ALGORITHM-FREE METHODS IN FUSION FRAME CONSTRUCTION

A Dissertation

presented to the Faculty of the Graduate School

University of Missouri

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

IAN CAMPBELL

Dr. Dan Edidin, Dissertation Supervisor

DECEMBER 2022

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

ALGORITHM-FREE METHODS IN FUSION FRAME CONSTRUCTION

presented by Ian Campbell, a candidate for the degree of Doctor of Philosophy of Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

Professor Dan Edidin

Professor Stamatis Dostoglou

Professor Stephen Montgomery-Smith

Professor Calin Chindris

This work is dedicated to the memory of my father, who was chiefly responsible for setting me on the path to its completion.

I would like to acknowledge the contributions, guidance, and support of Dr. Dan Edidin, whose patience with me was not misplaced; the Dissertation Committee collectively ; The National Science Foundation for their generous financial contributions; the University of Missouri; the Graduate School; Gwen, for always having a spare set of office keys; and finally Dr. Pete Casazza, whose research was the chief motivating factor for my own and whose guidance will be sorely missed.

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Abstract

Over the past decade numerous papers have been published with novel methods for proving the existence of and constructing fusion frames under various restrictions on the ambient Hilbert space, set of subspace dimensions, weights, and fusion frame operator eigenvalues. A unifying theme in many of these methods is their use of algorithmic, iterative, and recursive constructions – features which tend to obscure underlying patterns and inhibit deeper analysis. In this thesis we analyze various algorithms prominent in fusion frame construction and, to the greatest extent practicable, derive closed-form expressions describing the frames and fusion frames they generate. Additionally, we thoroughly analyze the process of iterating Naimark and spatial complements and develop explicit, closed-form expressions which illuminate underlying structural relationships and provide a relatively convenient method for classifying and constructing arbitrary tight fusion frames.

1 Introduction

This thesis is principally concerned with analyzing the most prominent methods for constructing fusion frames and, to the greatest extent practicable, eliminating the algorithmic, iterative, and recursive aspects of these approaches.

The first tool we will analyze, spectral tetris, was originally introduced in [7] to produce unit-norm, tight frames (UNTFs) and has subsequently been extended to produce frames with much weaker restrictions on the prescribed length of frame vectors and frame operator eigenvalues (see [3], [5]). Apart from its simplicity and general applicability, the spectral tetris algorithm is noteworthy because the frames it produces are optimally *sparse* ([2]) and therefore require minimal computational resources in real-world applications. The algorithm itself deterministically populates an empty synthesis matrix with 0s, 1s, and 2×2 submatrices to ensure the resulting matrix contains orthogonal rows and that its row and column vectors have specified norms. The matrices produced have column vectors which exhibit a high degree of orthogonality, making them ideal candidates for frame representations of fusion frames. Once generated, these frame vectors can grouped into equal-norm, orthogonal bases for the desired fusion frame subspaces as in [6] or further modified as in [7]. In section 3 the spectral tetris algorithm for producing equal-norm frames with arbitrary frame operator eigenvalues is investigated and closed-form expressions are obtained for various aspects of the resulting frame.

When the vectors of a Spectral Tetris Frame (STF) are used to directly form subspace bases then a very natural question to ask is what limitations exist on the dimensions of the resulting subspaces. In [6] it was shown that the sequence of subspace dimensions must be majorized by the sequence of subspace dimensions for an object termed the Reference Fusion Frame. The reference fusion frame is generated by an algorithm which takes as its input a frame generated by any spectral tetris algorithm and produces a sequence of subspaces whose dimensions are maximal with respect to majorization. In section 3 the reference fusion frame algorithm is investigated and closed-form expressions are developed for its sequence of subspace dimensions as well as alternative techniques for quickly identifying the vector groupings which produce them.

The final technique, which we term ‘NS-complementation’, involves iteratively applying a pair of highly involutory operations to tight fusion frames. The operations in question–Naimark and spatial complements–can be regarded as unitary completions of different components of the frame representation of a fusion frame, and their application has been extensively used to generate families of fusion frames with differing dimensional characteristics. For example, generating a fusion frame with the spectral tetris algorithm as in [6] generally requires that the total subspace dimension of the resulting fusion frame be at least twice the dimension of its ambient Hilbert space. The methodology employed in [1], however, utilizes the highly technical approach of constructing Littlewood-Richardson tableaux and requires precisely the opposite dimensional relationship. In either case, if the desired fusion frame does not meet the requirements of the construction technique then its spatial complement will. The existence of such dimensional restrictions for fusion frame construction techniques is a relatively common phenomenon which can be generally be overcome through the application of Naimark and spatial complements.

The degree to which these complement operations they can be considered involutions depends on their precise definitions as well as the properties of the specific fusion frames they are applied to. The approach used herein is to adopt the usual definitions for these complement operations with the added convention of discarding from the resulting fusion frame any trivial subspaces or subspaces whose associated weights are 0. Fusion frames which contain such vanishing subspaces are termed *degenerate* and an extensive analysis of their properties and structure is undertaken in section 4. Section 5 establishes that every tight fusion frame possesses a ‘minimal’ equivalent fusion frame which is, in some sense, canonical. In section 6 a pair of augmentation operations are introduced which serve as inverses to Naimark and spatial complements where they fail to be involutory and provides a relatively convenient method of classifying tight fusion frames.

1.1 Summary of Main Results

The most significant results from each section are abbreviated below. A tremendous amount of effort has been applied in this document to develop notation which is both clear and compact enough to express the relevant information and it is neither prudent nor feasible at this time to present these statements in full. Of the notations that have not yet been introduced, the floor, ceiling, and fractional part functions appear below and throughout the rest of this document as $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$, and $\{\cdot\}$, respectively; the '$\oplus$' symbol denotes a concatenation of sequences; $\mathbb{G}$ is the set of all compositions of Naimark and spatial complements; and the augmentation operations $\cdot_{N(\cdot)}$, $\cdot_{N(\cdot|\cdot)}$, $\cdot_{S(\cdot)}$, and $\cdot_{S(\cdot|\cdot)}$ are provided in definition 6.1.

Spectral Tetris Frame Components: (Proposition 3.1)

Closed-form expressions are derived for spectral tetris frames with equal-norm vectors and arbitrary eigenvalues. The expressions take row numbers, r, and frame operator eigenvalues, λ_r, as inputs and produce a complete description of the entries appearing in the corresponding row of the spectral tetris frame. Specifically, if N_r is the number of 1s in row r and the 2×2 submatrix spanning rows r and $r+1$ has the form $\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{R_r} & \sqrt{R_r}\\ \sqrt{C_{r+1}} & -\sqrt{C_{r+1}}\end{bmatrix}$, then

- $R_r = \{\sum_{i=1}^{r}\lambda_i\}$
- $C_{r+1} = \begin{cases} 0 & R_r = 0 \\ 2 - R_r & R_r \neq 0 \end{cases}$

- $N_r = \lfloor \lambda_r - C_r \rfloor$

Reference Fusion Frame Sequences: (Theorem 3.2)

Closed-form expressions are derived for the reference fusion frame sequences of tight, equal-norm, spectral tetris fusion frames. The expressions take as inputs the number of rows, n, and columns, M, of the spectral tetris frame and produce the corresponding reference fusion frame sequence, $\mathrm{RFFS}(n, M)$. Specifically,

- In general, $\mathrm{RFFS}(n, M) = \gcd(n, M) \cdot \mathrm{RFFS}\left(\frac{n}{\gcd(n,M)}, \frac{M}{\gcd(n,M)}\right)$

- In general, $\mathrm{RFFS}(n, M) = n^{\oplus \lfloor \frac{M}{n} \rfloor - 2} \oplus \mathrm{RFFS}\left(n, n\left\{\frac{M}{n}\right\} + 2n\right)$

- If $\gcd(n, M) = 1$ and $0 < \left\{\frac{M}{n}\right\} < \frac{1}{2}$, then

$$\mathrm{RFFS}(n, M) = n^{\oplus \lfloor \frac{M}{n} \rfloor - 2} \oplus \left(\left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor, n\left\{\frac{M}{n}\right\} - 1 \right)$$

- If $\gcd(n, M) = 1$ and $\frac{1}{2} < \left\{\frac{M}{n}\right\} < 1$, then

$$\mathrm{RFFS}(n, M) = n^{\oplus \lfloor \frac{M}{n} \rfloor - 2} \oplus \left(n\left\{\frac{M}{n}\right\} + 1, \left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil - 1 \right)$$

Necessary and Sufficient conditions for non-degeneracy: (Theorem 5.12)

Necessary and sufficient conditions for a fusion frame to be non-degenerate are developed. Specifically, an interval, (α, β), depending only on the number of subspaces is identified along with a pair of parameters, μ and ν, which must satisfy $\mu, \nu \in (\alpha, \beta)$. When this condition is met non-degeneracy is shown to be equivalent to the positivity of a set of four functions whose minimizers are known.

Double Degeneracy: (Theorem 6.5)

Loosely speaking, a fusion frame is said to be *degenerate* if it contains a vanishing subspace, *totally degenerate* if every subspace vanishes, and *doubly-degenerate* if it is degenerate in more than one way. It is shown that every doubly-degenerate fusion frame is totally degenerate.

NS-classification of tight fusion frames: (Theorems 6.7 and 7.2)

It is shown (Theorem 6.7) that for every tight fusion frame W which is not totally degenerate there corresponds a unique minimal fusion frame W_0 which possesses simultaneously the minimum values for its number of subspaces, ambient Hilbert space dimension, and total subspace dimension among all representatives of $\{W^G | G \in \mathbb{G}\}$. Further, such a W is shown (Theorem 7.2) to have a unique representation of the form

$$W \approx W_0^{G_1}{}_{X_1(x_1)}{}^{G_2}{}_{X_2(x_2)} \cdots {}^{G_q}{}_{X_q(x_q)}{}^{G_{q+1}}$$

Finally, every fusion frame W which is totally degenerate is shown (Theorem 7.2) to have a non-unique representation of the form

$$W \approx 0_{X_0(x_0|y)}{}^{G_1}{}_{X_1(x_1)} \cdots {}^{G_q}{}_{X_q(x_q)}{}^{G_{q+1}}$$

2 Background

2.1 Frames and Fusion Frames

The word 'frame' was first used with its current meaning by Duffin and Schaeffer in [10] for studying nonharmonic Fourier series. Frames have subsequently been shown to have broad application in data processing, signal recovery, and optimal packings [9]. For a general introduction to frame theory we recommend [8]. Formally, a **frame** Φ is a collection of vectors $\Phi = (\phi_i)_{i=1}^m$ in some ambient Hilbert space $\mathcal{H}^n$ which satisfy the **frame condition**: there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i=1}^{m} |\langle x, \phi_i \rangle|^2 \leq B\|x\|^2$$

for every $x \in \mathcal{H}^n$. The constants A and B are respectively called the **lower** and **upper frame bounds**, a frame is said to be A-**tight** or simply **tight** if $A = B$, and a frame is **Parseval** if $A = B = 1$. By a slight abuse of notation, the same symbol Φ is often used in literature to denote the matrix

$$\Phi = [\phi_i]_{i=1}^m = \begin{bmatrix} | & & | \\ \phi_1 & \cdots & \phi_m \\ | & & | \end{bmatrix}$$

This matrix is properly called the **synthesis operator** of the frame Φ. This notational convention should not cause confusion as it is always clear from context whether one is referring to the set/sequence of vectors $(\phi_i)_{i=1}^m$ or to the matrix/operator $[\phi_i]_{i=1}^m$ and each of these objects uniquely determines the other. In addition to the synthesis operator, for any frame Φ one has:

- The **analysis operator** Φ^*.
- The **frame operator** $S_\Phi = \Phi\Phi^*$.

- The **Gram matrix** $G_\Phi = \Phi^*\Phi$.

In frame theory one is frequently concerned with the properties and applications of these operators.

A fusion frame is a generalization of a frame wherein the primary objects of interest are weighted subspaces rather than individual vectors. The concept of a fusion frame was initially developed in [4] and subsequently refined in [11] to more efficiently model distributed sensing networks. Formally, a **fusion frame** W is a collection $W = (W_i, w_i)_{i=1}^m$ of positive weights $w_i > 0$ and subspaces W_i of some ambient Hilbert space $\mathcal{H}^n$ which satisfy the **fusion frame condition**: there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i=1}^m w_i^2 \|P_i x\|^2 \leq B\|x\|^2$$

for every vector $x \in \mathcal{H}^n$, where P_i is the orthogonal projection onto W_i. As with frames, the constants A and B are respectively called the **lower** and **upper fusion frame bounds**, a fusion frame is said to be A-**tight** or simply **tight** if $A = B$, and a fusion frame is **Parseval** if $A = B = 1$. One may alternatively define a fusion frame in terms of its projections, i.e. $W = (P_i, w_i)_{i=1}^m$, in which case the **fusion frame operator** S_W is defined as $S_W = \sum_{i=1}^m w_i^2 P_i$. Note that for a tight fusion frame the fusion frame condition becomes

$$A\|x\|^2 = \sum_{i=1}^m w_i^2 \|P_i x\|^2$$

which implies the following **fusion frame equation**:

$$An = \sum_{i=1}^m w_i^2 \dim(W_i)$$

which will be of singular importance in later analysis.

Frames and fusion frames are related in the following manner: if $\Phi = (\phi_i)_{i=1}^m$ is any collection of non-zero vectors in $\mathcal{H}^n$ then Φ is a frame for $\mathcal{H}^n$ if and only if $(\text{span}\{\phi_i\}, \|\phi_i\|)_{i=1}^m$ is a fusion frame for $\mathcal{H}^n$. Further, let $W = (W_i, w_i)_{i=1}^m$ be any collection of positive weights and subspaces and let the dimensions of the subspaces be given by $(k_i)_{i=1}^m = (\dim(W_i))_{i=1}^m$. Then W is a fusion frame for $\mathcal{H}^n$ if and only if $\Phi = (b_{ij})_{j=1}^{k_i}{}_{,i=1}^{m}$ is a frame for $\mathcal{H}^n$, where, for each i, $(b_{ij})_{j=1}^{k_i}$ is an equal-norm, orthogonal basis for W_i satisfying $\|b_{ij}\| = w_i$ for all j.

Any such frame $\Phi = (b_{ij})_{j=1}^{k_i}{}_{,i=1}^{m}$ consisting of appropriately weighted basis vectors of the subspaces of W is called a **frame representation** for W. A **synthesis operator** for a fusion frame W is likewise any matrix

$$\Phi = [b_{ij}]_{j=1}^{k_i}{}_{,i=1}^{m} = \left[\begin{array}{ccc|c|ccc} | & & | & & | & & | \\ b_{11} & \cdots & b_{1k_1} & \cdots & b_{m1} & \cdots & b_{mk_m} \\ | & & | & & | & & | \end{array}\right]$$

consisting of appropriately weighted and ordered orthogonal basis vectors for the subspaces $(W_i)_{i=1}^m$.

A frame representation for a given fusion frame is not unique since there are many choices of orthogonal bases for each subspace. Any two frame representations of the same fusion frame are, however, related in the following manner. For any positive integer p let $\mathbb{U}_p$ be the set of $p \times p$ unitary matrices. Then Φ and Φ' are frame representations of the same fusion frame W if and only if there exist $U_i \in \mathbb{U}_{k_i}$ such that

$$\Phi' = \Phi \bigoplus_{i=1}^m U_i = \Phi \begin{bmatrix} U_1 & & \\ & \ddots & \\ & & U_m \end{bmatrix}$$

Thus, every fusion frame uniquely determines an equivalence class of frame representations of the form given above, and vice versa. The **fusion frame operator** of W is the matrix $S_W = \Phi\Phi^*$, where Φ is any frame representation of W. The uniqueness of the fusion frame operator follows from the identity

$$\Phi\Phi^* = \Phi \left(\bigoplus_{i=1}^{m} U_i \right) \left(\bigoplus_{i=1}^{m} U_i \right)^* \Phi^* = \Phi' \Phi'^*$$

and this is clearly consistent with the alternative definition given previously.

2.2 Spectral Tetris and Fusion Frame Construction

Generally speaking, the problem of constructing a fusion frame W with specified properties is considered solved when one identifies an appropriate Hilbert space $\mathcal{H}^n$, set of subspaces $(W_i)_{i=1}^m$, and corresponding set of weights $(w_i)_{i=1}^m$. In practical terms this amounts to identifying a frame representation Φ for W, which is typically the most desirable solution from an applied perspective. Numerous techniques exist for constructing fusion frames which are generally applicable to a subset of fusion frames characterized by some restriction on their parameter values. For example, the first sufficiently general technique for constructing fusion frames was developed in [7] for constructing complex, tight, equal-weight, equal-dimension fusion frames. This approach used the *spectral tetris* algorithm to produce low dimensional synthesis operators for unit-norm, tight frames which were then modulated and reassembled to form the desired higher dimensional analysis operators. Subsequent generalizations of this algorithm in [6] were employed with additional techniques to produce fusion frames with significantly fewer restrictions on their allowed parameters. In particular, the *reference fusion frame algorithm* directly assigns vectors generated by any spectral tetris algorithm to subspaces in a manner which is optimal in the sense discussed below.

To begin, consider the problem of constructing (the synthesis operator of) an equal-norm frame $\Phi = (\phi_i)_{i=1}^m$ for $\mathbb{R}^n$ whose frame operator has eigenvalues $\lambda = (\lambda_i)_{i=1}^n$. We make the simplifying assumption that Φ is unit-norm since unit-norm and equal-norm frames are equivalent under a uniform scaling of frame vectors. It is necessary and sufficient for such a frame to have the following properties:

- the entries in each column of Φ must square-sum to 1
- the entries in row r of Φ must square-sum to λ_r
- the row vectors of Φ must be mutually orthogonal

To construct such a matrix, spectral tetris systematically populates an empty $n \times m$ matrix with three types of elements: 0s, 1s, and 2×2 blocks of the form

$$T(x) = \begin{bmatrix} \sqrt{\frac{x}{2}} & \sqrt{\frac{x}{2}} \\ \sqrt{1-\frac{x}{2}} & -\sqrt{1-\frac{x}{2}} \end{bmatrix}$$

according to the following rules:

1. Perform steps 2-5 by rows, beginning with row 1 and proceeding from top to bottom.
2. Beginning with the leftmost unused entry and proceeding from left to right, populate the matrix with 1s until any additional 1s would bring the square-sum of the current row (say, row r) above λ_r.
3. For an appropriate value of x, place the 2×2 block $T(x)$ so that its top left entry occupies the next unused entry in the current row. Here, x is chosen so that the square-sum of the current row (row r) is precisely λ_r.
4. Populate the remaining unused entries in the current row with 0s.

5. For each non-zero entry in the current row, populate the unused entries below it with 0s.

Example 2.1. In order to construct a unit-norm frame with eigenvalues $\lambda = (3.3, 3.5, 3.1, 1.1)$ in $\mathbb{R}^4$, note first that $\lambda_1 = 3.3$, therefore row 1 contains three 1s.

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Once no additional 1s can be used, the 2×2 block $T(0.3)$ is used so that the square-sum of row 1 is precisely 3.3.

$$\begin{bmatrix} 1 & 1 & 1 & \sqrt{.15} & \sqrt{.15} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{.85} & -\sqrt{.85} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The remaining entries in row 1 are populated with 0s, as well as the appropriate entries in subsequent rows.

$$\begin{bmatrix} 1 & 1 & 1 & \sqrt{.15} & \sqrt{.15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{.85} & -\sqrt{.85} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The current square-sum of row 2 is 1.7 and the required value is $\lambda_2 = 3.5$. Thus row 2 requires a single 1 and the block $T(0.8)$.

$$\begin{bmatrix} 1 & 1 & 1 & \sqrt{.15} & \sqrt{.15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{.85} & -\sqrt{.85} & 1 & \sqrt{.4} & \sqrt{.4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{.6} & -\sqrt{.6} & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The current square-sum of row 3 is 1.2 and the required value is $\lambda_3 = 3.1$. Thus row 3 requires a single 1 and the block $T(0.9)$.

$$\begin{bmatrix} 1 & 1 & 1 & \sqrt{.15} & \sqrt{.15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{.85} & -\sqrt{.85} & 1 & \sqrt{.4} & \sqrt{.4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{.6} & -\sqrt{.6} & 1 & \sqrt{.45} & \sqrt{.45} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{.55} & -\sqrt{.55} \end{bmatrix}$$

The spectral tetris algorithm produces frame vectors which exhibit a high degree of orthogonality: that is, it produces a set of vectors which can be partitioned into subsets of mutually orthogonal vectors in many ways. If one takes the vectors in each such subset to be basis vectors of a subspace then this collection of subspaces forms a Spectral Tetris Fusion Frame (STFF). For a given STF, however, there are typically many orthogonal partitions of its frame vectors and therefore many fusion frames that can be derived. The reference fusion frame algorithm of [6] assigns vectors to subspaces according to the following rules:

1. Perform steps 2-3 by frame vector, beginning at index 1 and proceeding in increasing order.

2. For each frame vector (say, ϕ_i), assign that vector to the orthogonal subspace with smallest index (say, W_j). If no such subspace exists, create a new one with the next unused subspace index and assign ϕ_i to that subspace.

