (c), recall from Theorem 2.33 that \(C \subset \mathcal{H}\) which is defined in (2.36). This shows that all the solutions contained in \(C\) are symmetric and supercritical. Moreover, they are monotonic as a consequence of the nodal properties established in Lemma 2.34.

Finally, we consider the stagnation limit claimed in part (a). It was already shown in Lemma 2.37 that the loss of compactness alternative in Theorem 2.33(a)(ii) does not occur. Thus, the blowup alternative (i) must happen:

\[
N(s) := \|w(s)\|_X + \frac{1}{\inf_R (w_p(s) + H_p)} + F(s) + \frac{1}{F(s) - F_{cr}} \to \infty, \quad \text{as } s \to \infty.
\]

From the bounds in Lemmas 2.35 and 2.38 this can be further refined to

\[
\|w(s)\|_X \to \infty, \quad \text{as } s \to \infty.
\]

By definition of \(X\) in (2.34) and Theorem 2.31 the above limit simplifies to \(\|w_p(s)\|_{C^0} \to \infty\).

We now translate this back to the physical variables. In Eulerian dimensionless form, it reads

\[
\inf_{\Omega(s)} \left( \frac{1}{\bar{\Omega}} - \bar{u}(s) \right) = \frac{1}{\sup_R \left| \sqrt{\rho} \partial_p h(s) \right|} \to 0 \quad \text{as } s \to \infty. \quad (2.136)
\]
Recall, that the dimensional and dimensionless Eulerian horizontal velocities are related by
\[
 u - c = \frac{m}{\sqrt{\rho_{\eta}d}}(\bar{u} - \bar{c}) = F\sqrt{\gamma d(\bar{u} - \bar{c})}.
\]  \hspace{1cm} (2.137)

Combining (2.136) with the bounds on the Froude number given in Theorem 2.14, we obtain
\[
 F(s)^2 \leq \frac{C}{\inf_{\hat{\Omega}(s)}(\bar{c} - \bar{u}(s))}.
\]

Taking the infimum of both sides of the equation (2.137) and combining with the inequality above results in the following
\[
 \inf_{\hat{\Omega}(s)}(c - u(s)) = F(s)\sqrt{\gamma d} \inf_{\hat{\Omega}(s)}(\bar{c} - \bar{u}(s)) \leq C \sqrt{\inf_{\hat{\Omega}(s)}(\bar{c} - \bar{u}(s))} \to 0,
\]
as \( s \to \infty \). Thus a point of horizontal stagnation develops in the limit. \( \blacksquare \)
Chapter 3

Orbital stability/instability

3.1 Introduction

The main object of study in the present chapter is internal water waves. As mentioned in Chapter 2, the formation of these waves is primarily due to variations in salinity and temperature in the water bulk. In Oceanography, internal waves are known to be immensely energetic. It is for this reason that they play a crucial role in the ocean dynamics: transporting and mixing nutrients inside the water.

In recent years, internal waves have gained a great deal of attention among mathematicians. A plethora of works have been devoted in proving existence of various types of internal traveling waves, see for instance [AT86, AT89, SS93, Nil17]. Largely for reasons of mathematical convenience, many of these results mainly dealt with flows that are irrotational; that is when vorticity is identically zero. Unlike irrotational flow, the number of literature that investigate fluid with vorticity is unsurprisingly outnumbered. This is mainly due to the mathematical complexity vorticity brings about to the already complicated free boundary problem. Despite this, rotational effects are inherently significant in various physical situations, for instance for fluids under stratification and temperature variation. Motivated by this observation, the
flows considered in this chapter will be rotational. For more reference on the physical relevance of vorticity, see the survey [Mar22] (on three-dimensional rotational flows): it discusses the implication and effects of vorticity for geophysical flow.

Besides showing existence of certain family of internal waves, the investigation on their dynamical aspects is unarguably crucial as well. The number of analytical and numerical works concerning their stability still is comparatively low. This, in particular, includes results on nonlinear stability which remain overall limited. Therefore, the primary contribution of this chapter is pertaining to the orbital stability of small-amplitude capillary-gravity internal waves. In particular, we present a criterion on the (conditional) stability of internal waves. This posits that the surface tension is present ($\sigma > 0$) on the internal interface.

Mathematically, the problem in question is formulated as follows. Let $(x, y) \in \mathbb{R}^2$ be a point in the standard Cartesian coordinates system. For $t \geq 0$, we assume that the fluid is confined in a time-dependent channel,

$$\Omega(t) := \Omega_+(t) \cup \Omega_-(t),$$

bounded above and below by infinitely-long rigid walls, $\{y = d_+\}$ and $\{y = -d_-\}$. The notations $(\cdot)_+$ and $(\cdot)_-$ denote the trace on the boundary of a quantity defined in the upper or lower layer, respectively. Both layers share a common internal interface $\mathcal{S} = \mathcal{S}(t)$ whose profile is given by the graph of an unknown function $y = \eta(t, x)$, where $\eta \to 0$ as $|x| \to \infty$. We assume that the density to be constant in each layer with $0 < \rho_+ \leq \rho_-$. More precisely, the upper and lower regions of the fluid can be written as

$$\Omega_+(t) = \{(x, y) \in \mathbb{R}^2 : \eta(t, x) < y < d_+\}$$

(3.1)
Figure 3.1: Configuration of the fluid domain. Two fluids of different densities are confined in an infinitely long channel.

and

\[ \Omega_-(t) = \{(x, y) \in \mathbb{R}^2 : -d_- < y < \eta(t, x)\}. \]  

Due to the incompressibility assumption in each layer, the velocity field is given by

\[ v = -(\psi_x)_\pm \text{ and } u = (\psi_y)_\pm, \]

for some \( \psi_\pm := \psi_\pm(t, x, y) \) known as the stream function. Furthermore, the presence of constant vorticity in each layer furnishes a (Poisson) equation

\[ \Delta \psi_\pm = -\omega_\pm \text{ in } \Omega_\pm, \]

where \( \omega_\pm = v_x - u_y. \)

Define a harmonic function \( \tilde{\psi}_\pm := \psi_\pm + \frac{\omega_\pm y^2}{2} \). Let \( \phi_\pm \) be its harmonic conjugate, namely,

\[ (\phi_x)_\pm = (\psi_y)_\pm + \omega_\pm y, \text{ and } (\phi_y)_\pm = -(\psi_x)_\pm. \]

Hence, the rotational incompressible Euler equations can be recast as follows. In the interior we have

\[ \Delta \phi_\pm = 0 \text{ in } \Omega_\pm(t), \]  

On the internal interface and rigid walls, the kinematic conditions read

\[
\begin{cases}
\eta_t = (\phi_\pm)_y - ((\phi_\pm)_x - \omega_\pm \eta) \eta_x & \text{on } y = \eta(x, t),
\\
(\phi_\pm)_y = 0 & \text{on } y = \pm d_\pm.
\end{cases}
\]
Finally, thanks to the Bernoulli condition, on the internal interface, we arrive at the jump condition

\[ \langle \rho \phi \rangle = - \left[ \frac{1}{2} \rho |\nabla \psi|^2 + g \rho \eta + \rho \omega \psi \right] - \sigma \left( \frac{\eta_x}{\sqrt{1 + (n_x)^2}} \right) \quad \text{on} \quad y = \eta(x, t), \quad (3.3c) \]

The notation \( [\cdot] := (\cdot)_+ - (\cdot)_- \) denotes the difference in trace between two quantities on the internal interface in the upper and lower regions, \( g > 0 \) is the gravitational constant, and \( \sigma > 0 \) is the coefficient of surface tension. The second equation in (3.3b) goes back to the assumption made on the continuity of the normal velocity across the internal interface. Meanwhile, equation (3.3c) owes itself to the Young-Laplace law which states the balance of the normal stress across the internal interface \( S \).

Instead of working with the unknown \( \phi_\pm \), let us define the trace of \( \phi_\pm \) on \( y = \eta(x, t) \)

\[ \xi_\pm(x, t) = \phi_\pm(x, \eta, t). \quad (3.4) \]

Using this new variable along with other mathematical tools, we can push the entire problem to the boundary, which in return makes it more tractable for analysis. Now, we are ready to state our main results. We record them in Theorem 3.1 and Theorem 3.6.

### 3.1.1 Statement of results

In order to fully state our result, we begin this subsection by stating a number of important terminology and parameters. The steady or traveling waves in the present context are solutions to the governing equations (3.3) that propagate via translation in \( x \)-coordinate with wave speed \( c \in \mathbb{R} \) without altering its shape. In the moving frame of reference, these solutions appear steady. As a result, the unknowns in (3.3)
(after switching \( \phi_\pm \) to \( \xi_\pm \) using (3.4)) can be written as

\[
\eta(t, x) = \eta_c(x - ct), \quad \xi_\pm(x - ct) = \xi_{c\pm}(x - ct),
\]

for some new profiles \( \eta_c, \xi_{c\pm} \). Here, we are interested in internal solitary waves. These are solutions to (3.3) whose profile \( \eta \) decays as \( |x| \to \infty \). As mentioned earlier, there have been various works devoted to the study of the existence of traveling waves in many regimes. For instance, in the gravity case when \( \sigma = 0 \), [AT86, BBT83, Mie95] and [AT86, AT89] concern the existence of solitary waves and periodic waves, respectively. It is crucial to note that when surface tension is neglected (i.e., \( \sigma = 0 \)), the water-wave problem (3.3) becomes ill-posed, see for example [Lan13]. Hence, when looking at questions pertaining to stability, one has to assume \( \sigma > 0 \) which is exactly the assumption made throughout this work. The idea of the proof for the stability is inspired by the recent paper of Chen and Walsh [CW22] which was based on the work of Mielke [Mie02]. We largely adopt their method which hinges on a variant of the Grillakis–Shatah–Strauss [GSS87] in [VWW20].

The existence of small-amplitude internal waves in the presence of surface tension was obtained, for instance, by [Kir22, SS93, Nil17]. However, none of those results considers vorticity in the flow. Since rotational flow is our main focus, we devote the first part of the work to pin down an existence theory for small-amplitude internal waves using a spatial dynamics method: we view the \( x \)-coordinate as a time-like variable. The process is closely inspired by the approach carried out by Nilsson [Nil17].

The existence of internal solitary waves in the present work depends on four dimensionless parameters, two of them are the Bond number \( \beta \) and the inverse square
of the Froude number denoted by \( \alpha \)

\[
\beta := \frac{\sigma}{d_+ \rho c^2}, \quad \alpha := \frac{-g \left[ \rho \right]}{\rho c^2}.
\]  

(3.5)

By definition, the Bond number \( \beta \) measures the strength of the surface tension. In view of (3.5), the Froude number (non-dimensionalized wave speed) can be thought as \( 1/\sqrt{\alpha} \).

Upon linearizing (3.3) at the trivial solution and inserting the plane-wave ansatz \( \eta = \exp(ik(x - ct)) \), we arrive at the following dispersion relation

\[
\alpha + \beta k^2 = \sum_{\pm} \frac{\rho_{\pm}}{\rho_-} k \coth\left( \frac{d_{\pm} k}{d_+} \right) + \left( \frac{\omega_+ d_+ \rho}{c} - \frac{\omega_- d_+}{c} \right).
\]  

(3.6)

In the aforementioned paper, when the Coriolis force is neglected and the surface tension is considered, one recovers back the dispersion relation here.

It can be checked easily that \( k = 0 \) is a root of (3.6) exactly when

\[
\beta = \beta_0 := \frac{1}{3} \left( \frac{\rho_+}{\rho_-} + \frac{d_-}{d_+} \right), \quad \alpha = \alpha_0 := \frac{\rho_+}{\rho_-} + \frac{d_+}{d_-} + \frac{\omega_+ d_+ \rho}{c} - \frac{\omega_- d_+}{c}.
\]  

(3.7)

We will regard \( \beta_0 \) as the critical Bond number separating two regimes: strong and weak surface tensions. It is known that the solitary waves in question will bifurcate from the two parameter values: \( \beta_0 \) and \( \alpha_0 \).

Apart from \( \beta \) and \( \alpha \), there are two other physical parameters that have to be considered when studying interfacial waves: the density ratio \( \varrho \) and the asymptotic height ratio \( d \) given as follows

\[
\varrho := \frac{\rho_+}{\rho_-}, \quad d := \frac{d_-}{d_+}.
\]  

(3.8)

These two parameters are known to play a role in classifying various types of small-amplitude internal waves. Nilsson [Nil17] proved that in the setting of irrotational
flow \( \omega_{\pm} = 0 \), if \( \varrho - 1/d^2 < 0 \) and \( O(1) \) as \( \alpha \searrow \alpha_0 \), then there exist waves of depression \( (\eta < 0) \). On the other hand, if \( \varrho - 1/d^2 > 0 \) and \( O(1) \) as \( \alpha \searrow \alpha_0 \), then waves of elevation exist \( (\eta > 0) \).

Below, we record our existence result of small-amplitude internal solitary wave solutions; this family of waves depends on various physical parameters mentioned earlier. It is important to highlight that, for this, instead of using (3.3), we use the steady incompressible Euler equations. That is, we hide the time dependency from the problem by switching to a moving reference frame.

**Theorem 3.1 (Existence).** Let \( \Pi_\epsilon = (\rho_{\pm,\epsilon}, d_{\pm,\epsilon}, \omega_{\pm,\epsilon}, \sigma_\epsilon, c_\epsilon) : 0 < \epsilon \ll 1 \) be a smooth parametrized curve where the Bond number satisfies the condition \( \beta > \beta_0 \) and \( \alpha = \alpha_0 + \epsilon^2 \). Suppose that

\[
\varrho - \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_+ \varrho}{c d} + \frac{\omega_+^2 d_+^2 \varrho}{3c^2} - \frac{\omega_-^2 d_+^2 \varrho}{3c^2} = O(1) \text{ as } \epsilon \searrow 0. \tag{3.9}
\]

Then for any \( k > 1/2 \) there exists a smooth curve of internal wave solutions

\[
\mathcal{C} = \{u_{\epsilon;\beta} : 0 < \epsilon \ll 1 \} \subset X_k. \tag{3.10}
\]

Every solution that exists on this curve obeys the following asymptotic surface profile:

\[
\eta_{\epsilon;\beta}(x) = \frac{d_+ \epsilon^2 \text{sech}^2 \left( \frac{\epsilon x}{2d_+ \sqrt{\beta - \beta_0}} \right)}{\varrho - \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_+ \varrho}{c d} + \frac{\omega_+^2 d_+^2 \varrho}{3c^2} - \frac{\omega_-^2 d_+^2 \varrho}{3c^2}} + O(\epsilon^3) \text{ in } X_k^k \text{ as } \epsilon \searrow 0. \tag{3.11}
\]

In dealing with the stability problem, we will have to work with the time-dependent governing equations (3.3).

**Theorem 3.2.** Fix the physical parameters: \( \rho_{\pm}, d_{\pm}, \omega_{\pm}, \sigma \). For any sufficiently small-amplitude solitary internal wave \( (\eta_c, \xi_c+, \xi_c-) \) with \( \beta_c > \beta_0 \) and \( \alpha_c = \alpha_0 + \epsilon^2 \), its
conditional orbital stability (instability) is determined by looking at the sign of the leading order derivative of a function $m$ (see (3.176)). The orbital stability can be understood as follows. For any $R > 0$ and $r > 0$, there exists $r_0 > 0$ such that, if $\eta, \xi_+, \xi_-$ is any solution of (3.3) defined on a time interval $[0, t_0)$ that satisfies the bound

$$\sup_{t \in [0, t_0)} \left( \| \eta(t) \|_{H^3} + \| \xi_+(t) \|_{H^{2+}_2 \cap H^1_2} + \| \xi_-(t) \|_{H^{2+}_2 \cap H^1_2} \right) < R,$$

and whose initial data satisfies

$$\| \eta(0) - \eta_c \|_{H^1} + \| \xi_+(0) - \xi_c^+ \|_{H^{1/2}} + \| \xi_-(0) - \xi_c^- \|_{H^{1/2}} < r_0,$$

then

$$\sup_{t \in [0, t_0)} \inf_{s \in \mathbb{R}} \left( \| \eta(t, \cdot - s) - \eta_c \|_{H^1} + \| \xi_+(t, \cdot - s) - \xi_c^+ \|_{H^{1/2}} + \| \xi_-(t, \cdot - s) - \xi_c^- \|_{H^{1/2}} \right) < r.$$

(3.14)

**Remark 3.3.** We obtain some regimes where orbital stability and instability occur.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_c - \beta_0 \ll 1$</td>
<td>$\eta &gt; 0, \omega_+ = 0, 2g(\varrho - 1) &lt; \frac{c\omega_-}{\varrho} \leq 0$</td>
<td>$\eta &lt; 0, \omega_- = 0, 2(1 - \varrho) &gt; \frac{c\omega_+}{\varrho} \geq 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta &lt; 0, \omega_+ = 0, c\omega_- \leq 0, 2(\varrho - 1) &gt; \frac{c\omega_-}{\varrho}$</td>
<td>$\eta &gt; 0, \omega_- = 0, c\omega_+ \geq 0, 2\frac{(\varrho - 1)}{\varrho} &lt; \frac{c\omega_+}{\varrho}$.</td>
</tr>
</tbody>
</table>

Note that the inequality (3.14) measures the distance between the translated solutions $(\eta, \xi_+, \xi_-)$ and the family of traveling waves. The regularity in (3.12) represents the regularity of local well-posedness for the Cauchy problem that is available. Furthermore, as we will see later, the regularity in (3.13) and (3.14) matches the regularity of the energy space.
3.1.2 Idea of the proof

In this section we will begin by outlining the idea for the existence of small-amplitude internal wave solutions. We began by writing the corresponding water wave problem in terms of a Hamiltonian formulation. It is obtained by viewing things as a variational problem. By means of a Legendre transform, we arrive at the desired Hamiltonian. Seeking for small-amplitude solutions, we then study the spectral properties of the linear part of the Hamiltonian system. Specifically, we look at the Hamiltonian $0^2$ resonance where a plethora of small-amplitude waves exist. It is crucial to note that the center manifold technique we implement requires us to work with linear boundary conditions. Therefore, an intermediate step needs to be taken to achieve the desired linearity on boundary conditions. At the end, we show that solitary waves of elevation or depression exist when

$$\varrho - \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_+}{c d} + \frac{\omega_+^2 d_+^2 \varrho}{3c^2} - \frac{\omega_-^2 d_+^2}{3c^2},$$

is positive or negative, respectively. Note that when $\omega_\pm = 0$, we recover the results in [Nil17].

Next, we shall discuss the idea in proving the stability of waves whose existence has just been proved. Similar to the fundamental approach in our existence theory, in studying the stability of the constructed waves, we again exploit the Hamiltonian structure of problem (3.3). In other words, abstractly, we can write (3.3) in the following way

$$\partial_t u = JDE(u),$$

where $u = u(t)$ is an unknown represented by $(\eta, \xi_+, \xi_-)$, $J$ being the skew-adjoint Poisson map, and $E$ is an energy functional. It is well known that the translation-
invariant nature of the problem gives rise to another conserved quantity known as the momentum $P$. By construction, it is clear that a travelling (steady) wave solution is a critical point of the augmented Hamiltonian formulated as follows

$$E_c := E - cP.$$  \hfill (3.17)

The main technique used to arrive at the stability/instability result is based on the seminal works of Grillakis, Shatah, and Strauss [GSS90a, GSS90b]. Their approach, known as the GSS methods, provides a systematic way of proving nonlinear stability/instability for systems that are invariant under continuous translation symmetry group and possess a Hamiltonian structure. Although direct application of GSS has been proven to work for various model equations, there are a number of technical conditions in the water wave problem that prevent a direct use of the GSS technique. For instance, the requirement on the isomorphic Poisson map $J$ is clearly satisfied anymore when considering the water wave problem in the present context. Further, the requirement on the global well-posedness of the Cauchy problem in the natural energy space of the problem is also not met. At this point, the full water wave problem is only known to be locally well-posed in a smoother space than the energy space.

In the recent work, Varholm, Wahlén, and Walsh obtained a variant of the GSS method with hypotheses sufficiently relaxed so that it can be applied directly to the water wave problem. Their framework allows for the Poisson map $J$ to only have a dense range. It also permits the mismatch between the local well-posedness space and the energy space that the water wave problem possess. Upon computing the spectrum of the linearized augmented Hamiltonian at a traveling wave, the general
theory only requires us to determine whether the moment-instability, a scalar-valued function, at a fixed wave speed is convex or concave. From there, the conclusions on stability or instability follow.

3.1.3 Outline

Now, we give a brief outline of the remaining portion of this chapter. In Section 3.2, we prove the existence of small-amplitude internal solitary waves. This uses a spatial dynamics approach that exploits the Hamiltonian structure derived from the Legendre transformation.

In Section 3.3, we state the assumptions needed in order to apply the version of GSS in [VWW20]. Section 3.4 discusses the reformulation the problem (3.3) as an abstract Hamiltonian equation using the Dirichlet–Neumann operator in the style of Benjamin and Bridges [BB97]. The most tedious portion of the work is designated to the analysis of the spectrum of the linearized augmented Hamiltonian at a small-amplitude traveling wave. This is the content of Section 3.5.

Finally, in Section 3.6, we show the sign condition for the second derivative of the moment instability $d(c)$. This is accomplished by imposing the long-wave scaling and using the knowledge on the asymptotic of the wave profiles in question.

