Analytic measures and Bochner measurability

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1 Introduction

Many authors have made great strides in extending the celebrated F. and M. Riesz Theorem to various abstract settings. Most notably, we have, in chronological order, the work of Bochner [1], Helson and Lowdenslager [7], de Leeuw and Glicksberg [4], and Forelli [6]. These formidable papers build on each other's ideas and provide broader extensions of the F. and M. Riesz Theorem. Our goal in this paper is to use the analytic Radon-Nikodým property and prove a representation theorem (Main Lemma 2.2 below) for a certain class of measure-valued mappings on the real line. Applications of this result yield the the main theorems from [4] and [6]. First, we will review briefly the results with which we are concerned, and describe our main theorem.

The F. and M. Riesz Theorem states that if a complex Borel measure μ on the circle is such that

$$\int_{-\pi}^{\pi} e^{-int} d\mu(t) = 0, \text{ for all } n < 0,$$

(i.e. μ is analytic) then μ is absolutely continuous with respect to Lebesgue measure. The first extension is due to Bochner [1] who used very elaborate methods to show that if the Fourier transform of a complex Borel measure on the two dimensional torus vanishes off a sector with opening strictly less than π, then the measure is absolutely continuous with respect to the two-dimensional Lebesgue measure. A few years later, Helson and Lowdenslager [7], and de Leeuw and Glicksberg [4] revisited this theorem and offered different proofs based on their abstract versions of the F. and M. Riesz theorem. The paper [7] is classical; it contains seminal work in harmonic analysis on ordered groups, an area of analysis that flourished in the decades that followed. In [7], a measure is called analytic if its Fourier transform vanishes on the negative characters, and their version of the F. and M. Riesz Theorem states:

if a measure μ is analytic, then its absolutely continuous part and its singular part, with respect to Haar measure on the group, are both analytic.

Looking at the F. and M. Riesz Theorem from a different perspective, de Leeuw and Glicksberg considered the setting of a compact abelian group G on which the real line $\mathbf{R}$ is acting by translation via a continuous homomorphism from $\mathbf{R}$ into G. Thus the dual homomorphism maps the dual group of G to $\mathbf{R}$. In this setting, analytic measures are those with Fourier transforms supported on the inverse image of the positive real line. The de Leeuw-Glicksberg version of the F. and M. Riesz Theorem states:

the Borel subsets of G on which an analytic measure vanishes identically is invariant under the action of $\mathbf{R}$.

De Leeuw and Glicksberg called a measure whose null sets are invariant under $\mathbf{R}$ quasi-invariant. With this terminology, their result states that every analytic measure is quasi-invariant.

The notion of quasi-invariance and analyticity were extended by Forelli [6] to the setting in which the real line is acting on a locally compact topological space. Since Forelli's setting is closest to ours, we will describe it in greater detail.

Forelli's main results Let Ω be a locally compact Hausdorff space, and let $T:\ t \mapsto T_t$ denote a representation of the real line $\mathbf{R}$ by homeomorphisms of the topological space Ω such that the mapping $(t, \omega) \mapsto T_t\omega$ is jointly continuous. The action of $\mathbf{R}$ on Ω induces, in a natural way, an action on the Baire measures on Ω. With a slight abuse of notation, if μ is a Baire measure and A is a Baire subset of Ω, we write $T_t\mu$ for the Baire measure whose value at A is $\mu(T_tA)$. Denote the Baire subsets by Σ, and the Baire measures by $M(\Omega, \Sigma)$, or simply $M(\Sigma)$. A measure ν in $M(\Sigma)$ is called quasi-invariant if the collection of subsets of Σ on which ν vanishes identically is invariant by T. That is, ν is quasi-invariant if $|\nu|(T_tA) = 0$ for all t if and only if $|\nu|(A) = 0$.

Using the representation T, one can define the spectrum of a measure in $M(\Sigma)$ (see (2) below), which plays the role of the support of the Fourier transform of a measure. A measure in $M(\Sigma)$ is then called analytic if its spectrum lies on the nonnegative real axis. With this terminology, Forelli's main result states that:

an analytic measure is quasi-invariant.

As a corollary of this result, Forelli [6, Theorem 4] showed that analytic measures translate continuously. That is,

if μ is analytic, then $t \mapsto T_t\mu$ is continuous from $\mathbf{R}$ *into $M(\Sigma)$.*

When Ω is the real line, and T_t stands for translation by t, a quasi-invariant measure, or a measure for which the mapping $t \mapsto T_t\mu$ is continuous is necessarily absolutely continuous with respect to Lebesgue measure. These facts were observed by de Leeuw and Glicksberg [4] and for these reasons the main results in [4] and [6] are viewed as extensions of the F. and M. Riesz Theorem.

Goals of this paper Although Forelli proves that analytic measures translate continuously as a consequence of quasi-invariance, it can be shown that, vice-versa, in the setting of Forelli's paper, the quasi-invariance of analytic measures is a consequence of the continuity of the mapping $t \mapsto T_t\mu$ (see Section 5 below). The latter approach is the one that we take in this paper. As we now describe, this approach has many advantages, and the main results of this paper cannot be obtained using Forelli's methods.

Let Σ denote a sigma algebra of subsets of a set Ω and let $M(\Sigma)$ denote the space of complex measures defined on Σ. Suppose that $T:\ t \mapsto T_t$ is a uniformly bounded group of isomorphisms of $M(\Sigma)$. Using the representation T, we can define the notion of analytic measures as in [6], or as described in Definition 1.2 below. For an analytic μ in $M(\Sigma)$, we ask: under what conditions on T is the mapping $t \mapsto T_t\mu$ continuous?

Clearly, if this mapping is to be continuous, then the following must hold: if ν is analytic such that for every $A \in \Sigma$, $T_t\nu(A) = 0$ for almost all $t \in \mathbf{R}$ then ν is the zero measure.

Our main results (Theorems 2.5 and 3.4 below) prove that the converse is also true. We call the property that we just described hypothesis (A) (see Definition 1.3 below), and show, for example, that if a representation T, given by mappings of the sigma algebra, satisfies hypothesis (A), then the mapping $t \mapsto T_t\mu$ is Bochner measurable whenever μ is analytic. Using this fact, we can derive with ease all the main properties of analytic measures that were obtained by Forelli [6]. By imposing the right conditions on T, we are able to use the analytic Radon-Nikodým property of the Banach space $M(\Sigma)$ to give short and perspicuous proofs which dispense with several unnecessary conditions on the representation. In particular, in many interesting situations, we do not even need the fact that the collection of operators $(T_t)_{t\in\mathbf{R}}$ forms a group under composition.

Notation and Definitions We use the symbols $\mathbf{Q}, \mathbf{R}$, and $\mathbf{C}$ to denote the rational numbers, the real numbers, and the complex numbers respectively. The circle group will be denoted by $\mathbf{T}$ and will be customarily parametrized as $\{e^{it} :\ 0 \leq t < 2\pi\}$. Our measure theory is borrowed from Hewitt and Ross [8]. In particular, the convolution of measures and functions is defined as in [8, §20]. We denote by $M(\mathbf{R})$ the Banach space of complex regular Borel measures on $\mathbf{R}$. The space of Lebesgue measurable integrable functions on $\mathbf{R}$ is denoted by $L^1(\mathbf{R})$, and the space of essentially bounded measurable functions by $L^\infty(\mathbf{R})$. The spaces $H^1(\mathbf{R})$ and $H^\infty(\mathbf{R})$ are defined as follows:

$$H^1(\mathbf{R}) = \left\{f \in L^1(\mathbf{R}) :\ \widehat{f}(s) = 0,\ s \leq 0\right\};$$

and

$$H^\infty(\mathbf{R}) = \left\{f \in L^\infty(\mathbf{R}) :\ \int_{\mathbf{R}} f(t)g(t)dt = 0 \text{ for all } g \in H^1(\mathbf{R})\right\}.$$

Let (Ω, Σ) denote a measurable space and let $\mathcal{L}^\infty(\Sigma)$ denote the bounded measurable functions on Ω. Denote by $M(\Sigma)$ the Banach space of countably additive complex measures on (Ω, Σ) with the total variation norm. Suppose that $T = (T_t)_{t\in\mathbf{R}}$ is a collection of uniformly bounded invertible isomorphisms of $M(\Sigma)$ with

$$\|T_t^{\pm 1}\| \leq c \tag{1}$$

for all $t \in \mathbf{R}$, where c is a positive constant. (Note that we do not require that $(T_t)^{-1} = T_{-t}$, but only that T_t be invertible.) The following definition determines the class of measures that we will be studying.

