TOPICS IN GEOMETRIC ANALYSIS WITH APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

A Dissertation
presented to
the Faculty of the Graduate School
University of Missouri-Columbia

In Partial Fulfillment
of the Requirements for the Degree

Doctor of Philosophy

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by
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JULY 2009

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

TOPICS IN GEOMETRIC ANALYSIS WITH APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

Presented by Simona Barb

A candidate for the degree of Doctor of Philosophy

And hereby certify that in their opinion it is worthy of acceptance.

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ACKNOWLEDGMENTS

I would like to express my appreciation and gratitude to Prof. Marius Mitrea. Without his guidance and extensive knowledge of the subject, none of what follows would have been possible. For sacrificing the countless hours to instruct and educate me, I will forever be extremely grateful.

To my committee members, Dr. Mitrea, Dr. Gesztesy, Dr. Pang and Dr. Kosztin, thank you for the honor you rendered in being part of the jury.

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Introduction

The current thesis has several major objectives.

One of these is the investigation of the mathematical tools and methods used to study problems in Geometrical Analysis which bridge between analytical properties and geometrical properties of basic entities (typically, subdomains of the Euclidean space, or various functions naturally associated with them, such as the distance function to the boundary). This is particularly useful in situations in which the geometry is variable, such as for

- minimal surfaces
- shape analysis and optimization
- engineering modeling
- continum mechanics (elasticity phenomena for beams, plates, shells, etc)
- free or moving boundary problems
- variational (PDE) problems

to give just a few examples. Often, a common feature is the fact that it is the underlying geometry which is the variable one wishes to study.

The basic principles used in the treatment of the aforementioned topics come from very different areas of applied and theoretical mathematics which have traditionally evolved in parallel directions. Consequently, one of our goals is to provide alternative, conceptually simpler approaches to some of the basic results in these fields. In order to be more specific, we need to introduce some notation. Let Σ be an oriented C^2 surface in $\mathbb{R}^n$, with canonical unit normal ν, and surface measure σ. Fix $\phi : \mathbb{R}^n \to \mathbb{R}$ a C^∞ function with compact support and such that

$$\Sigma \cap \operatorname{supp} \phi \text{ is a compact subset of } \Sigma. \tag{1.1}$$

Associated with these, consider the one-parameter variation of the original surface:

$$\Sigma_{\phi,\,t} := \{X + t\phi(X)\nu(X) :\ X \in \Sigma\} \quad \text{for} \quad t > 0 \quad \text{fixed.} \tag{1.2}$$

The surface Σ is called **minimal** if the following Euler-Lagrange equation is satisfied:

$$\frac{d}{dt}\left[\operatorname{Area}(\Sigma_{\phi,\,t})\right]\Big|_{t=0} = 0 \quad \text{for all } \phi. \tag{1.3}$$

In other words,

$$\Sigma \text{ is minimal} \Longleftrightarrow \Sigma \text{ is a critical point for the area function.} \tag{1.4}$$

Of course, it is of interest to have an intrinsec description of the above condition.

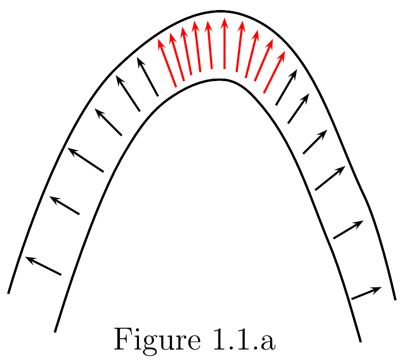

Figure 1.1.a

Remark 1.0.1. *As the picture above illustrates, the largest variation in the area of* $\Sigma_{\phi,\,t}$ *comes from portions of* Σ *where* Σ *is bending the most (i.e., where the Gauss mean curvature is largest). Likewise, portions where* Σ *is relatively flat induce little variation in the area of* $\Sigma_{\phi,\,t}$*.*

A precise answer is provided by the following classical theorem:

Theorem 1.0.2 (The First Variation Formula of Area)**.** *With the above notation and assumptions,* $\Sigma_{\phi,\,t}$ *continues to be a* C^2 *surface for* $|t|$ *small and*

$$\frac{d}{dt}\left[\operatorname{Area}(\Sigma_{\phi,\,t})\right]\Big|_{t=0} = \int_{\Sigma} \phi\, \mathcal{G}\, d\sigma, \tag{1.5}$$

where $\mathcal{G}$ is the Gauss mean curvature of Σ. In particular,

$$\Sigma \text{ is minimal} \Longleftrightarrow \text{the Gauss mean curvature } \mathcal{G} \text{ of } \Sigma \text{ vanishes.} \tag{1.6}$$

As an example, let $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ be an open set and suppose $\varphi : \mathcal{O} \to \mathbb{R}$ is a given C^2 function. Upon denoting by Σ the graph of φ, then for every $x' \in \mathcal{O}$, the Gauss mean curvature $\mathcal{G}$ of Σ at the point $(x', \varphi(x'))$ is

$$\begin{aligned} \mathcal{G}(x', \varphi(x')) &= \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} \left[\frac{\partial_j \varphi(x')}{\sqrt{1 + \|\nabla \varphi(x')\|^2}} \right] \\ &= \frac{1}{\sqrt{1 + \|\nabla \varphi(x')\|^2}} \left[\operatorname{Tr} H_\varphi(x') - \frac{\nabla \varphi(x') \cdot \big(H_\varphi(x') \nabla \varphi(x')\big)}{1 + \|\nabla \varphi(x')\|^2} \right], \end{aligned} \tag{1.7}$$

where $H_\varphi(x')$ is the Hessian matrix of φ at x'. Thus, the minimal surface equation for a graph takes the familiar form

$$\operatorname{div}\left(\frac{\nabla \varphi(x')}{\sqrt{1 + \|\nabla \varphi(x')\|^2}} \right) = 0. \tag{1.8}$$

Returning for a moment to (1.4), it is worth recalling that not any critical point is necessarily a local extremum, so the term "minimal" is somewhat misleading (albeit time-honored). It is therefore of interest to consider, besides the Euler-Lagrange equation (1.3), the Jacobi second variation operator

$$\frac{d^2}{dt^2} \left[\text{Area}\, (\Sigma_{\phi,\,t}) \right] \Big|_{t=0}. \tag{1.9}$$

A minimal surface Σ is called **stable** if

$$\frac{d^2}{dt^2} \left[\text{Area}\, (\Sigma_{\phi,\,t}) \right] \Big|_{t=0} \geqslant 0 \quad \text{for all } \phi. \tag{1.10}$$

Once again, it is of interest to be able to intrinsically decribe the class of minimal surfaces which are stable. In this regard, the key result is the following:

Theorem 1.0.3 (The Second Variation Formula of Area)**.** *In the context of Theorem 1.0.2 we have*

$$\frac{d^2}{dt^2} \left[\text{Area}\, (\Sigma_{\phi,\,t}) \right] \Big|_{t=0} = \int_\Sigma \Big[2\phi^2 \Big(\sum_{1 \leqslant j < k \leqslant n-1} \lambda_j \lambda_k \Big) + \|\nabla_{tan} \phi\|^2 \Big] \, d\sigma, \tag{1.11}$$

where $\lambda_1, ..., \lambda_{n-1}$ are the principal curvatures of Σ, and $\nabla_{tan}\phi := \nabla \phi - (\partial_\nu \phi)\nu$ is the tangential gradient of ϕ on Σ.

A few comments are in order • For a C^2 surface Σ in $\mathbb{R}^n$,

$$\int_{\Sigma} \|\nabla_{tan}\phi\|^2\,d\sigma = -\int_{\Sigma} (\Delta_\Sigma \phi)\phi\,d\sigma, \tag{1.12}$$

where Δ_Σ is the Laplacian on Σ, granted that (1.1) holds. As a result, we may write

$$\frac{d^2}{dt^2}[\mathrm{Area}\,(\Sigma_{\phi,\,t})]\Big|_{t=0} = \int_{\Sigma} \Big[2\phi^2\Big(\sum_{1\leqslant j<k\leqslant n-1} \lambda_j\lambda_k\Big) - (\Delta_\Sigma\phi)\phi\Big]\,d\sigma. \tag{1.13}$$

• If $\mathcal{G}=0$, equation (5.427) becomes

$$\frac{d^2}{dt^2}[\mathrm{Area}\,(\Sigma_{\phi,\,t})]\Big|_{t=0} = -\int_{\Sigma} \Big[\phi^2\Big(\sum_{j=1}^{n-1} \lambda_j^2\Big) + \phi(\Delta_\Sigma\phi)\Big]\,d\sigma. \tag{1.14}$$

We present new, self-contained proofs of these results which have the attractive feature that they completely avoid the heavy differential geometrical jargon which typically accompanies much of the work on this topic. Of course, this requires that we pedantically develop a number of alternative tools, some of which we would now like to elaborate upon. One such tool is a distinguished extension of the unit normal of a domain:

Theorem 1.0.4 (Extension of the Outward Unit Normal of a Domain)**.** *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a domain of class C^k, for some $k \in \mathbb{N}\cup\{\infty\}$, $k \geqslant 2$. Then there exists U open neighborhood of $\partial\Omega$, and there exists is a vector-valued function of class C^{k-1} in U and a vector field*

$$N = (N_1, ..., N_n) : U \longrightarrow \mathbb{R}^n \tag{1.15}$$

which has the following properties:

(1) $\|N(X)\| = 1$ for every $X \in U$;

(2) $N\Big|_{\partial\Omega} = \nu$, the outward unit normal to Ω;

(3) $\partial_j N_k = \partial_k N_j$ in U, for all $j,\, k \in \{1, ..., n\}$;

(4) for every $j \in \{1, ..., n\}$, the directional derivative $D_N N_j$ vanishes in U.

Other significant properties of this distinguished extension of the unit normal are as follows.

Theorem 1.0.5. *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a domain of class C^2 with outward unit normal ν. Denote by N the distinguished extension of ν to an open neighborhood*

U of $\partial\Omega$ described in Theorem 1.0.4. Then

(1) The Jacobian matrix DN is symmetric at all points in U, and its restriction to $\partial\Omega$ depends only on the domain Ω itself.

(2) One has $(DN(X))N(X) = 0$ for every $X \in U$. In particular,

$$(DN(X))\nu(X) = 0 \qquad \text{for every } X \in \partial\Omega. \tag{1.16}$$

(3) One has $(DN(X)v) \cdot w = \mathrm{II}_X(v, w)$, the second fundamental form of $\partial\Omega$, for every point $X \in \partial\Omega$ and any vectors v, w tangent to $\partial\Omega$ at X.

(4) For every $X \in \partial\Omega$ one has $\operatorname{Tr}(DN(X)) = \mathcal{G}(X)$, the Gauss mean curvature of $\partial\Omega$ at X.

(5) If $X \in \partial\Omega$ and $\lambda_1(X), ..., \lambda_n(X)$ are the eigenvalues of the symmetric matrix $DN(X)$, then $\lambda_n(X) = 0$ and

$$\mathcal{G}(X) = \lambda_1(X) + \cdots + \lambda_{n-1}(X). \tag{1.17}$$

The key idea in the proof of Theorem 1.0.4 is to take

$$N(X) := (\nabla d)(X), \quad \forall\, X \in U, \text{ with } d := \textbf{the signed distance to } \partial\Omega. \tag{1.18}$$

Given an arbitrary set $\Omega \subseteq \mathbb{R}^n$, the **signed distance** (to its boundary) is the function $d : \mathbb{R}^n \longrightarrow \mathbb{R}$, defined by

$$d(X) := \begin{cases} +\operatorname{dist}(X, \partial\Omega) & \text{if } X \in \Omega, \\ -\operatorname{dist}(X, \partial\Omega) & \text{if } X \in \Omega^c. \end{cases} \tag{1.19}$$

In this scenario, the focus becomes the analytical properties of this signed distance function, such as its differentiability. In this regard, we prove that

Theorem 1.0.6. *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a domain of class C^k, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geqslant 2$. Then there exists an open set $U \subseteq \mathbb{R}^n$, containing $\partial\Omega$, with the property that*

$$\textit{the signed distance function } d \textit{ is of class } C^k \textit{ in } U. \tag{1.20}$$

In the proof of the above theorem, the following formula plays a most significant role:

$$d(X) = \nu(\mathcal{P}(X)) \cdot (\mathcal{P}(X) - X), \qquad \forall\, X \in U, \tag{1.21}$$

where

$$\mathcal{P}(X) := \text{the unique nearest point on } \partial\Omega \text{ to } X. \tag{1.22}$$

Note that, for an arbitrary set $\Omega \subset \mathbb{R}^n$, there is no guarantee that $\mathcal{P}(X)$ is well-defined. This is related to the smoothness of $\partial\Omega$. Nonetheless, even if $\partial\Omega$ is very smooth, the nearest boundary point function $\mathcal{P}$ is unambigously defined only near $\partial\Omega$. The geometrical property which provides a natural environment where the "projection" $\mathcal{P}$ is well defined is the so-called *uniform ball condition.*

Theorem 1.0.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a C^1 domain satisfying a UIBC (with radius r). Then for every $X \in \Omega$ such that the dist$(X, \partial\Omega) < r$, there exists a unique point $X^* \in \partial\Omega$ with the property that*

$$\operatorname{dist}(X, \partial\Omega) = \|X - X^*\|. \tag{1.23}$$

Furthermore, an analogous result holds for the case of C^1 domains in $\mathbb{R}^n$ satisfying a UEBC granted that, this time, $X \in (\overline{\Omega})^c$.

Here and elsewhere, we have used the following definition.

Definition 1.0.8. *Fix a proper, non-empty, arbitrary subset D of $\mathbb{R}^n$.*

(1) We say that D satisfies a **uniform exterior ball condition** *(UEBC) if there exists a number $r > 0$ such that for any $X^* \in \partial D$ there exists a point $X \in \mathbb{R}^n$ for which $B(X, r) \cap D = \varnothing$ and $X^* \in \partial B(X, r)$.*

(2) We say that D satisfies a **uniform interior ball condition** *(UIBC) if $\mathbb{R}^n \setminus D$ satisfies a uniform exterior ball condition.*

(3) We say that $D \subseteq \mathbb{R}^n$ satisfies a **uniform two-sided ball condition** *if D satisfies both a UEBC and a UIBC.*

Informally speaking, a UEBC (respectively, UIBC) for a set Ω is equivalent with the requirement that one can roll a ball of fixed radius along the boundary of Ω, on the outside (respectively, inside) of Ω. Therefore, a natural question to ask is

$$\begin{gathered}\textbf{To what extent does a two sided ball condition} \\ \textbf{characterize the smoothness of a domain}?\end{gathered} \tag{1.24}$$

The answer to this question is provided by the following theorem:

Theorem 1.0.9. *Any domain Ω of class $C^{1,1}$ in $\mathbb{R}^n$ satisfies a uniform two-sided ball condition. Conversely, if Ω is an open, bounded, nonempty subset of $\mathbb{R}^n$ which satisfies a uniform two-sided ball condition then then Ω is a domain of class $C^{1,1}$.*

Recall that a domain is said to be of class $C^{1,1}$ if its boundary is locally given by graphs of differentiable functions, with Lipschitz gradients (a function is **Lipschitz** if it it does not distort distances by more than a fixed (finite) multiplicative factor). We wish to elaborate on the methodology used in the proof of this theorem. The simpler direction is **Analysis**$\Longrightarrow$**Geometry**, i.e., showing that if Ω is of class $C^{1,1}$ then it satisfies a uniform two-sided ball condition. To this end, here is a picture which illustrate what happens in this case:

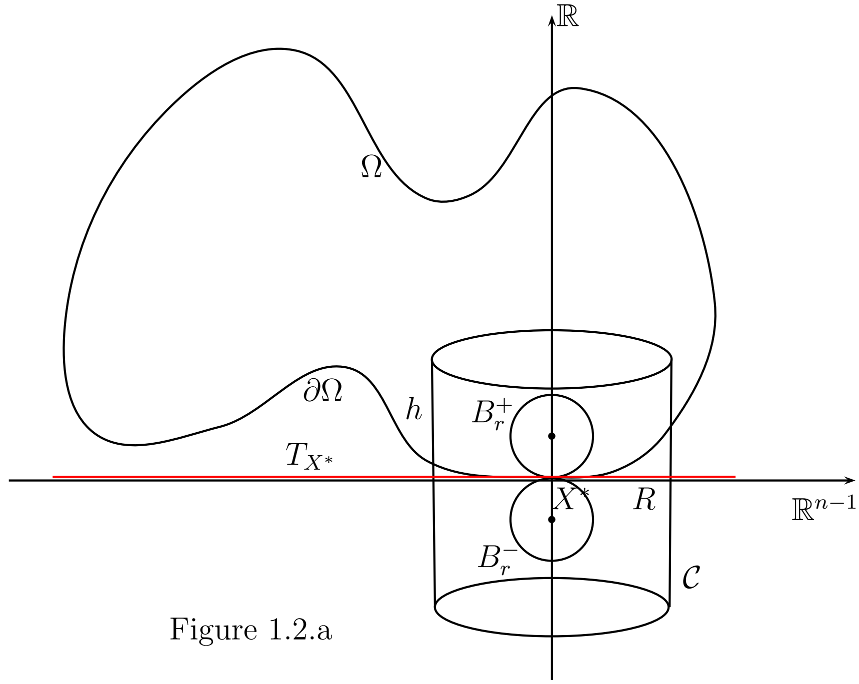

Figure 1.2.a

The idea is that, if $\partial\Omega$ coincides with the graph of a $C^{1,1}$ function φ near $X^* \in \partial\Omega$, then the rate at which $\partial\Omega$ bends at X^* is controlled in terms of the Lipschitz constant $\nabla\varphi$, whereas the rate at which the boundaries of the balls $B_r^{\pm}$ bend at X^* can be made as large as we wish by taking r small enogh. In turn, this guarantees that $B_r^+ \subseteq \Omega$ and $B_r^- \subseteq \Omega^c$, hence Ω satisfies a uniform two-sided ball condition. The difficult direction is the converse one, namely **Geometry**$\Longrightarrow$**Analysis**. That is, we wish to establish that if Ω is an open, bounded, nonempty subset of $\mathbb{R}^n$ which satisfies a uniform two-sided ball condition then Ω is a domain of class $C^{1,1}$. Recall that this means that we need to show that near each point $X^* \in \partial\Omega$, the boundary of Ω coincides with the graph of a function φ of class $C^{1,1}$. Below we outline the main steps of the argument.

<u>`Step I.` *Definition of the function* φ.</u> Fix $X^* \in \partial\Omega$ arbitrary and denote by $B_R^{\pm}$ the interior and exterior balls at X^*, where $R > 0$ is the common vales of the radii of these balls. These balls have a common $(n-1)$-dimensional tangent plane H passing through X^*). Make a translation and a rotation such that X^* becomes the origin in $\mathbb{R}^n$ and the plane H becomes the horizontal $(n-1)$-dimensional plane $\mathbb{R}^{n-1} \times \{0\}$). Below we make the convention that B_{n-1} will denote an $(n-1)$-dimensional ball in the plane $H \equiv \mathbb{R}^{n-1}$. Fix a small number $\lambda \in (0,1)$ to be specified later. Our goal is to find a $C^{1,1}$ function $\varphi : B_{n-1}(0', \lambda R) \to \mathbb{R}$ (where $0'$ is the origin in $\mathbb{R}^{n-1}$) which, among other things has the property that its graph coincide with $\partial\Omega$ in a neighborhood of $X^* = 0$. This situation is depicted in the following picture:

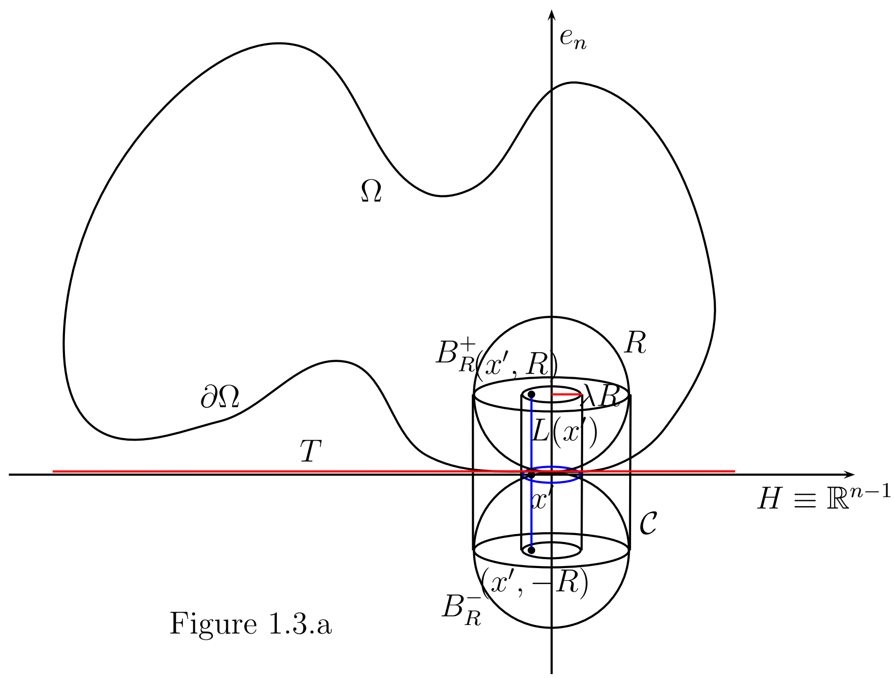

Figure 1.3.a

In the context of the above picture, we shall do the natural thing and define the function $\varphi : B_{n-1}(0', \lambda R) \longrightarrow \mathbb{R}$ such that the graph of φ coincides with the boundary of Ω inside the cylinder $\mathcal{C}$. Concretely, for each $x' \in B_{n-1}(0', \lambda R)$, we take $\varphi(x')$ to be the height of the vertical segment emerging from x' until it intersects the boundary of Ω. In particular, this implies that $\varphi(0') = 0$. Of course, we need to make sure that the function defined in the manner described above is well-defined (i.e., we need to show that the Vertical Line Test is not violated). With this goal in mind, for each $x' \in B_{n-1}(0', \lambda R)$ define $L(x') = [(x', R),\ (x', -R)]$, i.e., $L(x')$ is the line segment with end-points (x', R) and $(x', -R)$. At this stage, we make the following the claim.

$$\lambda > 0 \text{ small} \implies \#(L(x') \cap \partial\Omega) = 1, \qquad \forall\, x' \in B_{n-1}(0', \lambda R). \tag{1.25}$$

To prove this, we first note that Proposition 5.1.40 ensures that

$$L(x') \cap \partial\Omega \neq \varnothing, \qquad \forall\, x' \in B_{n-1}(0', \lambda R). \tag{1.26}$$

Granted this, we only need to show that $L(x')$ intersects $\partial\Omega$ only once. To justify this, fix a point $x' \in B_{n-1}(0', \lambda R)$, and consider the vertical line segment $L(x')$ as before. From what we have proved so far we know that exists a point $u \in L(x')$ such that $u \in \partial\Omega$ (since the segment intersects the boundary at least once). Consider the interior and the exterior balls $B^{\pm}(u)$ as in Uniform Interior Ball Condition/Uniform Exterior Ball Condition for this new point $u \in \partial\Omega$. If we can show that the portion of $L(x')$ lying above u is contained in the interior, or the exterior, ball (which, in turn, are included in Ω and Ω^c, respectively), and that the portion of $L(x')$ lying below u is contained in the exterior, or interior, ball (which, in turn, are included in Ω^c and in Ω, respectively), then there can be no other points from the boundary of Ω on $L(x')$ other than u. Any other possible combination will fit into the pattern above. In fact, we only need to check that the aforementioned inclusion properties apply only to the portion of $L(x')$ not already contained in the "original" interior/exterior balls (i.e., the interior and exterior balls touching at the origin). With this goal in mind, we write $u = (u', u_n)$ and we let $L[u']$ denote the portion of $L(u')$ not contained in the interior/exterior balls touching at origin. In Figure 1.4.a, this is the segment $[A, B]$. Our goal then becomes showing that the points A, B belong to the interior/exterior balls touching at u. In the picture above v is the direction of the line joining the centers of the "new" balls (the interior/exterior balls touching at u). This is a somewhat delicate issue which requires a considerable amount of attention. Here, we omit the details.

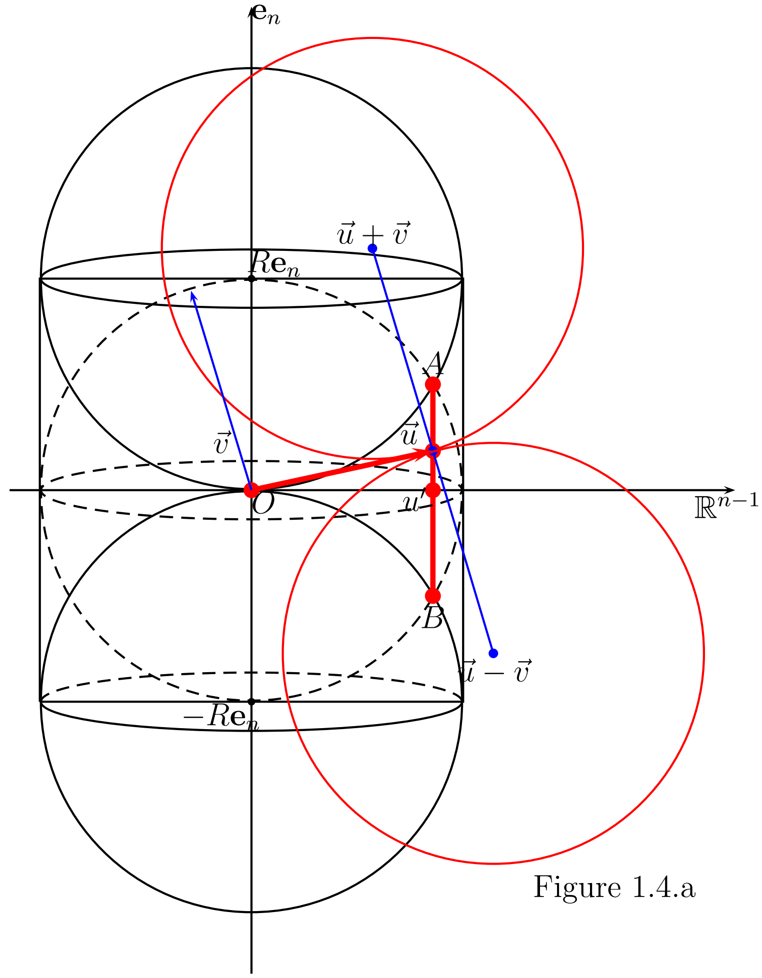

Figure 1.4.a

<u>`Step II.` *The function φ is differentiable*.</u> Granted the uniform two-sided ball condition satisfied by Ω as well as the specific way in which φ has been defined, it is possible to show that φ is continuous. To proceed, we need a *geometric differentiability criterion*. To state such a criterion, we first recall the following definition. Given a vector $v \in \mathbb{R}^n$ and an angle $\theta \in (0, \pi)$, we denote by $\mathcal{C}_n(X^*, v, \theta)$ the open, solid circular **cone** in $\mathbb{R}^n$ with vertex at $X^* \in \mathbb{R}^n$, whose axis is along v, and has aperture θ. That is,

$$\mathcal{C}_n(X^*, v, \theta) = \{X \in \mathbb{R}^n : (X - X^*) \cdot v > (\cos\theta)\|X - X^*\|\|v\|\}. \tag{1.27}$$

Returning to the mainstream discussion, we prove and utilize the following theorem:

Theorem 1.0.10 (The "Point of Impact" Differentiability Criterion)**.** *Assume that $U \subseteq \mathbb{R}^n$ is a an arbitrary set, and that $X^* \in U^\circ$. Given a function $f : U \to \mathbb{R}$, we denote by G_f the graph of f, i.e., $G_f := \{(X, f(X)) : X \in U\} \subseteq \mathbb{R}^{n+1}$. Then f is differentiable at the point X^* if and only if f is continuous at X^* and there exists a non-horizontal vector $N \in \mathbb{R}^{n+1}$ (i.e., satisfying $N \cdot \mathbf{e}_{n+1} \neq 0$) with the following significance. For every* *angle $\theta \in (0, \pi/2)$ there exists $\delta > 0$ with the property that $G_f \cap B_{n+1}((X^*, f(X^*)), \delta)$ lies in between the cones $\mathcal{C}_{n+1}((X^*, f(X^*)), N, \theta)$ and $\mathcal{C}_{n+1}((X^*, f(X^*)), -N, \theta)$, i.e.,*

$$\begin{aligned} & G_f \cap B_{n+1}((X^*, f(X^*)), \delta) \\ & \qquad \subseteq \mathbb{R}^{n+1} \setminus \Big[\mathcal{C}_{n+1}((X^*, f(X^*)), N, \theta) \cup \mathcal{C}_{n+1}((X^*, f(X^*)), -N, \theta)\Big]. \end{aligned} \tag{1.28}$$

If this happens, then necessarily N is a scalar multiple of $(\nabla f(X^*), -1) \in \mathbb{R}^{n+1}$.

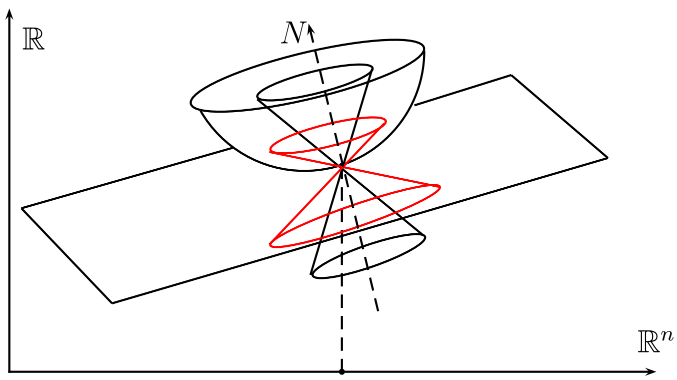

Figure 1.5.a

It is clear that the Point of Impact Differentiability Criterion from Theorem 1.0.10 readily gives that the function $\varphi : B_{n-1}(0', \lambda R) \to \mathbb{R}$ is differentiable at every point in its domain. In fact, the very last part in the statement of Theorem 1.0.10 also gives that

$$\frac{v}{R} = \frac{(-\nabla\varphi(u'), 1)}{\sqrt{1 + \|\nabla\varphi(u')\|^2}}. \tag{1.29}$$

This formula plays a vital role in the proof of the fact that the function $\nabla\varphi$ is Lipschitz, which is too technically involved step to be discussed in detail here. Once the differentiablity properties of the signed distance function (as well as the nearest boundary point function) have been clarified, a number of useful consequences can be derived. One such consequence is the following theorem.