3.2 Existence theory

We start this section by proving the existence of small amplitude solitary water waves. We closely follow the approach presented in [Nil17]. Since we are interested in traveling waves that propagate via translation with wave speed $c \in \mathbb{R}$, we impose a change of variables $(t, x, y) \mapsto (x - ct, y)$. The main governing equation (3.3) can then be
stated in terms of the relative streamfunction $\psi_\pm$ and posed in the moving reference frame:

\[
\begin{align*}
\Delta \psi_\pm &= -\omega_\pm \quad \text{in } \Omega_\pm, \\
\psi_\pm &= \mp m_\pm \quad \text{on } y = \pm d_\pm, \\
\psi_\pm &= 0 \quad \text{on } y = \eta(x), \\
\left[ \frac{1}{2} \rho |\nabla \psi|^2 + g \rho \eta \right] = -\sigma \left( \frac{\eta_x}{\langle \eta_x \rangle} \right)_x + Q \quad \text{on } y = \eta(x),
\end{align*}
\]

for some constants $m_\pm$ together with the following asymptotic condition

\[\eta \to 0 \quad \text{as } x \to \infty.\]

We define the non-dimensional variables

\[
(x', y') = \frac{1}{d_+}(x, y), \quad \eta'(x') = \frac{\eta(x)}{d_+}, \quad \psi'_\pm(x', y') = \frac{\psi_\pm(x, y)}{d_+c}.
\]

For the rest of the section, we will drop the $'$ for notational convenience. The problem now reads

\[
\begin{align*}
\Delta \psi_+ &= -\frac{\omega_+ d_+}{c} \quad \text{for } \eta(x) < y < 1, \\
\Delta \psi_- &= -\frac{\omega_- d_+}{c} \quad \text{for } d < y < \eta(x), \\
\psi_+ &= \frac{-m_+}{c d_+} \quad \text{on } y = 1, \\
\psi_- &= \frac{m_-}{c d_+} \quad \text{on } y = -d, \\
\psi_\pm &= 0 \quad \text{on } y = \eta(x), \\
\left[ \frac{1}{2} \rho c^2 |\nabla \psi|^2 + g \rho d_+ \eta \right] + \frac{\sigma}{d_+} \left( \frac{\eta_x}{\langle \eta_x \rangle} \right)_x = Q \quad \text{on } y = \eta(x),
\end{align*}
\]

where $Q$ is the hydraulic head constant.
Next, to obtain the harmonic function \( \tilde{\psi} \), we subtract the shear flow from \( \psi \):

\[
\tilde{\psi}_\pm := \psi_\pm + \frac{\omega_\pm dy^2}{2c}.
\]

Clearly, \( \tilde{\psi}_\pm \) is harmonic in both layers, that is

\[
\nabla^2 \tilde{\psi} = 0. \quad (3.21)
\]

Moreover, the boundary conditions on the rigid walls and internal interface become

\[
\tilde{\psi}_+ = 0 \quad \text{on } y = 1,
\]

\[
\tilde{\psi}_- = 0 \quad \text{on } y = -d,
\]

and

\[
\tilde{\psi}_\pm = \frac{\omega_\pm dy^2}{2c} \quad \text{on } y = \eta(x),
\]

\[
\left[ \frac{1}{2} \rho c^2 |\nabla \tilde{\psi}|^2 - c \rho \omega d_+ \eta \partial_y \tilde{\psi} + \frac{1}{2} \rho \omega^2 d_+^2 \eta^2 + g \rho d_+ \eta \right] = -\frac{\sigma}{d_+} \left( \frac{\eta_x}{\langle \eta_x \rangle} \right) x + Q \quad \text{on } y = \eta(x). \quad (3.22)
\]

Rather than working with \( \tilde{\psi} \), we will use its harmonic conjugate \( \phi \) to reformulate \( (3.21) \), \( (3.22) \), and \( (3.23) \). One can view \( \phi \) as the velocity potential. Using the fact that \( \phi_{\pm x} = \tilde{\psi}_{\pm y} \) and \( \phi_{\pm y} = -\tilde{\psi}_{\pm x} \), the equations now read

\[
\begin{cases}
\Delta \phi_\pm = 0 & \text{in } \Omega_\pm, \\
\phi_{+y} = 0 & \text{on } y = 1, \\
\phi_{-y} = 0 & \text{on } y = -d, \\
\phi_{\pm y} = \phi_{\pm x} \eta_x - \frac{\omega_\pm d_+ \eta_x}{c} - \eta_x & \text{on } y = \eta(x), \\
\left[ \frac{1}{2} \rho c^2 |\nabla \phi|^2 - c \rho \omega d_+ \eta \partial_x \phi \\
+ \frac{1}{2} \rho \omega^2 d_+^2 \eta^2 + g \rho d_+ \eta \right] = -\frac{\sigma}{d_+} \left( \frac{\eta_x}{\langle \eta_x \rangle} \right) x + Q & \text{on } y = \eta(x).
\end{cases} \quad (3.24)
\]

Consider the following rescaling of the domain via the mapping \((x, y) \mapsto (x, z)\),
where

\[
\begin{cases}
  \frac{y - 1}{\eta(x) - 1} & \text{for } \eta(x) < y < 1, \\
  \frac{y + d}{\eta(x) + d} & \text{for } -d < y < \eta(x).
\end{cases}
\] (3.25)

As a result, we have the following change of variables formulas

\[
\begin{align*}
  \partial_y &= \frac{1}{\eta + d} \partial_z, & \partial_y &= \frac{1}{\eta - 1} \partial_z, \\
  \partial_x &= \partial_X - \frac{z \eta_x}{\eta - 1} \partial_z, & \partial_x &= \partial_X - \frac{z \eta_x}{\eta + d} \partial_z.
\end{align*}
\] (3.26)

Abusing notations, let us define \( \phi_{\pm}(x, z) := \phi_{\pm}(x, y) \). Under the new rescaling, equation (3.21) now reads

\[
\begin{align*}
  \phi_{+zz} - \frac{2z \eta_x}{\eta - 1} \phi_{+zz} - \frac{z \eta_{xx}}{\eta - 1} \phi_{+zz} + \frac{2z \eta_x^2}{(\eta - 1)^2} \phi_{+zz} + \frac{z^2 \eta_x^2}{(\eta - 1)^2} \phi_{+zz} + \frac{1}{(\eta - 1)^2} \phi_{+zz} &= 0 & 0 < z < 1, \\
  \phi_{-zz} - \frac{2z \eta_x}{\eta + d} \phi_{-zz} - \frac{z \eta_{xx}}{\eta + d} \phi_{-zz} + \frac{2z \eta_x^2}{(\eta + d)^2} \phi_{-zz} + \frac{z^2 \eta_x^2}{(\eta + d)^2} \phi_{-zz} + \frac{1}{(\eta + d)^2} \phi_{-zz} &= 0 & 0 < z < 1.
\end{align*}
\] (3.27, 3.28)

Moreover, in this new formulation, the boundary conditions on the rigid walls (3.22) and the internal interface (3.23) respectively become

\[
\phi_{\pm} = 0 \quad \text{on } z = 0,
\] (3.29)

and

\[
\begin{align*}
  \phi_{+} &= (\eta - 1) \left( \frac{-\omega_{+} d_{+} \eta x}{c} + \phi_{+} \eta_x - \frac{\eta_x^2 \phi_{+}}{\eta - 1} - \eta_x \right), \quad \text{on } z = \eta, \\
  \phi_{-} &= (\eta + d) \left( \frac{-\omega_{-} d_{+} \eta x}{c} + \phi_{-} \eta_x - \frac{\eta_x^2 \phi_{-}}{\eta + d} - \eta_x \right), \quad \text{on } z = \eta,
\end{align*}
\]

\[
\begin{align*}
 Q \left[ \frac{1}{2} \left( \phi_{+} - \frac{\eta_x \phi_{+}}{\eta - 1} \right)^2 + \frac{1}{2} \left( \phi_{+} \right)^2 - \frac{\omega_{+} d_{+} \eta}{c} \left( \phi_{+} - \frac{z \eta_x}{\eta - 1} \phi_{+} \right) + \frac{1}{2} \frac{\omega_{+} d_{+}^2 \eta^2}{c^2} \right] \\
 - \left[ \frac{1}{2} \left( \phi_{-} - \frac{\eta_x \phi_{-}}{\eta + d} \right)^2 + \frac{1}{2} \left( \phi_{-} \right)^2 - \frac{\omega_{-} d_{+} \eta}{c} \left( \phi_{-} - \frac{z \eta_x}{\eta + d} \phi_{-} \right) + \frac{1}{2} \frac{\omega_{-} d_{+}^2 \eta^2}{c^2} \right] \\
 = \alpha \eta - \beta \left( \frac{\eta_x}{\langle \eta_x \rangle} \right) + Q, \quad \text{on } z = \eta,
\end{align*}
\] (3.30)
where $\alpha, \beta$ and $\varrho$ are variables defined earlier in (3.5) and (3.8). The energy can be formulated as

$$E = K + V$$

$$= \frac{c^2 d_+^2 \rho_+}{2} \int_{\mathbb{R}} \int_0^1 \left( \left( \phi_{+x} - \frac{z \eta_x \phi_{+z}}{\eta - 1} - \frac{\omega_+ d_+(z(\eta - 1) + 1)}{c} \right)^2 + \left( \frac{\phi_{+z}}{(\eta - 1)} \right)^2 \right) (1 - \eta) \, dz \, dx$$

$$+ \frac{c^2 d_-^2 \rho_-}{2} \int_{\mathbb{R}} \int_0^1 \left( \left( \phi_{-z} - \frac{z \eta_x \phi_{-z}}{\eta + d} - \frac{\omega_- d_-(z(\eta + d) - d)}{c} \right)^2 + \left( \frac{\phi_{-z}}{(\eta + d)} \right)^2 \right) (d + \eta) \, dz \, dx$$

$$- \frac{1}{2} g d_+^3 \int_{\mathbb{R}} \eta^2 \, dx + \sigma d_+ \int_{\mathbb{R}} \left( \sqrt{1 + \eta_x^2} - 1 \right) \, dx.$$ 

Further, the momentum $P$ is given by

$$P = d_+^2 c \int_{\mathbb{R}} \int_0^1 \rho_+ \left( \phi_{+x} - \frac{z \eta_x \phi_{+z}}{\eta - 1} - \frac{\omega_+ d_+(z(\eta - 1) + 1)}{c} \right) (1 - \eta) \, dz \, dx$$

$$+ d_+^2 c \int_{\mathbb{R}} \int_0^1 \rho_- \left( \phi_{-z} - \frac{z \eta_x \phi_{-z}}{\eta + d} - \frac{\omega_- d_-(z(\eta + d) - d)}{c} \right) (d + \eta) \, dz \, dx.$$ 

In literature, solitary waves are known to be critical points of the functional $E - cP$.

For this reason, we will look at the Hamiltonian that arise from it. Explicitly, the functional takes form

$$E - cP =$$

$$d_+^2 c^2 \rho_- \left[ \frac{c}{2} \int_{\mathbb{R}} \int_0^1 \left( \left( \phi_{+x} - \frac{z \eta_x \phi_{+z}}{\eta - 1} - \frac{\omega_+ d_+(z(\eta - 1) + 1)}{c} \right)^2 \right) (1 - \eta) \, dz \, dx$$

$$+ \left( \frac{\phi_{+z}}{(\eta - 1)} \right)^2 \right) (1 - \eta) \, dz \, dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}} \int_0^1 \left( \left( \phi_{-z} - \frac{z \eta_x \phi_{-z}}{\eta + d} - \frac{\omega_- d_-(z(\eta + d) - d)}{c} \right)^2 \right) (d + \eta) \, dz \, dx$$

$$+ \left( \frac{\phi_{-z}}{(\eta + d)} \right)^2 \right) (d + \eta) \, dz \, dx$$

$$- \int_{\mathbb{R}} \left( \frac{1}{2} \alpha \eta^2 - \beta \left( \sqrt{1 + \eta_x^2} - 1 \right) + \left( \frac{\eta}{2} - 1 \right) - \frac{1}{2} (\eta + d) \right) \, dx.$$
Hence, we arrive at the following Lagrangian
\[
L = \frac{\varrho}{2} \int_0^1 \left( \left( \frac{\varphi_{+z}}{\eta - 1} - \frac{\varphi_{+z}}{\eta - 1} - \frac{\varphi_{+z}}{\eta - 1} + 1 \right)^2 + \left( \frac{\varphi_{+z}}{\eta - 1} \right)^2 \right) (1 - \eta) \, dz \\
+ \frac{1}{2} \int_0^1 \left( \left( \frac{\varphi_{-z}}{\eta - 1} - \frac{\varphi_{-z}}{\eta - 1} - \frac{\varphi_{-z}}{\eta - 1} + 1 \right)^2 + \left( \frac{\varphi_{-z}}{\eta - 1} \right)^2 \right) (d + \eta) \, dz \\
- \frac{1}{2} \alpha \eta^2 + \beta \left( \sqrt{1 + \eta^2} - 1 \right) - \frac{\varrho}{2} (1 - \eta) - \frac{1}{2} (\eta + d).
\]

Having computed the Lagrangian above, for the importance of later analysis, we derive variational derivatives of \( L \) with respect to \( \varphi_{+z}, \varphi_{-z}, \) and \( \eta_x \), respectively
\[
\Phi_+ := \frac{\delta L}{\delta \varphi_{+z}} = \varrho (1 - \eta) \left( \frac{\varphi_{+z}}{\eta - 1} - \frac{\varphi_{+z}}{\eta - 1} - \frac{\varphi_{+z}}{\eta - 1} + 1 \right), \\
\Phi_- := \frac{\delta L}{\delta \varphi_{-z}} = (\eta + d) \left( \frac{\varphi_{-z}}{\eta + d} - \frac{\varphi_{-z}}{\eta + d} - \frac{\varphi_{-z}}{\eta + d} + 1 \right), \\
\gamma := \frac{\delta L}{\delta \eta_x} = \int_0^1 \varrho \left( \frac{\varphi_{+z}}{\eta - 1} - \frac{\varphi_{+z}}{\eta - 1} - \frac{\varphi_{+z}}{\eta - 1} + 1 \right) \, z \varphi_{+z} \, dz \\
- \int_0^1 \left( \frac{\varphi_{-z}}{\eta + d} - \frac{\varphi_{-z}}{\eta + d} - \frac{\varphi_{-z}}{\eta + d} + 1 \right) \, z \varphi_{-z} \, dz + \beta \frac{\eta_x}{\eta_x}.
\]

Again, using the Lagrangian \( L \) [3.31] and variational derivatives [3.32], we derive the corresponding Hamiltonian in terms of \( u = (\eta, \gamma, \varphi_+, \Phi_+, \varphi_-, \Phi_-) \):
\[
H(u) = \int_0^1 \Phi_+ \varphi_{+z} \, dz + \int_0^1 \Phi_- \varphi_{-z} \, dz + \gamma \eta_x - L \\
= \int_0^1 \frac{1}{2 \varrho (1 - \eta)} \left( (\Phi_+ + \varrho (1 - \eta))^2 - \varrho^2 \varphi_{+z}^2 \right) \, dz \\
+ \int_0^1 \frac{1}{2 (\eta + d)} \left( (\Phi_- + (\eta + d))^2 - \varphi_{-z}^2 \right) \, dz + \int_0^1 \bar{\Phi}_+ \omega_+ d_+ (z (\eta - 1) + 1) \, dz + \int_0^1 \bar{\Phi}_- \omega_+ d_+ (z (\eta + d) - d) \, dz \\
- \sqrt{\beta^2 - \eta^2} + \beta - \frac{\alpha \eta^2}{2},
\]
where
\[
\bar{\gamma} = \gamma + \int_0^1 \frac{\Phi_+ z \varphi_{+z}}{\eta - 1} \, dz + \int_0^1 \frac{\Phi_- z \varphi_{-z}}{\eta + d} \, dz.
\]
To formalize this, for $s \geq 0$, we define the following product spaces

$$\mathcal{X}_s = \mathbb{R} \times \mathbb{R} \times H^{s+1}(0,1) \times H^{s+1}(0,1) \times H^{s+1}(0,1).$$

We would like to point out that the symbol $H^{s+1}$ represents the Sobolev space, not the Hamiltonian $H$. Further, we let $\widetilde{M} = \mathcal{X}_0$ be a manifold with $m \in \widetilde{M}$ and let $v = (\eta, \gamma, \phi_+, \Phi_+, \phi_-, \Phi_-) \in T_m \widetilde{M}$. On $T_m \widetilde{M} \times T_m \widetilde{M}$, consider the position independent symplectic form

$$\widehat{\Omega}(v, v^*) = \eta^* \gamma - \gamma^* \eta + \int_0^1 (\Phi_+^* \phi_+ - \phi_+^* \Phi_+) \, dz + \int_0^1 (\Phi_-^* \phi_- - \phi_-^* \Phi_-) \, dz. \quad (3.34)$$

One may observe that $(\widetilde{M}, \widehat{\Omega})$ is a symplectic manifold. The corresponding set

$$\hat{N} = \{ m \in \widetilde{M} : |\gamma| < \beta, -d < \eta < 1 \}$$

is a manifold domain of $\widetilde{M}$ where the Hamiltonian $H$ is a smooth functional on it (i.e., $H \in C^\infty(\hat{N}, \mathbb{R})$). Hence, the tuple $(\widetilde{M}, H, \widehat{\Omega})$ forms a Hamiltonian system. Via the symplectic form $(3.34)$ and standard computations, the associated Hamilton’s
equations read

\[ \dot{\eta} = \frac{\bar{\gamma}}{\sqrt{\beta^2 - \gamma^2}}, \]

\[ \dot{\gamma} = \int_0^1 \left[ -\frac{\varrho}{2(1-\eta)^2} \left( \Phi_+^2 - \phi_+^2 \right) + \frac{\varrho}{2} \right] \, dz \]

\[ + \int_0^1 \left[ \frac{1}{2(d+\eta)^2} \left( \Phi_-^2 - \phi_-^2 \right) - \frac{1}{2} \right] \, dz - \int_0^1 \frac{\Phi_+ \, \omega_+ \, \phi_+ \, d_+ \, z}{c} \, dz \]

\[ + \int_0^1 \frac{\Phi_- \, \omega_- \, \phi_- \, d_- \, z}{c} \, dz + \bar{\gamma} \left( \int_0^1 \frac{\varphi \, \phi_+ \, z}{(1-\eta)^2} \, dz + \int_0^1 \frac{\varphi \, \phi_- \, z}{(d+\eta)^2} \, dz \right) + \alpha \eta, \]

\[ \dot{\phi}_+ = \frac{1}{\eta - 1} \left( -\Phi_+ + \frac{(\eta - 1) + \bar{\gamma} z \phi_+}{\sqrt{\beta^2 - \gamma^2}} + \frac{\omega_+ d_+ (z(\eta - 1) + 1)}{c} \right), \]

\[ \dot{\Phi}_+ = \frac{1}{\eta - 1} \left( \frac{\bar{\gamma} z \Phi_+}{\sqrt{\beta^2 - \gamma^2}} + \varphi \phi_+ \right), \]

\[ \dot{\phi}_- = \frac{1}{\eta + d} \left( \Phi_- + (\eta + d) + \frac{\bar{\gamma} z \phi_-}{\sqrt{\beta^2 - \gamma^2}} + \frac{\omega_- d_- (z(\eta + d) - d)}{c} \right), \]

\[ \dot{\Phi}_- = \frac{1}{\eta + d} \left( \frac{\bar{\gamma} z \Phi_-}{\sqrt{\beta^2 - \gamma^2}} - \phi_- \right), \]

(3.35)

where the Hamiltonian vector field also satisfies the corresponding boundary conditions

\[ \varphi \phi_+(1) = -\frac{\bar{\gamma} \Phi_+(1)}{\sqrt{\beta^2 - \gamma^2}}, \]

\[ \phi_-(1) = \frac{\bar{\gamma} \Phi_-(1)}{\sqrt{\beta^2 - \gamma^2}}, \]

\[ \phi_{\pm}(0) = 0. \]  

(3.36)

In order to set a firmer ground for the analysis later, we define the product space

\[ Y_s = \mathbb{R} \times \mathbb{R} \times H^{s+1}(0, 1) \times H^{s+1}_0(0, 1) \times H^{s+1}(0, 1) \times H^{s+1}_0(0, 1), \]

(3.37)

where \( H^{s+1}_0(0, 1) = \{ f \in H^{s+1}(0, 1) : f(0) = f(1) = 0 \} \). Additionally, let us also
define these spaces

\[ M = \{ m \in \mathbb{R}^2 \times H^1(0, 1)^4 : \Gamma_+(0) = \Gamma_-(0) = \int_0^1 \bar{\phi}_+ \, dz = \int_0^1 \bar{\phi}_- \, dz = 0 \}, \]
\[ \tilde{M} = \{ m \in Y_0 : \int_0^1 \bar{\phi}_+ \, dz = \int_0^1 \bar{\phi}_- \, dz = 0 \}, \]
\[ \tilde{N} = \{ m \in \tilde{M} : |\gamma| < \beta, -d < \eta < 1 \}, \]

where \( m = (\eta, \gamma, \bar{\phi}_+, \Gamma_+, \bar{\phi}_-, \Gamma_-) \). Going back to the Hamilton’s equations (3.35), we can see that it has an equilibrium point

\[
\begin{pmatrix}
\eta \\
\gamma \\
\bar{\phi}_+ \\
\Phi_+ \\
\bar{\phi}_- \\
\Phi_-
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
\frac{\omega_+ d_+ \varrho (z - 1)}{c} - \varrho \\
\frac{w_- d_- d^2 (1 - z)}{c} - d
\end{pmatrix}.
\]

However, in preparation for the Hamiltonian reduction process on a center manifold, we need to shift the equilibrium point obtained before to the origin \((0, 0, 0, 0, 0, 0)\). To achieve that, we impose another change of variables for the unknowns \( \Phi_\pm, \bar{\phi}_\pm, \) and \( \chi_\pm \)

\[
\begin{align*}
\Gamma_+ &:= \int_0^z (\Phi_+ + \frac{\varrho - \omega_+ d_+ \varrho (s - 1)}{c}) \, ds, \\
\Gamma_- &:= \int_0^z (\Phi_- + \frac{d - w_- d_- d^2 (1 - s)}{c}) \, ds, \\
\bar{\phi}_+ &:= \bar{\phi}_+ - \chi_+, \\
\bar{\phi}_- &:= \bar{\phi}_- - \chi_-, \\
\chi_+ &:= \int_0^1 \phi_+ \, dz, \\
\chi_- &:= \int_0^1 \phi_- \, dz.
\end{align*}
\]
Obviously, this will give rise to a new formulation of the Hamiltonian equation. Precisely, one may think of this change of variables as a map that sends $(\eta, \gamma, \phi_+, \Phi_+, \phi_-, \Phi_-) \in \tilde{M}$ to $(\eta, \gamma, \bar{\phi}_+, \bar{\Gamma}_+, \bar{\phi}_-, \bar{\Gamma}_-, \chi_+, \chi_-) \in M \times \mathbb{R}^2$. Thus, in the new variables the new symplectic form $\tilde{\Omega}$ becomes

$$\tilde{\Omega}(v, v^*) = \omega^* \eta - \eta^* \omega + \int_0^1 \left( \Gamma_{\bar{+}} \bar{\phi}_+ - \bar{\phi}_+^* \Gamma_{\bar{+}} \right) dz + \int_0^1 \left( \Gamma_{\bar{-}} \bar{\phi}_- - \bar{\phi}_-^* \Gamma_{\bar{-}} \right) dz$$

$$+ \Gamma_{\bar{+}}(1) \chi_+ - \chi_+^* \Gamma_{\bar{+}}(1) + \Gamma_{\bar{-}}(1) \chi_- - \chi_-^* \Gamma_{\bar{-}}(1).$$  \hspace{1cm} (3.39)$$

As a result, the Hamiltonian in (3.33) in terms of the new arguments now reads

$$H = \int_0^1 \frac{1}{2 \theta(1 - \eta)} \left( \left( \Gamma_{\bar{+}} + \frac{\omega_+ d_+ \theta(z - 1)}{c} - \eta \theta \right)^2 - \theta \bar{\phi}_+^2 \right) dz$$

$$+ \int_0^1 \frac{1}{2(d + \eta)} \left( \left( \Gamma_{\bar{-}} + \frac{w_- d_+ d^2(1 - z)}{c} + \eta \right)^2 - \bar{\phi}_-^2 \right) dz$$

$$+ \frac{1}{c} \int_0^1 \left( \Gamma_{\bar{+}} - d + \frac{\omega_+ d_+ \theta(z - 1)}{c} \right) \omega_+ d_+ (z(\eta - 1) + 1) dz$$

$$+ \frac{1}{c} \int_0^1 \left( \Gamma_{\bar{-}} - d + \frac{\omega_- d_+ d^2(1 - z)}{c} \right) \omega_- d_+ (z(\eta + d) - d) dz$$

$$- \sqrt{\beta^2 - \gamma^2} + \beta - \frac{\alpha}{2} \eta^2 + \frac{\omega_+^2 d_+^2 \theta}{6c^2} + \frac{\omega_-^2 d_+^2 d^2}{6c^2} + \frac{\omega_+ d_+ \theta}{2c} - \frac{\omega_- d_+ d^2}{2c},$$  \hspace{1cm} (3.40)$$

where

$$\bar{\gamma} = \gamma + \int_0^1 \frac{z \bar{\phi}_+}{\eta - 1} \left( \Gamma_{\bar{+}} + \frac{\omega_+ d_+ \theta(z - 1)}{c} - \theta \right) dz + \int_0^1 \frac{z \bar{\phi}_-}{\eta + d} \left( \Gamma_{\bar{-}} + \frac{w_- d_+ d^2(1 - z)}{c} - d \right) dz.$$  

Notice that we have added the constants in the definition of the Hamiltonian so that $H(0) = 0$.

The new Hamiltonian structure (3.40) gives rise to the new Hamilton’s equations
which are given by
\[ \dot{\eta} = \bar{\gamma}(\beta^2 - \bar{\gamma}^2)^{-1/2}, \]
\[ \dot{\gamma} = \int_0^1 \left[ -\frac{\varrho}{2(1-\eta)^2} \left( \frac{\varphi(z-1) - \varrho}{\varrho^2} - \frac{\bar{\phi}_+^2}{\varrho^2} \right) + \frac{\varrho}{2} \right] dz \]
\[ + \int_0^1 \left[ \frac{1}{2(d+\eta)^2} \left( \frac{\varphi(z-1)}{\varrho^2} - \frac{\bar{\phi}_+^2}{\varrho^2} \right) - \frac{1}{2} \right] dz \]
\[ - \int_0^1 \left( \frac{\varphi(z-1)}{\varrho^2} - \frac{\bar{\phi}_+^2}{\varrho^2} \right) \frac{\omega_+ d_+}{c} dz \]
\[ - \int_0^1 \left( \frac{\varphi(z-1)}{\varrho^2} - \frac{\bar{\phi}_+^2}{\varrho^2} \right) \frac{\omega_- d_-}{c} dz \]
\[ + \frac{\bar{\gamma}}{\sqrt{\beta^2 - \bar{\gamma}^2}} \left( \int_0^z \frac{\varphi(z-1)}{\varrho^2} \left( \frac{\varphi(z-1)}{\varrho^2} - \frac{\bar{\phi}_+^2}{\varrho^2} \right) d\eta \right) + \alpha \eta, \]
\[ \dot{\bar{\phi}}_+ = \frac{1}{\eta - 1} \left( \frac{-\Gamma_+ + \Gamma_+(1)}{\varrho} - \frac{\omega_+ d_+}{c} (2z - 1) + \frac{\bar{\gamma}}{\sqrt{\beta^2 - \bar{\gamma}^2}} \frac{\varphi(z-1)}{\varrho} \right) \]
\[ + \frac{\omega_+ d_+}{c} (\eta - 1)(2z - 1), \]
\[ \dot{\Gamma}_+ = \frac{1}{\eta - 1} \left( \frac{\bar{\gamma} z}{\sqrt{\beta^2 - \bar{\gamma}^2}} (\Gamma_+ + \frac{\varphi(z-1)}{\varrho}) + \frac{\omega_+ d_+}{c} \right), \]
\[ \dot{\bar{\phi}}_- = \frac{1}{\eta + d} \left( \frac{-\Gamma_- + \Gamma_-(1)}{\varrho} - \frac{\omega_- d_-}{c} (2z - 1) + \frac{\bar{\gamma}}{\sqrt{\beta^2 - \bar{\gamma}^2}} \frac{\varphi(z-1)}{\varrho} \right) \]
\[ + \frac{\omega_- d_-}{c} (\eta + d)(2z - 1), \]
\[ \dot{\Gamma}_- = \frac{1}{\eta + d} \left( \frac{\bar{\gamma} z}{\sqrt{\beta^2 - \bar{\gamma}^2}} (\Gamma_- + \frac{\varphi(z-1)}{\varrho}) - \bar{\phi}_- \right), \]
along with
\[ \dot{\chi}_+ = \frac{1}{\eta - 1} \left( \frac{-\Gamma_+ (1)}{\varrho} + \eta + \frac{\bar{\gamma} \bar{\phi}_+(1)}{\sqrt{\beta^2 - \bar{\gamma}^2}} \right), \]
\[ \dot{\chi}_- = \frac{1}{\eta + d} \left( \Gamma_-(1) + \eta + \frac{\bar{\gamma} \bar{\phi}_-(1)}{\sqrt{\beta^2 - \bar{\gamma}^2}} \right), \]
and the boundary conditions

\[
\bar{\phi}_+(1) = -\frac{\bar{\gamma} (\Gamma_+(1) - d)}{\sqrt{\beta^2 - \gamma^2}}, \\
\bar{\phi}_-(1) = \frac{\bar{\gamma} (\Gamma_-(1) - d)}{\sqrt{\beta^2 - \gamma^2}}, \\
\bar{\phi}_{\pm}(0) = 0.
\] (3.43)

Note that, the two equations in (3.42) can be neglected since they can be recovered from the rest of the equations in (3.41).