3. Set $W_j^{(\text{new})} = \text{span}\{W_j^{(\text{old})}, \phi_i\}$.

Example 2.2. The subspace assignments of the reference fusion frame algorithm to the unit-norm frame in example 2.1 are as follows:

W_1	W_2	W_3	W_4	W_5	W_1	W_2	W_3	W_1	W_4	W_5
1	1	1	$\sqrt{.15}$	$\sqrt{.15}$	0	0	0	0	0	0
0	0	0	$\sqrt{.85}$	$-\sqrt{.85}$	1	$\sqrt{.4}$	$\sqrt{.4}$	0	0	0
0	0	0	0	0	0	$\sqrt{.6}$	$-\sqrt{.6}$	1	$\sqrt{.45}$	$\sqrt{.45}$
0	0	0	0	0	0	0	0	0	$\sqrt{.55}$	$-\sqrt{.55}$

Thus, $W_1 = \text{span}\{\phi_1, \phi_6, \phi_9\}$, $W_2 = \text{span}\{\phi_2, \phi_7\}$, $W_3 = \text{span}\{\phi_3, \phi_8\}$, $W_4 = \text{span}\{\phi_4, \phi_{10}\}$, and $W_5 = \text{span}\{\phi_5, \phi_{11}\}$.

The assignments of the reference fusion frame algorithm was shown to be optimal in the sense that the generated sequence of subspace dimensions *majorizes* all possible sequences of STFF subspace dimensions, where

Definition 2.3 (Majorization)**.** *Let a and b be two sequences of real numbers, not necessarily of equal length. Let $a^\downarrow$ and $b^\downarrow$ be rearrangements of a and b in weakly decreasing order and with 0s appended to the shorter sequence so that $a^\downarrow$ and $b^\downarrow$ are of equal length. If the length of $a^\downarrow$ and $b^\downarrow$ is ℓ then a is said to* ***majorize*** *b if and only if $\sum_{i=1}^{l} a_i^\downarrow = \sum_{i=1}^{\ell} b_i^\downarrow$ and the partial sums satisfy $\sum_{i=1}^{j} a_i^\downarrow \geq \sum_{i=1}^{j} b_i^\downarrow$ for every $j \in [\ell]$. This relationship is denoted by $a \succeq b$*

Majorization is an essential concept which arises in various aspects of fusion frame analysis. In this context, however, the significance of majorization is that it is always possible for a higher dimensional subspace to divest itself into a lower dimensional subspace without altering the fusion frame operator. More formally,

Lemma 2.4 ([1], Lemma 2.4)**.** *Let P and Q be two orthogonal projections with $rank(P) > rank(Q)$. Then there exist orthogonal projections P' and Q' such that $rank(P') = rank(P) - 1$, $rank(Q') = rank(Q) + 1$, and $P + Q = P' + Q'$.*

Thus, a given fusion frame can always be modified so as to produce a fusion frame with any sequence of subspace dimensions majorized by its own. It is therefore always desirable to identify fusion frames whose subspace dimensions are maximal with respect to majorization. Unfortunately, spectral tetris and the reference fusion frame algorithm are not capable of generating all possible sequences of subspace dimensions and alternative methods are required in these cases.

2.3 Naimark and Spatial Complements

We are now able to properly define Naimark and spatial complements. For any fusion frame $W = (W_i, w_i)_{i=1}^m$ of $\mathcal{H}^n$ the **spatial complement** of W is defined as $W' = (W_i^\perp, w_i)_{i=1}^m$. If W is tight and any subspace of W is not all of $\mathcal{H}^n$ then W' is also a fusion frame for $\mathcal{H}^n$. Further, if W is A-tight and W' is a fusion frame then W' is $(\sum w_i^2 - A)$-tight. A given fusion frame uniquely determines its spatial complement and vice versa. In terms of unitary completion, the frame representation of W' is constructed by assembling the unitary completions (under appropriate scaling) of the individual subspace blocks in the frame representation of W.

While the concept of a spatial complement lends itself naturally to fusion frames, Naimark complements are most naturally applied to frames. A Naimark complement of a fusion frame is a concept which is then inherited by fusion frames via their frame representations. A constructive definition for a Naimark complement of a frame is best stated in terms of its synthesis operator, therefore let $\Phi = (\phi_i)_{i=1}^m$ be an A-tight frame for $\mathcal{H}^n$ with singular value decomposition

$$\Phi = U \left[\begin{array}{c|c} \sqrt{A} I_n & 0 \end{array} \right] V^*$$

where $U \in \mathbb{U}_n$ and $V^* \in \mathbb{U}_m$. Then a **Naimark complement** of Φ is any matrix Φ'

of the form

$$\Phi' = U' \left[\begin{array}{c|c} 0 & \sqrt{A} I_{m-n} \end{array} \right] V^*$$

for some $U' \in \mathbb{U}_{m-n}$. If any vector $\phi_i \in \Phi$ is not orthogonal to the remaining frame vectors then Φ' is an A-tight frame for $\mathcal{H}^{m-n}$. Thus, a given frame will have many Naimark complements which are related in the following manner: two A-tight frames Φ' and Φ'' are Naimark complements of the same A-tight frame Φ if and only if there exists some $U \in \mathbb{U}_{m-n}$ such that $\Phi' = U\Phi''$.

To define a Naimark complement of a fusion frame, let $W = (W_i, w_i)_{i=1}^m$ be an A-tight fusion frame for $\mathcal{H}^n$ with subspace dimensions $(k_i)_{i=1}^m = (|W_i|)_{i=1}^m$. Then a **Naimark complement** of W is any collection W' of subspaces and weights of the form $W' = (\text{span}\{b'_{ij}\}_{j=1}^{k_i}, \|b'_{ij}\|)_{i=1}^m$ where $(b'_{ij})_{j=1}^{k_i}{}_{,i=1}^{m}$ is any Naimark complement of any frame representation $(b_{ij})_{j=1}^{k_i}{}_{,i=1}^{m}$ of W. If some subspace W_i of W is not orthogonal to the remaining subspaces of W then W' is an A-tight fusion frame for $\mathcal{H}^{M-n}$. As with frames, Naimark complements of fusion frames are related in the following manner: two A-tight fusion frames W' and W'' are Naimark complements of the same A-tight fusion frame W if and only if there exists a $U \in \mathbb{U}_{M-n}$ such that $W' = (UW''_i, w''_i)_{i=1}^m$. In terms of unitary completion, a Naimark complement of a fusion frame W is any fusion frame whose frame representation is a unitary completion (under appropriate scaling) of a frame representation of W.

Example 2.5. Denote by e_i the canonical basis vectors for $\mathbb{R}^3$ and let W be the fusion frame given by:

$$W_1 = \text{span}\{e_3\} \qquad\qquad w_1 = 1$$

$$W_2 = \text{span}\{e_3\} \qquad\qquad w_2 = \sqrt{2}$$

$$W_3 = \text{span}\{e_1, e_2\} \qquad\qquad w_3 = \sqrt{5}$$

One possible frame representation for W is

$$\Phi = \left[\begin{array}{c|c|cc} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -1 \\ 1 & -2 & 0 & 0 \end{array}\right]$$

The row vectors of Φ are orthogonal and have norm $\sqrt{5}$. This implies that W is a (5)-tight fusion frame and that the matrix Φ can be completed to form a scaled unitary matrix. One such completion is

$$\left[\begin{array}{c} \Phi \\ \hline \Phi' \end{array}\right] = \left[\begin{array}{c|c|cc} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -1 \\ 1 & -2 & 0 & 0 \\ \hline 2 & 1 & 0 & 0 \end{array}\right]$$

and a Naimark compliment of W is therefore given by

$$W_1' = \text{span}\{e_1'\} \qquad\qquad w_1' = \sqrt{2}$$

$$W_2' = \text{span}\{e_1'\} \qquad\qquad w_2' = 1$$

$$W_3' = 0 \qquad\qquad w_3' = 0$$

Since $w'_3 = 0$ the Naimark compliment $W' = (W'_i, w'_i)_{i=1}^3$ fails to be a fusion frame under our strict definition. This failure arose precisely because the frame bound $A = 5$ and the squared-weight $w_3^2 = 5$ were in agreement. For tight fusion frames this phenomenon is equivalent to $W_3 \perp W_i$ for all $i \neq 3$.

2.4 Notation

For the sake of consistency the notation and conventions we have used thus far are standardized below, along with some additional definitions. These will be used throughout the rest of this paper.

- The symbol W always denotes a fusion frame, m its number of subspaces, and n the dimension of its ambient Hilbert space.

- The subspaces of W are always given by $(W_i)_{i=1}^m$, the dimensions of these subspaces are always given by $k = (k_i)_{i=1}^m = (|W_i|)_{i=1}^m$, and the total subspace dimension is given by $M = \sum_{i=1}^m k_i$.

- The weights of W are always given by $w = (w_i)_{i=1}^m$, and the squared-weights by $w^2 = (w_i^2)_{i=1}^m$.

- If W is tight then it is always A-tight.

- The parameters μ and ν are always given by $\mu = \frac{1}{n}\sum_{i=1}^m k_i$, and $\nu = \frac{1}{A}\sum_{i=1}^m w_i^2$.

- W may be specified in the traditional manner, $W = (W_i, w_i)_{i=1}^m$, or simply by identifying its relevant parameters, e.g. "Let W be a tight fusion frame with parameters A, n, and μ ..."

- When dealing with multiple fusion frames we will, whenever possible, apply matching decorations to these symbols to identify their corresponding parameters. For example, if W is a fusion frame with parameters m, k and w and W'

is another fusion frame then its parameters are likewise identified by m', k' and w'.

- $\mathbf{1}_m$ is the all 1s vector $\mathbf{1}_m = (1)_{i=1}^m \in \mathbb{R}^m$ or m-element sequence of 1s. $\mathbf{0}_m$ is defined similarly.

- $I_n \in \mathcal{H}^{n\times n}$ always denotes the identity matrix and $\mathbb{U}_n \subset \mathcal{H}^{n\times n}$ the set of $n \times n$ unitary matrices.

- As operations the symbols $\lceil\cdot\rceil$, $\lfloor\cdot\rfloor$, $[\cdot]$, and $\{\cdot\}$ respectively denote the ceiling function, floor function, nearest integer map, and fractional part. The nearest integer map is taken to be multivalued when its argument has fractional part $\frac{1}{2}$ (e.g. $[\frac{13}{2}] = \{6, 7\}$). When its argument is indeterminate, the symbols '=' and '$\in$' may be used interchangeably (e.g. $6 = [x]$ and $7 \in [x]$ are both valid expressions).

3 Spectral Tetris and Spectral Tetris Fusion Frames

We begin this section by developing an alternative formulation for spectral tetris which is generally applicable to the problem of constructing equal-norm frames with arbitrary eigenvalues. This will allow us to generate non-iterative, closed-form expressions for these frames and many of their relevant properties. Once the relevant features of spectral tetris frames have been identified we will reduce the problem of constructing a reference fusion frame to 4 typical cases with predictable structures.

As an alternative formulation of the spectral tetris algorithm, let N_r be the 'number' of 1s appearing in row r of a spectral tetris frame Φ. If the square-sum of the entries in row r (the 'spectral weight') cannot be made equal to λ_r by adding only 1s then a 2×2 block must be used. This 2×2 block must be chosen so that the spectral weight of its top two entries bring the spectral weight of row r to the desired value.

The spectral weight of the top two entries of this block (the 'remainder') is denoted R_r. The spectral weight of the bottom two entries of this 2×2 block is 'carried' into the next row and is denoted C_{r+1}. Under this formulation every 2×2 block is of the form

$$T(R_r) = \begin{bmatrix} \sqrt{\frac{R_r}{2}} & \sqrt{\frac{R_r}{2}} \\ \sqrt{\frac{C_{r+1}}{2}} & -\sqrt{\frac{C_{r+1}}{2}} \end{bmatrix}$$

By convention, if no such 2×2 block is used then we set $R_r = C_{r+1} = 0$.

Proposition 3.1. *Let $\Phi = (\phi_i)_{i=1}^m$ be a spectral tetris frame for $\mathbb{R}^n$ and let N_r, R_r, and C_r be as above. If $\lfloor x \rfloor$ is the floor function then we have the following recursive definitions for C_r, N_r, and R_r:*

1. $C_1 = 0$
2. $N_r = \lfloor \lambda_r - C_r \rfloor$
3. $R_r = \lambda_r - N_r - C_r$
4. $C_{r+1} = \begin{cases} 0 & R_r = 0 \\ 2 - R_r & R_r \neq 0 \end{cases}$

Further, if $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x, then we have the following closed-form expression for R_r:

$$R_r = \left\{ \sum_{i=1}^{r} \lambda_i \right\}$$

Proof. There is no row 0, therefore nothing is carried forward into row 1 which implies (1). (3) follows trivially from the requirement that $\lambda_r = C_r + N_r + R_r$. The same requirement along with the fact that $R_r < 1$ implies (2). (4) follows from the definition of $T(x)$. Finally, the closed expression for R_r follows from the observations that the quantity $R_r + C_{r+1}$ must be an integer (either 0 or 2), $R_r < 1$ in general, and the calculation

$$\begin{aligned}\left\{\sum_{i=1}^{r} \lambda_i\right\} &= \left\{\sum_{i=1}^{r} C_i + N_i + R_i\right\} \\ &= \left\{C_1 + \left(\sum_{i=1}^{r-1} R_i + C_{i+1}\right) + R_r + \sum_{i=1}^{r} N_i\right\} \\ &= \{0 + \text{integer} + R_r + \text{integer}\} \\ &= R_r\end{aligned}$$

□

The closed expression for R_r may be substituted into the recursive definitions of C_r and N_r to obtain closed expressions for those quantities.

The remainder of this section is devoted to proving the following RFFS identification theorem:

Theorem 3.2 (RFFS identification)**.**

Let $\oplus$ denote a concatenation of sequences and define addition and scalar multiplication of sequences to be in agreement with addition and scalar multiplication of equivalent vectors. Then

1. *In general,*

$$RFFS(n, M) = gcd(n, M) \cdot RFFS\left(\frac{n}{gcd(n, M)}, \frac{M}{gcd(n, M)}\right)$$

2. *In general,*

$$RFFS(n, M) = n^{\oplus \lfloor \frac{M}{n} \rfloor - 2} \oplus RFFS\left(n, n\left\{\frac{M}{n}\right\} + 2n\right)$$

3. *If* $gcd(n,M)=1$ *and* $0<\left\{\frac{M}{n}\right\}<\frac{1}{2}$, *then*

$$RFFS(n,M)=n^{\oplus\lfloor\frac{M}{n}\rfloor-2}\oplus\left(\left\lfloor\frac{n}{2}\right\rfloor+1,\left\lceil\frac{n}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil,\left\lfloor\frac{n}{2}\right\rfloor,n\left\{\frac{M}{n}\right\}-1\right)$$

4. *If* $gcd(n,M)=1$ *and* $\frac{1}{2}<\left\{\frac{M}{n}\right\}<1$, *then*

$$RFFS(n,M)=n^{\oplus\lfloor\frac{M}{n}\rfloor-2}\oplus\left(n\left\{\frac{M}{n}\right\}+1,\left\lceil\frac{n}{2}\right\rceil,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{n}{2}\right\rceil-1\right)$$

The proof of theorem 3.2 will be accomplished in several steps. To begin, we have the following observations and terminology:

- each 1 in row r of the STF corresponds to the canonical basis vector e_r. Such entries are termed **singlets**.
- each 2×2 block spanning rows r and $r+1$ corresponds to a pair of column vectors, each of which is a linear combination of e_r and e_{r+1}. The two entries appearing in each column of the 2×2 block are collectively termed **doublets** and are further classified as either **odd** or **even** according to the parity of r.

We have the critically important yet essentially obvious:

Proposition 3.3. *The vectors corresponding to any collection of singlets and/or doublets are mutually orthogonal if and only if the rows of their entries in the STF are disjoint.*

The notable case of orthogonal vectors corresponding to doublets in the same 2×2 block does not occur, as the only such block is

$$T(1)=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

and the argument must always satisfy $R_r < 1$. Proposition 3.3 implies that the problem of partitioning the column vectors of a STF into subspace bases can be accomplished by simply rearranging the entries of a STF so that the singlets and/or doublets corresponding to a particular subspace all lie in the same column. With this in mind, we will initially reorganize the entries of a STF into what we term a **Spectral Tetris Chart** (STC) according to the following rules:

- A STC contains the same number of rows as the STF that generates it.
- Each row of a STC contains the same non-zero entries as the STF that generates it and all 0s in the STF are omitted from the STC.
- The singlets and/or doublets present in the STF are shifted to form a STC that consists of three sections: singlets occupying the leftmost columns, odd doublets occupying the next two columns, and even doublets occupying the final two columns.

When referring to a specific spectral tetris chart or spectral tetris frame, we will typically use the designations STC(n,M) and STF(n,M). The spectral tetris charts STC(4,9), STC(4,10), and STC(4,11) are provided below along with the STFs that generate them. The 2×2 blocks have been outlined for clarity.

STF(4,9) STC(4,9)

$$\begin{bmatrix} 1 & 1 & \sqrt{\frac{1}{8}} & \sqrt{\frac{1}{8}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{7}{8}} & -\sqrt{\frac{7}{8}} & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 1 \end{bmatrix} \rightarrow \begin{array}{cc|cc|cc|} 1 & 1 & \sqrt{\frac{1}{8}} & \sqrt{\frac{1}{8}} & & \\ & & \sqrt{\frac{7}{8}} & -\sqrt{\frac{7}{8}} & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} \\ & & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} \\ 1 & & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & & \end{array}$$

STF(4,10) $\qquad\qquad$ STC(4,10)

$$\begin{bmatrix} 1 & 1 & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} & 1 \end{bmatrix} \to \begin{array}{cc|cc|} 1 & 1 & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} \\ 1 & & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} \\ \hline 1 & 1 & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} \\ 1 & & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} \end{array}$$

STF(4,11) $\qquad\qquad$ STC(4,11)

$$\begin{bmatrix} 1 & 1 & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 1 & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} & 1 & \sqrt{\frac{1}{8}} & \sqrt{\frac{1}{8}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{7}{8}} & -\sqrt{\frac{7}{8}} & 1 \end{bmatrix} \to \begin{array}{cc|cc|cc} 1 & 1 & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & & \\ 1 & & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} \\ \hline 1 & & \sqrt{\frac{1}{8}} & \sqrt{\frac{1}{8}} & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} \\ 1 & & \sqrt{\frac{7}{8}} & -\sqrt{\frac{7}{8}} & & \end{array}$$

The charts STC$(4,9)$ and STC$(4,11)$ are examples of 2 of the 4 typical cases alluded to at the beginning of this section. Note that of these three charts the only one with missing doublets, STC$(4,10)$, is also the only one satisfying $\gcd(n,M) \neq 1$. This essential observation motivates theorem 3.2(1), whose proof we now provide.