Theorem 1.0.11 (The Collar Neighborhood Theorem). *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a domain of class C^k, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geqslant 2$, and denote by ν its outward unit normal. Then there exist $\varepsilon > 0$ and an open set $U \subseteq \mathbb{R}^n$ which contains $\partial\Omega$, such that the function*

$$F : \partial\Omega \times (-\varepsilon, \varepsilon) \longrightarrow U, \quad F(X, t) := X - t\nu(X), \quad \forall\, (X, t) \in \partial\Omega \times (-\varepsilon, \varepsilon), \tag{1.30}$$

is a homeomorphism. Moreover, F *is of class* C^{k-1}*, the function*

$$G : U \longrightarrow \partial\Omega \times (-\varepsilon, \varepsilon), \quad G(X) := (\mathcal{P}(X), d(X)) \quad \forall\, X \in U, \tag{1.31}$$

is well-defined, of class C^{k-1}*, and is an inverse for* F *in* (1.30)*. That is, one has*

$$X = \mathcal{P}(X) - d(X)\nu(\mathcal{P}(X)), \qquad \forall\, X \in U, \tag{1.32}$$

as well as

$$d(X - t\nu(X)) = t \quad \text{and} \quad \mathcal{P}(X - t\nu(X)) = X, \qquad \forall\, (X, t) \in \partial\Omega \times (-\varepsilon, \varepsilon). \tag{1.33}$$

Finally, in the case when Ω *is a* $C^{1,1}$ *domain in* $\mathbb{R}^n$*,* $n \geqslant 2$*, the above results continue to hold if "being of class* C^{k-1}*" is, in this situation, interpreted as "being Lipschitz."*

The last part in the statement of the theorem is rather delicate, and its proof relies on the following theorem:

Theorem 1.0.12. *Assume that* $\Omega \subseteq \mathbb{R}^n$*,* $n \geqslant 2$*, is a domain of class* $C^{1,1}$*. Then there exists a neighborhood* U *of* $\partial\Omega$ *with the property that the nearest boundary point function* $\mathcal{P} : U \to \partial\Omega$ *is Lipschitz.*

One nonstandard aspect of the proof of the above theorem is the fact that it requires a version of the Implicit Function Theorem in the class of Lipschitz functions. Classically, the Implicit Function Theorem is stated and proved only for C^1 functions. We nonetheless have the following result.

Theorem 1.0.13 (The Implicit Function Theorem for Lipschitz Functions)**.** *Let* $U \subseteq \mathbb{R}^n$ *and* $V \subseteq \mathbb{R}^m$ *be open sets. Fix* $X_0 \in U$*,* $Y_0 \in V$ *and assume that*

$$F : U \times V \longrightarrow \mathbb{R}^m \tag{1.34}$$

is a Lipschitz function for which

$$F(X_0, Y_0) = 0 \tag{1.35}$$

and which has the property that there exists a constant $K > 0$ *for which*

$$\|F(X, Y_1) - F(X, Y_2)\| \geqslant K\|Y_1 - Y_2\| \quad \text{for all} \quad (X, Y_j) \in U \times V, \quad j = 1, 2. \tag{1.36}$$

Then there exist an open set $W \subseteq \mathbb{R}^n$ *such that* $X_0 \in W$*, along with a Lipschitz function* $\varphi : W \to V$ *with the property that* $\varphi(X_0) = Y_0$ *and*

$$\{(X, Y) \in W \times V : \; F(X, Y) = 0\} = \{(X, \varphi(X)) : \; X \in W\}\,. \tag{1.37}$$

In particular,

$$F(X, \varphi(X)) = 0, \qquad \textit{for all} \quad X \in W. \tag{1.38}$$

Finally, the function φ is unique among all the mappings $\varphi : W \to V$ for which (1.38) *holds.*

Let us present a few applications. First, we consider an application to the the two-sided ball smoothness criterion presented earlier.

Theorem 1.0.14. *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, open, bounded, convex set and, for some $\varepsilon > 0$, consider*

$$\Omega_\varepsilon := \{X \in \mathbb{R}^n : \operatorname{dist}(X, \Omega) < \varepsilon\}. \tag{1.39}$$

Then Ω_ε is a convex domain of class $C^{1,1}$.

A sketch of proof is as follows. That Ω_ε is a convex set is elementary. Also, it is not difficult to see that

$$\partial\Omega_\varepsilon = \{X \in \mathbb{R}^n : \operatorname{dist}(X, \Omega) = \varepsilon\}. \tag{1.40}$$

In order to show that Ω_ε is a domain of class $C^{1,1}$, it suffices to show that it satisfies a uniform two-sided ball condition. A key observation in this respect is that if $X^* \in \partial\Omega_\varepsilon$ and $Y^* \in \partial\Omega$ are such that $\|X^* - Y^*\| = \varepsilon$ then

$$B\big(\tfrac{1}{2}(X^* + Y^*)\,,\, \tfrac{\varepsilon}{2}\big) \subseteq \Omega_\varepsilon \quad \text{ and } \quad B\big(\tfrac{3}{2}X^* - \tfrac{1}{2}Y^*\,,\, \tfrac{\varepsilon}{2}\big) \subseteq \mathbb{R}^n \setminus \Omega_\varepsilon. \tag{1.41}$$

Remark 1.0.15. *The conclusion in Theorem 1.0.14 is optimal. Indeed, if we take $n := 2$, $\Omega := [-1, 0] \times [1, 2]$ and $\varepsilon := 1$, then $0 \in \partial\Omega_1$ and the boundary of Ω_1 coincides near 0 with the graph of the function*

$$f : (-1, 1) \to \mathbb{R}, \qquad f(x) := \begin{cases} 1 - \sqrt{1 - x^2} & \textit{if } 0 < x < 1, \\ 0 & \textit{if } -1 < x \leqslant 0. \end{cases} \tag{1.42}$$

Note that while f is of class C^1, its derivative

$$f' : (-1, 1) \to \mathbb{R}, \qquad f'(x) = \begin{cases} \frac{x}{\sqrt{1-x^2}} & \textit{if } 0 < x < 1, \\ 0 & \textit{if } -1 < x \leqslant 0, \end{cases} \tag{1.43}$$

is a Lipschitz function which fails to be differentiable at 0.

In turn, the above theorem can be used to give a conceptually simple proof of the following useful approximation result (which appears without proof in Grisvard's 1984 PDE book):

Theorem 1.0.16. *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, open, bounded, convex set. Then there exist $\varepsilon_o > 0$ and two families of convex $C^{1,1}$ domains $\{\Omega_\varepsilon^\pm\}_{\varepsilon_o > \varepsilon > 0}$ satisfying*

$$\overline{\Omega_\varepsilon^+} \subset \Omega, \qquad \overline{\Omega} \subset \Omega_\varepsilon^-, \qquad \forall\, \varepsilon \in (0, \varepsilon_o), \tag{1.44}$$

and

$$\Omega = \bigcup_{\varepsilon_o > \varepsilon > 0} \Omega_\varepsilon^+ \quad \text{and} \quad 0 < \varepsilon' < \varepsilon'' < \varepsilon_o \Longrightarrow \overline{\Omega_{\varepsilon''}^+} \subset \Omega_{\varepsilon'}^+, \tag{1.45}$$

$$\overline{\Omega} = \bigcap_{\varepsilon_o > \varepsilon > 0} \Omega_\varepsilon^- \quad \text{and} \quad 0 < \varepsilon' < \varepsilon'' < \varepsilon_o \Longrightarrow \overline{\Omega_{\varepsilon'}^-} \subset \Omega_{\varepsilon''}^-. \tag{1.46}$$

Furthermore, with $\mathrm{Dist}\,[\cdot,\cdot]$ *denoting the Hausdorff distance function, we have*

$$\mathrm{Dist}\,[\partial\Omega, \partial\Omega_\varepsilon^\pm] = \varepsilon \quad \text{for every} \quad \varepsilon \in (0, \varepsilon_o). \tag{1.47}$$

An application to the circle of ideas pertaining to the Collar Neighborhood Theorem is as follows.

Theorem 1.0.17. *Let Ω be a C^2 domain in $\mathbb{R}^n$, and let ρ be a defining function for Ω (that is, ρ is a function of class C^1 which is positive in Ω, negative in $(\overline{\Omega})^c$, and $\nabla\rho \neq 0$ on $\partial\Omega$). Then there exists $\varepsilon > 0$ such that for every function $f \in C_0^\infty(U)$*

$$\int_U f(X)\, dX = \int_{-\varepsilon}^{\varepsilon} \Big(\int_{\Sigma_{\phi,\,t}} f\, \|\nabla\rho\|^{-1}\, d\sigma_t \Big)\, dt \tag{1.48}$$

where $\Sigma_{\phi,\,t}$ is the surface $\{\rho = t\}$ and σ_t is the surface measure on $\Sigma_{\phi,\,t}$.

One of the most important particular cases is as follows. Consider the situation when $\rho := d$, the signed distance for Ω. In this case, $\nabla d(X) = \nu(\mathcal{P}(X))$ which implies that $\|\nabla\rho\| = \|\nabla d\| = 1$ in U. Consequently, we obtain

$$\int_U f(X)\, dX = \int_{-\varepsilon}^{\varepsilon} \Big(\int_{\{d=t\}} f\, d\sigma_t \Big)\, dt, \tag{1.49}$$

for any nice function f in U. We wish to further specialize this formula. To do so, for each $r > 0$ small consider $U_r := \{x \in \Omega \,:\, dist(X,\, \partial\Omega) < r\}$. Also, assume that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function. With the help of (1.49) and the Fundamental Theorem of Calculus, we deduce that

$$\begin{aligned} \frac{d}{dr}\Big[\int_{U_r} f(X)\, dX\Big] &= \frac{d}{dr}\Big[\int_0^r \Big(\int_{\Sigma_{\phi,\,t}} f\, d\sigma_t\Big)\, dr\Big] \\ &= \int_{\Sigma_{\phi,\,r}} f\, d\sigma_r, \qquad \text{for } r > 0 \text{ small}. \end{aligned} \tag{1.50}$$

Next, introduce $\Omega_r := \{x \in \Omega \,:\, dist(X,\, \partial\Omega) > r\}$. Since, clearly, $\Omega = \Omega_r \cup U_r$ (cf. the picture below)

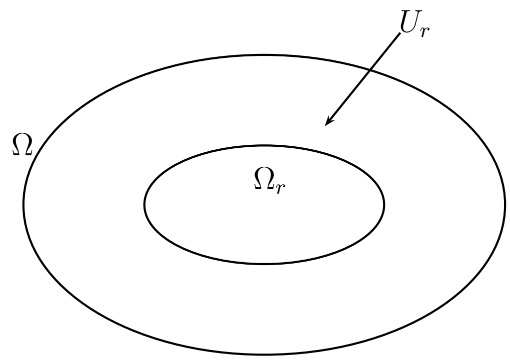

Figure 1.6.a

we may write

$$\int_{U_r} f(X)\,dX = \int_{\Omega} f(X)\,dX \;-\; \int_{\Omega_r} f(X)\,dX. \tag{1.51}$$

From (1.51), (1.50) and the fact that $\partial\Omega_r = \Sigma_{\phi,\,r}$ we therefore obtain

$$\frac{d}{dr}\Big[\int_{\Omega_r} f(X)\,dX\Big] = -\int_{\partial\Omega_r} f\,d\sigma_r. \tag{1.52}$$

In particular, taking $f \equiv 1$ in (1.52) yields the remarkable formula

$$\frac{d}{dr}[\mathrm{Vol}\,(\Omega_r)] = -\mathrm{Area}\,(\partial\Omega_r), \qquad \text{for } r > 0 \text{ small.} \tag{1.53}$$

Of course, a similar version is valid for $\Omega^r := \{x \in \Omega^c \,:\, dist(X,\, \partial\Omega) < r\}$. This time, however, we obtain

$$\frac{d}{dr}[\mathrm{Vol}\,(\Omega^r)] = \mathrm{Area}\,(\partial\Omega^r), \qquad \text{for } r > 0 \text{ small.} \tag{1.54}$$

It is interesting to compare this with the case when $\Omega := B(0,1) \subseteq \mathbb{R}^3$, in which case $\Omega_r = B(0,1-r)$ and $\Omega^r = B(0,1+r)$. In this situation, we have the well-known formulas

$$\begin{aligned} &\mathrm{Vol}\,(\Omega^r) = \tfrac{4\pi(1+r)^3}{3}, \qquad \mathrm{Vol}\,(\Omega_r) = \tfrac{4\pi(1-r)^3}{3}, \\ &\mathrm{Area}\,(\partial\Omega^r) = 4\pi(1+r)^2, \qquad \mathrm{Area}\,(\partial\Omega_r) = 4\pi(1-r)^2, \end{aligned} \tag{1.55}$$

which show that (1.53) and (1.54) hold in this special case. Another major objective of this thesis is to develop a calculus on surfaces, centered about a general integration by parts formulas for first-order differential operators on surfaces. Naturally, this extends Stoke's formula. Sir George Gabriel Stokes (1819 - 1903) was an Irish mathematician and physicist. He made important contributions to hydrodynamics, optics and mathematical physics. Stokes contributed to establishment of the science of hydrodynamics with the most famous equation in fluid mechanics, the Navier-Stokes equation. In optics he developed Stoke's Law for fluorescence, and studied ultraviolet light. In mathematics Stokes worked in numerical calculations of definite integrals, infinite series, and differential equations. Stoke's Theorem in differential geometry is a statement about the integration of differential forms which generalizes several theorems from vector calculus.

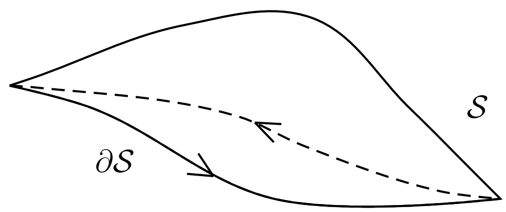

Figure 1.7.a

The classical Stokes Theorem, i.e.

$$\int_{\mathcal{S}} (\nu \cdot curl\vec{F})\, dS = \oint_{\partial\mathcal{S}} \tau \cdot \vec{F}\, ds, \tag{1.56}$$

relates the surface integral of the normal component of curl for a vector field $\vec{F}$ over a surface $\mathcal{S}$, in Euclidean three dimensional space, to the line integral of the tangential component of the vector field over the boundary of the surface. The Ostrogradsky-Gauss Theorem, The Fundamental Theorem of Calculus and the Green's Theorem are all special cases of this Stokes Theorem. The most general form of the Stokes Theorem using differential forms is more powerful than the special cases, of course, although the later are more accessible and are often considered more convenient in applications. A general first-order system of differential operators acting on a vector-valued function u will be:

$$P\, u := \left(\sum_{\beta} \left(\sum_{j=1}^{3} a_j^{\alpha\beta} \partial_j \right) u_\beta \right)_\alpha . \tag{1.57}$$

We define the symbol of P at $\xi \in \mathbb{R}^n$ as the matrix-valued function

$$\sigma(P;\xi) := \left(\sum_{j=1}^{3} a_j^{\alpha\beta} \xi_j \right)_{\alpha,\beta} , \tag{1.58}$$

and the adjoint of P, P^* acting on a function v, by

$$P^* v = \left(-\sum_{\alpha} \sum_{j=1}^{3} \partial_j \left(a_j^{\alpha\beta} v_\alpha \right) \right)_\beta . \tag{1.59}$$

We can then state and prove an explicit formula for integrating by parts large classes of first-order differential system operators over a surface $\mathcal{S}$, including boundary terms (for simplicity, this is stated here only in $\mathbb{R}^3$).

Theorem 1.0.18. *Let $\mathcal{S}$ be a smooth, compact surface in $\mathbf{R}^3$ with normal ν and boundary $\partial\mathcal{S}$. Denote by γ the unit normal to $\partial\mathcal{S}$. Let P be a first-order differential operator which is tangential, in the sense that the principal symbol of P vanishes along the normal to $\mathcal{S}$, i.e. $\sigma(P;\nu) = 0$ at all points on $\mathcal{S}$. Denote by P^* denote the adjoint in $\mathbf{R}^3$. Then for every u, v be $\mathcal{C}^1$ functions on $\mathcal{S}$,*

$$\int_{\mathcal{S}} \langle Pu, v\rangle \, dS = \int_{\mathcal{S}} \langle\, u,\, P^* v\,\rangle dS + \oint_{\partial\mathcal{S}} \langle \sigma(P;\gamma)u,\, v\rangle ds.$$

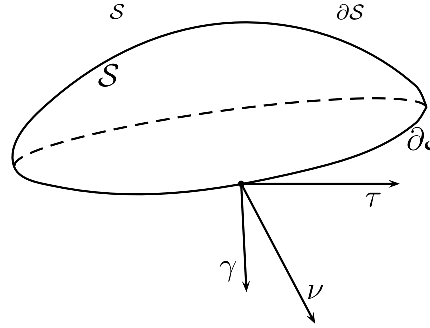

Figure 1.8.a

The usefulness of this result stems from the fact that all analytic objects involved, i.e., P, P^*, and $\sigma(P)$ are independent of the particular surface $\mathcal{S}$. Of course, this theorem contains as a particular case the classical Stokes formula. To see this, let N be the distingusihed extension of the unit normal ν to $\mathcal{S}$, as discussed earlier. Also, introcuce the first-order differential operator

$$P := N \cdot curl. \tag{1.60}$$

For a nice vector-valued function $\vec{u}$ we then have:

$$P\vec{u} = N \cdot (\nabla \times \vec{u})\,. \tag{1.61}$$

Consequently, for any $\xi \in \mathbb{R}^3$ we may compute

$$\sigma\,(P\,;\xi)\,\vec{u} = N \cdot (\xi \times \vec{u})\,, \tag{1.62}$$

and, hence, at any point in $\mathcal{S}$,

$$\sigma\,(P\,;\nu)\,\vec{u} = \nu \cdot (\nu \times \vec{u}) = 0, \tag{1.63}$$

since $N = \nu$ on $\mathcal{S}$.

To compute the adjoint operator on $\mathbf{R}^3$ consider the following integration by parts formula

$$\int_{\mathbf{R}^3} \langle Pu, \vec{v}\rangle\, dx = \int_{\mathbf{R}^3} N \cdot\ curl\,\vec{u}\, v\, dx = \int_{\mathbf{R}^3} \langle\, vN,\ curl\,\vec{u}\,\rangle\, dx$$

$$= \int_{\mathbf{R}^3} \langle\, curl\,(vN)\,,\ u\,\rangle\, dx, \tag{1.64}$$

so $P^* = curl\,(vN) = \nabla v \times N + v\, curl\, N$. However, we know that $\mathrm{curl}\, N = 0$ on $\mathcal{S}$. With this in mind, we can now apply our theorem to obtain

$$\int_{\mathcal{S}} (\nu \cdot \mathrm{curl}\,\vec{u})\ v\, dS = \int_{\mathcal{S}} \langle u,\ \nabla v \times \nu\,\rangle\, dS + \oint_{\partial\mathcal{S}} \nu \cdot (\gamma \times \vec{u})\ v\, ds. \tag{1.65}$$

In particular, this reduces to Stokes' formula when $v = 1$, since $\nabla v = 0$ and also $\nu \cdot (\gamma \times \vec{u}) = \tau \cdot \vec{u}$, where $\tau := \gamma \times \nu$ is the unit tangent along $\partial\mathcal{S}$.

Chapter 2

A First Look at Geometry of Surfaces

2.1 Geometry of Surfaces

Let $\mathbf{R}^n$ be the standard Euclidian space. We denote by $| x |= \sqrt{\sum_{j=1}^n x_j^2}$ the norm of a vector $x = (x_1, ..., x_n) \in \mathbf{R}^n$. Throughout the manuscript, "dot"will denote the scalar product.

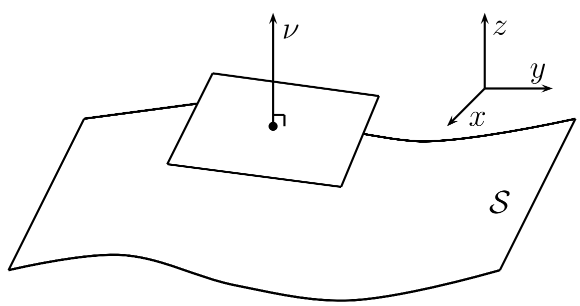

Figure 2.1: The surface $\mathcal{S}$, the tangent plane, and the normal ν.

Consider a point $x = (x_1, ..., x_{n-1}, x_n)$ on a surface $\mathcal{S}$ with unit normal ν. Let $\mathcal{S}$ be the graph of a function $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$. Then

$$\begin{aligned} \mathcal{S} &= \{(x', \varphi(x'))|\, x' \in \mathbf{R}^{n-1}\} \text{ and} \\ P(x') &= (x', \varphi(x')) \text{ where } x' = (x_1, x_2, ..., x_{n-1}), \end{aligned} \tag{2.1}$$

is the position function parametrization $P : \mathbf{R}^{n-1} \to \mathbf{R}^n$.

Lemma 2.1.1. *If* $1 \le j \le n-1$ *then* $\partial_{x_j} P(x')$ *is tangent to* $\mathcal{S}$ *at* $P(x')$.

Proof. $P(x') = (x_1...x_{n-1}, \varphi(x'))$ and $\partial_{x_j} P(x') = (0...1...0, \partial_{x_j}\varphi(x'))$ with 1 on j-th position. Then $\{\partial_{x_j} P(x')\}_{1\le j\le n-1}$ are linearly independent vectors. Consider $T_x\mathcal{S}$ the tangent plane to $\mathcal{S}$ at x. This plane is generated (spanned) by the vectors $\{\partial_{x_j} P(x')\}_{1\leq j\leq n-1}$, and his dimension is $n-1$. $\square$

In general if $v \in \mathbf{R}$, $D_v f := v \cdot \nabla f = \sum_{j=1}^n v_j(\partial_j f)$. If $v = \partial_{x_j} P(x')$ then

$$\begin{aligned}(D_v f)(P(x)) &= \partial_j P(x^{'}) \cdot (\nabla f)(P(x)) \\ &= \partial_j P(x') \cdot \Big(\partial_{x_1} f(P(x)), \partial_{x_2} f(P(x)), ..., \partial_{x_n} f(P(x))\Big) \\ &= \frac{\partial}{\partial x_j}[f(P(x'))] = \frac{\partial}{\partial x_j}[(f \circ P)(x')], \text{for each } 1 \leq j \leq n-1.\end{aligned} \tag{2.2}$$

Remark 2.1.2. *$D_v f$ is meaningful on $\mathcal{S}$ if f is defined on $\mathcal{S}$, and v is one of the vectors $\{\partial_{x_j} P(x')\}_{1\leq j\leq n-1}$. If $v \in T_x\mathcal{S}$ then $\exists(\lambda_j)_{1\leq j\leq n-1}$ real numbers such that $v = \sum_{j=1}^n \lambda_j[\partial_j f]$,*

$$\begin{aligned}(D_v f)(P(x')) &= v \cdot (\nabla f)(P(x')) = \sum_{j=1}^n \lambda_j \partial_j P(x') \cdot (\nabla f)(P(x')) \\ &= \sum_{j=1}^n \lambda_j \frac{\partial}{\partial x_j}[f(P(x'))].\end{aligned} \tag{2.3}$$

Remark 2.1.3. *For $f : \mathcal{S} \to \mathbf{R}$, $v \cdot \nu = 0$, and $D_v f$ is meaningful on $\mathcal{S}$.*

Lemma 2.1.4. *For every x', the unit normal is given by*

$$\nu(P(x')) = \frac{(\nabla\varphi(x'), -1)}{\sqrt{1+ \mid \nabla\varphi(x') \mid^2}}. \tag{2.4}$$

Proof. We need to check that $\nu \perp \partial_{x_j} P$, and $\mid \nu \mid = 1$. To this end, we observe that,

$$\nu(P(x')) = \frac{(\partial_{x_1}\varphi(x'),\ \partial_{x_2}\varphi(x'), ...,\ \partial_{x_{n-1}}\varphi(x'),\ -1)}{\sqrt{1+ \mid \nabla\varphi(x') \mid^2}}, \tag{2.5}$$

so that

$$\nu(P(x')) \cdot \partial_{x_j} P(x') = \frac{\partial_{x_j}\varphi(x')}{\sqrt{1+ \mid \nabla\varphi(x') \mid^2}} - \frac{\partial_{x_j}\varphi(x')}{\sqrt{1+ \mid \nabla\varphi(x') \mid^2}} = 0. \tag{2.6}$$

In general $v \cdot v = \mid v \mid^2$, $\forall\, v \in \mathbf{R}^n$. So

$$\nu(P(x')) \cdot \nu(P(x')) = \mid \nu(P(x') \mid^2 = 1. \tag{2.7}$$

This proves (2.1.4) $\square$

2.2 Mean Curvature

From previous lemma we have $\frac{\partial}{\partial x_j}[\nu(P(x'))\cdot\nu(P(x'))] = 0$, and $2\frac{\partial}{\partial x_j}[\nu(P(x'))]\cdot\nu(P(x')) = 0$, then $\frac{\partial}{\partial x_j}[\nu(P(x'))]$ is tangential to $\mathcal{S}$ at $P(x')$, for all $j \in \{1, ..., n-1\}$.

Remark 2.2.1. *$\exists\, b_{ij}$ real numbers $(1 \le j,\, k \le n-1)$ such that*

$$\frac{\partial}{\partial x_j}[\,\nu(P(x'))] = \sum_{k=1}^{n-1} b_{jk}[\,\partial_{x_k} P(x')], \qquad \forall j \in \{1, ..., n-1\}. \tag{2.8}$$

Lemma 2.2.2. *For $1 \le j \le n-1$,*

$$b_{jj} = \partial_{x_j}\left[\frac{\partial_{x_j}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}}\right]. \tag{2.9}$$

Proof. The left-hand side of (2.8) becomes:

$$\partial_{x_j}\nu(P(x')) = \left(\partial_{x_j}\left[\frac{\partial_{x_1}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}}\right], ..., \partial_{x_j}\left[\frac{-1}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}}\right]\right). \tag{2.10}$$

For the right-hand side of (2.8) we have:

$$\begin{aligned}\sum_{k=1}^{n-1} b_{jk}\,(0...1...0, \partial_{x_k}\varphi(x')) &= \sum_{k=1}^{n-1}(0...b_{jk}...0, b_{jk}\partial_{x_k}\varphi(x')) \\ &= \left(b_{j1}, b_{j2}, ...b_{jn-1}, \sum_{k=1}^{n-1} b_{jk}\partial_{x_k}\varphi(x')\right).\end{aligned} \tag{2.11}$$

This proves the lemma. □

Introduce

$$B := (b_{jk})_{1\le j,\,k\le n-1} = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1\,n-1} \\ b_{21} & b_{22} & ... & b_{2\,n-1} \\ b_{n-1\,1} & b_{n-1\,2} & ... & b_{n-1\,n-1}\end{pmatrix}. \tag{2.12}$$

Definition 2.2.3. *Define the Gauss Curvature:*

$$\mathcal{G} := Trace\, B = \sum_{k=1}^{n-1} b_{jj}. \tag{2.13}$$

Theorem 2.2.4. *There holds*

$$div\,\nu\, = \mathcal{G} \quad on \quad \mathcal{S}. \tag{2.14}$$

Prior to proving Theorem (2.2.4), we introduce a class of tangential vectors τ_{kj}. Concretely we set:

$$\tau_{jk} := \nu_k e_j - \nu_j e_k \in \mathbf{R}^n \tag{2.15}$$

with $e_j = (0, ..., 1, ...0)$ (i.e. 1 on j-th position), then

$$D_{\tau_{kj}} := \tau_{kj}\cdot\nabla = (\nu_k\partial_{x_j} - \nu_j\partial_{x_k}). \tag{2.16}$$

Lemma 2.2.5. *The vector τ_{kj} is tangent to $\mathcal{S}$, for every k, j.*

Proof. Note that $\tau_{kj} \cdot \nu = \nu_k \nu_j - \nu_j \nu_k = 0$ as wanted. $\square$

Proof. To prove Theorem 2.2.4, let us start with $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ and we have:

$$\begin{aligned} div\, \nu &= \sum_{j=1}^{n} \partial_{x_j} \nu_j = \sum_{j=1}^{n} \sum_{k=1}^{n} \nu_k \nu_k (\partial_{x_j} \nu_j) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} \nu_k (\nu_k \partial_{x_j} - \nu_j \partial_{x_k}) \nu_j + \sum_{j=1}^{n} \sum_{k=1}^{n} \nu_k \nu_j \partial_{x_k} \nu_j . \end{aligned} \tag{2.17}$$

The product $\nu_j \partial_{x_k} \nu_j$ could be written like $\frac{1}{2} \partial_{x_k} (\nu_j^2)$. If ν is extended in a neighborhood of $\mathcal{S}$ such that its norm is 1 in that neighborhood then $\frac{1}{2}\, \partial_{x_k}(\nu_j^2) = 0$ and the sum

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \nu_k \nu_j \partial_{x_k} \nu_j = 0. \tag{2.18}$$

Then

$$div\, \nu = \sum_{j=1}^{n} \sum_{k=1}^{n} \nu_k D_{\tau_{kj}} \nu_j . \tag{2.19}$$

At this stage we distinguish several cases: *Case* 1: $1 \le j, k \le n-1$. We have

$$\tau_{kj} = \frac{\partial_{x_k} \varphi(x')}{\sqrt{1 + \mid \nabla \varphi(x') \mid^2}} [\partial_{x_j} P(x')] - \frac{\partial_{x_j} \varphi(x')}{\sqrt{1 + \mid \nabla \varphi(x') \mid^2}} [\partial_{x_k} P(x')], \tag{2.20}$$

so that

$$\begin{aligned} D_{\tau_{kj}} f(P(x')) &= \frac{\partial_{x_k} \varphi(x')}{\sqrt{1 + \mid \nabla \varphi(x') \mid^2}} \cdot \frac{\partial}{\partial x_j} [f(P(x'))] \\ &\quad - \frac{\partial_{x_j} \varphi(x')}{\sqrt{1 + \mid \nabla \varphi(x') \mid^2}} \cdot \frac{\partial}{\partial x_j} [f(P(x'))]. \end{aligned} \tag{2.21}$$

Case 2: $k = n, \quad 1 \le j \le n-1$. In this situation,

$$\tau_{nj} = \nu_n e_j - \nu_j e_n = \frac{-1}{\sqrt{1 + \mid \nabla \varphi(x') \mid^2}} [\partial_{x_j} P(x')], \tag{2.22}$$

therefore

$$D_{\tau_{nj}} f(P(x')) = \frac{-1}{\sqrt{1 + \mid \nabla \varphi(x') \mid^2}} \cdot \frac{\partial}{\partial x_j} [f(P(x'))]. \tag{2.23}$$