We now proceed with linearizing (3.41) around the equilibrium point \((0, 0, 0, 0, 0, 0, 0)\).

This leads us to the linearized problem stated in terms of the operator \(\mathcal{L}\)

\[
\mathcal{L} \begin{bmatrix} \eta \\ \gamma \\ \bar{\phi}_+ \\ \Gamma_+ \\ \bar{\phi}_- \\ \Gamma_- \end{bmatrix} = \begin{bmatrix}
\frac{1}{\beta} \left( \gamma + \rho \bar{\phi}_+(1) + \int_0^1 \frac{2z \bar{\phi}_+ \omega_+ d_+ \rho}{c} \, dz + \int_0^1 \frac{2z \bar{\phi}_- \omega_- d_- d}{c} \, dz - \bar{\phi}_-(1) \right) \\
- \frac{1}{\beta} \left( \omega_+^2 d_+^2 \rho^2 + \omega_+ d_+ \rho^2 + \rho^2 \right) - \frac{1}{d^2} \left( \frac{\omega_+^2 d_+^2 c^4}{3c^2} - \omega_- d_- d^2 + d^2 \right) + \alpha \right] \eta \\
\frac{z(\rho - \omega_+ d_+ \rho (z - 1))}{\beta} \left( \gamma + \rho \bar{\phi}_+(1) + \int_0^1 \frac{2z \bar{\phi}_+ \omega_+ d_+ \rho}{c} \, dz \\
+ \int_0^1 \frac{2z \bar{\phi}_- \omega_- d_- d}{c} \, dz - \bar{\phi}_-(1) \right) - \rho \bar{\phi}_+ \\
\frac{\Gamma_+}{d} \frac{z(-d + \omega_- d_- d^2 (1 - z))}{d \beta} \left( \gamma + \rho \bar{\phi}_+(1) + \int_0^1 \frac{2z \bar{\phi}_+ \omega_+ d_+ \rho}{c} \, dz \\
+ \int_0^1 \frac{2z \bar{\phi}_- \omega_- d_- d}{c} \, dz - \bar{\phi}_-(1) \right) - \frac{\bar{\phi}_-}{d},
\end{bmatrix}
\] (3.44)
coupled with the linearized boundary conditions

\[\bar{\phi}_+(1) = \frac{1}{\beta} \left( \gamma + g\bar{\phi}_+(1) + \int_0^1 \left( \frac{2z\bar{\phi}_+\omega_+d_+d}{c} + \frac{2z\bar{\phi}_-\omega_-d_+d}{c} \right) dz - \bar{\phi}_-(1) \right),\]

\[\bar{\phi}_-(1) = -\frac{d}{\beta} \left( \gamma + g\bar{\phi}_+(1) + \int_0^1 \left( \frac{2z\bar{\phi}_+\omega_+d_+d}{c} + \frac{2z\bar{\phi}_-\omega_-d_+d}{c} \right) dz - \bar{\phi}_-(1) \right),\]

\[\bar{\phi}_\pm(0) = 0.\]  

(3.45)

Consider the eigenvalue problem \(Lu = \lambda u\), together with the boundary conditions (3.45). Upon setting \(\lambda = ik\), we obtain the dispersion relation

\[\alpha + \beta k^2 = \frac{k\rho}{\tanh(k)} + \frac{k}{\tanh(kd)} + \left( \frac{\omega_+d_+g}{c} - \frac{\omega_-d_+}{c} \right).\]  

(3.46)

We would like to mention that this dispersion relation is equivalent to and consistent with the dispersion relation obtained in (3.6)

Our next objective in the construction of small-amplitude solutions is to apply the center manifold approach due to Mielke \cite{Mie88b} to the system (3.44) and (3.45). For convenience, the theorem has been included in Appendix B, which is a version used, for instance, in \cite[Section 3]{Nil17}. Due to nonlinear boundary conditions (3.45), however, we are not able to naively implement the theorem right away. As an intermediate step, following a standard idea, we make a change of variables via the operator \(\mathcal{G}\) that linearizes the boundary conditions at the cost of complication of the problem in the bulk

\[\mathcal{G}(\eta, \gamma, \bar{\phi}_+, \Gamma_+, \bar{\phi}_-, \Gamma_-) = (\eta, \nu, \varphi_+, \Gamma_+, \varphi_-, \Gamma_-),\]  

(3.47)

where,

\[\nu = g\bar{\phi}_+(1) - \bar{\phi}_-(1),\]

\[\varphi_+ = g\bar{\phi}_+ + W \left( A[\Gamma_+](z) - \frac{g}{2}(z^2 - \frac{1}{3}) \right), \quad \varphi_- = \bar{\phi}_- - W \left( A[\Gamma_-](z) - \frac{d}{2}(z^2 - \frac{1}{3}) \right),\]  

(3.48)
\[
W = \frac{\tilde{\gamma}}{\sqrt{\beta^2 - \tilde{\gamma}^2}}
\]

\[
A[f](s) = \int_0^s sf_s \, ds - \int_0^1 \int_0^s sf_s(s) \, ds \, dz.
\]

One may check easily that \(\varphi_{\pm z}(1) = \varphi_{\pm z}(0) = 0\). Further, via the definition of \(\varphi_{\pm}\) in (3.48), we obtain

\[
\int_0^1 \varphi_{\pm} \, dz = 0 \quad \text{provided} \quad \int_0^1 \varphi_{\pm} \, dz = 0.
\]

It is also worth noting that the operator \(\mathfrak{G}\) is invertible in some neighborhood of the origin and its inverse is explicitly given by

\[
\mathfrak{G}^{-1} \begin{bmatrix} \eta \\ \psi \\ \varphi_+ \\ \Gamma_+ \\ \varphi_- \\ \Gamma_- \end{bmatrix} = \begin{bmatrix} \eta \\ \frac{\beta R}{\sqrt{1 + R^2}} - I - II \\ \varphi_+ - R A[\Gamma_+](z) - \frac{R}{2} (z^2 - \frac{1}{3}) \\ \Gamma_+ \\ \varphi_- + R A[\Gamma_-](z) - \frac{d}{2} (z^2 - \frac{1}{3}) \\ \Gamma_- \end{bmatrix},
\]

where

\[
I = \int_0^1 \frac{z}{\eta - 1} \left( \frac{\varphi_+}{\varphi} - R \varphi_+ (\Gamma_+ - d + \frac{\omega_+ d_+ d(z - 1)}{c}) \right) \left( \Gamma_+ - \frac{w_+ d_+ d(1 - z)}{c} (z - 1) \right) \, dz,
\]

\[
II = \int_0^1 \frac{z}{\eta + d} \left( \varphi_- - R \varphi_- (\Gamma_- - d + \frac{\omega_- d_+ d^2 (1 - z)}{c}) \right) \left( \Gamma_- - \frac{w_- d_+ d^2 (1 - z)}{c} (1 - z) \right) \, dz,
\]

\[
R = \frac{\varphi_+(1) - \varphi_-(1) - \nu}{A[\Gamma_+](1) + A[\Gamma_-](1) - \frac{\varphi + d}{4}}.
\]
In this new coordinate system, the Hamiltonian exhibits a new expression:

\[
H = \int_{0}^{1} \frac{1}{2\varrho(1 - \eta)} \left( (\Gamma_{+z} + \frac{\omega_{+d_{+}}d_{+}(z - 1)}{c} - \eta \varphi) - (\varphi_{+z} - Rz(\Gamma_{+z} - \varrho)) \right)^2 dz \\
+ \int_{0}^{1} \frac{1}{2(d + \eta)} \left( (\Gamma_{-z} + \frac{\omega_{-d_{+}}d_{+}^2(1 - z)}{c} + \eta) - (\varphi_{-z} + Rz(\Gamma_{-z} - d)) \right)^2 dz \\
+ \frac{1}{c} \int_{0}^{1} \left( \Gamma_{+z} - \varrho + \frac{\omega_{+d_{+}}d_{+}(z - 1)}{c} \right) \omega_{+d_{+}}(z(\eta - 1) + 1) \, dz \\
+ \frac{1}{c} \int_{0}^{1} \left( \Gamma_{-z} - d_{+} + \frac{\omega_{-d_{+}}d_{+}^2(1 - z)}{c} \right) \omega_{-d_{+}}(z(\eta + d) - d) \, dz \\
- \frac{\beta}{\sqrt{1 + R^2}} + \frac{\alpha}{2} \eta^2 + \frac{\omega_{+d_{+}}^2}{6c^2} + \frac{\omega_{-d_{+}}^2}{6c^2} + \frac{\omega_{+d_{+}}}{2c} - \frac{\omega_{-d_{+}}^2}{2c}.
\]

Consequently, the Hamilton’s equations now become

\[
\dot{\eta} = R,
\]

\[
\dot{\nu} = \frac{1}{\eta - 1} \left( -\Gamma_{+z}(1) + \frac{\omega_{+d_{+}}}{2c} + R \left[ -R(\Gamma_{+z} - \varrho) - \varphi_{+1} + R \left( A[\Gamma_{+}](1) - \frac{\varrho}{3} \right) \right] \right)
- \frac{1}{\eta + d} \left( -\Gamma_{-z}(1) + \frac{\omega_{-d_{+}}d_{+}^2}{2c} + R \left[ R(\Gamma_{-z} - d) - \varphi_{-1} + R \left( A[\Gamma_{-}](1) - \frac{d}{3} \right) \right] \right),
\]

\[
\dot{\varphi}_{+} = \frac{1}{\eta - 1} \left( -\Gamma_{+z} + \frac{\omega_{+d_{+}}d_{+}(2z - 1)}{2c} + R \left[ z\varphi_{+z} - Rz^2(\Gamma_{+z} - \varrho) - \varphi_{+1} + R \left( A[\Gamma_{+}](1) - \frac{\varrho}{3} \right) \right] \right)
+ \frac{\dot{\gamma}(1 + R^2)^{3/2}}{\beta} \left( A[\Gamma_{+}](z) - \frac{\varrho}{2}(z^2 - 1/3) \right) + \frac{RA[\varphi_{+z}](z)}{\eta - 1},
\]

\[
\dot{\Gamma}_{+} = \frac{1}{\eta - 1} \varphi_{+z},
\]

\[
\dot{\varphi}_{-} = \frac{1}{\eta + d} \left( -\Gamma_{-z} + \frac{\omega_{-d_{+}}d_{+}^2(1 - 2z)}{2c} + R \left[ z\varphi_{-z} + Rz^2(\Gamma_{-z} - d) - \varphi_{-1} + R \left( A[\Gamma_{-}](1) - \frac{d}{3} \right) \right] \right)
+ \frac{\dot{\gamma}(1 + R^2)^{3/2}}{\beta} \left( A[\Gamma_{-}](z) - \frac{d}{2}(z^2 - 1/3) \right) + \frac{RA[\varphi_{-z}](z)}{\eta + d},
\]

\[
\dot{\Gamma}_{-} = \frac{-1}{\eta + d} \varphi_{-z}.
\]

(3.50)
where

\[ \dot{\gamma} = (1 + R^2) \left( \frac{\omega_+ d_+ \varrho (z - 1)}{2 \varrho (\eta - 1)^2} - \varrho \right)^2 + \left( \frac{\omega_- d_- (1 - z)}{2 (\eta + d)^2} - d \right)^2 \right) + \frac{\varrho - 1}{2} + \alpha \eta. \]

Recall that we are interested in the solutions constraint to the condition \((\beta, \alpha) = (\beta, \alpha_0) + (0, \epsilon^2)\) with \(\beta > \beta_0\) where these parameters are defined in (3.5). It can be shown that the imaginary part of the spectrum of the linearized operator \(L\) consists of zero, which is an eigenvalue of (algebraic) multiplicity 2 when \(\alpha = \alpha_0\) and \(\beta = \beta_0\), as given by (3.7). The associated eigenvector and the generalized eigenvector, namely \(e_1\) and \(e_2\), of the zero eigenvalue are then computed. Explicitly, they take the form

\[
e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{1}{c} \omega_+ d_+ \varrho (z - z^2) \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ \beta - \frac{\varrho + d}{3} - \frac{(\omega_+ d_+ \varrho - \omega_- d_- d^2)}{12c} \\ \frac{z^2 - 1/3}{2} \\ -d (z^2 - 1/3) \end{pmatrix}.
\]

It is straightforward to check that \(L e_1 = 0\) and \(L e_2 = e_1\) with \(\tilde{\Omega}(e_1, e_2) = \beta - \frac{\varrho + d}{3} = \beta_*\), provided \(\alpha = \varrho + \frac{1}{d} + \frac{\omega_+ d_+ \varrho}{c} - \frac{\omega_- d_-}{c}\).

Let

\[ v_1 = \frac{e_1}{\sqrt{\beta_*}} \text{ and } v_2 = \frac{e_2}{\sqrt{\beta_*}}. \]

They, indeed, form a symplectic basis of the vector space spanned by the eigenvectors \(e_1\) and \(e_2\). Let us define \(f_i := d \tilde{\Theta}(0)(v_i)\). Upon applying the center manifold theorem along with Darboux’s theorem, we obtain a Hamiltonian system \((X^\mu_C, \Psi, \tilde{H}^\mu)\),

\[ X^\mu_C = \{ u_1 + r(u_1, \mu) : u_1 \in \tilde{U}_1 \} \]

and \(\tilde{U}_1\) is a neighborhood of 0 as stated in Appendix [B]. In the setting of the center manifold theorem found in Appendix [B] the space \(E = \tilde{M}\). We would like to note
here that all hyphothesis $H1 - H4$ are satisfied. As a conclusion, we obtain small bounded solutions that live on the two-dimensional center manifold. Precisely, every $u_1$ can be represented as

$$u_1 = (q, p) = qf_1 + pf_2,$$

where

$$f_1 = \frac{1}{\sqrt{\beta^*}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{1}{c} \omega_+ d_+ \theta (z - z^2) \end{pmatrix}, \quad f_2 = \frac{1}{\sqrt{\beta^*}} \begin{pmatrix} 0 \\ \frac{\theta + d}{3} \\ 0 \\ 0 \end{pmatrix}. \quad (3.52)$$

Upon completing some tedious computations, the Taylor expansion of the reduced Hamiltonian is given by

$$\tilde{H}^\mu(q, p) = \frac{1}{2} p^2 - \frac{1}{2 \beta^*} \epsilon^2 q^2 + \frac{\theta - \frac{1}{d^2} + \frac{\omega_+ d_+ \theta}{c} + \frac{\omega_- d_+ \theta}{cd} + \frac{\omega_+ d_+^2 \theta}{3c^2} - \frac{\omega_- d_+^2 \theta}{3c^2}}{2\beta^*} q^3$$

$$+ O(||(p, q)||\epsilon^2, p, q)||^2) + O(||(p, q)||^2|\epsilon^2, p, q||^2). \quad (3.53)$$

Therefore, we obtain the corresponding Hamilton’s equations:

$$q_x = p + O(||(p, q)||\epsilon^2, p, q||),$$

$$p_x = \frac{\epsilon^2 q}{\beta^*} + \frac{-\theta + \frac{1}{d^2} - \frac{\omega_+ d_+ \theta}{c} - \frac{\omega_- d_+ \theta}{cd} - \frac{\omega_+ d_+^2 \theta}{3c^2} + \frac{\omega_- d_+^2 \theta}{3c^2}}{2\beta^*} q^3$$

$$+ O(||p||\epsilon^2, p, q||) + O(||(p, q)||^2|\epsilon^2, p, q||^2). \quad (3.54)$$

Consider the following rescaling:

$$X = \frac{\epsilon}{\sqrt{\beta^*}} x, \quad q(x) = \beta^2 \epsilon^2 Q(X), \quad p(x) = \epsilon^3 \beta^{3/2} P(X). \quad (3.55)$$

Under these rescaling, the Hamilton’s equations in (3.54) read

$$Q_X = P + O(\epsilon),$$

$$P_X = Q + \frac{3K(\beta, d, \omega_+, \omega_-, \epsilon)}{2} Q^2 + O(\epsilon). \quad (3.56)$$
where
\[
K(\varrho, d, \omega_+, \omega_-, c) = \beta^3/2 \left( -\varrho + \frac{1}{d^2} - \frac{\omega_+ d_+ \varrho}{c} - \frac{\omega_- d_+}{cd} - \frac{\omega^2_+ d^2_+ \varrho}{3c^2} + \frac{\omega^2_+ d^2_+}{3c^2} \right). (3.57)
\]

Upon truncating the rescaled Hamilton’s equations in (3.56), we obtain
\[
Q_X = P, \quad P_X = Q + \frac{3K(\varrho, d, \omega_+, \omega_-, c)}{2}Q^2 \quad (3.58)
\]
which has solutions
\[
Q(X) = \frac{-\text{sech}^2(X/2)}{K(\varrho, d, \omega_+, \omega_-, c)},
\]
\[
P(X) = \frac{\text{sech}^2(X/2) \tanh(X/2)}{K(\varrho, d, \omega_+, \omega_-, c)}. (3.59)
\]

Thanks to the structure of the symplectic basis in (3.51), we then obtain the profile of \( \eta \) in the original variables
\[
\eta(x) = \frac{d_+ \epsilon^2 \text{sech}^2(\frac{\epsilon x}{2d_+ \sqrt{\beta_x}})}{-\varrho + \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_+}{cd} + \frac{\omega^2_+ d^2_+ \varrho}{3c^2} - \frac{\omega^2_+ d^2_+}{3c^2} + O(\epsilon^2)}. + O(\epsilon^3).
\]
Observe that, depending on the sign of the denominator in the expression above, we obtain a wave of depression or elevation. Hence, the proof of Theorem 3.1 is now complete.

### 3.3 Stability

#### 3.3.1 General theory

Having proved the existence of small-amplitude waves, in the remaining part of the work, we will investigate the aspect concerning their orbital stability/instability. For that, we are using the general theory stated in [VWW20], which is a variant of the well known GSS machinery introduced in [GSS90a, GSS90b]. As it is to any mathematical approach, there are a number of preliminary assumptions that first
have to hold before applying the theory. For the water wave problem, however, there are some conditions that obstruct a direct use of the classical GSS machinery. The variant in [VWW20] essentially solves these issues by weakening some of the requirements in GSS which then permits its application to the water wave problem. Although, the assumptions are weakened, the final conclusion of the both approaches in [VWW20] and [GSS90a, GSS90b] remain the same. We have outlined all the required hypothesis below and we will refer to them again later. For more in-depth and detailed explanations on the general theory, see [VWW20, Section 2] and the references therein.

**Assumption 1** (Spaces). Let $X, V, W$ be spaces defined by (3.87), (3.88), and (3.90). Assume there exists a constant $C > 0$ and $\theta \in (0, 1]$, so that the following inequality holds

$$\|u\|_V^3 \leq C \|u\|_X^{2+\theta} \|u\|_W^{1-\theta},$$

for any $u \in W$. (3.60)

Let $O \subset X$ be an open set where solutions live. Assume that $J : D(J) \subset X^* \to X$ is a closed linear operator and for any $u \in O \cap V$.

**Assumption 2** (Poisson map).

1. The domain $D(J)$ is dense in $X^*$.

2. $J$ is injective.

3. For each $u \in O \cap V$, $J(u)$ is skew-adjoint, that is

$$\langle J(u)v, w \rangle = -\langle v, J(u)w \rangle,$$

(3.61)

for all $v, w \in D(J)$.
**Assumption 3** (Derivative extension). Assume that there exist (extension) mappings \( \nabla E, \nabla P \in C^0(\mathcal{O} \cap \mathbb{V}; \mathbb{X}^*) \) of \( DE(u) \) and \( DP(u) \) respectively for all \( u \in \mathcal{O} \cap \mathbb{V} \).

Suppose also that there exists a family of affine maps \( T(s) : \mathbb{X} \to \mathbb{X} \) parameterized by \( s \) where the linear part \( dT(s) := T(s)u - T(s)0 \) enjoys a number of properties.

**Assumption 4** (Symmetry group). The symmetry group \( T(\cdot) \) satisfies

1. (Invariance) The neighborhood \( \mathcal{O} \), and the subspaces \( \mathbb{V} \) and \( \mathbb{W} \), are all invariant under the symmetry group. Moreover \( I^{-1}\mathcal{D}(J) \) is invariant under the linear symmetry group i.e. \( \mathcal{D}(J) \) is invariant under the adjoint \( dT^*(s) : \mathbb{X}^* \to \mathbb{X}^* \).

2. (Flow property) Assume \( T(0) = dT(0) = \text{Id}_X \), for any \( r, s \in \mathbb{R} \), we have

\[
T(s + r) = T(s)T(r), \quad dT(s + r) = dT(s)dT(r). \tag{3.62}
\]

3. (Unitary) The linear part \( dT(s) \) is a unitary operator on \( \mathbb{X} \) and an isometry on \( \mathbb{V} \) and \( \mathbb{W} \) for each \( s \in \mathbb{R} \).

4. (Strong continuity) The symmetry group is strongly continuous on \( \mathbb{X}, \mathbb{V}, \) and \( \mathbb{W} \).

5. (Affine part) The function \( T(\cdot)0 \) belongs to \( C^3(\mathbb{R}; \mathbb{W}) \) and there exists an increasing function \( \iota : [0, \infty) \to [0, \infty) \) such that

\[
\|T(s)0\|_W \neq \iota(\|T(s)0\|_X), \quad \text{for all} \ s \in \mathbb{R}. \tag{3.63}
\]

6. (Commutativity with \( J \)) For all \( s \in \mathbb{R} \),

\[
JIT(s) = T(s)JI. \tag{3.64}
\]
7. (Infinitesimal generator) The infinitesimal generator of $T$ is the affine mapping

$$T'(0)u = \lim_{s \to 0} \frac{T(s)u - u}{s} = dT'(0) + T'(0)0,$$  \hspace{1cm} (3.65)

with dense domain $\mathcal{D}(T'(0)) \subset X$ consisting of all $u \in X$ such that the limit above exists in $X$ (similarly for the spaces $V$ and $W$). Assume also that $\nabla P(u) \in \mathcal{D}(J)$ and that

$$T'(0)u = J(u)\nabla P(u),$$  \hspace{1cm} (3.66)

for all $u \in \mathcal{D}(T'(0)|_V) \cap \mathcal{O}$.

8. (Density) The subspace

$$\mathcal{D}(T'(0))|_W \cap \text{Rng}J$$  \hspace{1cm} (3.67)

is dense in $X$.

9. (Conservation) For all $u \in \mathcal{O} \cap V$, the energy is conserved by the flow of the symmetry group, meaning

$$E(u) = E(T(s)u), \quad \text{for all } s \in \mathbb{R}.$$  \hspace{1cm} (3.68)

We call $u \in C^1(\mathbb{R}; \mathcal{O} \cap W)$ to be a bound state of the abstract Hamiltonian (3.16), if $u$ is a solution of the form

$$u(t) = T(ct)U,$$  \hspace{1cm} (3.69)

for some $c \in \mathbb{R}$ and $U \in \mathcal{O} \cap W$.

We would like to point out that the water wave problem in the present setting satisfies Assumption 4. However, many of the requirements can be checked by going through a series of length, yet elementary, computations. Therefore, we will avoid
checking all the requirements in Assumption 4. Instead, we will focus more on showing that Assumption 4(8) holds: this is precisely one of requirements from the general theory in [GSS90a] that has been weakened and modified in [VWW20].