Definition 1.1 *Let $(T_t)_{t\in\mathbf{R}}$ be as above. A measure $\mu \in M(\Sigma)$ is called weakly measurable if for every $A \in \Sigma$, the mapping $t \mapsto T_t\mu(A)$ is Lebesgue measurable on $\mathbf{R}$.*

We next introduce our notion of analyticity. We will show at the end of this section that our notion of analyticity agrees with Forelli's notion in [6], when restricted to Forelli's setting.

Definition 1.2 *Let $(T_t)_{t\in\mathbf{R}}$ be a uniformly bounded collection of isomorphisms of $M(\Sigma)$. A weakly measurable $\mu \in M(\Sigma)$ is called weakly analytic if the mapping $t \mapsto T_t\mu(A)$ is in $H^\infty(\mathbf{R})$ for every $A \in \Sigma$.*

Our Main Theorem (Theorem 3.4 below) states that, under a certain condition on T that we described in our introduction, if μ is weakly analytic then the mapping $t \mapsto T_t\mu$ is Bochner measurable. This key property is presented in the following definition.

Definition 1.3 *Let $T = (T_t)_{t\in\mathbf{R}}$ be a uniformly bounded collection of isomorphisms of $M(\Sigma)$. Then T is said to satisfy hypothesis (A) if whenever μ is weakly analytic in $M(\Sigma)$, such that for every $A \in \Sigma$, $T_t\mu(A) = 0$ for almost all $t \in \mathbf{R}$, then μ must be the zero measure.*

We emphasize here that the set of $t's$ for which the equality $T_t\mu(A) = 0$ holds depends in general on A. Hypothesis (A) is crucial to our study. We offer two main sources of examples where it is satisfied. The first one is related to Forelli's setting [6].

Example 1.4 Suppose that Ω is a topological space and $(T_t)_{t\in\mathbf{R}}$ is a collection of homeomorphisms of Ω onto itself such that the mapping

$$(t, \omega) \mapsto T_t\omega$$

is jointly continuous. Let Σ denote the Baire subsets of Ω (hence Σ is the smallest σ-algebra such that all the continuous complex-valued functions are measurable with respect to Σ.) This is Forelli's setting, except that we do not require from Ω to be a locally compact Hausdorff space, and more interestingly, we do not assume (thus far) that $(T_t)_{t\in\mathbf{R}}$ forms a group. For any Baire measure μ, define $T_t\mu$ on the Baire sets by $T_t\mu(A) = \mu(T_t(A))$. Now, suppose that μ is such that $T_t\mu(A) = 0$ for almost all t, for any given Baire set A. Then it follows that for any bounded continuous function f that $\int f \circ T_t \, d\mu = 0$ for almost all t. Since the map $t \mapsto \int f \circ T_t \, d\mu$ is continuous, it follows that $\int f \, d\mu = 0$. Now suppose that $A = f^{-1}(0, \infty)$. Then $\mu(A) = \lim_{n\to\infty} \int \max\{0, \min\{f, 1\}\}^{1/n} \, d\mu = 0$. From this, it is easy to conclude that $\mu = 0$, and so T satisfies hypothesis (A).

Our second source of examples is given by the abstract Lebesgue spaces which provide ideal settings to study analytic measures, in the sense that the main results of this paper hold with very relaxed conditions on the representation. (See Theorem 2.5 and Remarks 3.8 and 4.4 below.)

Example 1.5 Suppose that Σ is countably generated. Then any uniformly bounded collection $(T_t)_{t\in\mathbf{R}}$ by isomorphisms of $M(\Sigma)$ satisfies hypothesis (A). The proof follows easily from definitions.

The next example will be used to construct counterexamples when a representation fails hypothesis (A). It also serves to illustrate the use of hypothesis (A).

Example 1.6 (a) Let Σ denote the sigma algebra of countable and co-countable subsets of $\mathbf{R}$. Define $\nu \in M(\Sigma)$ by

$$\nu(A) = \begin{cases} 1 & \text{if } A \text{ is co-countable,} \\ 0 & \text{if } A \text{ is countable.} \end{cases}$$

Let δ_t denote the point mass at $t \in \mathbf{R}$; and take $\mu = \nu - \delta_0$. Consider the representation T of $\mathbf{R}$ given by translation by t. Then it is easily verified that $\|\mu\| > 0$, whereas for every $A \in \Sigma$ we have that $T_t(\mu)(A) = 0$ for almost all $t \in \mathbf{R}$. Hence the representation T does not satisfy hypothesis (A).

The following generalization of (a) will be needed in the sequel.
(b) Let α be a real number and let Σ, μ, ν, δ_t, and T_t have the same meanings as in (a). Define a representation T^α by

$$T_t^\alpha = e^{i\alpha t} T_t.$$

Arguing as in (a), it is easy to see that T^α does not satisfy hypothesis (A).

Organization of the paper In the rest of this section, we introduce some notions from spectral synthesis of bounded functions and show how our definition of analytic measures compares to Forelli's notion. Section 2 contains our Main Lemma and some preliminary applications to generalized analyticity. Although this section does not contain our most general results, it shows the features of our new approach which is based on the analytic Radon-Nikodým property of Bukhvalov and Danilevich [3]. In Section 3, we deal with a one-parameter group acting on $M(\Sigma)$. Using results from Section 2, we derive our main application which concerns the Bochner measurability of the mapping $t \mapsto T_t\mu$. In Section 4, we specialize our study to representations that are defined by mappings of the sigma algebra and prove results concerning the Lebesgue decomposition of analytic measures. Finally in Section 5, we assume that the representation is given by point mappings and give a short and simple proof that analytic measures are quasi-invariant. The results of Sections 4 and 5 generalize their counterparts in Forelli's paper. We also show by examples that Forelli's approach cannot possibly imply the results of the earlier sections. Section 5 concludes with remarks about further extensions of our methods to the setting where $\mathbf{R}$ is replaced by any locally compact abelian group with an ordered dual group. These extensions combine the version of the F. and M. Riesz due to Helson and Lowdenslager [7] with the results of this paper.

Now let us discuss the definition of analyticity according to Forelli. We give this definition in our general setting of a representation T of $\mathbf{R}$ acting on $M(\Sigma)$. For a weakly measurable $\mu \in M(\Sigma)$, we let

$$\mathcal{J}(\mu) = \left\{ f \in L^1(\mathbf{R}) : \int_{\mathbf{R}} T_t\mu(A) f(s-t)\,dt = 0 \text{ for almost all } s \in \mathbf{R} \text{ for all } A \in \Sigma \right\}.$$

Define the T-spectrum of μ by

$$\operatorname{spec}_T(\mu) = \bigcap_{f \in \mathcal{J}(\mu)} \left\{ \chi \in \mathbf{R} : \quad \widehat{f}(\chi) = 0 \right\}. \tag{2}$$

The measure μ is called T-analytic if its T-spectrum is contained in $[0, \infty)$.

The result we need to equate Forelli's notion of analyticity with the notion we present is the following.

Proposition 1.7 *A measure $\mu \in M(\Sigma)$ is weakly analytic if and only if it is T-analytic.*

It follows almost immediately from the definitions that if μ is weakly analytic, then it is T-analytic. The converse is not so obvious, and requires the following notions. Our reference for the rest of this section is Rudin [10, Chapter 7].