Case 3: $j = n, \quad 1 \le k \le n-1$. In this case,

$$\tau_{kn} = \frac{1}{\sqrt{1 + \mid \nabla \varphi(x') \mid^2}} [\partial_{x_k} P(x')], \tag{2.24}$$

so

$$D_{\tau_{kn}} f(P(x')) = \frac{1}{\sqrt{1 + \mid \nabla \varphi(x') \mid^2}} \cdot \frac{\partial}{\partial x_k} [f(P(x'))]. \tag{2.25}$$

Case 4: $j = n, \quad k = n$. We can write

$$\tau_{kn} = \nu_n e_n - \nu_n e_n = 0, \tag{2.26}$$

hence, clearly,

$$D_{\tau_{kn}} f(P(x')) = 0. \tag{2.27}$$

As a consequence of the identities derived in Cases 1-4, we have:

$$\begin{aligned}
div\,\nu = &\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \times \\
&\times \left[\frac{\partial_{x_k}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \cdot \frac{\partial}{\partial x_j} \left(\frac{\partial_{x_j}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \right) \right. \\
&\qquad \left. - \frac{\partial_{x_j}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \cdot \frac{\partial}{\partial x_k} \left(\frac{\partial_{x_j}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \right) \right] \\
&+ \sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \left[\frac{1}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \cdot \frac{\partial}{\partial x_k} \left(\frac{-1}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \right) \right] \\
&+ \sum_{j=1}^{n-1} \left(\frac{-1}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \right)^2 \cdot \frac{\partial}{\partial x_j} \left[\frac{\partial_{x_j}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \right].
\end{aligned} \tag{2.28}$$

Therefore,

$$\begin{aligned}
div\,\nu = &\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{[\partial_{x_k}\varphi(x')]^2}{\left(\sqrt{1+\mid \nabla\varphi(x')\mid^2}\right)^4} \times \\
&\times \left[\partial^2_{x_j}\varphi(x') \cdot \sqrt{1+\mid \nabla\varphi(x')\mid^2} - \partial_{x_j}\varphi(x') \cdot \frac{\partial}{\partial x_j} \left(\sqrt{1+\mid \nabla\varphi(x')\mid^2} \right) \right] \\
&- \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x') \cdot \partial_{x_j}\varphi(x')}{\left(\sqrt{1+\mid \nabla\varphi(x')\mid^2}\right)^4} \times \\
&\times \left[\frac{\partial}{\partial x_k} \left(\partial x_j \varphi(x') \right) - \partial_{x_j}(\varphi(x')) \cdot \frac{\partial}{\partial x_k} \left(\sqrt{1+\mid \nabla\varphi(x')\mid^2} \right) \right] \\
&+ \sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')}{\left(\sqrt{1+\mid \nabla\varphi(x')\mid^2}\right)^4} \cdot \frac{\partial}{\partial x_k} \left(\sqrt{1+\mid \nabla\varphi(x')\mid^2} \right) \times \\
&+ \sum_{j=1}^{n-1} \frac{1}{\left(\sqrt{1+\mid \nabla\varphi(x')\mid^2}\right)^4} \times \\
&\times \left[\partial^2_{x_j}\varphi(x') \times \sqrt{1+\mid \nabla\varphi(x')\mid^2} - \partial_{x_j}\varphi(x') \cdot \frac{\partial}{\partial x_j} \left(\sqrt{1+\mid \nabla\varphi(x')\mid^2} \right) \right].
\end{aligned} \tag{2.29}$$

Using the fact that the sum is the same if the index j is replaced by the index k, after simplification the formula becomes:

$$\begin{aligned} div\,\nu &= \sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\frac{[\partial_{x_k}\varphi(x')]^2\cdot\partial^2_{x_j}\varphi(x')}{\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)^3}+\sum_{j=1}^{n-1}\frac{\partial^2_{x_j}\varphi(x')}{\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)^3} \\ &\quad -\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\frac{\partial_{x_k}\varphi(x')\cdot\partial_{x_j}\varphi(x')}{\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)^3}\cdot\frac{\partial}{\partial_{x_k}}\left(\partial_{x_j}\varphi(x')\right) \\ &= \sum_{j=1}^{n-1}\frac{\partial^2_{x_j}\varphi(x')}{\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)^3}\left[\sum_{k=1}^{n-1}\left(\partial_{x_k}\varphi(x')\right)^2+1\right] \\ &\quad -\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\frac{\partial_{x_k}\varphi(x')\cdot\partial_{x_j}\varphi(x')}{\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)^3}\cdot\frac{\partial}{\partial_{x_k}}\left(\partial_{x_j}\varphi(x')\right). \end{aligned}\tag{2.30}$$

Observing that

$$\sum_{k=1}^{n-1}\left(\partial_{x_k}\varphi(x')\right)^2=\mid\nabla\varphi(x')\mid^2\tag{2.31}$$

the formula becomes:

$$\begin{aligned} div\,\nu &= \sum_{j=1}^{n-1}\frac{\partial^2_{x_j}\varphi(x')}{\sqrt{1+\mid\nabla\varphi(x')\mid^2}\cdot(1+\mid\nabla\varphi(x')\mid^2)}\cdot\left[1+\mid\nabla\varphi(x')\mid^2\right] \\ &\quad -\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\frac{\partial_{x_k}\varphi(x')\cdot\partial_{x_j}\varphi(x')}{\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)^3}\cdot\frac{\partial}{\partial_{x_k}}\left(\partial_{x_j}\varphi(x')\right). \end{aligned}\tag{2.32}$$

Then

$$\begin{aligned} div\,\nu &= \sum_{j=1}^{n-1}\frac{\partial^2_{x_j}\varphi(x')}{\sqrt{1+\mid\nabla\varphi(x')\mid^2}} \\ &\quad -\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\frac{\partial_{x_k}\varphi(x')\cdot\partial_{x_j}\varphi(x')}{\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)^3}\cdot\frac{\partial}{\partial_{x_k}}\left(\partial_{x_j}\varphi(x')\right). \end{aligned}\tag{2.33}$$

Using the Quotient Rule for the sum of b_{jj}'s we can write:

$$\begin{aligned} \sum_{k=1}^{n-1}b_{jj} &= \sum_{k=1}^{n-1}\frac{\partial^2_{x_j}\varphi(x')}{\sqrt{1+\mid\nabla\varphi(x')\mid^2}} \qquad (2.34) \\ &\quad -\sum_{k=1}^{n-1}\frac{\partial_{x_j}\varphi(x')}{\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)^2}\cdot\frac{\partial}{\partial_{x_j}}\left[\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right]. \end{aligned}$$

With

$$\mid\nabla\varphi(x')\mid^2=\nabla\varphi(x')\cdot\nabla\varphi(x')=\sum_{k=1}^{n-1}\left[\partial_{x_k}\left(\varphi(x')\right)\right]^2,\tag{2.35}$$

and

$$\frac{\partial}{\partial x_j}\left[1+\sum_{k=1}^{n-1}\left(\partial_{x_k}\varphi(x')\right)^2\right]=2\sum_{k=1}^{n-1}\partial_{x_k}\varphi(x')\frac{\partial}{\partial_{x_j}}\partial_{x_k}\varphi(x'),\tag{2.36}$$

the formula for the mean curvature becomes:

$$\begin{aligned}\sum_{k=1}^{n-1} b_{jj} &= \sum_{k=1}^{n-1} \frac{\partial^2_{x_j}\varphi(x')}{\sqrt{1+\mid \nabla\varphi(x')\mid^2}} \\ &\quad -\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_j}\varphi(x')}{\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)^3}\,\partial_{x_k}\varphi(x')\cdot\frac{\partial}{\partial x_j}\left(\partial_{x_k}\varphi(x')\right).\end{aligned} \tag{2.37}$$

We know that $\sum_{j=1}^{n-1} b_{jj} = \mathcal{G}$ and comparing this with the formula for the divergence, we can finally conclude that:

$$div\,\nu = \sum_{k=1}^{n-1} b_{jj}, \tag{2.38}$$

as desired. This finishes the proof of Theorem 2.2.4. $\square$

Let us next discuss a special case. If

$$\nabla\varphi(0) = 0, \tag{2.39}$$

$$\varphi(0) = 0, \tag{2.40}$$

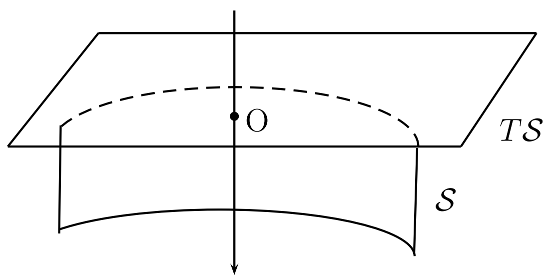

Figure 2.2: The surface $\mathcal{S}$ in the special case $\nabla\varphi(0) = 0, \varphi(0) = 0$.

from Lemma (2.2.2) we have:

$$\begin{aligned} b_{jj}(0) &= \frac{\left(\partial^2_{x_j}\varphi\right)(x')\sqrt{1+\mid\nabla\varphi(x')\mid^2}}{1+\mid\nabla\varphi(x')\mid^2}\ \Big|_{x'=0} \\ &\quad -\frac{\left(\partial^2_{x_j}\varphi\right)(x')\partial^2_{x_j}\left(\sqrt{1+\mid\nabla\varphi(x')\mid^2}\right)}{1+\mid\nabla\varphi(x')\mid^2}\ \Big|_{x'=0} \\ &= \frac{\left(\partial^2_{x_j}\varphi\right)(0)\cdot 1 - 0}{1} = \left(\partial^2_{x_j}\varphi\right)(0). \end{aligned} \tag{2.41}$$

Therefore the mean curvature becomes:

$$\mathcal{G}(0) = \sum_{j=1}^{n-1}\left(\partial^2_{x_j}\varphi\right)(0). \tag{2.42}$$

Invoking (2.37) the divergence can be written as:

$$
\begin{aligned}
(div\,\nu)\,(0) &= \sum_{k=1}^{n-1} \frac{\left(\partial^2_{x_j}\varphi\right)(0)}{\sqrt{1+\mid \nabla\varphi(0)\mid^2}} \\
&\quad - \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\left(\partial_{x_k}\varphi\right)(x')\left(\partial_{x_j}\varphi\right)(x')}{\left(\sqrt{1+\mid \nabla\varphi(x')\mid^2}\right)^3} \cdot \frac{\partial}{\partial_{x_k}}\left[\partial_{x_j}\varphi(x')\right]\mid_{x'=0} \\
&= \sum_{j=1}^{n-1}\left(\partial^2_{x_j}\varphi\right)(0). \qquad (2.43)
\end{aligned}
$$

The last step uses (2.41) and follows after some algebraic manipulation which we omit. In conclusion,

$$
\mathcal{G}(0) = (div\,\nu)\,(0), \qquad (2.44)
$$

which agrees with Theorem (2.2.4).

2.3 A Distinguished Extension of the Normal

Consider

$$\rho(x) := \begin{cases} dist(x, \mathcal{S}) & \text{if } x \text{ is above } \mathcal{S} \\ -\,dist(x, \mathcal{S}) & \text{if } x \text{ is below } \mathcal{S}, \end{cases} \tag{2.45}$$

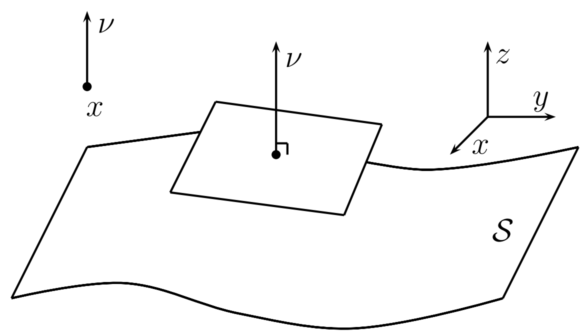

Figure 2.3: The surface $\mathcal{S}$, the tangent plane, and the normal ν.

Candidate:

$$\nu(x) := \frac{\nabla\rho(x)}{|\nabla\rho(x)|}\,,\ x \in \mathbf{R}^n. \tag{2.46}$$

Conclusion:

$$\rho(x) = 0 \Longleftrightarrow x \in \mathcal{S}. \tag{2.47}$$

If x is above $\mathcal{S}$:

$$\rho(x) = inf\,\{|\,x - y\,| \ : y \in \mathcal{S}\} \tag{2.48}$$

For $y = (y', \varphi(y'))$ this distance will be:

$$\rho(x) = inf\,\left\{|\,x - (y', \varphi(y'))\,| \ : y' \in \mathbf{R}^{n-1}\right\} \tag{2.49}$$

Consider a point x_0 on the surface $\mathcal{S}$. Then

$$x_0 = (x_0', \varphi(x_0')). \tag{2.50}$$

Lemma 2.3.1. *For t small, there holds*

$$t^2 = inf\,\left\{|\,x_0 + t\nu(x_0) - (y', \varphi(y'))\,|^2 \ : y' \in \mathbf{R}^{n-1}\right\} \tag{2.51}$$

Proof. Let $F : \mathbf{R}^{n-1} \to [0, \infty)$ be defined by

$$F(y') = |\,x_0 + t\nu(x_0) - (y', \varphi(y'))\,|^2\,. \tag{2.52}$$

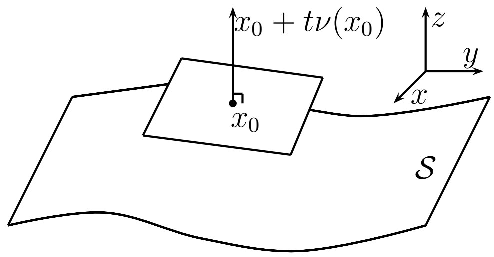

Figure 2.4: The surface $\mathcal{S}$, the tangent plane, the normal ν at a point x_0 on $\mathcal{S}$.

Then

$$F(y') = \sum_{j=1}^{n} \left(x_{0j} + t\nu_j(x_0) - y_j\right)^2 + \left(x_{0n} + t\nu_j(x_0) - \varphi(y')\right)^2. \tag{2.53}$$

Clearly, $inf F = t^2$, and $F(x_0') = t^2$. It is enough to prove that $F(y') \geq F(x_0')$. The Taylor series of F at x_0 will give us:

$$F(y') = F(x_0') + (y' - x_0') \cdot \nabla F(x_0') + \frac{1}{2}(HessF)(\xi)\,(y' - x_0')^2, \tag{2.54}$$

where $Hess\,F$ is the Hessian of F, and $\xi \in [x_0', y_0']$

$$(HessF)(\xi)\,(y' - x_0')^2 = \langle Hess\,F(\xi)\,(y' - x_0'),\,(y' - x_0')\rangle. \tag{2.55}$$

After differentiation:

$$\begin{aligned} \partial_{y_k} F(y') &= (-2)\,(x_{x_0} + t\nu_k(x_0) - y_k) \\ &\quad +2\,(x_{0n} + t\nu_n(x_0) - \varphi(y'))\,(\partial_{y_k}\varphi(y')). \end{aligned} \tag{2.56}$$

For $y' = x_0'$ we have:

$$\partial_{y_k} F(x_0') = -2t\nu_k(x_0) - 2t\nu_n(x_0)\partial_{y_k}\varphi(x_0'). \tag{2.57}$$

Using the definition for the normal on $\mathcal{S}$:

$$\nu_k(x_0) = \frac{\partial_{y_k}\varphi(x_0')}{\sqrt{1+ \mid \nabla\varphi(x') \mid^2}}, \tag{2.58}$$

and

$$\nu_n(x_0) = \frac{1}{\sqrt{1+ \mid \nabla\varphi(x') \mid^2}}, \tag{2.59}$$

the formula (2.57) becomes:

$$\partial_{y_k} F(x_0') = \frac{(-2t)}{\sqrt{1+ \mid \nabla\varphi(x') \mid^2}}\,[\partial_{y_k}\varphi(x_0') - \partial_{y_k}\varphi(x_0')] = 0, \tag{2.60}$$

then

$$\nabla F(x_0') = 0. \tag{2.61}$$

In the Taylor series expansion (2.54) $HessF \geq 0$ and $\nabla F(x_0) = 0$. Consequently,

$$F(y') \geq F(x_0'). \tag{2.62}$$

In order to finish the proof of Lemma (2.3.1) we need to know that the matrix $A := Hess\,F$ is non-negative definite. This is done in the next lemma. $\square$

Consider the $(n-1) \times (n-1)$ matrix:

$$A = (A_{jk})_{jk}\,. \tag{2.63}$$

with

$$A_{jk} = \left(\partial_{y_j}\partial_{y_k}F\right)(y'). \tag{2.64}$$

Lemma 2.3.2. *We have $\langle Az, z\rangle \geq 0$, for any $z \in \mathbf{R}^{n-1}$.*

Proof. It is enough to prove that:

$$\sum_{j,k=1}^{n-1} A_{jk} z_j z_k \geq 0 \quad \text{for} \quad \forall (z_j)_j \in \mathbf{R}^{n-1}. \tag{2.65}$$

Using the formula for $\partial_{y_k}F(y')$ we can write:

$$\begin{aligned}\partial_{y_j}\partial_{y_k}F(y') &= (-2)(-\delta_{jk}) + 2\left(\partial_{y_j}\varphi(y')\right)\left(\partial_{y_k}\varphi(y')\right) \\ &\quad +2\left(x_{0n} + t\nu_n(x_0) - \varphi(y')\right)\left(-\partial_{y_j}\partial_{y_k}\varphi(y')\right).\end{aligned} \tag{2.66}$$

Then:

$$\begin{aligned}\sum_{j,k=1}^{n-1} A_{jk} z_j z_k &= \sum_{j,k=1}^{n-1} 2\delta_{jk} z_j z_k + \sum_{j,k=1}^{n-1} 2\left(\partial_j\varphi\right)\left(\partial_k\varphi\right) z_j z_k \\ &\quad - \sum_{j,k=1}^{n-1} 2\left(x_{0n} + t\nu_n(x_0) - \varphi(y')\right)\left(\partial_{y_j}\partial_{y_k}\varphi\right) z_j z_k.\end{aligned} \tag{2.67}$$

We know that:

$$\sum_{j,k=1}^{n-1} \left(\partial_j\varphi\right) \cdot z_j = \nabla\varphi \cdot z, \tag{2.68}$$

then

$$\sum_{j,k=1}^{n-1} \left(\partial_j\varphi\right)\left(\partial_k\varphi\right) z_j z_k = \left(\nabla\varphi \cdot z\right)^2. \tag{2.69}$$

To finish the proof of Lemma (2.3.2) is enough to show:

$$2 \mid z \mid^2 + 2 (\nabla\varphi \cdot z)^2 - \sum_{j,k=1}^{n-1} (x_{0n} - \varphi(y')) \left(\partial_{y_j}\partial_{y_k}\varphi\right) z_j z_k \geq 0. \tag{2.70}$$

For y' near x'_0 we have $\varphi(y')$ near $\varphi(x'_0)$. With $\varphi(x'_0) = x_{0n}$ we have $\mid \varphi(y') - x_{0n} \mid$ small, and we can say:

$$\mid \varphi(y') - x_{0n} \mid \leq \varepsilon. \tag{2.71}$$

With

$$\mid z_j \mid \leq \mid z \mid \tag{2.72}$$

then

$$\mid z_k \mid\mid z_j \mid \leq \mid z \mid^2 . \tag{2.73}$$

Considering $\partial_{y_j}\partial_{y_k}\varphi \leq C$ with C a constant such that $C\varepsilon < \frac{1}{2}$ the formula (2.70) becomes:

$$\mid z \mid^2 + (\nabla\varphi \cdot z)^2 - \frac{1}{2} \mid z \mid^2 = \frac{1}{2} \mid z \mid^2 + (\nabla\varphi \cdot z)^2 , \tag{2.74}$$

which is bigger than 0. □

Summarizing,

$$\rho(x + t\nu(x)) = t \qquad \text{when } t \geq 0 \text{ is small and } x \in \mathcal{S}. \tag{2.75}$$

For $x \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}^n$ with Chain Rule we can write:

$$\begin{aligned} &\frac{\partial}{\partial x_j} \left[f\left(g_1(x_1 ... x_n), ..., g_n(x_1 ... x_n)\right)\right] \qquad (2.76) \\ = \;& \nabla f\left(g_1(x_1 ... x_n), ..., g_n(x_1 ... x_n)\right) \cdot \left(\frac{\partial g_1}{\partial x_j}(x_1 ... x_n), ..., \frac{\partial g_n}{\partial x_j}(x_1 ... x_n)\right). \end{aligned}$$

Apply this general fact to $\rho(x + t\nu(x)) = t$ implies

$$\frac{d}{dt}\left[\rho(x + t\nu(x))\right] = \frac{d}{dt}\left[t\right], \tag{2.77}$$

so that

$$\nabla\rho\left(x + t\nu(x)\right) \cdot \frac{d}{dt}\left(x + t\nu(x)\right) = 1, \tag{2.78}$$

or, in other words,

$$\nabla\rho\left(x + t\nu(x)\right) \cdot \nu(x) = 1. \tag{2.79}$$

If we consider $t = 0$, meaning the point is on the surface $\mathcal{S}$, then $\nabla\rho(x) \cdot \nu(x) = 1$ or:

$$\partial_\nu \rho(x) = 1 \qquad \text{for any} \quad x \in \mathcal{S}. \tag{2.80}$$

In general:

$$\nabla_{tan} f := \nabla f - (\nabla f \cdot \nu)\, \nu \tag{2.81}$$

or

$$\nabla f = \nabla_{tan} f + (\partial_\nu f)\, \nu. \tag{2.82}$$

Apply this general fact to ρ yields

$$\nabla \rho = \nabla_{tan} \rho + (\partial_\nu \rho)\, \nu. \tag{2.83}$$

Considering ρ on the surface $\mathcal{S}$ we have that $\rho \equiv 0$. This fact will give us the tangential component $\nabla_{tan}\rho = 0$, so

$$\nabla \rho\mid_{\mathcal{S}} = \nu, \tag{2.84}$$

from (2.83) and (2.80). These facts allow us to formulate the next proposition:

Proposition 2.3.3. *The vector field $\frac{\nabla\rho}{|\nabla\rho|}$ is an unitary extension of ν to a neighborhood of $\mathcal{S}$ in $\mathbf{R}^n$.*

From now on we adopt the convention that

$$\nu := \frac{\nabla \rho}{\mid \nabla \rho \mid} \tag{2.85}$$

is an unitary extension of the "old" ν.

Proposition 2.3.4. *For an unitary extension of the normal ν in a neighborhood of $\mathcal{S}$ we have the following conditions:*

$$(i) \quad \nu_j \partial_k - \nu_k \partial_j = 0 \text{ on } \mathcal{S} \quad \text{for } j = 1, 2, 3\,,$$

$$(ii) \quad \partial_\nu \nu_j = 0 \text{ on } \mathcal{S} \quad \text{for } j = 1, 2, 3.$$

Proof. Let us start with

$$curl\, \nu := \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}. \tag{2.86}$$

Remark 2.3.5. *If $\vec{a}$ and $\vec{b}$ are vectors in $\mathbb{R}^3$, then*

$$\vec{a} \times (\vec{a} \times \vec{b}) = -(\vec{a} \times \vec{b}) \times \vec{a}. \tag{2.87}$$

Remark 2.3.6. *If f is a scalar, and $\vec{F}$ is a vector-valued function in $\in \mathbb{R}^3$, the following identity holds:*

$$curl(f\vec{F}) = \nabla f \times \vec{F} + f\, curl\vec{F}. \tag{2.88}$$

By Remark 2.3.5, and Remark 2.3.6 we have:

$$curl(f\vec{F}) = \nabla f \times \vec{F} + f curl\vec{F}, \tag{2.89}$$

$$curl(\nabla f) = \vec{0}, \tag{2.90}$$

and using the vector product property (Remark 3.5.4)

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \tag{2.91}$$

the formula for $curl\,\nu$ becomes:

$$\begin{aligned} curl\,\nu &= curl\left(\mid \nabla\rho \mid^{-1} \nabla\rho\right) = \left(\mid \nabla\rho \mid^{-1}\right) \times \nabla\rho + \mid \nabla\rho \mid^{-1} curl(\nabla\rho) \\ &= -\nabla\rho \times \left(\mid \nabla\rho \mid^{-1}\right). \end{aligned} \tag{2.92}$$

If we consider this equation on $\mathcal{S}$, by Proposition (2.3.3), we can write:

$$curl\,\nu = (-\nu \times \nabla)\left(\mid \nabla\rho \mid^{-1}\right). \tag{2.93}$$

Note that $-\nu \times \nabla$ contains only tangential derivatives since $\nabla\rho$ is normal to $\mathcal{S}$. More concretely,

$$\begin{aligned} \nu \times \nabla &= (\nu_2\partial_3 - \nu_3\partial_2)\,\vec{i} + (\nu_3\partial_1 - \nu_1\partial_3)\,\vec{j} + (\nu_1\partial_2 - \nu_2\partial_1)\,\vec{k} \\ &= D_{\tau_{23}}\vec{i} + D_{\tau_{31}}\vec{j} + D_{\tau_{12}}\vec{k}, \end{aligned} \tag{2.94}$$

with $D_{\tau_{23}}$, $D_{\tau_{31}}$ and $D_{\tau_{12}}$ tangential derivatives (i.e., of the form $\vec{t} \cdot \nabla$ with $\vec{t}$ tangential to $\mathcal{S}$. (i.e. $\nu \cdot \vec{t} = 0$). For instance:

$$D_{\tau_{23}} = \nu_2\partial_3 - \nu_3\partial_2 = (0, -\nu_3, \nu_2) \cdot (\partial_1, \partial_2, \partial_3), \tag{2.95}$$

so that if $\vec{t} = (0, -\nu_3, \nu_2)$, then

$$\nu \cdot \vec{t} = (\nu_1, \nu_2\nu_3) \cdot (0, -\nu_3, \nu_2) = -\nu_2\nu_3 + \nu_3\nu_2 = 0. \tag{2.96}$$

Thus $(0, -\nu_3, \nu_2)$ is tangential.

Similar considerations apply for $D_{\tau_{31}}$ and $D_{\tau_{12}}$:

$$D_{\tau_{31}} = \nu_3\partial_1 - \nu_1\partial_3 = (\nu_3, 0, -\nu_1) \cdot (\partial_1, \partial_2, \partial_3). \tag{2.97}$$

Let $\vec{t} = (\nu_3, 0, -\nu_1)$ so that

$$\nu \cdot \vec{t} = (\nu_1, \nu_2 \nu_3) \cdot (\nu_3, 0, -\nu_1) = \nu_1\nu_3 - \nu_3\nu_1 = 0, \tag{2.98}$$

and

$$D_{\tau_{12}} = \nu_1 \partial_2 - \nu_2 \partial_1 = (-\nu_2, \nu_1, 0) \cdot (\partial_1, \partial_2, \partial_3), \tag{2.99}$$

and if $\vec{t} = (-\nu_2, \nu_1, 0)$ then

$$\nu \cdot \vec{t} = (\nu_1, \nu_2 \nu_3) \cdot (-\nu_2, \nu_1, 0) = -\nu_1\nu_2 + \nu_2\nu_1 = 0, \tag{2.100}$$

so $(\nu_3, 0, -\nu_1)$ and $(-\nu_2, \nu_1, 0)$ are tangential vectors.

Remark 2.3.7. *Tangential derivative applied to a function which is constat on $\mathcal{S}$ is equal to zero on $\mathcal{S}$.*

Going back to the formula for $curl\,\nu$ and using the fact that $\nabla\rho = \nu$ on $\mathcal{S}$, we can write $\mid \nabla\rho \mid^{-1} = 1$ on $\mathcal{S}$. Apply last remark to $curl\,\nu$:

$$(-\nu \times \nabla)\left(\mid \nabla\rho \mid^{-1}\right) = 0 \quad \text{on} \quad \mathcal{S} \quad \text{or} \quad curl\,\nu = 0 \quad \text{on} \quad \mathcal{S}. \tag{2.101}$$

This concludes the proof of (i) in Proposition 2.3.4.

Componentwise, we have: $\partial_1\nu_2 - \partial_2\nu_1 = 0$, $\partial_1\nu_3 - \partial_3\nu_1 = 0$ and $\partial_2\nu_3 - \partial_3\nu_2 = 0$ on $\mathcal{S}$.

