# PARTIAL CONNECTIONS AND CONTACT CONNECTIONS ON CONTACT MANIFOLDS

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**A Dissertation**

**presented to**

**the Faculty of the Graduate School**

**University of Missouri**

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

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by

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JULY 2022

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

PARTIAL CONNECTIONS AND CONTACT INSTANTONS ON CONTACT MANIFOLDS

presented by Nathapon Udomlertsakul, a candidate for the degree of Doctor of Philosophy of

Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

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Professor Jan Segert

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ACKNOWLEDGEMENTS

I would like to thank to my advisor Dr. Shuguang Wang for all the time he had been supporting me. Because of his guidance, I could enjoy my life at the University of Missouri even with the Covid-19 pandemic's situation. Without the support from the committee members, the dissertation would never be completed. Hence I would like to thank Dr. Jan Segert, Dr. Jianguo Sun, and Dr. Zhenbo Qin for all of the support. I sincerely appreciate the help from the people in the MU Mathematics department for many things. Additionally, I would like to express my gratitude to my financial sponsor, DPST (Development and Promotion of Science and Technology Talents Project, a Thai government scholarship). At last but not least, I am forever grateful for the long-distance support from my parents.

# Contents

Partial connections and contact instantons on contact manifolds.

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ABSTRACT

We study partial connections that are defined on a vector bundle $E$ over a contact distribution $H$ of a contact manifold $(M^{2m+1}, \theta)$ by adapting the Rumin complex of the exterior derivative in a contact case. Full connections will be investigated in a new manner using the partial connection's point of view (One can view it as a reduction method). The alternative one to one correspondence between a full connection $\overline{D}$ and a partial connection $D$ is introduced by linking with some $B \in EndE$, i.e. $\overline{D} = \overline{D}(D, B)$. For instance, we provide the new constructions of the Tanaka Webster connection and the Tanaka canonical connection through the suitable pair $(D, B)$. The contact instanton equation and Hermitian-Einstein connection over a contact manifold are explored using the above correspondence. Consequently, we prove the existence of a solution of the $B$-inhomogeneous Yang-Mills equation $\overline{D}^*\overline{F} = mB\theta$. This resembles the Tian instanton and Hermitian-Einstein connection over a Kahler manifold. For the applications, results of Dragomir-Urakawa in CR manifolds are covered into contact manifolds.

# Chapter 1

# Introduction

The study of partial connections on a contact manifold $(M^{2m+1}, H, \theta)$ took place before, in 1994, Rumin [Rum94] proposed the exact complex of an exterior derivative over a contact distribution. This work was subsequently cited in Gromov's publication [Gro96]. He discusses the c-c metrics on a contact manifold and also works on differential forms over the contact distribution in one section of the study. These are the motivations for a deep drive into partial connections on contact manifolds. The definition of a partial connection in the direction of an integrable complex distribution which is investigated over the direction of the Reeb vector field and $\widetilde{H}^{0,1} := H^{0,1} \oplus (\xi \otimes_{\mathbb{R}} \mathbb{C})$ is used by Biswas and Schumacher [BS10] to work on the Sasakian manifold. This thesis will be based on the situation where we can reduce the information from full connections to partial connections over the contact distribution. To define a partial connection on the non-integrable distribution, a new domain of curvature is required, which will be discussed in the study. This study is focused on the term $B \in EndE$ in order to link partial connections and full connections together. The technique is induced in the Udomlertsakul-Wang paper [UW22]. The instanton and Hermitian-Einstein connections have lately gained popularity in the contact manifold; Biswas and Schumacher [BS10] prove the existence of the Hermitian-Einstein structure of a stable Sasakian holomorphic vector bundle on Sasakian manifolds, Baraglia and Hekmati [BH14] introduce the contact instanton on K-contact 5-manifold and also study the moduli space of contact instantons on K-contact 5-manifolds, and Wang [Wan14] generally develops the instanton gauge theory for an arbitrary smooth manifold with foliations. In this thesis, we introduce the $B$ instanton and the Hermitian-Einstein connection in term of a pair of partial connection and $B$, we also give the relations among these connections.

In Chapter 2, we provide a fairly thorough analysis of the Rumin complex and investigate the narration of the $H$-partial connection $D$ with its extension $\overline{D}$. Originally, partial connections were first established as so-called connections on a fibration along a distribution on the base by Yuri I. M. [Yur84] . Since we are concentrating on a partial connection $D$ over a contact distribution $H$, then we must modify the curvature's domain due to that $H$ is not integrable, i.e. $[X, Y] \notin H$ for some $X, Y \in H$. In contrast to Biswas and Schumacher [BS10], which focuses on a partial connection over an integrable distribution, this will be different. The extended partial connection over a higher degree will be introduced. Additionally, we research the new connection $\widetilde{D}$ which connects the two complexes of partial connections in the same manner as the Rumin connection on the Rumin complex of an exterior derivative. In the later part of the chapter, a full connection $\overline{D}$ is discussed in view of $H$-partial connection and the term $B \in EndE$ by starting from the simple case which is of 3 dimensions then we progress to the higher dimension, $n \geqslant 5$. Hodge $*'$-operator is the name given to the Hodge $*$-operator, which will be algebraically explored on the domain of a contact distribution rather than a full tangent bundle.

In Chapter 3, we give the new constructions of Tanaka's types of connections, including the Tanaka-Webster connections in both real differential forms and complex differential forms, and Tanaka canonical connections. These will be built along the way of a partial connection using the uniqueness of the extension of $(D, B)$ to verify the existence and uniqueness of Tanaka's types of connections. At the end of the chapter, there is a comment made on the complexity of $B$ whether it is preferable to the condition on a full connection.

In Chapter 4, the definitions of a Hermitian-Einstein connection and structure will be reviewed for the case of Kahler manifolds, as well as the picture of Tian [Tia00], the relationship among $\Omega$-instantons, Hermitian-Einstein connections, and Yang-Mills connections. The Hermitian-Einstein connections over contact manifolds that Wang proposes in the Informal notes [Wan21] are then taken into account. Furthermore, we demonstrate that the Tanaka canonical connection is one of the examples of the Hermitian-Einstein connection over a contact manifold.

In the last chapter, we give the relationship among contact instantons, Hermitian-Einstein connections, and Yang-Mills connections in a similar way to Tain [Tia00], except we are working on a contact manifold. Firstly, we write a local picture of the differential forms type $(p, q)$ as a tool. Next, we study the operator $\star : \bigwedge^2 TM^* \longrightarrow \bigwedge^2 TM^*$ that will play a crucial role in the contact instanton equation. The eigen decomposition of the $\star$ operator is discussed explicitly in the algebraic picture (fiber-wise form). After we have sufficient tools, the $B$ contact instanton equation defined as $\star(\overline{F} - Bd\theta) = -(\overline{F} - Bd\theta)$ and the $B$ inhomogeneous Yang-Mill equation defined by $\overline{D}^*\overline{F} = mB\theta$ will be explored. We end by presenting the major theorem with its corollaries and applications.

**Theorem.** *Suppose $\overline{D}$ is $B$ contact instanton. Then*

$$\overline{D}^{*}\overline{F} = mB\theta - mJ(DB).$$

*Moreover, $\overline{D}$ is $B$-inhomogeneous Yang-Mills connection if and only if $DB = 0$.*

**Corollary.** *The $B$ contact instanton is a critical point of the functional $YM_B$ such that $YM_B(\overline{D}) := \int \|F_{\overline{D}} - mBd\theta\|^2$.*

Importantly, we generalize that the Dragomir-Urakawa's result [DU00] from a CR manifold to a contact manifold.

# Chapter 2

# Partial connections and curvatures

We start off by giving the definition of a contact manifold. It is worth noting that there are several definitions of a contact manifold; One by a distribution the other by a differential form. Since it is more practical to concentrate on a single differential form, we shall adopt the following definition for the contact manifold in this study.

**Definition 2.1.** Let $M^{2m+1}$ be an odd dimensional smooth manifold and $\theta \in \Gamma(T^*M)$. Then $(M, \theta)$ is called a contact manifold if $\theta \wedge (d\theta)^m$ is non vanishing everywhere.

For the definition based on a distribution, which is more general, it is referred through the distribution saying a subbundle $H \subset TM$ instead of a differential form and it allows one to have several contact forms. The brief explanation will be given as follows. For a smooth manifold $M$ and a subbundle $H \subset TM$, define $\omega \in \Gamma(\wedge^2 H^* \otimes H')$ by $\omega(X, Y) := \widetilde{\omega}([X, Y])$, where $\widetilde{\omega}$ is the projection of $TM$ to $H'(:= TM/H)$. Let $M$ be a smooth manifold and $H$ any subbundle of $TM$. $H$ is called a contact structure if $dim H' = 1$ and $\omega$ is pointwise non-degenerate. $(M, H)$ is called a contact manifold if $M$ is an odd dimensional (2m+1) manifold and $H$ is a contact structure. For a given contact manifold $(M, H)$ in this sense, we suppose that $p'$ is trivial then there is a differential form degree one $\theta$ such that $\theta \wedge (d\theta)^m$ is non-vanish everywhere and $ker\theta\mid_x = H_x,\ \forall x \in M$.

Now we return to the contact manifold in the sense of a differential form. One defines a 1 dimensional subspace, $\{X \in T_xM \mid d\theta(X, T_xM) = 0\}$. Let $\xi_x \in \{X \in T_xM \mid d\theta(X, T_xM) = 0\}$ such that $\theta(\xi) = 1$ at $x$. By this, we have the Reeb vector field $\xi \in \Gamma(TM)$ of the contact structure $\theta$, i.e. $\theta(\xi) = 1$ and $d\theta(\xi, X) = 0, \forall X \in \Gamma(TM)$.

A partial connection, its extension, and its features will be the primary topics of discussions in this chapter. We firstly establish the scenario we will be working before moving on. Let $(M, \theta)$ be a contact manifold with dimension $2m + 1$, where $\theta$ is a contact 1-form. Denote the contact $2m$ dimensional distribution and the Reeb vector field by $H$ and $\xi$ respectively. Since $TM = H \oplus < \xi >$, then we have

$$\wedge^r TM = \wedge^r H \oplus < \xi > \wedge(\wedge^{r-1} H). \tag{2.1}$$

By the same argument, we also have that

$$\wedge^r TM^* = \wedge^r H^* \oplus < \theta > \wedge(\wedge^{r-1} H^*) \tag{2.2}$$

, where $\wedge^r H^* \subset \wedge^r TM^*$ means the element in $\wedge^r H^*$ trivially extent to $\wedge^r TM^*$, i.e. $\Omega \in \wedge^r H^*$ means $\Omega(\omega) = 0$ for $\omega \in \wedge^r TM^*$ and $\omega \notin \wedge^r H^*$. This decomposition allows us to further investigate the partial connection.

## 2.1 Exterior derivatives on a contact distribution and theirs complexes

In this section, we will introduce the new complexes based on the partial derivative in Wang's Informal notes [Wan21] that motivated by Rumin complexes [**?**]. With the preceding decomposition, the exterior derivative $d : \bigwedge^r(T^*M) \longrightarrow \bigwedge^{r+1}(T^*M)$ can be in decomposed in two parts referred to as the horizontal part and the vertical part, $d'$ and $d''$ respectively.

$$d\alpha = d'\alpha - \theta \wedge d''\alpha, \tag{2.3}$$

where $\alpha \in \Gamma(\bigwedge^r(T*M))$ and $d'\alpha - \theta \wedge d''\alpha \in \Gamma(\wedge^{r+1}TM^*) = \gamma(\wedge^{r+1}H^*) \oplus \theta \wedge \gamma(\wedge^r H^*)$.

**Note 2.2.** $d'\alpha$ have the same degree as of the differential form $d\alpha$; however, the degree of $d''\alpha$ is less than the degree of $d\alpha$ by one. Moreover, $\theta \wedge d\alpha = \theta(d'\alpha + \theta \wedge d''\alpha) = \theta \wedge d'\alpha$.

**Theorem 2.3.** *$d' : \Gamma(\bigwedge^r(T*M)) \longrightarrow \gamma(\wedge^{r+1}H^*), d'' : \Gamma(\bigwedge^r(T*M)) \longrightarrow \gamma(\wedge^r H^*)$ are complied with the Leibnitz rule*

*Proof.* Let $\alpha, \beta \in \Gamma(\bigwedge^r(T*M))$. Consider

$$\begin{aligned}
&d'(\alpha \wedge \beta) + \theta \wedge d''(\alpha \wedge \beta) \\
&\quad = d(\alpha \wedge \beta) \\
&\quad = d\alpha \wedge \beta + (-1)^{deg\alpha}\alpha \wedge d\beta \\
&\quad = d'\alpha \wedge \beta - \theta \wedge (d''\alpha \wedge \beta) + (-1)^{deg\alpha}\alpha \wedge d'\beta - \theta \wedge ((-1)^{deg\alpha}\alpha \wedge d''\beta) \\
&\quad = [d'\alpha \wedge \beta + (-1)^{deg\alpha}\alpha \wedge d'\beta] - \theta \wedge [d''\alpha \wedge \beta + (-1)^{deg\alpha}\alpha \wedge d''\beta].
\end{aligned}$$

Then $d'(\alpha \wedge \beta) = d'\alpha \wedge \beta + (-1)^{deg\alpha}\alpha \wedge d'\beta$ and $d''(\alpha \wedge \beta) = d''\alpha \wedge \beta + (-1)^{deg\alpha}\alpha \wedge d''\beta$. Hence the Leibnitz rule applies with the two operators $d', d''$. ∎

**Theorem 2.4.** *$d'$ and $d''$ are commutable. Moreover, $(d')^2 = d\theta \wedge d$.*

*Proof.* Let $\alpha \in \Gamma(\bigwedge^r(T^*M))$. Since $d^2 = 0$, (2.3), and by the Leibnitz rule, then we have that

$$0 = d^2(\alpha) = d(d\alpha)$$

$$= d(d'\alpha - \theta \wedge d''\alpha)$$

$$= d'(d'\alpha - \theta \wedge d''\alpha) - \theta \wedge d''((d'\alpha - \theta \wedge d''\alpha))$$

$$= d'^2\alpha - d'\theta \wedge d''\alpha + \theta \wedge d'd''\alpha - \theta \wedge d''d'\alpha + \theta \wedge d''\theta \wedge d''\alpha + \theta \wedge \theta \wedge d''^2\alpha$$

$$= d'^2\alpha - d\theta \wedge d''\alpha + \theta \wedge (d'd''\alpha - \wedge d''d'\alpha).$$

Since $d\theta \in \Gamma(\bigwedge^2 H^*)$, then we have an orthogonal decomposition as (2.3). Hence $d'^2\alpha - d\theta \wedge d''\alpha = 0$ or saying $d'^2 = d\theta \wedge d''$ and $d'd''\alpha - \wedge d''d'\alpha = 0$ which is equivalent to that $d', d''$ are commute. ∎

**Remark 2.5.** We can always choose the local basis $\{\alpha_i, \beta_i, \theta\}$ in such a way that $d\theta = \sum \alpha_i \wedge \beta_i$. This mean that $d\theta$ vanishes on the $\theta$-component. One can write $d'\theta = d\theta$.

We are interested in a partial connection over the contact distribution $H$ in order to gain the numbers of the information. Hence the complexes will be base on $\bigwedge^r(H^*)$. We encounter the obstruction that it is nearly impossible to come up with the complex for $d'$ on $\bigwedge^r(H^*)$ with its property $d' \circ d' \neq 0$(unlike $d^2 = 0$). Therefore, one must modify the domain of $d'$. Two alternative complex creations will be presented. The first method will be restricting the space of $\Gamma((\bigwedge^r H^*)$ to the space of solutions $\{d\alpha \in \Gamma((\bigwedge^{r+1} H^*)\}$, equivalently $i_\xi d\alpha = 0$ and considering $d$ as the operator. The second method by Rumin is to quotient the space to $\Gamma(\bigwedge^r H^*)/ < \theta, d\theta >$ while retaining the operator; however, the complex will stop at $r = m$. Additionally he creates another rumin connection to link these two complexes together. Those methods will be clarified below. For the convenience, we denote $\Omega^r_M := \Gamma((\bigwedge^r T^*M)$ and $\Omega^r_H := \Gamma((\bigwedge^r H^*)$.