Given $\phi \in L^\infty(\mathbf{R})$, define its ideal by

$$\mathcal{J}(\phi) = \left\{ f \in L^1(\mathbf{R}) : \quad f * \phi = 0 \right\}.$$

One definition of the spectrum of ϕ is (see [10, Chapter 7, Theorem 7.8.2])

$$\sigma(\phi) = \bigcap_{f \in \mathcal{J}(\phi)} \left\{ \chi \in \mathbf{R} : \quad \widehat{f}(\chi) = 0 \right\}. \tag{3}$$

A set $S \subset \mathbf{R}$ is called a set of spectral synthesis if whenever $\phi \in L^\infty(\mathbf{R})$ with $\sigma(\phi) \subset S$, then ϕ can be approximated in the weak-* topology of $L^\infty(\mathbf{R})$ by linear combinations of characters from S. (See [10, Section 7.8].) With this definition, the following proposition follows easily.

Proposition 1.8 *Let S be a nonvoid closed subset of $\mathbf{R}$ that is a set of spectral synthesis. If $\phi \in L^\infty(\mathbf{R})$ with $\sigma(\phi) \subset S$, then*

$$\int_{\mathbf{R}} f(x) g(x) dx = 0$$

for all g in $L^1(\mathbf{R})$ such that $\widehat{g} = 0$ on $-S$.

The proof of Proposition 1.7 now follows immediately from the fact that $[0, \infty)$ is a set of spectral synthesis (see [10, Theorem 7.5.6]).

2 The Main Lemma

In our proofs we use the notions of Bochner measurability and Bochner integrability. A function f from a σ-finite measure space (Ω, Σ, μ) to a Banach space X is Bochner measurable if it satisfies one of the following two, equivalent, conditions:

- $f^{-1}(A) \in \Sigma$ for any open subset A of X, and there is a set $E \in \Sigma$ such that $\mu(\Omega \setminus E) = 0$ and $f(E)$ is separable;
- there is a sequence of simple functions $f_n : \Omega \to X$ such that $f_n \to f$ a.e.

Furthermore, if $\int \|f\| d\mu < \infty$, then we say that f is Bochner integrable, and it is possible to make sense of $\int f d\mu$ as an element of X. In particular, if $P : X \to Y$ is a bounded linear operator between two Banach spaces, then $P\left(\int f d\mu\right) = \int P f d\mu$. We refer the reader to [9, Section 3.5].

In this section we prove our Main Lemma about the Bochner measurability of functions defined on $\mathbf{R}$ with values in a Banach space with the analytic Radon-Nikodým property. This property of Banach spaces was introduced by Bukhvalov (see, for example [2]) to extend the basic properties of functions in the Hardy spaces on the disc to vector-valued functions.

Let $\mathcal{B}$ denote the Borel subsets of $\mathbf{T}$, and let X denote a complex Banach space. A vector-valued measure $\mu : \mathcal{B} \to X$ of bounded variation (in symbols, $\mu \in M(\mathcal{B}, X)$) is called analytic if

$$\int_0^{2\pi} e^{-int} d\mu(t) = 0 \quad \text{for all } n < 0.$$

Analytic measures were extensively studied by Bukhvalov and Danilevich (see for example [3]). We owe to them the following definition.

Definition 2.1 *A complex Banach space X is said to have the analytic Radon-Nikodým property (ARNP) if every X-valued analytic measure μ in $M(\mathcal{B}, X)$ has a Radon-Nikodým derivative — that is, there is a Bochner measurable X-valued function f in the space of Bochner integrable functions, $L^1(\mathbf{T}, X)$, such that*

$$\mu(A) = \int_A f dt$$

for all $A \in \mathcal{B}$.

Like the ordinary Radon-Nikodým property, the analytic Radon-Nikodým property is about the existence of a Bochner measurable derivative for vector-valued measures. However, the difference between the two properties, due to the fact that ARNP concerns only analytic measures, makes the class of Banach spaces with the ARNP strictly larger than the class of Banach spaces with the Radon-Nikodým property. In this paper, all we need from this theory is the basic fact that $M(\Sigma)$ has ARNP. Here, as before, $M(\Sigma)$ denotes the Banach space of complex measures on an arbitrary σ-algebra Σ of subsets of a set Ω. According to [3, Theorem 1], a Banach lattice X has ARNP if and only if c_0 does not embed in X. (Here, as usual, c_0 denotes the linear space of complex sequences tending to zero at infinity, and Banach lattices can be real or complex.) Since $M(\Sigma)$ is a Banach lattice that does not contain a copy of c_0, it follows that $M(\Sigma)$ has the analytic Radon-Nikodým property. (To see that c_0 does not embed in $M(\Sigma)$, note that $M(\Sigma)$ is weakly sequentially complete, but that c_0 is not. See [5, Theorem IV.9.4].)

Before we state our lemma, we describe the setting in which it will be used. This will clarify its statement and proof.

Let E denote the subspace of $\mathcal{L}^\infty(\Sigma)$ consisting of the bounded simple functions on Ω. The subspace E embeds isometrically in $M(\Sigma)^*$. Then E is a norming subspace of $M(\Sigma)^*$ for $M(\Sigma)$. It is also easy to verify that every weak-* sequentially continuous linear functional on E is given by point evaluation. That is, if $L : E \to \mathbf{C}$ is weak-* sequentially continuous, then there is a measure $\mu \in M(\Sigma)$ such that

$$L(\alpha) = \int_\Omega \alpha \, d\mu \tag{4}$$

for every $\alpha \in E$. To verify this fact, it is enough to show that the set function given by

$$\mu(A) = L(1_A) \tag{5}$$

defines a measure in $M(\Sigma)$, and this is a simple consequence of the weak-* sequential continuity of L.

Let $T = (T_t)_{t\in\mathbf{R}}$ be a family of uniformly bounded isomorphisms of $M(\Sigma)$ such that (1) holds. Suppose that $\mu \in M(\Sigma)$ is weakly analytic, and let $f(t) = T_t\mu$ for all $t \in \mathbf{R}$. Then $\|f(t)\| \le c\|\mu\|$ where c is as in (1), and for all $\alpha \in E$, the function $t \mapsto \alpha(f(t))$ is in $H^\infty(\mathbf{R})$. With this setting in mind, we state and prove our Main Lemma.

Main Lemma 2.2 *Suppose that X is a complex Banach space with the analytic Radon-Nikodým property, and that E is a norming subspace of X^*, the Banach dual space of X. Suppose that for every weak-* sequentially continuous functional*

$$L:\ E \to \mathbf{C} \tag{6}$$

there is an element $x \in X$ such that

$$L(\alpha) = \alpha(x) \tag{7}$$

for all $\alpha \in E$. Let $f:\ \mathbf{R} \to X$ be such that

$$\sup_t \|f(t)\| < \infty \tag{8}$$

and

$$t \mapsto \alpha(f(t)) \tag{9}$$

is a Lebesgue measurable function in $H^\infty(\mathbf{R})$ for all $\alpha \in E$. Then there is a Bochner measurable function, essentially bounded

$$g:\ \mathbf{R} \to X \tag{10}$$

such that for every $\alpha \in E$, we have

$$\alpha(g(t)) = \alpha(f(t)) \tag{11}$$

for almost all $t \in \mathbf{R}$. (The set of t's for which (11) holds may depend on α.)

Lemma 2.3 *Suppose that X is a Banach space, and that $G : \mathbf{T} \to X$ is a Bochner integrable function for which there is a constant c such that for all Borel sets $A \subset \mathbf{T}$*

$$\left\| \int_A G(\theta)\frac{d\theta}{2\pi} \right\| \le c\lambda(A). \tag{12}$$

(Here $\lambda(A)$ denotes the Lebesgue measure of A). Then G is essentially bounded.