To prove part (ii) of Proposition 2.3.4 we start with:

$$\nu = \frac{\nabla\rho}{\mid \nabla\rho \mid} \text{ in a neighborhood of } \mathcal{S}. \tag{2.102}$$

Then

$$\begin{aligned}\partial_\nu \nu_j &= \nu \cdot \nabla\nu_j = \nu \cdot \nabla\left(\frac{\partial_j\rho}{\mid \nabla\rho \mid}\right) = \frac{\nabla\rho}{\mid \nabla\rho \mid} \cdot \nabla\left(\frac{\partial_j\rho}{\mid \nabla\rho \mid}\right) \\ &= \sum_{k=1}^{n} \frac{\partial_k\rho}{\mid \nabla\rho \mid}\partial_k\left(\frac{\partial_j\rho}{\mid \nabla\rho \mid}\right) = \sum_{k=1}^{n} \frac{\partial_k\rho}{\mid \nabla\rho \mid}\left[\frac{\partial_k\partial_j\rho}{\mid \nabla\rho \mid} + \partial_j\rho\,\partial_k\left(\mid \nabla\rho \mid\right)\right] \\ &= \sum_{k=1}^{n} \frac{\partial_k\rho}{\mid \nabla\rho \mid}\frac{\partial_k\,\partial_j\rho}{\mid \nabla\rho \mid} + \sum_{k=1}^{n} \frac{\partial_k\rho\partial_j\rho}{\mid \nabla\rho \mid}\partial_k\left(\mid \nabla\rho \mid\right).\end{aligned} \tag{2.103}$$

Consequently,

$$\begin{aligned}\partial_\nu \nu_j = \quad & \sum_{k=1}^{n} \frac{1}{2}\partial_j\left[\left(\frac{\partial_k\rho}{\mid \nabla\rho \mid}\right)^2\right] - \sum_{k=1}^{n} \frac{\partial_k\rho}{\mid \nabla\rho \mid} \cdot \partial_k\rho \cdot \partial_j\left(\mid \nabla\rho \mid^{-1}\right) \\ &+ \sum_{k=1}^{n} \partial_k\rho\left(\nu_j\partial_k - \nu_k\partial_j\right)\left(\mid \nabla\rho \mid^{-1}\right) + \sum_{k=1}^{n} \partial_k\nu_k\partial_j\left(\mid \nabla\rho \mid\right).\end{aligned} \tag{2.104}$$

Using part (i) the third sum is zero on $\mathcal{S}$. For a component of ν we have $\nu_k = \frac{\partial_k\rho}{|\nabla\rho|}$, then:

$$\partial_\nu \nu_j = \frac{1}{2}\partial_j\left[\frac{|\nabla\rho|^2}{|\nabla\rho|^2}\right] - |\nabla\rho|\ \partial_j\left(|\nabla\rho|^{-1}\right) + \frac{|\nabla\rho|^2}{|\nabla\rho|}\partial_j\left(|\nabla\rho|^{-1}\right). \tag{2.105}$$

After simplification and observation that $\partial_j 1 = 0$ we can conclude that:

$$\partial_\nu \nu_j = 0 \quad \text{on} \quad \mathcal{S}. \tag{2.106}$$

This finish the proof of Proposition 2.3.4. □

Proposition 2.3.8. *The tangential components of the gradient consists of only tangential derivatives.*

Proof. Recall that

$$\nabla_{tan} f = \nabla f - (\partial_\nu f)\,\nu = (\partial_1 f - \partial_\nu f\,\nu_1,\ \partial_2 f - \partial_\nu f\,\nu_2,\ \partial_3 f - \partial_\nu f\,\nu_3). \tag{2.107}$$

Thus,

$$\begin{aligned}\partial_1 f - \partial_\nu f \nu_1 &= \partial_1 f - (\nu\cdot\nabla f)\,\nu_1 = \partial_1 f - \left(\sum_{j=1}^{3}\nu_j\partial_j f\right)\nu_1\\ &= \partial_1 f - (\nu_1\partial_1 f + \nu_2\partial_2 f + \nu_3\partial_3 f)\,\nu_1 = \vec{t}\cdot\nabla f.\end{aligned} \tag{2.108}$$

with the vector $\vec{t} = (t_1, t_2, t_3)$ with components: $t_1 = 1 - \nu_1^2$, $t_2 = -\nu_1\nu_2$, and $t_3 = -\nu_1\nu_3$. Consequently,

$$\vec{t}\cdot\nu = \left(1-\nu_1^2\right)\nu_1 - \nu_1\nu_2^2 - \nu_1\nu_3^2 = \nu_1\left(1-\nu_1^2-\nu_2^2-\nu_3^2\right) = 0. \tag{2.109}$$

So $\vec{t}$ is tangential.

Same calculation for the other components:

$$\begin{aligned}\partial_2 f - \partial_\nu f \nu_2 &= \partial_2 f - (\nu\cdot\nabla f)\,\nu_2 = \partial_2 f - \left(\sum_{j=1}^{3}\nu_j\partial_j f\right)\nu_2\\ &= \partial_2 f - (\nu_1\partial_1 f + \nu_2\partial_2 f + \nu_3\partial_3 f)\,\nu_2 = \vec{t}\cdot\nabla f,\end{aligned} \tag{2.110}$$

with the vector $\vec{t} = (t_1, t_2, t_3)$, with components: $t_1 = -\nu_1\nu_2$, $t_2 = 1 - \nu_2^2$, and $t_3 = -\nu_2\nu_3$. But

$$\vec{t}\cdot\nu = -\nu_1^2\nu_2 + \left(1-\nu_2^2\right)\nu_2 - \nu_1\nu_3^2 = \nu_2\left(1-\nu_1^2-\nu_2^2-\nu_3^2\right) = 0. \tag{2.111}$$

So $\vec{t}$ is tangential. Then

$$\begin{aligned}\partial_3 f - \partial_\nu f \nu_3 &= \partial_3 f - (\nu\cdot\nabla f)\,\nu_3 = \partial_3 f - \left(\sum_{j=1}^{3}\nu_j\partial_j f\right)\nu_3\\ &= \partial_3 f - (\nu_1\partial_1 f + \nu_2\partial_2 f + \nu_3\partial_3 f)\,\nu_1 = \vec{t}\cdot\nabla f.\end{aligned} \tag{2.112}$$

with the vector $\vec{t} = (t_1, t_2, t_3)$ with components: $t_1 = -\nu_1\nu_3$, $\quad t_2 = -\nu_1\nu_2$, and $\quad t_3 = 1-\nu_3^2$. But

$$\vec{t}\cdot\nu = -\nu_1^2\nu_3 - \nu_3\nu_2^2 - \left(1-\nu_3^2\right)\nu_3 = \nu_3\left(1-\nu_1^2-\nu_2^2-\nu_3^2\right) = 0. \tag{2.113}$$

So $\vec{t}$ is tangential. ☐

Remark 2.3.9. *If $\vec{t}$ is tangent to $\mathcal{S}$, then directional derivative along $\vec{t}$: $D_{\vec{t}} = \vec{t}\cdot\nabla$ when applied to a function, only depends on the values of that function on $\mathcal{S}$.*

Examples of tangential operators:

∇_{tan}, $\nu\times\nabla$, D_τ, $\nu_k\partial_j - \nu_j\partial_k$, $\nu\cdot\text{curl}$, $div - \partial_\nu\langle\cdot,\nu\rangle$, $div\,\pi(\cdot)$.

Proposition 2.3.10. *We observe that $(div\,\nu)/_{\mathcal{S}}$ does not depend on the particular extension of the normal ν.*

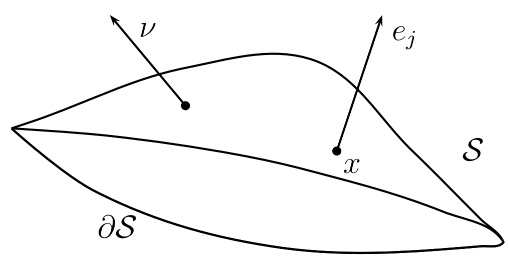

Figure 2.5: The surface $\mathcal{S}$, the normal ν, and the position vector at a point x on the surface $\mathcal{S}$.

Proof. Let e_j be the vector: (0, ... 1, 0..., 0), with 1 on the j-th position, from the definition of derivative we have $\partial_j f(x) = \lim_{h\to\infty}\frac{f(x+h\,e_j)-f(x)}{h}$, so we obtain

$$\begin{aligned} div\,\nu &= \sum_{j=1}^{n}\partial_j\nu_j = \sum_{j=1}^{n}\sum_{k=1}^{n}\nu_k\nu_k\partial_j\nu_j \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\nu_k[\nu_k\partial_j - \nu_j\partial_k]\,\nu_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\nu_k\nu_j(\partial_k\nu_j). \end{aligned} \tag{2.114}$$

With $D_{\tau_k j} = \nu_k\partial_j - \nu_j\partial_k$, which is a tangential operator, and $\sum_{j=1}^{n}\nu_j(\partial_k\nu_j) = \frac{1}{2}\partial_k(\nu_j^2)$, that give us $\sum_{k=1}^{n}\nu_k\partial_k\nu$ which is 0 on $\mathcal{S}$, we can write:

$$div\,\nu = \sum_{j=1}^{n}\sum_{k=1}^{n}\nu_k D_{\tau_k j}. \tag{2.115}$$

as desired. ☐

2.4 The Curvature Matrix

Proposition 2.4.1. *Let $\mathcal{S}$ be a surface in $\mathbf{R}^3$ and fix an unit field ν in a neighborhood $\mathcal{U}$ of $\mathcal{S}$ which extends the unit normal to $\mathcal{S}$. Then, for the 3×3 matrix valued function*

$$\mathcal{R}(x) := \nabla\nu(x) = \left(\partial_k \nu_j(x)\right)_{j,k} \qquad x \in \mathcal{U}, \tag{2.116}$$

the following are true:
$$\begin{array}{llll} (i) & \mathcal{R}\,\nu = 0 & in & \mathcal{U}; \\ (ii) & Tr(\mathcal{R}) = \mathcal{G} & in & \mathcal{U}; \\ (iii) & \mathcal{R}^t = \mathcal{R} & on & \mathcal{S}. \end{array}$$

Proof. We write:

$$\mathcal{R}\,\nu = \begin{pmatrix} \nu_1(\partial_1\nu_1) + \nu_2(\partial_1\nu_2) + \nu_3(\partial_1\nu_3) \\ \nu_1(\partial_2\nu_1) + \nu_2(\partial_2\nu_2) + \nu_3(\partial_2\nu_3) \\ \nu_1(\partial_1\nu_3) + \nu_2(\partial_2\nu_3) + \nu_3(\partial_3\nu_3) \end{pmatrix}. \tag{2.117}$$

Recall Proposition (2.3.4) part (i) on $\mathcal{S}$

$$(i) \quad \nu_j\partial_k - \nu_k\partial_j = 0 \text{ on } \mathcal{S} \quad \text{for } j = 1, 2, 3\,,$$

and we have:

$$\mathcal{R}\,\nu = \begin{pmatrix} \nu_1(\partial_1\nu_1) + \nu_2(\partial_2\nu_1) + \nu_3(\partial_3\nu_1) \\ \nu_1(\partial_1\nu_2) + \nu_2(\partial_2\nu_2) + \nu_3(\partial_3\nu_2) \\ \nu_1(\partial_1\nu_3) + \nu_2(\partial_2\nu_3) + \nu_3(\partial_3\nu_3 \end{pmatrix} = \begin{pmatrix} \partial_\nu\nu_1 \\ \partial_\nu\nu_2 \\ \partial_\nu\nu_3 \end{pmatrix}. \tag{2.118}$$

Recall Proposition (2.3.4) part (ii), on $\mathcal{S}$

$$(ii) \quad \partial_\nu\nu_j = 0 \text{ on } \mathcal{S} \quad \text{for } j = 1, 2, 3.$$

and we have:

$$\mathcal{R}\,\nu = 0. \tag{2.119}$$

To prove part (ii) in Proposition (2.4.1) we use Theorem (2.2.4):

$$Trace\mathcal{R} = \partial_1\nu_1 + \partial_2\nu_2 + \partial_3\nu_3 = div\ \nu = \mathcal{G}. \tag{2.120}$$

To prove part (iii) in Proposition (2.4.1) we use Proposition (2.3.4) part (ii). $\square$

Chapter 3

Analysis on Surfaces

3.1 First-Order Differential Operators in $\mathbf{R}^3$

Let us consider the vector valued function $\vec{u}$ and a scalar-valued f.

Examples of first-order differential operators:

Example 1: $P = div$. If we apply this operator to a vector $\vec{u}$ we have:

$$P\,\vec{u} = (\partial_1 \quad \partial_2 \quad \partial_3)_{1\times 3} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \sum_{j=1}^{3} \partial_j u_j \tag{3.1}$$

Example 2: $P = \nabla$. If we apply this operator to a function f we have:

$$Pf = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} (f)_{1\times 1} = (\partial_1 f \quad \partial_2 f \quad \partial_3 f)\,. \tag{3.2}$$

Example 3: $P = curl$. If we apply this operator to a function f we have:

$$P\,\vec{u} = (\partial_2 u_3 - \partial_3 u_2, \quad \partial_1 u_3 - \partial_3 u_1, \quad \partial_1 u_2 - \partial_2 u_1)\,. \tag{3.3}$$

Or

$$P\,\vec{u} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{3.4}$$

Example 4:

$$P = \begin{pmatrix} \partial_1 + 3\partial_2 & 0 & \partial_1 \\ 2\partial_1 - \partial_3 & -\partial_1 & -\partial_2 \\ 0 & 0 & \partial_3 \end{pmatrix}. \tag{3.5}$$

If we apply this operator to a vector $\vec{u}$ we have:

$$P\,\vec{u} = (\partial_1 u_1 + 3\partial_2 u_1 + \partial_1 u_3, \quad 2\partial_1 u_2 - \partial_3 u_2 - \partial_1 u_2 - \partial_2 u_2, \quad \partial_3 u_3) \tag{3.6}$$

Example 5: In general:

$$P = \left(\sum_{j=1}^{3} a_j^{\alpha\beta} \partial_j \right). \tag{3.7}$$

If we apply this operator to a vector $(u_\beta)_\beta$ we have:

$$P\ (u_\beta)_\beta = \left(\sum_\beta \left(\sum_{j=1}^{3} a_j^{\alpha\beta} \partial_j \right) u_\beta \right)_\alpha = \left(\sum_\beta \sum_{j=1}^{3} a_j^{\alpha\beta} \partial_j u_\beta \right)_\alpha . \tag{3.8}$$

The symbol for this first-order differential operator is given by the matrix-valued function (see [Ta], page 158):

$$\sigma\,(P;\xi) := \left(\sum_{j=1}^{3} a_j^{\alpha\beta} \xi_j \right)_{\alpha,\beta} \qquad \text{with } \xi = (\xi_j)_{1\le j\le 3}\,. \tag{3.9}$$

If we want to write the symbol for the operators from examples $1, 2$, and 3 we have:

Example 1:

$$\sigma\,(\nabla;\xi) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \xi. \tag{3.10}$$

Example 2:

$$\sigma\,(div;\xi) = \xi\,\cdot\,. \tag{3.11}$$

Example 3:

$$\sigma\,(curl;\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}. \tag{3.12}$$

3.2 Tangential Operators to Surface

Definition 3.2.1. *A first-order differential operator P is called (strongly) tangential to the surface $\mathcal{S}$ provided there exists an extended unit normal ν such that*

$$\sigma\left(P;\nu\right)=0 \quad \text{in a neighborhood of}\, \mathcal{S} \text{ in } \mathbf{R}^3. \tag{3.13}$$

Lemma 3.2.2. *The following operators are tangential:*

$$\begin{aligned}
&(i) && P_1=\nabla_{tan}\\
&(ii) && P_2=\nu\times\nabla\\
&(iii) && P_3=D_\tau\\
&(iv) && P_4=\nu_k\partial_j-\nu_j\partial_k\\
&(v) && P_5=\nu\cdot curl\\
&(vi) && P_6=div-\partial_\nu\langle\cdot,\nu\rangle\\
&(vii) && P_7=div\,\pi(\cdot).
\end{aligned} \tag{3.14}$$

Proof. We write

$$\sigma\left(P_1;\xi\right)=\sigma\left(\nabla_{tan};\xi\right)=\sigma\left(\nabla-\nu\partial_\nu;\xi\right)=\begin{pmatrix}\xi_1-\left(\sum_{j=1}^3\nu_j\xi_j\right)\nu_1\\ \xi_2-\left(\sum_{j=1}^3\nu_j\xi_j\right)\nu_2\\ \xi_3-\left(\sum_{j=1}^3\nu_j\xi_j\right)\nu_3\end{pmatrix}, \tag{3.15}$$

so that

$$\sigma\left(P_1;\nu\right)=\begin{pmatrix}\nu_1-\left(\sum_{j=1}^3\nu_j\nu_j\right)\nu_1\\ \nu_2-\left(\sum_{j=1}^3\nu_j\nu_j\right)\nu_2\\ \nu_3-\left(\sum_{j=1}^3\nu_j\nu_j\right)\nu_3\end{pmatrix}=\begin{pmatrix}\nu_1-\left(\nu_1^2+\nu_2^2+\nu_1^2\right)\nu_1\\ \nu_2-\left(\nu_1^2+\nu_2^2+\nu_1^2\right)\nu_2\\ \nu_3-\left(\nu_1^2+\nu_2^2+\nu_1^2\right)\nu_3\end{pmatrix}=0. \tag{3.16}$$

By Definition (3.3) P_1 is tangential near $\mathcal{S}$. Next,

$$\begin{aligned}
\sigma\left(P_2;\xi\right)&=\sigma\left(\nu\times\nabla;\xi\right)\\
&=\left(\nu_2\xi_3-\nu_3\xi_2\right)\vec{i}+\left(\nu_3\xi_1-\nu_1\xi_3\right)\vec{j}+\left(\nu_1\xi_2-\nu_2\xi_1\right)\vec{k},
\end{aligned} \tag{3.17}$$

so that

$$\sigma\left(P_2;\nu\right)=\left(\nu_2\nu_3-\nu_3\nu_2\right)\vec{i}\left(\nu_3\nu_1-\nu_1\nu_3\right)\vec{j}+\left(\nu_1\nu_2-\nu_2\nu_1\right)\vec{k}=0. \tag{3.18}$$

By Definition (3.3) P_2 is tangential near $\mathcal{S}$. Going further,

$$\sigma\left(P_3;\xi\right)=\sigma\left(D_{\tau_{jk}};\xi\right)=\left(\nu_k\xi_j-\nu_j\xi_k\right), \tag{3.19}$$

thus,

$$\sigma\left(P_3;\nu\right)=\left(\nu_k\nu_j-\nu_j\nu_k\right)=0. \tag{3.20}$$

By Definition (3.3) P_3 is tangential near $\mathcal{S}$. Turning our attention to P_4, we have

$$\sigma\left(P_4;\xi\right)=\nu_k\xi_j-\nu_j\xi_k\,, \tag{3.21}$$

so that

$$\sigma\left(P_4;\nu\right)=\nu_k\nu_j-\nu_j\nu_k=0. \tag{3.22}$$

By Definition (3.3) P_4 is tangential near $\mathcal{S}$.

For any vector-valued function $\vec{u}$ we can write:

$$\sigma\left(P_5;\xi\right)\vec{u}=\sigma\left(\nu\cdot curl\,\vec{u};\xi\right)=\sigma\left(\nu\cdot(\nabla\times\vec{u});\xi\right)=\nu\cdot(\xi\times\vec{u}), \tag{3.23}$$

hence

$$\sigma\left(P_5;\nu\right)=\nu\cdot(\nu\times\vec{u})=0. \tag{3.24}$$

By Definition (3.3) P_5 is tangential near $\mathcal{S}$. As for P_6,

$$\sigma\left(P_6;\xi\right)=\sigma\left(div-\partial_\nu\langle\vec{u},\nu\rangle;\xi\right)=\langle\xi,\vec{u}\rangle-\langle\nu,\xi\rangle\langle\vec{u},\nu\rangle. \tag{3.25}$$

Thus,

$$\sigma\left(P_6;\nu\right)=\langle\nu,\vec{u}\rangle-\langle\nu,\nu\rangle\langle\vec{u},\nu\rangle=\langle\nu,\vec{u}\rangle-1\cdot\langle\vec{u},\nu\rangle=0. \tag{3.26}$$

By Definition (3.3) P_6 is tangential near $\mathcal{S}$.

Finally, $\pi(\vec{u})=\vec{u}-\nu\langle\vec{u},\nu\rangle$ and

$$P_7=\sum_{j=1}^{3}\partial_j\left(u_j-\nu_j\langle\vec{u},\nu\rangle\right). \tag{3.27}$$

Then the symbol for P_7 will be:

$$\sigma\left(P_7;\xi\right)=\sum_{j=1}^{3}\xi_ju_j-\sum_{j=1}^{3}\sum_{k=1}^{3}\nu_j\nu_k\xi_ju_k=\langle\xi,\vec{u}\rangle-\langle\nu,\xi\rangle\langle\nu,\vec{u}\rangle, \tag{3.28}$$

so that

$$\sigma\left(P_7;\nu\right)=\langle\nu,\vec{u}\rangle-\langle\nu,\nu\rangle\langle\nu,\vec{u}\rangle=\langle\nu,\vec{u}\rangle-\rangle\langle\nu,\vec{u}\rangle=0. \tag{3.29}$$

By Definition (3.3) P_7 is tangential near $\mathcal{S}$. $\square$

Let $\mathcal{S}$ be a fixed C^2 in $\mathbf{R}^3$, with unit normal ν and consider the first-order tangential differential operators (Stoke's derivatives)

$$M_{jk}:=\nu_j\partial_k-\nu_k\partial_j,\qquad 1\leq j,k\leq 3. \tag{3.30}$$

Since ν is an extended normal in a neighborhood of $\mathcal{S}$, each operator M_{jk} extends accordingly in a neighborhood of $\mathcal{S}$.

Lemma 3.2.3. *The following formulas hold:*

$$(i)\ M_{jk} = M_{kj}, \quad \textit{for each}\ 1 \le j,k \le 3;$$

$$(ii)\ \partial_k = \sum_{j=1}^{3} \nu_j M_{jk} + \nu_k \partial_\nu, \quad \textit{for each}\ 1 \le j,k \le 3;$$

$$(iii)\ \sum_{j=1}^{3} M_{jk}\nu_j = \nu_j \mathcal{G}, \quad \textit{for each}\ 1 \le j,k \le 3.$$

Proof. To prove (i) we write:

$$M_{jk} := \nu_j\partial_k - \nu_k\partial_j = -\left(\nu_j\partial_k - \nu_k\partial_j\right) = -M_{jk}. \tag{3.31}$$

To prove (ii) we start from the right-hand side:

$$\sum_{j=1}^{3} \nu_j M_{jk} + \nu_k\partial_\nu = \sum_{j=1}^{3}\left(\nu_j^2\partial_k - \nu_j\nu_k\partial_j\right) + \nu_k\left(\nu_1\partial_1 + \nu_2\partial_2 + \nu_3\partial_3\right)$$

$$= \nu_1^2\partial_k + \nu_2^2\partial_k + \nu_3^2\partial_k - \nu_1\nu_k\partial_1 - \nu_2\nu_k\partial_2 - \nu_3\nu_k\partial_3 + \nu_k\nu_1\partial_1$$

$$+ \nu_k\nu_2\partial_2 + \nu_k\nu_3\partial_3 = \left(\nu_1^2 + \nu_2^2 + \nu_2^2\right)\partial_k = 1\cdot\partial_k = \partial_k.$$

To prove (iii) we start from the left hand side:

$$\sum_{j=1}^{3} M_{jk}\nu_k = \sum_{j=1}^{3}\left(\nu_j\partial_k - \nu_k\partial_j\right)\nu_k = \sum_{j=1}^{3}\left[\nu_j\left(\partial_k\nu_k\right) - \nu_k\left(\partial_j\nu_k\right)\right]$$

$$= \nu_j\left(\partial_1\nu_1 + \partial_2\nu_2 + \partial_3\nu_2\right) - \nu_1\partial_j\nu_1 - \nu_2\partial_j\nu_2 - \nu_3\partial_j\nu_3$$

$$= \nu_j \cdot div\,\nu - \frac{1}{2}\partial_j\left(|\ \nu\ |\right) = \nu_j \cdot div\,\nu - \frac{1}{2}\,\partial_j\left(1\right)$$

$$= \nu_j \cdot div\,\nu = \nu_j\,\mathcal{G}. \tag{3.32}$$

This finishes the proof of Lemma (3.2.3). ☐

Remark 3.2.4. *For tangential operators to a surface $\mathcal{S}$ the following holds:*

$$[P\,f]/\mathcal{S} = P[f/\mathcal{S}]. \tag{3.33}$$

3.3 Stokes Formula and Consequences

Recall Stoke's formula:

$$\int_{\mathcal{S}} (\nu \cdot curl\vec{F})\, dS = \oint_{\partial\mathcal{S}} \tau \cdot \vec{F}\, ds, \tag{3.34}$$

where $\vec{F}$ is a vector in $\mathbf{R}^3$, ν is the unit normal to the surface $\mathcal{S}$, γ is the outward unit normal to $\partial\mathcal{S}$, and τ the unit tangent to $\partial\mathcal{S}$.

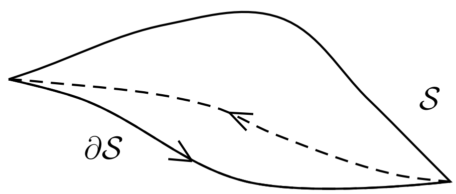

Figure 3.1: The surface $\mathcal{S}$ with the boundary $\partial\mathcal{S}$.

Let us apply M_{12} to a function f, with $f\ :\ \mathbf{R}\ \rightarrow\ \mathbf{R}$:

$$M_{12}f = \nu_1\partial_2 f - \nu_2\partial_1 f \tag{3.35}$$

Considering the vector $\vec{F} = (0, 0, f)$ we can write:

$$curl\vec{F} = (\partial_2 f - 0)\,\vec{i} + (0 - \partial_1 f)\,\vec{j} + (0)\,\vec{k} = (\partial_2 f,\ -\partial_1 f,\ 0) \tag{3.36}$$

and

$$\nu \cdot curl\vec{F} = \nu_1\partial_2 f - \nu_2\partial_1 f = M_{12}f. \tag{3.37}$$

Apply Stoke's formula to $M_{12}f$:

$$\int_{\mathcal{S}} M_{12}f\, dS = \int_{\mathcal{S}} \nu \cdot curl\vec{F}\, dS = \oint_{\partial\mathcal{S}} \tau \cdot \vec{F}\, ds. \tag{3.38}$$

If we write $\vec{\gamma} = \vec{\tau} \times \vec{\nu}$, the component τ_3 will be:

$$\tau_3 = (\tau \times \nu)_3 = \nu_1\gamma_2 - \nu_2\gamma_1 \tag{3.39}$$

and

$$\int_{\mathcal{S}} M_{12}f\, dS = \oint_{\partial\mathcal{S}} \tau_3 \cdot \vec{F}\, ds. \tag{3.40}$$

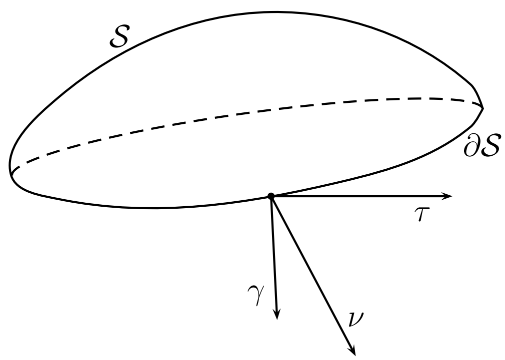

Figure 3.2: The surface $\mathcal{S}$ with the normal ν, the boundary $\partial\mathcal{S}$, and the normal to the boundary γ .

Let us consider another function $g : \mathbf{R} \to \mathbf{R}$ and apply M_{12} to $f \cdot g$:

$$\begin{aligned} M_{12}(f \cdot g) &= \nu_1 \partial_2 (fg) - \nu_2 \partial_1 (fg) \\ &= \nu_1 (\partial_2 f) g + \nu_1 (\partial_2 g) f - \nu_2 (\partial_1 f) g + \nu_2 (\partial_1 g) f + \nu_2 (\partial_1 g) f \\ &= f (\nu_1 \partial_2 g - \nu_2 \partial_1 g) + g (\nu_1 \partial_2 f - \nu_2 \partial_1 f) \\ &= g M_{12} f + f M_{12} g, \end{aligned} \tag{3.41}$$

$$\int_{\mathcal{S}} [(M_{12} f) g + f (M_{12} g)] \, dS = \int_{\mathcal{S}} M_{12}(fg) = \oint_{\partial\mathcal{S}} \tau_3 f g \, ds. \tag{3.42}$$

Consider similar formulas for M_{31} and M_{23}. Specifically, let us apply M_{31} to a function f, with $f : \mathbf{R} \to \mathbf{R}$:

$$M_{31} f = \nu_3 \partial_1 f - \nu_1 \partial_3 f \tag{3.43}$$

Considering the vector $\vec{F} = (0, f, 0)$ we can write:

$$curl \vec{F} = (-\partial_1 f) \vec{i} + (-0) \vec{j} + (\partial_3 f) \vec{k} = (-\partial_1 f, \ 0, \ \partial_3 f) \tag{3.44}$$

and

$$\nu \cdot curl \vec{F} = -\nu_3 \partial_3 f + \nu_1 \partial_3 f = M_{31} f. \tag{3.45}$$

If we write $\vec{\gamma} = \vec{\tau} \times \vec{\nu}$, the component τ_2 will be:

$$\tau_2 = (\tau \times \nu)_2 = \nu_3 \gamma_1 - \nu_1 \gamma_3 \tag{3.46}$$

and apply Stoke's formula to $M_{31} f$:

$$\int_{\mathcal{S}} M_{31} f \, dS = \oint_{\partial \mathcal{S}} \tau_2 \cdot \vec{F} \, ds. \tag{3.47}$$

Let us consider another function $g \; : \; \mathbf{R} \; \rightarrow \; \mathbf{R}$ and apply M_{31} to $f \cdot g$:

$$\begin{aligned} M_{31} \left(f \cdot g \right) &= \nu_3 \partial_1 (fg) - \nu_1 \partial_3 (fg) \\ &= \nu_3 \left(\partial_1 f \right) g + \nu_3 \left(\partial_1 g \right) f - \nu_1 \left(\partial_3 f \right) g - \nu_1 \left(\partial_3 g \right) f \\ &= f \left(\nu_3 \partial_1 g - \nu_1 \partial_3 g \right) + g \left(\nu_3 \partial_1 f - \nu_1 \partial_3 f \right) \\ &= g M_{31} f + f M_{31} g, \end{aligned} \tag{3.48}$$

$$\int_{\mathcal{S}} \left[\left(M_{31} f \right) g + f \left(M_{31} g \right) \right] \, dS = \int_{\mathcal{S}} M_{31} \left(fg \right) = \oint_{\partial \mathcal{S}} \tau_2 f g \, ds. \tag{3.49}$$

Let us apply M_{23} to a function f, with $f \; : \; \mathbf{R} \; \rightarrow \; \mathbf{R}$:

$$M_{23} f = \nu_2 \partial_3 f - \nu_3 \partial_2 f. \tag{3.50}$$

Considering the vector $\vec{F} = (f, 0, 0)$ we can write:

$$curl \vec{F} = (0) \, \vec{i} + \left(\partial_3 f \right) \vec{j} - \left(\partial_2 f \right) \vec{k} = (0, \; \partial_3 f, \; -\partial_2 f) \tag{3.51}$$

and

$$\nu \cdot curl \vec{F} = \nu_2 \partial_3 f - \nu_3 \partial_2 f = M_3 f. \tag{3.52}$$

If we write $\vec{\gamma} = \vec{\tau} \times \vec{\nu}$, the component τ_1 will be:

$$\tau_1 = \left(\tau \times \nu \right)_1 = \nu_2 \gamma_3 - \nu_3 \gamma_2 \tag{3.53}$$

and apply Stoke's formula to $M_{23} f$:

$$\int_{\mathcal{S}} M_{23} f \, dS = \oint_{\partial \mathcal{S}} \tau_1 \cdot \vec{F} \, ds. \tag{3.54}$$

Let us consider another function $g \; : \; \mathbf{R} \; \rightarrow \; \mathbf{R}$ and apply M_{23} to $f \cdot g$:

$$\begin{aligned} M_{23} \left(f \cdot g \right) &= \nu_2 \partial_3 (fg) - \nu_3 \partial_2 (fg) \\ &= \nu_2 \left(\partial_3 f \right) g + \nu_2 \left(\partial_3 g \right) f - \nu_3 \left(\partial_2 f \right) g - \nu_3 \left(\partial_2 g \right) f \\ &= f \left(\nu_2 \partial_3 g - \nu_3 \partial_2 g \right) + g \left(\nu_2 \partial_3 f - \nu_3 \partial_2 f \right) \\ &= g M_{23} f + f M_{23} g, \end{aligned} \tag{3.55}$$

$$\int_{\mathcal{S}} \left[\left(M_{23} f \right) g + f \left(M_{23} g \right) \right] \, dS = \int_{\mathcal{S}} M_{23} \left(fg \right) = \oint_{\partial \mathcal{S}} \tau_1 f g \, ds. \tag{3.56}$$

In conclusion, for every $1 \le j, k \le 3$,

$$\int\limits_{\mathcal{S}} \left[(M_{jk} f)\, g + f\, (M_{jk} g) \right] \, dS = \oint\limits_{\partial \mathcal{S}} (\nu_j \gamma_k - \nu_k \gamma_j)\, f g \, ds. \tag{3.57}$$

If the integral on the boundary of the surface is zero, the above formula reduces to:

$$\int\limits_{\mathcal{S}} (M_{jk} f)\, g = - \int\limits_{\mathcal{S}} f\, (M_{jk} g) \, dS. \tag{3.58}$$

3.4 General Integration by Parts Formulas on Surfaces

In this chapter we discuss a general integration by parts formula for first order-differential operators P which are tangent to a given surface in $\mathbf{R}^3$.