**Assumption 5** (Bound states). There exists a one-parameter family of bound state solutions $\{U_c : c \in I\}$ to the Hamiltonian system (3.16).

1. The mapping $c \in I \mapsto U_c \in O \cap W$ is of class $C^1$.

2. The non-degeneracy condition $T'(0)U_c \neq 0$ holds for every $c \in I$. Equivalently, $U_c$ is never a critical point of the momentum.

3. For all $c \in I$,

\[
U_c \in \mathcal{D}(T'''(0)) \cap \mathcal{D}(JIT'(0)),
\]

(3.70)

and

\[
JIT'(0)U_c \in \mathcal{D}(T'(0)|_W).
\]

(3.71)

4. It holds that $\lim \inf_{|s| \to \infty} \|T(s)U_c - U_c\|_X > 0$.

**Assumption 6** (Spectrum). The operator $D^2E_c(U_c) \in \text{Lin}(V, V^*)$ extends uniquely to a bounded linear operator $H_c : X \to X^*$ such that:

1. $I^{-1}H_c$ is a self-adjoint operator on $X$.

2. The spectrum of $I^{-1}H_c$ satisfies

\[
\text{spec}(I^{-1}H_c) = \{-\mu_c^2\} \cup \{0\} \cup \Sigma_c,
\]

(3.72)

where $-\mu_c^2 < 0$ is a simple eigenvalue that correspond to a unit eigenvector $\chi_c$, $0$ is a simple eigenvalue generated by $T$, and $\Sigma_C \subset (0, \infty)$ is bounded away from 0.
3.3.2 Notion on stability/instability

After outlining required assumptions for the general theory, we now proceed to carefully define the notion on stability/instability concerned here. At this point, the functions spaces that we are working with are still abstract and will be specified soon in the next subsection. Fix a bound state $U_c$ and radius $r > 0$, we define the following sets

$$U_r^X := \{ u \in O : \inf_{s \in \mathbb{R}} \| u - T(s)U_c \|_X < r \},$$

$$U_r^W := \{ u \in O \cap \mathbb{W} : \inf_{s \in \mathbb{R}} \| u - T(s)U_c \|_W < r \}.$$  \hspace{1cm} (3.73)

Fix $R > 0$, let $B_{R}^W$ denote the intersection between the ball of radius $R$ centered at the origin in $\mathbb{W}$ and the set $O$.

**Definition 3.4.** The bound state $U_C$ is conditionally orbitally stable provided that for any $r > 0$ and $R > 0$, there exists $r_0 > 0$ such that if $u : [0, t_0] \rightarrow B_{R}^W$ is a solution to (3.3) where $u(0) \in U_{r_0}^X$ then $u(t) \in U_r^X$ for all $t \in [0, t_0)$.

**Remark 3.5.** The term conditional here means that the stability only holds on a time interval in which the solutions exist and their growth in the norm of $\mathbb{W}$ is controlled. Additionally, we would like to point out that the bounding constant $r$ is independent of $t_0$. If, presumably, a global-in-time solution exists, then the conditional stability/instability can be interpreted classically.

From the general theory [GSS90a, GSS90b, VWW20], the conclusion on stability can be determined by looking at the sign of the second derivative of a scalar-valued function known as *moment instability*

$$d(c) := E_c(U_c) = E(U_c) - cP(U_c).$$  \hspace{1cm} (3.74)
This leads us to state the following theorem.

**Theorem 3.6** (Stability/Instability). Let all Assumptions 1–6 be satisfied. The bound state $U_c$ under consideration is conditionally orbitally stable (unstable), provided $d''(c) > 0$ ($d''(c) < 0$).

**Remark 3.7.** To make sense of the orbital instability, the water wave (Cauchy problem) needs to admit a unique local solution. In this context, the ill-posedness of the time-dependent problem can be interpreted as a case of instability.

### 3.4 Hamiltonian Formulation

#### 3.4.1 Nonlocal operators

We begin this section by reformulating the governing equations (3.3) in terms of variables restricted to the interface $\eta(x, t)$ in the spirit of Zakharov–Craig–Sulem. Although this way of formulating the problem forces us to work with some complicated non-local operators (pseudo-differential operators), it simplifies the problem by pushing all the unknowns to the boundary, in this case, the internal interface. The idea was then adopted by a number of authors studying internal waves, for instance, [BB97], [CG00].

Let $\xi_\pm$ be defined as the trace of $\phi_\pm$ on the interface $y = \eta(x, t)$ for the upper and lower regions of the fluid. It is clear that

$$\xi'_\pm = (\partial_x \phi_\pm)|_{y=\eta} + \eta'(\partial_y \phi_\pm)|_{y=\eta}. \quad (3.75)$$

Additionally, we denote $\mathcal{H}_\pm$ as the Hilbert transform acting on $\xi_\pm$:

$$\mathcal{H}_\pm(\eta)\xi_\pm = \tilde{\psi}_\pm(t, x, \eta) = \psi_\pm(t, x, \eta) + \frac{\omega_\pm}{2} \eta^2. \quad (3.76)$$
For later use, let us also introduce the Dirichlet–Neumann operator in $\Omega_+$ and $\Omega_-$ (for a fixed $\eta$):

$$G_\pm(\eta)\xi_\pm := \langle \eta' \rangle \left( N_+ \cdot \nabla H_\pm(\eta) \xi_\pm \right),$$

where $N_\pm$ is the outward unit normal relative to the domain $\Omega_\pm$ along the internal interface $S$ and the Japanese bracket $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$. Further, the notation $H_\pm(\eta)\xi_\pm$ denotes the harmonic extension of $\xi_\pm$ to $\Omega_\pm$ and uniquely solves

$$\begin{cases}
\Delta H_\pm(\eta)\xi_\pm = 0 & \text{in } \Omega_\pm, \\
H_\pm(\eta)\xi_\pm = \xi_\pm & \text{on } y = \eta(x,t), \\
\partial_y H_\pm(\eta)\xi_\pm = 0 & \text{on } y = \pm d_\pm.
\end{cases}$$

It is known that for $\eta \in H^{k_0+1/2}(\mathbb{R})$, the Dirichlet–Neumann operator $G_\pm(\eta)$ is an isomorphism $\dot{H}^k(\mathbb{R}) \rightarrow \dot{H}^{k-1}(\mathbb{R})$, where $\dot{H}^k$ denotes the usual homogeneous Sobolev space of order $k$ and $k \in [1/2 - k_0, 1/2 + k_0]$ for any real number $k_0 > 1/2$. The operator $H_\pm(\eta)$ is a bounded mapping from $H^k(\mathbb{R})$ to $H^{k+1/2}(\Omega_\pm)$ and from $\dot{H}^k(\mathbb{R})$ to $\dot{H}^{k+1/2}(\Omega_\pm)$.

**Remark 3.8.** The spaces $H^k$ and $\dot{H}^k$ are the Sobolev and homogeneous Sobolev spaces, respectively.

Using the operators $G_\pm(\eta)$, $H_\pm(\eta)$, and the new unknown $\xi_\pm$, the water wave problem can be pushed to the boundary $y = \eta(x,t)$. It now reads

$$\begin{cases}
\eta_t = \mp G_\pm(\eta)\xi_\pm + \omega_\pm \eta x & \text{on } y = \eta(x,t), \\
\left[ \rho \xi_t \right] = -g \eta [\rho] - \sigma \left( \frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right) x \\
- \sum_{\pm} \pm \frac{\rho_\pm}{2} \Gamma_\pm(\eta, \xi_\pm) \mp \rho_\pm \omega_\pm \eta x_\pm \mp \rho_\pm \mathcal{H}_\pm(\eta)\xi_\pm & \text{on } y = \eta(x,t).
\end{cases}$$

(3.78)
where
\[
\Gamma_\pm(\eta, \xi_\pm) = \left(\xi_{x_\pm}^2 - (\mathcal{G}_\pm(\eta)\xi_\pm)^2 \pm 2\eta_\pm\xi_{xx_\pm}\mathcal{G}_\pm(\eta)\xi_\pm\right) / (1 + \eta_\pm^2).
\] (3.79)

Next, in order to reformulate the problem in a Hamiltonian language, we introduce the variable \(\bar{\xi} := -[\rho\xi]\). Recall, from the kinematic boundary condition in (3.78) we have
\[
\mathcal{G}_-(\eta)\xi_- + \mathcal{G}_+(\eta)\xi_+ = \llbracket\omega\rrbracket \eta_{tx}.
\] (3.80)

This, together with the definition of \(\bar{\xi}\), gives us
\[
\pm \mathcal{G}_\pm \bar{\xi} = \mathcal{B}(\eta)\xi_\mp - \rho \llbracket\omega\rrbracket \eta_{tx},
\] (3.81)

where \(\mathcal{B}(\eta) := \rho_+ \mathcal{G}_-(\eta) + \rho_- \mathcal{G}_+(\eta)\). Following the property of \(\mathcal{G}_\pm(\eta)\), the operator \(\mathcal{B}(\eta)\) is also bounded and linear from \(H^k(\mathbb{R})\) to \(H^{k-1}(\mathbb{R})\) and from \(\dot{H}^k(\mathbb{R})\) to \(\dot{H}^{k-1}(\mathbb{R})\) for any \(\eta \in H^{k_0+1/2}(\mathbb{R})\) with \(k_0, k\) given as before. Moreover, \(\mathcal{B}(\eta)\) is an isomorphism from \(\dot{H}^k(\mathbb{R})\) to \(\dot{H}^{k-1}(\mathbb{R})\). Therefore, using (3.81) and solving for \(\xi_\pm\) we obtain
\[
\xi_\pm = \mp \mathcal{B}(\eta)^{-1} \mathcal{G}_\mp(\eta)\bar{\xi} + \rho \mp \mathcal{B}(\eta)^{-1} \llbracket\omega\rrbracket \eta_{tx}.
\] (3.82)

The kinematic and dynamic conditions now read
\[
\begin{cases}
\eta_t = \mathcal{A}(\eta)\bar{\xi} + \rho_+ \mathcal{G}_\pm(\eta)\mathcal{B}(\eta)^{-1} \llbracket\omega\rrbracket \eta_{tx} + \omega_\pm \eta_{tx} & \text{on } y = \eta(x, t), \\
\bar{\xi}_t = g_\eta \llbracket\rho\rrbracket + \sigma \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}}\right)_x \\
& + \sum_\pm \left[\pm \frac{\rho_\pm}{2} \Gamma_\pm(\eta, \xi_\pm) \mp \rho_\pm \omega_\pm \eta_{xx_\pm} \pm \rho_\pm \omega_\pm \mathcal{H}_\pm(\eta)\xi_\pm\right] & \text{on } y = \eta(x, t),
\end{cases}
\] (3.83)

where \(\mathcal{A}(\eta) := \mathcal{G}_\pm(\eta)\mathcal{B}(\eta)^{-1} \mathcal{G}_\mp(\eta)\).

The above equations can alternatively be written as
\[
\begin{cases}
\eta_t = \mathcal{A}(\eta)\bar{\xi} + \mathcal{G}_-(\eta)\mathcal{B}(\eta)^{-1} \rho_+ \llbracket\omega\rrbracket \eta_{tx} + \omega_- \eta_{tx} & \text{on } y = \eta(x, t), \\
\bar{\xi}_t = g_\eta \llbracket\rho\rrbracket + \sigma \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}}\right)_x + \frac{\rho_\mp \llbracket\nabla \psi\rrbracket^2}{2} - \eta_t \llbracket\rho \phi_y\rrbracket + \llbracket\rho \omega \psi\rrbracket & \text{on } y = \eta(x, t).
\end{cases}
\] (3.84)
We would like to mention that similar formulation involving constant (non-vanishing) vorticity for a one-fluid and two-fluid case can be found, for instance, in [Wah07] and [Com16, CI15] respectively.

**Remark 3.9.** The spaces $H^k$ and $\dot{H}^k$ are the Sobolev and homogeneous Sobolev spaces, respectively.

### 3.4.2 Function spaces

The formulation stated in (3.84) is crucial in helping us exploit the Hamiltonian structure of the problem. In light of that, let us informally introduce the required function spaces where the internal water wave problem will be posed. Fix $k \geq 1/2$, we define a product space for $u$:

$$X^k := X_1^k \times X_2^k := H^{k+1/2}(\mathbb{R}) \times \left( \dot{H}^k(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \right).$$  \hspace{1cm} (3.85)

For future reference, we will denote $X^{k+}$ to mean $X^{k+\epsilon}$ for any $0 < \epsilon \ll 1$, similarly for $H^{k+}$.

**Remark 3.10.** Note that the space $H^p(\mathbb{R}) \cap \dot{H}^q(\mathbb{R})$ is dense in $H^p(\mathbb{R})$ and $\dot{H}^q(\mathbb{R})$ for all $p, q \in \mathbb{R}$. We will use this fact to verify Assumption in the general theory.

Consider the following sequence of (continuously) embedded spaces

$$\mathbb{W} \hookrightarrow \mathbb{V} \hookrightarrow X,$$  \hspace{1cm} (3.86)

where $X$ is a Hilbert space, while $\mathbb{W}$ and $\mathbb{V}$ are reflexive Banach spaces. In practice, $\mathbb{W}$ will be the local well-posedness space of the internal water wave problem. Its regularity follows from the available results on local well-posedness. The space $X$ is
the natural energy space of the problem with \( X^* \) being its continuous dual:

\[
X := H^1(\mathbb{R}) \times \dot{H}^{1/2}(\mathbb{R}), \quad X^* := H^{-1}(\mathbb{R}) \times \dot{H}^{-1/2}(\mathbb{R}). \tag{3.87}
\]

Indeed, if \( u \in X \), then \( \nabla \phi_\pm \in L^2(\Omega_\pm) \). Furthermore, it also informs us that \( \eta \in H^1(\mathbb{R}) \) which ensures the finiteness of the potential energy. Both combined confirms that the energy is finite on \( X \). The space \( V \) is an intermediate space that lies between the spaces \( W \) and \( X \) where all the conserved quantities are smooth there.

The regularity of \( X \) turns out to be slightly insufficient for our analysis. This is because the map \( u \mapsto G_\pm(\eta) \) is not smooth with \( X \) being its domain. To fix this, the profile \( \eta \) has to be, at least, Lipschitz continuous and bounded away from the rigid walls \( \{y = \pm d_\pm\} \). In order to satisfy this level of smoothness condition, we define an intermediate space

\[
V := X^{1+} = H^{3/2+}(\mathbb{R}) \times \left( \dot{H}^{1+}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \right). \tag{3.88}
\]

along with a neighborhood

\[
\mathcal{O} := \{ (\eta, \tilde{\xi}) \in X : -d_- < \eta < d_+ \}, \tag{3.89}
\]

which makes sure that \( \eta \) is bounded away from the rigid walls. Moreover, observe that \( H^{3/2+}(\mathbb{R}) \hookrightarrow W^{1,\infty}(\mathbb{R}) \), meaning if \( (\eta, \tilde{\xi}) \in V \), then \( \eta \) is Lipschitz continuous.

Finally, since the current result on the Cauchy problem for water wave is not yet available with the level of regularity in \( V \), thus we define a smoother space

\[
\mathbb{W} := X^{5/2+} = H^{3+}(\mathbb{R}) \times \left( \dot{H}^{5/2+}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \right). \tag{3.90}
\]

It is worth mentioning that the work of Shatah and Zeng [SZ11] proves the local-wellposedness of the water wave problem at the same regularity as in \( \mathbb{W} \). Additionally,
we would like to point out that the regularity of these spaces is the same and largely follows from the recent work of Chen and Walsh [CW22].

Having specified the function spaces, we express the relationship between the trio function spaces $X$, $V$, and $W$ via an inequality recorded in the next lemma. Moreover, the content of the lemma shows that Assumption 1 in the general theory is satisfied.

**Lemma 3.11** (Interpolation). Consider the following spaces: $X$, $V$, and $W$ defined by (3.87), (3.88), and (3.90), respectively. Then there exists a constant $C > 0$ and $\theta \in (0, \frac{1}{4})$ such that

$$
\|u\|_V^3 \leq C \|u\|_X^{2+\theta} \|u\|_W^{1-\theta},
$$

for all $u \in W$.

Proof. This follows from the Gagliardo–Nirenberg interpolation theorem. ■

The above lemma shows that a small cubic term in $V$ norm can be bounded using a quadratic term in $X$. This fact is needed in the general theory when bounding some of the terms resulted from Taylor expanding functionals whose domain is $V \cap O$.

### 3.4.3 Hamiltonian Structure

In [BB97] Benjamin and Bridges formulated the internal water wave problem as a Hamiltonian system in the style of Zakharov–Craig–Sulem. Inspired by the aforementioned paper, we show that the water wave problem also exhibits a Hamiltonian structure (with a non-canonical Poisson map) that can be exploited for the stability analysis. Following the same idea as in [CW22, Section 3.3], we derive the energy
Observe that $E \in C^\infty(O \cap \mathbb{V}; \mathbb{R})$. Moreover, we will show that there is an extension mapping $\nabla E(u)$ of $DE(u)$ defined on the dual space $\mathbb{X}^*$ which is the content of the lemma below.

**Lemma 3.12 (Energy Extension).** There exists a mapping $\nabla E \in C^\infty(O \cap \mathbb{V}; \mathbb{X}^*)$ such that

$$\langle \nabla E(u), v \rangle_{\mathbb{X}^* \times \mathbb{X}} = DE(u)v, \text{ for all } u \in O \cap \mathbb{V}, v \in \mathbb{V}. \quad (3.92)$$

**Proof.** Fix $u = (\eta, \tilde{\xi}) \in O \cap \mathbb{V}$ and let $\dot{u} = (\dot{\eta}, \dot{\tilde{\xi}}) \in \mathbb{V}$ be given. Using the definition of the energy (3.91) with the self-adjointness properties of $A(\eta), G_\pm(\eta),$ and $B^{-1}(\eta)$, one can show that

$$D_\eta E(u) \dot{u} =$$

$$\frac{1}{2} \int_{\mathbb{R}} \xi \langle D A(\eta) \dot{\eta}, \dot{\tilde{\xi}} \rangle \, dx - \int_{\mathbb{R}} \left( g[\rho] \eta + \sigma \left( \frac{\eta'}{(\eta')'} \right) \right) \dot{\eta} \, dx$$

$$+ \int_{\mathbb{R}} \rho_+ \xi \langle D G_-(\eta) \dot{\eta}, B^{-1}(\eta) [\omega] \eta \rangle + \rho_+ G_-(\eta) \xi \langle D B^{-1}(\eta) \dot{\eta}, [\omega] \eta \rangle \, dx$$

$$+ \int_{\mathbb{R}} \rho_+ \xi G_-(\eta) B^{-1}(\eta) [\omega] (\eta \dot{\eta} + \eta \dot{\eta}) - \rho_+ \rho_- [\omega] (\eta \dot{\eta} + \eta \dot{\eta}) B^{-1}(\eta) [\omega] \eta \, dx$$

$$- \frac{1}{2} \int_{\mathbb{R}} \rho_+ \rho_- [\omega] \eta \xi \langle D B^{-1}(\eta) \dot{\eta}, [\omega] \eta \rangle + \left( \xi \omega_-(\eta \dot{\eta} + \eta \dot{\eta}) - \frac{\eta^2 \dot{\eta} [\rho \omega^2]}{2} \right) \, dx, \quad (3.93)$$

and

$$D_\xi E(u) \dot{u} = \int_{\mathbb{R}} \left( A(\eta) \tilde{\xi} + \omega_- \eta \right) \xi \, dx + \int_{\mathbb{R}} \rho_+ \xi G_-(\eta) B^{-1}(\eta) [\omega] \eta \, dx, \quad (3.94)$$

First of all, let us look at the expression in (3.94), it is easy to see that

$$A(\eta) \tilde{\xi}, \omega_- \eta \in L^2(\mathbb{R}).$$
Moreover, it is clear that the last integral in (3.94) is an element of the dual space $X^*$ acting on $\dot{u}$.

In (3.93), the first integral can be written as

$$\int_{\mathbb{R}} \tilde{\xi} \langle DA(\eta)\eta, \tilde{\xi} \rangle \, dx = \sum_{\pm} \rho_\pm \int_{\mathbb{R}} a_1^+(\eta, \theta_\pm) \theta_\pm' \eta \, dx + \sum_{\pm} \rho_\pm \int_{\mathbb{R}} (a_2^+(\eta_\pm, \theta_\pm)A(\eta)\tilde{\xi}) \dot{\eta} \, dx.$$

Since $u \in \mathcal{O} \cap \mathcal{V}$, we obtain

$$a_1^+(\eta, \theta_\pm), \ a_2^+(\eta_\pm, \theta_\pm) \in L^2(\mathbb{R}), \quad \theta_\pm \in H^1(\mathbb{R}),$$

where $\theta_\pm = A(\eta)G_\pm(\eta)^{-1}\tilde{\xi}$. Further, one can directly see that the second and last integral in (3.93) is an element of the dual space $X^*$ acting on $\dot{u}$.

Hence the extension $\nabla E(u)$ can be thought to have an $L^2$ gradient structure, that is,

$$\langle \nabla E(u), v \rangle_{X^* \times X} = (E'(u), v)_{L^2}, \quad (3.95)$$
where the $L^2$ gradient $E'(u) = (E'_\eta(u), E'_\xi(u))$ takes the form

$$E'_\eta(u) :=$$

$$\frac{1}{2} \sum_{\pm} \rho_{\pm} a_1^{\pm}(\eta, \theta_{\pm}) \theta'_{\pm} + \sum_{\pm} \rho_{\pm} (a_2^{\pm}(\eta_{\pm}, \theta_{\pm}) A(\eta) \xi) - \left( g [\rho] \eta + \sigma \left( \frac{\eta'}{\langle \eta' \rangle} \right) \right)$$

$$+ \rho_+ \left( a^-_1(\eta, B^{-1}(\eta) [\omega] \eta_x) \xi' + a^+_2(\eta, B^{-1}(\eta) [\omega] \eta_x) G_{+}(\eta) \xi \right)$$

$$- \rho_+ \sum_{\pm} \rho_{\pm} \left( a_1^{\pm}(\eta, B^{-1}(\eta) [\omega] \eta_x) B^{-1}(\eta) G_{-}(\eta) \xi \right)$$

$$- \rho_+ [\omega] \eta_x \left( B^{-1}(\eta) G_{+}(\eta) \xi \right) + \rho_+ [\omega] \eta \left( B^{-1}(\eta) [\omega] \eta_x \right)$$

$$+ \frac{1}{2} \rho_+ \rho_- \sum_{\pm} \rho_{\pm} \left( a_1^{\pm}(\eta, B^{-1}(\eta) [\omega] \eta_x) \left( B^{-1}(\eta) [\omega] \eta_x \right) \right)$$

$$+ a_2^{\pm}(\eta, B^{-1}(\eta) [\omega] \eta_x) B^{-1}(\eta) G_{+}(\eta) [\omega] \eta_x$$

$$- \omega_+ \xi' \eta - \frac{[\rho \omega^2]}{2} \eta^2,$$

$$E'_\xi(u) := A(\eta) \xi + \omega_- \eta \eta_x + \rho_+ G_{-}(\eta) B^{-1}(\eta) [\omega] \eta_x. \quad (3.96)$$

Hence, the proof is now complete. $\blacksquare$

The function space $\mathbb{X}$ (energy space) will be equipped with a symplectic form via

a Poisson map

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \sum_{\pm} \pm \rho_{\pm} \omega_{\pm} \partial_x^{-1} \end{pmatrix} : \text{Dom} \ J \subset \mathbb{X}^* \rightarrow \mathbb{X}, \quad (3.97)$$

where

$$\text{Dom} \ J := (H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})) \times (H^1(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R})). \quad (3.98)$$

Observe that from the structure of $J$ in (3.97), the difference in regularity, and homogeneity of the the spaces in (3.98), one can conclude that $J$ is not a bijective map. This fact clearly violates one of the assumption in the classical GSS approach.
This is one of the reasons why we instead use the relaxed GSS [VWW20] which only requires that $J$ to be an injection, as stated in Assumption 2. Due to the presence of vorticity, the Poisson map is not canonical anymore. In other words, it is different with the Poisson map obtained in [CW22]. When $\omega_+ = 0$ (one-layer fluid), we recover the map presented in [Wah07].

**Lemma 3.13** (Poisson map). The Poisson map $J$ (3.97) satisfies Assumption 2 in the general theory.

*Proof.* The density of Domain $J$ (3.98) in $X^*$ is a direct consequence of Remark 3.10. Further, the injectiveness and skew-adjointness of $J$ follows easily from the definition of $J$ in (3.97). □

**Theorem 3.14** (Hamiltonian formulation). Let $u = (\eta, \tilde{\xi}) \in \mathcal{O} \cap \mathbb{W}$. Consider the following Hamiltonian system given by

$$\partial_t u = JDE(u), \quad u|_{t=0} = u_0,$$

with $u_0 \in \mathcal{O} \cap \mathbb{W}$, where $E$ is given by (3.91) and $J$ is the Poisson map in (3.97). Then $u \in C^0([0, t_0); \mathcal{O} \cap \mathbb{W})$ is a (weak) solution to (3.99) if and only if the profiles $(\eta, \phi_\pm)$ solve the governing equations (3.3).