Proof. There is a function $H : \mathbf{T} \to X$ such that $H = G$ a.e., and the range of H is separable. Thus there is a countable sequence $\{\alpha_n\} \subset X^*$ such that $\|\alpha_n\| \le 1$ and $\|H(\theta)\| = \sup_n \alpha_n(H(\theta))$ for all $\theta \in \mathbf{T}$. From (12), it immediately follows that for every $n \in \mathbf{N}$ that

$$\int_A \alpha_n(H(\theta))\frac{d\theta}{2\pi} \le c\lambda(A),$$

from whence it follows that $\alpha_n(H(\theta)) \leq c$ a.e. Hence $\|H(\theta)\| = \sup_n \alpha_n(H(\theta)) \leq c$ a.e., and the result follows.

Proof of Main Lemma 2.2. Let $\phi(z) = i\frac{1-z}{1+z}$ be the conformal mapping of the unit disk onto the upper half plane, mapping $\mathbf{T}$ onto $\mathbf{R}$. Let $F = f \circ \phi$. For every $\alpha \in E$ we have that $\theta \mapsto \alpha(F(\theta)) \in H^\infty(\mathbf{T})$, since by assumption $\alpha(f(t)) \in H^\infty(\mathbf{R})$. Consequently, we have

$$\int_0^{2\pi} \alpha(F(\theta))e^{in\theta}\frac{d\theta}{2\pi} = 0 \text{ for all } n > 0. \tag{13}$$

For $\alpha \in E$, define a measure μ_α on the Borel subsets of $\mathbf{T}$ by

$$\mu_\alpha(A) = \int_A \alpha(F(\theta))\frac{d\theta}{2\pi}. \tag{14}$$

Then for all continuous functions h on $\mathbf{T}$, we have

$$\int_{\mathbf{T}} h(\theta)d\mu_\alpha(\theta) = \int_{\mathbf{T}} h(\theta)\alpha(F(\theta))\frac{d\theta}{2\pi}. \tag{15}$$

We now claim that for every $A \in \mathcal{B}(\mathbf{T})$, the mapping $\alpha \mapsto \mu_\alpha(A)$ is weak-* sequentially continuous. That is, if $\alpha_n \to \alpha$ in the weak-* topology of E, then $\mu_{\alpha_n}(A) \to \mu_\alpha(A)$. To prove this claim, note that if $\alpha_n \to \alpha$ weak-*, then we have that for all $\theta \in \mathbf{T}$,

$$\alpha_n(F(\theta)) \to \alpha(F(\theta)). \tag{16}$$

Also, by the uniform boundedness principle, we have that

$$\sup_n \|\alpha_n\| = M < \infty.$$

So, for all $\theta \in \mathbf{T}$, we have $|\alpha_n(F(\theta))| \leq \|\alpha_n\|\|F(\theta)\| \leq C$. Hence by bounded convergence, it follows from (16) that

$$\mu_{\alpha_n}(A) = \int_A \alpha_n(F(\theta))\frac{d\theta}{2\pi} \to \int_A \alpha(F(\theta))\frac{d\theta}{2\pi} = \mu_\alpha(A), \tag{17}$$

establishing the desired weak-* sequential continuity. By the hypothesis of the lemma, there is $\mu(A) \in X$ such that the mapping $\alpha \mapsto \mu_\alpha(A)$ is given by

$$\alpha \mapsto \mu_\alpha(A) = \alpha(\mu(A)) \text{ for all } \alpha \in E. \tag{18}$$

We now show that for all Borel subsets A of $\mathbf{T}$ that

$$\|\mu(A)\| \leq \int_A \|F(\theta)\|\frac{d\theta}{2\pi}. \tag{19}$$

This follows, because E is norming, and hence, given A, and $\epsilon > 0$, there is an $\alpha \in E$ with $\|\alpha\| \leq 1$ and $\|\mu(A)\| \leq |\alpha(\mu(A))| + \epsilon$, and because

$$|\alpha(\mu(A))| = \left|\int_A \alpha(F(\theta))\frac{d\theta}{2\pi}\right| \leq \int_A \|F(\theta)\|\frac{d\theta}{2\pi}.$$

From (19), it is easily seen that the set mapping $A \mapsto \mu(A)$ defines an X-valued measure of bounded variation on the Borel subsets of $\mathbf{T}$. Let n be a positive integer and let $\alpha \in E$. We have

$$\alpha\left(\int_{\mathbf{T}} e^{in\theta} d\mu(\theta)\right) = \int_{\mathbf{T}} e^{in\theta} d\mu_\alpha(\theta) = \int_{\mathbf{T}} e^{in\theta} \alpha(F(\theta)) \frac{d\theta}{2\pi} = 0.$$

And so, since E is norming, it follows that

$$\int_{\mathbf{T}} e^{in\theta} d\mu(\theta) = 0$$

for all $n > 0$. Now, appealing to the analytic Radon-Nikodým property of X, we find a Bochner integrable function $G: \quad \mathbf{T} \to X$ such that

$$\mu(A) = \int_A G(\theta) \frac{d\theta}{2\pi} \tag{20}$$

for all Borel subsets A of $\mathbf{T}$. Using (20), (18), and (14), we see that, for all $\alpha \in E$ and all $A \in \mathcal{B}$,

$$\int_A \alpha(G(\theta)) \frac{d\theta}{2\pi} = \alpha(\mu(A)) = \mu_\alpha(A) = \int_A \alpha(F(\theta)) \frac{d\theta}{2\pi}.$$

Since this holds for all $\alpha \in E$ and all $A \in \mathcal{B}$, we conclude that, for a given $\alpha \in E$,

$$\alpha(G(\theta)) = \alpha(F(\theta)) \quad \text{a.e. } \theta. \tag{21}$$

From (19), (20) and Lemma 2.3, it follows that G is essentially bounded. Let

$$g(t) = G\left(\phi^{-1}(t)\right). \tag{22}$$

Then g is Bochner measurable, essentially bounded, and for each $\alpha \in E$, for almost all $t \in \mathbf{R}$,

$$\alpha(g(t)) = \alpha(G(\phi^{-1}(t))) = \alpha(F(\phi^{-1}(t))) = \alpha(f(t)),$$

completing the proof.

When applied in the setting that we described before the lemma, we obtain the following important consequence.

Theorem 2.4 *Let $(T_t)_{t\in\mathbf{R}}$ be a one-parameter family of uniformly bounded operators on $M(\Sigma)$ satisfying (1), and let μ be a weakly analytic measure in $M(\Sigma)$. Then there is a Bochner measurable, essentially bounded function $g: \mathbf{R} \to M(\Sigma)$ such that, for every $A \in \Sigma$,*

$$g(t)(A) = T_t\mu(A),$$

for almost all $t \in \mathbf{R}$.

Note that the set of t's for which the equality in this theorem holds depends on A. Our goal in the next section is to establish this equality for all $A \in \Sigma$ and almost all $t \in \mathbf{R}$, under additional conditions on T. This will imply that the mapping $t \mapsto T_t\mu$ is Bochner measurable when μ is weakly analytic. However, when the sigma algebra is countably generated, this result is immediate without any further assumptions on the representation. We state it here for ease of reference.

Theorem 2.5 *Suppose that Σ is countably generated, and let $(T_t)_{t\in\mathbf{R}}$ be a one-parameter family of isomorphisms of $M(\Sigma)$ satisfying (1). Suppose that μ is a weakly analytic measure in $M(\Sigma)$. Then there is a Bochner measurable function $g: \mathbf{R} \to M(\Sigma)$ such that,*

$$g(t) = T_t\mu$$

for almost all $t \in \mathbf{R}$.

Proof. Suppose that $\Sigma = \sigma(\{A_n\}_{n=1}^{\infty})$, where the set $\{A_n\}_{n=1}^{\infty}$ is closed under finite unions and intersections. Apply Theorem 2.4 to obtain a Bochner measurable function g from $\mathbf{R}$ into $M(\Sigma)$ such that for almost all $t \in \mathbf{R}$ and all n we have $T_t\mu(A_n) = g(t)(A_n)$. Since Σ is the closure of $\{A_n\}_{n=1}^{\infty}$ under nested countable unions and intersections, the theorem follows.