**The first method** $(\widehat{\Omega}^r_H, d)$. Define $\widehat{\Omega}^r_H := \{\alpha \in \Omega^r_H \mid d\alpha \in \Omega^{r+1}_H\}$. Since $d''\alpha = 0$, it is clear that $d\alpha = d'\alpha$, where $\alpha \in \widehat{\Omega}^r_H$. Under the new circumstance, we can see that $d' \circ d' = d \circ d = 0$ over $\widehat{\Omega}^r_H$. Now we have the complex $(\widehat{\Omega}^r_H, d = d')$, where $\widehat{\Omega}^{2m}_H = 0$. To see why $\widehat{\Omega}^{2m}_H = 0$, one uses the fact that $d(\alpha) \in <\theta>$, $\forall \alpha \in \widehat{\Omega}^{2m}_H$. This complex is independent of the choice of $\{f\theta\}$ for any everywhere non-vanishing $f$, since we have $<\theta_1> = <\theta_2>$ for any two contact forms $\theta_1, \theta_2$ related by $\theta_1 = f\theta_2$ .

$$0 \longrightarrow \widehat{\Omega}^0_H C^\infty(M) \xrightarrow{d} \widehat{\Omega}^1_H \xrightarrow{d} \ldots \xrightarrow{d} \widehat{\Omega}^{2m-1}_H \xrightarrow{d} 0$$

**The second method** By Rumin [Rum94] in 1994. Firstly, he defines $\mathscr{S}^* := \{\theta \wedge \alpha + d\theta \wedge \beta \mid \alpha, \beta \in \Omega^*_M\}$. Let $I^r := \Omega^r_M / \mathscr{S}^*$. For the exterior derivative over $I^r$, he define $d_H : I^r \longrightarrow I^{r+1}$ such that $d_H([\alpha]) := [d\alpha]$. This is well defined, since $d_H(\mathscr{S}^*) = 0$. By Weil [Wei58], $L : \Omega^r_H \longrightarrow \Omega^{r+2}_H$ such that $L(\alpha) = d\theta \wedge \alpha$, $L$ is injective for $r \leqslant m-1$ and $L$ is surjective for $r \geqslant m-1$. In the result, $\Omega^i_M = <d\theta>$ for $i \geqslant m+1$. Hence $I^r = 0$ for $r \geqslant m+1$. Hence the complex will end at $r = m+1$

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \longrightarrow I^1(= H^*) \xrightarrow{d_H} I^2 \xrightarrow{d_H} \ldots \xrightarrow{d_H} I^m \xrightarrow{d} 0$$

He also considers another complex $(J^*, d_H)$, where $J^* = \{\alpha \in \Omega^r_M \mid \theta \wedge \alpha = 0 = d\theta \wedge \alpha\}$. The element in $J^*$ is always in the form of $\theta \wedge \alpha$, for some $\alpha \in \Omega^* M$. $d_H$ is defined by $d_H(\theta \wedge \alpha) := d(\theta \wedge \alpha)$. For $\alpha \in J^r$, we have that $d_H(\theta \wedge \alpha) = d\theta \wedge \alpha - \theta \wedge d\alpha = \theta \wedge d\alpha$, by that $d\theta \wedge \alpha = 0$. Again using Weil [Wei58], $J^r = 0, \forall r \leqslant m$. Then the complex begins at $r = m+1$,

$$0 \longrightarrow J^{m+1} \xrightarrow{d_H} J^{m+2} \xrightarrow{d_H} \ldots \xrightarrow{d_H} J^{2m} \xrightarrow{d} J^{2m+1}(= \Omega^n_M) \longrightarrow 0$$

**Note 2.6.** both $d_H$ on $I^r := \Omega^r_M/\mathscr{S}^*$ and $d_H$ on $J^* = \{\alpha \in \Omega^r_M \mid \theta \wedge \alpha = 0 = d\theta \wedge \alpha\}$ has the property that $d_H \circ d_H = 0$ by the definition induced from $d$. Moreover, $d_H([\alpha]) := [d\alpha] = [d'\alpha]$ on $I^r$ and $d_H(\theta \wedge \alpha) := d(\theta \wedge \alpha) = \theta \wedge d\alpha = \theta \wedge d'\alpha$. Hence $d_H$ can be described by $d'$ only.

Then he binds those two complexes together with the operator $\widetilde{d} : I^m \longrightarrow J^{m+1}$ satisfying

$$\widetilde{d}\alpha = d(\alpha - \theta \wedge L^{-1}d'\alpha). \tag{2.4}$$

Two arguments must be built in order to show that it is well defined; (1) $\widetilde{d}(\theta\wedge\alpha) = 0 = \widetilde{d}(d\theta \wedge \beta)$ and (2) $\widetilde{d}\alpha \in J^{m+1}$ for $\alpha \in I^m$. The first argument is immediately apparent from calculations. Recall that $\theta \wedge d\gamma = \theta \wedge d'\gamma$.

$$\begin{aligned}
\widetilde{d}(\theta \wedge \alpha) &= d(\theta \wedge \alpha - \theta L^{-1}d'(\theta \wedge \alpha)) \\
&= d(\theta \wedge \alpha - \theta L^{-1}(d\theta \wedge \alpha - \theta \wedge d'\alpha)) \\
&= d(\theta \wedge \alpha - \theta L^{-1}(d\theta \wedge \alpha - \theta \wedge d\alpha)) \\
&= d(\theta \wedge \alpha - \theta \wedge (\alpha - \theta \wedge \alpha^*) = 0.
\end{aligned}$$

By the fact that $d'd\theta = 0$, we then compute

$$\begin{aligned}
\widetilde{d}(d\theta \wedge \beta) &= d(d\theta \wedge \beta - \theta L^{-1}d'(d\theta \wedge \beta)) \\
&= d(d\theta \wedge \beta - \theta L^{-1}(d'd\theta \wedge \beta + d\theta \wedge d'\beta)) \\
&= d(d\theta \wedge \beta - \theta L^{-1}(d\theta \wedge d'\beta)) \\
&= d(d\theta \wedge \beta - \theta \wedge d'\beta) \\
&= d(d\theta \wedge \beta - \theta \wedge d\beta) = 0.
\end{aligned}$$

Hence $\widetilde{d}[\omega] = \widetilde{d}[\omega^*]$, where $[\omega] = [\omega^*]$. For the second argument, one applies the property that $L : \Omega_H^{m-1} \longrightarrow \Omega_H^{m+1}$ is isometric ($L \circ L^{-1} = id \mid_{\Omega_H^{m+1}}$).

$$\begin{aligned}
\theta \wedge \widetilde{d}\alpha &= \theta \wedge d(\alpha - \theta L^{-1} d'\alpha) \\
&= \theta \wedge (d\alpha - d\theta \wedge L^{-1} d'\alpha - \theta d(L^{-1} d'\alpha)) \\
&= \theta \wedge (d\alpha - L \circ L^{-1} d'\alpha - \theta d(L^{-1} d'\alpha)) \\
&= \theta \wedge d\alpha - \theta \wedge d'\alpha = 0.
\end{aligned}$$

Also, $d\theta \wedge \widetilde{d}\alpha = 0$ follows from $d\theta \wedge \widetilde{d}\alpha = d(\theta \wedge \widetilde{d}\alpha)$. Hence $\widetilde{d}\alpha \in J^{m+1}$.

**Remark 2.7.** The big difference between $\widetilde{d}$ and $d_H$ is that $\widetilde{d}$ is associated with both $d'$ and $d''$ while $d_H$ involves with $d'$ only. The fact that the elements in $J^{m+1}$ vanish on the $H$ component suggests that $\widetilde{d}$ operates the input in the way of modifying the $\theta$ component in order to get rid of the $H$ component. Hence the Rumin's concept will be to apply $d$ on some unique modifying form $\widetilde{\alpha} = \alpha + \theta \wedge \beta \in \Omega_M^m$ of $\alpha \in I^m$ so that $\widetilde{d}\alpha = d\widetilde{\alpha} \in J^{m+1}$. One can solve that $\beta = -L^{-1}d'\alpha$.

We will provide the explicit form of $\beta$ in the Rumin's idea in the followings. Since $\widetilde{d}\alpha \in J^{m+1}$, then $0 = \theta \wedge \widetilde{d}\alpha = \theta \wedge d\widetilde{\alpha} = \theta \wedge d(\alpha + \theta \wedge \beta)$.

$$\begin{aligned}
0 &= \theta \wedge d(\alpha + \theta \wedge \beta) \\
&= \theta \wedge d'(\alpha + \theta \wedge \beta) \\
&= \theta \wedge (d'\alpha + d\theta \wedge \beta + \theta \wedge d'\beta) \\
&= \theta \wedge d'\alpha + \theta \wedge L(\beta).
\end{aligned}$$

In the result, one have $d'\alpha = L(\beta)$. Since $L : \Omega_H^{m-1} \longrightarrow \Omega_H^{m+1}$ is bijective, then $\beta = -L^{-1}d'\alpha$. This is similar to how $\widetilde{d}$ was previously defined.

## 2.2 Partial Connections along $H$ and the extensions

In this section, we will present the definitions of a partial connection, curvature of a partial connection, and the theorems relating with the full connection. In the outset, these definitions will be introduced together with how it differ from the case of full connections. The main focus of the theorems will be on how a partial connection leads to (full)extension and vice versa. Additional element will be involved in the creation of such extensions; however, This element will added in the unique way depending on the curvature of the extension. Then we will move to the next section which is about the roles of this factor.

**Definition 2.8.** Given any contact manifold $(M, \theta)$, a vector bundle $E$ over $M$, and a contact distribution $H$, the $H$-partial connection on the vector bundle $E$ over the contact manifold $M$ is the smooth mapping $D : \Gamma(E) \longrightarrow \Gamma(H^* \otimes E)$ satisfying Leibniz rule

$$D(fs) = d_H f \otimes s + fDs, \ \textit{where } d_H f = d'f.$$

In general, the curvature $F : T^2M \otimes \Gamma(E) \longrightarrow \Gamma(E)$ is defined through a connection $D$ such that $F(X,Y)(u) = [D_X, D_Y](u) - D_{[X,Y]}u$. The general definition reveals the problem on the curvature's domain of a partial connection, since any contact distribution $H$ is far from being integrable. In order to define the curvature on a partial connection without loss of its property, $[X,Y]$ must still be a section of $H$. In this case, the domain will be restricted. One defines $K' := \{X \wedge Y \in \Gamma(\bigwedge^2 H) : X, Y \in \Gamma H \text{ and } [X,Y] \in \Gamma H\}$.

**Definition 2.9.** The *curvature* $F_D$ of $D$ is defined on $K'$ such that for any $X \wedge Y \in K'$,

$$F_D(X, Y) = F_D(X \wedge Y) = [D_X, D_Y] - D_{[X,Y]}. \tag{2.5}$$

Next, we will give information about the curvature in term of section. Originally, the curvature of a full connections can be described as a section of the vector bundle $(T^2M)^* \otimes EndE$, i.e., $F \in \Gamma((T^2M)^* \otimes EndE)$. Since the curvature of a partial connection is defined on the set $K'$, this implies that the curvatures of partial connections need not to be defined over the entire cotangent bundle or even $H^*$ contact distribution. In order to gain more information, the complex will be introduced along with the co-complex version of itself.

Recall the definition of Levi form $\mu_x : H_x \times H_x \longrightarrow T_xM/H_x$ such that $\mu_x(X, Y) := [[\widetilde{X}, \widetilde{Y}]_x]$, where $\widetilde{X}, \widetilde{Y}$ are the extended vector fields of tangent vectors $X, Y$ respectively. Since the Lie bracket is skew-symmetric, we can write $\mu_x : \bigwedge^2 H_x \longrightarrow T_xM/H_x$. By the definition of $K'$, one has that $K' = \Gamma K$, where $K = ker\mu$. Therefore we can view the curvature of a partial connection as a section of $K^* \otimes EndE$, i.e., $F_D \in \Gamma(K^* \otimes EndE)$. It is trivial that the following sequence is exact $0 \longrightarrow K \longrightarrow \wedge^2 H \longrightarrow \wedge^2 H/K \longrightarrow 0$. We will take the advantage of it to explicitly reveal $K^*$ for more convenient.

**Proposition 2.10.**

$$0 \longrightarrow \langle d\theta \rangle \longrightarrow \wedge^2 H^* \longrightarrow K^* \longrightarrow 0 \tag{2.6}$$

*is exact.*

*Proof.* Since $0 \longrightarrow K \longrightarrow \wedge^2 H \longrightarrow \wedge^2 H/K \longrightarrow 0$ is an exact sequence, then the dual sequence $0 \longrightarrow (\wedge^2 H/K)^* \longrightarrow \wedge^2 H^* \longrightarrow K^* \longrightarrow 0$ is also exact. To see that

$\wedge^2 H^*/K^* =< d\theta >$ is to check that $0 \neq d\theta \in \wedge^2 H^*/K^*$, since $K^*$ has co-dimension 1. For $X \wedge Y \in K$, we have that $d\theta(X \wedge Y) = X\theta(Y) - Y\theta(X) - \theta[X,Y] = 0$. This proves $\wedge^2 H^*/K^* =< d\theta >$.

∎

Since $(\wedge^2 H/K)^* =< d\theta >$, then we canonically have $K^* = \wedge^2 H^*/ < d\theta >$. By the definition of $I^2$ over the high dimensional manifold ($\geqslant 5$), we have that $I^2 = \Gamma(\wedge^2 H^*/ < d\theta >) = \Gamma K^*$ and $F_D \in \Gamma(K^* \otimes EndE)$. For our convenience, we will slightly abuse the notation of $I^2$ by using it as a bundle instead of a set of section. Therefore we have

$$F_D \in \Gamma(I^2 \otimes EndE).$$

The reason for the convenience is that we can generalize $D$ on the higher degree forms in term of $I^r$. In the case of $dim = 3$, every partial connection is flat, i.e., $F_D = 0$ due to the fact that $K' = \emptyset$ ($H$ is far away from integrable in the contact case).

In order to gain more of the curvature's properties, we needs to generalize $D$ to $D_H$ over the higher forms. By the generalization, we will have the main property, being similar to the case of full connections $F = D^2$ which is $F_D = D_H^2$. One defines $D_H : \Gamma(I^r \otimes E) \longrightarrow \Gamma(I^{r+1} \otimes E)$ for $0 \leqslant r < m$ such that

$$D_H(\alpha \otimes u) := d_H\alpha \otimes u + (-1)^r \alpha \wedge Du,$$

where $d_H$ is the generalized version of $d$ in the previous section. Similarly, $D_H : \Gamma(J^r \otimes E) \longrightarrow \Gamma(J^{r+1} \otimes E)$ for $m < r \leqslant n$ such that

$$D_H(\alpha \otimes u) := d_H\alpha \otimes u + (-1)^r \alpha \wedge Du.$$

It can be seen from the algebraic point of view that we have a basic property $F_D =$

$D_H^2 : \Gamma(E) \longrightarrow \Gamma(I^2 \otimes E)$ for $n > 3$. For the case of $n = 3$, one needs to define $\widetilde{D}$ and investigate more on $\widetilde{D} \wedge D_H$. Next, we will focus on defining $\widetilde{D} : \Gamma(I^m \otimes E) \longrightarrow \Gamma(J^{m+1} \otimes E)$ for not only 3 dimensional contact manifold, but also 5 dimensional contact manifold. The ideas are that

1. Create a well defined connection $\widetilde{D}$ over the domain $\Gamma(I^m \otimes E)$.

2. Narrow down the range from $\Gamma(\wedge^{m+1} H^* \otimes E)$ to $\Gamma(J^{m+1} \otimes E)$.

**Method 1** This will works only on 3 dimension, one gives

$$\widetilde{D}(\alpha \otimes u) := \widetilde{d}\alpha \otimes u - \alpha \widetilde{\wedge} Du.$$

Note that $\widetilde{d}$ is defined as before, where $\alpha \widetilde{\wedge} \beta := \theta \wedge d'f$ and $\alpha \wedge \beta = f d\theta$. This is well defined since the property of 3 dimensional contact manifold allows us to have $\bigwedge^2 H^* =< d\theta >$.

**Method 2** For $n \geqslant 5$, $\widetilde{D}$ can be defined through the idea of the extension $\overline{D}$ of $D$ in the similar manner to the Rumin exterior derivative $\widetilde{d}$. This concept depends on the choices of full connections. One starts by $\widetilde{D} : \Gamma(\wedge^m H^* \otimes E) \longrightarrow \theta \wedge \Gamma(\wedge^m H^* \otimes E)$,

$$\widetilde{D}\alpha = \overline{D}(\alpha - \theta \wedge L^{-1} \overline{D}' \alpha), \tag{2.7}$$

where $\overline{D}\alpha := \overline{D}'\alpha + \theta \wedge \overline{D}''\alpha$ such that $\overline{D}'$ is the extension of $\overline{D}' : \Gamma(E) \longrightarrow \Gamma(H^* \otimes E)$ which $\overline{D}' = D$. Then we prove that $\widetilde{D}$ satisfies the previous two ideas $(1.), (2.)$. The definition and the explanations of the extensions of a partial connection will be provided before we investigate $\widetilde{D}$ further.

**Definition 2.11.** $\overline{D} : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ is called an extension of a partial connection $D$, if $\overline{D} = \overline{D}' - \theta \wedge \overline{D}''$, where $\overline{D}' : \Gamma(E) \longrightarrow \Gamma(H^* \otimes E)$ satisfies $\overline{D}' = D$ and $\overline{D}'' : \Gamma(E) \longrightarrow \Gamma(E)$.

We may extend $\overline{D}' : \Omega^r_M(E) \longrightarrow \Omega^{r+1}_H(E)$ and $\overline{D}'' : \Omega^r_M(E) \longrightarrow \Omega^r_H(E)$ knowing there is the high degree extension of $\overline{D} : \Omega^r_M(E) \longrightarrow \Omega^{r+1}_M(E)$. These can be done in such a way that

$$\overline{D}'(\alpha \otimes u) := d'\alpha \otimes u + (-1)^r \alpha' \wedge \overline{D}' u$$

for $\alpha \otimes u \in \Gamma(\bigwedge^r T^*M \otimes E)$,and

$$\overline{D}''(\alpha \otimes u) := d''\alpha \otimes u - (-1)^r \alpha'' \otimes Du + \alpha' \wedge \overline{D}'' u$$

for $\alpha \otimes u \in \Gamma(\bigwedge^r T^*M \otimes E)$.

On $\Gamma(E)$, we have that $D' = D_H = D$. On the other hand, it is invalid for the domain $\Omega^r_M(E)$ for $1 < r < m$ by that $D_H$ is defined on $\Gamma(I^r \otimes E)$ for $r \neq m$. $D_H : \Gamma(I^r \otimes E) \longrightarrow \Gamma(I^{r+1} \otimes E)$ can be recovered by projecting $D' : \Omega^r_M(E) \longrightarrow \Omega^{r+1}_H(E)$ to $\Gamma(I^{r+1} \otimes E)$. By the fact that $D'(d\theta \wedge \alpha) = d\theta \wedge D'\alpha$, we have $D' = D_H : \Gamma(I^r \otimes E) \longrightarrow \Gamma(I^{r+1} \otimes E)$ to be well defined.