The starting point is the definition of the adjoint operator P^* of P on $\mathbf{R}^3$.

Definition 3.4.1. *Consider a first-order differential operator P. Then there exists P^* first-order differential operator such that for any u and v compactly supported vector-valued functions on $\mathbf{R}^3$, there holds:*

$$\int_{\mathbf{R}^3} \langle Pu, v\rangle \, dx = \int_{\mathbf{R}^3} \langle u, P^* v\rangle \, dx. \tag{3.59}$$

Example 1. Let $P = \nabla$. If u is a scalar function and $\vec{v} = \{v_1, v_2, v_3\}$ is a vector-valued function, then

$$\int_{\mathbf{R}^3} \langle \nabla u, \vec{v}\rangle \, dx = \int_{\mathbf{R}^3} \sum_{j=1}^{3} \partial_j u \, \vec{v}_j \, dx = -\int_{\mathbf{R}^3} \sum_{j=1}^{3} u \, \partial_j v_j \, dx = -\int_{\mathbf{R}^3} f \, div\,(\vec{g}) \, dx, \tag{3.60}$$

so $P^* = -div$.

Example 2. Let $P = curl$. If $\vec{u} = \{u_1, u_2, u_3\}$ and $\vec{v} = \{v_1, v_2, v_3\}$ are vector-valued function, then

$$\begin{aligned}
\int_{\mathbf{R}^3} \langle curl\,\vec{u}, \vec{v}\rangle \, dx &= \int_{\mathbf{R}^3} \langle [\partial_2 u_3 - \partial_3 u_2,\ \partial_3 u_1 - \partial_1 u_3,\ \partial_1 u_2 - \partial_2 u_1]\,,\ (v_1\,, v_2\,, v_3)\rangle dx \\
&= \int_{\mathbf{R}^3} [(\partial_2 u_3 - \partial_3 u_2)\, v_1 + (\partial_3 u_1 - \partial_1 u_3)\, v_2 + (\partial_1 u_2 - \partial_2 u_1)\, v_3] \, dx \\
&= \oint [v_1\nu_2 u_3 - v_1\nu_3 u_2 + v_2\nu_3 u_1 - v_2\nu_1 u_3 + v_3\nu_2 u_3 - v_3\nu_2 u_3] \, dx \\
&+ \int_{\mathbf{R}^3} (-u_3\partial_2 v_1 + u_2\partial_3 v_1 - u_1\partial_3 v_2 + u_3\partial_1 v_2 - u_2\partial_1 v_3 + u_1\partial_2 v_3] \, dx.
\end{aligned}$$

The integral over the boundary is zero so

$$\begin{aligned}
\int_{\mathbf{R}^3} \langle curl\,\vec{u}, \vec{v}\rangle \, dx &= \int_{\mathbf{R}^3} [u_1 (\partial_2 v_3 - \partial_3 v_2) + u_2 (\partial_3 v_1 - \partial_1 v_3) + u_3 (\partial_1 v_2 - \partial_2 v_1)] \, dx \\
&= \int_{\mathbf{R}^3} \langle \vec{u}, \nabla \times \vec{v}\rangle \, dx = \int_{\mathbf{R}^3} \langle \vec{u}, curl\,\vec{v}\rangle \, dx,
\end{aligned} \tag{3.61}$$

so $P^* = curl$.

Our next theorem gives an intrinsic description of P^* in terms of coefficients of P.

Theorem 3.4.2. *For u, and v $\mathcal{C}^1$ functions in $\mathbf{R}^3$, if*

$$P\,u = \left(\sum_{j=1}^{3}\sum_{\beta} a_j^{\alpha\beta}\partial_j u_\beta\right)_\alpha \quad \textit{then} \quad P^*\,v = -\left(\sum_{j=1}^{3}\sum_{\beta} a_j^{\alpha\beta}\partial_j v_\beta\right)_\beta . \tag{3.62}$$

Proof. In $\mathbf{R}^3$ the following holds:

$$\begin{aligned}
\int_{\mathbf{R}^3} \langle Pu, v\rangle dx &= \int_{\mathbf{R}^3} \left\langle \left(\sum_{j=1}^{3}\sum_{\beta} a_j^{\alpha\beta}\partial_j u_\beta\right)_\alpha, (v_\alpha)_\alpha \right\rangle dx \\
&= \int_{\mathbf{R}^3} \sum_{\alpha} \left(\sum_{j=1}^{3}\sum_{\beta} a_j^{\alpha\beta}\partial_j u_\beta\right) v_\alpha dx = \sum_{\alpha}\sum_{\beta}\sum_{j=1}^{3}\int_{\mathbf{R}^3} (\partial_j u_\beta)\left(a_j^{\alpha\beta} v_\alpha\right) dx \\
&= -\sum_{\alpha}\sum_{\beta}\sum_{j=1}^{3}\int_{\mathbf{R}^3} u_\beta \partial_j \left(a_j^{\alpha\beta} v_\alpha\right) dx \\
&= \int_{\mathbf{R}^3} \left\langle (u_\beta)_\beta, \left(-\sum_{\alpha}\sum_{j=1}^{3} \partial_j\left(a_j^{\alpha\beta} v_\alpha\right)\right)_\beta \right\rangle dx = \int_{\mathbf{R}^3} \langle\, u,\, P^* v\,\rangle.
\end{aligned} \tag{3.63}$$

as desired. ☐

It is very useful to point out that the class of differential operators tangent to a surface $\mathcal{S}$ is stable under taking the adjoint (in $\mathbf{R}^3$).

Proposition 3.4.3. *If P is a tangential differential operator to $\mathcal{S}$, then P^* is a tangential differential operator to $\mathcal{S}$ as well.*

Proof. To prove P^* tangential it is enough to prove $\sigma(P^*;\,\nu) = 0$. For this, we will try to write the symbol of P^* in terms of the symbol of P. Let us apply P^* to a $\mathcal{C}^1$ function v in $\mathbf{R}^3$. The symbol of P^* will be:

$$\begin{aligned}
\left(\sigma\left(P^*;\,\xi\right)v\right)_\beta &= -\left(\sum_{\alpha}\sum_{j=1}^{3}\left(a_j^{\alpha\beta}\right)^T \xi_j\, v_\alpha\right)_\beta = -\left(\sum_{\alpha}\left(\sum_{j=1}^{3}\left(a_j^{\alpha\beta}\right)^T \xi_j\right) v_\alpha\right)_\beta \\
&= -\left(\sum_{\alpha}\left(\sum_{j=1}^{3} a_j^{\alpha\beta}\xi_j\right)^T v_\alpha\right)_\beta .
\end{aligned} \tag{3.64}$$

If we write the symbol of P, using formula(3.8), we have:

$$\left(\sigma\left(P;\,\xi\right)u\right)_\alpha = \left(\sum_{\beta}\sum_{j=1}^{3} a_j^{\alpha\beta}\xi_j\, u_\beta\right)_\alpha = \left(\sum_{\beta}\left(\sigma(P;\,\xi)\right)_{\alpha\,\beta} u_\beta\right)_\alpha . \tag{3.65}$$

So we can write the symbol for P^*:

$$(\sigma(P^*;\, \xi)\, v)_\beta = -\left(\sum_\alpha (\sigma(P;\, \xi)_{\alpha\,\beta})^T \; v_\alpha\right)_\beta = -\left(\sigma(P;\, \xi)^T \, v\right)_\beta . \tag{3.66}$$

We choose the function v arbitrary,so the formula is true for any $\mathcal{C}^1$ function in $\mathbf{R}^3$:

$$\sigma(P^*;\, \xi) = -\sigma(P;\, \xi)^T . \tag{3.67}$$

With the observation that $\sigma(P;\, \xi) = 0$ will imply $\sigma(P;\, \xi)^T = 0$, we can conclude that P^* is a first-order differential operator tangential to $\mathcal{S}$. $\square$

The main theorem of this chapter, a culmination of the work done in the previous subchapters, gives an explicit formula for integration by parts first-order differential operator P over a surface $\mathcal{S}$, with boundary forms. The usefulness of this result comes from the fact that all analytic objects involved, P, P^*, $\sigma(P)$ are independent of $\mathcal{S}$.

Theorem 3.4.4. *Let $\mathcal{S}$ be a smooth, compact surface in $\mathbf{R}^3$ with normal ν and boundary $\partial\mathcal{S}$. Denote by γ the unit normal to $\partial\mathcal{S}$. Let u, v be $\mathcal{C}^1$ functions on $\mathcal{S}$, P be a tangential first-order differential operator, and let P^* denote the adjoint in $\mathbf{R}^3$. Then*

$$\int_{\mathcal{S}} \langle Pu, v\rangle \, dS = \int_{\mathcal{S}} \langle\, u,\, P^* v\,\rangle dS + \oint_{\partial\mathcal{S}} \langle \sigma(P;\gamma)u,\, v\rangle ds.$$

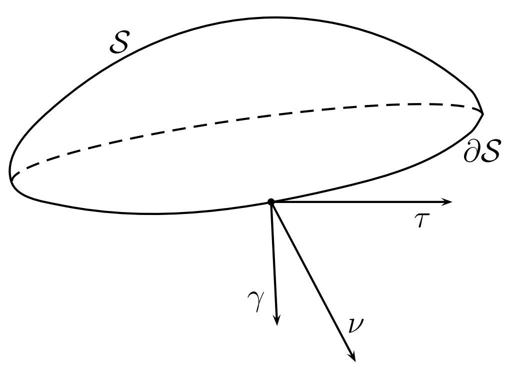

Figure 3.3: The surface $\mathcal{S}$ with the normal ν, the boundary $\partial\mathcal{S}$, and the normal to the boundary γ .

Proof. Recall that P is tangential to $\mathcal{S}$ if and only if $\sigma(P;\, \nu) = 0$.

Let us start with formula (3.8) for tangential operator to the surface $\mathcal{S}$:

$$P\, u = \left(\sum_{j=1}^{3} \sum_\beta a_j^{\alpha\beta} \partial_j u_\beta\right)_\alpha , \tag{3.68}$$

and after replacing ∂_j with the expression from Lemma (3.2.3), part (ii)

$$\partial_j = \sum_{j=1}^{3} \nu_k M_{kj} + \nu_j \partial_\nu, \quad \text{for each } 1 \leq j, k \leq 3, \tag{3.69}$$

we have:

$$\begin{aligned} P\,u &= \left(\sum_{j=1}^{3} \sum_{\beta} a_j^{\alpha\beta} \left(\sum_{k=1}^{3} \nu_k M_{kj} + \nu_j \partial_\nu \right) u_\beta \right)_\alpha \\ &= \left(\sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{\beta} a_j^{\alpha\beta} \nu_k M_{kj} u_\beta + \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{\beta} a_j^{\alpha\beta} \nu_j \partial_\nu u_\beta \right)_\alpha . \end{aligned} \tag{3.70}$$

Recall that $\sum_{j=1}^{3} a_j^{\alpha\beta} \nu_j = \sigma\,(P\,;\nu) = 0$, in an open neighborhood of $\mathcal{S}$ in $\mathbf{R}^3$ so:

$$P\,u = \left(\sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{\beta} a_j^{\alpha\beta} \nu_k M_{kj} u_\beta \right)_\alpha . \tag{3.71}$$

Next we want to find the adjoint operator for this general case:

$$\begin{aligned} \int_{\mathcal{S}} \langle Pu, v \rangle \, dS &= \int_{\mathcal{S}} \langle \left(\sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{\beta} a_j^{\alpha\beta} \nu_k M_{kj} u_\beta \right)_\alpha , (v_\alpha)_\alpha \rangle \, dS \\ &= \sum_\alpha \int_{\mathcal{S}} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{\beta} (M_{kj} u_\beta) \left(a_j^{\alpha\beta} \nu_k v_\alpha \right) dS \\ &= -\sum_\alpha \int_{\mathcal{S}} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{\beta} u_\beta M_{kj} \left[a_j^{\alpha\beta} \nu_k v_\alpha \right] dS \\ &\qquad + \oint_{\partial\mathcal{S}} \sum_{j=1}^{3} \sum_{k=1}^{3} (\nu_k \gamma_j - \nu_j \gamma_k) \left(u_\beta a_j^{\alpha\beta} \nu_k v_\alpha \right) ds. \end{aligned} \tag{3.72}$$

where dS, ds are the volume elements on $\mathcal{S}$, $\partial\mathcal{S}$, and ν, γ the outward unit vectors to $\mathcal{S}$, and $\partial\mathcal{S}$.

Let us simplify the first term in the previous relation using $\nu_j\, a_j^{\alpha\beta} = \sigma(P;\nu) = 0$,

$M_{kj}\,\nu_k = \nu_j\, div\, \nu$, and $\partial_\nu = \nu \cdot \nabla = \sum_{k=1}^n \nu_k \partial_k$:

$$-\sum_\alpha \int_{\mathcal{S}} \sum_{j=1}^3 \sum_{k=1}^3 \sum_\beta u_\beta M_{kj} \left[a_j^{\alpha\beta} \nu_k v_\alpha \right]$$

$$= -\sum_\alpha \int_{\mathcal{S}} \sum_{j=1}^3 \sum_{k=1}^3 \sum_\beta u_\beta \left[\nu_k v_\alpha \left(M_{kj} a_j^{\alpha\beta} \right) + a_j^{\alpha\beta} v_\alpha \left(M_{kj} \nu_k \right) + a_j^{\alpha\beta} \nu_k \left(M_{kj} v_\alpha \right) \right] dS$$

$$= -\sum_\alpha \int_{\mathcal{S}} \sum_{j=1}^3 \sum_{k=1}^3 \sum_\beta u_\beta \left[\nu_k v_\alpha \left(M_{kj} a_j^{\alpha\beta} \right) + a_j^{\alpha\beta} v_\alpha \nu_j\, div\, \nu + a_j^{\alpha\beta} \nu_k \left(M_{kj} v_\alpha \right) \right] dS$$

$$= -\sum_\alpha \sum_\beta \int_{\mathcal{S}} \left[\sum_{j=1}^3 \sum_{k=1}^3 u_\beta \nu_k v_\alpha \left(M_{kj} a_j^{\alpha\beta} \right) + \sum_{j=1}^3 \sum_{k=1}^3 u_\beta \nu_k a_j^{\alpha\beta} \left(M_{kj} v_\alpha \right) \right.$$

$$\left. + \sum_{j=1}^3 \sum_{k=1}^3 u_\beta \nu_j v_\alpha \nu_k \partial_k a_j^{\alpha\beta} \right] dS = -\sum_\beta \int_{\mathcal{S}} \sum_\alpha u_\beta \left[\sum_{j=1}^3 \sum_{k=1}^3 \nu_k v_\alpha \left(M_{kj} a_j^{\alpha\beta} \right) \right.$$

$$\left. + \sum_{j=1}^3 \sum_{k=1}^3 \nu_k a_j^{\alpha\beta} \left(M_{kj} v_\alpha \right) + \sum_{j=1}^3 \nu_j v_\alpha \partial_\nu a_j^{\alpha\beta} dS \quad + \sum_{j=1}^3 \nu_j a_j^{\alpha\beta} \partial_\nu v_\alpha \right] \tag{3.73}$$

so we can continue

$$-\sum_\alpha \int_{\mathcal{S}} \sum_{j=1}^3 \sum_{k=1}^3 \sum_\beta u_\beta M_{kj} \left[a_j^{\alpha\beta} \nu_k v_\alpha \right]$$

$$= -\sum_\beta \int_{\mathcal{S}} \sum_\alpha u_\beta \left[\sum_{j=1}^3 \sum_{k=1}^3 \nu_k M_{kj} \left(a_j^{\alpha\beta} v_\alpha \right) + \nu_j \partial_\nu \left(a_j^{\alpha\beta} v_\alpha \right) \right] dS$$

$$= -\sum_\beta \int_{\mathcal{S}} \sum_\alpha \sum_{j=1}^3 u_\beta \left[\sum_{k=1}^3 \nu_k M_{kj} + \nu_j \partial_\nu \left(a_j^{\alpha\beta} v_\alpha \right) \right] dS$$

$$= -\sum_\beta \int_{\mathcal{S}} \sum_\alpha \sum_{j=1}^3 u_\beta \partial_j \left(a_j^{\alpha\beta} v_\alpha \right) dS = \int_{\mathcal{S}} \langle\, (u_\beta)_\beta\,,\ \left((P^* v)_\beta \right)_\beta \rangle dS$$

$$= \int_{\mathcal{S}} \langle\, u,\ P^* v\, \rangle dS. \tag{3.74}$$

Next we use this simplification in formula(3.72) and we have:

$$\int_{\mathcal{S}} \langle Pu, v \rangle\, dS = \int_{\mathcal{S}} \langle\, u,\ P^* v\, \rangle + \oint_{\partial\mathcal{S}} \sum_{j=1}^3 \sum_{k=1}^3 \left(\nu_k \gamma_j - \nu_j \gamma_k \right) \left(u_\beta a_j^{\alpha\beta} \nu_k v_\alpha \right) ds$$

$$= \int_{\mathcal{S}} \langle\, u,\ P^* v\, \rangle + \oint_{\partial\mathcal{S}} \langle \sigma(P;\gamma) u,\ v \rangle ds. \tag{3.75}$$

This finishes the proof of the Main Theorem of this Chapter. □

(We apply this theorem to all the first-order tangential differential operators defined in Lemma (3.2.2)).

To illustrate the range of applicability of our main result we consider a number of concrete examples in detail. As expected, our formulas encompass several classical results in analysis, including Stoke's formula (cf Example 4).

Example 1. Let $P_1 = \nabla_{tan} = \nabla - \nu\,\partial_\nu$. For a scalar u and a vector-valued function $\vec{v}$ we have:

$$P_1 u = \nabla u - \partial_\nu u \cdot \nu = \nabla u - (\nu \cdot \nabla u)\ \nu, \tag{3.76}$$

$$\text{then} \quad \sigma\,(P_1;\xi)\,u = \xi u - (\nu\cdot\xi)\,u\,\nu = u\xi - u\,(\nu\cdot\xi)\,\nu$$

$$= u\,[\xi - (\nu\cdot\xi)\,\nu] = u\cdot\xi_{tan}, \tag{3.77}$$

$$\text{and} \quad \sigma\,(P_1;\nu)\,u = u\,\nu - u\,(\nu\cdot\nu)\,\nu = 0. \tag{3.78}$$

Remark 3.4.5. *If $\vec{a},\vec{b}\in\mathbb{R}^3$ and $|a|=1$, then*

$$\vec{b} - \langle\vec{b},\vec{a}\rangle\,\vec{a} = -\vec{a}\times(\vec{a}\times\vec{b}). \tag{3.79}$$

Remark 3.4.6. *If f is $\mathcal{C}^1$ function, and $\vec{F}$ is a vector-valued function we have:*

$$div\left(f\vec{F}\right) = \nabla f\cdot\vec{F} + f\,div\vec{F}. \tag{3.80}$$

The adjoint operator on $\mathbf{R}^3$ will be:

$$\int_{\mathbf{R}^3}\langle P_1 u,\vec{v}\rangle\,dx = \int_{\mathbf{R}^3}\langle\,\nabla u - (\partial_\nu u)\,\nu,\ \vec{v}\,\rangle dx = \int_{\mathbf{R}^3}\langle\,\nabla u,\ \vec{v}\rangle\,dx - \int_{\mathbf{R}^3}\partial_\nu u\langle\,\nu,\ \vec{v}\,\rangle dx \tag{3.81}$$

$$= \int_{\mathbf{R}^3}\langle\nabla u,\ \vec{v}\rangle dx - \int_{\mathbf{R}^3}\langle\,\nu,\ \nabla u\,\rangle\langle\,\nu,\ \vec{v}\,\rangle dx = -\int_{\mathbf{R}^3} u\,div\,\vec{v}\,dx$$

$$-\int_{\mathbf{R}^3}\langle\,\langle\,\nu,\ \vec{v}\rangle\nu,\ \nabla u\rangle\,dx = -\int_{\mathbf{R}^3} u\,div\,\vec{v}\,dx - \int_{\mathbf{R}^3} u\cdot div\,(\langle\nu,\ \vec{v}\rangle\,\nu)\ dx,$$

and applying Remark 3.4.6, and then Remark 3.4.5 we have:

$$\int_{\mathbf{R}^3}\langle P_1 u,\vec{v}\rangle\,dx = -\int_{\mathbf{R}^3} u\,div\,\vec{v}\,dx - \int_{\mathbf{R}^3} u\,[\nabla\,(\nu\cdot\vec{v})\cdot\nu + (\nu\cdot\vec{v})\ div\,\nu]\ dx$$

$$= -\int_{\mathbf{R}^3} u\,div\,\vec{v}\,dx - \int_{\mathbf{R}^3} u\,[(\nu\times curl\,\vec{v})\,\nu + (\vec{v}\times curl\,\nu)\,\nu$$

$$+\,[(\nu\cdot\nabla)\vec{v}\,]\nu + ((\vec{v}\cdot\nabla)\nu)\nu + (\nu\cdot\vec{v})\,div\,\nu]\ dx, \tag{3.82}$$

and considering $curl\,\nu = 0$, $div\,\nu = \mathcal{G}$, and $R := (\partial_j \nu_k)_{jk}$ on $\mathcal{S}$, the second integral in previous formula becomes:

$$\int\limits_{\mathbf{R}^3} \langle P_1 u, \vec{v}\rangle\, dx = \int\limits_{\mathbf{R}^3} u \left[(\partial_\nu \vec{v})\, \nu + \left(\sum_{j=1}^{3} \vec{v}_j \partial_j \nu_k \right)_k + (\nu \cdot \vec{v})\, \mathcal{G} \right] dx$$

$$= \int\limits_{\mathbf{R}^3} u \left[(\partial_\nu \vec{v})\, \nu + \langle \mathcal{R}\vec{v},\ \nu\rangle + (\nu \cdot \vec{v})\, \mathcal{G} \right] dx. \tag{3.83}$$

Recall from Proposition (2.4.1) that $\mathcal{R}\nu = 0$, and $\mathcal{R} = \mathcal{R}^t$ on $\mathcal{S}$ so:

$$\langle \mathcal{R}\vec{v},\ \nu\rangle = \langle \vec{v}, \mathcal{R}^t \nu\rangle = 0, \tag{3.84}$$

then the formula becomes:

$$\int\limits_{\mathbf{R}^3} \langle P_1 u, \vec{v}\rangle\, dx = \int\limits_{\mathbf{R}^3} u\, div\, \vec{v}\, dx - \int\limits_{\mathbf{R}^3} u \left[(\partial_\nu \vec{v})\, \nu + (\nu \cdot \vec{v}\,)\, \mathcal{G} \right]\, dx, \tag{3.85}$$

so $P_1^* \vec{v} = -div\, \vec{v} - (\partial_\nu \vec{v})\, \nu - (\nu \cdot \vec{v}\,)\, \mathcal{G}$.

Now we can apply Theorem (3.4.4) and write:

$$\int\limits_{\mathcal{S}} \langle u, [(\partial_\nu \vec{v}\,)\, \nu + \nu \cdot \vec{v}\,] \rangle\, dS = -\int\limits_{\mathcal{S}} u\ \left[(\partial_\nu \vec{v} \cdot \nu) + (\nu \cdot \vec{v})\, \mathcal{G} + div\, \vec{v}\, \right] dS$$

$$+ \oint\limits_{\partial\mathcal{S}} u\, \langle \gamma_{tan}, \vec{v}\, \rangle\, ds. \tag{3.86}$$

Example 2. Let $P_2 = \nu \times \nabla$. For a scalar u and a vector-valued function $\vec{v}$:

$$P_2 u = \nu \times \nabla u, \tag{3.87}$$

$$\text{then} \quad \sigma\,(P_2\,; \xi)\, u = \nu \times (\xi\, u) = u\, \nu \times \xi, \tag{3.88}$$

$$\text{and} \quad \sigma\,(P_2\,; \nu)\, u = u\, \nu \times \nu = 0. \tag{3.89}$$

The adjoint operator on $\mathbf{R}^3$ will be:

$$\int\limits_{\mathbf{R}^3} \langle P_2 u, \vec{v}\rangle\, dx = \int\limits_{\mathbf{R}^3} \langle\, \nu \times \nabla,\ \vec{v}\rangle dx = \int\limits_{\mathbf{R}^3} \langle \nabla u,\ \nu \times \vec{v}\rangle dx \tag{3.90}$$

$$= -\int\limits_{\mathbf{R}^3} \sum_{j=1}^{3} \partial_j u\, (\nu \times \vec{v})_j\, dx = -\int\limits_{\mathbf{R}^3} u\, div\, (\nu \times \vec{v})\, dx,$$

so $P_2^* \vec{v} = div(\nu \times \vec{v})$.

Remark 3.4.7. *If $\vec{F}$, $\vec{G}$ are vector-valued function the following holds:*

$$div\left(\vec{F} \times \vec{G} \right) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G}). \tag{3.91}$$

From the previous and knowing that $curl\,\nu = 0$ on $\mathcal{S}$ the formula becomes:

$$P_2^* \vec{v} = \vec{v} \cdot curl\,\nu - \nu \cdot curl\,\vec{v}. \tag{3.92}$$

Now we can apply Theorem (3.4.4):

$$\int\limits_{\mathcal{S}} \langle \nu \times \nabla u, \vec{v} \rangle \, dS = - \int\limits_{\mathcal{S}} u \, \langle \, \nu, \; curl\, \vec{v} \rangle dS + \oint\limits_{\partial \mathcal{S}} u \, \langle \nu \times \gamma, \vec{v} \rangle ds. \tag{3.93}$$

Example 3. Fix $1 \leq i, \; j \leq 3$, and let $P_3 = D_{\tau_{ij}}$. For u and v scalars we have:

$$P_3 u = \nu_i \partial_j u - \nu_j \partial_i u, \tag{3.94}$$

$$\text{then} \quad \sigma\,(P_3\,; \xi)\, u = \nu_i \xi_j u - \nu_j \xi_i u, \tag{3.95}$$

$$\text{and} \quad \sigma\,(P_3\,; \nu)\, u = \nu_i \nu_j u - \nu_j \nu_i u = 0. \tag{3.96}$$

The adjoint operator on $\mathbf{R}^3$ we write:

$$\begin{aligned} \int\limits_{\mathbf{R}^3} \langle P_3 u, \vec{v} \rangle \, dx &= \int\limits_{\mathbf{R}^3} (\nu_i \partial_j u - \nu_j \partial_i u) \; v \, dx = - \int\limits_{\mathbf{R}^3} \nu_i \; u \; \partial_j v \, dx \\ &+ \oint\limits_{\partial\, \mathbf{R}^3} \nu_i \; \nu_j \; u \, v \, ds + \int\limits_{\mathbf{R}^3} \nu_j \; u \; \partial_i v \, dx - \oint\limits_{\partial\, \mathbf{R}^3} \nu_j \; \nu_i \; u \, v \, ds \\ &= - \int\limits_{\mathbf{R}^3} u \, (\nu_i \partial_j v - \nu_j \partial_i v) \; dx = - \int\limits_{\mathbf{R}^3} \langle u, \; P_3 v \, \rangle \, dx, \end{aligned} \tag{3.97}$$

so $P_3^* = -P_3$.

Now we can apply Theorem (3.4.4):

$$\int\limits_{\mathcal{S}} \langle D_{\tau_{ij}} u, \; v \, \rangle \, dS = - \int\limits_{\mathcal{S}} \langle u, \; \nu_j \partial_k v - \nu_k \partial_j v \rangle \, dS + \oint\limits_{\partial \mathcal{S}} (\nu_j \gamma_k - \nu_k \gamma_j) \, u \, v \, ds. \tag{3.98}$$

Example 4. Fix $1 \leq i, \; j \leq 3$, and let $P_4 = M_{kj}$. For u and v scalars we have:

$$P_4 u = \nu_k \partial_j u - \nu_j \partial_k u, \tag{3.99}$$

$$\text{then} \quad \sigma\,(P_4\,; \xi)\, u = \nu_k \xi_j u - \nu_j \xi_k u, \tag{3.100}$$

$$\text{and} \quad \sigma\,(P_4\,; \nu)\, u = \nu_k \nu_j u - \nu_j \nu_k u = 0. \tag{3.101}$$

The adjoint operator on $\mathbf{R}^3$, like in previous example, will be $P_4^* = P_4$.

Now we can apply Theorem (3.4.4):

$$\int\limits_{\mathcal{S}} \langle M_{kj} u, \; v \, \rangle \, dS = \int\limits_{\mathcal{S}} \langle u, \; -M_{kj} v \rangle \, dS + \oint\limits_{\partial \mathcal{S}} (\nu_k \gamma_j - \nu_j \gamma_k) \, u, \; v \rangle \, ds. \tag{3.102}$$

Example 5. Let $P_5 = \nu \cdot curl$. For a vector-valued function $\vec{u}$ and a scalar v

$$P_5\vec{u} = \nu \cdot (\nabla \times \vec{u})\,, \tag{3.103}$$

$$\text{then} \quad \sigma\,(P_5\,;\xi)\,\vec{u} = \nu \cdot (\xi \times \vec{u})\,, \tag{3.104}$$

$$\text{and} \quad \sigma\,(P_5\,;\nu)\,\vec{u} = \nu \cdot (\nu \times \vec{u}) = 0. \tag{3.105}$$

To compute the adjoint operator on $\mathbf{R}^3$ consider:

$$\int\limits_{\mathbf{R}^3} \langle P_5 u, \vec{v}\rangle\,dx = \int\limits_{\mathbf{R}^3} \nu \cdot\ curl\,\vec{u}\,v\,dx = \int\limits_{\mathbf{R}^3} \langle\ v\nu,\ curl\,\vec{u}\ \rangle\,dx$$

$$= \int\limits_{\mathbf{R}^3} \langle\ curl\,(v\nu)\,,\ u\ \rangle\,dx, \tag{3.106}$$

so $P_5^* = curl\,(v\nu) = \nabla v \times \nu + v\,curl\,\nu$, then we have: $curl\,\nu = 0$ on $\mathcal{S}$.