*Proof.* Suppose that $u(t) = (\eta(t), \tilde{\xi}(t)) \in C^0([0, t_0); \mathcal{O} \cap \mathbb{W})$ is a weak solution to the abstract Hamiltonian (3.16). Using (3.82), we define

$$\phi_\pm := \mathcal{H}(\eta) \left( \mp \mathcal{B}(\eta)^{-1} \zeta_\pm(\eta) \tilde{\xi} + \rho_\pm \mathcal{B}(\eta)^{-1} [\omega] \eta \right).$$

Clearly, this satisfies the Laplace equation in (3.3a) and the boundary conditions on the walls $\{y = \pm d_\pm\}$ by definition of $\mathcal{H}_\pm$ in (3.77). Further, recalling the expression ...
for \( E'(u) \) in Lemma 3.12, we have

\[
\eta_t = E'_\xi(u) = A(\eta)\xi + \rho_+ G_-(\eta)B^{-1}(\eta) [\omega] \eta x + \omega_\eta \eta x,
\]

which holds in the distributional sense and it is equivalent to the kinematic boundary conditions in (3.84), therefore leads to the kinematic condition in (3.8b). Next, we claim that the Bernoulli condition is equivalent to

\[
\tilde{\xi}_t = -E'_\eta(u) - \sum_{\pm} \pm \rho_\pm \omega_\pm \partial_x^{-1} E'_\xi(u).
\]

We begin by writing the integrand in (3.91) in terms of \( \xi_\pm \) instead of \( \tilde{\xi} \):

\[
\frac{1}{2} \left[ \tilde{\xi} A(\eta) \tilde{\xi} + 2 \rho_+ [\omega] \eta x G_-(\eta) B^{-1}(\eta) \xi - \rho_+ \rho_- [\omega] \eta x B^{-1}[\omega] \eta x 
- g [\rho] \eta^2 + 2 \tilde{\xi} \omega_\eta \eta x - \eta^3 [\rho \omega^2] + 2 \sigma(\sqrt{1 + \eta^2} - 1) \right]
= \frac{1}{2} \left[ \rho_- \xi_- G_-(\eta) \xi_- + \rho_+ \xi_+ G_+(\eta) \xi_+ - 2 \rho_+ [\omega] \eta x 
- g [\rho] \eta^2 + 2 \tilde{\xi} \omega_\eta \eta x - \eta^3 [\rho \omega^2] + 2 \sigma(\sqrt{1 + \eta^2} - 1) \right]
= \frac{1}{2} \left[ \rho_- \xi_- G_-(\eta) \xi_- + \rho_+ \xi_+ G_+(\eta) \xi_+ - 2 [\rho \xi \omega] \eta x 
- g [\rho] \eta^2 - \eta^3 [\rho \omega^2] + 2 \sigma(\sqrt{1 + \eta^2} - 1) \right].
\]

Therefore, equivalently, the Hamiltonian (3.91) can be written as

\[
E(\xi_\pm, \eta) = \frac{1}{2} \int_\mathbb{R} \rho_- \xi_- G_-(\eta) \xi_- + \rho_+ \xi_+ G_+(\eta) \xi_+ - 2 [\rho \xi \omega] \eta x 
- g [\rho] \eta^2 - \eta^3 [\rho \omega^2] + 2 \sigma(\sqrt{1 + \eta^2} - 1) \, dx.
\]

Thanks to the derivative formula in the Appendix A for the operator \( G_\pm(\eta) \) with respect to \( \eta \):

\[
\int_\mathbb{R} \xi_\pm \langle D G_\pm(\eta) \eta_\pm, \xi_\pm \rangle \, dx = \int_\mathbb{R} \dot{\eta}(\mp \Gamma_\pm(\eta, \xi_\pm)) \, dx,
\]

where \( \Gamma_\pm \) is defined in (3.79). Due to formula (3.101), it follows that the Bernoulli equation is satisfied. ■
3.4.4 The symmetry group and momentum

It is well known that the internal water wave problem is invariant under the horizontal translations. For this reason, we define a one-parameter symmetry group:

\[ T(s)u := u(\cdot - s) \quad \text{for } u \in X. \]

In addition to that, this invariance also gives rise to another conserved quantity known as the momentum, \( P_\pm \) in each layer:

\[ P_\pm = \pm \int_R \left( \rho_\pm \eta_\pm \xi_\pm + \frac{1}{2} \rho_\pm \omega_\pm \eta_\pm^2 \right) dx. \]

Summing both momentum in each layers leads to the total momentum equation given by

\[ P(\eta, \tilde{\xi}) = -\int_R \left( \eta_\pm \tilde{\xi} - \frac{1}{2} \left[ \rho \omega \right] \eta_\pm^2 \right) dx. \quad (3.102) \]

Observe that \( P \) is a smooth functional in \( O \cap V \).

The following lemma shows that \( T \) and \( P \) satisfy a number of properties as required by Assumption 4.

**Lemma 3.15** (Conserved quantities and symmetry). *The energy \( E \), momentum \( P \), and the translation symmetry group \( T \) given above satisfy Assumptions 3 and 4.*

Specifically, the infinitesimal generator of \( T_X^k \) is the unbounded linear operator

\[ T'(0)|_{X^k} : \text{Dom} T'(0) \subset X^k \to X^k \]

\[ u \mapsto -\partial_x u \quad (3.103) \]

with dense domain \( \text{Dom} T'(0)|_{X^k} := X^{k+1} \), and

\[ T'(0)u = J \nabla P(u) \quad \text{for all } u \in O \cap \text{Dom} T'(0). \quad (3.104) \]
Proof. In light of Assumption 3, we have shown that the energy has an extension as stated by Lemma 3.12. Here, we will show that the momentum can also be extended. Let \( u = (\eta, \tilde{\xi}) \in \mathcal{O} \cap \mathcal{V} \) and \( \dot{u} = (\dot{\eta}, \dot{\tilde{\xi}}) \in \mathcal{V} \). Recalling the definition of \( P \) in (3.102) and computing its first variation yield

\[
DP(u)\dot{u} = \int_{\mathbb{R}} \left( \tilde{\xi}' + [\rho \omega] \eta \right) \dot{\eta} \, dx - \int_{\mathbb{R}} \eta' \tilde{\xi} \, dx =: \langle \nabla P(u), \dot{u} \rangle_{X^* \times X}. \tag{3.105}
\]

The above expression has an \( L^2 \) gradient

\[
P'(u) = (P'_\eta(u), P'_\xi(u)) = (\tilde{\xi} + [\rho \omega] \eta, -\eta').
\]

Further, it is obvious that \( \nabla P(u) \in \text{Dom } J \) for \( u \in \mathcal{O} \cap \mathcal{V} \). Observe that \( \text{Dom } T'(0) = \mathcal{X}^{3/2} \subset \mathcal{X} \). The expression (3.104) follows easily from the definition of \( J \) in (3.97) and \( T'(0) \) in (3.103).

As it is mentioned previously, most of the requirements in Assumption 4 can be shown to hold in a straightforward manner. Hence, we will omit the details here. However, we want to focus more on the point number 8 in Assumption 3.4.4. First, recall that

\[
\text{Rng } J = \left( H^1(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R}) \right) \times \left( H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \right).
\]

Therefore, by Lemma 3.15, we obtain

\[
\text{Dom } T'(0)|_W \cap \text{Rng } J = \left( H^{4+}(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R}) \right) \times \left( H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \cap \dot{H}^{7/2+}(\mathbb{R}) \right).
\]

Hence, by Remark 3.10, \( \text{Dom } T'(0)|_W \cap \text{Rng } J \) is dense in \( \mathcal{X} \).

### 3.4.5 Bound states

The existence result in Theorem 3.1 was obtained by fixing the value of \( \beta \) and letting the rest of the parameters vary. Unfortunately, when using the general theory, this
choice is not ideal: essentially, one might study the stability of two waves that solve
two different internal water wave problem. To avoid such degenerate scenario, instead,
we require a family of solutions parameterized only by the variable \( c \), known as bound
states, while fixing the rest of the physical parameters.

Let the parameters \((\rho_{\pm*}, d_{\pm*}, \omega_{\pm*}, \sigma_*, c_*)\) be fixed. We define
\[
(\beta_c, \alpha_c) := \left( \frac{\sigma_*}{d_{++}\rho_{-}\,c^2}, -g \sigma_* \rho_{-*} \frac{d_{++}}{\rho_{-*} c^2} \right), \quad \epsilon_c := \sqrt{\alpha_c - \alpha_0} \quad \text{for } |c - c_*| \ll 1. \tag{3.106}
\]
The pair \((\beta_c, \alpha_c)\) parameterizes a line segment joining the fixed point \((\beta_*, \alpha_*)\) to the
origin in the \((\beta, \alpha)\) plane. Meanwhile, \( \epsilon_c \) plays a role as a bifurcation parameter in
terms of \( c \). Throughout this week, \( \epsilon_c \) will be kept sufficiently small.

**Corollary 3.16 (Bound states).** Fix \((\rho_{\pm*}, d_{\pm*}, \omega_{\pm*}, \sigma_*, c_*)\) such that
\[
\rho_* - \frac{1}{d_*^2} + \frac{\omega_{++} d_{++} \rho_*}{c_*} + \frac{\omega_{-\gamma d_{++} \rho_*}}{c_d} + \frac{\omega_{++} d_{++}^2 \rho_*}{3c_*^2} - \frac{\omega_{-\gamma d_{++}^2}}{3c_*^2} \neq 0
\]
and the fixed non-dimensional parameters \((\beta_*, \alpha_*)\) satisfies the condition \( \beta_* > \beta_0 \) and
\( \alpha_* = \alpha_0 + \epsilon^2 \). Then there exists an open interval \( \mathcal{I} \ni c_* \) and a family of bound states
\( \{U_c\}_{c \in \mathcal{I}} \subset \mathcal{O} \cap \mathcal{W} \) having the non-dimensional parameter values \((\beta_c, \alpha_c)\). The free
surface of the bound states is

\[
\eta_c := \eta_{c, \beta_c} \quad \text{for } c \in \mathcal{I}.
\]
Moreover, the family of bound states \( \{U_c\} \) satisfies Assumption \([5]\).

**Proof.** Let \((\rho_{\pm*}, d_{\pm*}, \omega_{\pm*}, \sigma_*, c_*)\) be given. Suppose that \( \beta_* \neq \beta_0 \) and \( 0 < \alpha_* - \alpha_0 \ll 1 \).
For any \( c \) such that \( 0 < c - c_* \ll 1 \), then by Theorem \([3.1]\) the bound states can be taken
to be \( U_c := u_{\epsilon_c, \beta_c} \) where the surface profile depends on the parameter \( \beta_c, \alpha_c \). Observe
that from the explicit expression of the profile in Theorem \([3.1]\) it is exponentially
localized. Further, due to the translation invariance of the problem, the profile $\eta_c$ is of class $C^\infty$. Thus, $\eta_c \in X_1^k$ for all $k \geq 1/2$. Similarly, via the kinematic condition, $\tilde{\xi}_c$ is also smooth and belongs to $X_2^k$ for all $k \geq 1/2$. The first part of Assumption 5 now follows. Point number 3 also follows from the regularity considered here. From the knowledge on the expression of $P$ and the profile $\eta_c$, point number 2 and 4 in Assumption 5 clearly hold.

3.5 Spectral Analysis

If $u(t) = T(ct)U$ is a travelling wave solutions for any bound state solution $U \in \mathcal{O} \cap \mathcal{W}$ with a wave speed $c \in \mathbb{R}$, then by the Hamiltonian structure, Lemma 3.12, and Assumption 5(6), we have

$$\frac{du}{dt} = cT'(0)U = JDE(U). \quad (3.107)$$

Furthermore, recall that via (3.104), the infinitesimal generator of $T$ satisfies the following relation

$$T'(0)(u) = J\nabla P(u), \quad (3.108)$$

where the operator $T'(0)$ maps $u \mapsto -\partial_x u$. In concert with (3.107), they yield

$$\frac{du}{dt} = cT'(0)U = DE(U) = cDP(U) = DE(U). \quad (3.109)$$

The above equation leads us to define the following functional known as the augmented Hamiltonian for a fixed speed $c$:

$$E_c(u) = E(u) - cP(u). \quad (3.110)$$

Let $u_\ast = (\eta_\ast, \tilde{\xi}_\ast)$ be the critical point of the functional $E_c$, then $D\tilde{\xi}E(u_\ast) = D\tilde{\xi}P(u_\ast)$.

From the Kinematic conditions in (3.78) and (3.83), for traveling waves, we have
useful expressions for $\xi_{\pm}$ and $\tilde{\xi}$

$$\xi_{\pm} = \pm c G_{\pm}^{-1}(\eta) \eta_x \pm \omega_{\pm} G_{\pm}^{-1}(\eta) \eta \eta_x = \pm G_{\pm}^{-1}(\eta)(c \eta_x + \omega_{\pm} \eta \eta_x),$$

$$\tilde{\xi} = -c A^{-1}(\eta) \eta_x - \rho_+ \omega \ A^{-1}(\eta) G_{-}(\eta) B^{-1}(\eta) \eta \eta_x - \omega_- A^{-1}(\eta) \eta \eta_x.$$  

Inspired by the notation used in [CW22], we define the $a_{1}^{\pm}(\eta, \xi)$ and $a_{2}^{\pm}(\eta, \xi)$ as follows:

$$a_{1}^{\pm}(\eta, \phi) := \mp (\partial_{x} H_{\pm}(\eta) \phi)|_{y=\eta} \quad a_{2}^{\pm}(\eta, \phi) := -(\partial_{y} H_{\pm}(\eta) \phi)|_{y=\eta}.$$  

Note that these two quantities represent the horizontal and vertical velocities respectively when $\phi$ is being replaced with $\xi_{\pm}$. Further, in relations to $a_{1}^{\pm}$ and $a_{2}^{\pm}$, we define the following functions which represent the relative velocities in horizontal and vertical directions:

$$b_{1}^{\pm} := \mp a_{1}^{\pm}(\eta, \xi_{\pm}) - c - \omega_{\pm} \eta, \quad b_{2}^{\pm} := -a_{2}^{\pm}(\eta, \xi_{\pm}). \quad (3.111)$$  

For travelling waves, solutions, the Kinematic condition can now be written as follows:

$$b_{2}^{\pm} = (b_{1}^{\pm}) \eta_x.$$  

Via some standard computation (in the notes), we obtain

$$D \xi_{\pm}(\eta) \dot{\eta} = \mp G_{\pm}(\eta)^{-1}(b_{1}^{\pm} \dot{\eta})_x + b_{2}^{\pm} \dot{\eta} \quad (3.112)$$

Using the definition of $\tilde{\xi}$ and the formula for $D \xi_{\pm}(\eta)$, we can infer

$$D \tilde{\xi}(\eta) \dot{\eta} = \sum_{\pm} \rho_{\pm} G_{\pm}^{-1}(\eta) (b_{1}^{\pm} \dot{\eta})_x - \sum_{\pm} \pm \rho_{\pm} b_{2}^{\pm} \dot{\eta} =: S \dot{\eta} - T \dot{\eta}. \quad (3.113)$$

Let us now define a smooth functional known as the augmented potential $V_{c}^{\text{aug}}$ as follows:

$$V_{c}^{\text{aug}} := E_{c}(\eta, \tilde{\xi}(\eta)) = \min_{\xi} E_{c}(\eta, \tilde{\xi}).$$
In the rest of this article, we shall compute the spectrum of $D^2V^\text{aug}_c$, which will determine the spectrum of $D^2E_c$. The formula for $D^2V^\text{aug}_c$ is similar to the one in [].

**Lemma 3.17** (Second derivative of $V^\text{aug}_c$). For all $(\eta, \tilde{\xi}_*(\eta)) \in \mathcal{O} \cap \mathbb{V}$ and $\dot{\eta} \in H^{3/2+}$, we have the following formula

$$D^2V^\text{aug}_c(\eta)[\dot{\eta}, \dot{\eta}] = D^2E_c(\eta, \tilde{\xi}_*)[\dot{\eta}, \dot{\eta}] - \int_{\mathbb{R}} (S - T)\dot{\eta}A(\eta)(S - T)\dot{\eta} \, dx.$$  \hspace{1cm} (3.114)

**Proof.** We begin by differentiating in the direction of $\dot{\eta}$,

$$D^2V^\text{aug}_c(\eta)\dot{\eta} = D^2E_c(u_*)\dot{\eta} + D\xi E_c(u_*)D\tilde{\xi}_*(\eta)\dot{\eta} = D^2E_c(u_*)\dot{\eta},$$  \hspace{1cm} (3.115)

where $u_* = (\eta, \tilde{\xi}_*)$. Observe that the second term in the summation above has vanished due to its evaluation at $u_* = (\eta, \tilde{\xi}_*(\eta))$ which is a critical point of $E_c$.

Differentiating again in the direction of $\dot{\eta}$ gives us

$$D^2V^\text{aug}_c(\eta)[\dot{\eta}, \dot{\eta}] = D^2E_c(\eta)\dot{\eta} + D\xi D^2E_c(u_*)[D\tilde{\xi}_*(\eta)\dot{\eta}, \dot{\eta}]$$

$$= D^2E_c(\eta)[\dot{\eta}, \dot{\eta}] - D^2E_c(\eta)[D\tilde{\xi}_*(\eta)\dot{\eta}, D\tilde{\xi}_*(\eta)\dot{\eta}]$$  \hspace{1cm} (3.116)

From the definition of $E_c$ and the fact that the momentum is linear in $\tilde{\xi}$, we obtain

$$D^2E_c(\eta)[D\tilde{\xi}_*(\eta)\dot{\eta}, D\tilde{\xi}_*(\eta)\dot{\eta}] = \int_{\mathbb{R}} D\tilde{\xi}_*(\eta)\dot{\eta}A(\eta)D\tilde{\xi}_*(\eta)\dot{\eta} \, dx.$$  \hspace{1cm} (3.117)

Combining (3.117) and (3.113) leads us to the formula (3.114). \blacksquare

**Lemma 3.18** (Quadratic form). For all $(\eta, \tilde{\xi}_*(\eta)) \in \mathcal{O} \cap \mathbb{V}$ and $c \in \mathbb{R}$, there exists a self-adjoint operator $Q_c(\eta) \in \text{Lin}(X_1; X_1^*)$ such that

$$D^2V^\text{aug}_c(\eta)[\dot{\eta}, \dot{\zeta}] = \langle Q_c(\eta)\dot{\eta}, \dot{\zeta} \rangle_{X_1^* \times X_1},$$  \hspace{1cm} (3.118)
for all $\dot{\eta}, \dot{\zeta} \in \mathbb{V}_1$ and

$$Q_c(\eta) \dot{\eta} = -\left( \frac{\dot{x}}{(\eta_x)^3} \right)' - \left( g [\rho] + \sum_{\pm} \rho_{\pm} (\pm b_{1\pm}^\dagger (b_{2\pm}^\dagger)')' + \omega_{\pm} \eta (b_{2\pm}^\dagger)' - a_{1\mp}^\dagger (\eta, \mathbb{V}_\pm) (b_{2\mp}^\dagger)' \right) \dot{\eta}$$

$$- (\dot{[\rho_\xi \omega]} - \eta \dot{[\rho \omega^2]}) \dot{\eta}$$

$$- c \dot{[\rho \omega]} \dot{\eta} + \sum_{\pm} \rho_{\pm} b_{1\pm}^\dagger \left( G_{\pm}(\eta)^{-1} ((b_{1\pm}^\dagger)')' \right) + 2 \rho_{\pm} \dot{[\omega]} \dot{\eta} \mathbb{B}(\eta)^{-1} [\omega] \eta x$$

$$+ 2 \rho_{\pm} \dot{[\omega]} \eta \dot{\partial}_x \left( \mathbb{B}(\eta)^{-1} [\omega] (\dot{\eta}_x) \right)$$

$$- 4 \rho_{-\rho_{\pm}} \sum_{\pm} \left( a_{1\mp}^\dagger (\eta, \mathbb{V}_\pm) \mathbb{B}^{-1} [\omega] (\dot{\eta}_x) \right)$$

$$- 2 \rho_{+\rho_{\pm}} \sum_{\pm} \left( a_{1\mp}^\dagger (\eta, \mathbb{V}_\pm) \mathbb{B}^{-1} [\omega] (\dot{\eta}_x) \right).$$

(3.119)

**Proof.** First, it is clear that the Hamiltonian can alternatively be written in the following way

$$E(\eta, \xi, \dot{\xi}) = \frac{1}{2} \int_{\mathbb{R}} \dot{\xi} A(\eta) \dot{\xi} + \rho - \rho_{\pm} [\omega] \eta_\eta x \mathbb{B}^{-1} (\eta) [\omega] \eta x - 2 \rho_{\pm} \dot{\xi}_+ [\omega] \eta x + 2 \dot{\xi}_- \omega \eta x$$

$$- g [\rho] \eta^2 - \frac{\eta^3 \dot{[\rho \omega^2]}}{3} + 2 \sigma \sqrt{1 + \eta^2} - 1 \, dx.$$  

(3.120)

From (3.114), it is therefore useful to start doing an expansion on $D^2_\eta E_c(\eta, \xi, \dot{\xi})[\eta, \dot{\eta}]$. Using the expression of the Hamiltonian (3.120), we can see that

$$D^2_\eta E_c(\eta, \xi, \dot{\xi})[\eta, \dot{\eta}] = \frac{1}{2} \int_{\mathbb{R}} \dot{\xi} A(\eta) \dot{\xi} - \int_{\mathbb{R}} g [\rho] \eta^2 \, dx + \int_{\mathbb{R}} \left( \sigma \frac{\dot{x}}{(\eta_x)^3} \right) \, dx$$

$$- \int_{\mathbb{R}} c \dot{[\rho \omega]} \eta^2 \, dx - \int_{\mathbb{R}} [\omega \xi x \rho] \eta^2 \, dx - \int_{\mathbb{R}} \dot{[\rho \omega^2]} \eta \dot{\eta}^2 \, dx$$

$$- 2 \int_{\mathbb{R}} \rho_{-\rho_{\pm}} [\omega] \partial_x (\mathbb{B}^{-1}(\eta) [\omega] \eta x) \dot{\eta}^2 \, dx$$

$$- 2 \int_{\mathbb{R}} \rho_{+\rho_{\pm}} [\omega] \eta \dot{\partial}_x (\mathbb{B}^{-1}(\eta) [\omega] (\dot{\eta}_x)) \dot{\eta} \, dx$$

$$- 4 \int_{\mathbb{R}} \rho_{-\rho_{\pm}} [\omega] \eta x \dot{\partial}_x (\mathbb{B}^{-1}(\eta) \dot{\eta}, [\omega] \eta x) \dot{\eta} \, dx$$

$$+ \int_{\mathbb{R}} \rho_{+\rho_{\pm}} [\omega] \eta x \dot{\partial}_x (\mathbb{B}^{-1}(\eta) [\eta, \dot{\eta}], [\omega] \eta x) \, dx.$$  

(3.121)
It is important to note that when we let \( \omega_\pm = 0 \), we recover back the expression found in [CW22]. To arrive at the expression stated in (3.119), we first begin by looking at the term involving \( \tilde{\xi}_*(D^2A(\eta)[\dot{\eta}, \dot{\eta}]\tilde{\xi}_*) \) in (3.121). Since we are going to use some formula in [CW22], let us define the following variable

\[
\theta_\pm(u_*) = G_\pm(\eta)^{-1}A(\eta)\tilde{\xi} = \mp\xi_\pm \pm \rho_\pm B(\eta)^{-1}\lbrack \omega \rbrack \eta_x =: \mp\xi_\pm \pm \Upsilon_\mp.
\]

This implies

\[
a_1^\pm(\eta, \theta_\pm) = b_1^\pm + c + \omega_\pm \eta + a_1^\pm(\eta, \pm \Upsilon_\mp), \quad a_2^\pm(\eta, \theta_\pm) = \pm b_2^5.
\]

We further define the following expressions for later use:

\[
S_\pm(\eta)\zeta := G_\pm(\eta)^{-1}(b_1^\pm \zeta)', \quad T_\pm(\eta)\zeta := \pm b_2^\pm \zeta.
\]