**Proposition 2.12.** *The curvature of the extension $\overline{D}$, called $\overline{F}$, can be decomposed as $\overline{F} = F_H + \theta \wedge \widetilde{F}$, where $F_H \in \Gamma(\bigwedge^2 H^2 \otimes EndE)$ has the form $F_H = \overline{D}'^2 - d\theta \wedge \overline{D}''^2$ and $\widetilde{F} \in \Gamma(H^* \otimes EndE)$ satisfying $\widetilde{F} = [\overline{D}', \overline{D}'']$.*

*Proof.* By the unique decomposition 2.2, we have $\overline{F} = F_H + \theta \wedge \widetilde{F}$. Both $F_H$ and $\widetilde{F}$ can be computed explicitly by using the facts that $\overline{D} = D' + \theta \wedge D''$ and $\overline{F}(u) = \overline{D}^2(u)$.

$$\begin{aligned}
(F_H + \theta \wedge \widetilde{F})(u) = \overline{F}(u) = \overline{D}^2(u) \\
&= (\overline{D}' + \theta \wedge \overline{D}'')^2(u) \\
&= (\overline{D}'^2 + d\theta \wedge \overline{D}'' + \theta \wedge \overline{D}'\overline{D}'' + \theta \wedge \overline{D}''\overline{D}')(u) \\
&= (\overline{D}'^2 + d\theta \wedge \overline{D}'' + \theta \wedge ([\overline{D}', \overline{D}'']))(u).
\end{aligned}$$

By the uniqueness of the decomposition, we have that $F_H = \overline{D}'^2 - d\theta \wedge \overline{D}''^2$ and $\widetilde{F} = [\overline{D}', \overline{D}'']$. ∎

This tools allow us to analyze $\widetilde{D} : \Gamma(\wedge^m H^* \otimes E) \longrightarrow \theta \wedge \Gamma(\wedge^m H^* \otimes E)$. Wang [Wan21] proves the sufficient condition of the following sequence to be a sequential complex.

$$\Gamma(E) \xrightarrow{D_H} \ldots \xrightarrow{D_H} \Gamma(I^m \otimes E) \xrightarrow{\widetilde{D}} \Gamma(J^{m+1} \otimes E) \xrightarrow{D_H} \ldots \xrightarrow{D_H} \Omega^n_M(E).$$

**Lemma 2.13.** *[Wan21] $\widetilde{D} : \Gamma(\wedge^m H^* \otimes E) \longrightarrow \theta \wedge \Gamma(\wedge^m H^* \otimes E)$ defined as above satisfies the followings:*

*(i) $\widetilde{D}(d\theta \wedge \alpha) = \theta \wedge F_H(\alpha)$, for $\alpha \in \Gamma(\wedge^{m-2} H^* \otimes E)$.*

*(ii) $\theta \wedge \widetilde{D}\alpha = 0$, for $\alpha \in \Gamma(\wedge^m H^* \otimes E)$.*

*(iii) $d\theta \wedge \widetilde{D}\alpha = \theta \wedge F_H(\alpha)$, for $\alpha \in \Gamma(\wedge^m H^* \otimes E)$.*

**Theorem 2.14.** *[Wan21] $\widetilde{D} : \Gamma(\wedge^m H^* \otimes E) \longrightarrow \theta \wedge \Gamma(\wedge^m H^* \otimes E)$ defined as* (2.18) *can be descended (follow the two ideas above) to $\widetilde{D} : \Gamma(I^m \otimes E) \longrightarrow \Gamma(J^{m+1} \otimes E)$ for $n = 3$, and it required $F_H = 0$ for $n \geqslant 5$. In particular, the sequential complex is well defined for $n = 3$ and if $F_H = 0$, for $n \geqslant 5$.*

Now we can conclude that if $\overline{F} = F_H + \theta \wedge \widetilde{F}$, then $F_D = P \circ F_H$, where $P : \Gamma(\bigwedge^2 H^* \otimes E) \longrightarrow \Gamma(I^2 \otimes E)$ is the projection/quotient mapping to $\bigwedge^2 H^* / < d\theta >$ for any extension $\overline{D}$ of a partial connection $D$. The observation is that a partial connection $D$ alone can not identify $F_H$; however, there is an extra information together with a partial connection $D$ that can build the unique $F_H$. The investigation of this will be shown later. Moreover, $\widetilde{F}$ is dependent on $F_H$ alone. One can find the unique full extension $\overline{D}$ of $D$ from $D$ and $F_H$ in such a way that $\overline{F} = F_H + \theta \wedge \widetilde{F}$. The detail will be provided as followings.

**Proposition 2.15.** *For a given partial connection $D$ and any lifting $F' \in \Omega^2_H(EndE)$ of $F_D \in \Gamma(I^2 \otimes EndE)$, i.e., $F' = F_D$ under the quotient $\bigwedge^2 H^* \longrightarrow I^2$, there is a unique full connection $\overline{D}$ extending $D$ such that $F_H = F'$. In particular, $(D, F_H)$ determines $\widetilde{F}$ uniquely.*

*Proof.* The proof will be based on the local picture. Given any basis of E, one then have that $\overline{D}' = D = d' + A$ and $\overline{D}'' = d'' - X$ where $A \in \Gamma(H^* \otimes EndE)$ and $X \in \Gamma(EndE)$. This means locally $\overline{D} = d + A + \theta \wedge X$ and

$$
\begin{aligned}
\overline{F} &= d(A + \theta \wedge X) + (A + \theta \wedge X) \wedge (A + \theta \wedge X) \\
&= dA + d\theta \wedge X - \theta \wedge dX + A^2 + \theta \wedge [X, A] \\
&= (d'A + A^2 + d\theta \wedge X) + \theta \wedge (d''A - dX + [X, A]).
\end{aligned}
$$

Hence the lifting $F'$ of $F_D$ determines $X$ in the first term. By the above picture of $\overline{F}$, $\widetilde{F}$ is uniquely determined by $A, X$ which $A$ comes from $D$ and $X$ comes from $D, F_H$. Then there is a full connection extending $D$ such that $F_H = F'$. ∎

**Note 2.16.** $F_H = d'A + A^2 + d\theta \wedge X$ does not guarantee that $F_D = d'A + A^2$, since this might consist the term in $< d\theta >$.

**Corollary 2.17.** *For dim $\geqslant 5$,*

*(i)* $\overline{F} = 0$ *if and only if* $F_H = 0$ *for any full connection* $\overline{D}$.

*(ii) there is a one-to-one natural correspondence between flat partial connections and flat full connections.*

*Proof.* For $(i)$, the necessary condition is obvious. Hence we will simply focus on the converse part. This can be done by using the Bianchi identity over a full connection, $\overline{D}(\overline{F}) = 0$. Suppose $F_H = 0$,

$$
\begin{aligned}
0 &= \overline{D}(\overline{F}) \\
&= \overline{D}(0 + \theta \wedge \widetilde{F}) \\
&= d\theta \wedge \widetilde{F} - \theta \wedge \overline{D}(\widetilde{F}).
\end{aligned}
$$

Since $dim \geqslant 5$, then $d\theta \wedge \widetilde{F} \in \Gamma(\bigwedge^3 H^* \otimes EndE)$ will not always be zero. Then $\widetilde{F} = 0$ from the first term and the fact that $d\theta \wedge \widetilde{F} \in \Gamma(\bigwedge^3 H^* \otimes EndE)$. It implies that $\overline{F} = 0$ if $F_H = 0$. For $(ii)$, one can apply from proposition 2.15 together with $(i)$. For a flat partial connection $D$, there is the only way to lift $F_D = 0$ to $F_H = 0$. Then $\overline{F} = 0$ by $(i)$. For a flat full connection, it can be extracted to a unique partial connection $D$ such that $F_D = 0$ by the fact that $F_H = 0$. ∎

**Proposition 2.18.** *For* $\widetilde{D} : \Gamma(I^m \otimes E) \longrightarrow \Gamma(J^{m+1})$ *such that* $\widetilde{D}\alpha = \overline{D}(\alpha - \theta \wedge L^{-1}\overline{D}'\alpha)$ *is equivalently to*

$$\widetilde{D}\alpha = \theta \wedge (\overline{D}' L^{-1} \overline{D}' \alpha - \overline{D}'' \alpha)$$

*Proof.*

$$
\begin{aligned}
\widetilde{D}\alpha &= \overline{D}(\alpha - \theta \wedge L^{-1}\overline{D}'\alpha) \\
&= \overline{D}\alpha - d\theta \wedge L^{-1}\overline{D}'\alpha + \theta \wedge \overline{D} L^{-1}\overline{D}'\alpha \\
&= \overline{D}\alpha - (L \circ L^{-1})(\overline{D}'\alpha) + \theta \wedge \overline{D}' L^{-1}\overline{D}'\alpha \\
&= -\theta \wedge \overline{D}''\alpha + \theta \wedge \overline{D}' L^{-1}\overline{D}'\alpha \\
&= \theta \wedge (\overline{D}' L^{-1}\overline{D}'\alpha - \overline{D}''\alpha).
\end{aligned}
$$

∎

Since $(D, F_H)$ defines a unique extension $\overline{D}$ and $F_H$ consists the information of $D$, then it is possible to find another independence set of ingredients instead of $(D, F_H)$ to determine $\overline{D}$. The idea is choosing a metric on $\bigwedge^2 H^*$ that $F_H$ can be uniquely decomposed to $F_D + Bd\theta$, for some $B \in EndE$. With $D$ and $B$, one can recover $F_H$ from $F_H = F_D + Bd\theta$. Then we have the pair $(D, B)$ instead of $(D, F_H)$. Next, we will investigate the propositions about the relations between $D$ and $B$ when one wants to extend to some unique full connection. Before we move to this part, the case of 3 dimension will be introduced in order to express the idea of the simple scenario. In the case of $dim = 3$, we have $F_H = d\theta \otimes B$ since $F_D = 0$.

**Proposition 2.19.** *Let $D$ be a partial connection on a contact manifold $M$ with dimension 3. Write $F_H = d\theta \otimes B$ for a unique $B \in EndE$. In the sequence*

$$\Gamma(E) \xrightarrow{D_H} \Gamma(I^1 \otimes E) \xrightarrow{\widetilde{D}} \Gamma(J^2 \otimes E) \xrightarrow{D_H} \Omega^3_M(E).$$

*or*

$$\Gamma(E) \xrightarrow{D_H} \Gamma(H^* \otimes E) \xrightarrow{\widetilde{D}} \Gamma(\theta \wedge H^* \otimes E) \xrightarrow{D_H} \Gamma(\bigwedge^3 T^*M \otimes E),$$

*The following are true.*

*(i)* $(\widetilde{D} \circ D_H)(u) = \theta \wedge (\widetilde{F}(u) + D(Bu)))$, *for* $u \in \Gamma(E)$.

*(ii)* $(D_H \circ \widetilde{D})(u') = \theta \wedge (\widetilde{F}(u') - B(\overline{D}'u'))$, *for* $u' \in \Gamma(H^* \otimes E)$.

*Proof.* The propositions (2.18), (2.12), and the fact that $\overline{D}' = D_H = D$ on $\Gamma(E)$ allow straightforward computation.

$$(\widetilde{D} \circ D_H)(u) = \widetilde{D}(\overline{D}'u)$$

$$= \theta \wedge (\overline{D}' L^{-1} \overline{D}' \overline{D}' u - \overline{D}'' \overline{D}' u)$$

$$= \theta \wedge (\overline{D}' L^{-1} (F_H + d\theta \wedge \overline{D}'')(u) - \overline{D}'' \overline{D}' u)$$

$$= \theta \wedge (\overline{D}' L^{-1} (B d\theta + d\theta \wedge \overline{D}'')(u) - \overline{D}'' \overline{D}' u)$$

$$= \theta \wedge (\overline{D}' (B + \overline{D}'')(u) - \overline{D}'' \overline{D}' u)$$

$$= \theta \wedge (\overline{D}' (Bu) + [\overline{D}', \overline{D}''](u))$$

$$= \theta \wedge (D(Bu) + \widetilde{F}(u)),$$

$$\begin{aligned}
(D_H \circ \widetilde{D})(u') &= D_H(\theta \wedge (\overline{D} L^{-1} \overline{D}' u' - \overline{D}'' u')) \\
&= d_H \theta \wedge (\overline{D} L^{-1} \overline{D}' u' - \overline{D}'' u') - \theta \wedge D_H(\overline{D}' L^{-1} \overline{D}' u' - \overline{D}'' u') \\
&= -\theta \wedge D_H(\overline{D}' L^{-1} \overline{D}' u' - \overline{D}'' u') \\
&= -\theta \wedge \overline{D}' (\overline{D}' L^{-1} \overline{D}' u' - \overline{D}'' u') \\
&= -\theta \wedge (B d\theta + d\theta \wedge \overline{D}'')(L^{-1} \overline{D}' u') - \overline{D}' \overline{D}'' u') \\
&= -\theta \wedge ((B + \overline{D}'')(L \circ L^{-1})(\overline{D}' u') - \overline{D}' \overline{D}'' u') \\
&= -\theta \wedge (B(\overline{D}' u') + \overline{D}'' (\overline{D}' u) - \overline{D}' \overline{D}'' u') \\
&= -\theta \wedge (B(\overline{D}' u') - [\overline{D}', \overline{D}''] u') \\
&= \theta \wedge (\widetilde{F}(u') - B(\overline{D}' u')).
\end{aligned}$$

∎

**Remark 2.20.** In the higher dimensional case, the metric $g^H$ on $H$ is needed in such a way that $g^H$ induces the metric $g^H$ on $\bigwedge^2 H^*$ and that $K^*$ and $d\theta$ are orthogonal, i.e., $\bigwedge^2 H^* = K^* \oplus < d\theta >$ or saying $< d\theta >^{\perp} = K^*$. In 3 dimensional case, there is no need to pick such a metric on $H$ because $K^* = 0$ or $\bigwedge^2 H^* =< d\theta >$. One can write $F_H = F_D + B \otimes d\theta$ if such metric is given.

## 2.3 Descriptions of a full connection $\overline{D}$ in term of a partial connection $D$ and a $B \in EndE$

In 3 dimensional case, one can find $B$ explicitly from the full connection by considering the term $F_H = d\theta \otimes B$. Moreover, it is independent of the partial connection, e.g., $B$ does not change if $D$ changes and it is conversely true. In this section, the term of $B$ will be introduced. The case of flat connection will be represented along with the condition on a partial connection and $B \in EndE$. Later in the section, the impact of decomposing a full connection into $D, B$ will be exhibited in some special cases.

**Proposition 2.21.** *For a given metric $g^H$ on $H$ over a contact manifold $M$ with dim $\geqslant 5$, a partial connection $D$, and $B \in EndE$, $(D, B)$ can determine a unique full connection $\overline{D}$ such that $\overline{D}$ is an extension of $D$ and $F_H$ of $\overline{D}$ is $F_D + B \otimes d\theta$.*

*Proof.* The proof is directly from the orthonormal decomposition of $\bigwedge^2 H^*$ by the metric $g^H$. ∎

Note that the case of 3 dimensional contact manifold needs no metric on $\bigwedge^2 H^*$ in order to satisfy proposition (2.21).

**Proposition 2.22.** *For a given metric $g^H$ on $H$ over a contact manifold $M$ with dim $\geqslant 5$, a partial connection $D$, and $B \in EndE$, we set the induced full connection $\overline{D} = \overline{D}(D, B)$ from the previous proposition* (2.21). *Then*

*(i) $\overline{D}$ is flat if and only if $D$ is flat and $B = 0$.*

*(ii) If $D$ is flat, then $\widetilde{F} = -DB$*

*Proof.* For $(i)$, we know that having $D$ is flat and $B = 0$ is equivalently having $F_H = 0$. By corollary 2.17, one can get the result of $(i)$. For $(ii)$, one applies Bianchi identity.

$$0 = \overline{DF} = d\theta \wedge \overline{D}B + d\theta \wedge \widetilde{F} - \theta \wedge \overline{D}\widetilde{F}$$

$$= d\theta \wedge (DB + \widetilde{F}) - \theta \wedge (d\theta \otimes \overline{D}''B + \overline{D}\widetilde{F}).$$

Since dimension is greater than 5, then we have $DB + \widetilde{F} = 0$, or $\widetilde{F} = -DB$. ∎

**Proposition 2.23.** *For a high dimensional contact manifold $M$ with chosen metric $g^H$ on $H$ and a 3 dimensional contact manifold with no requirement of $g^H$, If $D$ is flat then $D\widetilde{F} = d\theta \otimes \overline{D}''B$*

*Proof.* By the previous proof of the proposition (2.22), we have

$$0 = \theta \wedge (d\theta \otimes \overline{D}''B + \overline{D}\widetilde{F})$$

$$= \theta \wedge (d\theta \otimes \overline{D}''B + \overline{D}'\widetilde{F} + \theta \wedge \overline{D}''\widetilde{F})$$

$$= \theta \wedge (d\theta \otimes \overline{D}''B + D\widetilde{F}).$$

Since $d\theta \otimes \overline{D}''B + D\widetilde{F} \in \bigwedge^2 H^*$, then $0 = d\theta \otimes \overline{D}''B + D\widetilde{F}$ or $d\theta \otimes \overline{D}''B = D\widetilde{F}$. ∎

**Corollary 2.24.** *Let a partial connection $D$ be flat over a contact manifold $(M, g^H)$ with dim $\geqslant 5$. $\widetilde{F} = 0$ if and only if $B$ is a flat section with respect to $D$. In addition, $\widetilde{F} = 0$ for the extension induced by any linear mapping $B \in EndE$, i.e., $B = \lambda I$ where $I$ is identity and $\lambda$ is a constant by a local point of views.*

*Proof.* The proof are straightforward from (2.22) and that $DB = 0$ as a flat section with respect to $D$. Since $DB = 0$ for any $B$ a constant section, then $\widetilde{F} = 0$. ∎

One of the reasons for considering $(D, B)$ is that $B \in EndE$ is an independent variable with any partial connection $D$ which is a huge different than the case of extension $\overline{D}$ induced by $D$ and $F_H$. One can write $\overline{D}(D, B)$ for easy recall of the introduced full connection $\overline{D}$ by two independent variables. This works for all contact manifolds with dimension greater or equal to 3; however, it still requires the metric on $H$ in the manifold with $dim \geqslant 5$ in order to orthogonally decompose $< d\theta > \oplus < d\theta >^{\perp}$. One takes advantage of considering $(D, B)$ by finding an explicit relation between a flat partial connection and its flat extension. Moreover, the curvatures $\overline{F}_{(D,B_1)}$ and $\overline{F}_{(D,B_2)}$ can be delicately compared by focusing on $B$ part.