Proof of 3.2(1). Let $n' = \frac{n}{\gcd(n,M)}$, $M' = \frac{M}{\gcd(n,M)}$, and r be any multiple of n'. Then we have

$$R_r = \left\{r\frac{M}{n}\right\} = \left\{r \cdot \frac{M'}{n'}\right\} = \{\text{integer} \cdot M'\} = \{\text{integer} \cdot \text{integer}\} = 0$$

Since $R_r = 0$ we must also have $C_{r+1} = 0 = C_1$ by proposition 3.1(1) and 3.1(4). Further, since STF(n,M) is tight we must have $\lambda_1 = \lambda_{r+1}$. It follows that $N_1 = N_{r+1}$

and $R_1 = R_{r+1}$ by proposition 3.1(2) and 3.1(3). By induction on the row number we have $C_i = C_{i+r}$, $N_i = N_{i+r}$, and $R_i = R_{i+r}$ for any integer i such that $0 \leq i < n - r$. As the quantities C_i, N_i, and R_i are periodic in i we may set $s = \gcd(n, M)$ and write $\mathrm{STF}(n, M)$ in block form as

$$\mathrm{STF}(n,M) = \begin{bmatrix} \boxed{\mathrm{STF}\left(\frac{n}{s}, \frac{M}{s}\right)} & & 0 \\ & \ddots & \\ 0 & & \boxed{\mathrm{STF}\left(\frac{n}{s}, \frac{M}{s}\right)} \end{bmatrix} = \mathrm{STF}\left(\frac{n}{s}, \frac{M}{s}\right)^{\oplus s}$$

and $\mathrm{STC}(n, M)$ as

$$\mathrm{STC}(n,M) = \left.\begin{matrix} \boxed{\mathrm{STC}\left(\frac{n}{s}, \frac{M}{s}\right)} \\ \vdots \\ \boxed{\mathrm{STC}\left(\frac{n}{s}, \frac{M}{s}\right)} \end{matrix}\right\} s \text{ times}$$

Evidently, all singlets and doublets in $\mathrm{STC}(n, M)$ are confined to some copy of $\mathrm{STC}(\frac{n}{s}, \frac{M}{s})$ in the above representation. By proposition 3.3 and the definition of majorization, the maximal sequence of subspace dimensions that can be constructed from $\mathrm{STC}(n, M)$ is the sum of maximal sequences that can be formed from each copy of $\mathrm{STC}(\frac{n}{s}, \frac{M}{s})$. By definition of the reference fusion frame sequence, we have

$$\mathrm{RFFS}(n,M) = \mathrm{RFFS}\left(\frac{n}{s}, \frac{M}{s}\right) + \cdots + \mathrm{RFFS}\left(\frac{n}{s}, \frac{M}{s}\right) = s \cdot \mathrm{RFFS}\left(\frac{n}{s}, \frac{M}{s}\right)$$

Recalling that $s = \gcd(n, M)$, the theorem is proved.

□

Observe also that the charts STC$(4,9)$ and STC$(4,11)$ satisfy $N_1 = 2$, $N_n = 1$, and $2n < M < 3n$. This, too, is no coincidence:

Proposition 3.4. *If* $\gcd(n, M) = 1$ *and* $2n < M < 3n$ *then*

1. $N_1 = 2$

2. $N_r \in \{0, 1\}$ *for* $1 < r < n$

3. $N_n = 1$

Proof.

1. By proposition 3.1(2) we have $N_1 = \lfloor \lambda_1 - C_1 \rfloor = \left\lfloor \frac{M}{n} \right\rfloor$. Trivially, $2n < M < 3n$ implies $\left\lfloor \frac{M}{n} \right\rfloor = 2$.

2. If $\gcd(n, M) = 1$ then proposition 3.1 implies both $R_r \neq 0$ unless $r = n$ and $C_r \neq 0$ unless $r = 1$. Recalling that $\left\lfloor \frac{M}{n} \right\rfloor = 2$, we have the following calculation for N_r whenever $1 < r < n$:

$$\begin{aligned} N_r &= \lambda_r - R_r - C_r \\ &= \lambda_r - R_r - (2 - R_{r-1}) \\ &= \tfrac{M}{n} - 2 + R_{r-1} - R_r \\ &= \left\{ \tfrac{M}{n} \right\} + R_{r-1} - R_r \end{aligned}$$

The quantities $\left\{ \frac{M}{n} \right\}$, R_r, and R_{r-1} are all bounded below by 0 and above by 1. As N_r is an integer it can only be either 0 or 1.

3. Repeating the calculation in (2) for $r = n$ and noting that $R_n = 0$ we have

$$N_n = \left\{ \tfrac{M}{n} \right\} + R_{n-1}$$

Since $\left\{\frac{M}{n}\right\}$ and R_{n-1} are both bounded below by 0 and above by 1, the integer N_n must be 1.

$\square$

Thus, a given STC with $\gcd(n, M) = 1$ and $2n < M < 3n$ will always have six columns: one column with singlets in rows 1, n, and possibly others; one column with a singlet appearing only in row 1; two columns of $\lfloor \frac{n}{2} \rfloor$ odd doublets; and two columns of $\lfloor \frac{n-1}{2} \rfloor$ even doublets.

The final step in determining a Reference Fusion Frame consists of shifting the singlets of a STC so that number of singlets and/or doublets present in each column form a sequence which is maximal with respect to majorization. The following example illustrates this process for 4 frames which are examples of the typical cases alluded to at the beginning of this section.

Example 3.5. The shifted charts for STC(4, 9), STC(4, 11), STC(5, 11), and STC(5, 13) which produce maximal sequences are provided below, along with their subspace assignments and corresponding RFFSs.

$$
\begin{array}{cc|cc|cc|}
 & & \multicolumn{4}{c}{\mathrm{STC}(4,9)} \\
\cline{3-4}
1 & 1 & \sqrt{\frac{1}{8}} & \sqrt{\frac{1}{8}} & & \\
\cline{5-6}
 & & \sqrt{\frac{7}{8}} & -\sqrt{\frac{7}{8}} & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} \\
\cline{3-4}
 & & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} \\
\cline{5-6}
1 & & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & & \\
\cline{3-4}
\end{array}
\rightarrow
\begin{array}{|cc|cc|}
\multicolumn{1}{c}{W_3} & \multicolumn{1}{c}{W_4} & \multicolumn{1}{c}{W_1} & \multicolumn{1}{c}{W_2} \\
\hline
\sqrt{\frac{1}{8}} & \sqrt{\frac{1}{8}} & 1 & 1 \\
\cline{3-4}
\sqrt{\frac{7}{8}} & -\sqrt{\frac{7}{8}} & \sqrt{\frac{2}{8}} & \sqrt{\frac{2}{8}} \\
\cline{1-2}
\sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & \sqrt{\frac{6}{8}} & -\sqrt{\frac{6}{8}} \\
\cline{3-4}
\sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 1 & \\
\cline{1-2}
\end{array}
\rightarrow
\begin{array}{c}
\mathrm{RFFS}(4,9) \\
(3, 2, 2, 2)
\end{array}
$$

STC$(4,11)$

1	1	$\sqrt{\frac{3}{8}}$	$\sqrt{\frac{3}{8}}$		
1		$\sqrt{\frac{5}{8}}$	$-\sqrt{\frac{5}{8}}$	$\sqrt{\frac{2}{8}}$	$\sqrt{\frac{2}{8}}$
1		$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{6}{8}}$	$-\sqrt{\frac{6}{8}}$
1		$\sqrt{\frac{7}{8}}$	$-\sqrt{\frac{7}{8}}$		

$\rightarrow$

RFFS$(4,11)$

W_1	W_3	W_4	W_2	W_5
1	$\sqrt{\frac{3}{8}}$	$\sqrt{\frac{3}{8}}$	1	
1	$\sqrt{\frac{5}{8}}$	$-\sqrt{\frac{5}{8}}$	$\sqrt{\frac{2}{8}}$	$\sqrt{\frac{2}{8}}$
1	$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{6}{8}}$	$-\sqrt{\frac{6}{8}}$
1	$\sqrt{\frac{7}{8}}$	$-\sqrt{\frac{7}{8}}$		

$\rightarrow (4,2,2,2,1)$

STC$(5,11)$

1	1	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$		
		$\sqrt{\frac{9}{10}}$	$-\sqrt{\frac{9}{10}}$	$\sqrt{\frac{2}{10}}$	$\sqrt{\frac{2}{10}}$
		$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{8}{10}}$	$-\sqrt{\frac{8}{10}}$
		$\sqrt{\frac{7}{10}}$	$-\sqrt{\frac{7}{10}}$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{4}{10}}$
1				$\sqrt{\frac{6}{10}}$	$-\sqrt{\frac{6}{10}}$

$\rightarrow$

RFFS$(5,11)$

W_3	W_4	W_1	W_2
$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	1	1
$\sqrt{\frac{9}{10}}$	$-\sqrt{\frac{9}{10}}$	$\sqrt{\frac{2}{10}}$	$\sqrt{\frac{2}{10}}$
$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{8}{10}}$	$-\sqrt{\frac{8}{10}}$
$\sqrt{\frac{7}{10}}$	$-\sqrt{\frac{7}{10}}$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{4}{10}}$
1		$\sqrt{\frac{6}{10}}$	$-\sqrt{\frac{6}{10}}$

$\rightarrow (3,3,3,2)$

STC$(5,13)$

1	1	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{3}{10}}$		
1		$\sqrt{\frac{7}{10}}$	$-\sqrt{\frac{7}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$
		$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{9}{10}}$	$-\sqrt{\frac{9}{10}}$
1		$\sqrt{\frac{6}{10}}$	$-\sqrt{\frac{6}{10}}$	$\sqrt{\frac{2}{10}}$	$\sqrt{\frac{2}{10}}$
1				$\sqrt{\frac{8}{10}}$	$-\sqrt{\frac{8}{10}}$

$\rightarrow$

RFFS$(5,13)$

W_1	W_3	W_4	W_2	W_5
1	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{3}{10}}$	1	
1	$\sqrt{\frac{7}{10}}$	$-\sqrt{\frac{7}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$
	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{9}{10}}$	$-\sqrt{\frac{9}{10}}$
1	$\sqrt{\frac{6}{10}}$	$-\sqrt{\frac{6}{10}}$	$\sqrt{\frac{2}{10}}$	$\sqrt{\frac{2}{10}}$
1			$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{3}{10}}$

$\rightarrow (4,3,2,2,2)$

That the Reference Fusion Frame Algorithm produces fusion frames with these vector assignments we leave as an exercise for the reader. Note that all possible doublets are present in the above charts since in each case we have $\gcd(n, M) = 1$. This limits the useful manipulations we can perform to

- shifting one or two singlets from row 1 to the columns of even doublets
- shifting the singlet in row n to the columns of even (resp. odd) doublets if n is even (resp. odd)

and this will hold in all cases with $\gcd(n, M) = 1$ and $2n < M < 3n$. Theorem 3.2(1) addresses the case where $\gcd(n, M) \neq 1$ and spectral tetris generally requires $M \geq 2n$. For the remaining case of $M \geq 3n$ we have

Lemma 3.6. *Suppose n and M are positive integers with $M \geq 3n$. Then* $\mathrm{RFFS}(n, M+n) = (n) \oplus \mathrm{RFFS}(n, M)$.

Proof. Let $\Phi = \mathrm{STF}(n, M)$ and $\Phi' = \mathrm{STF}(n, M+n)$. Since Φ and Φ' are tight their associated eigenvalues are $\lambda_r = \frac{M}{n}$ and $\lambda'_r = \frac{M+n}{n} = \lambda_r + 1$. By proposition 3.1 we have $N'_r = N_r + 1$, $R'_r = R_r$, and $C'_r = C_r$. Thus, $\mathrm{STC}(n, M)$ and $\mathrm{STC}(n, M+n)$ differ only in that $\mathrm{STC}(n, M+n)$ has an additional full column of 1s. This column of 1s is associated with a full-rank subspace of $\mathbb{R}^n$, therefore $\mathrm{RFFS}(n, M+n)$ must be of the form $(n) \oplus \mathrm{RFFS}(n, M)$.

□

As a corollary we have:

Proof of 3.2(2). $\left\lfloor \frac{M}{n} \right\rfloor - 2$ applications of lemma 3.6 imply

$$\mathrm{RFFS}(n, M) = n^{\oplus \left\lfloor \frac{M}{n} \right\rfloor - 2} \oplus \mathrm{RFFS}\left(n, M - n\left(\left\lfloor \frac{M}{n} \right\rfloor - 2\right)\right)$$

Theorem 3.2(2) then follows from the observation that

$$M - n\left\lfloor \frac{M}{n} \right\rfloor = n\left(\frac{M}{n} - \left\lfloor \frac{M}{n} \right\rfloor\right) = n\left\{\frac{M}{n}\right\}$$

□

In light of theorem 3.2(1) and 3.2(2), we may reduce the problem of calculating RFFS(n, M) to the special case where $\gcd(n, M) = 1$ and $2n < M < 3n$. In this case RFFS(n, M) can be determined based solely on the size of $\frac{M}{n}$ (equivalently, the size of $\left\{\frac{M}{n}\right\}$) and evidently the parity of n. The parity of n determines whether the final singlet can be paired with odd or even doublets, while the size of $\left\{\frac{M}{n}\right\}$ determines how and which singlets are shifted. In general, odd doublets can always form a pair of subspaces each of dimension $\lfloor \frac{n}{2} \rfloor$ and even doublets can form a pair of subspaces of dimension $\lfloor \frac{n-1}{2} \rfloor$. When the number of singlets appearing in the first column of the STC is below these thresholds, majorization demands as many as possible be reassigned to columns of doublets to further grow their already larger subspaces. When the number of singlets in the first column is below these thresholds, majorization demands that none be shifted since this collection already forms the largest possible subspace. In all cases the lone singlet in the second column is shifted to a column of even doublets. We conclude this section with the proof of the remaining portions of theorem 3.2:

Proof of theorem 3.2(3) and 3.2(4). Consider the chart STC(n, M) with $\gcd(n, m) = 1$ and $2n < M < 3n$. The total spectral weight of any 2×2 block is always 2 and there are always $n - 1$ such blocks since $\gcd(n, M) = 1$. It follows that there are $M - 2(n-1) - 1 = M - 2n + 1$ singlets in the first column. If subspace assignment is carried out in the order singlets and doublets currently appear then the sequence of subspace dimensions is given by

$$\left(M-2n+1,1,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n-1}{2}\right\rfloor,\left\lfloor\frac{n-1}{2}\right\rfloor\right)$$

After shifting the lone singlet in column 2 to the first column of even doublets, we have

$$\left(M-2n+1,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n+1}{2}\right\rfloor,\left\lfloor\frac{n-1}{2}\right\rfloor\right)$$

If $M-2n+1=n\{\frac{M}{n}\}+1$ is sufficiently large then no further rearrangements are necessary, and this sequence is equivalent to that given in 3.2(4) after reordering. If $M-2n+1=n\{\frac{M}{n}\}+1$ is not sufficiently large, then the additional singlet in row 1 is shifted to the second column of even doublets:

$$\left(M-2n,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n+1}{2}\right\rfloor,\left\lfloor\frac{n+1}{2}\right\rfloor\right)$$

and the singlet in row n is shifted to the first column of odd doublets if n is odd:

$$\left(M-2n-1,\left\lfloor\frac{n+2}{2}\right\rfloor,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n+1}{2}\right\rfloor,\left\lfloor\frac{n+1}{2}\right\rfloor\right)$$

or to the first column of even doublets if n is even:

$$\left(M-2n-1,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n+3}{2}\right\rfloor,\left\lfloor\frac{n+1}{2}\right\rfloor\right)$$

After reordering and accounting for differences in parity, these sequences are both equivalent to that given in 3.2(3).

$\square$

4 Degeneracy

We begin this section by recalling that the definition of a fusion frame for $\mathcal{H}^n$ forbids the inclusion of 0 among its weights but does not forbid the inclusion of the trivial subspace $W_i = 0$. This is problematic from the standpoint of uniqueness since a trivial subspace may have any associated weight without affecting the properties of the fusion frame. Additionally, difficulties arise immediately when considering the frame representation and Naimark complement of such a fusion frame. Modifying the definition of a fusion frame to forbid the inclusion of such trivial subspaces is equally problematic, as the spatial complement of a full-rank subspace $W_i = \mathcal{H}^n$ is 0. Similarly, if a fusion frame contains a subspace which is orthogonal to all other subspaces then its Naimark complement contains the 0 subspace. One would thus encounter no shortage of scenarios where the Naimark or spatial complement of a fusion frame fails to be a fusion frame.

To circumvent these technical issues we will make use of special notation. Here and throughout the rest of this paper the symbols $\cdot^N$ and $\cdot^S$ will be used to denote objects arising from Naimark and spatial complementation, respectively. For example, if $W = (W_i, w_i)_{i\in I}$ is an A-tight fusion frame for $\mathcal{H}^n$ and W' is one of its Naimark complements, then in general we may write $W' = (W_i^N, w_i^N)_{i\in I^N}$ is an A^N-tight fusion frame for $\mathcal{H}^{n^N}$. As there are many Naimark complements for a given fusion frame, we must emphasize that the symbols W_i^N, w_i^N, k_i^N, etc. always refer to the subspaces, weights, subspace dimensions, etc. of the *same fixed* Naimark complement of W. The exception to this general usage rule occurs when applying $\cdot^N$ and $\cdot^S$ to the parameter m and the fusion frame itself, for which we have:

Definition 4.1. *Let $W = (W_i, w_i)_{i\in I}$ be a tight fusion frame for $\mathcal{H}^n$ with $|I| = m$. Then*

1. $W^N := (W_i^N, w_i^N)_{i\in I^N}$ *and* $m^N := |I^N|$*, where* $I^N = \{i \in I \mid w_i^N \neq 0\}$.

2. $W^S := (W_i^S, w_i^S)_{i\in I^S}$ *and* $m^S := |I^S|$*, where* $I^S = \{i \in I \,|\, k_i^S \neq 0\}$.

Therefore when applying Naimark and spatial complements to obtain new fusion frames it is our convention to automatically discard any subspaces which vanish. 'Vanishing' in this context is taken to mean the subspace dimension is zero under a spatial complement ($k_i^S = 0$) or its associated weight is zero under a Naimark complement ($w_i^N = 0$), as the alternative quantities (k_i^N and w_i^S) are assumed to always be preserved.

Example 4.2. Let W be a tight fusion frame with parameters $n = 5$, $m = 6$, $k = (5, 4, 3, 3, 1, 1)$ and let W' be its spatial complement as defined in section 2. Since $k_1 = 5 = n$, it must be that $W_1 = \mathcal{H}^n$ and therefore $W_1' = 0$. This implies that W' has parameters $n' = 5$, $m' = 6$, and $k' = (0, 1, 2, 2, 4, 4)$. W^S, however, has parameters $n^S = 5$, $m^S = 5$, and $k^S = (1, 2, 2, 4, 4)$. Further, W^{SS} has parameters $n^{SS} = 5$, $m^{SS} = 5$, and $k^{SS} = (4, 3, 3, 1, 1)$. Thus, W and W^{SS} have a different number of subspaces and so are not equivalent in any sense. Finally, $W_1^S = 0$ by definition, however W_1^S is not included among the subspaces of W^S. Objects such as W_1^{SS}, w_1^{SN}, k_1^{SNS}, etc. are therefore not defined.

The following proposition compactly codifies several elementary properties of tight fusion frames and illustrates the usefulness of this notation:

Proposition 4.3. *Let* $W = (W_i, w_i)_{i=1}^m$ *be a tight fusion frame for* $\mathcal{H}^n$*. Then*

1. *The following are equivalent:*

   (a) $W_i^S = 0$.

   (b) $W_i = \mathcal{H}^n$.

2. *The following are equivalent:*

   (a) $W_i^N = 0$.

(b) $W_i \perp W_j$ *for all* $j \neq i$.

3. *The following are equivalent:*

   (a) *For all* $i \in [m]$, $W_i^S = 0$.

   (b) *For all* $i \in [m]$, $W_i = \mathcal{H}^n$.

   (c) W^S *is not a fusion frame.*

4. *The following are equivalent:*

   (a) *For all* $i \in [m]$, $W_i^N = 0$.

   (b) $\bigoplus_{i \in [m]} W_i = \mathcal{H}^n$.

   (c) W^N *is not a fusion frame.*

5. *If* $m \geq 2$ *then the following are mutually exclusive:*

   (a) *There exists an* $i \in [m]$ *such that* $W_i^S = 0$.

   (b) *There exists an* $i \in [m]$ *such that* $W_i^N = 0$.

Proposition 4.3 also codifies the structural relationship of vanishing subspaces to the remaining elements of the fusion frame, both of which may be considered trivial. If $W_i^S = 0$ and W^{SS} is defined then W can be constructed by appending a copy of $\mathcal{H}^n$ to W^{SS}. Similarly, if $W_i^N = 0$ and W^{NN} is defined then W can be constructed by embedding W^{NN} in $\mathcal{H}^n$ and adjoining its orthogonal compliment. Any fusion frame which eventually exhibits either of these extreme orientations under complement operations is said to be *degenerate*. Formally,

Definition 4.4. *Let* $\mathbb{G}$ *be the set of words on* $\{N, S\}$ *and* $W = (W_i, w_i)_{i=1}^m$ *be a tight fusion frame for* $\mathcal{H}^n$. *Then* W *is said to be* ***degenerate*** *if there exist a* $G \in \mathbb{G}$ *and* $i \in [m]$ *such that* $W_i^G = 0$.