Now we can apply Theorem (3.4.4):

$$\int\limits_{\mathcal{S}} (\nu \cdot curl\,\vec{u})\ v\,dS = \int\limits_{\mathcal{S}} \langle u,\ \nabla v \times \nu\ \rangle\,dS + \oint\limits_{\partial\mathcal{S}} \nu \cdot (\gamma \times \vec{u})\ v\,ds. \tag{3.107}$$

Remark 3.4.8. *In the last formula if we replace $v = 1$, ∇v will be zero, and knowing $\nu \times \gamma = \tau$ we obtain the Stokes Formula:*

$$\int\limits_{\mathcal{S}} (\nu \cdot curl\,\vec{u})\ dS = \oint\limits_{\partial\mathcal{S}} \langle \nu,\ \gamma \times \vec{u}\ \rangle\,ds = \oint\limits_{\partial\mathcal{S}} \tau \cdot \vec{u}\,ds.$$

Example 6. Let $P_6 = div - \partial_\nu\langle\cdot,\nu\rangle$. For a vector-valued function u and a scalar v we have:

$$P_6\vec{u} = div\,\vec{u} - \partial_\nu\langle\vec{u},\nu\rangle, \tag{3.108}$$

$$\text{then} \quad \sigma\,(P_6\,;\xi)\,\vec{u} = \langle\xi,\ \vec{u}\rangle - \langle\nu,\ \xi\rangle\langle\vec{u},\ \nu\rangle, \tag{3.109}$$

$$\text{and} \quad \sigma\,(P_6\,;\nu)\,\vec{u} = \langle\nu,\ \vec{u}\rangle - \langle\nu,\ \nu\rangle\langle\vec{u},\ \nu\rangle = \langle\nu,\ \vec{u}\rangle - \langle\nu,\ \vec{u}\rangle = 0. \tag{3.110}$$

The adjoint operator on $\mathbf{R}^3$ will be:

$$\int\limits_{\mathbf{R}^3} \langle P_6 u, \vec{v}\rangle\,dx = \int\limits_{\mathbf{R}^3} \langle div\,\vec{u} - \langle\nu,\ \nabla\langle\nu,\ \vec{u}\rangle\rangle,\ v\ \rangle\,dx$$

$$= -\int\limits_{\mathbf{R}^3} \langle\vec{u},\ \nabla v\ \rangle\,dx - \int\limits_{\mathbf{R}^3} \langle v\,\nu,\ \nabla\langle\nu,\ \vec{u}\rangle\rangle\,dx$$

$$= -\int\limits_{\mathbf{R}^3} \langle\nu,\ \nabla v\ \rangle\,dx + \int\limits_{\mathbf{R}^3} div\,(v\,\nu)\,\langle\nu,\ \vec{u}\rangle\,dx. \tag{3.111}$$

Let us apply Remark 3.4.6 and the formula becomes:

$$\int_{\mathbf{R}^3} \langle P_6 u, \vec{v}\rangle\, dx = -\int_{\mathbf{R}^3} \langle \nu,\ \nabla v\ \rangle\, dx + \int_{\mathbf{R}^3} \langle \nabla v,\ \nu\ \rangle \langle \nu,\ \vec{u}\ \rangle\, dx - \int_{\mathbf{R}^3} v\, div\, \nu\, \langle \nu,\ \vec{u}\ \rangle\, dx$$

$$= -\int_{\mathbf{R}^3} \langle \nu,\ \nabla v\ \rangle\, dx + \int_{\mathbf{R}^3} \langle \nu \langle \nabla v,\ \nu\ \rangle,\ \vec{u}\ \rangle\, dx - \int_{\mathbf{R}^3} \langle v\, div\, \nu \nu,\ \vec{u}\ \rangle\, dx$$

$$= -\int_{\mathbf{R}^3} \langle \nu, \nabla v\rangle dx + \int_{\mathbf{R}^3} \langle \vec{u}, (\partial_\nu v)\, \nu\rangle dx + \int_{\mathbf{R}^3} \langle \vec{u}, (v\, div\, \nu)\, \nu\rangle dx, \tag{3.112}$$

so with $div\, \nu = \mathcal{G}$ we have:

$$P_6^* v = -\nabla v + (\partial_\nu v)\, \nu + \mathcal{G}\, v\, \nu. \tag{3.113}$$

Now we can apply Theorem (3.4.4):

$$\int_{\mathcal{S}} \langle P_6 \vec{u}, \vec{v}\rangle\, dS = \int_{\mathcal{S}} [-\langle \nabla v,\ \vec{u}\ \rangle + (\partial_\nu v)\, \langle \nu,\ \vec{u}\ \rangle + \mathcal{G}\, v\, \langle \nu,\ \vec{u}\ \rangle]\, dS$$

$$+ \oint_{\partial\mathcal{S}} [\langle \gamma,\ \vec{u}\ \rangle - \langle \nu,\ \gamma\rangle \langle \vec{u},\ \nu\rangle]\ v\, dS, \tag{3.114}$$

and invoking $\langle \nu,\ \gamma\rangle = 0$ we finally have:

$$\int_{\mathcal{S}} \langle [div\, \vec{u} - \partial_\nu \langle \vec{u}, \nu\rangle]\, ,\ \vec{v}\rangle\, dS = \int_{\mathcal{S}} [-\langle \nabla v,\ \vec{u}\ \rangle + (\partial_\nu v)\, \langle \nu,\ \vec{u}\ \rangle + \mathcal{G}\, v\, \langle \nu,\ \vec{u}\ \rangle]\, dS$$

$$+ \oint_{\partial\mathcal{S}} [\langle \gamma,\ \vec{u}\ \rangle]\ v\, dS. \tag{3.115}$$

Example 7. Let $P_7 = div\, \pi(\cdot)$, then for a vector-valued function $\vec{u}$ and a scalar v we have:

$$P_7 \vec{u} = div\, \pi(\vec{u}), \tag{3.116}$$

with $\pi(\vec{u}) = -\nu \times (\nu \times \vec{u}) = \vec{u} - \langle \nu,\ \vec{u}\rangle\, \nu$ from Remark 3.4.5.

Then

$$P_7 \vec{u} = div\, [-\nu \times (\nu \times \vec{u})] = \sum_{j=1}^{3} \partial_j \left(u_j - \nu_j \langle \nu,\ \vec{u}\rangle\right) = \sum_{j=1}^{3} \partial_j u_j - \sum_{j=1}^{3} \langle \nu,\ \vec{u}\rangle \partial_j \nu_j$$

$$- \sum_{j=1}^{3} \nu_j \partial_j \langle \nu,\ \vec{u}\rangle = \sum_{j=1}^{3} \partial_j u_j - \sum_{j=1}^{3} \langle \nu,\ \vec{u}\rangle \partial_j \nu_j - \sum_{j=1}^{3} \sum_{k=1}^{3} \nu_j \partial_j \left(\nu_k u_k\right)$$

$$= \sum_{j=1}^{3} \partial_j u_j - \sum_{j=1}^{3} \langle \nu,\ \vec{u}\rangle \partial_j \nu_j - \sum_{j=1}^{3} \sum_{k=1}^{3} \nu_j \nu_k \partial_j u_k - \sum_{j=1}^{3} \sum_{k=1}^{3} \nu_j u_k \partial_j \nu_k, \tag{3.117}$$

and we have:

$$\sigma\,(P_7\,;\xi)\,\vec{u} = \sum_{j=1}^{3}\xi_j u_j - \sum_{j=1}^{3}\sum_{k=1}^{3}\nu_j\nu_k\xi_j u_k = \langle \xi,\ \vec{u}\rangle - \langle \nu,\ \nu\ \rangle\langle \nu,\ \vec{u}\rangle, \tag{3.118}$$

and

$$\sigma\,(P_7\,;\nu)\,\vec{u} = \langle \nu,\ \vec{u}\rangle - \langle \nu,\ \nu\ \rangle\langle \nu,\ \vec{u}\ \rangle = 0. \tag{3.119}$$

The adjoint operator on $\mathbf{R}^3$ will be:

$$\begin{aligned}\int\limits_{\mathbf{R}^3}\langle P_7\vec{u}, v\rangle\,dx = \int\limits_{\mathbf{R}^3}\langle div\,(\pi(\vec{u}))\,,\ v\rangle &= \int\limits_{\mathbf{R}^3}\sum_{j=1}^{3}\partial_j\,(u_j - \nu_j\langle \nu,\ \vec{u}\rangle)\ v\,dx \\ &= -\int\limits_{\mathbf{R}^3}\langle \pi(\vec{u}),\ \nabla v\rangle\,dx = -\int\limits_{\mathbf{R}^3}\langle -\nu\times(\nu\times\vec{u})\,,\ \nabla v\ \rangle\,dx \\ &= -\int\limits_{\mathbf{R}^3}\langle \nu\times\vec{u}, \nu\times\nabla v\ \rangle\,dx = \int\limits_{\mathbf{R}^3}\langle \vec{u},\ \nu\times(\nu\times\nabla v)\rangle\,dx \\ &= -\int\limits_{\mathbf{R}^3}\langle \vec{u},\ \pi\,(\nabla v)\rangle\,dx, \end{aligned} \tag{3.120}$$

so $P_7^* v = -\pi\,(\nabla v)$.

Now we can apply Theorem (3.4.4):

$$\int\limits_{\mathcal{S}}\langle div\,\pi(\vec{u}), v\rangle\,dS = -\int\limits_{\mathcal{S}}\langle \vec{u},\ \pi\,(\nabla v)\rangle\,dS + \oint\limits_{\partial\mathcal{S}}\langle \gamma,\ \pi\vec{u}\ \rangle\,v\,ds. \tag{3.121}$$

3.5 Further Applications

Of special interest for us are the Günter derivatives (see [Gü]) $\mathcal{D} := (\mathcal{D}_1,\, \mathcal{D}_2,\, ...,\, \mathcal{D}_n)$ defined for any $1 \le j \le n$ by:

$$\mathcal{D}_j := \partial_j - \nu_j\, \partial_\nu, \qquad j = 1,\, 2,\, ...,\, n. \tag{3.122}$$

Then we set:

$$\mathcal{D}\, f\ := (\mathcal{D}_1 f,\, \mathcal{D}_2 f,\, ...,\, \mathcal{D}_n f), \qquad \text{for scalar-valued functions,} \tag{3.123}$$

$$\mathcal{D} \cdot \vec{u} := \sum_{j=1}^{n} \mathcal{D}_j\, u_j \qquad \text{for vector-valued functions.} \tag{3.124}$$

Some of the most basic properties of this system of differential operators are listed in the next proposition.

Proposition 3.5.1. *The following relations are valid:*

$(i)\quad \mathcal{D}_j = \sum_{k=1}^{n} \nu_k \mathcal{M}_{kj},\qquad$ *for each* $\ 1 \le j \le n;$

$(ii)\quad \mathcal{M}_{kj} = \nu_j \mathcal{D}_k - \nu_k\, \mathcal{D}_j \qquad$ *for each* $\ 1 \le j \le n;$

$(iii)\quad \sum_{k=1}^{n} \nu_j \mathcal{D}_j = 0;$

$(iv)\quad [\mathcal{D}_j,\, \mathcal{D}_k] = \nu_j(\nabla \nu_k \cdot \nabla) - \nu_k(\nabla \nu_j \cdot \nabla)\quad$ *on* $\mathcal{S}$ *for each* $\ 1 \le j \le n;$

$(v)\quad \sum_{k=1}^{n} \mathcal{D}_j \nu_j = \mathcal{G};$

$(vi)\quad$ *for every* $\mathcal{C}^1$ *functions* $f,\ g$ *on* $\mathcal{S}$, *and every* $\ 1 \le j \le n,$

$$\int_{\mathcal{S}} (\mathcal{D}_j\ f)\ g\, dS = \int_{\mathcal{S}} [-f\,(\mathcal{D}_j\, g) + \nu_j\, \mathcal{G}\ f\ g\,]\ dS + \oint_{\partial \mathcal{S}} \gamma_j\ f\ g\, ds,$$

where $r = (r_1, ..., r_n)$ is the unit normal to $\partial \mathcal{S}$.

Proof. Recall $\mathcal{M}_{kj} = \nu_j \partial_k - \nu_k \partial_j,\ \ 1 \le j,\ k \le 3$ from formula (3.30).

For part (i) we have:

$$\mathcal{D}_j = \sum_{k=1}^{n} \nu_k\, \mathcal{M}_{kj} = \sum_{k=1}^{n} \nu_k\, (\nu_k \partial_j - \nu_j\, \partial_k) = \sum_{k=1}^{n} \left(\nu_k^2 \partial_j - \nu_k \nu_j\, \partial_k\right)$$

$$= \partial_j \sum_{k=1}^{n} \nu_k^2 - \nu_j \sum_{k=1}^{n} \nu_k \partial_k = \partial_j - \nu_j \langle \nu,\, \nabla \rangle = \partial_j - \nu_j\, \partial_\nu = \mathcal{D}_j. \tag{3.125}$$

For part (ii)we have:

$$\mathcal{M}_{kj} = \nu_j \partial_k - \nu_k \partial_j = \nu_j \left[\partial_k - \nu_k (\nu \cdot \nabla)\right] - \nu_k \left[\partial_j - \nu_j (\nu \cdot \nabla)\right]$$

$$= \nu_j \partial_k - \nu_j \nu_k (\nu \cdot \nabla) - \nu_k \partial_j + \nu_k \nu_j (\nu \cdot \nabla) = \mathcal{M}_{jk}. \tag{3.126}$$

For part (iii) we have:

$$\sum_{j=1}^{n} \nu_j \, \mathcal{D}_j = \sum_{j=1}^{n} \nu_j \left[\partial_j - \nu_j (\nu \cdot \nabla)\right] = \sum_{j=1}^{n} \left[\nu_j \partial_j - \nu_j \nu_j (\nu \cdot \nabla)\right]$$

$$= \sum_{j=1}^{n} \nu_j \partial_j - (\nu \cdot \nabla) \sum_{j=1}^{3} \nu_j \nu_j = \sum_{j=1}^{n} \nu_j \partial_j - \sum_{j=1}^{n} \nu_j \partial_j = 0. \tag{3.127}$$

For part (iv) we have:

$$\mathcal{D}_j \, \mathcal{D}_k = (\partial_j - \nu_j \partial_\nu)\,(\partial_k - \nu_k \partial_\nu)$$

$$= \partial_j \partial_k - \partial_j (\nu_k \partial_\nu) - (\nu_j \partial_\nu)(\nu_k) + (\nu_j \partial_\nu)(\nu_k \partial_\nu)$$

$$= \partial_j \partial_k - (\partial_j \nu_k) \partial_\nu - \nu_k (\partial_j \partial_\nu) - \nu_j (\partial_\nu \partial_k) + \nu_j (\partial_\nu \nu_k) \partial_\nu + \nu_j \nu_k (\partial_\nu)(\partial_\nu)$$

$$= \partial_j \partial_k - (\partial_j \nu_k) \partial_\nu - \nu_k \partial_j \left(\sum_{l=1}^{n} \nu_l \partial_l \right) - \nu_j \left(\sum_{l=1}^{n} \nu_l \partial_l \right) \partial_k + \nu_j \nu_k (\partial_\nu)^2$$

$$= \partial_j \partial_k - (\partial_j \nu_k) \partial_\nu - \nu_k \sum_{l=1}^{n} \partial_j (\nu_l \partial_l) - \nu_j \sum_{l=1}^{n} \nu_l \partial_l \partial_k + \nu_j \nu_k (\partial_\nu)^2$$

$$= \partial_j \partial_k - (\partial_j \nu_k) \partial_\nu - \sum_{l=1}^{n} \left[\nu_k \left(\partial_j \nu_l\right) \partial_l + \nu_k \nu_l \partial_j \partial_k + \nu_j \nu_l \partial_l \partial_k\right]. \tag{3.128}$$

The previous expression is symmetric in j and k, so we can write:

$$\mathcal{D}_k \, \mathcal{D}_j = \partial_k \partial_j - (\partial_k \nu_j) \partial_\nu - \sum_{l=1}^{n} \left[\nu_j \left(\partial_k \nu_l\right) \partial_l + \nu_j \nu_l \partial_k \partial_j + \nu_k \nu_l \partial_l \partial_j\right]. \tag{3.129}$$

Let us write the difference:

$$\mathcal{D}_j \, \mathcal{D}_k - \mathcal{D}_k \, \mathcal{D}_j = \partial_j \partial_k - (\partial_j \nu_k) \partial_\nu - \sum_{l=1}^{n} \left[\nu_k \left(\partial_j \nu_l\right) \partial_l + \nu_k \nu_l \partial_j \partial_k + \nu_j \nu_l \partial_l \partial_k\right]$$

$$- \partial_k \partial_j + (\partial_k \nu_j) \partial_\nu + \sum_{l=1}^{n} \left[\nu_j \left(\partial_k \nu_l\right) \partial_l - \nu_j \nu_l \partial_k \partial_j - \nu_k \nu_l \partial_l \partial_j\right]. \tag{3.130}$$

Recall Proposition (2.3.4) part (i): $\nu_j \partial_k = \nu_k \partial_j$ on $\mathcal{S}$ and the formula becomes:

$$\mathcal{D}_j\, \mathcal{D}_k - \mathcal{D}_k\, \mathcal{D}_j = -\sum_{l=1}^{n} \nu_k\, (\partial_l \nu_j)\, \partial_l + \sum_{l=1}^{n} \nu_j\, (\partial_l \nu_k)\, \partial_l$$

$$= \nu_j\, (\partial_1 \nu_k \partial_1 + \partial_2 \nu_k \partial_2 + ... + \partial_n \nu_k \partial_n) - \nu_k\, (\partial_1 \nu_j \partial_1 + \partial_2 \nu_j \partial_2 + ... + \partial_n \nu_j \partial_n)$$

$$= \nu_j\, (\nabla \nu_k \cdot \nabla) - \nu_k\, (\nabla \nu_j \cdot \nabla) = 0. \tag{3.131}$$

This conclude the proof for part (iv).

For part (v) we have:

$$\sum_{j=1}^{n} \mathcal{D}_j \nu_j = \sum_{j=1}^{n} [\partial_j - \nu_j \partial_\nu](\nu_j) = \sum_{j=1}^{n} \partial_j \nu_j - \sum_{j=1}^{n} \nu_j \partial_\nu \nu_j. \tag{3.132}$$

Recall Proposition (2.3.4) part (ii): $\partial_\nu \nu_j = 0$ on $\mathcal{S}$, and Theorem (2.2.4) then formula becomes:

$$\sum_{j=1}^{n} \mathcal{D}_j \nu_j = \sum_{j=1}^{n} \partial_j \nu_j = div\, \nu = \mathcal{G}. \tag{3.133}$$

To prove part (vi) let us first find the adjoint operator for $\mathcal{D}_j$ in $\mathbf{R}^3$. If u, v are scalars we can write:

$$Pu = \mathcal{D}_j u = \partial_j u - \nu_j(\nu \cdot \nabla u), \tag{3.134}$$

$$\sigma(P;\, \xi) u = \xi_j u - \nu_j(\nu \cdot \xi u), \tag{3.135}$$

$$\sigma(P;\, \nu) u = \nu_j u - \nu_j(\nu \cdot \nu u) = \nu_j u - \nu_j u \langle \nu,\, \nu \rangle = 0, \tag{3.136}$$

so

$$\int_{\mathbf{R}^3}(Pu)v\,dx=\int_{\mathbf{R}^3}[\partial_j u-\nu_j(\nu\cdot\nabla)]\,v\,dx=\int_{\mathbf{R}^3}v\,\partial_j u\,dx-\int_{\mathbf{R}^3}v\,\nu_j(\nu\cdot\nabla)\,dx$$

$$=-\int_{\mathbf{R}^3}u\partial_j v\,dx-\int_{\mathbf{R}^3}\langle(v\nu_j)\nabla u,\ \nu\rangle\,dx=-\int_{\mathbf{R}^3}v\partial_j u\,dx-\int_{\mathbf{R}^3}\sum_{k=1}^{3}(v\nu_j\partial_k u)\,\nu_k\,dx$$

$$=-\int_{\mathbf{R}^3}v\partial_j u\,dx-\int_{\mathbf{R}^3}\sum_{k=1}^{3}(v\nu_j\nu_k)\,\partial_k u\nu_k\,dx$$

$$=-\int_{\mathbf{R}^3}v\,\partial_j u\,dx-\int_{\mathbf{R}^3}\sum_{k=1}^{3}\left[(u\nu_j\partial_k v)\,\nu_k+(uv\partial_k\nu_j)\nu_j+uv\nu_j\partial_k\nu_k\right]\,dx$$

$$=-\int_{\mathbf{R}^3}v\partial_j u\,dx-\int_{\mathbf{R}^3}(u\nu_j\nabla v)\,\nu+\int_{\mathbf{R}^3}uv\sum_{k=1}^{3}(\partial_k\nu_j)\nu_k+\int_{\mathbf{R}^3}uv\nu_j\sum_{k=1}^{3}\partial_k\nu_k\,dx$$

$$=-\int_{\mathbf{R}^3}u\,(\mathcal{D}_j v)+\int_{\mathbf{R}^3}u\,v[\nu\cdot\nabla\nu_j+\nu_j\cdot div\,\nu]\,dx$$

$$=-\int_{\mathbf{R}^3}\left[u\,(\mathcal{D}_j v)+u\,v(\partial_\nu\nu_j+\nu_j\mathcal{G})\right]\,dx, \tag{3.137}$$

then the adjoint operator for $\mathcal{D}_j$ will be:

$$P^*=\mathcal{D}_j+\partial_\nu\nu_j+\nu_j\mathcal{G}. \tag{3.138}$$

To prove part (vi) from (3.5.1) we recall $\partial_\nu\nu_j=0$ on the surface $\mathcal{S}$, and with integration by parts we have:

$$\int_{\mathcal{S}}(\mathcal{D}_j\,f)\,g\,dS=\int_{\mathcal{S}}\left[-f\,(\mathcal{D}_j\,g)+fg\nu_j\,\mathcal{G}\,f\,g\,\right]\,dS+\oint_{\partial\mathcal{S}}[\gamma_j\,f-\nu_j(\nu\cdot\gamma\,u)]g\,ds.$$

$$=\int_{\mathcal{S}}\left[-f\,(\mathcal{D}_j\,g)+f\,g\,\nu_j\,\mathcal{G}\,f\,g\,\right]\,dS+\oint_{\partial\mathcal{S}}fg[\gamma_j-\nu_j(\nu\cdot\gamma)]\,g\,ds, \tag{3.139}$$

and with the observation that the scalar product $\nu\cdot\gamma=0$ we have:

$$\int_{\mathcal{S}}(\mathcal{D}_j\,f)\;g\,dS=\int_{\mathcal{S}}\left[-f\,(\mathcal{D}_j\,g)+fg\nu_j\,\mathcal{G}\right]\,dS+\oint_{\partial\mathcal{S}}fg\gamma_j\,ds. \tag{3.140}$$

This conclude the proof of Proposition (3.5.1). $\square$

Proposition 3.5.2. *For any vector-valued function $\vec{u}$ the following holds:*

$$P_1\vec{u} = \mathcal{D}\cdot\vec{u} = div\,\vec{u} - \partial_\nu\,\langle \nu,\ \vec{u}\rangle + \langle \mathcal{R}\vec{u},\ \nu\rangle, \tag{3.141}$$

$$P_2\vec{u} = div(\pi\vec{u}) = \mathcal{D}\cdot(\pi\vec{u}) -\ \langle \nabla_\nu \nu,\ \vec{u}\rangle, \tag{3.142}$$

where $\mathcal{R} = \left(\partial_k \nu_j\right)_{j,k}$ like in Proposition(2.4.1)*.*

Proof. First let us check that we can write $\mathcal{D}\cdot\vec{u}$ in the following form:

$$\mathcal{D}\cdot\vec{u} = \sum_{j=1}^{n} \mathcal{D}_j u_j = \mathcal{D}\cdot(\pi\vec{u}) + \mathcal{G}\,\langle \nu,\ \vec{u}\rangle. \tag{3.143}$$

This formula follows from straightforward calculation:

$$\begin{aligned}
\mathcal{D}\cdot(\pi\vec{u}) &= \sum_{j=1}^{n} \mathcal{D}_j(\pi\vec{u})_j = \sum_{j=1}^{n} \mathcal{D}_j(u_j - \nu_j\langle \nu,\ \vec{u}\rangle) \\
&= \sum_{j=1}^{n} \mathcal{D}_j u_j - \sum_{j=1}^{n} \langle \nu,\ \vec{u}\rangle \mathcal{D}_j \nu_j - \sum_{j=1}^{n} \nu_j \mathcal{D}_j \langle \nu,\ \vec{u}\rangle,
\end{aligned} \tag{3.144}$$

and with Proposition (3.5.1) $\sum_{j=1}^{n} \nu_j \mathcal{D}_j = 0$ we have:

$$\begin{aligned}
\mathcal{D}\cdot(\pi\vec{u}) &= \sum_{j=1}^{n} \mathcal{D}_j u_j - \sum_{j=1}^{n} \langle \nu,\ \vec{u}\rangle (\partial_j - \nu_j\langle \nu,\ \nabla\rangle)\nu_j \\
&= \sum_{j=1}^{n} \mathcal{D}_j u_j - \langle \nu,\ \vec{u}\rangle \sum_{j=1}^{n} \partial_j \nu_j - \langle \nu,\ \vec{u}\rangle \sum_{j=1}^{n}\ \nu_j \langle \nu,\ \nabla \nu_j\rangle \\
&= \sum_{j=1}^{n} \mathcal{D}_j u_j - \mathcal{G}\langle \nu,\ \vec{u}\rangle - \langle \nu,\ \vec{u}\rangle \frac{1}{2} \sum_{j=1}^{n} \partial_\nu (\nu_j)^2 \\
&= \sum_{j=1}^{n} \mathcal{D}_j u_j - \mathcal{G}\langle \nu,\ \vec{u}\rangle - \langle \nu,\ \vec{u}\rangle \frac{1}{2}\, \partial_\nu \left(\sum_{j=1}^{n} \nu_j^2 \right).
\end{aligned} \tag{3.145}$$

We know $\sum_{j=1}^{n} \nu_j^2 = 1$, and this give us $\partial_\nu 1 = 0$, so the previous formula becomes:

$$\mathcal{D}\cdot(\pi\vec{u}) = \sum_{j=1}^{n} \mathcal{D}_j u_j - \mathcal{G}\,\langle \nu,\ \vec{u}\rangle, \tag{3.146}$$

and this proves formula (3.143).

With this formula P_1 becomes:

$$\begin{aligned}
&\sum_{j=1}^{n} \mathcal{D}_j u_j = \sum_{j=1}^{n} \left(\partial_j - \nu_j \langle \nu,\ \nabla\rangle\right) u_j = \sum_{j=1}^{n} \partial_j u_j - \sum_{j=1}^{n} [\partial_\nu(\nu_j u_j) - (\partial_\nu \nu_j) u_j] \\
&= \sum_{j=1}^{n} \partial_j u_j - \sum_{j=1}^{n} \partial_\nu(\nu_j u_j) \quad + \sum_{j=1}^{n} (\partial_\nu \nu_j) u_j] = div\,\vec{u} - \partial_\nu\,\langle \nu,\ \vec{u}\rangle + \sum_{j=1}^{n}\sum_{k=1}^{n} \nu_k \partial_k \nu_j u_j \\
&= div\,\vec{u} - \partial_\nu\,\langle \nu,\ \vec{u}\rangle + \langle \mathcal{R}\vec{u},\ \nu\rangle.
\end{aligned} \tag{3.147}$$

This conclude the proof of the first identity in Proposition (3.5.2).