**Proposition 2.25.** *Let $D$ be a partial connection such that its local is $D = d' + A$ and $\overline{D}(D, B) = d + A + \theta \otimes X$. Then*

*(i)* $X = B - P(d'A + A \wedge A)$, *where $P$ projects to the coefficient of $d\theta$,*

*(ii)* $\widetilde{F} = d''A - d'X + [X, A]$.

*Proof.* By considering the curvature $\overline{F}$,

$$\begin{aligned}
F_H - \theta \wedge \widetilde{F} = \overline{F} & \\
&= (d + A + \theta \otimes X)(d + A + \theta \otimes X) \\
&= dA + A \wedge A + d(\theta \otimes X) + \theta \wedge [A, X] \\
&= (d'A + A \wedge A + d\theta \otimes X) - \theta(-dX + d''A - [X, A]).
\end{aligned}$$

Since $F_H = F_D + Bd\theta$, then $B = X + P(d'A + A \wedge A)$. Also, we get from the above that $\widetilde{F} = d''A - d'X + [X, A]$. ∎

**Proposition 2.26.** *For a partial connection $D$,*

$$\widetilde{F}_{(D,B_1)} = \widetilde{F}_{(D,B_2)} + D(B_2 - B_1).$$

*Proof.* This can be done directly using the proposition (2.25). We denote $\overline{D}(D, B_1) = d + A + \theta \otimes X_1$ and $\overline{D}(D, B_2) = d + A + \theta \otimes X_2$. By $(i)$ in the proposition (2.25), we can conclude that

$$X_2 - X_1 = B_2 - B_1. \tag{2.8}$$

Using the equation (2.8) with the proposition $(ii)$ in (2.25), it yields

$$\begin{aligned}
\widetilde{F}_{(D,B_1)} &= (d''A - d'X_1 + [X_1, A]) \\
&= (d''A - d'X_2 + [X_2, A]) + d'(X_2 - X_1) - [B_2 - B_1, A] \\
&= \widetilde{F}_{(D,B_2)} + d'(B_2 - B1) - [B_2 - B_1, A] \\
&= \widetilde{F}_{(D,B_2)} + D(B_2 - B_1).
\end{aligned}$$

This proves the proposition. ∎

## 2.4 Hodge star operator $*'$ on a $\bigwedge^r H^*$

In this section, we will introduce a Hodge star operator $*'$ on a contact distribution $H$ over a metric $g$. The $L^2$ adjoint of a full connection $\overline{D}$ will be investigated in term of $*'$. These will be done through the lens of the fiberwise method. On a manifold with the almost contact metric structure $(M^{2m+1}, \theta, J, g)$,where $g(X, JY) = d\theta(X, Y)$. The metric $g$ is a compatible metric over $H$, i.e., $g(X, Y) = g(JX, JY)$ for $X, Y \in H$. This compatibility implies that $g(X, JX) = 0$. Hence one can choose the local orthonormal basis $\{\alpha_1^*, \beta_1^*, ..., \alpha_m^*, \beta_m^*\}$ of $H^*$ in such a way that $\beta_i^* = J\alpha_i^*$. In contact manifold, there is a well-known local coordinate $(x_1, y_1, ..., x_m, y_m, z)$ satisfying $dz = \Sigma dx_i \wedge dy_i$. We will take the advantage of both properties into the algebraic points of view. For the convenience, we will abuse the notation by writing $\alpha, \beta$ instead of $\alpha^*, \beta^*$.

**Proposition 2.27.** *Let $(M^{2m+1}, \theta)$ be a contact manifold with the associated metric $g$ such that $g(X, JY) = d\theta(X, Y)$, then there is an fiberwise orthonormal basis $\{\alpha_1, \beta_1, ..., \alpha_m, \beta_m, \theta\}$ of $T_xM$ such that*

*(i)* $\beta_i = J\alpha_i$,

*(ii)* $d\theta = \Sigma\alpha_i^* \wedge \beta_i^*$.

*Proof.* Since $g$ is a compatible metric over $H$, then we can choose the fiberwise orthonormal basis $\{\alpha_1, \beta_1, ..., \alpha_m, \beta_m\}$ of $H$ such that $\beta_i = J\alpha_i$. By the construction of $g$, we have that $d\theta(u, v) = -g(-Ju, v)$. Hence $d\theta = \Sigma\alpha_i^* \otimes \beta_i^* - \beta_i^* \otimes \alpha_i^* = \Sigma\alpha_i^* \wedge \beta_i^*$. ∎

This is the case of fiberwise space, not over the local picture. For the local case or local basis section, it turns out that the proposition (2.27) is not obvious. All of the computations will be computed algebraically in tangent vector spaces. Now, let's recall the definition of hodge star operator on a vector space $V$.

**Definition 2.28.** For a given vector space $V$ with the orthonormal (order) basis $\{e_1, ..., e_n\}$ and the inner product $<, >$, the Hodege $*$-operator is a linear mapping $* : \bigwedge^r V \longrightarrow \bigwedge^{n-r} V$ for any $r = 1, ..., n$ such that $\alpha \wedge *\beta :=< \alpha, \beta > vol$, where $<, >$ is the induced on $\bigwedge^r V$ and $vol = e_1 \wedge e_2 \wedge ... \wedge e_n$.

**Remark 2.29.** $*^2 = (-1)^{r(n-r)}$.

**Proposition 2.30.** *Let $(M^{2m+1}, \theta, J, g)$ be a contact manifold and the Hodge$-*'$ operator on $\Gamma(\bigwedge^r H^*)$. Then*

*(i)* $*'d\theta = \frac{d\theta^{m-1}}{(m-1)!}$,

*(ii)* $*'1 = \frac{d\theta^m}{m!}$.

*Proof.* The proof will be done over the algebraic picture in order to use the proposition (2.27). Since $d\theta = \Sigma_{i=1}^m \alpha_i^* \wedge \beta_i^*$, then $d\theta^r = r!\Sigma\gamma_{i_1} \wedge ... \wedge \gamma_{i_r}$, where $\gamma_i = \alpha_i^* \wedge \beta_i^*$. Hence we have that $d\theta^m = m! vol$. This means $vol = \frac{d\theta^m}{m!}$, since $vol = \gamma_1 \wedge ... \wedge \gamma_m$. This proves $(ii)$. In order to see $(i)$, one can derive from the fact that $g(d\theta, d\theta) = m$. This leads us to $d\theta \wedge *'d\theta = g(d\theta, d\theta)vol = \frac{md\theta^m}{m!}$. This implies $*'d\theta = \frac{d\theta^{m-1}}{(m-1)!}$. ∎

**Note 2.31.** $J$ in the almost contact metric structure can be extended to $J$ over $TM$ by $J^2 = -I + \theta \otimes \xi$, where $\theta(\xi) = 1$. Moreover $g$ can also be extended to $TM$ by $g(X, \xi) = \theta(X)$ for $X \in \Gamma(TM)$. Here one have the oriented orthonormal basis $\{\alpha_1, \beta_1, ..., \alpha_m, \beta_m, \theta\}$ of $T^*M$.

For the convenience, we abuse the notation of $\alpha_i^*, \beta_i^*$ by writing $\alpha_i, \beta_i$ instead. Now we also have the Hodge $*$-operator on $T^*M$. The next proposition will show the relationship between $*$ and $*'$.

**Proposition 2.32.** *Let $*$ be the Hodge star operator over $\bigwedge^r T^*M$ and $*'$ the Hodge star operator on $\bigwedge^r H*$. The followings are true.*

*(i)* $* : \bigwedge^r H^* \longrightarrow \theta \wedge (\bigwedge^{2m-r} H^*)$ *is given by* $*\alpha = (-1)^r \theta \wedge *'\alpha$.

*(ii)* $* : \theta \wedge (\bigwedge^{r-1} H^*) \longrightarrow \bigwedge^{2m-r+1=n-r} H^*$ *is given by* $*(\theta \wedge \alpha) = *'\alpha$.

*In particular,* $*d\theta = \frac{\theta \wedge d\theta^{m-1}}{(m-1)!}$.

*Proof.* First, we observe that $vol = \gamma_1 \wedge ... \wedge \gamma_m \wedge \theta$ for $T^*M$. By the definition of hodge star operator, we have that

$$\alpha \wedge *\alpha = g_{\bigwedge^r T^*M}(\alpha, \alpha) vol_{T^*M} \ , for\ \alpha \in \bigwedge^r T^*M \tag{2.9}$$

$$\alpha \wedge *'\alpha = g_{\bigwedge^r H^*}(\alpha,\alpha) vol_{H^*} \ , for \ \alpha \in \bigwedge^r H^*, \tag{2.10}$$

where $vol_{T^*M} = vol_{H^*} \wedge \theta$. For $(i)$, one can wedge $\theta$ on the right of the equation 2.10 then apply 2.9 along the way,

$$\begin{aligned}
\alpha \wedge (*'\alpha) \wedge \theta &= g_{\bigwedge^r H^*}(\alpha,\alpha) vol_{H^*} \wedge \theta \\
&= g_{\bigwedge^r T^*M}(\alpha,\alpha) vol_{T^*M} \\
&= \alpha \wedge *\alpha
\end{aligned}$$

Hence $*\alpha = (-1)^r \theta \wedge *'\alpha$. Note that $g_{\bigwedge^r H^*}(\alpha,\alpha) = g_{\bigwedge^r T^*M}$ is due to $\alpha \in \bigwedge^r T^*M$. Similar to the proof of $(i)$, one can prove $(ii)$ by starting from the equation 2.9 then apply 2.10 along the way,

$$\begin{aligned}
(\theta \wedge \alpha) \wedge *(\theta \wedge \alpha) &= g_{\bigwedge^r T^*M}(\theta \wedge \alpha, \theta \wedge \alpha) vol_{T^*M} \\
&= g_{\bigwedge^r H^*}(\alpha,\alpha) vol_{H^*} \wedge \theta \\
&= \alpha \wedge (*'\alpha) \wedge \theta \\
&= \theta \wedge \alpha \wedge *'\alpha
\end{aligned}$$

Hence $*(\theta \wedge \alpha) = *'\alpha$. Since the degree of $\alpha \wedge *'\alpha$ is even, we can move $\theta$ from the right to the left in the last line. To compute $*d\theta$, we apply the result $(i)$. Since $d\theta \in \bigwedge^2 H^*$, then $*d\theta = (-1)^2 \theta \wedge *'d\theta = \frac{\theta \wedge d\theta^{m-1}}{(m-1)!}$. ∎

One of the advantages of the Hodge $*$-operator is that it can be applied to the adjoint operator of a vector space, or even a vector bundle. The goal of the story is to determine the adjoint operator of the full connection over $\bigwedge^r T^*M$ in term of Hodge $*'$-operator and its partial connection.

**Proposition 2.33.** *For a given contact metric manifold* $(M^{2m+1}, \theta, J, g)$*, let* $L : \bigwedge^r H^* \longrightarrow \bigwedge^{r+2} H^*$ *be a linear operator defined by* $L(\alpha) = d\theta \wedge \alpha$*. Then the pointwise adjoint operator* $L^* : \bigwedge^{r+2} H^* \longrightarrow \bigwedge^r H^*$ *is determined by*

$$L^* = *'L*'$$

*Proof.* Let $\alpha \in \bigwedge^r H^*$ and $\beta \in \bigwedge^k H^*$. Then

$$\begin{aligned} g(L\alpha, v)vol_{H^*} &= L\alpha \wedge *'\beta \\ &= d\theta \wedge \alpha \wedge *'\beta \\ &= (-1)^2 \alpha \wedge d\theta \wedge *'\beta \\ &= \alpha \wedge *'(*'L *' \beta) \\ &= g(\alpha, (-1)^{(2m-r)r}(*'L*')\beta)vol_{H^*}. \end{aligned}$$

Hence $L^* = (-1)^r *' L*'$. ∎

We will denote $L^*$ by $\Lambda$ for the convenience.

**Lemma 2.34.** *For a given contact metric manifold* $(M^{2m+1}, \theta, J, g)$*, the adjoint of* $L$*,* $\Lambda : \bigwedge^2 H^* \longrightarrow \bigwedge^0 H^* (= C^\infty(M))$ *satisfies*

$$ker\Lambda =< d\theta >^\perp .$$

*Proof.* We know that $g(L\alpha, \beta) = g(\alpha, \Lambda\beta)$ pointwisely for $\alpha \in \bigwedge^0 H^*$, $\beta \in \bigwedge^2 H^*$ and also $L(\cdot) = d\theta \wedge (\cdot)$. Then we have that $\beta \in < d\theta >^\perp$ if and only if $g(\alpha, \Lambda\beta) = (L\alpha, \beta) = 0$. This implies $ker\Lambda =< d\theta >^\perp$ . ∎

**Proposition 2.35.** *For a given contact metric manifold* $(M^{2m+1}, \theta, J, g)$ *and a metric* $g^H$ *on* $H$ *as the set up in the remark* (2.20)*, we have that* $\Lambda\alpha = g^H(\alpha, d\theta)$ *for* $\alpha \in \bigwedge^2 H^*$*.*

*Proof.* Decompose $\alpha = g^H(\alpha, d\theta)d\theta + \beta$, where $\beta \in < d\theta >^{\perp}$. By the lemma 2.34,

$$\begin{aligned}
\Lambda(\alpha) &= \Lambda g^H(\alpha, d\theta)d\theta + \beta) \\
&= \Lambda(g^H(\alpha, d\theta)d\theta) \\
&= (-1)^2 *' L *' (g^H(\alpha, d\theta)d\theta) \\
&= g^H(\alpha, d\theta) *' L *' d\theta \\
&= g^H(\alpha, d\theta) *' d\theta \wedge \frac{d\theta^{m-1}}{(m-1)!} \\
&= g^H(\alpha, d\theta) *' \frac{d\theta^{m}}{(m-1)!} \\
&= g^H(\alpha, d\theta).
\end{aligned}$$

∎

With the property of $\Lambda$, we can easily identify $B$ from the full connection $\overline{D}$ : $\Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$. This can be achieved by taking $\Lambda$ operator over the $F_H$ part of the curvature $\overline{F}$ of $\overline{D}$.

**Corollary 2.36.** *Given a contact manifold $M$ and $\overline{D}(D, B)$ the full connection induced by $D, B$ as previously described. Then*

$$\Lambda(F_H) = B.$$

In the following context, we will use $\Lambda$ as the operator on the domain $\bigwedge^2 H^*$. Hence one may extend naturally to $\Lambda : \bigwedge^2 T^*M \longrightarrow \bigwedge^0 H^*$ and still preserve the same property by $\Lambda(\bigwedge^2 T^*M \big/ \bigwedge^2 H^*) := 0$. We will often use this notation in the next chapter.

Next, we consider the case of $L^2$ adjoint of the connection $D : \Gamma(\bigwedge^r T^*M \otimes E) \longrightarrow \Gamma(\bigwedge^r r + 1T^*M \otimes E)$ over a contact manifold $(M, \theta, J, g)$. One defines the $L^2$ inner product $<,>$ on $\Gamma(\bigwedge^r T^*M \otimes E)$ by

$$< \alpha \otimes s_1, \beta \otimes s_2 >:= \int_M \overline{g}(\alpha_x \otimes s_1(x), \beta_x \otimes s_2(x)) vol,$$

where $\overline{g}$ is the metric on $\bigwedge^r T^*M \otimes E$ and $\alpha \otimes s_1, \beta \otimes s_2 \in \Gamma(\bigwedge^r T^*M \otimes E)$. The metric $\overline{g}$ here is defined by $\overline{g}(\alpha_x \otimes s_1(x), \beta_x \otimes s_2(x)) := g(\alpha_x, \beta_x) < s_1(x), s_2(x) >$, where $< s_1(x), s_2(x) >$ is the inner product on $E$. The Hodge star operator is originally defined over differential forms; however, one can extended to the domain $\Gamma(\bigwedge^r T^*M \otimes E)$ by

$$*(\alpha \otimes s) := (*\alpha) \otimes s,$$

for $\alpha_x \in \bigwedge^r T^*_x M, s_x \in E_x$. Moreover this also works for $*'$ over $\Gamma(\bigwedge^r H^* \otimes E)$. With this extension, It is easily to verify that $D^* = (-1)^r * D*$ for a connection $D : \Omega^{r-1}_M(E) \longrightarrow \Omega^r_M(E)$

**Proposition 2.37.** *Let $\overline{D}^*$ be the $L^2$ adjoint connection of a connection $\overline{D}$, $\overline{D}^* = (-1)^r * \overline{D} * : \Omega^r_M(E) \longrightarrow \Omega^{r-1}_M(E)$. Also $D^* = - *' D *'$ for a partial connection. Then*

*(i) the restriction of $\overline{D}^*$ to $_{\Omega^r_H(E)}$ is that $\overline{D}^*\alpha = D^*\alpha + \theta \wedge \Lambda\alpha$,*

*(ii) the restriction of $\overline{D}^*$ to $\theta \wedge \Omega^{r-1}_H(E)$ is that $\overline{D}^*(\theta \wedge \alpha) = (-1)^k \overline{D}''^* \alpha - \theta \wedge D^*\alpha$, where $\overline{D}''^* = - *' \overline{D}''^*$.*

*Proof.* For $(i), (ii)$, these can be computed directly using the proposition (2.32). Let $\alpha \in \Gamma(\Omega^r_H(E))$.

$$\begin{aligned}
\overline{D}^*\alpha &= (-1)^r * \overline{D} * \alpha \\
&= (-1)^r * \overline{D}((-1)^r \theta \wedge *'\alpha) \\
&= *(d\theta \wedge *'\alpha + \theta \wedge D(*'\alpha))
\end{aligned}$$

$$= (-1)^r *' d\theta \wedge *'\alpha + *'D *' \alpha$$

$$= \Lambda\alpha + D^*\alpha.$$

Hence $(i)$ is proved. Now consider $\theta \wedge \alpha$, where $\alpha \in \Gamma(\Omega_H^{r-1}(E))$.

$$\overline{D}^*(\theta \wedge \alpha) = (-1)^r * \overline{D} * (\theta \wedge \alpha)$$

$$= (-1)^r * \overline{D} *' \alpha$$

$$= (-1)^r * (D' *' \alpha - \theta \wedge D'' *' \alpha)$$

$$= \theta \wedge *'D *' \alpha - (-1)^r *' D'' *' \alpha$$

$$= -\theta \wedge D^*\alpha + (-1)^r D''^*\alpha.$$

Hence $(ii)$ is proven. ∎

# Chapter 3

# Applications to Tanaka-Webster connections and Tanaka canonical connections

In this Chapter, we study the alternative construction of Tanaka-Webster connection and Tanaka canonical connection on the strongly pseudoconvex CR manifold using the technique of decomposition over contact distribution $H$ which is introduced in the previous chapter, i.e., $T^*M = H^* \oplus < \theta >$. Recall that the full connection $\overline{D} : \Gamma(E) \longrightarrow \Gamma(TM \otimes E)$ is uniquely determined by $H-$partial connection $D : TM \longrightarrow H^* \otimes TM$ and the independent term $B \in EndE$. In the original definitions [Tan75], Tanaka-webster connection and Tanaka canonical connection are defined as a full connection with conditions. The main difference is that we will start by creating the canonical partial connection and then find the suitable $B \in EndE$ in such a way that the axioms of Tanaka's versions are all satisfied. We will recall many of the following definitions: CR manifold, strongly pseudoconvex CR manifold, Tanaka-Webster connection in the Tanaka version, and Tanaka canonical connection in the Tanaka version. For the convenience, we write $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ and $T^{\mathbb{C}}M = T^*M \otimes \mathbb{C}$.