One would like to define $\mathbb{G}$ as a group of two involutions; however, as example 4.2 demonstrates, the action of $\cdot^N$ and $\cdot^S$ is only guaranteed to be involutory when applied to non-degenerate fusion frames. For the sake of notational convenience we will nonetheless retain the formalism of exponents applied to such groups, i.e. $(NS)^{-1} = S^{-1}N^{-1} = SN$, $(SN)^2 = SNSN$, etc., and employ $\emptyset$ as the identity element, i.e. $W^{\emptyset} = W$.

The vanishing of subspaces under $\cdot^N$ and $\cdot^S$ will play a central role in our analysis of tight fusion frames. Additionally, definition 4.1 implies that a tight fusion frame with m subspaces satisfies m^N, $m^S \leq m$, suggesting a partial order. We therefore define:

Definition 4.5. *Let W and W' be arbitrary tight fusion frames. Then*

1. *We write $W \approx W'$ if and only if there exists a unitary U such that if $W = (W_i, w_i)_{i=1}^m$ then $W' = (UW_i, w_i)_{i=1}^m$. In this case, W and W' are said to be* ***unitarily equivalent****.*

2. *We write $W \gtrsim W'$ if and only if there exists a $G \in \mathbb{G}$ such that $W^G \approx W'$.*

3. *We write $W \sim W'$ if and only if $W \gtrsim W'$ and $W' \gtrsim W$. In this case, W and W' are said to be* ***NS-equivalent****.*

4. *The* ***NS-equivalence class*** *of a tight fusion frame W is given by $[W]_{\text{NS}} = \left\{W^G | W \sim W^G\right\}$. For any parameter x of W the* ***NS-equivalence class*** *of x is given by $[x]_{\text{NS}} = \left\{x^G | W \sim W^G\right\}$.*

5. *The symbols $\lesssim$, $>$, and $<$ are similarly defined according to the usual rules for partial ordering.*

6. *We write $W \gtrsim 0$ if and only if there exists a $G \in \mathbb{G}$ such that W^G is not a fusion frame (i.e. all subspaces of W vanish). In this case W is said to be* ***totally degenerate****.*

The majority of the remainder of this section is devoted to systematically developing a model which can readily identify vanishing subspaces of a given fusion frame.

Let W be a tight fusion frame with parameters n, m, A, k, and w. Recall $w^2 := (w_i^2)_{i=1}^m$ and assume that W is non-degenerate. Then the parameters n, A, k, and w^2 transform according to

$$\begin{array}{ccccc} n & \overset{.S}{\longleftarrow} & n & \overset{.N}{\longrightarrow} & \sum k_i - n \\ n\mathbf{1}_m - k & \overset{.S}{\longleftarrow} & k & \overset{.N}{\longrightarrow} & k \\ \sum w_i^2 - A & \overset{.S}{\longleftarrow} & A & \overset{.N}{\longrightarrow} & A \\ w^2 & \overset{.S}{\longleftarrow} & w^2 & \overset{.N}{\longrightarrow} & A\mathbf{1}_m - w^2 \end{array}$$

Since these transformations are linear we will model them as such. With this in mind, recall that a real vector v is said to be **positive** (denoted $v > 0$) if its components are positive and **non-negative** (denoted $v \geq 0$) if its components are non-negative. Define, then, the positive vectors

$$\kappa = (n, k_1, \ldots, k_m)^{\mathrm{T}}$$

$$\omega = (A, w_1^2, \ldots, w_m^2)^{\mathrm{T}}$$

By construction, $0 < \kappa \in \mathbb{N}^{m+1} \subset \mathbb{R}^{m+1}$ and $0 < \omega \in \mathbb{R}^{m+1}$. Writing out the transformations of κ and ω in block-matrix form yields the following equations, in which we utilize the matrices $\mathcal{N}$ and $\mathcal{S}$:

$$\mathcal{N} := \left[\begin{array}{c|c} -1 & \mathbf{1}_m^{\mathrm{T}} \\ \hline \mathbf{0}_m & I_m \end{array}\right] \quad \text{and} \quad \mathcal{S} := \left[\begin{array}{c|c} 1 & \mathbf{0}_m^{\mathrm{T}} \\ \hline \mathbf{1}_m & -I_m \end{array}\right]$$

$$\kappa^N = \left[\begin{array}{c} n \\ \hline k \end{array}\right]^N = \left[\begin{array}{c} \sum_i k_i - n \\ \hline k \end{array}\right] = \left[\begin{array}{c|c} -1 & \mathbf{1}_m^{\mathrm{T}} \\ \hline \mathbf{0}_m & I_m \end{array}\right] \left[\begin{array}{c} n \\ \hline k \end{array}\right] = \mathcal{N}\kappa$$

$$\kappa^S = \left[\begin{array}{c} n \\ \hline k \end{array}\right]^S = \left[\begin{array}{c} n \\ \hline n\mathbf{1}_m - k \end{array}\right] = \left[\begin{array}{c|c} 1 & \mathbf{0}_m^{\mathrm{T}} \\ \hline \mathbf{1}_m & -I_m \end{array}\right] \left[\begin{array}{c} n \\ \hline k \end{array}\right] = \mathcal{S}\kappa$$

$$\omega^N = \left[\begin{array}{c} A \\ \hline w^2 \end{array}\right]^N = \left[\begin{array}{c} A \\ \hline A\mathbf{1}_m - w^2 \end{array}\right] = \left[\begin{array}{c|c} 1 & \mathbf{0}_m^{\mathrm{T}} \\ \hline \mathbf{1}_m & -I_m \end{array}\right] \left[\begin{array}{c} A \\ \hline w^2 \end{array}\right] = \mathcal{S}\omega$$

$$\omega^S = \left[\begin{array}{c} A \\ \hline w^2 \end{array}\right]^S = \left[\begin{array}{c} \sum_i w_i^2 - A \\ \hline w^2 \end{array}\right] = \left[\begin{array}{c|c} -1 & \mathbf{1}_m^{\mathrm{T}} \\ \hline \mathbf{0}_m & I_m \end{array}\right] \left[\begin{array}{c} A \\ \hline w^2 \end{array}\right] = \mathcal{N}\omega$$

(note the juxtaposition of $\mathcal{N}$ and $\mathcal{S}$ between κ and ω)

By construction $\mathcal{N}$ and $\mathcal{S}$ describe the transformation of parameters of non-degenerate fusion frames and can only serve as accurate models of $\cdot^N$ and $\cdot^S$ when their domains and ranges are restricted to positive vectors. We can, however, conclude a great deal by analyzing what happens when the range restriction is violated. Additionally, analyzing violations of this range restriction will greatly simplify further analysis. Observe, then, the following:

Observation 4.6. Let κ and ω be arbitrary positive vectors, i.e. $\kappa, \omega > 0$ but are not necessarily parameter vectors of a tight fusion frame. Then

1. If $\mathcal{N}\kappa \not> 0$ then one of the following holds:

   (a) $n > \sum k_i$. In this case κ is not a parameter vector for a tight fusion frame.

   (b) $n = \sum k_i$. If κ is a parameter vector for a tight fusion frame W in $\mathcal{H}^n$, then $\bigoplus W_i = \mathcal{H}^n$ and $W \gtrsim 0$.

   (c) $k_i \leq 0$ for some k_i. This is impossible since $\kappa > 0$ by assumption.

2. If $\mathcal{N}\omega \not> 0$ then one of the following holds:

(a) $A > \sum w_i^2$. In this case ω is not a parameter vector for a tight fusion frame.

(b) $A = \sum w_i^2$. In general we have $A = \sum \frac{k_i}{n} w_i^2$, so if ω is a parameter vector for a tight fusion frame W in $\mathcal{H}^n$ then $W_i = \mathcal{H}^n$ for all i and $W \gtrsim 0$.

(c) $w_i^2 \leq 0$ for some w_i. This is impossible since $\omega > 0$ by assumption.

3. If $\mathcal{S}\kappa \not> 0$ then one of the following holds:

(a) $n \leq 0$. This is impossible since $\kappa > 0$ by assumption.

(b) $k_i > n$ for some k_i. In this case κ is not a parameter vector for a tight fusion frame.

(c) $k_i = n$ for some k_i. If κ is a parameter vector for a tight fusion frame W in $\mathcal{H}^n$, then $W_i^S = 0$. Further,

i. If $k_i = k_j$ for all $j \in [m]$ then $W \gtrsim 0$.

ii. If $k_i \neq k_j$ for some $j \in [m]$ then $W > W'$ for some W'.

4. If $\mathcal{S}\omega \not> 0$ then one of the following holds:

(a) $A \leq 0$. This is impossible since $\omega > 0$ by assumption.

(b) $w_i^2 > A$ for some w_i. In this case ω is not a parameter vector for a tight fusion frame.

(c) $w_i^2 = A$ for some w_i. In general we have $A\|x\|^2 = \sum w_i^2 \|P_i x\|^2$, so if ω is a parameter vector for a tight fusion frame W in $\mathcal{H}^n$ then $W_i \perp W_j \forall j \neq i$ and $W_i^N = 0$. Further,

i. If $w_i = w_j$ for all $j \in [m]$ then $W \gtrsim 0$.

ii. If $w_i \neq w_j$ for some $j \in [m]$ then $W > W'$ for some W'.

The majority of the preceding observations can be further consolidated as follows:

Observation 4.7. Let $0 < x \in \mathbb{R}^{m+1}$ be arbitrary. Then

1. If $\mathcal{N}x$ or $\mathcal{S}x \not\geq 0$ then x is not a parameter vector (either κ-type or ω-type) of a tight fusion frame.

2. If $0 \leq \mathcal{N}x \not> 0$ or $0 \leq \mathcal{S}x \not> 0$ then x is a parameter vector (either κ-type or ω-type) of a tight fusion frame W only if W is degenerate.

Evidently we may draw the same conclusions regarding existence and degeneracy whether analyzing the action of $\mathcal{N}$ or $\mathcal{S}$ on either κ or ω. We now turn our attention to the matrix

$$\mathcal{G} := \mathcal{N}\mathcal{S} = \left[\begin{array}{c|c} m-1 & -\mathbf{1}_m^{\mathrm{T}} \\ \hline \mathbf{1}_m & -I_m \end{array}\right]$$

with the immediate goal of understanding how parameters transform under iterations of $\mathcal{G}$ and $\mathcal{G}^{-1}$. Since the conclusions we may draw by applying $\mathcal{G}$ to κ and ω are identical with respect to degeneracy, we will largely limit further analysis to κ-type parameter vectors.

Lemma 4.8. *The characteristic polynomial and eigenspaces of $\mathcal{G}$ are given by*

$$p_{\mathcal{G}}(x) = (x+1)^{m-1}(x-\theta^{-1})(x-\theta)$$
$$E_\theta = \operatorname{span}\left(\begin{smallmatrix}\beta \\ \mathbf{1}_m\end{smallmatrix}\right), \quad E_{\theta^{-1}} = \operatorname{span}\left(\begin{smallmatrix}\alpha \\ \mathbf{1}_m\end{smallmatrix}\right), \quad E_{-1} = (0) \oplus \mathbf{1}_m^{\perp}$$

where $\alpha = \frac{m-\sqrt{m(m-4)}}{2}$, $\beta = \frac{m+\sqrt{m(m-4)}}{2}$, *and* $\theta = \frac{m-\sqrt{m(m-4)}}{m+\sqrt{m(m-4)}}$.
Further, $\mathcal{G}^p\kappa$ *is given by*

$$\mathcal{G}^p\kappa = \frac{n\delta}{m}\left(\frac{\beta-\mu}{\beta}\theta^p\left[\frac{\beta}{\mathbf{1}_m}\right] + \frac{\mu-\alpha}{\alpha}\theta^{-p}\left[\frac{\alpha}{\mathbf{1}_m}\right]\right) + (-1)^p P_{E_{-1}}\kappa \tag{1}$$

where $\delta = \sqrt{\frac{m}{m-4}}$ *and* $\mu = \frac{1}{n}\sum_i k_i$.

Note that for $m \leq 3$ the eigenvalues of $\mathcal{G}$ are not real and for $m = 4$ they are identical. For these reasons it will be our **general assumption** going forward that $m \geq 5$. The proof of lemma 4.8 as well as a complete description of fusion frames with $m \leq 4$ subspaces are provided in the appendices.

We now wish to extract the transformed values of n, M, and k_i using equation (1). We also note at this time that iterations of $\mathcal{G}$ and $\mathcal{G}^{-1}$ do not satisfactorily describe the full range of parameter transformations since they are models for $\cdot^{NS}$ and $\cdot^{SN}$. Consequently, no integer power of $\mathcal{G}$ can model transformations such as $\cdot^{N}$, $\cdot^{S}$, $\cdot^{NSN}$, etc. We therefore define:

Definition 4.9. *Let $p \in \mathbb{Z}$. Then*

$$n^{(p)} := \langle \mathcal{G}^p \kappa, e_1 \rangle \qquad n^{(p+1/2)} := \langle \mathcal{S}\mathcal{G}^p \kappa, e_1 \rangle \tag{2}$$

$$M^{(p)} := \langle \mathcal{G}^p \kappa, \boldsymbol{1}_{m+1} - e_1 \rangle \qquad M^{(p+1/2)} := \langle \mathcal{S}\mathcal{G}^p \kappa, \boldsymbol{1}_{m+1} - e_1 \rangle \tag{3}$$

$$k_i^{(p)} := \langle \mathcal{G}^p \kappa, e_{i+1} \rangle \qquad k_i^{(p+1/2)} := \langle \mathcal{S}\mathcal{G}^p \kappa, e_{i+1} \rangle \tag{4}$$

The half-integer notation is useful for completeness but is never actually necessary for performing calculations. Indeed, since $\mathcal{S}$ fixes n and $\mathcal{N}$ fixes k, we have the following identities for $p \in {}^1\!/_2\mathbb{Z}$:

$$n^{(p)} = n^{(\lfloor p \rfloor)} \tag{5}$$

$$M^{(p)} = M^{(\lceil p \rceil)} \tag{6}$$

$$k_i^{(p)} = k_i^{(\lceil p \rceil)} \tag{7}$$

Returning now to our calculations, applying $\langle \cdot , e_1 \rangle$ to both sides of (1) yields

$$\begin{aligned} n^{(p)} &= \frac{n\delta}{m}\left(\frac{\beta-\mu}{\beta}\theta^p\beta + \frac{\mu-\alpha}{\alpha}\theta^{-p}\alpha\right) + (-1)^p\cdot 0 \\ &= \frac{n\delta}{m}\big((\beta-\mu)\theta^p + (\mu-\alpha)\theta^{-p}\big) \\ &= \frac{n(\beta-\mu)}{\beta-\alpha}\,\theta^p + \frac{n(\mu-\alpha)}{\beta-\alpha}\,\theta^{-p} \end{aligned} \tag{8}$$

Applying $\langle \cdot , \mathbf{1}_{m+1} - e_1 \rangle$ to both sides of (1) yields

$$\begin{aligned} M^{(p)} &= \frac{n\delta}{m}\left(\frac{\beta-\mu}{\beta}\theta^p m + \frac{\mu-\alpha}{\alpha}\theta^{-p} m\right) + (-1)^p\cdot 0 \\ &= \frac{n\delta}{m}\big(\alpha(\beta-\mu)\theta^p + \beta(\mu-\alpha)\theta^{-p}\big) \\ &= \frac{n\alpha(\beta-\mu)}{\beta-\alpha}\,\theta^p + \frac{n\beta(\mu-\alpha)}{\beta-\alpha}\,\theta^{-p} \end{aligned} \tag{9}$$

Applying $\langle \cdot , e_{i+1} \rangle$ to both sides of (1) yields:

$$\begin{aligned} k_i^{(p)} &= \frac{n\delta}{m}\left(\frac{\beta-\mu}{\beta}\theta^p + \frac{\mu-\alpha}{\alpha}\theta^{-p}\right) + (-1)^p(k_i - \tfrac{M}{m}) \\ &= \frac{n(\beta-\mu)}{\beta(\beta-\alpha)}\theta^p + \frac{n(\mu-\alpha)}{\alpha(\beta-\alpha)}\theta^{-p} + (-1)^p\left(k_i - \tfrac{n}{m}\mu\right) && (10) \\ &= \frac{M^{(p)}}{m} + (-1)^p\left(k_i - \tfrac{n}{m}\mu\right) && (11) \end{aligned}$$

In the interest of completeness, recall $\nu = \frac{1}{A}\sum_i w_i^2$. Then the analogues of (8) and (10) for A and w_i^2 are given by

$$A^{(p)} = \frac{A(\beta-\nu)}{\beta-\alpha}\,\theta^p + \frac{A(\nu-\alpha)}{\beta-\alpha}\,\theta^{-p} \tag{12}$$

$$\left(w_i^2\right)^{(p)} = \left(w_i^{(p)}\right)^2 = \frac{A(\beta-\nu)}{\beta(\beta-\alpha)}\theta^p + \frac{A(\nu-\alpha)}{\alpha(\beta-\alpha)}\theta^{-p} + (-1)^p\left(w_i^2 - \tfrac{A}{m}\nu\right) \tag{13}$$

Equations (8)-(10) along with (14)-(16) from the appendix are generalizations of those found in [7], Theorem 16.

We must now take care to emphasize the distinction between, for example, $n^{(p)}$ as given in definition 4.9, $n^{(p)}$ as given in equation (8), and expressions such as $n^{(SN)^p}$. The expression $n^{(SN)^p}$ is defined only for integer values of p and properly means "the dimension of the ambient Hilbert space of a fusion frame after p iterations of spatial-then-Naimark complementation if vanishing subspaces are discarded." $n^{(p)}$ is defined for integer and half-integer values of p in definition 4.9 and denotes how the dimension of the ambient Hilbert space *would* transform after p iterations of spatial-then-Naimark complementation *if no subspaces were to vanish.* For integer values of p one has $n^{(p)} = n^{(SN)^p}$ whenever $W \sim W^{(SN)^p}$. Similarly, one has $n^{(p+1/2)} = n^{(SN)^p S}$ whenever $W \sim W^{(SN)^p S}$. If we regard the right hand side of (8) as a function $f(p)$ then f has a natural extension to $p \in \mathbb{R}$, however this function and $n^{(p)}$ are in agreement only for integer values of p.

Having obtained expressions for $n^{(p)}$, $M^{(p)}$, and $k_i^{(p)}$ our next goal is to identify when these quantities are 0 and when they achieve minimal values. With this in mind, note the right hand sides of (8)-(10) are functions of p of the form $a\theta^p + b\theta^{-p} + (-1)^p D$. It is prudent at this point to analyze the closely related function $f(x) = a\theta^{cx+d} + b\theta^{-cx-d} + D$.

Lemma 4.10. *Let $f(x) = a\theta^{cx+d} + b\theta^{-cx-d} + D$ and $[x]$ be the nearest integer map, which we take to be multi-valued in the event x has fractional-part $\frac{1}{2}$. If $\theta > 0$ then*

1. $f(x) = 0 \Leftrightarrow x = \frac{1}{c}\log_\theta\left(\frac{-D}{2a} \pm \sqrt{\left(\frac{D}{2a}\right)^2 - \frac{b}{a}}\right) - \frac{d}{c}$

2. *Further, if $a, b > 0$ then*

   (a) $\operatorname*{argmin}_{x\in\mathbb{R}} f(x) = \frac{1}{2c}\log_\theta\left(\frac{b}{a}\right) - \frac{d}{c}$

   (b) $\operatorname*{argmin}_{p\in\mathbb{Z}} f(p) = \left[\frac{1}{2c}\log_\theta\left(\frac{b}{a}\right) - \frac{d}{c}\right]$

Proof. (1) and (2a) are elementary applications of calculus and the quadratic formula.