To prove P_2 let us start from the right-hand-side, and with formula (3.143) we have:

$$P_2 u = \mathcal{D} \cdot (\pi \vec{u}) - \langle \nabla_\nu \nu,\ \vec{u} \rangle$$

$$= div\, \vec{u} - \partial_\nu \langle \nu,\ \vec{u} \rangle + \langle \mathcal{R}\vec{u},\ \nu \rangle - \mathcal{G}\langle \nu,\ \vec{u} \rangle - \langle \nabla_\nu \nu,\ \vec{u} \rangle \tag{3.148}$$

Let us check that the following holds:

$$\langle \nabla_\nu \nu,\ u \rangle = \langle \mathcal{R}\vec{u},\ \nu \rangle. \tag{3.149}$$

This equality follows from a straightforward calculation:

$$\langle \nabla_\nu \nu,\ u \rangle = \sum_{j=1}^{n} \partial_\nu \nu_j u_j = \sum_{j=1}^{n} \sum_{k=1}^{n} \nu_k \partial_k \nu_j u_j = \langle \nu,\ \mathcal{R}\vec{u} \rangle = \langle \mathcal{R}\vec{u},\ \nu \rangle. \tag{3.150}$$

So the formula (3.148) becomes:

$$P_2 u = div\, \vec{u} - \partial_\nu \langle \nu,\ \vec{u} \rangle + \langle \mathcal{R}\vec{u},\ \nu \rangle - \mathcal{G}\langle \nu,\ \vec{u} \rangle - \langle \mathcal{R}\vec{u},\ \nu \rangle$$

$$= \sum_{j=1}^{n} \partial_j u_j - \nu \cdot \nabla \langle \nu,\ \vec{u} \rangle - \sum_{j=1}^{n} \partial_j \nu_j \langle \nu,\ u \rangle = \sum_{j=1}^{n} \partial_j u_j - \sum_{j=1}^{n} \nu_j \partial_j \left(\langle \nu,\ \vec{u} \rangle \right)$$

$$- \sum_{j=1}^{n} \partial_j \nu_j \langle \nu,\ \vec{u} \rangle = \sum_{j=1}^{n} \partial_j u_j - \sum_{j=1}^{n} \partial_j \left(\nu_j \langle \nu,\ \vec{u} \rangle \right)$$

$$= \sum_{j=1}^{n} \partial_j \left(u_j - \nu_j \langle \nu,\ \vec{u} \rangle \right) = div(\pi \vec{u}). \tag{3.151}$$

This finishes the proof of Proposition (3.5.2). □

Theorem 3.5.3. *For any vector field $\vec{u}$ tangential to a surface $\mathcal{S}$ the following identities hold:*

$$div_{\mathcal{S}} \vec{u} := div(\pi(\vec{u}))/_{\mathcal{S}} = \mathcal{D} \cdot \vec{u} = \sum_{j=1}^{n} \mathcal{D}_j u_j. \tag{3.152}$$

Also, for any real-valued function f on $\mathcal{S}$ we can write:

$$grad_{\mathcal{S}} f := \pi \left(\nabla f \right)/_{\mathcal{S}} = \mathcal{D} f = \left(\mathcal{D}_1 f,\ \mathcal{D}_2 f,\ \ldots,\ \mathcal{D}_n f \right), \tag{3.153}$$

$$div_{\mathcal{S}}\ grad_{\mathcal{S}}\ f = \mathcal{D} \cdot (\mathcal{D} f) = \sum_{j=1}^{n} \mathcal{D}_j^2 f. \tag{3.154}$$

Proof. From Proposition (3.5.2) we have:

$$div(\pi(\vec{u}))/_{\mathcal{S}} = P_2 \vec{u}/_{\mathcal{S}} = \left(\mathcal{D} \cdot (\pi \vec{u}) - \langle \nabla_\nu \nu,\ \vec{u} \rangle \right)/_{\mathcal{S}}. \tag{3.155}$$

With formula (3.143) and (3.149), and recall that $\vec{u}$ is tangential to $\mathcal{S}$ give us the scalar product $\langle \nu,\ \vec{u}\rangle = 0$, the formula becomes:

$$div(\pi(\vec{u}))/_{\mathcal{S}} = \left(\mathcal{D}\cdot\vec{u} - \mathcal{G}\,\langle \nu,\ \vec{u}\rangle - \langle \mathcal{R}\vec{u}, \nu\rangle\right)/_{\mathcal{S}}. \tag{3.156}$$

Using the equality $\langle \mathcal{R}\vec{u},\ \nu\rangle = \langle \vec{u},\ \mathcal{R}^t\nu\rangle$, Proposition (2.4.1) with $\mathcal{R}\nu = 0$ and $\mathcal{R} = \mathcal{R}^t$, and $\langle \nu,\ \vec{u}\rangle = 0$ on $\mathcal{S}$ we can write:

$$div(\pi(\vec{u}))/_{\mathcal{S}} = \left(\mathcal{D}\cdot\vec{u} - \langle \vec{u}, \mathcal{R}\nu\rangle\right)/_{\mathcal{S}} = \mathcal{D}\cdot\vec{u}. \tag{3.157}$$

Remark 3.5.4. *If $\vec{a}$ and $\vec{b}$ are vectors in $\mathbb{R}^3$, then*

$$\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a}). \tag{3.158}$$

For the second expression in Theorem (3.5.3), and using Remark 3.5.4 we have:

$$\begin{aligned}\pi\left(\nabla f\right)/_{\mathcal{S}} &= -\nu\times(\nu\times\nabla f)/_{\mathcal{S}} = (\nu\times\nabla f)\times\nu/_{\mathcal{S}}\\ &= \sum_{j=1}^{3}\left(\partial_j f - \nu_j\langle \nu,\ \nabla f\rangle\right) = \nabla f - \nu\langle \nu,\ \nabla f\rangle.\end{aligned} \tag{3.159}$$

Recall formula (3.151), and the formula becomes:

$$\pi\left(\nabla f\right)/_{\mathcal{S}} = \begin{pmatrix}\partial_1 f - \nu_1\langle \nu,\ \nabla f\rangle\\ \partial_2 f - \nu_2\langle \nu,\ \nabla f\rangle\\ \partial_3 f - \nu_3\langle \nu,\ \nabla f\rangle\end{pmatrix}. \tag{3.160}$$

On the other side we can write:

$$\mathcal{D}f = (\mathcal{D}_1 f,\ \mathcal{D}_2 f,\ \mathcal{D}_3 f) = \left(\partial_1 f - \nu_1\langle \nu,\ \nabla f\rangle,\ \partial_2 f - \nu_2\langle \nu,\ \nabla f\rangle,\ \partial_3 f - \nu_3\langle \nu,\ \nabla f\rangle\right), \tag{3.161}$$

and this will conclude the second expression in Theorem (3.5.3).

In order to prove the third expression in (3.5.3) let us use the first, and the second expressions in the same theorem:

$$div_{\mathcal{S}}\ grad_{\mathcal{S}}\ f = \mathcal{D}\cdot(\mathcal{D}f) = \sum_{j=1}^{n}\mathcal{D}_j(\mathcal{D}_j f) = \sum_{j=1}^{n}\mathcal{D}_j^2 f. \tag{3.162}$$

This establishes Theorem (3.5.3). □

Next we want to write a relation between the Laplace operator in $\mathbf{R}^3$ applied to a scalar function restricted to a surface $\mathcal{S}$ in terms of the Laplace operator on the surface itself.

Let us start by introducing the notation:

$$\left(\partial_\nu^2 f\right)/_{\mathcal{S}} = \partial_\nu(\partial_\nu f) = \sum_{j,\,k=1}^{n}\nu_j\nu_k\left(\partial_j\partial_k f\right)/_{\mathcal{S}}. \tag{3.163}$$

Corollary 3.5.5. *For any $\mathcal{C}^2$ scalar function f, defined in a neighborhood of $\mathcal{S}$, there* *holds:*

$$(\triangle_{\mathbf{R}^3} f)/_{\mathcal{S}} = \triangle_{\mathcal{S}}(f/_{\mathcal{S}}) + \mathcal{G}(\partial_\nu f)/_{\mathcal{S}} + \left(\partial_\nu^2 f\right)/_{\mathcal{S}}. \tag{3.164}$$

Proof. Recall (3.2.4), and write the Laplace operator on the surface $\mathcal{S}$:

$$\triangle_{\mathcal{S}}(f/_{\mathcal{S}}) = \sum_{j=1}^{n} \mathcal{D}_j^2 f/_{\mathcal{S}} = \sum_{j=1}^{n} (\partial_j - \nu_j \langle \nu, \nabla\rangle)(\partial_j - \nu_j \langle \nu, \nabla\rangle) f/_{\mathcal{S}} \tag{3.165}$$

$$= \sum_{j=1}^{n} [\partial_j(\partial_j f) - \partial_j(\nu_j \langle \nu, \nabla\rangle f) - (\nu_j \langle \nu, \nabla\rangle)(\partial_j f) + \nu_j \langle \nu, \nabla\rangle (\nu_j \langle \nu, \nabla\rangle f)]/_{\mathcal{S}}$$

$$= \sum_{j=1}^{n} \left[\partial_j^2 f - \partial_j(\nu_j \partial_\nu f) - \nu_j \partial_\nu(\partial_j f) - \nu_j \partial_\nu(\nu_j \partial_\nu f)\right]/_{\mathcal{S}}$$

$$= \sum_{j=1}^{n} \left[\partial_j^2 f - (\partial_j \nu_j)(\partial_\nu f) - \nu_j \partial_j(\partial_\nu f) - \nu_j \partial_\nu(\partial_j f) + \nu_j(\partial_\nu \nu_j)(\partial_\nu f)\right.$$

$$\left. + \nu_j^2 \partial_\nu(\partial_\nu f)\right]/_{\mathcal{S}}.$$

We know that $\sum_{j=1}^n \partial_j \nu_j = \mathcal{G}$ on $\mathcal{S}$, $\sum_{j=1}^n \partial_\nu \nu_j = 0$ on $\mathcal{S}$, $\sum_{j=1}^n \nu_j^2 = 1$, and $\partial_\nu = \sum_{k=1}^n \nu_k \partial_k$, so the formula becomes:

$$\triangle_{\mathcal{S}}(f/_{\mathcal{S}}) = (\triangle_{\mathbf{R}^3} f)/_{\mathcal{S}} - (\mathcal{G}\partial_\nu f)/_{\mathcal{S}} - \sum_{j,\,k=1}^{n} \nu_j \nu_k \partial_k(\partial_j f)/_{\mathcal{S}}$$

$$= (\triangle_{\mathbf{R}^3} f)/_{\mathcal{S}} - (\mathcal{G}\partial_\nu f)/_{\mathcal{S}} - (\partial_\nu^2 f)/_{\mathcal{S}}. \tag{3.166}$$

If we rewrite this equality, Corollary (3.5.5) is proved. □

Next, we specialize our discussion to the case of a sphere.

Proposition 3.5.6. *Let $\mathcal{S} = \partial B(0, r)$. Then the following identity holds:*

$$\mathcal{G} = \frac{2}{r}. \tag{3.167}$$

Proof: From Theorem (2.2.4) we have $\mathcal{G} = div\ \nu$, so

$$\mathcal{G} = div\ \nu = div\left(\frac{x}{|x|}\right) = \sum_{j=1}^{3} \partial_j\left(\frac{x}{|x|}\right) = \sum_{j=1}^{3} \frac{(\partial_j x_j)|x| - x_j \partial_j(|x|)}{|x|^2}$$

$$= \frac{\sum_{j=1}^3 \partial_j x_j}{|x|} - \frac{\sum_{j=1}^3 x_j \frac{1}{2|x|} 2x_j}{|x|^2}. \tag{3.168}$$

With $\partial_j x_j = 1$, and $\sum_{j=1}^3 x_j^2 = |x|$ the equality becomes:

$$\mathcal{G} = \frac{\sum_{j=1}^3 1}{|x|} - \frac{1}{|x|} = \frac{2}{|x|}.$$

Since $|x| = r$, we are done. $\square$

Let us consider $|x| = r$ and $\partial_r := \partial_\nu$, then we have:

$$\partial_r = \nu \cdot \nabla = \frac{x}{|x|} \cdot \nabla = \sum_{j=1}^{3} \frac{x_j}{|x|} \partial_j. \tag{3.169}$$

Then the Corollary (3.5.5) becomes:

$$\triangle_{\mathbf{R}^3}(f/_{\mathcal{S}}) = \triangle_{\mathcal{S}}(\, f/_{\mathcal{S}}) + \frac{2}{r}\,\partial_r f/_{\mathcal{S}} + \partial_r^2 f/_{\mathcal{S}}. \tag{3.170}$$

We can write the Laplace operator on the sphere of radius r in terms of the Laplace operator on the unit sphere (see also Example 3, formula 3.188):

$$\triangle_{S_r^2}(f(rx)) = \frac{1}{r^2} \triangle_{S_1^2}(f(rx)), \qquad x \in S_1^2, \tag{3.171}$$

$$\triangle_{\mathbf{R}^3}(f/_{\mathcal{S}}) = \partial_r^2 f/_{\mathcal{S}} + \frac{2}{r}\partial_r f/_{\mathcal{S}} + \frac{1}{r^2}\triangle_{S^2} f/\mathcal{S}. \tag{3.172}$$

This reproves the formula for Laplace operator in spherical polar coordinates on $\mathbf{R}^3$ from [Ta], page 338. We now return to the case of a general surface.

Corollary 3.5.7. *For any $\mathcal{C}^1$ vector-valued function $\vec{u}$, defined in a neighborhood of $\mathcal{S}$, the following identities hold:*

$$(div\ \vec{u})/_{\mathcal{S}} = div_{\mathcal{S}}(\pi\vec{u}/_{\mathcal{S}}) + \mathcal{G}\langle \vec{u}/_{\mathcal{S}},\ \nu\rangle + \langle (\nabla_\nu \vec{u})/_{\mathcal{S}},\ \nu\rangle. \tag{3.173}$$

Proof. To prove the identity, let us recall first the last line in formula (3.149):

$$div\ \vec{u} - \partial_\nu\ \langle \nu,\ \vec{u}\rangle - \langle \mathcal{R}\vec{u},\ \nu\rangle = div(\pi\vec{u}). \tag{3.174}$$

Then

$$div\ \vec{u} = \partial_\nu\ \langle \nu,\ \vec{u}\rangle + \langle \mathcal{R}\vec{u},\ \nu\rangle + div(\pi\vec{u}). \tag{3.175}$$

Thus, we just need to show $\partial_\nu\ \langle \nu,\ \vec{u}\rangle = \nabla\langle \nu,\ \vec{u}\rangle$. So

$$\begin{aligned}
\partial_\nu\ \langle \nu,\ \vec{u}\rangle &= \nu \cdot \nabla \left(\sum_{k=1}^{n} u_k \nu_k\right) = \sum_{j=1}^{n} \nu_j \partial_j \left(\sum_{k=1}^{n} u_k \nu_k\right) \\
&= \sum_{j,\,k=1}^{n} \left[\nu_j \nu_k \partial_j u_k + \nu_j u_k \partial_j \nu_k\right] = \sum_{j,\,k=1}^{n} \left[\nu_k \nu_j \partial_j u_k + u_k \nu_j \partial_j \nu_k\right] \\
&= \sum_{k=1}^{n} \left[\nu_k \left(\nu \cdot \nabla u_k\right) + u_k \left(\nu \cdot \nabla \nu_k\right)\right],
\end{aligned} \tag{3.176}$$

with $\nu \cdot \nabla u_k = \partial_\nu u_k$, and $\nu \cdot \nabla \nu_k = 0$ on $\mathcal{S}$ we have:

$$\partial_\nu\ \langle \nu,\ \vec{u}\rangle = \sum_{k=1}^{n} \nu_k \left(\partial_\nu u_k\right) = \nabla\langle \nu,\ \vec{u}\rangle. \tag{3.177}$$

Then the desired identity becomes:

$$(div\,\vec{u})/_{\mathcal{S}} = div(\pi\vec{u})/_{\mathcal{S}} + \mathcal{G}\langle\vec{u},\ \nu\rangle/_{\mathcal{S}} + \langle\nu,\ \nabla_\nu\vec{u}\rangle/_{\mathcal{S}}. \tag{3.178}$$

Next let us look at some applications of the properties of tangential operators to a surface $\mathcal{S}$:

Example 1. Let $\mathcal{S}$ be the xy-plane in $\mathbf{R}^3$. Then

$$\triangle_{\mathcal{S}} = \partial_x^2 + \partial_y^2, \qquad \triangle_{\mathbf{R}^3} = \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{3.179}$$

$$\mathcal{G} = div\,\nu = 0, \qquad \partial_\nu = (0,\ 0,\ 1)\cdot\nabla = \partial_z, \qquad \partial_\nu^2 = \partial_z^2. \tag{3.180}$$

Applying Corollary (3.5.5) we have:

$$\partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_x^2 + \partial_y^2) + 0 + \partial_z^2. \tag{3.181}$$

Example 2. Let $\mathcal{S} = \mathcal{S}_r^2$ with $r > 0$. Then $\nu(x) := \frac{x}{|x|}$ for any $x \in \mathbf{R}^3\backslash\{0\}$. Let $\omega \in\ S^2$, then

$$f(x) = f(r\omega) \qquad\qquad \text{for } x = r\omega, \quad r = |x|. \tag{3.182}$$

The partial derivative with respect to r becomes:

$$\begin{aligned}\partial_r[f(r\omega)] &= \partial_r[f(r\omega_1,\ r\omega_2,\ \dots,\ r\omega_n)]\\ &= (\partial_1 f)(r\omega)\cdot\omega_1 + \ \dots\ + (\partial_n f)(r\omega)\cdot\omega_n = \nabla f(r\omega)\cdot\omega.\end{aligned} \tag{3.183}$$

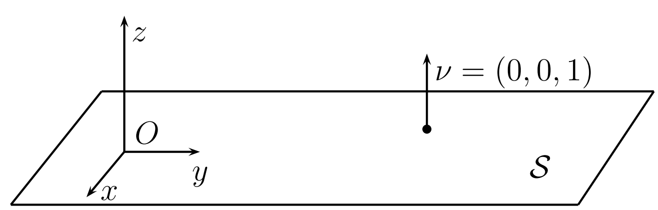

Figure 3.4: The surface $\mathcal{S}$ with the normal ν.

$$\partial_r f = \nabla f\cdot\frac{x}{|x|} = \frac{x}{|x|}\cdot\nabla f. \tag{3.184}$$

$$\partial_r = \sum_{j=1}^{n}\frac{x_j}{|x|}\partial_j. \tag{3.185}$$

Example 3.

$$\left(\triangle_{\mathbf{R}^n} f\right)/_{\mathcal{S}} = \triangle_{\mathcal{S}}\left(f/_{\mathcal{S}}\right) + \mathcal{G}\left(\partial_\nu f\right)/_{\mathcal{S}} + \left(\partial_\nu^2 f\right)/_{\mathcal{S}}. \tag{3.186}$$

For $\mathcal{S} = S_r^2$, $x \in S_r^2$, $x = r\,\omega$:

$$\left(\triangle_{\mathbf{R}^3} f\right)(x) = \triangle_{S_r^2}\left(f/_{S_r^2}\right)(x) + \frac{2}{r}\left(\partial_r f\right)(x) + (\partial_r^2 f)(x). \tag{3.187}$$

$$\triangle_{S_r^2}\left(f/_{S_r^2}\right) = \frac{1}{r^2}\triangle_{S^2}(f(r)). \tag{3.188}$$

$$\left(\triangle_{\mathbf{R}^3} f\right)(x) = \frac{\partial^2}{\partial r^2}[f(r\omega)] + \frac{2}{r}\frac{\partial}{\partial r}[f(r\omega)] + \frac{1}{r^2}\triangle_{S^2} f(r\,\cdot)](\omega), \tag{3.189}$$

where $f(r\,\cdot)$ is a function defined on S^2.

Example 4. Maxwell System

Let $\vec{E}$ and $\vec{H}$ be two vectors and $\partial\Omega = \mathcal{S}$, then

$$\begin{cases} \nabla \times \vec{E} - k\,\vec{H} = 0 \quad \text{in} \quad \Omega \\ \nabla \times \vec{H} + k\,\vec{E} = 0 \quad \text{in} \quad \Omega \\ \quad \nu \times \vec{E} = \vec{f}, \end{cases} \tag{3.190}$$

with $\vec{f}$ tangential, given on $\mathcal{S}$ (in other words $\nu \cdot \vec{f} = 0$ on $\mathcal{S}$). Then

$$\pi\vec{E} = -\nu \times (\nu \times \vec{E}) = -\nu \times \vec{f}, \tag{3.191}$$

and by Remark 3.4.5

$$\pi\vec{f} = -\nu \times (\nu \times \vec{f}) = \vec{f} - (\nu \cdot \vec{f})\nu = \vec{f}. \tag{3.192}$$

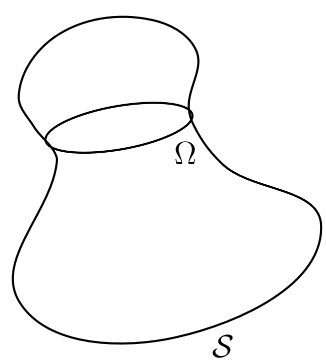

Figure 3.5: The surface $\mathcal{S} = \partial\Omega$ (The Maxwell System).

Next let us write $div_{\mathcal{S}}\vec{f}$:

$$div_{\mathcal{S}}\vec{f} = div_{\mathcal{S}}(\pi\vec{f}) = div(\vec{f})/_{\mathcal{S}} + \mathcal{G}\,\vec{f} \cdot \nu + \langle \partial_\nu \vec{f},\, \nu \rangle$$

$$= div(\nu \times \vec{E})/_{\mathcal{S}} + \partial_\nu[\langle \vec{f},\, \nu \rangle] - \langle \vec{f},\, \partial_\nu \nu \rangle, \tag{3.193}$$

and after we simplified $\vec{f} \cdot \nu = 0$, $\partial_\nu \nu = 0$, and with Remark 3.4.7 the formula becomes:

$$div_{\mathcal{S}} \vec{f} = \vec{E} \cdot (\nabla \times \nu)/_{\mathcal{S}} - \nu \cdot (\nabla \times \vec{E})/_{\mathcal{S}} = -k\, \nu \cdot \vec{H}/_{\mathcal{S}}, \tag{3.194}$$

where $\nabla \times \nu = 0$.

Chapter 4

Geometry of Surfaces Revisited

4.1 The Implicit Function Theorem for Lipschitz Functions

In this subsection we present a version of the classical Implicit Function Theorem in a more general setting than the classical framework of functions of class C^1, namely functions which are only Lipschitz (among other things). Before proceeding with the statement and proof of the main result we have to state the Invariance of Domain Theorem and the definition of Lipschitz function.

Definition 4.1.1. *Call a function $f : U \to V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$,* **Lipschitz (on U)** *if there exists $M > 0$ with the property that*

$$\|f(X) - f(Y)\| \leqslant M\|X - Y\| \qquad \text{for every } X, Y \in U. \tag{4.1}$$

The smallest number M satisfying the above estimate, i.e.,

$$\mathrm{Lip}\,(f; U) := \sup\left\{\frac{\|f(X) - f(Y)\|}{\|X - Y\|} \,:\, X, Y \in U,\, X \neq Y\right\}, \tag{4.2}$$

is called the **Lipschitz constant of f**.

Remark 4.1.2. *A function is Lipschitz if it does not distort distances by more than a fixed (finite) multiplicative factor.*

Theorem 4.1.3 (Invariance of Domain)**.** *Let U be an open subset of $\mathbb{R}^n$ and assume that $f : U \to \mathbb{R}^n$ is an injective, continuous function. Then $V = f(U)$ is open and $f : U \to V$ is a homeomorphism. In particular $f^{-1} : V \to U$ is a continuous function.*

This theorem, due to the Dutch topologist L.E.J. Brouwer, was originally published in 1912; see [1]. The proof (cf., e.g., [6]) uses tools from algebraic topology which are beyond the scope of this paper.

Theorem 4.1.4 (The Implicit Function Theorem for Lipschitz Functions)**.** *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets. Fix $X_0 \in U$, $Y_0 \in V$ and assume that*

$$F : U \times V \longrightarrow \mathbb{R}^m \tag{4.3}$$

is a Lipschitz function for which

$$F(X_0, Y_0) = 0 \tag{4.4}$$

and which has the property that there exists a constant $K > 0$ for which

$$\|F(X, Y_1) - F(X, Y_2)\| \geqslant K\|Y_1 - Y_2\| \quad \textit{for all} \quad (X, Y_j) \in U \times V, \quad j = 1, 2. \tag{4.5}$$

Then there exist an open set $W \subseteq \mathbb{R}^n$ such that $X_0 \in W$, along with a Lipschitz function $\varphi : W \to V$ with the property that $\varphi(X_0) = Y_0$ and

$$\{(X, Y) \in W \times V :\ F(X, Y) = 0\} = \{(X, \varphi(X)) :\ X \in W\}. \tag{4.6}$$

In particular,

$$F(X, \varphi(X)) = 0, \qquad \textit{for all} \quad X \in W. \tag{4.7}$$

Finally, the function φ is unique among all the mappings $\varphi : W \to V$ for which (4.7) *holds.*

Proof. By the Lipschitzianity of F, there exists $M \geqslant 0$ such that

$$\|F(X_1, Y_1) - F(X_2, Y_2)\| \leqslant M\|(X_1, Y_1) - (X_2, Y_2)\|, \tag{4.8}$$

for every (X_1, Y_1), (X_2, Y_2) in $U \times V$.

For some $\varepsilon \in (0, 1)$ small (to be specified later) consider now the function

$$f : U \times V \to \mathbb{R}^{m+n}, \quad f(X, Y) := (X, \varepsilon F(X, Y)) \ \text{ for } \ (X, Y) \in U \times V. \tag{4.9}$$

We will first show that f is a Lipschitz function. To this end, for every (X_1, Y_1), (X_2, Y_2)

in $U \times V$ we estimate (based on (4.9) and (4.8))

$$\begin{aligned}
\|f(X_1, Y_1) - f(X_2, Y_2)\|^2 &= \|(X_1 - X_2, \varepsilon(F(X_1, Y_1) - F(X_2, Y_2))\|^2 \\
&= \|X_1 - X_2\|^2 + \varepsilon^2 \|F(X_1, Y_1) - F(X_2, Y_2)\|^2 \\
&\leqslant \|X_1 - X_2\|^2 + \varepsilon^2 M^2 \|(X_1, Y_1) - (X_2, Y_2)\|^2 \\
&= (1 + \varepsilon^2 M^2)\|X_1 - X_2\|^2 + \varepsilon^2 M^2 \|Y_1 - Y_2\|^2 \\
&\leqslant (1 + M^2)\|(X_1, Y_1) - (X_2, Y_2)\|^2. \qquad (4.10)
\end{aligned}$$

Thus,

$$\begin{gathered}
\|f(X_1, Y_1) - f(X_2, Y_2)\| \leqslant \sqrt{1 + M^2}\|(X_1, Y_1) - (X_2, Y_2)\|, \\
\text{for all} \quad (X_1, Y_1),\ (X_2, Y_2) \in U \times V,
\end{gathered} \qquad (4.11)$$

which shows that f is Lipschitz on $U \times V$, with constant $\leqslant \sqrt{1 + M^2}$. In particular, since Lipschitz functions are continuous functions, f is a continuous function.

Next, we establish a Lipschitz bound from below for f. Specifically, given two points (X_1, Y_1), (X_1, Y_2) in $U \times V$, using (4.5), the triangle inequality, and the fact that, in general, $(a + b)^2 \leqslant 2(a^2 + b^2)$ for every $a, b \in \mathbb{R}$, we write

$$\begin{aligned}
\|Y_1 - Y_2\|^2 &\leqslant K^{-2}\|F(X_1, Y_1) - F(X_1, Y_2)\|^2 \qquad (4.12) \\
&\leqslant K^{-2}(\|F(X_1, Y_1) - F(X_2, Y_2)\| + \|F(X_2, Y_2) - F(X_1, Y_2)\|)^2 \\
&\leqslant 2K^{-2}(\|F(X_1, Y_1) - F(X_2, Y_2)\|^2 + \|F(X_2, Y_2) - F(X_1, Y_2)\|^2).
\end{aligned}$$

Since F is Lipschitz (cf. (4.8)), estimate (4.12) implies

$$\frac{K^2}{2}\|Y_1 - Y_2\|^2 \leqslant \|F(X_1, Y_1) - F(X_2, Y_2)\|^2 + M^2\|X_2 - X_1\|^2 \qquad (4.13)$$

and, further,

$$\frac{K^2\varepsilon^2}{2}\|Y_1 - Y_2\|^2 - M^2\varepsilon^2\|X_2 - X_1\|^2 \leqslant \varepsilon^2\|F(X_1, Y_1) - F(X_2, Y_2)\|^2. \qquad (4.14)$$

Recall now from (4.10) that

$$\varepsilon^2\|F(X_1, Y_1) - F(X_2, Y_2)\|^2 = \|f(X_1, Y_1) - f(X_2, Y_2)\|^2 - \|X_1 - X_2\|^2 \qquad (4.15)$$

which, when used back in (4.14), gives

$$\begin{aligned}
&\frac{K^2\varepsilon^2}{2}\|Y_1 - Y_2\|^2 - M^2\varepsilon^2\|X_2 - X_1\|^2 \\
&\qquad \leqslant \|f(X_1, Y_1) - f(X_2, Y_2)\|^2 - \|X_1 - X_2\|^2. \qquad (4.16)
\end{aligned}$$

In other words, for every (X_1, Y_1), (X_2, Y_2) in $U \times V$,

$$\frac{K^2\varepsilon^2}{2}\|Y_1 - Y_2\|^2 + (1 - M^2\varepsilon^2)\|X_2 - X_1\|^2 \leqslant \|f(X_1, Y_1) - f(X_2, Y_2)\|^2. \tag{4.17}$$

Next we fix ε such that $0 < \varepsilon < M^{-1}$ and take $C_0 := \sqrt{\min\left\{\frac{K^2\varepsilon^2}{2}, 1 - \varepsilon^2 M^2\right\}}$. Thus, $C_0 > 0$ is well-defined and

$$C_0^2(\|X_1 - X_2\|^2 + \|Y_1 - Y_2\|^2) \leqslant \|f(X_1, Y_1) - f(X_2, Y_2)\|^2. \tag{4.18}$$

Taking square-roots we obtain

$$\begin{gathered} C_0\|(X_1, Y_1) - (X_2, Y_2)\| \leqslant \|f(X_1, Y_1) - f(X_2, Y_2)\|, \\ \text{for all } \ (X_1, Y_1), (X_2, Y_2) \in U \times V. \end{gathered} \tag{4.19}$$

This is the desired Lipschitz lower bound for f alluded to earlier. In particular, this shows that f is injective, and (since, f is also continuous), by the Invariance of Domains Theorem (cf. Theorem 4.1.3) the following properties hold:

$$\mathcal{O} := f(U \times V) \text{ is an open subset of } \mathbb{R}^{m+n}, \tag{4.20}$$

$$f : U \times V \to \mathcal{O} \text{ is bijective (i.e., one-to-one and onto)}, \tag{4.21}$$

$$\text{the inverse function } f^{-1} : \mathcal{O} \to U \times V \text{ is continuous.} \tag{4.22}$$

In addition, (4.4) and (4.9) ensure that

$$(X_0, Y_0) \in U \times V \Longrightarrow (X_0, 0) = f(X_0, Y_0) \in \mathcal{O} \ \text{ and } \ f^{-1}(X_0, 0) = (X_0, Y_0). \tag{4.23}$$

Next, consider the coordinate projection functions

$$\begin{aligned} \pi_1 &: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n, \quad \pi_1(X, Y) := X, \\ \pi_2 &: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m, \quad \pi_2(X, Y) := Y, \end{aligned} \tag{4.24}$$

so that $Z = (\pi_1(Z), \pi_2(Z))$ for every $Z \in \mathbb{R}^n \times \mathbb{R}^m$. Define

$$W := B(X_0, r) \subset \mathbb{R}^n, \tag{4.25}$$

where $r > 0$ is so small that

$$\begin{gathered} W \subseteq U, \quad X \in W \Longrightarrow (X, 0) \in \mathcal{O}, \\ \pi_2(f^{-1}(X, 0)) \in V \ \text{ for every } \ X \in W. \end{gathered} \tag{4.26}$$

Since X_0 is contained in U which is open, $(X_0, 0)$ is contained in $\mathcal{O}$ which is open, $\pi_2(f^{-1}(X_0, 0)) = Y_0$ is contained in V which is open, and since both π_2 and f^{-1} are continuous functions, this is certainly possible. After this preamble, introduce

$$\varphi : W \longrightarrow V, \qquad \varphi(X) := \pi_2(f^{-1}(X, 0)) \quad \text{for } \ X \in W. \tag{4.27}$$

It follows that φ is well-defined, and since $f^{-1}(X_0, 0) = (X_0, Y_0)$ then $\varphi(X_0) = Y_0$.