**Definition 3.1.** Greenfield [Gre68] For a given $M$ a $n$-dimensional smooth manifold and $\mathcal{H}$ a $l$-dimensional complex subbundle of $T^{\mathbb{C}}M$, $(M, H)$ is called a CR manifold of real dimension $n$ and CR-dimension $l$ if $\mathcal{H}_x \cap \overline{\mathcal{H}_x} = 0$ for all $x \in M$ and $\mathcal{H}$ is involutive, i.e., $[X, Y] \in \mathcal{H}$, $\forall X, Y \in \mathcal{H}$.

**Note 3.2.** There will be the unique $2l$ dimensional subbundle $H \subset TM$ such that $H^{\mathbb{C}} = \mathcal{H} \oplus \overline{\mathcal{H}}$ and the unique almost complex structure $I$ on $H$, i.e., $I : H \longrightarrow H$ and $I^2 = -i_d$, for any CR manifold $(M, \mathcal{H})$. Moreover, $\mathcal{H} = \{X - iIX \mid X \in H\}$. In the case of Tanaka, the CR manifold will be an orientable connected CR manifold with a real dimension $2m + 1$ and a CR-dimension $m$. We will refer to this when the CR manifold is mentioned from now on. Define $E_x := \{f \in T_x^*M \mid f(H_x) = 0\}$. Since $H_x$ is $2m$ dimensional vector space , then $E$ is a line bundle over $M$. $E$ is a trivial bundle by the fact that $M$ is orientable connected. Hence There is a nowhere-vanishing $\theta \in \Gamma(T^*M)$ such that $H \in ker(\theta)$.

**Definition 3.3.** $(M, H, \theta)$ is called a strongly pseudoconvex CR manifold if the Levi form,

$$L_\theta(X, Y) := -d\theta(X, IY), \quad X, Y \in H$$

is positive definite, i.e., $L_\theta(X, X) > 0$ for $0 \neq X \in H_x$, $\forall x \in M$.

Now, if $(M, H, \theta)$ is a strongly pseudoconvex CR manifold, then $d\theta_x$ contains $\Sigma_i a_i e_i^* \wedge (Je_i)^*$, where $\{e_i, Je_i\}_{i=1}^m$ is an orthonormal basis of $H_x$ and $a_{ii} \neq 0$. Hence the existence of natural volume $\theta \wedge (d\theta)^m$ is guaranteed. One can obtain that strongly pseudoconvex CR manifold $(M, H, \theta)$ is a contact manifold with a contact form $\theta$. We will denote $\xi$ as a Reeb vector field of the strongly pseudoconvex CR manifold $(M, H, \theta)$. Define $J$ as the extension of $I : H \longrightarrow H$ such that $J^2 = -i_d + \theta \otimes \xi$ and $g$ a canonical Riemannian metric over $M$ defined by $g_\theta(X, Y) = L_\theta(X, Y), g_\theta(\xi, X) = 0,$

and $g_\theta(\xi, \xi) = 1$, for $X, Y \in H$. One can write $g_\theta(X, Y) = -d\theta(X, Y) + \theta(X)\theta(Y)$, for $X, Y \in TM$. With all arguments, a strongly pseudoconvex CR manifold $(M, H, \theta)$ carries a contact metric structure $(\theta, \xi, J, g)$. Therefore, we may write $(M, H, \theta, J, g)$ or $(M, H, \xi, J, g)$ for any strongly pseudoconvex CR manifold $M$.

## 3.1 Tanaka-Webster connections (Real differential forms)

This section provides the original real version of the Tanaka-Webster connection and the alternative construction trough a partial connection and $B \in EndE$.

**Definition 3.4.** Tanaka-Webster connection, induced in Tanaka book [Tan75] is an unique covariant derivative $\nabla$ on the strongly pseudoconvex CR manifold $(M, H, \xi, J, g)$ satisfying the followings:

(i) $\nabla_X Y \in \Gamma(H)$ for $Y \in \Gamma(H), X \in \Gamma(TM)$,

(ii) $\nabla\xi = 0, \nabla J = 0, \nabla\omega = 0$, where $\omega = -d\theta$ (then $\nabla\theta = \nabla g$)

(iii) Torsion satisfies $T(X, Y) = -\omega(X, Y)\xi$, and $T(\xi, JY) = -JT(\xi, Y)$, for all $X, Y \in H_x, x \in M$.

The Tanaka-webster connection described above is induced as a full connection on $TM \longrightarrow M, \nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$. The ideas of the alternative construction are that the we will create a partial connection that is compatible with a metric $g$, and then use the property $(iii)$ in order to pick a perfect $B \in EndTM$. In the conclusion, $\overline{D}(D, B)$ will be a unique Tanaka-Webster connection. Many definitions and some remarks will be introduced along the way of the construction so that we can see the relationships among the definitions.

Before we move to the definition of the CR manifold being holomorphic, we introduce some notations such as $H_{0,1}, H_{1,0},\ H^{0,1}$, and $H^{1,0}$. Since $I$ is an almost complex structure on $H$, then its complicification can be decomposed to the direct sum, $H \otimes \mathbb{C} = H_{0,1} \oplus H_{1,0}$, where $H_{1,0}$ and $H_{0,1}$ are the eigenbundle of $i$ and $-i$ over $I$ respectively. Also $H^* \otimes \mathbb{C} = H^{0,1} \oplus H^{1,0}$ dually. Note that $I : H \otimes \mathbb{C} \longrightarrow H \otimes \mathbb{C}$ such that $I(X \otimes \alpha) = IX \otimes \alpha$. The conjugation is also defined on $H \otimes \mathbb{C}$ by $\overline{X \otimes \alpha} = X \otimes \overline{\alpha}$, for $X \in \Gamma(H)$.

**Definition 3.5.** Let $E$ be a complex vector bundle over a compact strongly pseudoconvex CR manifold $(M, H, \theta, \xi)$. the complex vector bundle $E$ is holomorphic, if there is a Cauchy-Riemann operator $\overline{\partial}$ on $E$, $\overline{\partial} : \Gamma(E) \to \Gamma(H^{0,1} \otimes E)$, such that

1 $\overline{\partial}_{\overline{X}}(f \cdot u) = \overline{\partial}_{\overline{X}}(f) \cdot u + f \cdot \overline{\partial}_{\overline{X}}(u), \overline{X} \in \Gamma H_{0,1}, u \in \Gamma(E)$,

2 $\overline{\partial}_{\overline{X}}(\overline{\partial}_{\overline{Y}} u) - \overline{\partial}_{\overline{Y}}(\overline{\partial}_{\overline{X}} u) - \overline{\partial}_{[\overline{X},\overline{Y}]} u = 0, \overline{X}, \overline{Y} \in \Gamma H_{0,1}, u \in \Gamma(E)$.
(This is an integrability condition of $\overline{\partial}$)

**Theorem 3.6.** *Let $M$ be a strongly pseudoconvex CR manifold. Define $\overline{\partial} : H_{1,0} \longrightarrow (H^{0,1} \otimes H_{1,0})$ such that*

$$Y \mapsto \overline{\partial}_{\overline{X}} Y := [\overline{X}, Y]_{H_{1,0}},$$

*where $(\cdots)_{H_{1,0}}$ refers to a projecting to $H_{1,0}$. Then $\overline{\partial}$ is a connection.*

*Proof.* we write $[\ ,\ ]$ as $[\ ,\ ]_{H_{1,0}}$ for convenience. In order to show that this is a connection, we will prove the Leibnitz's rule, i.e., $\overline{\partial}_{\overline{X}}(f \cdot Y) = df(\overline{X})Y \cdot Y + f \cdot \overline{\partial}_{\overline{X}}(Y)$ and $\overline{\partial}_{f\overline{X}} Y = f \overline{\partial}_{\overline{X}} Y$. For $\overline{\partial}_{f\overline{X}} Y = f \overline{\partial}_{\overline{X}} Y$, we see that

$$\overline{\partial}_{f\overline{X}} Y = [f\overline{X}, Y]_{H_{1,0}}$$

$$= (f\overline{X}(Y) - Y(f\overline{X})) \mid_{H_{1,0}}$$

$$= (f\overline{X}(Y) - Y(f) \cdot \overline{X} + fY(\overline{X})) \mid_{H_{1,0}}$$

$$= f[\overline{X}, Y]_{H_{1,0}}.$$

Since $Y(f) \cdot \overline{X} \in H_{0,1}$, then it is zero under the projection of $H_{1,0}$. For the Libnitz's rule, it can be examined by the following computation.

$$\begin{aligned}
\overline{\partial}_{\overline{X}}(f \cdot Y) &= [\overline{X}, f \cdot Y]_{H_{1,0}} \\
&= (\overline{X}(f \cdot Y) - fY(\overline{X}))_{H_{1,0}} \\
&= ((\overline{X}f)Y + f\overline{X}(Y) - fY(\overline{X}))_{H_{1,0}} \\
&= (df(\overline{X})Y + f[\overline{X}, Y])_{H_{1,0}} \\
&= (df(\overline{X})Y)_{H_{1,0}} + f \cdot \overline{\partial}_{\overline{X}}(Y) \\
&= (\overline{\partial}_{\overline{X}}f)Y + f \cdot \overline{\partial}_{\overline{X}}(Y).
\end{aligned}$$

∎

**Note 3.7.** Let $(M, H, \xi, J, g)$ be a strongly pseudoconvex CR manifold. There is the induced the Hermitian metric $g$ on $TM \otimes \mathbb{C}$ defined by

$$g_x(X_1 \otimes \alpha_1, X_2 \otimes \alpha_2) :=< \alpha_1, \alpha_2 > g_x(X_1, X_2)\cdot,$$

where $< \cdot, \cdot >$ is a usual Hermitian metric on a complex space. One can define the compatibility of a connection $D$ with a metric $g$ by saying $Dg = 0$ or equivalently $Xg(Y, Y) = g(D_X Y, Y) + g(Y, D_{\overline{X}} Y)$.

**Theorem 3.8.** *Let $(M, H, \xi, J, g)$ be a strongly pseudoconvex CR manifold. There is a unique connection $D : \Gamma(H_{1,0}) \longrightarrow \Gamma(H^{\mathbb{C}} \otimes H_{1,0})$ such that $Dg = 0$ and $D \mid_{H_{0,1}} = \overline{\partial}$, where $\overline{\partial}$ is defined in* (3.6).

*Proof.* We begin the construction by defining $D : H_{1,0} \longrightarrow H^{0,1} \otimes H_{1,0}$ in such a way that $D_{\overline{X}} Y = \overline{\partial}$, for $\overline{X} \in \Gamma(H_{0,1})$, where $\overline{\partial}$ is defined as above. Then $D_X Y$ will be canonically induced for $X \in \Gamma(H_{1,0})$ in order to have the compatibility with the Hermitian matrix, e.g. $D_X Y$ can be achieved by considering $Xg(Y, Y) = g(D_X Y, Y) + g(Y, D_{\overline{X}} Y)$. Since $H^{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$, then one can constructs the canonical connection $D : \Gamma(H_{1,0}) \longrightarrow \Gamma(H^{\mathbb{C}} \otimes H_{1,0})$, where $Dh = 0$ and $D\mid_{H_{0,1}} = \overline{\partial}$. ∎

Since $H_{1,0}$ is $\mathbb{C}$ linearly isomorphic to $(H, I)$, which can be viewed as a real subbundle $H$, then the connection $D$ in (3.8) is allowed to be the connection under the real vector bundle $H \longrightarrow M$, i.e., $D : \Gamma(H) \longrightarrow \Gamma(H^{\mathbb{C}} \otimes H)$. Also, we have the connection $D : \Gamma(H) \longrightarrow \Gamma(H^* \otimes H)$ by restricting to the real part of differential form. Moreover, setting $D\xi = 0$ will create the connection $D : \Gamma(TM) \longrightarrow \Gamma(H^* \otimes TM)$. The next theorem will be the last step of the construction of the Tanaka-Webster connection. This will be based on the technique of a partial connection using the proposition (2.21). First, we need some tools in the Tanaka-Webster connection, which we obtain in the following lemma.

**Lemma 3.9.** *Let $\nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$ be the Tanaka-Webster connection. Then it satisfies*

$$\nabla_\xi X = \frac{1}{2}(J([\xi, JX]) + [\xi, X])$$

*, for $X \in \Gamma(TM)$.*

*Proof.* Since $\nabla\xi = 0 = \nabla J =$, then and $T(\xi, JX) = -JT(\xi, X)$, then

$$\begin{aligned} T(\xi, JX) &= \nabla_\xi JX - \nabla_{JX}\xi - [\xi, JX] \\ &= \nabla_\xi JX - [\xi, JX] \end{aligned}$$

$$
\begin{aligned}
&= (\nabla_\xi J)X + J(\nabla_\xi X) - [\xi, JX] \\
&= J(\nabla_\xi X) - [\xi, JX] \qquad (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
-JT(\xi, X) &= -J(\nabla_\xi X - \nabla_X \xi - [\xi, X]) \\
&= -J(\nabla_\xi X + [\xi, X]) \\
&= -J(\nabla_\xi X) - J[\xi, X]. \qquad (3.2)
\end{aligned}
$$

By the property of the Tanaka-Webster connection, $T(\xi, JX) = -JT(\xi, X)$ we have that $J(\nabla_\xi X) = \frac{1}{2}(-J[\xi, X] + [\xi, JX])$. Hence $\nabla_\xi X = \frac{1}{2}([\xi, X] + J[\xi, JX])$ ∎

**Theorem 3.10.** *Let $(M, H, \xi, J, g)$ be a $n(\geqslant 3)$ dimensional strongly pseudoconvex CR manifold. Then the Tanaka-Webster connection $\nabla$ is induced by the unique canonical partial connection $D$ that is compatible with $g$ and $D\xi = 0$ and a unique $B \in End(TM)$, where $B = \Lambda Q + \frac{n}{2}J\mathscr{L}_\xi(J) \in End(H)$ and $Q := [D_X, D_Y] - D_{D_XY - D_YX} \in EndH$ for $X, Y \in \Gamma(H^*)$. Note that $\mathscr{L}$ is a Lie derivative.* $(\nabla = \overline{D}(D, B))$

*Proof.* One needs to verify that the restriction of the Tanaka-Webster connection $\nabla$ to $H^*$ is the partial connection $D$ referred in (3.8) and find $B$ from the axioms of Tanaka-Webster connection: $T(\xi, IY) = -JT(\xi, Y)$,and $T(X, Y) = -\omega(X, Y)\xi$ for all $X, Y \in H$. Then one can use the proposition (2.21) to confirm that $\overline{D}(D, B)$ is the unique connection satisfying this conditions. Hence $\nabla = \overline{D}(D, B)$ is the Tanaka-Webster connection by the uniqueness. One needs no metric on $\bigwedge^2 H^*$ in order to have proposition (2.21) for 3 dimensions; however, one will have an induced metric on $\bigwedge^2 H^2$ such that $\bigwedge^2 H^* =< d\theta > \oplus < d\theta >^\perp$ from choosing a metric $g$ for 5 dimension. Let $\nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$ be the Tanaka-Webster connection(by the original definition). By the theorem (3.8), the Tanaka-Webster connection $\nabla$ is restricted to the unique partial connection $D$ that is compatible with $g$ and $D\xi = 0$. To find $B \in End(TM)$ such that $\nabla = \overline{D}(D, B)$, we apply the lemma (3.9) in the computation of $\Lambda F$, where $F$ is the curvature of $\nabla$. Also, we will use the Tanaka's technique [Ura94] that $F(X,Y)Z = Q(X,Y)Z + \nabla_{T(X,Y)}Z$, where $Q(X,Y)Z := D_X D_Y Z - D_Y D_X Z - D_{D_X Y}Z + D_{D_Y X}Z$ and $\Lambda\nabla_{d\theta\wedge\xi} = n\nabla_\xi$. Let $X, Y, X \in \Gamma(H)$.

$$
\begin{aligned}
B(Z) &= (\Lambda F)(Z) \\
&= \Lambda(Q + \nabla_{T(\cdot,\cdot)})Z \\
&= \Lambda(Q + \nabla_{-\omega(\cdot,\cdot)\xi})Z \\
&= \Lambda(Q + \nabla_{d\theta\wedge\xi})Z \\
&= \Lambda Q(Z) + n\nabla_\xi Z \\
&= \Lambda Q(Z) + \frac{n}{2}(J([\xi, JZ]) + [\xi, Z]) \\
&= \Lambda Q(Z) + \frac{n}{2} J\mathscr{L}_\xi(J)(Z).
\end{aligned}
$$

By the proposition (2.21), we have that $\nabla = \overline{D}(D, B)$ is the Tanaka-Webster connection. ∎

## 3.2 Tanaka-Webster connections (Complex differential forms)

In this section, we work on the Tanaka-Webster connection $\nabla : \Gamma(T_{\mathbb{C}}M) \longrightarrow \Gamma(T^{\mathbb{C}}M \otimes T_{\mathbb{C}}M)$ which is induced in the Urakawa's paper [Ura94].