For (2b), let $x_0 = \frac{1}{2c}\log_\theta\left(\frac{b}{a}\right) - \frac{d}{c}$, $p_0 = \operatorname{argmin}_{p\in\mathbb{Z}} f(p)$, and $\epsilon = p_0 - x_0$. Then

$$\begin{aligned} f(p_0) &= f(x_0+\epsilon) \\ &= a\theta^{cx_0+d}\theta^{c\epsilon} + b\theta^{-cx_0-d}\theta^{-c\epsilon} + D \\ &= a\sqrt{\tfrac{b}{a}}\,\theta^{c\epsilon} + b\sqrt{\tfrac{a}{b}}\,\theta^{-c\epsilon} + D \\ &= \sqrt{ab}(\theta^{c\epsilon} + \theta^{-c\epsilon}) + D \qquad (*) \end{aligned}$$

Now, f is strictly convex since $a, b > 0$, therefore p_0 is limited in value to either $\lfloor x_0 \rfloor$, $\lceil x_0 \rceil$, or both if x_0 has fractional-part $\frac{1}{2}$. This implies that there are two possible values for ϵ and $(*)$ is minimized by choosing the one smallest in absolute value. Therefore $p_0 = [x_0]$. □

As an immediate application of lemma 4.10 we have

Observation 4.11. Let W be a degenerate fusion frame and consider the problem of identifying its vanishing subspaces. To fix the idea, suppose $W \sim W^G > W^{GS}$ or $W \sim W^G > W^{GN}$ for some $G \in \mathbb{G}$. Clearly this implies that either $k_i \neq 0 = k_i^{GS}$ or $w_i \neq 0 = w_i^{GN}$ for some $i \in [m]$. Now, let $k_{\min} = \min_i\{k_i\}$, $k_{\max} = \max_i\{k_i\}$, $w_{\min} = \min_i\{w_i\}$, and $w_{\max} = \max_i\{w_i\}$. By inspection of (10) and (13) we have

$$\underset{k_i}{\operatorname{argmin}}\left\{k_i^{(p)}\right\} = \begin{cases} k_{\min} & p \text{ is even} \\ k_{\max} & p \text{ is odd} \end{cases}$$

$$\underset{w_i}{\operatorname{argmin}}\left\{w_i^{(p)}\right\} = \begin{cases} w_{\min} & p \text{ is even} \\ w_{\max} & p \text{ is odd} \end{cases}$$

When attempting to identify vanishing subspaces it is therefore sufficient to limit our analysis to the parameters $k_{\min}$, $k_{\max}$, $w_{\min}$, and $w_{\max}$. Further, let

$$f_k^+(s) = \frac{n(\beta-\mu)}{\beta(\beta-\alpha)}\theta^{2s-1} + \frac{n(\mu-\alpha)}{\alpha(\beta-\alpha)}\theta^{-2s+1} - k_{\max} + \tfrac{n}{m}\mu$$

$$f_k^-(s) = \frac{n(\beta-\mu)}{\beta(\beta-\alpha)}\theta^{2s} + \frac{n(\mu-\alpha)}{\alpha(\beta-\alpha)}\theta^{-2s} + k_{\min} - \tfrac{n}{m}\mu$$

$$f_w^+(s)^2 = \frac{A(\beta-\nu)}{\beta(\beta-\alpha)}\theta^{2s-1} + \frac{A(\nu-\alpha)}{\alpha(\beta-\alpha)}\theta^{-2s+1} - w_{\max}^2 + \tfrac{A}{m}\nu$$

$$f_w^-(s)^2 = \frac{A(\beta-\nu)}{\beta(\beta-\alpha)}\theta^{2s} + \frac{A(\nu-\alpha)}{\alpha(\beta-\alpha)}\theta^{-2s} + w_{\min}^2 - \tfrac{A}{m}\nu$$

By inspection we have $f_k^+(s) = k_{\max}^{(2s-1)}$, $f_k^-(s) = k_{\min}^{(2s)}$, $f_w^+(s) = w_{\max}^{(2s-1)}$, and $f_w^-(s) = w_{\min}^{(2s)}$ for $s \in \mathbb{Z}$. It is therefore sufficient to limit our analysis to the functions f_k^+, f_k^-, f_w^+, and f_w^-. Finally, these functions are all of the form $a\theta^{cx+d} + b\theta^{-cx-d} + D$, therefore the results of lemma 4.10 apply.

We conclude this section with necessary and sufficient conditions for non-degeneracy.

Theorem 4.12. *Let W be a tight fusion frame with parameters μ and ν, let $[\cdot]$ be the nearest integer map, and define f_k^+, f_k^-, f_w^+, and f_w^- as in observation 4.11. Then a necessary condition for W to be non-degenerate is*

$$\mu, \nu \in (\alpha, \beta)$$

Further, if $\mu, \nu \in (\alpha, \beta)$ then the quantities

$$x_0 = \tfrac{1}{4}\log_\theta\left(\tfrac{\mu-\alpha}{\beta-\mu}\right) \qquad y_0 = \tfrac{1}{4}\log_\theta\left(\tfrac{\nu-\alpha}{\beta-\nu}\right)$$

are well defined, and W is non-degenerate if and only if

$$f_k^+\left([x_0+\tfrac{1}{2}]\right),\ f_k^-([x_0]),\ f_w^+\left([y_0+\tfrac{1}{2}]\right),\ f_w^-([y_0]) > 0$$

Proof. Assume W is non-degenerate and let $n^{(p)} = a\theta^p + b\theta^{-p}$. Proceding by contradiction, suppose $\mu \notin [\alpha, \beta]$. Then either a or b is negative. It follows that there exists a $p \in \mathbb{Z}$ such that $n^{(p)} < 0$. This implies that W is degenerate by observation 4.6, contradicting our assumption of non-degeneracy. We must therefore have $\mu \in [\alpha, \beta]$. A similar argument implies $\nu \in [\alpha, \beta]$.

Suppose now that $\mu \in \{\alpha, \beta\}$. Then either α or β is rational which requires $m = 4$. By general assumption $m \geq 5$ so it cannot be that $\mu \in \{\alpha, \beta\}$. It follows that $\mu \in (\alpha, \beta)$.

Finally, suppose $\nu \in \{\alpha, \beta\}$ and let $(w_i^2)^{(p)} = a\theta^p + b\theta^{-p} + (-1)^p D_i$. Since $\nu \in \{\alpha, \beta\}$ we must have either $a = 0$ or $b = 0$ by equation (13). If $a = 0$ or $b = 0$ then $\lim_{p \to L} a\theta^p + b\theta^{-p} = 0$ for either $L = -\infty$ or $L = +\infty$. It follows that, unless $D_i = 0$ for every i, there exists a $p \in \mathbb{Z}$ and $i \in [m]$ such that $(w_i^2)^{(p)} < 0$. This implies W is degenerate by observation 4.7, contradicting our assumption of non-degeneracy. Alternatively, if $D_i = 0$ for every i then W is equal-weight, therefore $w = w_0 \mathbf{1}_m$ for some $w_0 > 0$. This gives $\nu = \frac{1}{A} \sum_i w_i^2 = \frac{m w_0^2}{A} \in \{\alpha, \beta\}$. By the fusion frame equation we have

$$nA = \sum_{i=1}^{m} k_i w_i^2 = w_0^2 \sum_{i=1}^{m} k_i = w_0^2 M$$

and therefore $\nu = \frac{nm}{M} \in \{\alpha, \beta\}$. This again requires that either α or β is rational. We conclude that W is non-degenerate only if $\nu \in (\alpha, \beta)$.

If $\mu, \nu \in (\alpha, \beta)$ then the functions f_k^+, f_k^-, f_w^+, and f_w^- all have well defined minima over $\mathbb{Z}$ which are achieved at the integers specified in lemma 4.10(2b). By observation 4.11, these minima are all positive if and only if W is non-degenerate. □

5 Minimality

Given a tight fusion frame W there are evidently an infinity of transformations afforded by Naimark and spatial complementation. Broadly speaking, the difficulty of constructing a tight fusion frame with a given set of parameters is most heavily dependent on the number of subspaces, m; the dimension of the ambient Hilbert space, n; and the total subspace dimension, M. When attempting to directly construct a fusion frame it is therefore often desirable to consider NS-equivalent fusion frames with smaller values for these parameters. Once found, the simplified fusion frame can be used to reconstruct the desired fusion frame in a deterministic way. This approach begs the question of *which* NS-equivalent fusion frame to analyze, as there are potentially a significant number with more desirable values of m, n, or M. As the figure below partially suggests and as will be shown, in all cases the question is either moot or has a unique answer.

Theorem 5.1. *Let W be a tight fusion frame with parameters n, M, μ and suppose $\mu \in (\alpha, \beta)$. Then there exists a unique $q \in {}^1\!/_2\mathbb{Z}$ such that $q \in \operatorname{argmin}_{p\in{}^1\!/_2\mathbb{Z}} \left\{n^{(p)}\right\}$ and $q \in \operatorname{argmin}_{p\in{}^1\!/_2\mathbb{Z}} \left\{M^{(p)}\right\}$. Further, $q = 0$ if and only if $2 \le \mu \le \frac{m}{2}$.*

Proof. Observe that $n^{(p)}$ has the form $f(p) = a\theta^p + b\theta^{-p}$ as in lemma 4.10 with $c = 1$ and $d = D = 0$. Since $\mu \in (\alpha, \beta)$ by assumption we have $a, b > 0$ and can apply lemma 4.10(2b). Setting $x_0 = \frac{1}{2}\log_\theta\left(\frac{\mu-\alpha}{\beta-\mu}\right)$ and $p_0 = \operatorname{argmin}_{p\in{}^1\!/_2\mathbb{Z}} \left\{n^{(p)}\right\}$ we have

$$p_0 = \left[\frac{1}{2}\log_\theta\left(\frac{b}{a}\right)\right] = \left[\frac{1}{2}\log_\theta\left(\frac{\mu-\alpha}{\beta-\mu}\right)\right] = [x_0]$$

Next, observe $M^{(p)}$ also has the form $a\theta^p + b\theta^{-p}$ as in lemma 4.10. Setting $r_0 = \operatorname{argmin}_{r\in{}^1\!/_2\mathbb{Z}} \left\{M^{(r)}\right\}$ we have

$$r_0 = \left[\frac{1}{2}\log_\theta\left(\frac{b}{a}\right)\right] = \left[\frac{1}{2}\log_\theta\left(\frac{\beta(\mu-\alpha)}{\alpha(\beta-\mu)}\right)\right] = \left[\frac{1}{2}\log_\theta\left(\frac{\mu-\alpha}{\beta-\mu}\right) + \frac{1}{2}\right] = \left[x_0 + \tfrac{1}{2}\right]$$

We now identify the unique $q \in {}^1\!/_2\mathbb{Z}$ which simultaneously minimizes $n^{(q)}$ and $M^{(q)}$. There are three cases to consider:

1. Suppose x_0 is an integer. Then $p_0 = x_0$ and $r_0 = x_0 + \frac{1}{2} \pm \frac{1}{2}$. In this case, the

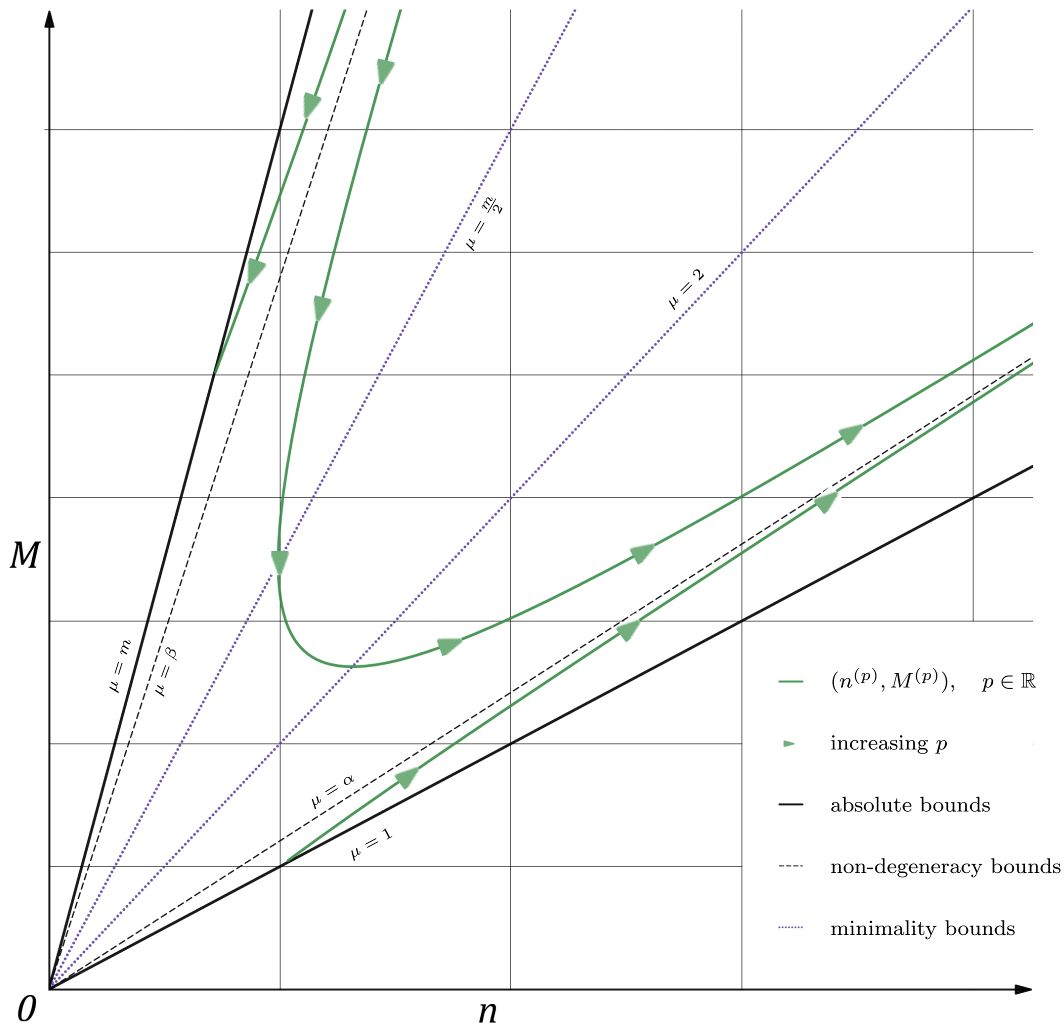

Figure 1: Relationship between Hilbert space dimension (n) and total subspace dimension (M) for a tight fusion frame with $n^{(p)}$ and $M^{(p)}$ (equations (8) and (9)) extended to $p \in \mathbb{R}$. All fusion frames require $\mu \in [1, m]$, non-degeneracy requires $\mu \in (\alpha, \beta)$, and minimality is equivalent to $\mu \in [2, \frac{m}{2}]$.

only value of q which simultaneously minimizes $n^{(q)}$ and $M^{(q)}$ is $q = x_0$.

2. Suppose x_0 is a half-integer. Then $p_0 = x_0 \pm \frac{1}{2}$ and $r_0 = x_0 + \frac{1}{2}$. In this case, the only value of q which simultaneously minimizes $n^{(q)}$ and $M^{(q)}$ is $q = x_0 + \frac{1}{2}$.

3. Suppose $x_0 \notin {}^1\!/_2\mathbb{Z}$. Then both p_0 and r_0 are single-valued and we must have either $r_0 = p_0$ or $r_0 = p_0 + 1$. In either case, setting $q = \frac{r_0 + p_0}{2} \in {}^1\!/_2\mathbb{Z}$ and

applying identities (5) and (6) yields

$$n^{(q)} = n^{(\lfloor q \rfloor)} = n^{(p_0)} = \min_{p\in\mathbb{Z}} \left\{ n^{(p)} \right\}$$

$$M^{(q)} = M^{(\lceil q \rceil)} = M^{(r_0)} = \min_{p\in\mathbb{Z}} \left\{ M^{(p)} \right\}$$

It remains to show that $q = 0 \Leftrightarrow 2 \leq \mu \leq \frac{m}{2}$, which we now demonstrate.

$$\begin{aligned} q = 0 &\Leftrightarrow 0 \in [x_0] \text{ and } 0 \in \left[x_0 + \tfrac{1}{2}\right] \\ &\Leftrightarrow -\tfrac{1}{2} \leq x_0 \leq 0 \\ &\Leftrightarrow -\tfrac{1}{2} \leq \tfrac{1}{2} \log_\theta \left(\tfrac{\mu-\alpha}{\beta-\mu} \right) \leq 0 \\ &\Leftrightarrow \tfrac{\alpha}{\beta} \leq \tfrac{\mu-\alpha}{\beta-\mu} \leq 1 \\ &\Leftrightarrow \alpha\beta - \alpha\mu \leq \beta\mu - \alpha\beta \text{ and } \mu - \alpha \leq \beta - \mu \\ &\Leftrightarrow 2m \leq (\alpha+\beta)\mu \text{ and } 2\mu \leq \alpha + \beta \\ &\Leftrightarrow 2 \leq \mu \text{ and } \mu \leq \tfrac{m}{2} \end{aligned}$$

□

Corollary 5.2. *Let W be a tight fusion frame with parameters n, M, and $m \geq 5$. If $2 \leq \mu \leq \frac{m}{2}$ then $n = \min[n]_{\mathrm{NS}}$ and $M = \min[M]_{\mathrm{NS}}$.*

Proof. Since $m \geq 5$ the constants α, β are well defined and satisfy $\alpha < 2 < \frac{m}{2} < \beta$. By theorem 5.1 we have

$$n \geq \min[n]_{\mathrm{NS}} \geq \min_{p\in\mathbb{Z}} \left\{ n^{(p)} \right\} = n^{(0)} = n$$

$$M \geq \min[M]_{\mathrm{NS}} \geq \min_{p\in\mathbb{Z}} \left\{ M^{(p)} \right\} = M^{(0)} = M$$

□

Corollary 5.3. *Let W be a non-degenerate tight fusion frame with parameters n and M. Then there exists, up to unitary equivalence, a unique $W' \in [W]_{\mathrm{NS}}$ with parameters n' and M' such that $n' = \min[n]_{\mathrm{NS}}$ and $M' = \min[M]_{\mathrm{NS}}$. Further, $W \approx W'$ if and only if $2 \leq \mu \leq \frac{m}{2}$.*

Proof. Since W is non-degenerate by assumption, for any parameter x of W we have $[x]_{\mathrm{NS}} = \{x^{(p)}\}_{p\in\mathbb{Z}}$. Further, as W is non-degenerate we have $\mu \in (\alpha, \beta)$ by theorem 4.12 and can therefore apply the results of theorem 5.1. By theorem 5.1 and identities (5) and (6) there exists a unique $q \in {}^1\!/\!_2\mathbb{Z}$ satisfying

$$n^{(q)} = n^{(\lfloor q \rfloor)} = \min_{p\in\mathbb{Z}}\{n^{(p)}\} = \min[n]_{\mathrm{NS}}$$

$$M^{(q)} = M^{(\lceil q \rceil)} = \min_{p\in\mathbb{Z}}\{M^{(p)}\} = \min[M]_{\mathrm{NS}}$$

Finally, the minimal fusion frame W' is given by

$$W' \approx \begin{cases} W^{(SN)^q} & q \text{ is an integer} \\ W^{(SN)^{\lfloor q \rfloor}S} & q \text{ is a half-integer} \end{cases}$$

so by theorem 5.1 we have $W \approx W' \Leftrightarrow q = 0 \Leftrightarrow 2 \leq \mu \leq \frac{m}{2}$. □

Theorem 5.1 and its corollaries accomplish several things. First, they assert the existence of a minimal element of the NS-equivalence class of non-degenerate fusion frames (corollary 5.3) and certain degenerate fusion frames (corollary 5.2). Second, they provide an elementary criterion for identifying this minimal element, $2 \leq \mu \leq \frac{m}{2}$. Finally, they demonstrate in their proofs how to readily obtain such a minimal element from any other member of the NS-equivalence class using lemma 4.10.

Our next goal will be to establish a certain uniqueness in the ways in which a fusion frame can be degenerate, effectively extending proposition 4.3(5). This uniqueness will allow us to extend the partial order $\gtrsim$ on fusion frames to a total order on the majority of NS-equivalence classes. Combined with corollary 5.3, this will provide a powerful tool for classifying and constructing fusion frames. Before proceeding along these lines we require the following lemma which identifies several objects invariant under Naimark and spatial complementation.