3 Analyticity of measures and Bochner measurability

In this section, we prove our main result which states that if μ is weakly analytic, then the mapping $t \mapsto T_t\mu$ is Bochner measurable from $\mathbf{R}$ into $M(\Sigma)$. As an immediate consequence of this result we will obtain that the Poisson integral of a weakly analytic measure converges in $M(\Sigma)$ to the measure, and we also obtain that the mapping $t \mapsto T_t\mu$ is continuous. Both of these results are direct analogues of classical properties of analytic measures on the real line.

The proofs in this section require the use of convolution. To define this operation and to derive its basic properties, we will need additional conditions on the representation T. We start by stating these conditions, setting in the process the notation for this section. These conditions are automatically satisfied in the case of a representation by mappings of the given sigma-algebra.

We let $T = (T_t)_{t\in\mathbf{R}}$ be a one-parameter group of isomorphisms of $M(\Sigma)$ for which (1) holds, satisfying hypothesis (A). Here, as before, $M(\Sigma)$ is the Banach space of countably additive complex measures on an arbitrary sigma algebra Σ of subsets of a set Ω. We will suppose throughout this section that the adjoint of T_t maps $\mathcal{L}^\infty$ into itself; in symbols:

$$T_t^*: \mathcal{L}^\infty(\Sigma) \to \mathcal{L}^\infty(\Sigma). \tag{23}$$

Although this property will not appear explicitly in the proofs of the main results, we use it at the end of this section to establish basic properties of convolutions of Borel measures on $\mathbf{R}$ with weakly measurable μ in $M(\Sigma)$. More explicitly, suppose that $\nu \in M(\mathbf{R})$ and μ is weakly measurable. Define a measure $\nu *_T \mu$ (or simply $\nu * \mu$, when there is no risk of confusion) in $M(\Sigma)$ by

$$\nu *_T \mu(A) = \int_{\mathbf{R}} T_{-t}\mu(A)d\nu(t),$$

for all $A \in \Sigma$. It is the work of a moment to show that this indeed defines a measure in $M(\Sigma)$. We have:

- for all $t \in \mathbf{R}$, $T_t(\nu *_T \mu) = \nu *_T (T_t\mu)$;

- the measure $\nu *_T \mu$ is weakly measurable;
- for $\sigma, \nu \in M(\mathbf{R})$, and μ weakly measurable, $\sigma *_T (\nu *_T \mu) = (\sigma * \nu) *_T \mu$.

For clarity's sake, we postpone the proofs of these results until the end of the section, and proceed towards the main results.

Recall the definition of the Poisson kernel on $\mathbf{R}$: for $y > 0$, let

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2} ,$$

for all $x \in \mathbf{R}$. Let μ be a weakly analytic measure in $M(\Sigma)$, and let g be the Bochner measurable function defined on $\mathbf{R}$ with values in $M(\Sigma)$, given by Lemma 2.2. Form the Poisson integral of g as follows

$$P_y * g(t) = \int_{\mathbf{R}} g(t - x) P_y(x) dx, \tag{24}$$

where the integral exists as a Bochner integral. Because the function g is essentially bounded, we have the following result whose proof follows as in the classical setting for scalar-valued functions.

Proposition 3.1 *With the above notation, we have that*

$$\lim_{y \to 0} P_y * g(t) = g(t) \tag{25}$$

for almost all $t \in \mathbf{R}$.

We can now establish basic relations between the Poisson integral of the function g and the measure μ.

Lemma 3.2 *For all $t \in \mathbf{R}$, we have*

$$P_y * g(t) = P_y * T_t \mu.$$

Proof. For $A \in \Sigma$, we have

$$\begin{aligned} P_y * g(t)(A) &= \int_{\mathbf{R}} g(s)(A) P_y(t - s) ds \\ &= \int_{\mathbf{R}} (T_s \mu)(A) P_y(t - s) ds \\ &= \int_{\mathbf{R}} (T_{t-s} \mu)(A) P_y(s) ds \\ &= P_y * (T_t \mu)(A). \end{aligned}$$

Since this holds for all $A \in \Sigma$, the lemma follows.

Lemma 3.3 *Let t_0 be any real number such that (25) holds. Then, for all $t \in \mathbf{R}$, we have*

$$\lim_{y \to 0} P_y * T_t \mu = T_{t - t_0} (g(t_0)) ,$$

in $M(\Sigma)$.

Proof. Since $P_y * g(t_0) \to g(t_0)$, it follows that

$$T_{t-t_0}\left(P_y * g(t_0)\right) \to T_{t-t_0}\left(g(t_0)\right).$$

Using Lemmas 3.2 and the basic properties of convolutions, we get

$$T_{t-t_0}\left(P_y * g(t_0)\right) = T_{t-t_0}\left(P_y * T_{t_0}\mu\right) = P_y * T_t\mu,$$

establishing the lemma.

We can now prove the main result of this section.

Main Theorem 3.4 *Suppose that $T = (T_t)_{t\in\mathbf{R}}$ is a group of isomorphisms of $M(\Sigma)$ satisfying hypothesis (A) and such that (1) and (23) hold. Let μ be a weakly analytic measure, and let g be the Bochner measurable function on $\mathbf{R}$ constructed from μ as in Theorem 2.4. Then for almost all $t \in \mathbf{R}$, we have*

$$T_t\mu = g(t).$$

Consequently, the mapping $t \mapsto T_t\mu$ is Bochner measurable.

Proof. It is enough to show that the equality in the theorem holds for all $t = t_0$ where (25) holds. Fix such a t_0, and let $A \in \Sigma$. Since the function $t \mapsto T_t\mu(A)$ is bounded on $\mathbf{R}$, it follows from the properties of the Poisson kernel that

$$P_y * (T_t\mu)(A) \to T_t\mu(A) \text{ for almost all } t \in \mathbf{R}.$$

But by Lemma 3.3, we have

$$P_y * (T_t\mu)(A) \to T_{t-t_0}\left(g(t_0)\right)(A) \text{ for all } t \in \mathbf{R}.$$

Hence

$$T_t\mu(A) = T_{t-t_0}(g(t_0))(A)$$

for almost all $t \in \mathbf{R}$. It is clear from Lemma 3.3 that the measure $T_{-t_0}g(t_0)$ is weakly analytic, since it is the strong limit in $M(\Sigma)$ of weakly analytic measures. Applying hypothesis (A), we infer that

$$\mu = T_{-t_0}g(t_0).$$

Applying T_{t_0} to both sides of the last equality completes the proof.

From Theorem 3.4 we can derive several interesting properties of analytic measures, which, as the reader may check, are equivalent to the F. and M. Riesz Theorem in the classical setting. We start with a property of the Poisson integral of weakly analytic measures.

Theorem 3.5 *Let T and μ be as in Theorem 3.4. Then,*

$$\lim_{y\to 0} P_y * \mu = \mu$$

in the $M(\Sigma)$-norm.

Proof. Let $t_0 \in \mathbf{R}$ be such that $P_y * g(t_0) \to g(t_0)$ in the $M(\Sigma)$-norm (recall (25)). We have

$$T_{-t_0}\left(P_y * g(t_0)\right) \to T_{-t_0}\left(g(t_0)\right)$$

in the $M(\Sigma)$-norm. But $T_{-t_0}\left(P_y * g(t_0)\right) = P_y * \mu$, and $T_{-t_0} g(t_0) = \mu$, and the theorem follows.

The following generalizes Theorem 4 of Forelli [6] which in turn is a generalization of Theorem (3.1) of de Leeuw and Glicksberg [4].