Recall that if $(M, H, \xi, J, g)$ is a strongly pseudoconvex CR manifold $M$, we have the real Levi form $L$ and the real bilinear form $\omega$. Then one can naturally extends $L$

to a complex bilinear forms on $T^{\mathbb{C}}M$ by

$$L(X_1 \otimes \alpha_1, X_2 \otimes \alpha_2) := \alpha_1 \alpha_2 L(X_1, X_2)$$

, for $X_1, X_2 \in T_x M, x \in M$. Similar to $L$, one can extend $\omega$ to a complex bilinear form. Moreover, the almost complex structure $J$ can also be extended to the complexification by

$$J(X \otimes \alpha) := J(X) \otimes \alpha. \tag{3.3}$$

**Definition 3.11.** Tanaka-Webster connection $\nabla : \Gamma(TM) \longrightarrow \Gamma(T^{\mathbb{C}}M \otimes TM)$ on the strongly pseudoconvex CR manifold $(M, H, \xi, J, g)$ induced in the Urakawa's paper is the unique affine connection satisfying the followings:

(i) $\nabla_X Y \in \Gamma(H)$ for $Y \in \Gamma(H), X \in \Gamma(T^{\mathbb{C}}M)$

(ii) $\nabla \xi = 0, \nabla J = 0, \nabla \omega = 0$, where $\omega = -d\theta$ (then $\nabla \theta = \nabla g$)

(iii) Torsion satisfies $T(X, Y) = -\omega(X, Y)\xi$, and $T(\xi, JY) = -JT(\xi, Y)$, for all $X, Y \in H_x, x \in M$.

The Riemannian metric $g$ over $H$ can be extended naturally to $H_{\mathbb{C}}$ by

$$g(X_1 \otimes \alpha_1, X_2 \otimes \alpha_2) := \alpha_1 \overline{\alpha_2} g(X_1, X_2). \tag{3.4}$$

For the oriented orthonormal basis of $(H_{\mathbb{C}})$, one can choose the basis $\{e_i\}_1^{2m}$ in such a way that $g(e_i, \overline{e_j}) = \delta_{ij}$. This basically means that one can naturally consider $\{\alpha_1 \otimes 1, \beta_1 \otimes 1, ..., \alpha_m \otimes 1, \beta_m \otimes 1\}$ as the orthonormal basis of $H^{\mathbb{C}}$, for any orthonormal basis $\{\alpha_1, \beta_1, ..., \alpha_m, \beta_m\}$ of $H^*$. Then he extends the Tanaka-Webster connection to $\nabla : \Gamma(T_{\mathbb{C}}M) \longrightarrow \Gamma(T^{\mathbb{C}}M \otimes T_{\mathbb{C}}M)$, where the first condition of Tanaka-Webster connection will be changed to $\nabla_X(\Gamma(H_{1,0})) \subset \Gamma(H_{1,0})$ and $\nabla_X(\Gamma(H_{0,1})) \subset \Gamma(H_{0,1})$, for any $X \in T^{\mathbb{C}}M$.

**Definition 3.12.** Tanaka-Webster connection (Complex bilinear map) $\nabla : \Gamma(T_{\mathbb{C}}M) \longrightarrow \Gamma(T^{\mathbb{C}}M \otimes T_{\mathbb{C}}M)$ on the strongly pseudoconvex CR manifold $(M, H, \xi, J, g)$ induced in Urakawa paper is the unique affine connection satisfying the followings:

(i) $\nabla_X(\Gamma(H_{1,0})) \subset \Gamma(H_{1,0})$, $\nabla_X(\Gamma(H_{0,1})) \subset \Gamma(H_{0,1})$, for any $X \in \Gamma(T_{\mathbb{C}}M)$

(ii) $\nabla\xi = 0, \nabla J = 0, \nabla\omega = 0$, where $\omega = -d\theta$ (then $\nabla\theta = \nabla g$)

(iii) Torsion satisfies $T(X,Y) = -\omega(X,Y)\xi$, and $T(\xi, JY) = -JT(\xi, Y)$, for all $X, Y \in (H_{\mathbb{C}})_x, x \in M$.

To see the result like the proposition (3.10), we can follow the guideline of its construction.

**Theorem 3.13.** *Let $(M, H, \xi, J, g)$ be a strongly pseudoconvex CR manifold. There is a unique connection $D : \Gamma(H_{\mathbb{C}}) \longrightarrow \Gamma(H^{\mathbb{C}} \otimes H_{\mathbb{C}})$ satisfying the following:*

*(i)* $Dg = 0$

*(ii)* $\nabla_X(\Gamma(H_{1,0})) \subset \Gamma(H_{1,0})$, *for any* $X \in \Gamma(T_{\mathbb{C}}M)$

*(iii)* $\nabla_X(\Gamma(H_{0,1})) \subset \Gamma(H_{0,1})$, *for any* $X \in \Gamma(T_{\mathbb{C}}M)$

*Proof.* By proposition (3.8), There is a unique connection $D : \Gamma(H_{1,0}) \longrightarrow \Gamma(H^{\mathbb{C}} \otimes H_{1,0})$ such that $Dg = 0$ and $D(\Gamma(H_{1,0})) \subset \Gamma(H_{1,0})$. Instead of using the isomorphism between $H_{0,1}$ and $(H, J)$ to get the real distribution, this time we will extend to the whole $H_{\mathbb{C}}$. This can be done by extending $D$ to a map $D : \Gamma(H_{\mathbb{C}}) \longrightarrow \Gamma(H^{\mathbb{C}} \otimes H_{\mathbb{C}})$

by

$$D(\overline{X}) = \overline{D(X)}),$$

for $\overline{X} \in \Gamma(H_{0,1})$ or $X \in \Gamma(H_{1,0})$. This is well-defined since $H_{1,0} = \overline{H^{0,1}}$. Hence we have a unique connection $D : \Gamma(H_{\mathbb{C}}) \longrightarrow \Gamma(H^{\mathbb{C}} \otimes H_{\mathbb{C}})$ satisfying; $Dg = 0$, $D(\Gamma(H_{1,0})) \subset \Gamma(H_{1,0})$, and $\nabla_X(\Gamma(H_{0,1})) \subset \Gamma(H_{0,1})$. ∎

With this theorem (3.13), it allows us to have the environment on the Tanaka-Webster connection (Complex bilinear map) in a similar way to the real Tanaka-Webster connection case with the theorem (3.8). Hence we will have that

**Theorem 3.14.** *Let $(M, H, \xi, J, g)$ be a strongly pseudoconvex CR manifold with dimension $\geqslant 3$. Then the Tanaka-Webster connection (Complex bilinear map) $\nabla$ is induced by the unique canonical partial connection $D$ in* (3.13)*, and $B = \Lambda Q + \frac{n}{2} iJ\mathscr{L}_\xi(J) \in End(H)$, where $Q := [D_X, D_Y] - D_{D_X Y - D_Y X} \in End H$ for $X, Y \in \Gamma(H^{\mathbb{C}})$ and $\mathscr{L}$ a Lie derivative.* $(\nabla = \overline{D}(D, B))$

*Proof.* The proof is similar to the proof in theorem (3.10), except that we have $\Lambda\nabla_{d\theta\wedge\xi} = in\nabla_\xi$ not $\Lambda\nabla_{d\theta\wedge\xi} = n\nabla_\xi$. ∎

## 3.3 Tanaka canonical connections

**Definition 3.15.** Let $(E, h) \longrightarrow M$ be a holomorphic vector bundle over a a strongly pseudoconvex CR manifold $(M, H, \xi, J, g)$, where $h$ is a Hermitian metric on $E$. There exists a unique Hermitian connection $D : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ such that $\Lambda F = 0$, where $F$ is the curvature of $D$.

**Theorem 3.16.** *Let $\nabla$ be the Tanaka canonical connection on a holomorphic vector bundle $E$ over a compact strongly pseudoconvex CR manifold $M^{2n+1}$. Then $\nabla =$*

$\overline{D}(D, 0)$, *where the partial connection $D$ is induced in the same way as in the real Tanaka-Webster case.*

*Proof.* The main part is that $B = 0$ by the definition of Tanaka canonical connection. The creation of the partial connection is produced in a similar way to the Tanaka-Webster connection case, e.g., the partial connection is automatically the Cauchy-Riemann operator (the partial connection of the Tanaka-Webster connection is induced by an almost Cauchy-Riemann operator without the integrability). ∎

As we can see, the benefit of proposition (2.21) is that it may reduce some of the conditions for being the arbitrary special canonical full connection $\overline{D}$ by categorizing the information into two independent ingredients;

(i) the reduced information in a partial connection $D : \Gamma(E) \longrightarrow \Gamma(H^* \otimes TM)$ of $\overline{D}$,

(ii) the reduced information in $B \in End(E)$.

Then one might be able to find some redundant condition in a partial connection or in $B$ term. In Tanaka-Webster connection's situation, we can see that we do not use the condition $D\omega = 0$ to create such a partial connection due to the uniqueness of compatibility with metric $g$. On the other hand, the proposition (2.21) may produce the very complicated $B$ that is hard to use or apply into others theorems. For example, the term $B$ in Tanaka-Webster is very complicated since the term $Q$ is still a mystery (whether it is already in the easiest form).

# Chapter 4

# Contact Hermitian-Einstein connections

In this chapter, the contact Hermitian-Einstein connection will be introduced by starting from the Hermitian-Einstein or Hermitian Yang-Mills connection defined on the Kahler manifold, and then the proposed Himitian Einstein for the almost contact manifold. People often use two different definitions of the Hermitian-Einstein connection on a Kahler manifold:

(i) the Hermitian-Einstein connection defined over the holomorphic vector bundle,

(ii) the Hermitian-Einstein connection defined over the complex vector bundle.

The one over the holomorphic bundle will be a unique connection, since it is one to one corresponding to a Cauchy operator. The second one over the complex vector bundle might allow more than one Hermitian-Einstein connection. We will give the information though the distinct sources; Tian [Tia00] and Kobayashi [Kob87].

## 4.1 Hermitian-Einstein connections over a Kahler manifold

Here we will start with the version in the Kobayashi's work based on the following schemes:

(i) $(M, g)$ is the Hermitian manifold,

(ii) $(E, h)$ is the holomorphic Hermitian vector bundle of rank $r$ over $M$.

Recall that there is the canonical Hermitian connection $D$ such that $Dh = 0$ and $D'' = d''$. Since $M$ is the Hermitian manifold, then there is a fundamental form $\Phi$ by the fact that there is a $1-1$ correspondence between the Hermitian forms and the fundamental forms(If it is closed, then it will be the Kahler form). With the fundamental form, we have the operator $L : \Omega_M^{m-1} \longrightarrow \Omega_M^{m+1}$ such that $L(u) = \Phi \wedge u$. In particular, $L : \Omega_M^{p,q} \longrightarrow \Omega_M^{p+1,q+1}$. Let $\Lambda : \Omega_M^{p,q} \longrightarrow \Omega_M^{p-1,q-1}$ be the adjoint of $L$.

**Definition 4.1.** The mean curvature $K$ of $\pi : E \longrightarrow M$, as set up above, is defined by

$$K = i\Lambda F$$

, where $F$ is the curvature of the canonical Hermitian connection $D$.

**Definition 4.2.** $(E, h)$ is an Einstein hermitian vector bundle over $(M, g)$ if

$$K = ci_d, \tag{4.1}$$

for a constant $c$.

We can see that the Einstein condition is dependent only on a holomorphic vector bundle. On the other hand, the way to define Einstein's condition over a complex vector bundle is based on the choice of a holomorphic structure, which is not unique. Now we will take a look at the Tian version and some of the applications of the Hermitian-Einstein connection.

In the Tian's paper, the version will be based on the following schemes:

(i) $(M, \omega)$ is a Kahler manifold,

(ii) $E$ is a complex vector bundle of rank $r$ over $M$.

**Definition 4.3.** The connection $D$ of $E$ is called a Hermitian-Yang-Mills connection if

(i) $D$ is the unitary connection,

(ii) $F_D^{1,1} \cdot \omega = \lambda I_d$,

(iii) $F_D^{0,2} = 0$,

where $\lambda = \frac{m(C_1(E)\cdot[\omega]^{n-1})}{r[\omega]^m}$.

**Note 4.4.** $(i)$ represents the Hermitian connection property, $(ii)$ represents the equation (4.1), and $(iii)$ means the existence of a holomorphic structure.

In the Tian's paper, he introduces the relationship between Hermitian-Yang-Mills connections and Yang-Mills connections through the following propositions.

**Proposition 4.5.** *For a given complex vector bundle $E$ over a Kahler manifold $(M, \omega)$, if $C_1(E)$ is a type of $(1,1)$, then $D$ is a Hermitian-Yang-Mills connection if and only if $D$ satisfies the equation*

$$\Omega \wedge (F_D - \frac{1}{r} tr(F_D) I_d) = - * (F_D - \frac{1}{r} tr(F_D) I_d), \tag{4.2}$$

*where $\Omega = \frac{\omega^{m-2}}{(m-2)!}$.*

**Proposition 4.6.** *If $D$ is the unitary connection on $\pi : E \longrightarrow (M, \omega)$ such that $tr(F_D)$ is a harmonic 2 form and satisfies the equation (4.2), then $D$ is a Yang-Mills connection.*

With these two propositions, we can adjust the relations to the contact case in a similar manner. In the contact case, it is required the distinct versions of the equation (4.2) and the special definition of Hermitian connection since there is the extra term so called contact form. Hence, in the next section, we will introduce the idea and propose the Hermitian connection's definition for a contact manifold and end Chapter. The new equation functioning in a same way with the equation (4.2) on the contact case will be introduced as the contact instanton equation in the next Chapter along with the relationship of Hermitian connections and Yang-Mills connections.

## 4.2 Hermitian-Einstein connections over a contact manifold

In Wang's Informal notes [**?**], he proposed the definition of the Hermitian-Einstein connection as follows:

**Definition 4.7.** Let $E$ be a Hermitian vector bundle over a contact manifold $M$. A connection $\overline{D}(D, B)$ is called a Hermitian-Einstein connection if

(i) $\overline{D}(D, B)$ is a Hermitian connection,

(ii) $\widetilde{F} = 0$,

(iii) $DB = 0$.

One of the ideas is from the Urakawa's paper, which emphasizes the roles of Tanaka connections in the same manner with the Hermitian-Einstein connection of a holomorphic bundle over a compact Kahler manifold under the condition that $i_\xi \overline{F} = 0$, where $\overline{F}$ is the curvature of the Hermitian connection. In this context, we translate $i_\xi F = 0$ into $\widetilde{F} = 0$. The first condition has the similar intuitive; however, the third axiom, $DB = 0$, turns out to be more general than the Tanaka caconical connection, i.e. $\Lambda\overline{F} = 0$ or $B = 0$.

**Example 4.8.** The Tanaka canonical connection is the Hermitian-Einstein connection under the condition that the curvature $F$ is of $(1,1)$ type.

*Proof.* We know that $\widetilde{F} = 0$ if and only if $F$ is of $(1,1)$ type. For $DB = 0$, it is directly from the fact that $\Lambda F = 0 = B$. ∎

# Chapter 5

# Contact instantons and Yang-Mills connections

This chapter will introduce the definition of a $B$ contact instanton and also introduce the tools in the Wang's Informal notes in order to express the relationship of $B$ contact instantons and some inhomogeneous Yang-Mills connections. Moreover, the inhomogeneous Yang-Mills functional will be considered. Firstly, we will start with the definition and properties of the $\star$ operator over the domain $\bigwedge^2 H^*$, which is the main ingredient of the $B$ contact instanton equation. The $B$ contact instanton that will be shown in this thesis is similar to the case of the $\Omega-$anti-self-dual instanton (on Kahler manifold) in Tian [Tia00], where analogy of $\Omega$ for the contact instanton on a contact manifold is different.

Before proceeding further, consider the local picture of $(p,q)$ differential forms on $\bigwedge^r H^*$ with respect to $(M,\theta,J,g)$ in order to gain tools when some complicated computations are required.