Lemma 5.4 (NS-invariants). *Let $W = (W_i, w_i)_{i=1}^m$ be a tight fusion frame with parameters k and w. Then*

1. *For all $i, j \in [m]$ and all $G \in \mathbb{G}$ such that $W \sim W^G$ we have*

$$\left\{\left|W_i \cap W_j^{\perp}\right| \ , \ \left|W_i^{\perp} \cap W_j\right|\right\} = \left\{\left|W_i^G \cap W_j^{G\perp}\right| \ , \ \left|W_i^{G\perp} \cap W_j^G\right|\right\}$$

2. *For all $i, j \in [m]$ and all $p \in {}^1\!/_2\mathbb{Z}$ we have*

$$\begin{aligned} |k_i - k_j| &= \left|k_i^{(p)} - k_j^{(p)}\right| \\ |w_i^2 - w_j^2| &= \left|w_i^{2(p)} - w_j^{2(p)}\right| \end{aligned}$$

Proof (1). We show only that the lemma holds for $G = S$ and $G = N$. The full lemma then follows by induction.

Trivially we have $W_i^S \cap W_j^{S\perp} = W_i^{\perp} \cap W_j^{\perp\perp} = W_i^{\perp} \cap W_j$, so by symmetry the lemma holds for $G = S$.

For the case $G = N$ we must take care to avoid ambiguity. Fix an equal-norm, orthogonal basis B of $W_i \cap W_j^{\perp}$ such that $\|b\| = w_i$ for all $b \in B$. Let Φ be a frame representation for W such that $B \subset \Phi$. Then b^N is well-defined and $b^N \in W_i^N$ for all $b \in B$. Additionally, $b \perp W_j$ implies $b^N \perp W_j^N$, therefore $|W_i \cap W_j^{\perp}| \leq |W_i^N \cap W_j^{N\perp}|$. Repeating this process for a basis B' of $W_i^N \cap W_j^{N\perp}$ yields $|W_i^N \cap W_j^{N\perp}| \leq |W_i^{NN} \cap W_j^{NN\perp}|$. By assumption we have $W \sim W^N$ and therefore $W^{NN} \approx W$. It follows that

$$\left|W_i \cap W_j^{\perp}\right| \leq \left|W_i^N \cap W_j^{N\perp}\right| \leq \left|W_i^{NN} \cap W_j^{NN\perp}\right| = \left|W_i \cap W_j^{\perp}\right|$$

Therefore $|W_i \cap W_j^{\perp}| = |W_i^N \cap W_j^{N\perp}|$ and the lemma holds for $G = N$. $\square$

Proof (2). Fix any $p \in {}^1\!/_2\mathbb{Z}$ and let $q = \lceil p \rceil$. Then identity (7) and equation (11) give

$$\begin{aligned}
\left|k_i^{(p)} - k_j^{(p)}\right| &= \left|k_i^{(q)} - k_j^{(q)}\right| \\
&= \left|\frac{M^{(q)}}{m} + (-1)^q(k_i - \tfrac{n}{m}\mu) - \left(\frac{M^{(q)}}{m} + (-1)^q(k_j - \tfrac{n}{m}\mu)\right)\right| \\
&= \left|(-1)^q(k_i - \tfrac{n}{m}\mu) - (-1)^q(k_j - \tfrac{n}{m}\mu)\right| \\
&= \left|k_i - \tfrac{n}{m}\mu - k_j + \tfrac{n}{m}\mu\right| \\
&= |k_i - k_j|
\end{aligned}$$

The analogues of (7) and (11) provide a similar result for w_i^2. $\square$

A fusion frame W is said to be **doubly-degenerate** if there exist $X_1, X_2 \in \{N, S\}$ and $G_1, G_2 \in \mathbb{G}$ such that

$$W^{G_1X_1} < W^{G_1} \sim W \sim W^{G_2} > W^{G_2X_2}$$

As NS-equivalence is transitive and $\emptyset \in \mathbb{G}$, double-degeneracy is equivalent to

$$W^{X_1} < W \sim W^G > W^{GX_2}.$$

Theorem 5.5 (Double Degeneracy)**.** *Let W be a doubly-degenerate fusion frame with $W^{X_1} < W \sim W^G > W^{GX_2}$ for some $X_1, X_2 \in \{N, S\}$ and $G \in \mathbb{G}$. Further, let $J_1 = \{i | W_i^{X_1} = 0\}$ and $J_2 = \{j | W_j^{GX_2} = 0\}$. Then*

1. $X_1 = X_2$

2. $J_1 = J_2$

3. $W \gtrsim 0$

Informally, "every doubly-degenerate fusion frame is totally degenerate."

Proof (1). Let W be doubly-degenerate as above, suppose $i \in J_1$, $j \in J_2$, and denote by L and R the invariant sets in lemma 5.4(1):

$$L := \{|W_i \cap W_j^\perp|, |W_i^\perp \cap W_j|\} = \{|W_i^G \cap W_j^{G\perp}|, |W_i^{G\perp} \cap W_j|\} =: R.$$

Application of proposition 4.3(1) and 4.3(2) yields

$$\begin{aligned} X_1 = S \quad &\Rightarrow \quad L = \{|W_j^\perp|, 0\} \\ X_2 = S \quad &\Rightarrow \quad R = \{0, |W_i^{G\perp}|\} \\ X_1 = N \quad &\Rightarrow \quad L = \{|W_i|, |W_j|\} \\ X_2 = N \quad &\Rightarrow \quad R = \{|W_i^G|, |W_j^G|\}. \end{aligned}$$

Therefore the only permissible combinations are $X_1 = X_2 = N$ and $X_1 = X_2 = S$.

□

Proof (2). In light of (1), if W is doubly-degenerate then there exist $i, j \in [m]$, $p \in \mathbb{Z}$ such that one of the following hold:

$$\begin{aligned} k_i^S = 0 = k_j^{(SN)^p}, \quad p \leq -1 \\ w_i^N = 0 = w_j^{(SN)^p}, \quad p \geq 1 \end{aligned}$$

These equations imply $k_j = k_{\min}$ (resp. $w_j = w_{\min}$) for p even, and $k_j = k_{\max}$ (resp. $w_j = w_{\max}$) for p odd. It follows that $J_1 = J_2$ is equivalent to p being odd.

There are evidently 4 cases to consider based on the sign and parity of p. The results and intermediate steps of a case-by-case analysis are provided in the table below, which utilizes the quantities x_0, y_0 from theorem 4.12:

$$x_0 := \tfrac{1}{4}\log_\theta\left(\tfrac{\mu-\alpha}{\beta-\mu}\right) \qquad y_0 := \tfrac{1}{4}\log_\theta\left(\tfrac{\nu-\alpha}{\beta-\nu}\right)$$

A sketch of the analysis is also provided, however computational details are left to the reader.

Table 1: Summary of analysis for theorem 5.5

X	p		zeros		minimizers over $\mathbb{Z}$		bounds on p
S	odd	≤ -1	$f_k^+(\frac{p+1}{2})$	$f_k^+(1)$	$[x_0+\frac{3}{4}] \leq \frac{p+1}{2}$	$1 \leq [x_0+\frac{3}{4}]$	$0 \leq x_0+\frac{1}{4} \leq \frac{p+1}{2}$
S	even	≤ -2	$f_k^-(\frac{p}{2})$	$f_k^+(1)$	$[x_0+\frac{1}{4}] \leq \frac{p}{2}$	$1 \leq [x_0+\frac{3}{4}]$	$0 \leq x_0+\frac{1}{4} \leq \frac{p+1}{2}$
N	odd	≥ 1	$f_w^+(0)$	$f_w^+(\frac{p+1}{2})$	$[y_0+\frac{3}{4}] \leq 0$	$\frac{p+1}{2} \leq [y_0+\frac{3}{4}]$	$\frac{p-1}{2} \leq y_0+\frac{1}{4} \leq 0$
N	even	≥ 2	$f_w^+(0)$	$f_w^-(\frac{p}{2})$	$[y_0+\frac{3}{4}] \leq 0$	$\frac{p}{2} \leq [y_0+\frac{1}{4}]$	$\frac{p-1}{2} \leq y_0+\frac{1}{4} \leq 0$

Note first that this table is complete based on the possible values of X (column 1) and parities of p (column 2). The permissible values of p (column 2) follow from the given parity and the first sentence of this proof. The vanishing functions (f_k^+, f_k^-, f_w^+, or f_w^-) and their zeros (column 3) likewise follow from the opening sentence and their definitions in observation 4.11. As the functions f_k^+, f_k^- (resp. f_w^+, f_w^-) are monotonic whenever $\mu \notin (\alpha, \beta)$ (resp. $\nu \notin (\alpha, \beta)$), we may conclude that $\mu \in (\alpha, \beta)$ (resp. $\nu \in (\alpha, \beta)$) since they have multiple roots. This implies that these functions are strictly convex and therefore their minima over $\mathbb{Z}$ can be constrained relative to the location of their zeros (column 4). The resulting inequalities can then be solved to provide additional bounds on p (column 5).

By inspection of the final column, if $X = S$ then $p \geq -1$. As $p \leq -1$ by assumption, it follows that $p = -1$ and W satisfies $W^S < W \sim W^N > W^{NS}$. Similarly, if $X = N$ then $p \leq 1$. As $p \geq 1$ by assumption, it follows that $p = 1$ and

$W^N < W \sim W^S > W^{SN}$.

□

Proof (3). In light of (2), if W is doubly degenerate then either $f_k^+(0) = f_k^+(1)$ or $f_w^+(0) = f_w^+(1)$ based on the value of X. We consider each case individually. Diagrams of the doubly-degenerate fusion frames described below are provided in Appendix B, which the reader is encouraged to consult at this time.

If $X = S$ then (2) implies that $f_k^+(0) = f_k^+(1)$. Solving this equation yields $\mu = 2$. We may therefore conclude that W has one subspace (say, W_1) of dimension n and the total dimension of the remaining subspaces is also n. In other words

$$k_1 = n = \sum_{i=2}^{m} k_i$$

Now, consider the tight fusion frame $W^{SS} = (W_i, w_i)_{i=2}^m$. The total subspace dimension of this fusion frame and the dimension of its ambient Hilbert space are both n, therefore these subspaces must be mutually orthogonal. It follows that they all vanish under a Naimark complement and so $W^{SSN} = 0$.

If $X = N$ then (2) implies that $f_w^+(0) = f_w^+(1)$. Solving this equation yields $\nu = 2$. We may therefore conclude that W has one subspace (say, W_1) with squared-weight A and the total squared-weights of the remaining subspaces is also A. In other words

$$w_1^2 = A = \sum_{i=2}^{m} w_i^2$$

Now, consider the tight fusion frame W^{NN}. The total squared-weights of this fusion frame and its frame bound are both equal to A, therefore each subspace has full rank. It follows that all subspaces vanish under a spatial complement and so $W^{NNS} = 0$.

□

Theorem 5.5 asserts that if a tight fusion frame W is degenerate (but not totally degenerate) then it can only be degenerate in one way. This carries with it several essentially obvious yet critical consequences:

Corollary 5.6. *Let $W \gtrsim\!\!\!\!/\; 0$ be a tight fusion frame. Then*

1. *$\{W^G\}_{G\in\mathbb{G}}$ can be decomposed into at most m NS-equivalence classes given by*

$$\left\{W^G\right\}_{G\in\mathbb{G}} = \bigsqcup_{i=0}^{q} \left[W^{G_0\cdots G_i}\right]_{\mathrm{NS}}$$

   where $q \leq m-1$ and $G_0 = \emptyset$.

2. *The ordering on the set of NS-equivalence classes $\left\{\left[W^G\right]_{\mathrm{NS}}\right\}_{G\in\mathbb{G}}$ induced by $\gtrsim$ is a total order, i.e.*

$$[W]_{\mathrm{NS}} = \left[W^{G_0}\right]_{\mathrm{NS}} > \left[W^{G_0G_1}\right]_{\mathrm{NS}} > \cdots > \left[W^{G_0\cdots G_q}\right]_{\mathrm{NS}}$$

   and all members of the 'minimal' equivalence class $\left[W^{G_0\cdots G_q}\right]_{\mathrm{NS}}$ are necessarily non-degenerate.

We note that the use of the word 'minimal' in reference to $\left[W^{G_0\cdots G_q}\right]_{\mathrm{NS}}$ is appropriate for at least two reasons. Most obviously, $\left[W^{G_0\cdots G_q}\right]_{\mathrm{NS}}$ is minimal in the sense of the ordering induced by $\gtrsim$. This follows in part from the second reason, that the number of subspaces common to the members of each equivalence class also exhibit the same ordering under $\geq$:

$$m = m^{G_0} > m^{G_0G_1} > \cdots > m^{G_0\cdots G_q}$$

There is another sense in which $[W^{G_0\cdots G_q}]_{\mathrm{NS}}$ can be considered minimal, as the following lemma suggests:

Lemma 5.7. *Let $W \gtrsim\!\!\!\!/\; 0$ be a degenerate fusion frame with $W > W^X$ for some $X \in \{N, S\}$. Then there exists a $G \in \mathbb{G}$ such that $n^{XG} \leq \min[n]_{\mathrm{NS}}$ and $M^{XG} \leq \min[M]_{\mathrm{NS}}$.*

Proof. We examine 5 cases based on the possible values of $X \in \{N, S\}$ and $\mu \in [1, m]$. The results are summarized in the table below.

Table 2: Summary of analysis for lemma 5.7

X	μ	p	$y^{(p)}$	$\min[n]_{\text{NS}}$	$\min[M]_{\text{NS}}$	XG
S	$\in [1, 2)$	-	-	-	-	-
S	$\in [2, m]$	≤ 0	dec.	$n^{(0)} = n$	$M^{(0)} = M$	SS
N	$\in [1, \frac{m}{2})$	≥ 0	dec.	$n^{(0)} = n$	$M^{(0)} = M$	NN
N	$\in [\frac{m}{2}, m-2]$	≥ 0	inc.	$n^{(1/2)} = n^S$	$M^{(1/2)} = M^S$	NNS
N	$\in (m-2, m]$	-	-	-	-	-

For each case we identify the values of p satisfying $y^{(p)} \in [y]_{\text{NS}}$ and the corresponding behavior of $y^{(p)}$ for $y \in \{n, M\}$ (column 3). These may be determined from figure 1 and the given values of X and μ since, for any such parameter y, theorem 5.5 implies that $[y]_{\text{NS}}$ is given by

$$X = S \Rightarrow [y]_{\text{NS}} = \{y, y^N, y^{NS}, \dots\} = \{y^{(p)}\}_{p \leq 0}$$
$$X = N \Rightarrow [y]_{\text{NS}} = \{y, y^S, y^{SN}, \dots\} = \{y^{(p)}\}_{p \geq 0}$$

The minimums $\min[y]_{\text{NS}}$ (column 4) may be determined by applying the requirement $2 \leq \mu \leq \frac{m}{2}$ from theorem 5.1 to the appropriate transformation of μ. It remains to demonstrate the impossibility of rows 1 and 5 as well as certify the indicated values of XG (column 5) satisfy $y^{XG} \leq \min[y]_{\text{NS}}$.

Suppose $X = S$. Then W has s vanishing subspaces, each of dimension n, for some integer $s \geq 1$. As $W \ngtrsim 0$ by assumption it follows that W^{SS} is a tight fusion frame which satisfies

$$M^{SS} = M - ns < M = \min[M]_{\text{NS}}$$
$$n^{SS} = n = \min[n]_{\text{NS}}$$

certifying row 2. Further, as $1 \leq \mu^G \leq m^G$ for every $G \in \mathbb{G}$ we must have

$$1 \leq \mu^{SS} = \frac{M^{SS}}{n^{SS}} = \frac{M - ns}{n} = \mu - s \leq \mu - 1$$

. Rearranging yields $2 \leq \mu$, certifying row 1.

Suppose instead that $X = N$. Then W has s vanishing subspaces whose total subspace dimension is t for some integers $1 \leq s \leq t$. As $W \not\gtrsim 0$ by assumption it follows that W^{NN} is a tight fusion frame which satisfies

$$M^{NN} = M - t < M = \min[M]_{\text{NS}}$$
$$n^{NN} = n - t < n = \min[n]_{\text{NS}}$$

certifying row 3.

Suppose further that W^{NN} has r full-rank subspaces of dimension $n^{NN} = n - t$ for some integer $r \geq 0$. As $W \not\gtrsim 0$ by assumption it follows that W^{NNS} is a tight fusion frame which satisfies

$$\begin{aligned}
M^{NNS} &= (m^{NN} - r)n^{NN} - M^{NN} \\
&= (m - s - r)(n - t) - (M - t) \\
&= (nm - M) - (ns - t) - nr - t(m - s - r) \\
&< M^S - 0 - 0 - 0 = M^S = \min[M]_{\text{NS}} \\
n^{NNS} &= n^{NN} = n - t < n = n^S = \min[n]_{\text{NS}}
\end{aligned}$$

certifying row 4. Finally, as W satisfies $n^G \leq M^G$ for all $G \in \mathbb{G}$ we must have

$$\begin{aligned}
0 &\leq M^{NNS} - n^{NNS} \\
&= (m^{NN} - r)n^{NN} - M^{NN} - n^{NN} \\
&= (m - s - r)(n - t) - (M - t) - (n - t) \\
&= n(m - s - r - 1) - t(m - s - r - 2) - M \\
&\leq n(m - 2) - M
\end{aligned}$$

Rearranging yields $\mu \leq m - 2$, certifying row 5. □

Theorem 5.5 implies that, for every tight fusion frame W which is degenerate but not totally degenerate, there exists a unique NS-equivalence class $[W']_{\text{NS}} \subset \{W^G\}_{G \in \mathbb{G}}$ of non-degenerate fusion frames with a minimal number of subspaces, m'. Corollary 5.3 establishes that, for any NS-equivalence class of non-degenerate fusion frames $[W']_{\text{NS}}$, there exists a unique element $W'' \in [W']_{\text{NS}}$ which possesses simultaneously the minimum dimension for its ambient Hilbert space, n'', and minimum total subspace dimension, M'', among all representatives of $[W']_{\text{NS}}$. We conclude this section with a theorem establishing that M'' and n'' are, in fact, minimums over the larger set of fusion frames $\{W^G\}_{G \in \mathbb{G}} \supset [W']_{\text{NS}}$.

Theorem 5.8. *Let W be a tight fusion frame with parameters m, n, and M. If $W \not\gtrsim 0$ then there exists a unique $W_0 \lesssim W$ with parameters m_0, n_0, and M_0 such that*

$$m_0 = \min_{G \in \mathbb{G}} \left\{ m^G \right\}$$

$$n_0 = \min_{G \in \mathbb{G}} \left\{ n^G \right\}$$

$$M_0 = \min_{G \in \mathbb{G}} \left\{ M^G \right\}.$$

Further, there exists a unique $G \in \mathbb{G}$ of minimal length such that $W_0 \approx W^G$.