Upon using the formula in Chen and Walsh paper, we can infer that

\[
\frac{1}{2} \int_{\mathbb{R}} \tilde{\xi}_*(D^2A(\eta)[\dot{\eta}, \dot{\eta}]\tilde{\xi}_*) \, dx = \sum_\pm \mp \rho_\pm \int_{\mathbb{R}} \left[(b_1^\pm)'b_2^\pm + \omega_\pm \eta' b_2^\pm + \omega_\pm \eta'b_2^\pm + (a_1^\pm(\eta, \pm \Upsilon_\mp))'b_2^\pm \right] \eta^2
\]

\[
+ T\dot{\eta}G_\pm(\eta)T\dot{\eta} \right] \, dx
\]

\[
+ \int_{\mathbb{R}} (-\dot{\eta}M(u_*)\dot{\eta} + \dot{\eta}N(u_*)\dot{\eta}) \, dx. \tag{3.122}
\]

To compute the terms on the second row in (3.122), we would need to define the following expressions:

\[
L_\pm(u_*)\dot{\eta} = T\dot{\eta} - S\dot{\eta} - G_\pm(\eta)^{-1}\partial_x(c\eta + \omega_\pm \eta \dot{\eta} + a_1^\pm(\eta, \pm \Upsilon_\mp)\dot{\eta})
\]

\[
L(\eta)\dot{\eta} = T\dot{\eta} - S\dot{\eta} - A(\eta)^{-1}\partial_x(c\eta + \omega_\pm \eta \dot{\eta} + a_1^\pm(\eta, \pm \Upsilon_\mp)\dot{\eta}). \tag{3.123}
\]

Furthermore,

\[
\int_{\mathbb{R}} \dot{\eta}M\dot{\eta} \, dx = \sum_\pm \rho_\pm \int_{\mathbb{R}} (b_1^\pm + c + \omega_\pm \eta + a_1^\pm(\eta, \pm \Upsilon_\mp))(L_\pm\dot{\eta})' \pm b_2^\pm G_\pm(\eta)L_\pm\dot{\eta} \, dx
\]

\[
= \sum_\pm \rho_\pm \int_{\mathbb{R}} L_\pm\dot{\eta}G_\pm(\eta)L_\pm\dot{\eta} \, dx \tag{3.124}
\]

146
Via the definition of $L_\pm$ in (3.123), we can infer

$$\int_R \dot{\eta} \mathcal{M} \dot{\eta} \, dx = \sum_{\pm} \rho_\pm \int_R (S_{\pm} \dot{\eta} \mathcal{G}_{\pm}(\eta) \mathcal{S}_{\pm} \dot{\eta} - 2S_{\pm} \dot{\eta} \mathcal{G}_{\pm}(\eta) \mathcal{T}_{\pm} \dot{\eta} + \mathcal{T}_{\pm} \dot{\eta} \mathcal{G}_{\pm}(\eta) \mathcal{T}_{\pm} \dot{\eta}) \, dx$$

$$+ \int_R (\partial_x (c \dot{\eta} + \omega \pm \eta \dot{\eta} + a_1^\pm (\eta, \pm \Upsilon \pm) \dot{\eta}) \mathcal{A}(\eta)^{-1} \partial_x (c \dot{\eta} + \omega \pm \eta \dot{\eta} + a_1^\pm (\eta, \pm \Upsilon \pm) \dot{\eta})) \, dx$$

$$+ \int_R 2\partial_x (c \dot{\eta} + \omega \pm \eta \dot{\eta} + a_1^\pm (\eta, \pm \Upsilon \pm) \dot{\eta})(S - T) \dot{\eta} \, dx.$$

(3.125)

For the importance of simplification later, we display the following formula

$$\int_R S_{\pm} \dot{\eta} \mathcal{G}_{\pm}(\eta) \mathcal{T}_{\pm} \dot{\eta} \, dx = \pm \int_R \mathcal{G}_{\pm}(\eta)^{-1} (b_1^\pm \dot{\eta}') \mathcal{G}_{\pm}(\eta)(b_2^\pm \dot{\eta}) \, dx$$

$$= \pm \int_R (b_1^\pm \dot{\eta}')(b_2^\pm \dot{\eta}) \, dx$$

(3.126)

$$= \pm \frac{1}{2} \int_R ((b_1^\pm)'(b_2^\pm) - (b_1^\pm')(b_2^\pm)) \, \dot{\eta}^2 \, dx.$$

At last, following the formula in Appendix A, we obtain

$$\int_R \dot{\eta} \mathcal{N} \dot{\eta} \, dx = \int_R (\mathcal{T} \dot{\eta} - \mathcal{S} \dot{\eta} - \mathcal{A}(\eta)^{-1} \partial_x (c \dot{\eta} + \omega \pm \eta \dot{\eta} + a_1^\pm (\eta, \pm \Upsilon \pm) \dot{\eta})) \mathcal{A}(\eta)$$

$$= \int_R (D \tilde{\xi}(\eta) \dot{\eta} \mathcal{A}(\eta) D \tilde{\xi}(\eta) \dot{\eta} + 2\partial_x (c \dot{\eta} + \omega \pm \eta \dot{\eta} + a_1^\pm (\eta, \pm \Upsilon \pm) \dot{\eta})(S - T) \dot{\eta}$$

$$+ (\partial_x (c \dot{\eta} + \omega \pm \eta \dot{\eta} + a_1^\pm (\eta, \Upsilon \pm) \dot{\eta}) \mathcal{A}(\eta)^{-1} \partial_x (c \dot{\eta} + \omega \pm \eta \dot{\eta} + a_1^\pm (\eta, \pm \Upsilon \pm) \dot{\eta})) \, dx.$$

(3.127)

Combining together all the computations above and the formula in (3.114), we get
\[ D^2 \mathcal{V}_{e}^{\text{ang}}(\eta)[\dot{\eta}, \dot{\eta}] \]

\[ = D^2_{\eta} E_{e}(\eta, \tilde{\xi}_x)[\dot{\eta}, \dot{\eta}] - \int_{\mathbb{R}} (S - T) \dot{\eta} A(\eta)(S - T) \dot{\eta} \, dx \]

\[ = \int_{\mathbb{R}} \left( \sigma_{\frac{\dot{\eta}_x}{\eta_x}}^2 - \left( g \rho + \sum_{\pm} \rho_{\pm} \left( \pm b_{\pm}^x (b_{\pm}^x)' + \omega_{\pm} \eta (b_{\pm}^x)' \mp a_{\pm}^x (\eta, \mp \gamma_{\pm} (b_{\pm}^x)') \right) \right) \eta^2 \]

\[ - \left( \| \rho \xi \omega \| - \eta \| \rho \omega \| - c \| \rho \omega \| \right) \eta^2 - \sum_{\pm} \rho_{\pm} \eta_{\pm} \dot{\eta}_{\pm} \eta_{\pm} \eta \right) \, dx \]

\[-2 \int_{\mathbb{R}} \rho_{- \rho_{+}} \left[ \omega \right] \partial_x \left( B^{-1}(\eta) \left[ \omega \right] \eta_{\eta} \right) \, \eta^2 \, dx \]

\[-2 \int_{\mathbb{R}} \rho_{- \rho_{+}} \left[ \omega \right] \eta \partial_x \left( B^{-1}(\eta) \left[ \omega \right] \eta_{\eta} \right) \, \eta \, dx \]

\[-4 \int_{\mathbb{R}} \rho_{- \rho_{+}} \left[ \omega \right] \eta \partial_x (DB^{-1}(\eta) \eta, \left[ \omega \right] \eta_{\eta}) \, \eta \, dx \]

\[ + \int_{\mathbb{R}} \rho_{- \rho_{+}} \left[ \omega \right] \eta \eta_{\eta} (DB^{-1}(\eta) \left[ \eta, \eta \right], \left[ \omega \right] \eta_{\eta}) \, dx. \]

(3.128)

To compute the last two integrals in (3.128), we need to derive the formula for

\[ \langle DB^{-1}(\eta) \eta, \left[ \omega \right] \eta_{\eta} \rangle \text{ and } \langle D^2 B^{-1}(\eta)[\dot{\eta}, \dot{\eta}], \left[ \omega \right] \eta_{\eta} \rangle. \]

We begin with the expansion of the formula below

\[ \langle DB(\eta) \dot{\eta}, \tilde{\xi} \rangle = \rho_{+} \langle DG_{-}(\eta) \dot{\eta}, \tilde{\xi} \rangle + \rho_{-} \langle DG_{+}(\eta) \dot{\eta}, \tilde{\xi} \rangle \]

\[ = \rho_{+} \left( -\partial_x (a_{1}^x (\eta, \tilde{\xi}) \eta) + G_{-}(\eta)a_{2}^x (\eta, \tilde{\xi}) \dot{\eta} \right) \]

\[ + \rho_{-} \left( -\partial_x (a_{1}^x (\eta, \tilde{\xi}) \eta) + G_{+}(\eta)a_{2}^x (\eta, \tilde{\xi}) \dot{\eta} \right). \]

(3.129)

Using the formula

\[ \langle DB^{-1}(\eta) \dot{\eta}, \tilde{\xi} \rangle = -B^{-1} \langle DB(\eta) \dot{\eta}, B^{-1} \tilde{\xi} \rangle \]

in combination with (3.129), we derive the representation formula for the Fréchet derivative of \( B^{-1}(\eta) \) (for any given \( \zeta \)),

148
\[
\int_{\mathbb{R}} \zeta \langle DB^{-1}(\eta) \dot{\eta}, \tilde{\xi} \rangle \, dx = - \int_{\mathbb{R}} B^{-1} \left( \rho_+ \left( - \partial_x (a_1^-(\eta, B^{-1}\tilde{\xi}) \dot{\eta} \right) \right) \, dx \\
- \int_{\mathbb{R}} B^{-1} \left( \rho_- \left( - \partial_x (a_2^+ (\eta, B^{-1}\tilde{\xi}) \dot{\eta} \right) \right) \, dx \\
= - \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \left( a_{1}^{\pm}(\eta, B^{-1}\tilde{\xi}) (B^{-1}\zeta)_x \right) \dot{\eta} \\
+ \left( a_2^-(\eta, B^{-1}\tilde{\xi}) B^{-1}\mathcal{G}_\pm(\eta) \dot{\xi} \right) \dot{\eta} \, dx.
\]

(3.130)

From (3.130), we arrive the expression in the second row from the bottom of equation (3.119).

Finally, it remains to show that the last integral in (3.128) should give rise to the last expression in (3.119). First, recall that \( B(\eta) := \sum_{\pm} \rho_{\pm} \mathcal{G}_\pm(\eta) \). Exploiting the second derivative formula for \( \mathcal{G}_\pm(\eta) \) we obtain

\[
\int_{\mathbb{R}} \tilde{\xi} \langle D^2 B(\eta)[\dot{\eta}, \dot{\eta}], \tilde{\xi} \rangle \, dx = \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \tilde{\xi} \langle D^2 \mathcal{G}_\pm(\eta)[\dot{\eta}, \dot{\eta}], \tilde{\xi} \rangle \, dx \\
= \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} a_4^{\pm}(\eta, \tilde{\xi}) \dot{\eta}^2 + 2a_2^{\pm}(\eta, \tilde{\xi}) \dot{\eta} \mathcal{G}_\pm(\eta) (a_1^{\pm}(\eta, \tilde{\xi}) \dot{\eta}) \dot{\xi} \, dx.
\]

(3.131)

Additionally, we have the following identity

\[
D^2 B(\eta)[\dot{\eta}, \dot{\eta}] = -B(\eta) D^2 B^{-1}(\eta)[\dot{\eta}, \dot{\eta}] B(\eta) + 2D B(\eta)[\dot{\eta}] B^{-1}(\eta) D B(\eta)[\dot{\eta}].
\]

After rearranging terms and applying (3.131), we obtain

\[
\int_{\mathbb{R}} \rho_{-}\rho_{+} \left[ \omega \right] \eta \eta_x \langle D^2 B^{-1}(\eta)[\dot{\eta}, \dot{\eta}], \left[ \omega \right] \eta \eta_x \rangle \, dx \\
= \int_{\mathbb{R}} 2\rho_{+}\rho_{-} \left( \sum_{\pm} a_1^{\pm}(\eta, \gamma_{\pm}) \dot{\eta} \right)' B^{-1}(\eta) \left( \sum_{\pm} a_1^{\pm}(\eta, \gamma_{\pm}) \dot{\eta} \right)' \, dx.
\]

(3.132)

Lemma 3.19 (Continuous spectrum). Let \( u = (\eta, \tilde{\xi}) \in \mathcal{O} \cap \mathbb{V} \) be given. Then the operator \( \mathcal{Q}_c(\eta) \) is a self-adjoint operator on \( L^2(\mathbb{R}) \) with domain \( H^2(\mathbb{R}) \). Further, the
spectrum of this operator is equal to the one of $Q_c(0)$, that is $[\tau_*, +\infty]$, where

$$
\tau_* = \begin{cases} 
-g[\rho] \left( 1 - \frac{\alpha_0}{\alpha} \right) & \text{for } \beta \geq \beta_0, \\
-g[\rho] \left( 1 - \frac{1}{\alpha} \max_{\xi \in \mathbb{R}} \left( \sum_\pm \frac{\rho_\pm}{\rho_-} d_+ \xi \coth(d_+ \xi) - \beta d_+ \xi^2 + \left( \frac{\omega_+ d_+ \rho}{c} - \frac{\omega_- d_+}{c} \right) \right) \right) & \text{for } \beta < \beta_0.
\end{cases}
$$

(3.133)

Proof. The fact that $Q_c(\eta)$ is self-adjoint on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$ follows directly from the regularity of $\eta$. Also, since $\eta(x) \to 0$ as $x \to \infty$, then the continuous spectrum of $Q_c(\eta)$ coincides with that of $Q_c(0)$. Since $Q_c(0)$ is translation invariant, hence the whole spectrum is continuous. The symbol of $Q_c(0)$ is

$$
q_c(\xi) = -g[\rho] \left[ 1 - \frac{1}{\alpha} \left( \sum_\pm \frac{\rho_\pm}{\rho_-} d_+ \xi \coth(d_+ \xi) - \beta d_+ \xi^2 + \left( \frac{\omega_+ d_+ \rho}{c} - \frac{\omega_- d_+}{c} \right) \right) \right].
$$

(3.134)

The continuous spectrum results from looking at the range of the mapping $\xi \mapsto q_c(\xi)$, and $\tau_* = \min\{q_c(\xi) : \xi \in \mathbb{R}\}$. For $\beta \geq \beta_0$, the minimum is attained at $\xi = 0$. But for $\beta < \beta_0$, the minimum is attained at some value $\xi \neq 0$. 

### 3.5.1 Rescaled operator

Now, we will make a use of a long-wave rescaling to obtain the information of the leading-order form of the operator $Q_c(\eta)$. Assume $\beta > \beta_0$ and $\alpha = \alpha_0 + \epsilon^2$, consider the following rescaling operator:

$$
S_\epsilon := f \left( \frac{\epsilon}{d_+} \right).
$$

It is clear to see that $S_\epsilon$ is an isomorphism on $H^k(\mathbb{R})$. Note also that

$$
\partial_x S_\epsilon = \frac{\epsilon}{d_+} S_\epsilon \partial_x, \quad \partial_x S_\epsilon^{-1} = \frac{d_+}{\epsilon} S_\epsilon^{-1} \partial_x.
$$

This shows that $\partial_x S_\epsilon$ and $\partial_x S_\epsilon^{-1}$ are uniformly bounded in $\text{Lin}(H^{k+1}, H^k)$.
Lemma 3.20 (Expansion of $\tilde{Q}_ε(\eta_ε)$). The operator $\tilde{Q}_ε(\eta_ε)$ admits the following expansion

$$\tilde{Q}_ε(\eta_ε) = \tilde{Q}_ε(0) + \tilde{R}_ε,$$

where

$$\tilde{R}_ε = -3 \left( \frac{d}{d^2} - \frac{1}{d^2} + \frac{\omega_+ d_+ \rho}{c} + \frac{\omega_- d_- \rho}{cd} + \frac{\omega_+^2 d_+^2 \rho}{3c^2} - \frac{\omega_-^2 d_-^2 \rho}{3c^2} \right) \bar{\eta} + O(ε). \quad (3.135)$$

Proof. Let $Q_ε$ be the operator obtained from evaluating operator $Q_ε$ at $\eta_ε$

$$Q_ε(\eta_ε) := -\partial_x \left( \sigma \frac{\partial_x}{\langle \eta'_ε \rangle^3} \right) - \left( g [\rho] + \sum_{\pm} (\pm \rho \pm b_{1ε}^±(b_{2ε}^±)'') + [\rho \xi' \omega] - \eta_ε [\rho \omega^2] + c [\rho \omega] \right)$$

$$+ \sum_{\pm} ρ_{±} b_{1ε}^± \partial_x G_{±}(\eta_ε)^{-1} \partial_x b_{1ε}^±$$

$$- \sum_{\pm} ρ_{±} (\mp \omega_{±} η(b_{2ε}^±)' - a_{1ε}^±(\eta, \Upsilon_±)(b_{2ε}^±)')$$

$$+ 2ρ_+ρ_- [\omega] \partial_x B(\eta_ε)^{-1} [\omega] \eta_ε' \eta_ε' + 2ρ_+ρ_- [\omega] \eta_ε \partial_x (B(\eta_ε)^{-1} [\omega] (\eta_ε x))$$

$$− 4ρ_−ρ_+ \sum_{\pm} (a_{1ε}^±(\eta_ε, \Upsilon_±)B^{-1} [\omega] (\eta_ε x))$$

$$− 2ρ_+ρ_- \sum_{\pm} a_{1ε}^±(\eta_ε, \Upsilon_±) \partial_x \left( B^{-1}(\eta_ε) \left( \sum_{\pm} a_{1ε}^±(\eta_ε, \Upsilon_±) \bar{η} \right) \right). \quad (3.136)$$

Following the same rescaling argument [CW22], the rescaled surface tension gives us

$$\frac{-1}{ε^2} \frac{d_+}{c^2 ρ_-} S_{ε}^{-1} \partial_x \left( \sigma \frac{\partial_x}{\langle \eta'_ε \rangle^3} \right) S_ε = -\partial_x \left( \frac{\beta}{(ε^3(\bar{η}_ε' + \bar{r}_ε'))^3} \partial_x \right). \quad (3.137)$$

Further, let us define the non-dimensionalized and rescaled relative velocity field

$$b_{1ε}^± := cS_ε b_{1ε}^±, \quad b_{2ε}^± := cS_ε b_{2ε}^±. \quad (3.138)$$
Since $b_{2\epsilon}^\pm = \eta'_\epsilon(b_{1\epsilon}^\pm)$, then we have $\tilde{b}_2^\pm = c^3 \tilde{\eta}' b_1^\pm$. Using this, we obtain the following rescaled expression of the second and the third terms:

$$
\frac{-1}{\epsilon^2} \frac{d_+}{e^2} S_\epsilon^{-1} \left( g \left[ \rho \right] + c \left[ \rho \omega \right] + \sum \left( \pm \rho_\epsilon b_{1\epsilon}^\pm (b_{2\epsilon}^\pm)' \right) \right) S_\epsilon = \frac{\alpha}{\epsilon^2} - \frac{1}{\epsilon^2} \left( \frac{\omega_+ d_+ g}{c} - \frac{\omega_- d_+}{c} \right) - \epsilon^2 \sum \left( \pm \frac{\rho_\epsilon}{\rho_\epsilon - 1} (\tilde{\eta}' b_1^\pm)' \right).
$$

(3.139)

For later use in dealing with the non-local terms in $Q_\epsilon(\eta_\epsilon)$, we define the following two operators:

$$
\tilde{M}_\epsilon^\pm(\eta_\epsilon) := \frac{d_+}{\epsilon^2} S_\epsilon^{-1} \partial_x G_\pm(\eta_\epsilon)^{-1} \partial_x S_\epsilon
$$

(3.140)

and

$$
\tilde{Z}_\epsilon(\eta_\epsilon) := \frac{d_+}{\epsilon^2} S_\epsilon^{-1} \partial_x B(\eta_\epsilon)^{-1} \partial_x S_\epsilon
$$

(3.141)

For any $f \in H^{k+2}$, we have

$$
\mathcal{F}(\tilde{M}_\epsilon^\pm(0)f)(\xi) = \frac{d_+}{\epsilon^2} \frac{\epsilon}{d_+} \mathcal{F}(\partial_x G_\pm(\eta_\epsilon)^{-1} \partial_x S_\epsilon f)(\frac{\epsilon}{d_+} \xi) = \frac{d_+}{\epsilon^2} \epsilon \frac{\xi}{d_+} \mathcal{F}(\hat{f})(\xi),
$$

(3.142)

where $m_\pm := -\xi \coth(d_\pm \xi)$ is the symbol for $\partial_x G_\pm(0)^{-1} \partial_x$. Thus, $\tilde{M}_\epsilon^\pm(0)$ is a Fourier multiplier with a symbol given by

$$
\tilde{m}_\epsilon^\pm(\xi) := \frac{-1}{\epsilon^2} \epsilon \frac{\xi}{\tanh(d_\pm \xi/d_+)}. \tag{3.143}
$$

Additionally, for any $f \in H^{k+2}$, we have

$$
\mathcal{F}(\tilde{Z}_\epsilon(0)f)(\xi) = \frac{d_+}{\epsilon^2} \frac{\epsilon}{d_+} \mathcal{F}(\partial_x B(\eta_\epsilon)^{-1} \partial_x S_\epsilon f)(\frac{\epsilon}{d_+} \xi) = \frac{d_+}{\epsilon^2} \epsilon \frac{\xi}{d_+} \mathcal{F}(\hat{f})(\xi), \tag{3.144}
$$

where $b_\pm := -\xi/(\rho_+ \tanh(d_\pm \xi) + \rho_- \tanh(d_+ \xi))$ is the symbol for $\partial_x B(0)^{-1} \partial_x$. Thus, $\tilde{Z}_\epsilon(0)$ is a Fourier multiplier with a symbol given by
\[ \tilde{b}_\epsilon^\pm (\xi) := -\frac{1}{\epsilon^2} \frac{\epsilon \xi}{(\rho_+ \tanh(\epsilon \xi) + \rho_- \tanh(\epsilon \xi))}. \] (3.145)

As a result we get

\[ \| \epsilon^2 \tilde{M}_\epsilon^\pm (0) + \frac{d_+}{d_\pm} \|_{\text{Lin}(H^{k+2}, H^k)} \leq \| \frac{1}{\xi^2} \left( \epsilon^2 \tilde{m}_\epsilon^\pm + \frac{d_+}{d_\pm} \right) \| \lesssim \epsilon^2, \] (3.146)

and

\[ \| \epsilon^2 \tilde{Z}_\epsilon (0) + \frac{d_+}{(\rho_- d_+ + \rho_+ d_-)} \|_{\text{Lin}(H^{k+2}, H^k)} \leq \| \frac{1}{\xi^2} \left( \epsilon^2 \tilde{b}_\epsilon^\pm + \frac{d_+}{(\rho_- d_+ + \rho_+ d_-)} \right) \| \lesssim \epsilon^2. \] (3.147)

In particular, we obtain

\[ \tilde{Q}_\epsilon (0) = \frac{1}{\epsilon^2} \left( -\epsilon^2 \beta \partial_x^2 + \alpha - \left( \frac{\omega_+ d_+ g}{c} - \frac{\omega_- d_-}{c} \right) + \sum_\pm \frac{\rho_\pm}{\rho_-} \epsilon^2 \tilde{M}_\epsilon^\pm (0) \right). \] (3.148)

We are now ready to carefully analyze the remainder operator:

\[ \tilde{R}_\epsilon := \tilde{Q}_\epsilon (\eta_\epsilon) - \tilde{Q}_\epsilon (0) \]

\[ = -\beta \partial_x \left( \frac{\beta}{\epsilon^3 (\not{\eta} + \not{\eta}')}^3 - 1 \right) \partial_x - \epsilon^2 \sum_\pm \frac{\rho_\pm}{\rho_-} \tilde{b}_1^\pm (\tilde{\eta} \tilde{b}_1^\pm)' \]
\[ + \sum_\pm \frac{\rho_\pm}{\rho_-} \left( \tilde{b}_1^\pm \tilde{M}_\epsilon^\pm (\eta_\epsilon) \tilde{b}_1^\pm - \tilde{M}_\epsilon^\pm (0) \right) - \frac{d_+}{\epsilon^2 c^2 \rho_-} [\rho \omega, \omega] + \frac{[\rho \omega_\xi]}{c^2 \rho_-} \tilde{\eta}. \] (3.149)

It is straightforward to see that

\[ -\beta \partial_x \left( \frac{\beta}{\epsilon^3 (\not{\eta} + \not{\eta}')}^3 - 1 \right) \partial_x = O(\epsilon^9) \quad \text{in } \text{Lin} (H^{k+2}, H^k). \] (3.150)