Theorem 3.6 *Let μ and T be as in Theorem 3.5. Then the mapping $t \mapsto T_t\mu$ is uniformly continuous from $\mathbf{R}$ into $M(\Sigma)$.*

Proof. It is easily seen that for each $y > 0$, the map $t \mapsto P_y * g(t)$ is continuous. By Lemma 3.2, it follows that the map $t \mapsto P_y * T_t\mu = T_t(P_y * \mu)$ is continuous. By Theorem 3.5 and (1), we see that $T_t(P_y * \mu) \to T_t\mu$ uniformly in t, and the result follows.

Theorem 3.7 *Let T be a representation of $\mathbf{R}$ acting on $M(\Sigma)$ and satisfying hypothesis (A), (1), and (23). Suppose that P is a bounded linear operator from $M(\Sigma)$ into itself that commutes with T_t for each $t \in \mathbf{R}$. If μ is a weakly analytic measure in $M(\Sigma)$, then $\mathrm{spec}_T(P\mu)$ is contained in $\mathrm{spec}_T\mu$. In particular, $P\mu$ is weakly analytic.*

Proof. Clearly, it is sufficient to show that $\mathcal{J}(\mu) \subset \mathcal{J}(P\mu)$. So, suppose that $f \in \mathcal{J}(\mu)$. Define the measure $\nu = f *_T \mu$. Now, the map $t \mapsto T_t\mu$ is Bochner measurable, and hence the map $t \mapsto f(t)T_{-t}\mu$ is Bochner integrable. By properties of the Bochner integral, it follows that

$$\int_{\mathbf{R}} f(t)T_{-t}\mu dt = \nu.$$

From the definition of $\mathcal{J}(\mu)$, we see that for each $A \in \Sigma$

$$T_s\nu(A) = \int_{\mathbf{R}} f(t)T_{s-t}\mu(A)dt = 0 \quad \text{a.e. } s.$$

Hence by hypothesis (A), it follows that $\nu = 0$. Thus, once again, using the properties of the Bochner integral, we see that

$$\begin{aligned} \int_{\mathbf{R}} f(s-t)T_tP\mu dt &= \int_{\mathbf{R}} f(t)T_{s-t}P\mu dt \\ &= PT_s\left(\int_{\mathbf{R}} f(t)T_{-t}\mu dt\right) \\ &= PT_s\nu = 0. \end{aligned}$$

Hence, for every $A \in \Sigma$, we have that

$$\int_{\mathbf{R}} f(s-t)T_t(P\mu)(A)dt = 0,$$

that is, $f \in \mathcal{J}(P\mu)$.

Remark 3.8 When Σ is countably generated, using Theorem 2.5, instead of the Main Theorem of this section, we can derive a version of Theorem 3.7 without the additional condition on the adjoint (23), and more interestingly, without assuming that T is a representation by a group of isomorphisms on $M(\Sigma)$. The hypotheses of Theorem 2.5 are enough to derive these results.

We end this section with the proofs of the properties of convolutions that we stated at the outset of this section. Throughout the rest of this section, we use the following notation: μ is a weakly measurable element in $M(\Sigma)$; ν and σ are regular Borel measures in $M(\mathbf{R})$; $T = (T_t)_{t\in\mathbf{R}}$ is a one-parameter group of operators on $M(\Sigma)$ satisfying hypothesis (A) and such that (1) and (23) hold. The convolution of μ and ν is defined on the sigma-algebra Σ by

$$\nu *_T \mu(A) = \int_{\mathbf{R}} T_{-t}\mu(A)d\nu(t), \tag{26}$$

for all $A \in \Sigma$. When there is no risk of confusion we will simply write $\nu * \mu$ for $\nu *_T \mu$.

Using dominated convergence, it is easy to check that (26) defines a measure in $M(\Sigma)$, and that $\|\nu * \mu\| \leq c\|\mu\|\|\nu\|$, where c is the as in (1).

Lemma 3.9 *Suppose that $f \in \mathcal{L}^\infty(\Sigma)$. Then the mapping $t \mapsto \int_\Omega f d(T_t\mu)$ is Lebesgue measurable on $\mathbf{R}$. Furthermore,*

$$\int_{\mathbf{R}}\int_\Omega f d(T_{-s})\mu d\nu(s) = \int_\Omega f d\nu * \mu. \tag{27}$$

Proof. It is sufficient to prove the lemma in the case when f is a simple function, and then it is obvious.

Corollary 3.10 *For all $t \in \mathbf{R}$, we have*

$$T_t(\nu * \mu) = \nu * (T_t\mu).$$

Moreover, the measure $\nu * \mu$ is weakly measurable.

Proof. For $A \in \Sigma$, we have

$$\begin{aligned}
\nu * (T_t\mu)(A) &= \int_{\mathbf{R}} (T_{-s+t}\mu)(A)d\nu(s) \\
&= \int_{\mathbf{R}}\int_\Omega T_t^* 1_A dT_{-s}\mu d\nu(s) \\
&= \int_{\mathbf{R}} T_t^* 1_A d\nu * \mu \quad \text{(by Lemma 3.9)} \\
&= \int_{\mathbf{R}} 1_A dT_t(\nu * \mu) \\
&= T_t(\nu * \mu)(A).
\end{aligned}$$

To prove the second assertion, note that $t \mapsto T_t(\nu * \mu)(A) = \nu * (s \mapsto T_s\mu(A))(t)$, and so the function $t \mapsto T_t(\nu * \mu)(A)$ is Lebesgue measurable, being the convolution of a measure in $M(\mathbf{R})$ and a bounded measurable function on $\mathbf{R}$.

Corollary 3.11 *With the above notation, we have*

$$(\sigma * \nu) * \mu = \sigma * (\nu * \mu).$$

Proof. For $A \in \Sigma$, we have

$$\begin{aligned}
(\sigma * \nu) * \mu(A) &= \int_{\mathbf{R}} (T_{-s}\mu)(A) d(\sigma * \nu)(s) \\
&= \int_{\mathbf{R}} \int_{G} (T_{(-s-t)}\mu)(A) d\nu(t) d\sigma(s) \quad \text{(by [8, Theorem (19.10)])} \\
&= \int_{\mathbf{R}} (\nu * (T_{-s}\mu))(A) d\sigma(s) \\
&= \int_{\mathbf{R}} T_{-s}(\nu * \mu)(A) d\sigma(s) \text{ (by Corollary 3.10)} \\
&= \sigma * (\nu * \mu)(A),
\end{aligned}$$

and the lemma follows.

4 Lebesgue decomposition of analytic measures

In their extension of the F. and M. Riesz Theorem to compact abelian groups, Helson and Lowdenslager [7] realized that while an analytic measure may not be absolutely continuous with respect to Haar measure, its absolutely continuous part and singular part are both analytic. This property was then generalized by de Leeuw and Glicksberg [4] and Forelli [6] to the Lebesgue decomposition of analytic measures with respect to quasi-invariant measures, which take the place of Haar measures on arbitrary measure spaces. In this section, we derive our version of this result as a simple corollary of Theorem 3.7. We then derive a version of this theorem in the case when Σ consists of the Baire subsets of a topological space, without using the group property of the representation.

The setting for this section is as follows. Let $T = (T_t)_{t\in\mathbf{R}}$ denote a one-parameter group given by mappings of a sigma algebra Σ. With a slight abuse of notation, we will write

$$T_t\mu(A) = \mu(T_t(A))$$

for all $t \in \mathbf{R}$, all $A \in \Sigma$, and all $\mu \in M(\Sigma)$. Note that conditions (1) and (23) are satisfied. In addition to these properties, we suppose that T satisfies hypothesis (A), and so the results of the previous section can be applied in our present setting.

Definition 4.1 *Let T be as above, and let $\nu \in M(\Sigma)$ be weakly measurable. We say that ν is quasi-invariant if, for all $t \in \mathbf{R}$, ν and $T_t\nu$ are mutually absolutely continuous.*

The following is a generalization to arbitrary measure spaces of Theorem 5 of Forelli [6].