## 5.1 Local differential forms of type (p,q)

This section will be very useful for one to explicitly compute the differential forms in a very convenient point of views. Let $(M^{2m+1},\theta,J,g)$ be a manifold with an almost contact structure. Considering the complexification $H^{\mathbb{C}}$, one can decompose the bundle in to the two eigen(bundle)spaces from the almost complex structure $J : H^{\mathbb{C}} \longrightarrow H^{\mathbb{C}}$. We note that $J$ over $H^{\mathbb{C}}$ is naturally induced from $J$ over $H$ by (3.3) and still $J^2 = i_d$. Since $H^{\mathbb{C}}$ is a complex bundle, then one can easily solve that the eigenvalues of $J$ over $H^{\mathbb{C}}$ are $i$ and $-i$. We write $H^{\mathbb{C}} = \bigwedge^{1,0} \oplus \bigwedge^{0,1}$, where $\bigwedge^{1,0}$ and $\bigwedge^{0,1}$ are the eigenbundle of $i, -i$ respectively.

In the proposition (2.27), we have the oriented orthogonal basis $\{\alpha_1, \beta_2, ..., \alpha_m, \beta_m\}$ of the real vector space $(H_x^*, g)$, where $\beta_k = J(\alpha_k)$, for $k = 1, ..., m$. With (3.4), the basis can also be represented as the oriented orthogonal basis of the complex bundle $(H^{\mathbb{C}}, g)$. One may abuse the notation by replacing $\alpha_k = \alpha_k \otimes 1$ and $\beta_k = \beta_k \otimes 1$ for more convenient. All of the followings will be based on the fiber-wise point of views.

**Proposition 5.1.** *Given $(M, \theta, J, g)$ with the oriented orthogonal basis $\{\alpha_1, \beta_2, ..., \alpha_m, \beta_m\}$ in the proposition (2.27). Then $H^{\mathbb{C}} = \bigwedge^{1,0} \oplus \bigwedge^{0,1}$, where $\bigwedge^{1,0}$ and $\bigwedge^{0,1}$ are the eigenspace of $i, -i$ respectively. Fiber-wise, we have that;*

*(i)* $\bigwedge^{1,0} =< \alpha_k - i\beta_k >_{k=1,...,m}$,

*(ii)* $\bigwedge^{0,1} =< \alpha_k + i\beta_k >_{k=1,...,m}$.

*Proof.* We can compute it directly that $\alpha_k - i\beta_k$ is the eigenvector of $i$ by

$$\begin{aligned} J(\alpha_k - i\beta_k) &= J\alpha_k - iJ\beta_k \\ &= \beta_k + i\alpha_k \\ &= i(\alpha_k - i\beta_k). \end{aligned}$$

Also, $\alpha_k + i\beta_k$ is the eigenvector of $-i$.

$$J(\alpha_k + i\beta_k) = J\alpha_k + iJ\beta_k$$

$= \beta_k - i\alpha_k$

$= -i(\alpha_k + i\beta_k).$

Since $\{\alpha_k - i\beta_k, \alpha_k + i\beta_k\}$ are orthogonal to each others, then we can assume that there are only two eigenvectors and $H^{\mathbb{C}} = \bigwedge^{0,1} \oplus \bigwedge^{1,0}$,where $\bigwedge^{1,0}$ is the eigenspace generated by $< \alpha_k - i\beta_k >_{k=1,...,m}$ and $\bigwedge^{0,1}$ is the eigenspace generated by $< \alpha_k + i\beta_k >_{k=1,...,m}$.

∎

Note that $H_x$ can be decomposed in to the direct summation of all eigenspaces with respect to $J$ by the fact that $J$ is an isometric linear operator.

**Proposition 5.2.** *Given $(M, \theta, J, g)$ with the oriented orthogonal basis $\{\alpha_1, \beta_2, ..., \alpha_m, \beta_m\}$ of the $(H^*, g)$, where $\beta_k = J(\alpha_k)$, Then $\bigwedge^{\mathbb{C}} H^* = \bigwedge^{2,0} \oplus \bigwedge^{0,2} \oplus \bigwedge^{1,1}$. we have that;*

*(i)* $\bigwedge^{1,1} =< \alpha_k \wedge \alpha_l + \beta_k \wedge \beta_l, \alpha_k \wedge \beta_l + \alpha_l \wedge \beta_k >_{k \leqslant l} \otimes \mathbb{C},$

*(ii)* $\bigwedge^{2,0} \oplus \bigwedge^{0,2} =< \alpha_k \wedge \alpha_l - \beta_k \wedge \beta_l, \alpha_k \wedge \beta_l - \alpha_l \wedge \beta_k >_{k \leqslant l} \otimes \mathbb{C}.$

*Proof.* We will use the proposition (5.1) to compute it explicitly. For $(i)$,

$$\overset{1,1}{\bigwedge} = \overset{1,0}{\bigwedge} \wedge \overset{0,1}{\bigwedge} \oplus \overset{0,1}{\bigwedge} \wedge \overset{1,0}{\bigwedge}$$

$$=< (\alpha_k - i\beta_k) \wedge (\alpha_l + i\beta_l) > \oplus < (\alpha_k + i\beta_k) \wedge (\alpha_l - i\beta_l) >$$

$$=< \alpha_k \wedge \alpha_l + i\alpha_k \wedge \beta_l - i\beta_k \wedge \alpha_l + \beta_k \wedge \beta_l >$$

$$\oplus < \alpha_k \wedge \alpha_l - i\alpha_k \wedge \beta_l + i\beta_k \wedge \alpha_l + \beta_k \wedge \beta_l >$$

$$=< (\alpha_k \wedge \alpha_l + \beta_k \wedge \beta_l) + i(\alpha_k \wedge \beta_l + \alpha_l \wedge \beta_k) >$$

$$\oplus < (\alpha_k \wedge \alpha_l + \beta_k \wedge \beta_l) - i(\alpha_k \wedge \beta_l + \alpha_l \wedge \beta_k) >$$

$$=< \alpha_k \wedge \alpha_l + \beta_k \wedge \beta_l, \alpha_k \wedge \beta_l + \alpha_l \wedge \beta_k > \otimes \mathbb{C}.$$

For $(ii)$,

$$\bigwedge^{2,0} \oplus \bigwedge^{0,2} =< (\alpha_k - i\beta_k) \wedge (\alpha_l - i\beta_l) > \oplus < (\alpha_k + i\beta_k) \wedge (\alpha_l + i\beta_l) >$$

$$=< (\alpha_k \wedge \alpha_l - \beta_k \wedge \beta_l) - i(\alpha_k \wedge \beta_l - \alpha_l \wedge \beta_k) >$$

$$\oplus < (\alpha_k \wedge \alpha_l - \beta_k \wedge \beta_l) + i(\alpha_k \wedge \beta_l - \alpha_l \wedge \beta_k) >$$

$$= (< \alpha_k \wedge \alpha_l - \beta_k \wedge \beta_l > \oplus < \alpha_k \wedge \beta_l - \alpha_l \wedge \beta_k >) \otimes \mathbb{C}.$$

∎

One can observe that $\bigwedge^{1,1}$ can be generated by 3 types of elements, $\bigwedge^{1,1} =< \alpha_k \wedge \alpha_l + \beta_k \wedge \beta_l, \alpha_k \wedge \beta_l + \alpha_l \wedge \beta_k, \alpha_k \wedge \beta_k >_{k<l} \otimes \mathbb{C}$.

## 5.2 The $\star$ operator on a contact distribution

**Definition 5.3.** Given a contact manifold $(M^{2m+1}, \theta, g, J)$, where $m \geqslant 2$. Define $\Theta \in \bigwedge^{2m-3} TM^*$ in such a way that $\Theta = \theta \wedge d\theta^{m-2}/(m-2)!$. The $\star$ operator is a mapping in $\bigwedge^2 TM^*$. $\star : \bigwedge^2 TM^* \longrightarrow \bigwedge^2 TM^*$ is such that $\star\alpha := *(\Theta \wedge \alpha)$.

Since $\alpha = \alpha^H + \theta \wedge \alpha \in \bigwedge^2 H^* \oplus \theta \wedge H^*$, then $\star\alpha = \star\alpha^H = *(\theta \wedge d\theta^{m-2}/(m-2)! \wedge \alpha)$. By Lemma4.1 in Wang's note, the image must be in $\bigwedge^2 H^*$. Hence we can rewrite that $\star : \bigwedge^2 TM^* \longrightarrow \bigwedge^2 H^*$ and vanish on the vertical term. For the convenience, we begin computing $\star d\theta$ before going to the general case $\gamma \in \bigwedge^2 H^*$.

$$\star d\theta = *((\theta \wedge d\theta^{m-2}/(m-2)!) \wedge d\theta)$$

$$= *(\theta \wedge (d\theta)^{m-1}/(m-2)!)$$

$$= (m-1)d\theta = *^{'}(d\theta)^{m-1}/(m-2)!$$

$$= (m-1)d\theta.$$

We consider the orthonormal oriented basis $\{\alpha_1, \beta_1, ..., \alpha_m, \beta_m, \theta\}$ in such a way that $d\theta = \Sigma_k \alpha_k \wedge \beta_k$. For an arbitrary $\gamma \in \bigwedge^2 H^*$, which is represented by $\gamma = \Sigma_{k<l}(a_{kl}\alpha_k \wedge \alpha_l + b_{kl}\beta_k \wedge \beta_l) + \Sigma_{k,l} c_{kl}\alpha_k \wedge \beta_l$, we have that

$$\begin{aligned}
\star\gamma &= *(\Theta \wedge \gamma) \\
&= *(\theta \wedge d\theta^{m-2}/(m-2)! \wedge \gamma) \\
&= *^{'}(d\theta^{m-2}/(m-2)! \wedge \gamma) \\
&= *^{'}[(\Sigma_{k<l}(\alpha_1 \wedge \beta_1 \wedge ... \wedge \widehat{\alpha_k \wedge \beta_k} \wedge ... \wedge \widehat{\alpha_l \wedge \beta_l} \wedge ... \wedge \alpha_n \wedge \beta_n) \\
&\quad \wedge (\Sigma_{k<l}(a_{kl}\alpha_k \wedge \alpha_l + b_{kl}\beta_k \wedge \beta_l) + \Sigma_{k,l} c_{kl}\alpha_k \wedge \beta_l)] \\
&= -[\Sigma_{k<l}(b_{kl}\alpha_k \wedge \alpha_l + a_{kl}\beta_k \wedge \beta_l)] - [\Sigma_{k \neq l} c_{kl}\alpha_l \wedge \beta_k] + [\Sigma_{l \neq k} c_{jj}\alpha_k \wedge \beta_k].
\end{aligned} \tag{5.1}$$

For $m = 2$, the $\star$ operator behaves in the same way $*^{'}$ does. We know from the above procedure that $\star\gamma = *^{'}(d\theta^{m-2}/(m-2)! \wedge \gamma)$. Hence $\star\gamma = *^{'}\gamma$ when $m = 2$.

We observe the decomposition $\bigwedge^2 H^* = S_1 \oplus S_2 \oplus S_3$, where $S_1, S_2$, and $S_3$ are defined to be the spaces generated by $\{\alpha_k \wedge \alpha_l, \beta_k \wedge \beta_l\}$, $\{\alpha_k \wedge \beta_l\}_{k \neq l}$, and $\{\alpha_k \wedge \beta_k\}$ respectively. By this decomposition, it allows us to see the behavior of the $\star$ operator in the similar way as the isometric involution. Let $J$ be an isometry on $\bigwedge^2 H^*$ such that

$$\begin{aligned}
J(\gamma) &= J(\Sigma_{k<l}(a_{kl}\alpha_k \wedge \alpha_l + b_{kl}\beta_k \wedge \beta_l) + \Sigma_{k,l} c_{kl}\alpha_k \wedge \beta_l) \\
&:= \Sigma_{k<l}(a_{kl}J(\alpha_k) \wedge J(\alpha_l) + b_{kl}J(\beta_k) \wedge J(\beta_l)) + \Sigma_{k,l} c_{kl}J(\alpha_k) \wedge J(\beta_l) \\
&= \Sigma_{k<l}(a_{kl}\beta_k \wedge \beta_l + b_{kl}\alpha_k \wedge \alpha_l) - \Sigma_{k,l} c_{kl}\beta_k \wedge \alpha_l.
\end{aligned}$$

After applying $J$ again, one can easily acquire that $J$ is an involution, i.e. $J^2$ is identity. Now, the information is collected and we have $\star = -J$ over $S_1$ and $S_2$.

Then it implies that the $\star$ operator is an isometric involution on $S_1 \oplus S_2$. This leads us to the following proposition.

**Proposition 5.4.** *Let $(M, \theta, J, g)$ be a contact manifold with $n = 2m+1$ dimension, where $m \geqslant 2$. Define $S_1, S_2$ and $S_3$ as the above, the following are true.*

*(i) For $m = 2$, $\star$ is an isometric involution on $\bigwedge^2 H^* = S_1 \oplus S_2 \oplus S_3$.*

*(ii) For $m > 2$, $\star$ is an isometric involution on $< d\theta >^{\perp} = S_1 \oplus S_2 \oplus S_3^-$, where $S_3^- := \{\gamma = \Sigma_k c_k \alpha_k \wedge \beta_k \mid \bar{c} := \Sigma_k c_k = 0\}$.*

*Proof.* We already know that the $\star$ operator is an isometric involution on $S_1 \oplus S_2$ from the above information. Hence we will investigate only on $S_3$. Given $\gamma = \Sigma_k c_k \alpha_k \wedge \beta_k \in S_3$, One computes $\star^2(\gamma) = \Sigma_k((m-2)\bar{c} + c_k)\alpha_k \wedge \beta_k$. If $m = 2$, then $(i)$ is automatically true. For $(ii)$, case of $m > 2$, $\star$ is an isometric involution on $S_3^- := \{\gamma = \Sigma_k c_k \alpha_k \wedge \beta_k \mid \bar{c} = 0\}$, which is Equivalently to $S_3^- = \{\gamma \in S_3 \mid < \gamma, d\theta >= 0\}$. This mean $S_3^- =< d\theta >^{\perp} \cap S_3$ can be obtained by the definition of $S_3^-$, i.e. $\bar{c} = 0$. ∎

We have only considered on the case of $m \geqslant 2$, since $m = 1$ is the trivial case where $\bigwedge^2 H^* =< d\theta >$. Since $\star$ is an isometric involution on $\bigwedge^2 H^*$ for $m = 2$ and on $< d\theta >^{\perp}$ for $m >$,then we can study the properties of the eigenspaces of eigenvalues $\pm 1$ of $\star$ on those subbundle.

**Proposition 5.5.** *Let $\bigwedge^+$ and $\bigwedge^-$ be $\pm 1$ eigenspaces of eigenvalues $\{\pm 1\}$ of the $\star$ operator on $\bigwedge^2 H^*$.*

*For $m = 2$,*

$$\bigwedge^2 H^* = \bigwedge^+ \oplus \bigwedge^-, \text{ where } d\theta \in \bigwedge^+$$

*For $m > 2$, the $\star$ operator has $\{1, -1, m-1\}$ as eigenvalues.*

$$\bigwedge^2 H^* = \bigwedge^+ \oplus \bigwedge^- \oplus < d\theta >$$

*, with $\star d\theta = (m-1)d\theta$,*

*Proof.* One can get $(i)$ directly from the previous proposition. For $m = 2$, $\star$ is an involution on the entire $\bigwedge^2 H^*$. Hence $\bigwedge^2 H^* = \bigwedge^+ \oplus \bigwedge^-$. From our computation, $\star d\theta = (m-1)d\theta$, we have that $\star d\theta = (m-1)d\theta = d\theta$, or equivalently $d\theta \in \bigwedge^+$ for $m = 2$. For $m > 2$, $d\theta$ is the eigenvector of the eigenvalue $m - 1 \neq 1$, and we have the decomposition $\bigwedge^2 H^* = \bigwedge^+ \oplus \bigwedge^- \oplus < d\theta >$. ∎

**Proposition 5.6.** *Let $L$ be the mapping $L : C^\infty(M) \longrightarrow \Omega^2_H$ such that $L(\alpha) = d\theta \wedge \alpha$, and $\Lambda : \Omega^2_H \longrightarrow C^\infty(M)$ be a pointwise adjoint of $L$. Then*

*(i)* $\bigwedge^- \subset (imL)^\perp = ker(\Lambda)$. *In particular,* $\bigwedge^- = (\bigwedge^{1,1})_{\mathbb{R}} \cap ker(\Lambda)$,

*(ii)* *For $m > 2$,* $\bigwedge^+ = (\bigwedge^{2,0} \oplus \bigwedge^{0,2})_{\mathbb{R}}$,

*For $m = 2$,* $\bigwedge^+ = (\bigwedge^{2,0} \oplus \bigwedge^{0,2})_{\mathbb{R}} \oplus < d\theta >$,

*(iii)* $d\theta \in (\bigwedge^{1,1})_{\mathbb{R}}$.