In view of (3.111) and (3.138), we know that

\[ \tilde{b}_1^\pm = \frac{1}{c} S^{-1}_\epsilon \left( \partial_x \phi_{\epsilon \xi} |_{\mathcal{F} - c - \omega_{\pm} \eta_\epsilon} \right). \] (3.151)
Therefore, the second term in the remainder operator (3.149)

\[ \xi'_{\epsilon \pm} = \pm \partial_{x} G_{\pm}(\eta_{c})^{-1} (c n'_{\epsilon} + \omega_{\pm} \eta \eta'_{\epsilon}) \]

\[ = \pm \partial_{x} \left[ G_{\pm}(0)^{-1} (c n'_{\epsilon} + \omega_{\pm} \eta \eta'_{\epsilon}) + \langle D G_{\pm}(0)^{-1} \eta_{c}, (c n'_{\epsilon} + \omega_{\pm} \eta \eta'_{\epsilon}) \rangle \right] + O(\epsilon^{0}) \]

\( (\partial_{x} \phi_{\epsilon \pm})|_{y} = \frac{1}{1 + (\eta')^{2}} (\xi'_{\pm} \pm \eta'_{c} G_{\pm}(\eta_{c}) \xi_{\pm}) \)

\[ = \pm \partial_{x} \left[ G_{\pm}(0)^{-1} (c n'_{\epsilon} + \omega_{\pm} \eta \eta'_{\epsilon}) + \langle D G_{\pm}(0)^{-1} \eta_{c}, (c n'_{\epsilon} + \omega_{\pm} \eta \eta'_{\epsilon}) \rangle \right] + O(\epsilon^{0}) \]

Combining the formula

\[ \langle D G_{\pm}(0)^{-1} \eta_{c}, f \rangle = -G_{\pm}(0)^{-1} \langle D G_{\pm}(0) \eta_{c}, G_{\pm}(0)^{-1} f \rangle \]

with the first derivative formula for \( G_{\pm}(\eta) \), we can infer that

\[ \langle D G_{\pm}(0) \eta_{c}, G_{\pm}(0)^{-1} \partial_{x} S f \rangle = \pm \partial_{x} S \epsilon^{4} \left( \tilde{M}_{\epsilon}(0) f \right) \eta \pm \epsilon^{3} G_{\pm}(0) S_{c}(\tilde{\eta} \partial_{x} f) \]

\[ (3.153) \]

Patterning the computation done in [CW22], therefore we can say that

\[ \tilde{b}_{1}^{\pm} = -1 \mp \epsilon^{2} \frac{d}{d_{\pm}} \tilde{\eta} - \epsilon^{2} \frac{\omega_{\pm} d_{+}}{c} \tilde{\eta} - \epsilon^{4} \frac{d^{2}}{d_{\pm}^{2}} \tilde{\eta}^{2} + O(\epsilon^{6}). \]

\[ (3.154) \]

Therefore, the second term in the remainder operator (3.149)

\[ \epsilon^{2} \sum_{\pm} \pm \rho_{\pm} \tilde{b}_{1}^{\pm} (\eta' \tilde{b}_{1}^{\pm}) = \epsilon^{2} (1 - \rho) \tilde{\eta}'' + \epsilon^{4} \left( \frac{1}{d} - \rho + \frac{\omega_{\pm} d_{+}}{c} - \frac{\omega_{\pm} d_{+}}{c} \right) \left[ 2 \tilde{\eta}'' + (\tilde{\eta}')^{2} \right] \]

\[ + O(\epsilon^{6}). \]

\[ (3.155) \]

Utilizing the expansion that we obtain in (3.154) gives us

\[ \tilde{b}_{1}^{\pm} \tilde{M}_{\epsilon}(\eta_{c}) \tilde{b}_{1}^{\pm} = \tilde{M}_{\epsilon}(\eta_{c}) \pm \epsilon^{2} \frac{d}{d_{\pm}} \left( \tilde{\eta} \tilde{M}_{\epsilon}(\eta_{c}) + \tilde{M}_{\epsilon}(\eta_{c}) \tilde{\eta} \right) \]

\[ + \epsilon^{2} \frac{\omega_{\pm} d_{+}}{c} \left( \tilde{\eta} \tilde{M}_{\epsilon}(\eta_{c}) + \tilde{M}_{\epsilon}(\eta_{c}) \tilde{\eta} \right) \]

\[ + \epsilon^{4} \frac{d^{2}}{d_{\pm}^{2}} \left( \tilde{\eta} \tilde{M}_{\epsilon}(\eta_{c}) \tilde{\eta} + \tilde{\eta}^{2} \tilde{M}_{\epsilon}(\eta_{c}) + \tilde{M}_{\epsilon}(\eta_{c}) \tilde{\eta}^{2} \right) \]

\[ + \epsilon^{4} \frac{\omega_{\pm} d_{+}}{c} \left( \pm 2 \frac{d}{d_{\pm}} + \frac{\omega_{\pm} d_{+}}{c} \right) \tilde{\eta} \tilde{M}_{\epsilon}(\eta_{c}) \tilde{\eta} + O(\epsilon^{6}) \text{ in Lin}(H^{k+2}, H^{k}). \]

\[ (3.156) \]
Furthermore, for $f \in H^{k+2}$ with $\|f\|_{H^{k+2}} = 1$, we have

$$
\left( \tilde{\mathcal{M}}^\pm_\epsilon (\eta_c) - \tilde{\mathcal{M}}^\pm_\epsilon (0) \right) f = \frac{d_+}{\epsilon^2} S^{-1}_\epsilon \partial_x (G^\pm_\epsilon (0^{-1}) \partial_x S_\epsilon f) \\
= \frac{d_+}{\epsilon^2} S^{-1}_\epsilon \partial_x (DG^\pm_\epsilon (0)^{-1} \eta_c, \partial_x S_\epsilon f) \\
+ \frac{d_+}{2\epsilon^2} S^{-1}_\epsilon \partial_x (D^2 G^\pm_\epsilon (0)^{-1} [\eta_c, \partial_x S_\epsilon f]) + O(\epsilon^4)
$$

(3.157)

in $H^k$. In order to understand the third term in (3.135) better, further expansion needs to be done to the equation above. In [CW22] such computation has been done carefully. From that we can infer

$$
\tilde{\mathcal{M}}^\pm_\epsilon (\eta_c) - \tilde{\mathcal{M}}^\pm_\epsilon (0) f = \mp \frac{d_+}{2} \tilde{\eta} f + \epsilon^2 \partial_x (\tilde{\eta} \partial_x f) - \epsilon^2 \frac{d_+^3}{d_+^2} \tilde{\eta}^2 f + O(\epsilon^3),
$$

(3.158)

in $H^k$. Using this, the expression in (3.156) becomes

$$
\tilde{b}_1^\pm \tilde{\mathcal{M}}^\pm_\epsilon (\eta_c) \tilde{b}_1^\pm f = \tilde{\mathcal{M}}^\pm_\epsilon (0) f \mp \epsilon^2 \frac{d_+}{d_+^2} \tilde{\eta} \tilde{b}_1^\pm f \\
+ \epsilon^2 \frac{\omega_+ d_+}{c} \left( \tilde{\eta} \tilde{\mathcal{M}}^\pm_\epsilon (0) f + \tilde{\mathcal{M}}^\pm_\epsilon (0) \tilde{\eta} f \right) \mp \frac{d_+^2}{d_+^2} \tilde{\eta} f + O(\epsilon^2).
$$

(3.159)

Finally, replacing $\tilde{\mathcal{M}}^\pm_\epsilon (0)$ with $-d_+/(d_+\epsilon^2)$ as in (3.146), we obtain

$$
\tilde{b}_1^\pm \tilde{\mathcal{M}}^\pm_\epsilon (\eta_c) \tilde{b}_1^\pm f = \tilde{\mathcal{M}}^\pm_\epsilon (0) f \mp 3 \frac{d_+^2}{d_+^2} \tilde{\eta} f - 2 \frac{\omega_+ d_+^2}{d_+ c} \tilde{\eta} f + O(\epsilon^2)
$$

(3.160)

Hence, the third term in (3.149) becomes

$$
\sum_{\pm} \frac{\rho_\pm}{\rho_-} \left( \tilde{b}_1^\pm \tilde{\mathcal{M}}^\pm_\epsilon (\eta_c) \tilde{b}_1^\pm - \tilde{\mathcal{M}}^\pm_\epsilon (0) \right) f
$$

$$
= 3 \sum_{\pm} \frac{\rho_\pm d_+^2}{\rho_-} \tilde{\eta} f - 2 \sum_{\pm} \frac{\rho_\pm \omega_+ d_+^2}{\rho_- d_+ c} \tilde{\eta} f + O(\epsilon^2)
$$

(3.161)

$$
= -3 \left( q - \frac{1}{d_+^2} \right) \tilde{\eta} f - 2 \left( \frac{\omega_+ d_+^2}{c} + \frac{\omega_- d_+}{cd} \right) \tilde{\eta} f + O(\epsilon^2).
$$

The next term to analyze is $[\rho \xi'_\epsilon \omega]$. For that, we are going to exploit the expression in (3.152). Via the same type expansion procedure, one can infer

$$
\xi'_\epsilon = \pm \frac{d_+^2}{c \rho_- d_+} + O(\epsilon^2)
$$

in $H^k$. 155
Hence,

\[- \rho \xi' \omega = - \left( \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_-}{cd} \right). \tag{3.162} \]

Another term in $Q_{\epsilon}(\eta_{\epsilon})$ that we still have yet to handle is $-\eta_{\epsilon} \rho \omega^2$. It is straightforward to see that

\[- \frac{1}{\epsilon^2 c^2 \rho} S_{\epsilon}^{-1} \eta_{\epsilon} \rho \omega^2 S_{\epsilon} = - \frac{1}{\epsilon^2 c^2 \rho} \epsilon^2 d_+ \rho \omega^2 + O(\epsilon) = - \frac{\omega^2_+ \rho d_+^2}{c^2} + d_+ \rho \omega + O(\epsilon). \tag{3.163} \]

Finally, it remains to deal with all the expressions of the last four rows in (3.136). Again, via the same analysis as before and using the fact that $\tilde{\eta}_{\epsilon}(0) = -d_+/(\epsilon^2 d_\pm)$, one can show that those terms are lower order (sufficiently small).

Combining (3.161), (3.162), and (3.163) we arrive at (3.135). The proof is then complete.

Lemma 3.21 (Limiting rescaled operator). Consider the rescaled operator $\mathcal{Q}_{\epsilon}(\eta_{\epsilon})$.

Assume that $\beta > \beta_0$ and $\alpha = \alpha_0 + \epsilon^2$. Then for any $k > 1/2$ and $\zeta \in H^{k+2}$, we have

\[ \| \mathcal{Q}_{\epsilon}(\eta_{\epsilon}) \zeta - \mathcal{Q}_0 \zeta \|_{H^k} \to 0 \quad \text{as} \quad \epsilon \searrow 0, \]

where the operator

\[ \mathcal{Q}_0 = -(\beta - \beta_0) \partial_x^2 + 1 - 3 \left( \varrho - \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_-}{cd} + \frac{\omega^2_+ d_+^2 \varrho}{3c^2} - \frac{\omega^2_- d_-^2}{3c^2} \right) \tilde{\eta} \tag{3.164} \]

Proof. Let $k > 1/2$. Recall from the expansion of $Q_{\epsilon}$ in Lemma 3.20, we have

$\mathcal{Q}_{\epsilon}(\eta_{\epsilon}) = \mathcal{Q}_{\epsilon}(0) + \mathcal{R}_{\epsilon}$. It is also important to note that $\mathcal{R}_{\epsilon}$ has a uniform limit at linear map from $H^{k+1}$ to $H^k$ as $\epsilon \searrow 0$. Moreover, the operator $\mathcal{Q}_{\epsilon}(0)$ is a Fourier multiplier in $H^{k+2}$ (as $\mathcal{M}_\epsilon(0)$ is). Precisely,
Further the symbol \( \tilde{q} \) can be re-written in the following way

\[
\tilde{q}_\epsilon(\xi) = \epsilon^2(\beta - \beta_0)\xi^2 + \frac{\alpha - \alpha_0}{\epsilon^2} + \frac{1}{\epsilon^2} \left( \beta_0(\epsilon\xi)^2 + \alpha_0 - \frac{[\rho\omega d_+]}{\rho - c} - \sum \frac{\rho_+ \rho_-}{\rho_+ - \rho_-} \epsilon\xi \coth \left( \frac{d_+ \epsilon\xi}{d_+ \epsilon} \right) \right)
\]

as \( \epsilon \xi \to 0 \). Since \( \alpha = \alpha_0 + \epsilon^2 \), hence for each fixed \( \xi \in \mathbb{R} \), we obtain

\[
\tilde{q}_\epsilon(\xi) \to (\beta - \beta_0)\xi^2 + 1 \text{ as } \epsilon \searrow 0.
\]

Combining this result with the expression for \( \tilde{R}_\epsilon \) in (3.135), we obtain the formula of \( \tilde{Q}_0 \).

### 3.5.2 Spectrum of the linearized augmented potential

Having established the limiting behavior above, we will now analyze the spectrum of the operator \( Q_\epsilon(\eta_\epsilon) \). Recall that, the operator \( Q_\epsilon(\eta_\epsilon) \) only converges point-wise to the operator \( Q_0(0) \) whose essential spectrum is \([0, \infty)\). This is certainly creates a challenge in deducing the spectrum of \( Q_\epsilon(\eta_\epsilon) \). Thanks to the rescaled operator \( \tilde{Q}_\epsilon(\eta_\epsilon) \) introduced earlier. It is known that such rescaled operator converges point-wise to \( \tilde{Q}_0 \), which has a gap between 0 and the positive essential spectrum.

**Lemma 3.22** (Spectrum of the Limiting Operator \( \tilde{Q}_0 \)). *Let the assumptions in Lemma 3.21 hold. The limiting rescaled operator \( \tilde{Q}_0 \) satisfies*

\[
\text{ess spec } \tilde{Q}_0 = [1, \infty), \quad \text{spec } \tilde{Q}_0 = \{-\tau^2, 0\} \cup \tilde{\Lambda},
\]

(3.167)
where the first two eigenvalues \(-\tilde{\tau}^2 < 0\) and 0 are both simple eigenvalues with the corresponding eigenfunctions \(\tilde{g}_1\) and \(\tilde{g}_2 = \tilde{\eta}'\), respectively; and there exists \(\tau_* > 0\) such that \(\tilde{\Lambda} \subset\).

Proof. The spectra condition above is a classic result and can be found, for instance, in the surveys of Pava \cite{Pav09}. It is important to mention that the proof of that is not trivial by any means. The fact that \(-\tilde{\tau}^2\) and 0 are simple follows from the result which says that: the Wronskian of any two solutions in \(L^2\) of the eigenvalue problem \(\tilde{Q}_0 f = \tilde{\tau}\) must be 0.

**Theorem 3.23.** Suppose that the assumptions of Lemma 3.21 hold. For each \(a \in (0, \tau_*)\), there exists some \(\epsilon_0 > 0\) such that for any \(\epsilon \in (0, \epsilon_0)\) the operator \(Q_\epsilon(\eta_\epsilon)\) satisfies

\[
\text{ess spec } Q_\epsilon(\eta_\epsilon) \subset [\epsilon^2 c^2 \rho_- / d_+, \infty), \quad \text{spec } Q_\epsilon(\eta_\epsilon) = \{-\tau^2, 0\} \cup \Lambda,
\]

where \(\Lambda \subset [ac^2 c^2 \rho_- / d_+, \infty), \) and

\[
\tau^2 = \frac{\epsilon^2 c^2 \rho_-}{d_+} \tilde{\tau}^2 + o(\epsilon^2) \quad \text{as } \epsilon \downarrow 0.
\]

The first two eigenvalues \(\tau_1 := -\tilde{\tau}^2 < 0\) and \(\tau_2 := 0\) are both simple with the corresponding eigenfunctions given by \(g_i := Sc_i \tilde{g}_i + o(1)\) in \(H^k\) as \(\epsilon \downarrow 0\).

Proof. The proof of this theorem can be done in a similar manner as the one in \cite[Theorem 3.2]{CW22} which is mainly inspired by the proof of \cite[Theorem 4.3]{Mic02}. Therefore, we opt to avoid rewriting it here.

**Lemma 3.24** (Extension of \(D^2E_c\)). Let \(\{U_c\}\) be a family of bound states, then the operator \(D^2E_c(U_c)\) can be extended uniquely to a bounded linear operator \(H_c : X \to X^*\)
such that

\[ D^2 E_c(U_c)[\dot{u}, \dot{v}] = \langle H_c \dot{u}, \dot{v} \rangle, \quad \text{for all } \dot{u}, \dot{v} \in V, \]  \hspace{1cm} (3.168)

and \( I^{-1}H_c \) is self-adjoint on \( X \).

\textbf{Proof.} Let \( U_c = (\eta_c, \xi_c) \) be a bound state and \( \dot{u} = (\eta, \dot{\xi}) \in V \) be given. From Lemma 3.17 and Lemma 3.18 we know that

\[ D^2 E_c(U_c)[\dot{u}, \dot{u}] = D^2 V^{\text{aug}}(\eta_c)[\dot{\eta}, \dot{\eta}] + \int_{\mathbb{R}} (S_c - T_c) \dot{\eta} A(\eta_c) (S_c - T_c) \dot{\eta} \, dx 
+ 2D \xi D \eta E_c(U_c)[\dot{\eta}, \dot{\xi}] + D^2 E_c(U_c)[\dot{\xi}, \dot{\xi}] 
= \langle Q_C(\eta_c) \dot{\eta}, \dot{\eta} \rangle + \int_{\mathbb{R}} (S_c - T_c) \dot{\eta} A(\eta_c) (S_c - T_c) \dot{\eta} \, dx 
+ 2 \int_{\mathbb{R}} \dot{\xi} G_-(\eta_c) B^{-1}(\eta_c) \omega (\eta_c \dot{\eta}) \, dx 
+ 2 \int_{\mathbb{R}} \dot{\xi} \langle D G_-(\eta_c) \dot{\eta}, B^{-1}(\eta_c) \omega \eta_c \dot{\eta} \rangle \, dx 
+ 2 \int_{\mathbb{R}} G_-(\eta_c) \dot{\xi} \langle D B_-(\eta_c) \dot{\eta}, \omega \eta_c \dot{\eta} \rangle \, dx 
+ 2 \int_{\mathbb{R}} \dot{\xi} \omega_-(\eta \eta_c) \, dx + 2c \int_{\mathbb{R}} \dot{\eta} \dot{\xi} \, dx + \int_{\mathbb{R}} \dot{\xi} A(\eta_c) \dot{\xi} \, dx. \]

Expanding some of the terms using the first derivative formula for \( G_-(\eta) \) and \( B^{-1}(\eta) \) yields

\[ D^2 E_c(U_c)[\dot{u}, \dot{u}] = \langle Q_C(\eta_c) \dot{\eta}, \dot{\eta} \rangle + \int_{\mathbb{R}} (S_c - T_c) \dot{\eta} A(\eta_c) (S_c - T_c) \dot{\eta} \, dx + 2c \int_{\mathbb{R}} \dot{\eta} \dot{\xi} \, dx + \int_{\mathbb{R}} \dot{\xi} A(\eta_c) \dot{\xi} \, dx 
+ 2 \sum_{\pm} \rho_\pm \int_{\mathbb{R}} \left( b_1^+ \eta_c \omega_\pm \eta_c \pm a_1^+ (\eta_c, \lambda_\pm) \right) \left( G_{\pm}(\eta_c)^{-1} A(\eta_c) \dot{\xi} \right) \, dx 
+ 2 \sum_{\pm} \rho_\pm \int_{\mathbb{R}} \pm b_2^\pm \eta A(\eta_c) \dot{\xi} \, dx 
+ 2 \int_{\mathbb{R}} \dot{\xi} G_-(\eta_c) B^{-1}(\eta_c) \omega (\eta_c \dot{\eta}) \, dx + 2 \int_{\mathbb{R}} \dot{\xi} \langle D G_-(\eta_c) \dot{\eta}, B^{-1}(\eta_c) \omega \eta_c \dot{\eta} \rangle \, dx 
+ 2 \int_{\mathbb{R}} G_-(\eta_c) \dot{\xi} \langle D B_-(\eta_c) \dot{\eta}, \omega \eta_c \dot{\eta} \rangle \, dx. \]
Upon cancellation and regrouping, we get
\[ \begin{align*}
D^2 E_c(U_c)[\dot{u}, \dot{u}] &= \langle Q_C(\eta_c) \dot{\eta}, \dot{\eta} \rangle + \int_{\mathbb{R}} \left( (S_c - T_c) \dot{\eta} + \dot{\xi} \right) A(\eta_c) \left( (S_c - T_c) \dot{\eta} + \dot{\xi} \right) \, dx \\
+ 2 \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} g_{\pm}(\eta_c)^{-1} \left( a_1^{-1}(\eta_c, \Upsilon_{\pm}) \dot{\eta} \right)' A(\eta_c) \dot{\xi} \, dx - \int_{\mathbb{R}} 2(a_1^{-1}(\eta_c, \Upsilon_+)^{-1} \dot{\eta})' \dot{\xi} \, dx. 
\end{align*} \]

Using the fact that
\[ \mathcal{A}(\eta)^{-1} = \rho_+ \mathcal{G}_+(\eta)^{-1} + \rho_- \mathcal{G}_-(\eta)^{-1}, \]
the expression on the second row \((3.169)\) vanishes. Therefore, we obtain
\[ \begin{align*}
D^2 E_c(U_c)[\dot{u}, \dot{u}] &= \langle Q_C(\eta_c) \dot{\eta}, \dot{\eta} \rangle + \int_{\mathbb{R}} \left( (S_c - T_c) \dot{\eta} + \dot{\xi} \right) A(\eta_c) \left( (S_c - T_c) \dot{\eta} + \dot{\xi} \right) \, dx. 
\end{align*} \]

From the expression above, \(D^2 E_c(U_c)\) extends to an element in \(X^*\).

**Theorem 3.25** (Spectrum). Let \(\{U_c\}\) be one family of bound states given in the Corollary \((3.16)\). Then
\[ \text{spec } I^{-1} H_c = \{-\mu_c^2, 0\} \cup \Sigma_c, \]
where \(-\mu_c^2 < 0\) is a simple eigenvalue that corresponds to a unique eigenvector \(\chi_c\); 0 is simple eigenvalue generated by \(T\); and \(\Sigma_c \subset (0, \infty)\) is uniformly bounded away from 0.

**Proof.** The following proof relies on the nice structure of \(H_c\) in Lemma \((3.24)\) therefore \(I^{-1} H_c\) and the idea presented in [Mie02, Proposition 5.3] and [CW22, Theorem 4.1.3]. Consider the following operator
\[ Q_c(\eta_c) + (\lambda - \tau_c^2)\langle \cdot, g_{1c} \rangle g_{1c} + \lambda \langle \cdot, \eta'_c \rangle \eta'_c. \]
One can easily check that for \(\lambda > 0\), \(-\tau_c^2\) is the negative eigenvalue of \(Q_c(\eta_c)\) and \(g_{1c}\) is the corresponding eigenfunction. Recall that \(A(\eta_c)\) is a positive definite operator.
Therefore, using the expression for $H_c$ from Lemma 3.24, we can conclude that
\[
(H_c u, u)_{X^* \times X} + (\alpha - \tau_i^2)(I^{-1}(g_{1e}, 0), u)^2_{X^* \times X} + \lambda(I^{-1}(\eta'_c, 0), u)^2_{X^* \times X} \geq \|u\|_{X}^2,
\]
for any $u \in X$. Hence $I^{-1}H_c$ is positive definite on a codimension 2 space. Further, we know that $T'(0)U_c \in \text{Ker } H_c$. Further, from (3.170), for $u = (g_{1e}, (SC - T_c)g_{1e})$, we have
\[
(H_c u, u)_{X^* \times X} = (Q_c(\eta_c)g_{1e}, g_{1e})_{X^* \times X} = -\tau_i^2 < 0.
\]
Hence, we can conclude that $I^{-1}H_c$ has a one-dimensional kernel generated by $T'(0)U_c$, a one-dimensional negative definite subspace. Moreover, it is positive definite on the orthogonal complement. This then proves Assumption 6.

3.6 Proof of theorem

In this section, we will prove the (conditional) orbital stability of the bound states presented in Corollary (3.16). As mentioned previously, it requires us to check the sign of the second derivative of the scalar valued function $d$.

**Theorem 3.26** (Stability for strong surface tension). Fix $c_*$ such that $0 < \alpha_{c_*} - \alpha_0 \ll 1$, the bound states $U_{c_*}$ are conditionally orbitally stable.