Theorem 4.2 *Let T be as above, and let μ and σ be weakly measurable in $M(\Sigma)$ such that σ is quasi-invariant and μ is weakly analytic. Write $\mu = \mu_a + \mu_s$ for the Lebesgue decomposition of μ with respect to σ. Then the spectra of μ_a and μ_s are contained in the spectrum of μ. In particular, μ_a and μ_s are both weakly analytic in $M(\Sigma)$.*

Proof. Define P on $M(\Sigma)$ by $P(\eta) = \eta_s$, where η_s is the singular part of η in its Lebesgue decomposition with respect to σ.

It is easy to see that the quasi-invariance of ν is equivalent to the fact that for all $A \in \Sigma$, we have $|\nu|(A) = 0$ if and only if $T_t|\nu|(A) = 0$ for all $t \in \mathbf{R}$. Consequently,

$$P \circ T_t(\eta) = T_t \circ P(\eta).$$

Now apply Theorem 3.7.

Example 4.3 Consider Examples 1.6 (a) and (b). Clearly, the measure $\mu = (\nu - \delta_0)$ is weakly analytic with respect to the representations T^α, for any α, since, for every $A \in \Sigma$, $T_t^\alpha \mu(A) = 0$ for almost all $t \in \mathbf{R}$, and so the function $t \mapsto T_t^\alpha \mu(A)$ is trivially in $H^\infty(\mathbf{R})$. However, $\mu_s = -\delta_0$, and hence $T_t^\alpha \mu_s = -e^{i\alpha}\delta_t$, which is not weakly analytic if $\alpha < 0$.

Remark 4.4 Using Remark 3.8, we see that Theorem 4.2 also holds under the hypotheses of Theorem 2.5.

We can use Remark 4.4 to show that, on topological spaces where the action of $\mathbf{R}$ is given by jointly continuous point mappings of the underlying space, Theorem 4.2 holds even if we dispense with the group property of the representation.

Theorem 4.5 *Let Ω be a topological space, and let $(T_t)_{t\in\mathbf{R}}$ be a family of homeomorphisms of Ω such that $(t, \omega) \mapsto T_t\omega$ is jointly continuous. Suppose that μ and ν are Baire measures such that ν is quasi-invariant, and write $\mu = \mu_a + \mu_s$ for the Lebesgue decomposition of μ with respect to ν. If μ is weakly analytic, then the spectra of μ_a and μ_s are contained in the spectrum of μ. In particular, μ_a and μ_s are both weakly analytic in $M(\Sigma)$.*

Proof. It is enough to consider μ_s. Let Σ denote the sigma algebra of Baire subsets of Ω, and let $A \in \Sigma$. We want to show that the mapping $t \mapsto T_t\mu_s(A)$ is in $H^\infty(\mathbf{R})$. We will reduce the problem to a countably generated subsigma algebra of Σ that depends on A, then use Remark 4.4.

A simple argument shows that for each $C \in \Sigma$, there exist a countable collection of continuous function $\{f_n : \Omega \to \mathbf{R}\}$ such that C is contained in the minimal sigma-algebra for which the functions f_n are measurable. Furthermore, for each continuous function, we have that $f \circ T_t \to f \circ T_{t_0}$ pointwise as $t \to t_0$. Hence, we see that C is an element of

$$\begin{aligned}\Sigma(C) &= \sigma\left(\left(f_n \circ T_{r_1}^{\pm 1} T_{r_2}^{\pm 1} \ldots T_{r_k}^{\pm 1}\right)^{-1}(-r, \infty) : n \in \mathbf{N},\, r_1, r_2, \ldots, r \in \mathbf{R}\right) \\ &= \sigma\left(\left(f_n \circ T_{r_1}^{\pm 1} T_{r_2}^{\pm 1} \ldots T_{r_k}^{\pm 1}\right)^{-1}(-r, \infty) : n \in \mathbf{N},\, r_1, r_2, \ldots, r \in \mathbf{Q}\right).\end{aligned}$$

Clearly $\Sigma(C)$ is countably generated, and invariant under T_t and T_t^{-1} for all $t \in \mathbf{R}$.

Let B denote the support of μ_s, and let $\Sigma(A,B) = \sigma\left(\Sigma(A), \Sigma(B)\right)$. Again, we have that $\Sigma(A,B)$ is countably generated and invariant by all $T_t^{\pm 1}$. Let $\mu_s|_{\Sigma(A,B)}$, $\mu|_{\Sigma(A,B)}$, and $\nu|_{\Sigma(A,B)}$ denote the restrictions of μ_s, μ, and ν to the $\Sigma(A,B)$, respectively. It is clear that $\nu|_{\Sigma(A,B)}$ is quasi-invariant and that $\mu|_{\Sigma(A,B)}$ is weakly analytic, where here we are restricting the definitions to the smaller sigma algebra $\Sigma(A,B)$. By Remark 4.4, the measure $\left(\mu|_{\Sigma(A,B)}\right)_s$ is weakly analytic. But, $\mu_s|_{\Sigma(A,B)} = \left(\mu|_{\Sigma(A,B)}\right)_s$, and hence $t \mapsto T_t\mu_s(A)$ is in $H^\infty(\mathbf{R})$, completing the proof of the theorem.

5 Quasi-invariance of analytic measures

In this last section, we use some of the machinery that we have developed in the previous sections to give a simpler proof of a result of de Leeuw, Glicksberg, and Forelli, that asserts that analytic measures are quasi-invariant. We will show by an example that unless the action is restricted to point mappings of the underlying topological space, such a result may fail. Thus, the various results that we obtained in Sections 2 and 3 for more general representations of $\mathbf{R}$ cannot be obtained by the methods of Forelli [6] which imply the quasi-invariance of analytic measures.

As in the previous sections, we start by describing the setting for our work. Here $(T_t)_{t\in\mathbf{R}}$ denotes a group of homeomorphisms of a topological space Ω, and Σ stands for the Baire subsets of Ω. Given a Baire measure μ, we let $T_t\mu$ be the measure defined on the Baire subsets $A \in \Sigma$ by $(T_t\mu)(A) = \mu(T_tA)$. Applying Theorem 3.5, we have that $P_y * \mu \to \mu$ in the $M(\Sigma)$ norm. Using the Jordan decomposition of the measure μ, we see that $P_y * |\mu| \to |\mu|$. Hence, from the proof of Theorem 3.6, we see that the mapping $t \mapsto T_t|\mu|$ is also continuous. Define a measure ν in $M(\Sigma)$, by

$$\nu(A) = \frac{1}{\pi}\int_{-\infty}^{\infty} T_t|\mu|(A)\frac{dt}{1+t^2}, \tag{28}$$

for all $A \in \Sigma$. Note that $\nu = P_1 * |\mu|$.

Lemma 5.1 *For all $t \in \mathbf{R}$, we have $T_t|\mu| << \nu$, and hence $T_t\mu << \nu$.*

Proof. Let $A \in \Sigma$. Since the mapping $t \mapsto T_t|\mu|(A)$ is continuous and nonnegative, the lemma follows easily.

Lemma 5.2 *Let $h(t,\omega)$ denote the Radon-Nikodým derivative of $T_t\mu$ with respect to ν. Then the mapping $t \mapsto h(t,\cdot)$ is continuous from $\mathbf{R}$ into $L^1(\nu)$. Consequently it is Bochner measurable, and hence $(t,\omega) \mapsto h(t,\omega)$ is jointly measurable on $\mathbf{R}\times\Omega$.*

Proof. The first part of the lemma follows once we establish that the real and imaginary parts, $t \mapsto \Re h(t,\cdot)$ and $t \mapsto \Im h(t,\cdot)$, are continuous. We deal with the first function only; the second is handled similarly. Let $B = \{\omega \in \Omega : \ \Re(h(t,\omega) - h(t',\omega)) > 0\}$. We have

$$\begin{aligned}
&\|\Re(h(t,\cdot) - h(t',\cdot))\|_{L^1(\nu)} = \int_\Omega |\Re(h(t,\omega) - h(t',\omega))| d\nu(\omega) \\
&= \int_B \Re(h(t,\omega) - h(t',\omega)) d\nu(\omega) + \int_{\Omega\setminus B} \Re(h(t',\omega) - h(t,\omega)) d\nu(\omega) \\
&\leq |T_t(|\mu|)(B) - T_{t'}(|\mu|)(B)| + |T_t(|\mu|)(\Omega\setminus B) - T_{t'}(|\mu|)(\Omega\setminus B)|.
\end{aligned}$$

The continuity follows now from Lemma 5.1. To complete the proof of the lemma, note that since $t \mapsto h(t,\cdot)$ is Bochner measurable, it is the limit of simple functions $h_n(t,\cdot)$ each of which is jointly measurable on $\mathbf{R} \times \Omega$.