Proof. If W is non-degenerate then this theorem follows directly from corollary 5.3. Suppose, then, that W is degenerate. By corollary 5.6 there exists an NS-equivalence class of non-degenerate fusion frames $[W']_{\text{NS}}$ such that $[W]_{\text{NS}} > [W']_{\text{NS}}$ and $m' = \min_{G \in \mathbb{G}} \left\{ m^G \right\}$. Further, there exists a positive integer q, a sequence $(X_i)_{i=1}^q$ in $\{N, S\}$, and a sequence $(G_i)_{i=1}^q$ in $\mathbb{G}$ such that

$$W \sim W^{G_1} > W^{G_1 X_1} \sim W^{G_1 X_1 G_2} > \cdots > W^{G_1 X_1 \cdots G_q X_q} \in [W']_{\text{NS}}.$$

The integer q and sequence $(X_i)_{i=1}^q$ are unique by theorem 5.5. Further, each G_i is unique when chosen to be of minimal length since $W \sim W^X$ implies $W \approx W^{XX}$ for $X \in \{N, S\}$. By corollary 5.3 there exists a $G_{q+1} \in \mathbb{G}$ such that

$$n^{G_1 X_1 \cdots G_q X_q G_{q+1}} = \min \left[n^{G_1 X_1 \cdots G_q X_q} \right]_{\text{NS}}$$

$$M^{G_1 X_1 \cdots G_q X_q G_{q+1}} = \min \left[M^{G_1 X_1 \cdots G_q X_q} \right]_{\text{NS}}.$$

G_{q+1} is also unique when chosen to be of minimal length, therefore so is $G =$

$G_1X_1 \cdots G_qX_qG_{q+1}$. Repeated application of lemma 5.7 gives

$$n^{G_1X_1\cdots G_qX_qG_{q+1}} = \min\left[n^{G_1X_1\cdots G_qX_q}\right]_{\text{NS}} \leq \cdots \leq \min\left[n^{G_1X_1}\right]_{\text{NS}} \leq \min[n]_{\text{NS}}$$
$$M^{G_1X_1\cdots G_qX_qG_{q+1}} = \min\left[M^{G_1X_1\cdots G_qX_q}\right]_{\text{NS}} \leq \cdots \leq \min\left[M^{G_1X_1}\right]_{\text{NS}} \leq \min[M]_{\text{NS}}.$$

By corollary 5.6 the sets $\{n^G\}_{G\in\mathbb{G}}$ and $\{M^G\}_{G\in\mathbb{G}}$ are partitioned by these equivalence classes, therefore $n_0 = n^{G_1X_1\cdots G_qX_qG_{q+1}}$, $M_0 = M^{G_1X_1\cdots G_qX_qG_{q+1}}$, and $W_0 \approx W^{G_1X_1\cdots G_qX_qG_{q+1}}$. □

6 Construction and Classification

By construction the action of $\cdot^N$ and $\cdot^S$ on non-degenerate fusion frames is a group of two involutions. This is not true for degenerate fusion frames, however, since in general we do not have $W^{NN} \approx W$ and $W^{SS} \approx W$. For example, let $I = \left\{i|W_i^S = 0\right\}$ and assume $0 < |I| < m$. To recover W one must adjoin $|I|$ subspaces with appropriate weights and dimension n to W^{SS}. The weights (or squared-weights) of these subspaces must be specified, however the subspace dimensions are determined solely based on the dimension of the ambient Hilbert space, $n^{SS} = n$. The new frame bound can be derived from the new weights according to the fusion frame equation $An = \sum_i k_i w_i^2$. If W is totally degenerate and $|I| = m$ then W^{SS} is undefined, however W can trivially be specified by a set of weights and Hilbert space dimension, n. Similarly, if $J = \{j|W_j^N = 0\}$ and $0 < |J| < m$ then to recover W one must adjoin $|J|$ orthogonal subspaces with appropriate dimensions and weight $\sqrt{A}$ to W^{NN}. The dimensions of these subspaces must be specified, however the weights are determined solely based on the frame bound, $A^{NN} = A$. The dimension of the ambient Hilbert space is likewise derived from the new subspace dimensions according to $An = \sum_i k_i w_i^2$. If W is totally degenerate and $|J| = m$ then W^{NN} is undefined, however W can trivially be specified by a sequence of subspace dimensions and a frame bound, A. In either case, recovering W requires only specifying a particular sequence of subspace dimensions or weights and an additional value for the Hilbert space dimension or frame bound in the event of total degeneracy. We therefore define

Definition 6.1 (NS-augmentation)**.** *Let $W = (W_i, w_i)_{i=1}^m$ be an A-tight fusion frame for $\mathcal{H}^n$ with $k = (|W_i|)_{i=1}^m$.*

1. *A* ***Naimark augment*** *of W is any fusion frame of the form*

$$W_{N(a)} := (W_i, w_i)_{i=1}^m \cup (\mathcal{H}^{a_1}, \sqrt{A}) \cup \cdots \cup (\mathcal{H}^{a_s}, \sqrt{A})$$

   where $a = (a_i)_{i=1}^s$ is any sequence of positive integers and the ambient Hilbert space for $W_{N(a)}$ has the decomposition $\mathcal{H}^n \oplus \mathcal{H}^{a_1} \oplus \cdots \oplus \mathcal{H}^{a_s}$. Alternatively, the squared-weights may be explicitly stated with the notation $W_{N(a)} = W_{N(a|A)}$.

2. *A* ***spatial augment*** *of W is any fusion frame of the form*

$$W_{S(a)} := (W_i, w_i)_{i=1}^m \cup (\mathcal{H}^n, \sqrt{a_1}) \cup \cdots \cup (\mathcal{H}^n, \sqrt{a_s})$$

   where $a = (a_i)_{i=1}^s$ is any sequence of positive real numbers and the ambient Hilbert space for $W_{S(a)}$ is $\mathcal{H}^n$. Alternatively, the subspace dimensions may be explicitly stated with the notation $W_{S(a)} = W_{S(a|n)}$.

The extended notations $\cdot_{N(\cdot|\cdot)}$ and $\cdot_{S(\cdot|\cdot)}$ may also be applied to 0 with the same interpretation as (1) and (2) above.

Combining these definitions with the results of the previous section, we have

Theorem 6.2 (Tight Fusion Frame Classification)**.** *For every tight fusion frame $W \not\gtrsim 0$ there exists a unique positive integer q, a unique sequence $(G_i)_{i=1}^{q+1}$ of minimal length elements of $\mathbb{G}$, a unique sequence $(X_i)_{i=1}^q$ in $\{N, S\}$, and a sequence $(x_i)_{i=1}^q$ of* *sequences* $(x_{ij})_{j=1}^{s_i}$ *of positive numbers such that*

$$W \approx W_0^{G_1}{}_{X_1(x_1)}{}^{G_2}{}_{X_2(x_2)} \cdots {}^{G_q}{}_{X_q(x_q)}{}^{G_{q+1}}$$

and the parameters m_0*,* n_0*, and* M_0 *of* W_0 *are simultaneously minimized among all members of* $\{W^G | G \in \mathbb{G}\}$*.*

For every tight fusion frame $W \gtrsim 0$ *there exists a positive integer* q*, a sequence* $(G_i)_{i=1}^{q+1}$ *in* $\mathbb{G}$*, a sequence* $(X_i)_{i=0}^{q}$ *in* $\{N, S\}$*, a sequence* $(x_i)_{i=0}^{q}$ *of sequences* $(x_{ij})_{j=1}^{s_i}$ *of positive numbers, and a positive number* y *such that*

$$W \approx 0_{X_0(x_0|y)}{}^{G_1}{}_{X_1(x_1)} \cdots {}^{G_q}{}_{X_q(x_q)}{}^{G_{q+1}}$$

To demonstrate the broad applicability of the operations given in definition 6.1, we have produced below a complete table of maximal, equal-weight fusion frames in dimensions 3 through 8. This is similar to what was done in [1], but with a complete description in terms of definition 6.1. Of the 58 possible sequences of subspace dimensions, only 7 are not totally degenerate.

Table 3: NS classification of tight, equal-norm fusion frames for $3 \leq n \leq 8$

$n = 3$		
M	max elements	NS classification
3	(3)	$0_{S(\mathbf{1}_1\|3)}$
4	(1,1,1,1)	$0_{S(1/4\,\mathbf{1}_4\|1)}{}^{N}$
5	(2,1,1,1)	$0_{S(1/5\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/5\,\mathbf{1}_1)}{}^{N}$
6	(3,3)	$0_{S(1/2\,\mathbf{1}_2\|3)}$

$n = 4$		
4	(4)	$0_{S(\mathbf{1}_1\|4)}$
5	(1,1,1,1,1)	$0_{S(1/5\,\mathbf{1}_5\|1)}{}^{N}$
6	(2,2,2)	$0_{S(1/3\,\mathbf{1}_3\|2)}{}^{N}$
7	(3,1,1,1,1)	$0_{S(1/7\,\mathbf{1}_4\|1)}{}^{N}{}_{S(3/7\,\mathbf{1}_1)}{}^{N}$
	(2,2,2,1)	$0_{S(1/7\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/7\,\mathbf{1}_1)}{}^{NSN}$
8	(4,4)	$0_{S(1/2\,\mathbf{1}_2\|4)}$

$n = 5$		
5	(5)	$0_{S(\mathbf{1}_1\|5)}$
6	(1,1,1,1,1,1)	$0_{S(1/6\,\mathbf{1}_6\|1)}{}^{N}$
7	(2,2,1,1,1)	$0_{S(1/7\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/7\,\mathbf{1}_2)}{}^{N}$
8	(3,2,1,1,1)	$0_{S(1/8\,\mathbf{1}_3\|1)}{}^{N}{}_{S(1/4\,\mathbf{1}_1)}{}^{N}{}_{S(3/8\,\mathbf{1}_1)}{}^{N}$
	(2,2,2,2)	$0_{S(1/8\,\mathbf{1}_4\|1)}{}^{NSN}$
9	(4,1,1,1,1,1)	$0_{S(1/9\,\mathbf{1}_5\|1)}{}^{N}{}_{S(4/9\,\mathbf{1}_1)}{}^{N}$
	(3,2,2,2)	$0_{S(1/9\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/9\,\mathbf{1}_1)}{}^{NSNSN}$
10	(5,5)	$0_{S(1/2\,\mathbf{1}_2\|5)}$

$n = 6$		
6	(6)	$0_{S(\mathbf{1}_1\|6)}$
7	(1,1,1,1,1,1,1)	$0_{S(1/7\,\mathbf{1}_7\|1)}{}^{N}$
8	(2,2,2,2)	$0_{S(1/4\,\mathbf{1}_4\|2)}{}^{N}$
9	(3,3,3)	$0_{S(1/3\,\mathbf{1}_3\|3)}{}^{N}$
10	(4,2,2,2)	$0_{S(1/5\,\mathbf{1}_3\|2)}{}^{N}{}_{S(2/5\,\mathbf{1}_1)}{}^{N}$
11	(5,1,1,1,1,1,1)	$0_{S(1/11\,\mathbf{1}_6\|1)}{}^{N}{}_{S(5/11\,\mathbf{1}_1)}{}^{N}$
	(4,2,2,2,1)	${W_0}^{N}$
	(3,3,3,2)	$0_{S(1/11\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/11\,\mathbf{1}_1)}{}^{N(SN)^3}$
12	(6,6)	$0_{S(1/2\,\mathbf{1}_2\|6)}$

$n = 7$		
M	max elements	NS classification
7	(7)	$0_{S(\mathbf{1}_1\|7)}$
8	(1,1,1,1,1,1,1,1)	$0_{S(1/8\,\mathbf{1}_8\|1)}{}^{N}$
9	(2,2,2,1,1,1)	$0_{S(1/9\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/9\,\mathbf{1}_3)}{}^{N}$
10	(3,3,1,1,1,1)	$0_{S(1/10\,\mathbf{1}_4\|1)}{}^{N}{}_{S(3/10\,\mathbf{1}_2)}{}^{N}$
	(3,2,2,2,1)	$0_{S(1/10\,\mathbf{1}_3\|1)}{}^{N}{}_{S(1/5\,\mathbf{1}_1)}{}^{NS}{}_{S(3/10\,\mathbf{1}_1)}{}^{N}$
11	(4,3,1,1,1,1)	$0_{S(1/7\,\mathbf{1}_4\|4/7)}{}^{N}{}_{S(3/7\,\mathbf{1}_1)}{}^{N}{}_{S(4/7\,\mathbf{1}_1)}{}^{N}$
	(4,2,2,2,1)	$0_{S(1/11\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/11\,\mathbf{1}_1)}{}^{NSN}{}_{S(4/11\,\mathbf{1}_1)}{}^{N}$
12	(5,2,2,1,1,1)	$0_{S(1/12\,\mathbf{1}_3\|1)}{}^{N}{}_{S(1/6\,\mathbf{1}_2)}{}^{N}{}_{S(5/12\,\mathbf{1}_1)}{}^{N}$
	(4,3,3,1,1)	${W_0}^{N}$
	(3,3,3,3)	$0_{S(1/12\,\mathbf{1}_4\|1)}{}^{NSNSN}$
13	(6,1,1,1,1,1,1,1)	$0_{S(1/13\,\mathbf{1}_7\|1)}{}^{N}{}_{S(6/13\,\mathbf{1}_1)}{}^{N}$
	(5,2,2,2,2)	${W_0}^{N}$
	(4,3,3,3)	$0_{S(1/13\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/13\,\mathbf{1}_1)}{}^{N(SN)^4}$
14	(7,7)	$0_{S(1/2\,\mathbf{1}_2\|7)}$

$n = 8$		
8	(8)	$0_{S(\mathbf{1}_1\|8)}$
9	(1,1,1,1,1,1,1,1,1)	$0_{S(1/9\,\mathbf{1}_9\|1)}{}^{N}$
10	(2,2,2,2,2)	$0_{S(1/5\,\mathbf{1}_5\|2)}{}^{N}$
11	(3,2,2,2,2)	$0_{S(1/11\,\mathbf{1}_4\|1)}{}^{NS}{}_{S(3/11\,\mathbf{1}_1)}{}^{N}$
	(3,3,2,1,1,1)	$0_{S(1/11\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/11\,\mathbf{1}_1)}{}^{N}{}_{S(3/11\,\mathbf{1}_2)}{}^{N}$
12	(4,4,4)	$0_{S(1/3\,\mathbf{1}_3\|4)}{}^{N}$
13	(5,3,2,1,1,1)	$0_{S(1/13\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/13\,\mathbf{1}_1)}{}^{N}\cdots$
		$\cdots{}_{S(3/13\,\mathbf{1}_1)}{}^{N}{}_{S(5/13\,\mathbf{1}_1)}{}^{N}$
	(5,2,2,2,2)	$0_{S(1/13\,\mathbf{1}_4\|1)}{}^{NSN}{}_{S(5/13\,\mathbf{1}_1)}{}^{N}$
	(4,4,2,2,1)	${W_0}^{NS}$
14	(6,2,2,2,2)	$0_{S(1/7\,\mathbf{1}_4\|2)}{}^{N}{}_{S(3/7\,\mathbf{1}_1)}{}^{N}$
	(5,3,3,2,1)	${W_0}^{N}$
	(4,4,4,2)	$0_{S(1/7\,\mathbf{1}_3\|2)}{}^{N}{}_{S(2/7\,\mathbf{1}_1)}{}^{NSN}$
15	(7,1,1,1,1,1,1,1,1)	$0_{S(7/15\,\mathbf{1}_8\|1)}{}^{N}{}_{S(7/15\,\mathbf{1}_1)}{}^{N}$
	(6,2,2,2,2,1)	${W_0}^{N}$
	(5,3,3,2,2)	${W_0}^{N}$
	(4,4,4,3)	$0_{S(1/15\,\mathbf{1}_3\|1)}{}^{N}{}_{S(2/15\,\mathbf{1}_1)}{}^{N(SN)^5}$
16	(8,8)	$0_{S(1/2\,\mathbf{1}_2\|8)}$

A Fusion frames with $m \leq 4$ subspaces

The main body of this paper addresses fusion frames with $m \geq 5$ subspaces. The remaining cases with $m \leq 4$ subspaces are presented below in the form of 'NS-trees,' wherein we demonstrate how the parameters of such fusion frames transform under complement operations. For a given fusion frame the parameters n, A, k, and w^2 are listed as elements of a block matrix

$$\left[\begin{array}{c|c} n & A \\ \hline k & w^2 \end{array}\right] = \left[\begin{array}{c|c} n & A \\ \hline k_1 & w_1^2 \\ \vdots & \vdots \\ k_m & w_m^2 \end{array}\right]$$

and arrows are used to indicate degeneracy and NS-equivalence. For example, if W and W^S are tight fusion frames with respective parameter matrices X and X^S, then $X \overset{.S}{\longleftrightarrow} X^S$ indicates $W \sim W^S$ and $X \overset{.S}{\longrightarrow} X^S$ indicates $W > W^S$. The '$-$' symbol denotes that the corresponding subspace has vanished and '0' that all subspaces have vanished. Ellipses indicate that no subspaces will ever vanish under continued complement operations, and if a given NS-tree becomes equivalent to a previously listed one because of vanishing subspaces then the corresponding tree is given. That these are the only possible parameter configurations can be deduced by reasoning similar to that used in the final steps of the proof of theorem 5.5, which we leave to the reader.

For $m = 1$ there is only one NS-tree, T1, which is given below for arbitrary values of n and A. Such fusion frames are the reason proposition 4.3(5) requires $m \geq 2$ and are the only fusion frames satisfying $W^N < W \sim W^G > W^{GS}$; i.e. they are the only fusion frames which violate $X_1 = X_2$ in theorem 5.5.

T1:

$$0 \xleftarrow{.S} \left[\begin{array}{c|c} n & A \\ \hline n & A \end{array}\right] \xrightarrow{.N} 0$$

For $m = 2$ there are two possible NS-trees, corresponding to the cases $n = k_1 + k_2$ (T2-1) and $n = k_1 = k_2$ (T2-2).

T2-1:

$$0 \xleftarrow{.N} \left[\begin{array}{c|c} n & A \\ \hline k_2 & A \\ k_1 & A \end{array}\right] \overset{.S}{\longleftrightarrow} \left[\begin{array}{c|c} n & A \\ \hline k_1 & A \\ k_2 & A \end{array}\right] \xrightarrow{.N} 0$$

T2-2:

$$0 \xleftarrow{.S} \left[\begin{array}{c|c} n & A \\ \hline n & w_1^2 \\ n & w_2^2 \end{array}\right] \overset{.N}{\longleftrightarrow} \left[\begin{array}{c|c} n & A \\ \hline n & w_2^2 \\ n & w_1^2 \end{array}\right] \xrightarrow{.S} 0$$

For $m = 3$ there are three possible NS-trees, corresponding to the cases $n = k_1 + k_2 + k_3$ (T3-1), $n = k_1 = k_2 + k_3$ (T3-2), and $n = k_1 = k_2 = k_3$ (T3-3).

T3-1:

$$0 \xleftarrow{.N} \left[\begin{array}{c|c} n & A \\ \hline k_1 & A \\ k_2 & A \\ k_3 & A \end{array}\right] \overset{.S}{\longleftrightarrow} \left[\begin{array}{c|c} n & 2A \\ \hline n - k_1 & A \\ n - k_2 & A \\ n - k_3 & A \end{array}\right] \overset{.N}{\longleftrightarrow} \left[\begin{array}{c|c} n & 2A \\ \hline n - k_1 & A \\ n - k_2 & A \\ n - k_3 & A \end{array}\right] \overset{.S}{\longleftrightarrow} \left[\begin{array}{c|c} n & A \\ \hline k_1 & A \\ k_2 & A \\ k_3 & A \end{array}\right] \xrightarrow{.N} 0$$

T3-2:

$$\text{T2-1} \ni \left[\begin{array}{c|c} n & A-w_1^2 \\ \hline - & - \\ k_3 & A-w_1^2 \\ k_2 & A-w_1^2 \end{array}\right] \xleftarrow{.S} \left[\begin{array}{c|c} n & A \\ \hline n & w_1^2 \\ k_2 & A-w_1^2 \\ k_3 & A-w_1^2 \end{array}\right] \xleftrightarrow{.N} \left[\begin{array}{c|c} n & A \\ \hline n & A-w_1^2 \\ k_2 & w_1^2 \\ k_3 & w_1^2 \end{array}\right] \xrightarrow{.S} \left[\begin{array}{c|c} n & w_1^2 \\ \hline - & - \\ k_3 & w_1^2 \\ k_2 & w_1^2 \end{array}\right] \in \text{T2-1}$$

T3-3:

$$0 \xleftarrow{.S} \left[\begin{array}{c|c} n & A \\ \hline n & w_1^2 \\ n & w_2^2 \\ n & w_3^2 \end{array}\right] \xleftrightarrow{.N} \left[\begin{array}{c|c} 2n & A \\ \hline n & A-w_1^2 \\ n & A-w_2^2 \\ n & A-w_3^2 \end{array}\right] \xleftrightarrow{.S} \left[\begin{array}{c|c} 2n & A \\ \hline n & A-w_1^2 \\ n & A-w_2^2 \\ n & A-w_3^2 \end{array}\right] \xleftrightarrow{.N} \left[\begin{array}{c|c} n & A \\ \hline n & w_1^2 \\ n & w_2^2 \\ n & w_3^2 \end{array}\right] \xrightarrow{.S} 0$$

If $m = 4$ then the matrix

$$\mathcal{G} = \left[\begin{array}{c|c} 3 & -\mathbf{1}_4^{\mathrm{T}} \\ \hline \mathbf{1}_4 & -I_4 \end{array}\right]$$

is defective with the following properties:

$$\begin{aligned} \text{characteristic polynomial} \qquad & p_{\mathcal{G}}(x) = (x+1)^3(x-1)^2 \\ \text{eigenspace} \qquad & E_{-1} = (0) \oplus \mathbf{1}_4^{\perp} \\ \text{eigenvector} \qquad & v_1 = (2,1,1,1,1)^{\mathrm{T}} \\ \text{generalized eigenvector} \qquad & v_2 = (1,0,0,0,0)^{\mathrm{T}} \\ \text{generalized eigenspace} \qquad & E_{\text{gen}} = \operatorname{span}\{v_1, v_2\} \end{aligned}$$