**Proof.** Having confirmed all the assumptions, The conclusion on (conditional) orbital stability is drawn by showing $d''(c_*) > 0$. Recall that since $U_{c_*}$ is a critical point of $E_c$, we therefore have
\[
d'(c_*) = -P(U_{c_*}).
\]
We need to show that $d'(c)$ is strictly increasing at $c = c_*$. We start by defining a rescaling operator:
\[
S_c f := f\left(\frac{\epsilon_c}{d_+ \sqrt{3 - \beta_0}}\right).
\]
Using the asymptotics for the free surface, we define
\[ \eta_c =: \epsilon_c^2 d_+ S_c(\tilde{\eta}_c + \tilde{r}_c), \quad \text{for } \tilde{r}_c = O(\epsilon_c) \]

Via the definition of \( P \) in (3.102), we can express \( d'(c) \) as follows
\[
d'(c) = c \int_{\mathbb{R}} \eta_c \partial_x (A(\eta_c)^{-1} \eta'_c) \, dx + \int_{\mathbb{R}} \eta_c \partial_x (A(\eta_c)^{-1} \mathcal{G}_-(\eta_c) B(\eta_c)^{-1} \rho_+ \|\omega\| \eta_c \eta'_c) \, dx \\
+ \int_{\mathbb{R}} \eta_c \omega- \partial_x (A(\eta_c)^{-1} \eta_c \eta'_c) \, dx - \int_{\mathbb{R}} \frac{1}{2} \|\rho \omega\| \eta_c^2 \, dx \\
= cc_\epsilon^4 d_+^2 \int_{\mathbb{R}} S_c(\tilde{\eta}_c + \tilde{r}_c) \partial_x (A(\eta_c)^{-1} \partial_x S_c(\tilde{\eta}_c + \tilde{r}_c)) \, dx \\
+ \epsilon_\epsilon^3 d_+^3 \int_{\mathbb{R}} S_c(\tilde{\eta}_c + \tilde{r}_c) \partial_x (\mathcal{G}_+(\eta_c)^{-1} \rho_+ \|\omega\| S_c(\tilde{\eta}_c + \tilde{r}_c) \partial_x S_c(\tilde{\eta}_c + \tilde{r}_c)) \, dx \\
+ \epsilon_\epsilon^3 d_+^3 \int_{\mathbb{R}} S_c(\tilde{\eta}_c + \tilde{r}_c) \omega- \partial_x (A(\eta_c)^{-1} S_c(\tilde{\eta}_c + \tilde{r}_c) \partial_x S_c(\tilde{\eta}_c + \tilde{r}_c)) \, dx \\
- \epsilon_\epsilon^4 d_+^2 \int_{\mathbb{R}} \frac{1}{2} \|\rho \omega\| (S_c(\tilde{\eta}_c + \tilde{r}_c))^2 \, dx \\
(3.172) \]

Undoing the scaling,
\[
= cc_\epsilon^3 d_+^3 \sqrt{\beta_c - \beta_0} \int_{\mathbb{R}} (\tilde{\eta}_c + \tilde{r}_c) S_c^{-1} \partial_x (A(\eta_c)^{-1} \partial_x S_c(\tilde{\eta}_c + \tilde{r}_c)) \, dx \\
+ \frac{\epsilon_\epsilon^3 d_+^4}{2} \sqrt{\beta_c - \beta_0} \int_{\mathbb{R}} (\tilde{\eta}_c + \tilde{r}_c) S_c^{-1} \partial_x (\mathcal{G}_+(\eta_c)^{-1} \rho_+ \|\omega\| \partial_x (S_c(\tilde{\eta}_c + \tilde{r}_c))^2) \, dx \\
+ \frac{\epsilon_\epsilon^3 d_+^4}{2} \omega- \frac{\sqrt{\beta_c - \beta_0}}{\sqrt{2}} \int_{\mathbb{R}} (\tilde{\eta}_c + \tilde{r}_c) S_c^{-1} \partial_x (A(\eta_c)^{-1} \partial_x (S_c(\tilde{\eta}_c + \tilde{r}_c))^2) \, dx \\
- \epsilon_\epsilon^4 d_+^3 \sqrt{\beta_c - \beta_0} \int_{\mathbb{R}} \frac{1}{2} (\tilde{\eta}_c + \tilde{r}_c) \|\rho \omega\| (\tilde{\eta}_c + \tilde{r}_c) \, dx \\
(3.173) \]

Using the idea in [CW22], we define \( \tilde{\mathcal{M}}^\pm_c(\eta_c) := d_+ S_c^{-1} \partial_x S_c(\eta_c)^{-1} \partial_x S_c \). Following similar line of argument as in Lemma 3.20 we find that
\[
\|\tilde{\mathcal{M}}^\pm_c(0) + \frac{d_+}{d_\pm}\|_{\text{Lin}(H^{k+2},H^k)} \lesssim \epsilon_c^2, \quad \|\tilde{\mathcal{M}}^\pm_c(\eta_c) - \tilde{\mathcal{M}}^\pm_c(0)\|_{\text{Lin}(H^{2},L^2)} \lesssim \epsilon_c^2, \quad (3.174) \]

which yields
\[
= -c \epsilon_c^3 d_+^2 \sqrt{\beta_c - \beta_0} \sum_{\pm} \rho_\pm \frac{d_+}{d_\pm} \int_{\mathbb{R}} (\tilde{\eta}_c)^2 \, dx - \frac{\epsilon_\epsilon^3 d_+^3}{2} \sqrt{\beta_c - \beta_0} \|\rho \omega\| \int_{\mathbb{R}} (\tilde{\eta}_c)^2 \, dx + O(\epsilon_c^4) \\
= -c \left( \rho + \frac{1}{d} + \frac{\omega_+ d_+}{2c} - \frac{\omega_+ d_+}{2c} \right) \left[ \epsilon_c^3 \rho_\pm d_+^2 \sqrt{\beta_c - \beta_0} \int_{\mathbb{R}} (\tilde{\eta}_c)^2 \, dx \right] + O(\epsilon_c^4). \quad (3.175) \]
Recall that

\[ \tilde{\eta}_c = \frac{\text{sech}^2 (x/2)}{\rho - \frac{1}{d^2} + \frac{\omega_+ d_+}{c} + \frac{\omega_- d_+}{cd} + \frac{\omega_+^2 d_+^2 \rho}{3c^2} - \frac{\omega_- d_+^2}{3c^2}}. \]

Thus,

\[ d'(c) = -\left( \frac{\rho + \frac{1}{d} + \frac{\omega_+ d_+}{2c} - \frac{\omega_- d_+}{2c}}{\rho - \frac{1}{d^2} + \frac{\omega_+ d_+}{c} + \frac{\omega_- d_+}{cd} + \frac{\omega_+^2 d_+^2 \rho}{3c^2} - \frac{\omega_- d_+^2}{3c^2}} \right)^2 \int_{\mathbb{R}} \text{sech}^2 \left( \frac{x}{2} \right) dx + O(\epsilon_c^4). \]

For simplicity, let us define the following variables:

- \( \mathfrak{A} := \rho + \frac{1}{d} + \frac{\omega_+ d_+}{2c} - \frac{\omega_- d_+}{2c} \),
- \( \mathfrak{B} := \rho - \frac{1}{d^2} + \frac{\omega_+ d_+}{c} + \frac{\omega_- d_+}{cd} + \frac{\omega_+^2 d_+^2 \rho}{3c^2} - \frac{\omega_- d_+^2}{3c^2} \).

Differentiating \( d'(c) \) with respect to \( c \) yields

\[ d''(c) = \frac{-1}{\mathfrak{B}^4 \sqrt{\beta_c - \beta_0}} \left[ \mathfrak{B}^2 \left( \mathfrak{A}'e_+^3 (\beta_c - \beta_0) + \mathfrak{A}e_+^3 (\beta_c - \beta_0) + 3\mathfrak{A}e_+^2 e'_c (\beta_c - \beta_0) + \frac{\mathfrak{A}e_+^3 c_0}{2} \right) \right. \]

\[ \left. - 2\mathfrak{A}\mathfrak{B} \mathfrak{B}'e_+^2 (\beta_c - \beta_0) \right] \int_{\mathbb{R}} \text{sech}^2 \left( \frac{x}{2} \right) dx + O(\epsilon_c^4). \]

From the above expression of \( d''(c) \) along with definitions of \( \mathfrak{A}, \mathfrak{B} \) and \( \epsilon_c \) including their derivatives, we conclude that for any \( c \in I \), orbital stability (\( d''(c) > 0 \)) is obtained in each of the three cases below.

**Case 1.** Any internal wave with \( \beta_c - \beta_0 \ll 1 \).

**Case 2.** Any internal wave of elevation with \( \omega_+ = 0 \) and \( 2g(\rho - 1) < \frac{c \omega_+}{g} \leq 0 \).

**Case 3.** Any internal wave of depression with \( \omega_- = 0 \) and \( 2\left( \frac{1 - \rho}{\rho} \right) > \frac{c \omega_-}{g} \geq 0 \).

Moreover, we derive some cases where orbital instability (\( d''(c) < 0 \)) occur:
Case 1. Any internal wave of depression with \( \omega_+ = 0 \), \( c\omega_- \leq 0 \), and \( 0 > 2(q - 1) \geq \frac{c\omega_-}{g} \).

Case 2. Any internal wave of elevation with \( \omega_- = 0 \), \( c\omega_+ \geq 0 \), and \( 2\frac{(q - 1)}{\rho} \leq \frac{c\omega_+}{g} \).
Appendices
Appendix A

Global Bifurcation

To keep the presentation reasonably self-contained, this appendix collects two important results from the literature that are used in part one of the present work. We begin with a theorem that contains the maximum principle, Hopf boundary lemma and Serrin edge point lemma. Notably, this includes versions that allow for the “bad sign” of the zeroth order term in the operator provided the sign of the solution is known (see, for instance, [Fra00], [GNN79] and [Ser71]).

Theorem A.1. Let \( \Omega \subset \mathbb{R}^2 \) be a connected, open set (possibly unbounded), consider the second-order operator

\[
L := \sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j + \sum_{i} b_i(x) \partial_i + c(x)
\]

where \( \partial_i \) denotes the spatial derivative in \( x_i \) coordinate and the coefficients \( a_{ij}, b_i, c \) are of class \( C^0(\overline{\Omega}) \). We also assume that \( L \) is uniformly elliptic; that is there exists \( \lambda > 0 \) with

\[
\sum_{ij} a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \ x \in \overline{\Omega},
\]

and \( a_{ij} \) being symmetric. Let \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) be a classical solution of \( Lu = 0 \) in \( \Omega \).
(a) (Strong maximum principle) Suppose $u$ attains its maximum value on $\Omega$ at a point in the interior of $\Omega$. If $c \leq 0$ in $\Omega$, or if $\sup_\Omega u = 0$, then $u$ is a constant function.

(b) (Hopf boundary lemma) Suppose that $u$ attains a maximum on $\Omega$ at a point $x^* \in \partial \Omega$ for which there exists an open ball $B \subset \Omega$ such that $\overline{B} \cap \partial \Omega = \{x^*\}$. Assume either $C \leq 0$ in $\Omega$ or else $\sup_B u = 0$. Then $u$ is a constant function or

$$\nu \cdot \nabla u(x^*) > 0,$$

where $\nu$ is the outward unit normal to $\Omega$ at $x^*$.

(c) (Serrin edge point lemma) Let $x^* \in \partial \Omega$ be an “Edge point” in the sense that near $x^*$ consists of two transversally intersecting $C^2$-hypersurfaces $\{\gamma(x) = 0\}$ and $\{\sigma = 0\}$. Suppose that $\gamma, \sigma < 0$ in $\Omega$. If $u \in C^2(\overline{\Omega})$, $u > 0$, $u(x^*) = 0$. Assume further that $a_{ij} \in C^2$ around the neighborhood of $x^*$,

$$B(x^*) = 0, \quad \text{and} \quad \partial_r B(x^*) = 0$$

for every differential $\partial_r$ tangential to $\{\gamma = 0\} \cap \{\sigma = 0\}$ at $x^*$. Then for any unit vector $s$ outward from $\Omega$ at $x^*$, either

$$\partial_s u(x^*) < 0 \text{ or } \partial^2_s u(x^*) < 0.$$

Secondly, we record here the abstract analytic global bifurcation result from [CWW18, Theorem 6.1].

**Theorem A.2** (Chen, Walsh, Wheeler [CWW18]). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, with $\mathcal{U} \subset \mathcal{X} \times \mathbb{R}$ an open set. Suppose that $\mathcal{F} = \mathcal{F}(x, \lambda) : \mathcal{U} \to \mathcal{Y}$ is real analytic.
Assume that there exists a continuous local curve $\mathcal{C}_{loc}$ of solutions to $\mathcal{F}(x, \lambda) = 0$ parametrized as

$$\mathcal{C}_{loc} := \{(\tilde{x}(\lambda), \lambda) : 0 < \lambda < \lambda_*\},$$

where $\lambda_* > 0$ and the map $\tilde{x} : (0, \lambda_*) \to \mathcal{U}$ is continuous. If

$$\tilde{x}(\lambda) \to 0 \in \partial \mathcal{U} \text{ as } \lambda \to 0^+ \text{ and } \mathcal{F}_x(\tilde{x}(\lambda), \lambda) : \mathcal{X} \to \mathcal{Y} \text{ is invertible for all } \lambda,$$

then the local curve $\mathcal{C}_{loc}$ of the nonlinear operator $\mathcal{F}(x, \lambda) = 0$ is contained in a global $C^0$ curve $\mathcal{C}$ parameterized as

$$\mathcal{C} := \{(x(s), s) : 0 < s < \infty\} \subset \mathcal{F}^{-1}(0),$$

for some continuous $(0, \infty) \ni s \mapsto (x(s), s) \in \mathcal{U} \times \mathcal{I}$ and exhibiting the following properties

(a) One of the following alternatives must hold:

(i) (Blowup) as $s \to \infty$,

$$N(s) := \|x(s)\|_X + \frac{1}{\text{dist}(x(s), \partial \mathcal{U})} + \lambda(s) + \frac{1}{\text{dist}(x(s), \partial \mathcal{I})} \to \infty.$$

(ii) (Loss of compactness) There exists a sequence of $s_n \to \infty$ such that $\sup_n N(s_n) < \infty$ but $\{x(s_n)\}$ does not have subsequences converging in $\mathcal{X}$.

(b) Fix parameter $s^* \in (0, \infty)$, around the neighborhood $(x(s^*), \lambda(s^*)) \in \mathcal{C}$, we can reparametrize $\mathcal{C}$ so that $s \mapsto (x(s), \lambda(s))$ is real-analytic.

(c) For all $s \gg 1$, $(x(s), \lambda(s)) \notin \mathcal{C}_{loc}$.
Appendix B

Stability

The first part of appendix records the center manifold reduction theorem introduced in [Mic95] that was implemented in, for example, [Nil17]. It is used in proving the existence of small-amplitude internal water waves.

**Theorem B.1** (Center manifold). Consider the differential equation of the form

\[ \dot{u} = Lu + F(u, \mu), \quad (B.1) \]

where the unknown \( u \in E \) for some Hilbert space \( E \), \( \mu \in \mathbb{R}^n \) is a parameter and \( L : \mathcal{D}(L) \subset E \to E \) is a closed linear operator. Assume that the differential equation \( (B.1) \) is Hamilton’s equations that correspond to the Hamiltonian system \((E, \Omega, H)\) with 0 being its fix point. Moreover, assume also the following:

**H1** The space \( E \) has two closed and \( L \)-invariant subspaces, namely \( E_1 \) and \( E_2 \) such that

\[ E = E_1 \oplus E_2, \]

\[ u_1 = L_1 u_1 + F_1(u_1 + u_2, \mu), \tag{B.2} \]

\[ u_2 = L_2 u_2 + F_2(u_1 + u_2, \mu), \]

where \( L_i = L|_{\mathcal{D}L_i \cap E_i} : \mathcal{D}L_i \cap E_i \to E_i \), for \( i = 1, 2 \) and \( F_1 = PF \), \( F_2 = (I-P)F \),

where the operator \( P \) is a projection of \( E \) onto \( E_1 \).
H2  $E_1$ is a finite dimensional Hilbert space and the spectrum of $L_1$ is purely imaginary.

H3  The imaginary axis lies in the resolvent of $L_2$ and

$$\| (L_2 - iaI)^{-1} \| \leq \frac{C}{1 + |a|}, \quad \text{for } a \in \mathbb{R}. \quad (B.3)$$

H4  There exists $k \in \mathbb{N}$ and neighborhoods $\Lambda \subset \mathbb{R}^n$ and $U \subset D(L)$ of 0 such that $F$ is $k + 1$ continuously differentiable on $U \times \Lambda$ and the derivatives of $F$ are all bounded and uniformly continuous on $U \times \Lambda$ with

$$F(0, \mu_0) = 0, \quad dF[0, \mu_0] = 0.$$ 

Under the hypothesis H1-H4 there exist neighborhoods $\tilde{\Lambda} \subset \Lambda$ and $\tilde{U}_1 \subset U \cap E_1$, $\tilde{U}_2 \subset U \cap E_2$ of zero and a reduction function $r : \tilde{U}_1 \times \tilde{\Lambda} \rightarrow \tilde{U}_2$ with the following properties. The reduction function $r$ is $k$ times continuously differentiable on $\tilde{U}_1 \times \tilde{\Lambda}$ and the derivatives of $r$ are bounded and uniformly continuous on $\tilde{U}_1 \times \tilde{\Lambda}$ with

$$r(0, \mu_0) = 0, \quad dr[0, \mu_0] = 0.$$ 

The graph

$$X_C^\mu = \{ u_1 + r(u_1, \mu) \in \tilde{U}_1 \times \tilde{U}_2 : u_1 \in \tilde{U}_1 \},$$

is a Hamiltonian center manifold for (B.1) with the following properties:

- Through every point in $X_C^\mu$ there passes a unique solution of (B.1) that stays on $X_C^\mu$ as long as it remains in $\tilde{U}_1 \times \tilde{U}_2$. We say that $X_C^\mu$ is a locally invariant manifold of (B.1).
• Every small bounded solution $u(x), x \in \mathbb{R}$ of (B.1) that satisfies $u_1(x) \in \tilde{U}_1$ and $u_2(x) \in \tilde{U}_2$ lies completely in $X_C^\mu$.

• Every solution $u_1$ of the reduced equation

$$\dot{u}_1 = L_1 u_1 + F_1(u_1 + r(u_1, \mu), \mu), \quad (B.4)$$

generates a solution

$$u(x) = u_1(x) + r(u_1(x), \mu)$$

of (B.1).

• $X_C^\mu$ is a symplectic submanifold $E$, and the flow determined by the Hamiltonian system $X_C^\mu, \tilde{\Omega}, \tilde{H}$, where the tilde denotes the restriction to $X_C^\mu$, coincides with the flow on $X_C^\mu$ determined by $(E, \Omega, H)$. The reduced equation (B.4) represents Hamilton’s equations for $(X_C^\mu, \tilde{\Omega}, \tilde{H})$.

• If (B.1) is reversible, that is if there exists a linear symmetry $S$ that anti-commutes with the right hand side of (B.1), then the reduction function $r$ can be chosen so that it commutes with $S$.

Many of the computations in the present work make use the first and second derivative formulas of the non-local operators $G_\pm(\eta)$ and $A(\eta)$. We record them in a series of lemmas below. The derivation of them can be found in [CW22].

**Lemma B.2** (First Derivatives). Let $\eta, \tilde{\xi} \in \mathcal{O} \cap \mathcal{V}, \dot{\eta} \in \mathcal{V}_1$ and be given.

(a) The Fréchet derivative of $G_\pm(\eta)$ admits the representation formula

$$\int_{\mathbb{R}} \zeta \langle DG_\pm(\eta) \dot{\eta}, \tilde{\xi} \rangle \, dx = \int_{\mathbb{R}} \left( a_1^\pm(\eta, \tilde{\xi}) \zeta' + a_2^\pm(\eta, \tilde{\xi}) G_\pm(\eta) \zeta \right) \dot{\eta} \, dx, \quad (B.5)$$
with
\[ a_1^±(\eta, \tilde{\xi}) := \frac{1}{1 + (\eta')^2} \left( \mp \tilde{\xi}' - \eta' G_±(\eta) \tilde{\xi} \right) \]
\[ a_2^±(\eta, \tilde{\xi}) := \frac{1}{1 + (\eta')^2} \left( \pm G_±(\eta) \tilde{\xi} - \eta' \tilde{\xi}' \right) \]  \hfill (B.6)

(b) The Fréchet derivative of \( A(\eta) \) admits the representation formula
\[ \int_R \zeta \langle D A(\eta) \hat{\eta}, \hat{\xi} \rangle \, dx \]
\[ = \sum_± \rho_\pm \left( a_1^±(\eta, A(\eta) G_±(\eta)^{-1} \tilde{\xi}) (\dot{A}(\eta) G_±(\eta)^{-1} \zeta)' \right) \hat{\eta} \, dx \]
\[ + \sum_± \rho_\pm \int_R \left( a_2^±(\eta, A(\eta) G_±(\eta)^{-1} \tilde{\xi} A(\eta) \zeta) \right) \hat{\eta} \, dx. \]  \hfill (B.7)

**Lemma B.3** (Second derivative of \( G_± \)). For all \( u = (\eta, \tilde{\xi}) \in O \cap V \) and \( \dot{\eta} \in V_1 \), it holds that
\[ \int_R \tilde{\xi} \langle D^2 G_±(\eta)[\dot{\eta}, \dot{\eta}], \tilde{\xi} \rangle \, dx \]
\[ = \int_R \left( a_4^±(u) \dot{\eta}^2 + 2a_2^±(u) \dot{\eta} G_±(\eta) (a_2^±(u) \dot{\eta}) \right) \, dx, \]  \hfill (B.8)

where
\[ a_4^±(u) := -2a_1^±(u) a_2^±(u), \]  \hfill (B.9)

and \( a_1^±, a_2^± \) are given by (B.6).

**Lemma B.4** (Second derivative formula of \( A \)). For all \( u = (\eta, \tilde{\xi}) \in O \cap V \) and \( \dot{\eta} \in V_1 \), it holds that
\[ \int_R \tilde{\xi} \langle D^2 A(\eta)[\dot{\eta}, \dot{\eta}], \tilde{\xi} \rangle \, dx \]
\[ = \int_R \left( a_4(u) \dot{\eta} + 2 \sum_± \rho_± a_2^±(u, \theta_±) G_±(\eta) (a_2^±(u, \theta_±) \dot{\eta}) - 2 \mathcal{M}(u) \dot{\eta} + 2 \mathcal{N}(u) \dot{\eta} \right) \, dx, \]  \hfill (B.10)

where
\[ \theta_±(u) := G_±(\eta)^{-1} A(\eta) \tilde{\xi}, \quad a_4(u) := \sum_± \rho_± a_4^±(u, \theta_±) \]  \hfill (B.11)
\( \mathcal{L}_\pm(u) \dot{\eta} := -G_\pm(\eta)^{-1} (a_1^\pm(\eta, \theta_\pm) \dot{\eta})' + a_2^\pm(\eta, \theta_\pm) \dot{\eta}, \quad \mathcal{L}(u) := \sum_\pm \rho_\pm \mathcal{L}_\pm(u) \)

\( \mathcal{M}(u) \dot{\eta} := \sum_\pm \rho_\pm \left( a_1^\pm(\eta, \theta_\pm) (\mathcal{L}_\pm(u) \dot{\eta})' + a_2^\pm(\eta, \theta_\pm) G_\pm(\eta) \mathcal{L}_\pm(u) \dot{\eta} \right) \)  \hspace{1cm} (B.12)

\( \mathcal{N}(u) \dot{\eta} := \sum_\pm \rho_\pm \left( a_1^\pm(\eta, \theta_\pm) (\mathcal{A}(\eta) G_\pm(\eta)^{-1} \mathcal{L}(u) \dot{\eta})' + a_2^\pm(\eta, \theta_\pm) \mathcal{A}(\eta) \mathcal{L}(u) \dot{\eta} \right) . \)
Bibliography


Klaus Kirchgässner. Wave-solutions of reversible systems and applications. 


[Whe15a] Miles H. Wheeler. The Froude number for solitary water waves with 

[Whe15b] Miles H. Wheeler. Solitary water waves of large amplitude generated by 

[Yih65] Chia-Shun Yih. Dynamics of nonhomogeneous fluids. The Macmillan Co.,
VITA

Daniel Sinambela was born in Tanjung Enim, South Sumatra, Indonesia to Kaspar Sinambela and Ruslina Napitupulu in 1994. He came to the University of Missouri, Columbia in 2012 to pursue an undergraduate degree in Mathematics; he received his B.S. in Mathematics, in Fall 2015. Soon after that in Spring 2016, he decided to continue his education and do a Masters degree in Mathematics at the same institution. Under the guidance of Dr. Samuel Walsh, he defended his thesis titled “Asymptotic properties of deep water solitary waves with compactly supported vorticity” and received his M.A. in Mathematics. Currently, he is a PhD candidate in Mathematics at the University of Missouri, Columbia, mentored by Dr. Samuel Walsh. His research interest is in the area of nonlinear partial differential equations, pertaining to the questions of existence and stability of certain family of solutions. Particularly, he studies traveling water waves governed by the incompressible Euler equations. Upon graduation, he will begin his Postdoctorate Fellowship at New York University in Abu Dhabi, in September 2022.