*Proof.* For $(i)$, one observes that $Im(L) =< d\theta >$ by the definition. By 5.5, $\bigwedge^- \subset < d\theta >^\perp$ for $m \geqslant 2$. Also, by the fact that $Im(L)^\perp = ker(\Lambda)$. Then $\bigwedge^- \in Im(L)^\perp = ker(\Lambda)$. To see what $\bigwedge^-$ looks like, one can compute from the equation (5.1) and find the suitable coefficients of $\gamma \in \bigwedge^-$. Since $\star\gamma = -\gamma$, then we have that

$$-(\Sigma_{k<l}(a_{kl}\alpha_k \wedge \alpha_l + b_{kl}\beta_k \wedge \beta_l) + \Sigma_{k,l}c_{kl}\alpha_k \wedge \beta_l)$$

$$= -\gamma$$

$$= \star\gamma$$

$$= -[\Sigma_{k<l}(b_{kl}\alpha_k \wedge \alpha_l + a_{kl}\beta_k \wedge \beta_l)] - [\Sigma_{k\neq l}c_{kl}\alpha_l \wedge \beta_k] + [\Sigma_{l\neq k}c_{ll}\alpha_k \wedge \beta_k].$$

By comparing the coefficients, one has that $a_{kl} = b_{kl}$, $c_{kl} = c_{lk}$, and $-c_{kk} = \Sigma_{l\neq k}c_{ll}$. This implies that

$$\overset{-}{\bigwedge} =< \alpha_k \wedge \alpha_l + \beta_k \wedge \beta_l, \alpha_k \wedge \beta_l + \alpha_l \wedge \beta_k > \oplus S_3^-. \tag{5.2}$$

Now, we consider the case of $\bigwedge^+$.

$$\Sigma_{k<l}(a_{kl}\alpha_k \wedge \alpha_l + b_{kl}\beta_k \wedge \beta_l) + \Sigma_{k,l}c_{kl}\alpha_k \wedge \beta_l$$

$$= \gamma$$

$$= \star\gamma$$

$$= -[\Sigma_{k<l}(b_{kl}\alpha_k \wedge \alpha_l + a_{kl}\beta_k \wedge \beta_l)] - [\Sigma_{k\neq l}c_{kl}\alpha_l \wedge \beta_k] + [\Sigma_{l\neq k}c_{ll}\alpha_k \wedge \beta_k].$$

Then we have that $a_{kl} = -b_{kl}$, $c_{kl} = -c_{lk}$, and $c_{kk} = \Sigma_{l\neq k}c_{ll}$. We observe that $c_{kk} = 0, \forall k$ if $m > 2$ and $c_{11} = c_{22}$ for $m = 2$. This implies

$$\overset{+}{\bigwedge} =< \alpha_k \wedge \alpha_l - \beta_k \wedge \beta_l, \alpha_k \wedge \beta_l - \alpha_l \wedge \beta_k > \tag{5.3}$$

for $m > 2$ with the extra term $< d\theta >$ for $m = 2$. Next we will show that $\bigwedge^- = (\bigwedge^{1,1})_{\mathbb{R}} \cap ker(\Lambda)$. By the propositions (5.2), (2.34), and (5.4), we have that $\bigwedge^- = (\bigwedge^{1,1})_{\mathbb{R}} \cap ker(\Lambda)$. For $(ii)$, we already know that $(\bigwedge^{2,0} \oplus \bigwedge^{0,2})_{\mathbb{R}} =< \alpha_k\wedge\alpha_l - \beta_k\wedge\beta_l, \alpha_k\wedge\beta_l - \alpha_l \wedge \beta_k >$ from the proposition (5.2). This is same as $\bigwedge^-$ by considering (5.3). Hence one have that $\bigwedge^+ = (\bigwedge^{2,0} \oplus \bigwedge^{0,2})_{\mathbb{R}}$, for $m > 2$ and $\bigwedge^+ = (\bigwedge^{2,0} \oplus \bigwedge^{0,2})_{\mathbb{R}} \oplus < d\theta >$ for $m = 2$. For $(iii)$, It is directly from $\bigwedge_{\mathbb{R}}^{1,1} =< \alpha_k \wedge \alpha_l + \beta_k \wedge \beta_l, \alpha_k \wedge \beta_l + \alpha_l \wedge \beta_k >_{k<l} \oplus S_3$.

∎

## 5.3 Relationships between $B$-inhomogeneous Yang-Mills connections and contact instantons with the application on Tanaka canonical connections in Urakawa's paper

In Tian's paper [Tia00](2000), The star operator is used to define a contact instanton in the same way to $\Omega-$anti-self-dual instanton(on Kahler manifold) is. Let we introduce a Yang-Mills connection and a contact instanton. Also, we provide the relationship between Yang-Mills connections and contact instantons later on. For a Riemannian manifold $M$, we define the Yang-Mills functional, or the energy functional $YM(D) = \|F_D\|^2 = \int_M |F_D|^2 Vol_M$. The Yang-Mill equation $d_D^* F_D = 0$ is the Euler-Lagrange equation of the Yang-Mills functional. The critical point of the Yang-Mills functional, equivalently the solution of Yang-Mill equation is called a Yang-Mills connection.

The Kahler case was introduced introduced already in Chapter of Hermitian-Einstein connections. To investigate the contact case, we can see the similar parts between anti-self-dual instantons on a Kahler manifold and $B$ contact instantons o an contact manifold.

For the contact case, the analogous of fundamental form is $\Theta = \theta \wedge d\theta^{m-2}/(m-2)!$.

**Definition 5.7.** For a given vector bundle $E \longrightarrow M$ with $B \in End(E)$, the full connection $\overline{D}$ is called a $B$-contact instanton if its curvature $\overline{F}$ satisfies

$$\star(\overline{F} - Bd\theta) = -(\overline{F} - Bd\theta).$$

Equivalently,

$$\star\overline{F} = -\overline{F} + mBd\theta$$

by $d\theta = (m-1)d\theta$.

Note that $\overline{F} = F_H + \theta \wedge \widetilde{F} = F_\perp + \Lambda F_H d\theta + \theta \wedge \widetilde{F}$. By proposition 5.5, it allows us to decompose $F_\perp$ again into $F_\perp^+ + F_\perp^- \in \bigwedge^+ \oplus \bigwedge^-$. This means $\overline{F} = F_H + \theta \wedge \widetilde{F}$, where $F_H = F_\perp^+ + F_\perp^- + \Lambda F_H d\theta$.

**Lemma 5.8.** *$F_H \in \bigwedge^{1,1}$ if and only if $F_\perp^+ = 0$.*

*Proof.* We can see that $F_H = F_\perp^+ + F_\perp^- + \Lambda F_H d\theta$. Suppose $F_H \in \bigwedge^{1,1}$. Then $F_\perp^+ + F_\perp^- + \Lambda F_H d\theta \in \bigwedge^{1,1}$. We know that $F_\perp^- \in \bigwedge^- \subset \bigwedge^{1,1}$ and $\Lambda F_H d\theta \in \bigwedge^{1,1}$ from 5.6. Hence we must have $F_\perp^+ \in \bigwedge^{1,1}$. By 5.6 again, we have $F_\perp^+ \in \bigwedge^+ \subset \bigwedge^{2,0} \oplus \bigwedge^{0,2}$. Then $F_\perp^+ = 0$. Suppose $F_\perp^+ = 0$. Then $F_H = F_\perp^- + \Lambda F_H d\theta$. By 5.6, $F_\perp^- + \Lambda F_H d\theta \in \bigwedge^{1,1}$. Hence $F_H \in \bigwedge^{1,1}$. ∎

**Proposition 5.9.** *$\overline{D}$ is a $B$ contact instanton if and only if $\widetilde{F} = 0, \Lambda F_H = B$, and $F_H \in \bigwedge^{1,1}$.*

*Proof.* Since the image of $\star$ is in $\wedge^2 H^*$, then $\widetilde{F} = 0$. To check $\Lambda F_H = B$, we will investigate the $B$ contact instanton equation, $\star(\overline{F} - Bd\theta) = -(\overline{F} - Bd\theta)$. By this, we can conclude that $\overline{F} - Bd\theta \in \bigwedge^-$. Hence $\overline{F} - Bd\theta \in ker(\Lambda) = (ImL)^\perp$ by 5.9. This implies that $\Lambda F_H = B$. We want to show that $F_H \in \bigwedge^{1,1}$, this is equivalence to that the restriction of $F_\perp$ to $\bigwedge^+$ is 0. By $\widetilde{F} = 0$, we can consider

$$\star(\overline{F} - Bd\theta) = -(\overline{F} - Bd\theta)$$

$$\star(F_\perp) = -F_\perp.$$

This means $F_\perp \in \bigwedge^-$. Hence $F_\perp^+ = 0$. For the converse part, we observe that $\star(\overline{F} - Bd\theta) = -(\overline{F} - Bd\theta)$ is equivalently to say $\overline{F} - Bd\theta$ is an eigenvector of -1,

$\overline{F} - Bd\theta \in \bigwedge^{-}$. We will show the converse part by proving that $\overline{F} - Bd\theta \in \bigwedge^{-} \otimes \mathbb{C}$. We know that if $\widetilde{F} = 0$ and $\Lambda F_H = B$, then $\overline{F} - Bd\theta = F_H + \theta \wedge \widetilde{F} - Bd\theta = (F_{\perp} + \Lambda F_H d\theta + \theta \wedge \widetilde{F}) - Bd\theta = F_{\perp} \in ker(\Lambda)$. Suppose $F_H \in \bigwedge^{1,1}$, then $F_{\perp} \in \bigwedge^{1,1}$. Hence $\overline{F} - Bd\theta \in \bigwedge^{1,1} \cap ker(\Lambda) = \bigwedge^{-}$ by the proposition (5.6). ∎

The above definition of a $B$ contact instanton is viewed in term of full connection, and the proof of the previous proposition avoids working on only the term of $H$-partial connection. It turns out that the proposition 5.9 suggests that $B$ contact instanton $\overline{D}$ must be extended by some $H$-partial connection $D$ in such a way that $\overline{D} = \overline{D}(D, B$. Hence we can define it on $\overline{D}(D, B)$ for any $H$-partial connection $D$ as the alternative definition of the original version. To work on the alternative definition of $B$ contact instanton, one defines the $\star$ self dual and $\star$ anti self dual connection. Note that $F_D = F_D^+ + F_D^- \in \bigwedge^{+} \oplus \bigwedge^{-}$ for any $H$-partial connection $D$ by the proposition (2.3).

**Definition 5.10.** The $H$-partial connection $D$ is call a $\star$ self dual if $\star(F_D^-) = 0$, and is called a $\star$ anti self dual if $\star(F_D^+) = 0$.

**Theorem 5.11.** *$\overline{D}(D, B)$ is $B$ a contact instanton if and only if $D$ is a $\star$ anti self dual and $\widetilde{F} = 0$.*

*Proof.* The proof is directly from that $F_{\perp}^+ = F_D^+$, $\star(F_D^+) = 0$, and together with lemma (5.8). ∎

The above theorem delights us the advantage of decomposing a full connection into $H$-partial connection and some $B \in EndE$. In this case, we can find the sufficient and necessary condition of the $H$-partial connection being a $B$ contact instanton $\overline{D}(D, B)$.

Basically we reduce the condition from a full connection to a contact distribution's point of views.

**Definition 5.12.** A full connection $\overline{D}$ is called a $B$-inhomogeneous Yang-Mills connection if $\overline{D}^*\overline{F} = mB\theta$.

It is a homogeneous Yang-Mills connection if $B = 0$.

**Lemma 5.13.** *$d\theta^{m-1} \wedge F_\perp = 0$.*

*Proof.*

$$\begin{aligned} d\theta^{m-1} \wedge F_\perp &= (m-1)! *' (d\theta) \wedge F_\perp \\ &= (m-1)! < d\theta, F_\perp > vol_H \\ &= 0. \end{aligned}$$

∎

**Theorem 5.14.** *Suppose that $\overline{D}$ is $B$ contact instanton. Then*

$$\overline{D}^*\overline{F} = mB\theta - mJ(DB).$$

*Moreover, $\overline{D}$ is a $B$-inhomogeneous Yang-Mills connection if and only if $DB = 0$.*

*Proof.* By the $B-$instanton definition, we have that $-\overline{F} + mBd\theta = -*(\Theta \wedge \overline{F}) = -*(\Theta \wedge F_H)$. By taking the full connection and Hodge star operator $*$ on both sides, $\overline{D}*\overline{F} = -d\theta \wedge F_H + \Theta \wedge \overline{DF} + (m\overline{D}B \wedge \theta \wedge d\theta^{m-1} + mBd\theta^m)/(m-1)!$. With Bianchi identity $\overline{DF} = 0$ and the next lemma, $\overline{D}*\overline{F} = -Bd\theta^m/(m-2)! + mDB \wedge \theta \wedge d\theta^{m-1}/(m-1)! + mBd\theta^m/(m-1)! = Bd\theta^m/(m-1)! + mDB \wedge \theta \wedge d\theta^{m-1}/(m-1)!$. Hence

$$\overline{D}^*\overline{F} = (-1)^2 * \overline{D} * \overline{F}$$

$$= *(Bd\theta^m/(m-1)! + mDB \wedge \theta \wedge d\theta^{m-1}/(m-1)!)$$
$$= \frac{m!B\theta}{(m-1)!} - m * (\theta \wedge DB \wedge d\theta^{m-1}/(m-1))!$$
$$= mB\theta - \frac{m}{(m-1)!} *' (DB \wedge d\theta^{m-1})$$
$$= mB\theta - \frac{m}{(m-1)!} *' (DB \wedge (m-1)!(\Sigma_k(\alpha_1 \wedge \beta_1 \wedge ... \wedge \widehat{\alpha_k \wedge \beta_k} \wedge ... \wedge \alpha_m \wedge \beta_m)).$$

Since $DB \in \Gamma(H^* \otimes End(E))$, then each element must be in $\{\alpha_k, \beta_k\}$. After taking the $*'$ operator, the result will be the conjugate of itself, that is $J(DB)$. Hence $\overline{D}^*\overline{F} = mB\theta - mJ(DB)$. It is clear that if $DB = 0$, then $\overline{D}$ is a $B$-inhomogeneous Yang-Mills connection by previous definition. ∎

**Corollary 5.15.** *(Full connection picture) For a given contact manifold $M$ and a vector bundle $E$, we suppose that $D$ is a full connection such that $F_H \in \bigwedge_{\mathbb{R}}^{1,1}$ and $\widetilde{F} = 0$. Then $D$ is a Yang-Mills connection if and only if $\Lambda F = 0$.*

*Proof.* This is directly from proposition 5.14, when we apply $B = 0$. ∎

**Corollary 5.16.** *(H-partial connection picture) For a given contact manifold $M$ and a vector bundle $E$, we suppose $D$ is $H$-partial connection and $\star$ anti self dual and $\widetilde{F}_{\overline{D}(D,B)} = 0$. Then $\overline{D}(D,B)$ is a Yang-Mills connection if and only if $B = 0$.*

The theorem 5.14 generalizes Urakawa's theorem [Ura94] in the situation that $B = 0$ through the corollary 5.15. In Urakawa paper, the theorem based on CR manifold which is the special case of a contact manifold states that the sufficient and necessary condition for being Yang Mills connection under the assumption $F \in \Lambda^{1,1}H^*$ is that the Hermitian connection must be the Tanaka's canonical connection. The full stating theorem of Urakawa and how the theorem 5.14 covers Urakawa theorem by corollary 5.15 are provided below.

**Theorem 5.17.** *Urakawa [Ura94] For a given $2m+1$ dimensional compact strongly pseudoconvex CR manifold $(M, \xi)$ and a holomorphic vector bundle $(E, h)$ over $M$, suppose $D$ is the Hermitian connection such that $F \in \bigwedge^{1,1}$. Then $D$ is a Yang-Mills connection if and only if $D$ is a Tanaka canonical connection.*

The condition $F \in \Lambda^{1,1}H^*$ in Urakawa's theorem implies that $\widetilde{F} = 0$ and $F_H \in \bigwedge^{1,1}$. By 5.15, the necessary and sufficient condition of $D$ being Yang-Mills connection is that $\Lambda F = 0$. By the definition of the Tanaka canonical connection, Hermitian connection is the Tanaka canonical connection if and only if it satisfies that $\Lambda F = 0$. Hence Urakawa's theorem is the special case of the theorem 5.14 for $B = 0$. Moreover, our proof applies the decomposition technique $\overline{D}(D, B)$ which is unrelated to the Tanaka canonical connection and that makes a huge different.

At last, we shall give the functional related to a $B$-inhomogeneous Yang-Mills connection. Since a $B$-inhomogeneous Yang-Mills connection is defined in different ways compared to the homogeneous Yang-Mills connection, we expect the functional must be in the form which includes the $B$ term inside the integral. We consider a $B$ Yang-Mills functional, $YM_B$, over a full connection as $YM_B(\overline{D}) := \int \|F_{\overline{D}} - mBd\theta\|^2$

**Corollary 5.18.** *The $B$ contact instanton is a critical point of the functional $YM_B$.*

*Proof.* The idea of the proof is that we deform a $B$ contact instanton and see how the integral reacts when it comes closed to the $B$ contact instanton itself. Suppose $\varphi \in \Gamma(T^*M \otimes E)$. Define $\delta_t := YM_B(\overline{D} + t\varphi)$. We can simplify by

$$\begin{aligned}
\delta_t &= \int \|F_{\overline{D}+t\varphi} - mBd\theta\|^2 \\
&= \int \|\overline{F} - mBd\theta\|^2 + 2t < \overline{F} - mBd\theta, \overline{D}\varphi > +t^2(...)
\end{aligned}$$

In order to get the critical point of the functional, we calculate $0 = \frac{d\delta_t}{dt} \mid_{t=0}$. Hence we have that $0 =< \overline{F} - mBd\theta, \overline{D}\varphi >$. Since $\varphi \in \Gamma(T^*M \otimes E)$ is arbitrary, then $0 =< \overline{D}^*(\overline{F} - mBd\theta), \varphi >, \forall \varphi \in \Gamma(T^*M \otimes E)$. This implies that $\overline{D}^*(\overline{F}) = \overline{D}^*(mBd\theta)$. Next, we compute $\overline{D}^*(mBd\theta)$. By the proposition (2.37), we have that

$$\begin{aligned}
\overline{D}^*(mBd\theta) &= D^*(mBd\theta) + \theta \wedge \Lambda(mBd\theta) \\
&= mB\theta + D^*(mBd\theta) \\
&= mB\theta - *' D *' (mBd\theta) \\
&= mB\theta - *' D(\frac{mBd\theta^{m-1}}{(m-1)!}) \\
&= mB\theta - *' m(DB)(\frac{d\theta^{m-1}}{(m-1)!}) - *' mBd(\frac{d\theta^{m-1}}{(m-1)!}) \\
&= mB\theta - mJ(DB).
\end{aligned}$$

Since $\overline{D}^*(\overline{F}) = mB\theta - mJ(DB)$ is a necessary condition of being a $B$-instanton, then we conclude that a $B$-instanton is the critical point of functional $YM_B$. ∎

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