One more property of the function h is needed before establishing the quasi-invariance of analytic measures.

Lemma 5.3 *Suppose that $\mu \in M(\Sigma)$ is weakly analytic, and let $h(t,\omega)$ be as in Lemma 5.2. Then there is a Baire subset Ω_0 of Ω such that $\nu(\Omega \setminus \Omega_0) = 0$, and for all $\omega \in \Omega_0$, the function $t \mapsto h(t,\omega)$ is in $H^\infty(\mathbf{R})$.*

Proof. By Lemma 5.2, the function $t \mapsto h(t,\cdot)$ is bounded and Bochner measurable. Hence, for any $g(t)$ in $H^1(\mathbf{R})$, the function $h(t,\omega)g(t)$ is in $L^1(\mathbf{R}, L^1(\nu))$. Moreover, for any Baire subset $A \in \Sigma$, we have

$$\begin{aligned} \int_A \int_{\mathbf{R}} h(t,\omega)g(t)dtd\nu(\omega) &= \int_{\mathbf{R}} \int_A h(t,\omega)g(t)d\nu(\omega)dt \\ &= \int_{\mathbf{R}} T_t\mu(A)g(t)dt = 0 \end{aligned} \tag{29}$$

since μ is weakly analytic. Since (29) holds for every $A \in \Sigma$, we conclude that the function $\omega \mapsto \int_{\mathbf{R}} h(t,\omega)g(t)dt = 0$ for ν-almost all ω. Hence there is a subset $\Omega_g \in \Sigma$ such that $\nu(\Omega \setminus \Omega_g) = 0$ and $\int_{\mathbf{R}} h(t,\omega)g(t)dt = 0$ for all $\omega \in \Omega_g$. Since $H^1(\mathbf{R})$ is separable, it contains a countable dense subset, say $\{g_n\}$. Let $\Omega_0 = \bigcap \Omega_{g_n}$. Then, $\nu(\Omega \setminus \Omega_0) = 0$, and for $\omega \in \Omega_0$ and all $g \in H^1(\mathbf{R})$ we have that $\int_{\mathbf{R}} h(t,\omega)g(t)dt = 0$ which proves the lemma.

We now come to the main result of this section. In addition to the preliminary lemmas that we have just established, the proof uses the fact that a function in $H^\infty(\mathbf{R})$ is either zero almost everywhere or is not zero almost everywhere.

Theorem 5.4 *Suppose that $t \mapsto T_t$ is a one-parameter group of homeomorphisms of a topological space Ω with the property that $(t,\omega) \mapsto T_t\omega$ is continuous, and let Σ denote the sigma-algebra of Baire subsets of Ω. Suppose that $\mu \in M(\Sigma)$ is weakly analytic. Then μ is quasi-invariant.*

Proof. We use the notation of the previous lemma. For $\omega \in \Omega_0$ and $A \in \Sigma$, we have

$$\begin{aligned} T_t|\mu|(A) &= T_{t+s}|\mu|\,(T_{-s}A) \\ &= \int_{T_{-s}A} |h(t+s,\omega)|d\nu(\omega). \end{aligned}$$

Hence,

$$\begin{aligned} T_t|\mu|(A) &= \int_{\mathbf{R}} T_t|\mu|(A)\frac{ds}{\pi(1+s^2)} \\ &= \int_{\mathbf{R}} \int_{T_{-s}A} |h(t+s,\omega)|d\nu(\omega)\frac{ds}{\pi(1+s^2)} \\ &= \int_{\Omega} \int_{C_\omega} |h(t+s,\omega)|\frac{ds}{\pi(1+s^2)}d\nu(\omega) \\ &= \int_{\Omega_0} \int_{C_\omega} |h(t+s,\omega)|\frac{ds}{\pi(1+s^2)}d\nu(\omega), \end{aligned} \tag{30}$$

where $C_\omega = \{s \in \mathbf{R} : \quad \omega \in T_{-s}A\}$. Since for $\omega \in \Omega_0$, the function $t \mapsto h(t,\omega)$ is in $H^\infty(\mathbf{R})$, it follows that this function is either zero t-a.e., or not zero t-a.e. Let $\Omega_1 = \{\omega \in \Omega : \quad h(t,\omega) = 0, \text{for almost all } t \in \mathbf{R}\}$. Then $\Omega_1 \in \Sigma$, and from (30) we have that

$$T_t|\mu|(A) = \int_{\Omega_0 \setminus \Omega_1} \int_{C_\omega} |h(t+s,\omega)| \frac{ds}{\pi(1+s^2)} d\nu(\omega). \tag{31}$$

Hence, $T_t|\mu|(A) = 0$ if and only if

$$\int_{C_\omega} |h(t+s,\omega)| \frac{ds}{\pi(1+s^2)} = 0$$

for ν-almost all $\omega \in \Omega_0 \setminus \Omega_1$. Since the integrand is strictly positive except on a set of zero measure, this happens if and only if the Lebesgue measure of C_ω is zero for ν-almost all $\omega \in \Omega_0 \setminus \Omega_1$. But since this last condition does not depend on t, we see that $T_t|\mu|(A) = 0$ if and only if $|\mu|(A) = 0$.

That Theorem 5.4 does not hold for more general representations is demonstrated by the following example.

Example 5.5 Let $\Omega = \{0,1\}$, and let Σ consist of the power set of Ω. Denote by δ_0 and δ_1 the point masses at 0 and 1, respectively. Then δ_0 and δ_1 form a basis for $M(\Sigma)$, and every element in $M(\Sigma)$ will be represented as a vector in this basis. For $t \in \mathbf{R}$, define T_t by the matrix

$$T_t = \begin{pmatrix} e^{4it}\cos t & e^{4it}\sin t \\ -e^{4it}\sin t & e^{4it}\cos t \end{pmatrix}.$$

Note that T satisfies hypothesis (A). Also, it is easy to verify that δ_0 is weakly analytic. However, $T_{\pi/2}\delta_0 = -\delta_1$, and δ_0 and δ_1 are not mutually absolutely continuous. Hence δ_0 is weakly analytic but not quasi-invariant.

Final Remarks The approach that we took to the F. and M. Riesz Theorem can be carried out in the more general setting where $\mathbf{R}$ is replaced by any locally compact abelian group G with an ordered dual group Γ, and where the notion of analyticity is defined as in [7] using the order structure on the dual group. With the exceptions of $G = \mathbf{R}$ and $G = \mathbf{T}$, the main result of §3, concerning the Bochner measurability of $t \mapsto T_t\mu$, fails even in the nice setting of a regular action of G by translation in $M(G)$. However, we can prove a weaker result that states that a weakly analytic measure is strongly analytic, that is, whenever $\delta \in M(\Sigma)^*$ (the Banach dual of $M(\Sigma)$), then the map $t \mapsto \delta(T_t\mu)$ is in $H^\infty(G)$. This result, in turn, implies the versions of the F. and M. Riesz Theorems proved by Helson and Lowdenslager [7] for actions of locally compact abelian groups on abstract measure spaces. This work is done in a separate paper by the authors.

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