The corresponding parameter transformations may be computed via the Jordan decomposition of $\mathcal{G}$ and are given by

$$n^{(p)} = n + n(2-\mu)p \tag{14}$$

$$M^{(p)} = M + 2n(2-\mu)p \tag{15}$$

$$k_i^{(p)} = \tfrac{1}{4}M + \tfrac{1}{2}n(2-\mu)p + (-1)^p(k_i - \tfrac{1}{4}M) \tag{16}$$

$$A^{(p)} = A\nu + 2A(2-\nu)p \tag{17}$$

$$(w_i^2)^{(p)} = \tfrac{1}{4}A\nu + \tfrac{1}{2}A(2-\nu)p + (-1)^p(w_i^2 - \tfrac{1}{4}A\nu) \tag{18}$$

The degeneracy conditions are calculated using (16) and (18) and are given by

$$p \text{ even } \Rightarrow \begin{cases} k_i^{(p)} = 0 \Leftrightarrow p = \frac{2k_i}{n(\mu-2)} \\ (w_i^2)^{(p)} = 0 \Leftrightarrow p = \frac{2w_i^2}{A(\nu-2)} \end{cases} \tag{19}$$

$$p \text{ odd } \Rightarrow \begin{cases} k_i^{(p)} = 0 \Leftrightarrow p = \frac{2k_i - n\mu}{n(\mu-2)} \\ (w_i^2)^{(p)} = 0 \Leftrightarrow p = \frac{2w_i^2 - A\nu}{A(\nu-2)} \end{cases} \tag{20}$$

In general we have bounds $1 \leq \mu, \nu \leq m$ so by inspection of (14) and (17) there exists a $p \in \mathbb{Z}$ such that $n^{(p)} < 0$ and/or $A^{(p)} < 0$ unless $\mu = \nu = 2$. It follows that any tight fusion frame W with $m = 4$ subspaces is degenerate unless $\mu = \nu = 2$. Since all fusion frames with $m \leq 3$ subspaces are totally degenerate, W is also totally degenerate unless $\mu = \nu = 2$. If $\mu = \nu = 2$ then the NS-tree for W is a closed loop:

$$
\begin{array}{ccc}
\left[\begin{array}{c|c} n & A \\ \hline k_1 & w_1^2 \\ k_2 & w_2^2 \\ k_3 & w_3^2 \\ k_4 & w_4^2 \end{array}\right] & \stackrel{.S}{\longleftrightarrow} & \left[\begin{array}{c|c} n & A \\ \hline n-k_1 & w_1^2 \\ n-k_2 & w_2^2 \\ n-k_3 & w_3^2 \\ n-k_4 & w_4^2 \end{array}\right] \\
\updownarrow .N & & \updownarrow .N \\
\left[\begin{array}{c|c} n & A \\ \hline k_1 & A-w_1^2 \\ k_2 & A-w_2^2 \\ k_3 & A-w_3^2 \\ k_4 & A-w_4^2 \end{array}\right] & \stackrel{.S}{\longleftrightarrow} & \left[\begin{array}{c|c} n & A \\ \hline n-k_1 & A-w_1^2 \\ n-k_2 & A-w_2^2 \\ n-k_3 & A-w_3^2 \\ n-k_4 & A-w_4^2 \end{array}\right]
\end{array}
$$

B Doubly-degenerate fusion frames

The NS-trees for the doubly-degenerate fusion frames described in the proof of theorem 5.5 are given below. An explanation of the symbols used was provided in Appendix A.

If $W^S < W \sim W^G > W^{GS}$ then $n = k_1 = \sum_{i\geq 2} k_i$ and the NS-tree for W is of the form

$$\begin{array}{ccccccccc}
\cdots \overset{.N}{\longleftrightarrow} & \left[\begin{array}{c|c} n & (m-2)(A-w_1^2) \\ \hline - & - \\ n-k_2 & A-w_1^2 \\ \vdots & \vdots \\ n-k_m & A-w_1^2 \end{array}\right] & \overset{.S}{\longleftarrow} & \left[\begin{array}{c|c} n & A \\ \hline n & w_1^2 \\ k_2 & A-w_1^2 \\ \vdots & \vdots \\ k_m & A-w_1^2 \end{array}\right] & \overset{.N}{\longleftrightarrow} & \left[\begin{array}{c|c} n & A \\ \hline n & A-w_1^2 \\ k_2 & w_1^2 \\ \vdots & \vdots \\ k_m & w_1^2 \end{array}\right] & \overset{.S}{\longrightarrow} & \left[\begin{array}{c|c} n & (m-2)w_1^2 \\ \hline - & - \\ n-k_2 & w_1^2 \\ \vdots & \vdots \\ n-k_m & w_1^2 \end{array}\right] & \overset{.N}{\longleftrightarrow} \cdots \\
 & \big\updownarrow .S & & & & & & \big\updownarrow .S & \\
0 \overset{.N}{\longleftarrow} & \left[\begin{array}{c|c} n & A-w_1^2 \\ \hline - & - \\ k_2 & A-w_1^2 \\ \vdots & \vdots \\ k_m & A-w_1^2 \end{array}\right] & & & & & & \left[\begin{array}{c|c} n & w_1^2 \\ \hline - & - \\ k_2 & w_1^2 \\ \vdots & \vdots \\ k_m & w_1^2 \end{array}\right] \overset{.N}{\longrightarrow} 0 &
\end{array}$$

If $W^N < W \sim W^G > W^{GN}$ then $A = w_1^2 = \sum_{i \geq 2} w_i^2$ and the NS-tree for W is of the form

$$\cdots \overset{.S}{\longleftrightarrow} \left[\begin{array}{c|c} (m-2)(n-k_1) & A \\ \hline - & - \\ n-k_1 & A-w_2^2 \\ \vdots & \vdots \\ n-k_1 & A-w_m^2 \end{array}\right] \xleftarrow{.N} \left[\begin{array}{c|c} n & A \\ \hline k_1 & A \\ n-k_1 & w_2^2 \\ \vdots & \vdots \\ n-k_1 & w_m^2 \end{array}\right] \overset{.S}{\longleftrightarrow} \left[\begin{array}{c|c} n & A \\ \hline n-k_1 & A \\ k_1 & w_2^2 \\ \vdots & \vdots \\ k_1 & w_m^2 \end{array}\right] \xrightarrow{.N} \left[\begin{array}{c|c} (m-2)k_1 & A \\ \hline - & - \\ k_1 & A-w_2^2 \\ \vdots & \vdots \\ k_1 & A-w_m^2 \end{array}\right] \overset{.S}{\longleftrightarrow} \cdots$$

$$\Big\updownarrow {.N} \qquad\qquad\qquad\qquad \Big\updownarrow {.N}$$

$$0 \xleftarrow{.S} \left[\begin{array}{c|c} n-k_1 & A \\ \hline - & - \\ n-k_1 & w_2^2 \\ \vdots & \vdots \\ n-k_1 & w_m^2 \end{array}\right] \qquad\qquad \left[\begin{array}{c|c} k_1 & w_1^2 \\ \hline - & - \\ k_1 & w_2^2 \\ \vdots & \vdots \\ k_1 & w_m^2 \end{array}\right] \xrightarrow{.S} 0$$

C Proof of lemma 4.8

Proof. To determine the characteristic polynomial and eigenspaces of the matrix $\mathcal{G}$, we compute

$$\det(\lambda I_{m+1} - \mathcal{G}) = \det \left[\begin{array}{c|cccc} \lambda - m + 1 & 1 & 1 & \cdots & 1 \\ \hline -1 & \lambda + 1 & & & \\ -1 & & \lambda + 1 & & \\ \vdots & & & \ddots & \\ -1 & & & & \lambda + 1 \end{array}\right]$$
$$= (\lambda - m + 1)(\lambda + 1)^m + \sum_{j=2}^{m+1} (-1)^{j-1}(1) \det(M_{1j})$$

where we have written separately the first term in the determinant expansion and used M_{1j} as the minor of $\lambda I_{m+1} - \mathcal{G}$ obtained by omitting row 1 and column j. For $j \geq 2$ the determinant of M_{1j} contains a single nonzero term given by

$$\det(M_{1j}) = (-1)^j(-1)(\lambda + 1)^{m-1}$$

This yields

$$\begin{aligned} \det(\lambda I_{m+1} - \mathcal{G}) &= (\lambda - m + 1)(\lambda + 1)^m + \sum_{j=2}^{m+1} (-1)^{j-1}(1)(-1)^j(-1)(\lambda + 1)^{m-1} \\ &= (\lambda - m + 1)(\lambda + 1)^m + m(\lambda + 1)^{m-1} \\ &= (\lambda + 1)^{m-1}((\lambda - m + 1)(\lambda + 1) + m) \\ &= (\lambda + 1)^{m-1}(\lambda^2 + (2 - m)\lambda + 1) \\ &= (\lambda + 1)^{m-1} \left(\lambda - \frac{m+\sqrt{m(m-4)}}{m-\sqrt{m(m-4)}}\right) \left(\lambda - \frac{m-\sqrt{m(m-4)}}{m+\sqrt{m(m-4)}}\right) \end{aligned}$$

In order to compute $\mathcal{G}^p \kappa$, we define the constants

$$\alpha = \frac{m-\sqrt{m(m-4)}}{2},\ \beta = \frac{m+\sqrt{m(m-4)}}{2},\ \gamma = \frac{2}{\sqrt{m-4}},\ \delta = \sqrt{\frac{m}{m-4}},\ \text{and}\ \theta = \frac{m-\sqrt{m(m-4)}}{m+\sqrt{m(m-4)}}$$

and vectors

$$a = \frac{1}{\sqrt{m\alpha}}\begin{bmatrix}\alpha \\ \hline \mathbf{1}_m\end{bmatrix},\ b = \frac{1}{\sqrt{m\beta}}\begin{bmatrix}\beta \\ \hline \mathbf{1}_m\end{bmatrix},\ \text{and}\ c = \gamma b - \delta a$$

Note that $\|a\| = \|b\| = \|c\| = 1$, $E_{-1} \perp \text{span}\{a, b\}$, $c \in \text{span}\{a, b\}$, and $c \perp b$. Taken together, these facts allow us to decompose an arbitrary vector $x \in \mathbb{R}^{m+1}$ as

$$x = P_b x + P_c x + P_{E_{-1}} x$$

We now have the following calculation for $\mathcal{G}^p x$:

$$\begin{aligned}
\mathcal{G}^p x &= \mathcal{G}^p(P_b x + P_c x + P_{E_{-1}} x) \\
&= \langle x, b\rangle \mathcal{G}^p b + \langle x, c\rangle \mathcal{G}^p c + \mathcal{G}^p P_{E_{-1}} x \\
&= \langle x, b\rangle \mathcal{G}^p b + \langle x, \gamma b - \delta a\rangle (\gamma \mathcal{G}^p b - \delta \mathcal{G}^p a) + \mathcal{G}^p P_{E_{-1}} x \\
&= (\langle x, b\rangle + \langle x, \gamma^2 b - \delta\gamma a\rangle)\mathcal{G}^p b + \langle x, \delta^2 a - \delta\gamma b\rangle \mathcal{G}^p a + \mathcal{G}^p P_{E_{-1}} x \\
&= \langle x, (1+\gamma^2) b - \delta\gamma a\rangle \mathcal{G}^p b + \langle x, \delta^2 a - \delta\gamma b\rangle \mathcal{G}^p a + \mathcal{G}^p P_{E_{-1}} x \\
&= \langle x, \delta^2 b - \delta\gamma a\rangle \theta^p b + \langle x, \delta^2 a - \delta\gamma b\rangle \theta^{-p} a + (-1)^p P_{E_{-1}} x \\
&= \delta\langle x, \delta b - \gamma a\rangle \theta^p b + \delta\langle x, \delta a - \gamma b\rangle \theta^{-p} a + (-1)^p P_{E_{-1}} x
\end{aligned}$$

Rearranging and setting $x = \kappa$ we have

$$
\begin{aligned}
\frac{1}{\delta}(\mathcal{G}^p-(-1)^pP_{E_{-1}})\kappa &= \langle\kappa,\delta b-\gamma a\rangle\theta^p b+\langle\kappa,\delta a-\gamma b\rangle\theta^{-p}a \\
&= \left\langle \left[\frac{n}{k}\right], \frac{\delta}{\sqrt{m\beta}}\left[\frac{\beta}{\mathbf{1}_m}\right]-\frac{\gamma}{\sqrt{m\alpha}}\left[\frac{\alpha}{\mathbf{1}_m}\right]\right\rangle \frac{\theta^p}{\sqrt{m\beta}}\left[\frac{\beta}{\mathbf{1}_m}\right]+ \\
&\qquad + \left\langle \left[\frac{n}{k}\right], \frac{\delta}{\sqrt{m\alpha}}\left[\frac{\alpha}{\mathbf{1}_m}\right]-\frac{\gamma}{\sqrt{m\beta}}\left[\frac{\beta}{\mathbf{1}_m}\right]\right\rangle \frac{\theta^{-p}}{\sqrt{m\alpha}}\left[\frac{\alpha}{\mathbf{1}_m}\right] \\
&= \left\langle \left[\frac{n}{k}\right], \frac{\delta}{m\beta}\left[\frac{\beta}{\mathbf{1}_m}\right]-\frac{\gamma}{m\sqrt{m}}\left[\frac{\alpha}{\mathbf{1}_m}\right]\right\rangle \theta^p\left[\frac{\beta}{\mathbf{1}_m}\right]+ \\
&\qquad + \left\langle \left[\frac{n}{k}\right], \frac{\delta}{m\alpha}\left[\frac{\alpha}{\mathbf{1}_m}\right]-\frac{\gamma}{m\sqrt{m}}\left[\frac{\beta}{\mathbf{1}_m}\right]\right\rangle \theta^{-p}\left[\frac{\alpha}{\mathbf{1}_m}\right] \\
&= \frac{\delta}{m}\left\langle \left[\frac{n}{k}\right], \frac{1}{\beta}\left[\frac{\beta}{\mathbf{1}_m}\right]-\frac{2}{m}\left[\frac{\alpha}{\mathbf{1}_m}\right]\right\rangle \theta^p\left[\frac{\beta}{\mathbf{1}_m}\right]+ \\
&\qquad + \frac{\delta}{m}\left\langle \left[\frac{n}{k}\right], \frac{1}{\alpha}\left[\frac{\alpha}{\mathbf{1}_m}\right]-\frac{2}{m}\left[\frac{\beta}{\mathbf{1}_m}\right]\right\rangle \theta^{-p}\left[\frac{\alpha}{\mathbf{1}_m}\right] \\
&= \frac{\delta}{m}\left(\frac{n\beta+M}{\beta}-\frac{2n\alpha+2M}{m}\right)\theta^p\left[\frac{\beta}{\mathbf{1}_m}\right]+ \\
&\qquad + \frac{\delta}{m}\left(\frac{n\alpha+M}{\alpha}-\frac{2n\beta+2M}{m}\right)\theta^{-p}\left[\frac{\alpha}{\mathbf{1}_m}\right] \\
&= \frac{\delta}{m}\left(\frac{nm\beta+mM-2nm-2\beta M}{m\beta}\right)\theta^p\left[\frac{\beta}{\mathbf{1}_m}\right]+ \\
&\qquad + \frac{\delta}{m}\left(\frac{nm\alpha+mM-2nm-2M\alpha}{m\alpha}\right)\theta^{-p}\left[\frac{\alpha}{\mathbf{1}_m}\right] \\
&= \frac{\delta n}{m^2\beta}(m\beta+m\mu-2m-2\beta\mu)\theta^p\left[\frac{\beta}{\mathbf{1}_m}\right]+ \\
&\qquad + \frac{\delta n}{m^2\alpha}(m\alpha+m\mu-2m-2\alpha\mu)\theta^{-p}\left[\frac{\alpha}{\mathbf{1}_m}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\delta n}{m^2\beta}(m(\beta-2)+\mu(m-2\beta))\theta^p \left[\frac{\beta}{\mathbf{1}_m}\right] + \\
&\qquad + \frac{\delta n}{m^2\alpha}(\mu(m-2\alpha)+m(\alpha-2))\theta^{-p} \left[\frac{\alpha}{\mathbf{1}_m}\right] \\
&= \frac{\delta n}{m^2\beta}(\beta(\beta-\alpha)-\mu(\beta-\alpha))\theta^p \left[\frac{\beta}{\mathbf{1}_m}\right] + \\
&\qquad + \frac{\delta n}{m^2\alpha}(\mu(\beta-\alpha)-\alpha(\beta-\alpha))\theta^{-1} \left[\frac{\alpha}{\mathbf{1}_m}\right] \\
&= \frac{\delta n(\beta-\alpha)}{m^2\beta}(\beta-\mu)\theta^p \left[\frac{\beta}{\mathbf{1}_m}\right] + \frac{\delta n(\beta-\alpha)}{m^2\alpha}(\mu-\alpha)\theta^p \left[\frac{\alpha}{\mathbf{1}_m}\right] \\
&= \frac{\delta n(\beta-\alpha)}{m^2}\left(\frac{\beta-\mu}{\beta}\theta^p \left[\frac{\beta}{\mathbf{1}_m}\right] + \frac{\mu-\alpha}{\alpha}\theta^{-p} \left[\frac{\alpha}{\mathbf{1}_m}\right]\right) \\
&= \frac{n}{m}\left(\frac{\beta-\mu}{\beta}\theta^p \left[\frac{\beta}{\mathbf{1}_m}\right] + \frac{\mu-\alpha}{\alpha}\theta^{-p} \left[\frac{\alpha}{\mathbf{1}_m}\right]\right)
\end{aligned}
$$

Rearranging, we have

$$
\mathcal{G}^p\kappa = \frac{n\delta}{m}\left(\frac{\beta-\mu}{\beta}\theta^p \left[\frac{\beta}{\mathbf{1}_m}\right] + \frac{\mu-\alpha}{\alpha}\theta^{-p} \left[\frac{\alpha}{\mathbf{1}_m}\right]\right) + (-1)^p P_{E_{-1}}\kappa
$$

as desired. □

References

[1] Marcin Bownik, Kurt Luoto, and Edward Richmond. A combinatorial characterization of tight fusion frames. *Pacific Journal of Mathematics*, 275(2):257–294, May 2015.

[2] Robert Calderbank, Andreas Heinecke, Gitta Kutyniok, and Ali Pezeshki. Sparse fusion frames: Existence and construction. *Adv. Comput. Math.*, 35:1–31, 07 2011.

[3] Peter Casazza, Andreas Heinecke, Keri Kornelson, Yang Wang, and Zhengfang Zhou. Necessary and sufficient conditions to perform spectral tetris, 2012.

[4] Peter Casazza and Gitta Kutyniok. Frame of subspaces. *Contemporary Math.*, 345:87–114, 01 2004.

[5] Peter Casazza and Jesse Peterson. Weighted fusion frame construction via spectral tetris. *Advances in Computational Mathematics*, 40, 07 2012.

[6] Peter G. Casazza, Matthew Fickus, Andreas Heinecke, Yang Wang, and Zhengfang Zhou. Spectral tetris fusion frame constructions. In Manos Papadakis, Dimitri Van De Ville, and Vivek K. Goyal, editors, *SPIE Proceedings*. SPIE, sep 2011.

[7] Peter G. Casazza, Matthew Fickus, Dustin G. Mixon, Yang Wang, and Zhengfang Zhou. Constructing tight fusion frames. *Applied and Computational Harmonic Analysis*, 30(2):175–187, 2011.

[8] Peter G. Casazza and Richard G. Lynch. A brief introduction to hilbert space frame theory and its applications, 2015.

[9] P.G. Casazza and G. Kutyniok, editors. *Finite Frames: Theory and Applications*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, MA, 2013.

[10] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic fourier series. *Transactions of the American Mathematical Society*, 72(2):341–366, 1952.

[11] Gitta Kutyniok and Shidong Li. Fusion frames and distributed processing. *Applied and Computational Harmonic Analysis*, 25:114–132, 07 2008.

Ian Campbell graduated from Jefferson City High School in 2003. He served as an Electrician's Mate in the US Navy from 2003 until 2013 and completed 4 deployments while stationed in Pearl Harbor onboard the USS Olympia. After completing his enlistment Ian received a bachelor's degree in mathematics from the University of Wisconsin-Madison in 2015 and a PhD in mathematics from the University of Missouri-Columbia in 2022.