Going further, we may invoke (4.21) and (4.26) in order to write

$$
\begin{aligned}
X \in W \Longrightarrow (X,0) \in \mathcal{O} \;\; \Longrightarrow \\
(X,0) &= f(f^{-1}(X,0)) \qquad (4.28)\\
&= f(\pi_1(f^{-1}(X,0)),\ \pi_2(f^{-1}(X,0)))\\
&= (\pi_1(f^{-1}(X,0)),\ \varepsilon\, F(\pi_1(f^{-1}(X,0)), \varphi(X))),
\end{aligned}
$$

where the last line uses the definitions of f and φ from (4.9) and (4.27), respectively. Thus,

$$\pi_1(f^{-1}(X,0)) = X \quad \text{and} \quad F(\pi_1(f^{-1}(X,0)), \varphi(X)) = 0, \quad \forall\, X \in W. \qquad (4.29)$$

In concert, these entail $F(X, \varphi(X)) = 0$ for every $X \in W$, proving (4.7). In turn, this justifies the inclusion

$$\{(X, \varphi(X)) :\ X \in W\} \subseteq \{(X,Y) \in W \times V :\ F(X,Y) = 0\}. \qquad (4.30)$$

Thus, as far as (4.6) is concerned, there remains to prove that

$$\{(X,Y) \in W \times V :\ F(X,Y) = 0\} \subseteq \{(X, \varphi(X)) :\ X \in W\}. \qquad (4.31)$$

However, if the pair $(X,Y) \in W \times V$ is such that $F(X,Y) = 0$ then we have $X \in W$ and $f(X,Y) = (X,0) \in \mathcal{O}$. Since $(X,Y) \in U \times V$, we may conclude that $(X,Y) = f^{-1}(X,0)$ which implies $Y = \pi_2(f^{-1}(X,0))$. Thus, by definition, we obtain $\varphi(X) = Y$, proving (4.31).

Let us now check that the function φ in (4.27) is Lipschitz. For this, given any $X_1, X_2 \in W$ we note that $(X_j, \varphi(X_j)) \in W \times V \subseteq U \times V$, for $j = 1, 2$, so we may write

$$f(X_j, \varphi(X_j)) = (X_j, \varepsilon F(X_j, \varphi(X_j))) = (X_j, 0), \qquad j = 1, 2, \qquad (4.32)$$

where the last equality is based on (4.7). Having established this, we then estimate, upon recalling (4.19),

$$
\begin{aligned}
C_0\|\varphi(X_1) - \varphi(X_2)\| &\leqslant C_0\|(X_1, \varphi(X_1)) - (X_2, \varphi(X_2))\|\\
&\leqslant \|f(X_1, \varphi(X_1)) - f(X_2, \varphi(X_2))\|\\
&= \|(X_1, 0) - (X_2, 0)\| = \|X_1 - X_2\|. \qquad (4.33)
\end{aligned}
$$

Thus,

$$\|\varphi(X_1) - \varphi(X_2)\| \leqslant C_0^{-1}\|X_1 - X_2\| \quad \text{for every} \quad X_1, X_2 \in W, \tag{4.34}$$

proving that φ in (4.27) is indeed a Lipschitz function.

Finally, we are left with showing that if $\varphi, \psi : W \to V$ are two functions for which $F(X, \varphi(X)) = F(X, \psi(X)) = 0$ for every $X \in W$, then necessarily $\varphi = \psi$. Indeed, this property and (4.5) allow us to write

$$K\|\varphi(X) - \psi(X)\| \leqslant \|F(X, \varphi(X)) - F(X, \psi(X))\| = 0, \quad \forall\, X \in W, \tag{4.35}$$

so $\varphi(X) = \psi(X)$ for every $X \in W$. This concludes the proof of the theorem. □

4.2 The Cross Product of $n-1$ Vectors in $\mathbb{R}^n$

Let

$$\mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0), \quad \mathbf{k} = (0,0,1) \tag{4.36}$$

be the standard orthonormal basis in $\mathbb{R}^3$. Given two vectors

$$v_1 = (v_{11}, v_{12}, v_{13}) \in \mathbb{R}^3 \quad \text{and} \quad v_2 = (v_{21}, v_{22}, v_{23}) \in \mathbb{R}^3,$$

their cross product is defined as the formal determinant

$$v_1 \times v_2 = \det \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix} \tag{4.37}$$

with the understanding that this determinant is expanded along the last row, i.e.,

$$v_1 \times v_2 = (v_{12}v_{23} - v_{13}v_{22})\mathbf{i} - (v_{11}v_{23} - v_{13}v_{21})\mathbf{j} + (v_{11}v_{22} - v_{12}v_{21})\mathbf{k}. \tag{4.38}$$

The first task for us in this subsection is to define a higher dimensional version of the cross product. Compared with the case of two vectors in $\mathbb{R}^3$, our version of the cross product in $\mathbb{R}^n$ (where $n \geqslant 3$) will, this time, involve $n-1$ vectors. Recall that, as before, $\mathbf{e}_1, ..., \mathbf{e}_n$ are the vectors of the standard orthonormal basis in $\mathbb{R}^n$.

Definition 4.2.1. *Assume that $n \geqslant 3$. Consider $n-1$ vectors in $\mathbb{R}^n$, denoted by $v_1 = (v_{11}, ..., v_{1n}), \dots, v_{n-1} = (v_{n-1\,1}, ..., v_{n-1\,n})$. Their* **cross product** *is then defined as*

$$v_1 \times v_2 \times \cdots \times v_{n-1} := \det \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \dots & \vdots \\ v_{n-1\,1} & v_{n-1\,2} & \dots & v_{n-1\,n} \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \end{pmatrix}, \tag{4.39}$$

where the determinant is understood as computed by formally expanding it with respect to the last row, the result being a vector in $\mathbb{R}^n$. More precisely,

$$\begin{aligned} & v_1 \times ... \times v_{n-1} \\ & := \sum_{j=1}^{n} (-1)^{j+1} \det \begin{pmatrix} v_{11} & ... & v_{j-1} & v_{j+1} & ... & v_{1n} \\ \vdots & ... & \vdots & \vdots & ... & \vdots \\ v_{n-1\,1} & ... & v_{n-1\,j-1} & v_{n-1\,j+1} & ... & v_{n-1\,n} \end{pmatrix} \mathbf{e}_j. \end{aligned} \tag{4.40}$$

Below we summarize some of the main properties of the cross product.

Proposition 4.2.2. *Assume that $n \geqslant 3$. Then the cross product introduced in Defini-*

tion 4.2.1 enjoys the following properties:

(i) $(v_1 \times v_2 \times \cdots \times v_{n-1}) \cdot v_n$ *is the (oriented) volume of the parallelopiped spanned by the vectors* $v_1, \ldots, v_n$ *in* $\mathbb{R}^n$.

(ii) The vector $v_1 \times v_2 \times \cdots \times v_{n-1}$ *is perpendicular to each vector* $v_1, \ldots, v_{n-1}$.

(iii) If A *is an* $n \times n$ *invertible matrix and* $v_1, \ldots, v_{n-1}$ *are* $n-1$ *vectors in* $\mathbb{R}^n$

$$Av_1 \times \cdots \times Av_{n-1} = (\det A)(A^{-1})^\top (v_1 \times \cdots \times v_{n-1}), \tag{4.41}$$

where, as in the past, "$\top$*" stands for transposition of matrices.*

(iv) If $\mathcal{R}$ *is a rotation in* $\mathbb{R}^n$, *then*

$$\|\mathcal{R}v_1 \times \cdots \times \mathcal{R}v_{n-1}\| = \|v_1 \times \cdots \times v_{n-1}\|. \tag{4.42}$$

(v) For every permutation σ *of the set* $\{1, ..., n-1\}$, *one has*

$$v_{\sigma(1)} \times v_{\sigma(2)} \times \cdots \times v_{\sigma(n-1)} = \operatorname{sign}(\sigma)\, v_1 \times v_2 \times \cdots \times v_{n-1}. \tag{4.43}$$

(vi) The cross product is linear in each of the individual vectors involved. That is, if $j \in \{1, ..., n-1\}$,

$$\begin{aligned} & v_1 \times v_2 \times \ldots v_{j-1} \times (w_j + w_j') \times v_j \cdots \times v_{n-1} \\ & \quad = v_1 \times v_2 \times \ldots v_{j-1} \times w_j \times v_j \cdots \times v_{n-1} \\ & \qquad + v_1 \times v_2 \times \ldots v_{j-1} \times w_j' \times v_j \cdots \times v_{n-1}, \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} & v_1 \times v_2 \times \ldots v_{j-1} \times (\lambda v_j) \times v_j \cdots \times v_{n-1} \\ & \quad = \lambda\, v_1 \times v_2 \times \ldots v_{j-1} \times v_j \times v_j \cdots \times v_{n-1}. \end{aligned} \tag{4.45}$$

(vii) If $A = \big(a_{jk}\big)_{1 \leqslant j,k \leqslant n-1} \in \mathcal{M}_{(n-1)\times(n-1)}$, *then*

$$\Big(\sum_{k=1}^{n-1} a_{1k} v_k\Big) \times \cdots \times \Big(\sum_{k=1}^{n-1} a_{n-1\,k} v_k\Big) = (\det A)\, v_1 \times v_2 \times \cdots \times v_{n-1}. \tag{4.46}$$

(viii) One has $\|v_1 \times \cdots \times v_{n-1}\| \leqslant \|v_1\| \cdots \|v_n\|$, *for every* $v_1, ..., v_{n-1} \in \mathbb{R}^n$.

(ix) For every $v_1, ..., v_{n-1} \in \mathbb{R}^n$, one has

$$\|v_1 \times \cdots \times v_{n-1}\| = \sqrt{\det (V^\top V)}, \tag{4.47}$$

where $V := (v_1\, v_2 \ldots v_{n-1}) \in \mathcal{M}_{n\times(n-1)}$ is the matrix whose columns are $v_1, ..., v_{n-1}$.

Proof. From (4.39) we have that

$$v_1 \times \ldots \times v_{n-1} = \sum_{j=1}^{n} (-1)^{j+1} \det A_j \mathbf{e}_j, \tag{4.48}$$

where

$$A_j := \begin{pmatrix} v_{11} & \ldots & v_{1\,j-1} & v_{1\,j+1} & \ldots & v_{1n} \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ v_{n-1\,1} & \ldots & v_{n-1\,j-1} & v_{n-1\,j+1} & \ldots & v_{n-1\,n} \end{pmatrix} \tag{4.49}$$

Then by (4.48) and (4.39),

$$\begin{aligned} (v_1 \times \ldots \times v_{n-1}) \cdot v_n &= \left[\sum_{j=1}^{n} (-1)^{j+1} (\det A_j)\mathbf{e}_j\right] \cdot v_n = \sum_{j=1}^{n} (-1)^{j+1} v_{nj} \det A_j \\ &= \det \begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & \ldots & \vdots \\ v_{n-1\,1} & \ldots & v_{n-1\,n} \\ v_{n1} & \ldots & v_{nn} \end{pmatrix}. \end{aligned} \tag{4.50}$$

As in (4.39), this determinant is to be expanded by the last row. Moreover, this determinant is known to be the oriented volume of the parallelopiped spanned by the vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$. This finishes the proof of (i).

For (ii), observe that for every fixed $j \in \{1, 2, \ldots, n-1\}$, we have by (4.50)

$$(v_1 \times \ldots \times v_{n-1}) \cdot v_j = \det \begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & & \vdots \\ v_{j1} & \ldots & v_{jn} \\ \vdots & & \vdots \\ v_{n-1\,1} & \ldots & v_{n-1\,n} \\ v_{j1} & \ldots & v_{jn} \end{pmatrix} = 0, \tag{4.51}$$

since the above matrix has two identical rows. This finishes (ii).

Turning our attention to (iii), fix $v_1, ..., v_{n-1} \in \mathbb{R}^n$ and let $v_n \in \mathbb{R}^n$ be arbitrary. Using that if $A \in \mathcal{M}_{m\times n}$ we have that

$$(AX) \cdot Y = (A^\top Y) \cdot X \quad \text{for every} \quad X \in \mathbb{R}^n \quad \text{and} \quad Y \in \mathbb{R}^m, \tag{4.52}$$

we may write

$$(A^\top (Av_1 \times \ldots \times Av_{n-1})) \cdot v_n = (Av_1 \times \ldots \times Av_{n-1}) \cdot (Av_n), \tag{4.53}$$

so that, by (4.50), we have

$$\begin{aligned}
(Av_1 \times \ldots \times Av_{n-1}) \cdot (Av_n) &= \det \begin{pmatrix} \sum_{i=1}^{n} v_{1i} a_{1i} & \ldots & \sum_{i=1}^{n} v_{1i} a_{ni} \\ \vdots & & \vdots \\ \sum_{i=1}^{n} v_{ni} a_{1i} & \ldots & \sum_{i=1}^{n} v_{ni} a_{ni} \end{pmatrix} \\
&= \det \left[\begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \ldots & v_{nn} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \ldots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix} \right]. \\
&= \det \begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \ldots & v_{nn} \end{pmatrix} \cdot \det(A^\top) \\
&= (\det A)(v_1 \times \ldots \times v_{n-1}) \cdot v_n. \qquad (4.54)
\end{aligned}$$

Hence,

$$(A^\top(Av_1 \times \ldots \times Av_{n-1})) \cdot v_n = (\det A)(v_1 \times \ldots \times v_{n-1}) \cdot v_n \text{ for all } v_n \in \mathbb{R}^n. \qquad (4.55)$$

In concert with (4.55), and the fact that for any vectors $X, Y \in \mathbb{R}^n$,

$$X = Y \iff X \cdot Z = Y \cdot Z \quad \forall\, Z \in \mathbb{R}^n. \qquad (4.56)$$

then forces

$$A^\top(Av_1 \times \ldots \times Av_{n-1}) = (\det A)(v_1 \times \ldots \times v_{n-1}) \text{ for all } v_1, \ldots, v_{n-1} \in \mathbb{R}^n. \qquad (4.57)$$

Lemma 4.2.3. *One has* $(A^{-1})^\top = (A^\top)^{-1}$ *for every* $A \in \mathcal{M}_{n\times n}$ *invertible.*

Proof. We need to show $(A^{-1})^\top A^\top = I_{n\times n}$ and $A^\top(A^{-1})^\top = I_{n\times n}$. To this end, write

$$(A^{-1})^\top A^\top = (AA^{-1})^\top = (I_{n\times n})^\top = I_{n\times n}. \qquad (4.58)$$

The same type of argument shows $A^\top(A^{-1})^\top = I_{n\times n}$. Applying $(A^\top)^{-1}$ to both sides of (4.57) and using Lemma 4.2.3 gives

$$Av_1 \times \ldots \times Av_{n-1} = (\det A)(A^{-1})^\top(v_1 \times \ldots \times v_{n-1}).$$

This concludes (iii) of Proposition 4.2.2.

Turning to the proof of (iv) we recall that, by (iii),

$$\mathcal{R}v_1 \times \ldots \times \mathcal{R}v_{n-1} = (\det \mathcal{R})(\mathcal{R}^{-1})^\top(v_1 \times \ldots \times v_{n-1}). \qquad (4.59)$$

as desired. ☐

Remark 4.2.4. *If $\mathcal{R}:\mathbb{R}^n \to \mathbb{R}^n$ is a rotation about the origin, then $\mathcal{R}^\top = \mathcal{R}^{-1}$, $|\det \mathcal{R}| = 1$. Also, show that any rotation is distance-preserving and angle-preserving, which comes down to*

$$\mathcal{R}(X)\cdot\mathcal{R}(Y) = X\cdot Y, \qquad \forall\, X, Y \in \mathbb{R}^n, \tag{4.60}$$

$$\|\mathcal{R}(X)\| = \|X\|, \qquad \forall\, X \in \mathbb{R}^n. \tag{4.61}$$

Taking the norm and using the properties of rotations specified in Remark 4.2.4, this yields

$$\begin{aligned}\|\mathcal{R}v_1 \times \ldots \times \mathcal{R}v_{n-1}\| &= |\det \mathcal{R}|\,\|\mathcal{R}(v_1\times\ldots\times v_{n-1})\| \\ &= \|v_1\times\ldots\times v_{n-1}\|\,. \end{aligned}\tag{4.62}$$

Moving on, (v) and (vi) are direct consequences of (4.40) and standard properties of determinants. In order to prove (vii) we write

$$\begin{aligned}&\Big(\sum_{k=1}^{n-1} a_{1k}v_k\Big)\times\cdots\times\Big(\sum_{k=1}^{n-1} a_{n-1\,k}v_k\Big)\\ &\quad= \sum_{\substack{\sigma\, permutation\\ of\,\{1,\ldots,n-1\}}} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n-1\sigma(n-1)}\, v_{\sigma(1)}\times v_{\sigma(2)}\times\cdots\times v_{\sigma(n-1)}\\ &\quad= \sum_{\substack{\sigma\, permutation\\ of\,\{1,\ldots,n-1\}}} \operatorname{sign}(\sigma)\, a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n-1\sigma(n-1)}\, v_1\times v_2\times\cdots\times v_{n-1}\\ &\quad= (\det A)\, v_1\times v_2\times\cdots\times v_{n-1},\end{aligned}\tag{4.63}$$

as wanted. As far as (viii) is concerned, we have

$$\begin{aligned}\|v_1\times\cdots\times v_{n-1}\| &= \sup_{v_n\in S^{n-1}} |(v_1\times\cdots\times v_{n-1})\cdot v_n| = \sup_{v_n\in S^{n-1}}\left|\det\begin{pmatrix} v_1\\ \vdots\\ v_n\end{pmatrix}\right|\\ &\leqslant \sup_{v_n\in S^{n-1}} [\|v_1\|\cdots\|v_n\|] = \|v_1\|\cdots\|v_{n-1}\|,\end{aligned}\tag{4.64}$$

where for inequality in (4.64) we have used (4.65) from Theorem 4.2.5 below.

Theorem 4.2.5. [`Hadarmard's Inequality`] *Let $X_1,\ldots,X_n \in \mathbb{R}^n$ and suppose the matrix $A \in \mathcal{M}_{n\times n}$ is such that $A = (X_1\, X_2 \ldots X_n)$ or $A = \begin{pmatrix} X_1\\ X_2\\ \vdots\\ X_n\end{pmatrix}$. Then*

$$|\det A| \leqslant \|X_1\|\ldots\|X_n\|. \tag{4.65}$$

Theorem 4.2.6 (Cauchy-Binet Formula)**.** *Let $m, n \in \mathbb{N}$, $n \geqslant m$, and assume that $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times m}$ are two given matrices. For each $S = (k_1, ..., k_m) \in \{1, ..., n\}^m$ increasing, i.e., with $1 \leqslant k_1 < k_2 < \cdots < k_m \leqslant n$, we write A_S for the $m \times m$ matrix whose columns are $\mathrm{col}_{k_1} A, ..., \mathrm{col}_{k_m} A$, and we write B^S for the $m\times m$ matrix whose rows are $\mathrm{row}_{k_1} B, ..., \mathrm{row}_{k_m} B$. Then*

$$\det(AB) = \sum_{\substack{S\in\{1,...,n\}^m \\ S\ increasing}} \det(A_S)\det(B^S). \tag{4.66}$$

Proof. Fix $n, m \in \mathbb{N}$ along with a $m \times n$ matrix $A = (a_{ik})_{i,k}$ and a $n \times m$ matrix $B = (b_{kj})_{k,j}$. For each $S = (k_1, ..., k_m) \in \{1, ..., n\}^m$ we write $A_{(k_1,...,k_m)}$ for A_S and $B^{(k_1,...,k_m)}$ for B^S. In this notation, we have to show that

$$\det(AB) = \sum_{1\leqslant k_1<k_2<\cdots<k_m\leqslant n} \det[A_{(k_1,...,k_m)}]\det[B^{(k_1,...,k_m)}]. \tag{4.67}$$

Since $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, and since the determinant is multilinear in the columns of a matrix, we may write

$$\begin{aligned}
\det(AB) &= \begin{vmatrix} \sum_{k_1=1}^{n} a_{1k_1}b_{k_11} & \cdots & \sum_{k_m=1}^{n} a_{mk_m}b_{k_mm} \\ \vdots & \ddots & \vdots \\ \sum_{k_1=1}^{n} a_{mk_1}b_{k_11} & \cdots & \sum_{k_m=1}^{n} a_{mk_m}b_{k_m1} \end{vmatrix} \\
&= \det\Big[\sum_{k_1=1}^{n} b_{k_11}\,\mathrm{col}_{k_1}A\,,\ \sum_{k_2=1}^{n} b_{k_22}\,\mathrm{col}_{k_2}A\,,\ \cdots\,,\ \sum_{k_m=1}^{n} b_{k_mm}\,\mathrm{col}_{k_m}A\Big] \\
&= \sum_{k_1,...,k_m=1}^{n} b_{k_22}\cdots b_{k_mm}\det\Big[\mathrm{col}_{k_1}A\,,\ \mathrm{col}_{k_2}A\,,\ \cdots,\mathrm{col}_{k_m}A\Big].
\end{aligned} \tag{4.68}$$

By using the fact that $\det\Big[\mathrm{col}_{k_1}A\,,\ \mathrm{col}_{k_2}A\,,\ \cdots,\mathrm{col}_{k_m}A\Big] = 0$ if $(k_1, k_2, ..., k_m)$ contains any repetition, and

$$\det\Big[\mathrm{col}_{k_{\sigma(1)}}A, \cdots, \mathrm{col}_{k_{\sigma(m)}}A\Big] = \mathrm{sign}(\sigma)\det\Big[\mathrm{col}_{k_1}A, \cdots, \mathrm{col}_{k_m}A\Big] \tag{4.69}$$

for every $\sigma \in S_m$, the permutation group on m elements, we may further transform the last expression in (4.68) as

$$\sum_{k_1,\ldots,k_m=1}^{n} \det\left[A_{(k_1,k_2,\ldots,k_m)}\right] b_{k_1 1} b_{k_2 2} \cdots b_{k_m m}$$

$$= \sum_{1\leqslant k_1<\cdots<k_m\leqslant n} \det\left[A_{(k_1,k_2,\ldots,k_m)}\right] \sum_{\sigma\in S_m} \operatorname{sign}(\sigma)\, b_{k_{\sigma(1)}1} b_{k_{\sigma(2)}2} \cdots b_{k_{\sigma(m)}m}$$

$$= \sum_{1\leqslant k_1<k_2<\cdots<k_m\leqslant n} \det\left[A_{(k_1,\ldots,k_m)}\right] \det\left[B^{(k_1,\ldots,k_m)}\right]. \tag{4.70}$$

Hence, (4.67) follows. $\square$

Remark 4.2.7. *Let $m, n \in \mathbb{N}$, $n \geqslant m$, and assume that $A \in \mathcal{M}_{m\times n}$ is an arbitrary matrix. Use the Cauchy-Binet formula (cf. Theorem 4.2.6 and the terminology introduced there) to show that*

$$\det(AA^\top) = \sum_{\substack{S\in\{1,\ldots,n\}^m \\ S\,\textit{increasing}}} [\det(A_S)]^2. \tag{4.71}$$

Finally, consider (ix). From (4.48) we have

$$\|v_1 \times \ldots \times v_{n-1}\| = \sum_{j=1}^{n} \left(\det A_j\right)^2, \tag{4.72}$$

where A_j's are as in (4.49), whereas from (4.71) we have

$$\sum_{j=1}^{n} \left(\det A_j\right)^2 = \det(V^\top V), \tag{4.73}$$

where $V := (v_1\, v_2 \ldots v_{n-1}) \in \mathcal{M}_{n\times(n-1)}$. Now, formula (4.47) follows from this and (4.72). The proof of the proposition is therefore completed. $\square$

Lemma 4.2.8. *Assume that $n \geqslant 3$. Then the vectors $v_1, v_2, \ldots, v_{n-1} \in \mathbb{R}^n$ are linearly independent if and only if $v_1 \times \cdots \times v_{n-1} \neq 0$.*

Proof. If, say, $v_{n-1} = \sum_{j=1}^{n-2} \lambda_j v_j$, then $v_1\times\cdots\times v_{n-1} = v_1\times\cdots\times v_{n-2}\times\left(\sum_{j=1}^{n-2} \lambda_j v_j\right) = 0$. This proves the right-to-left implication. As for the opposite one, if $v_1, v_2, \ldots, v_{n-1}$ are linearly independent then there exists $v_n \in \mathbb{R}^n$ such that $v_1, \ldots, v_{n-1}, v_n$ is a basis in $\mathbb{R}^n$. Using this and part (i) in Proposition 4.2.2, we may write

$$\begin{aligned} 0 \;&\neq\; \text{the (oriented) volume of the parallelopiped spanned by } v_1, \ldots, v_n \\ &=\; (v_1 \times v_2 \times \cdots \times v_{n-1}) \cdot v_n \end{aligned} \tag{4.74}$$

which gives $v_1 \times \cdots \times v_{n-1} \neq 0$ (after using the fact that for any vectors $X, Y \in \mathbb{R}^n$,

$X \cdot X = \|X\|^2$,for all $X \in \mathbb{R}^n$. we have that $X = Y$ implies that $X \cdot Z = Y \cdot Z$ for all $Z \in \mathbb{R}^n$). $\square$

Remark 4.2.9. *Let us also record a variation of Proposition 4.2.2. Assume that $n \geqslant 3$ and that $v_1, v_2, ..., v_{n-1} \in \mathbb{R}^n$ are given vectors, then, for any matrix $A = \left(a_{jk}\right)_{1 \leqslant j,k \leqslant n-2} \in \mathcal{M}_{(n-2)\times(n-2)}$, one has*

$$\Big(\sum_{k=1}^{n-2} a_{1k} v_k\Big) \times \cdots \times \Big(\sum_{k=1}^{n-2} a_{n-2\,k} v_k\Big) \times v_{n-1} = (\det A)\, v_1 \times v_2 \times \cdots \times v_{n-1}. \tag{4.75}$$

4.3 Parametrizations

Definition 4.3.1. *Assume that* $n \geqslant 2$ *and let* $\mathcal{O}$ *be an open subset of* $\mathbb{R}^{n-1}$. *A function*

$$P = (P_1, P_2, \dots, P_n) : \mathcal{O} \longrightarrow \mathbb{R}^n \tag{4.76}$$

is called a C^k **parametrization**, *where* $k \in \mathbb{N} \cup \{\infty\}$, *provided*

$$P : \mathcal{O} \longrightarrow \mathbb{R}^n \text{ is an injective function, of class } C^k, \text{ and} \tag{4.77}$$

$$\operatorname{rank}[DP(u)] = n - 1, \text{ for all } u = (u_1, \dots, u_{n-1}) \in \mathcal{O}, \tag{4.78}$$

where DP *is the Jacobian matrix of* P, *i.e.,*

$$DP(u) = \left(\frac{D(P_1, \dots, P_n)}{D(u_1, \dots, u_{n-1})} \right)(u), \qquad u \in \mathcal{O}. \tag{4.79}$$

Definition 4.3.2. *(i) The vectors* $X_1, ...X_N \in \mathbb{R}^n$ *are called* **linearly independent** *provided for any real numbers* $\lambda_1, ..., \lambda_N$ *the following implication holds:*

$$\lambda_1 X_1 + \cdots \lambda_N X_N = 0 \Longrightarrow \lambda_1 = \cdots \lambda_N = 0. \tag{4.80}$$

The vectors $X_1, ..., X_N \in \mathbb{R}^n$ *are called* **linearly dependent** *provided they are not linearly independent.*

(ii) The **linear span** *of the vectors* $X_1, ..., X_N \in \mathbb{R}^n$ *is the set*

$$\{\lambda_1 X_1 + \cdots + \lambda_N X_N : \lambda_1, ..., \lambda_N \in \mathbb{R}\}. \tag{4.81}$$

(iii) Call $V \subset \mathbb{R}^n$ *a* **linear subspace** *of* $\mathbb{R}^n$ *provided* V *is stable under addition of vectors, as well as multiplication of vectors by real scalars. The* **dimension** *of the linear subspace* V, *denoted* $\dim V$, *is the maximum number of linearly independent vectors in* V.

(iv) The vectors $X_1, ...X_N \in V$ *are called a* **basis** *of the linear subspace* V *of* $\mathbb{R}^n$, *provided they are linearly independent and their linear span is* V *(in which case, one necessarily has* $N = \dim V$*).*

Definition 4.3.3. *Let* $A \in \mathcal{M}_{m\times n}$ *be a given matrix. The maximal number of linearly independent rows of* A *(cf Definition 4.3.2) is called the* **row-rank** *of* A. *Also the maximal number of linearly independent columns of* A *is called the* **column-rank** *of* A.

Remark 4.3.4. *It is useful to note that, when interpreted in the sense of column-rank* *(cf. Definition 4.3.3), condition* (4.78) *translates into saying that for every* $u \in \mathcal{O}$*, there exists* $j \in \{1, 2, \ldots, n\}$ *such that*

$$det\left(\frac{D(P_1, \ldots, P_{j-1}, P_{j+1}, \ldots, P_n)}{D(u_1, \ldots, u_{n-1})}\right)(u) \neq 0, \tag{4.82}$$

or, equivalently,

$$\begin{gathered} \forall\, u \in \mathcal{O} \Longrightarrow \nabla P_1(u), \ldots, \nabla P_{j-1}(u), \nabla P_{j+1}(u), \ldots, \nabla P_n(u) \\ \text{are } n-1 \text{ linearly independent vectors in } \mathbb{R}^{n-1}. \end{gathered} \tag{4.83}$$

Convention. *For a vector-valued function* $g = (g_1, ..., g_n)$ *we agree to abbreviate*

$$(\partial_i g_1, ..., \partial_i g_n) =: \partial_i g. \tag{4.84}$$

On the other hand, when viewed in the sense of row-rank (see Definition 4.3.3), condition (4.78) *becomes equivalent to (recall the convention made in* (4.84)*)*

$$\begin{gathered} \forall\, u \in \mathcal{O} \Longrightarrow \partial_1 P(u), \ldots, \partial_{n-1} P(u) \\ \text{are } n-1 \text{ linearly independent vectors in } \mathbb{R}^{n}. \end{gathered} \tag{4.85}$$

Remark 4.3.5. *It is clear from definition that if* $P : \mathcal{O} \to \mathbb{R}^n$ *is a* C^k *parametrization then, for every open subset* O *of* $\mathcal{O}$*, the restriction* $P|_O : O \to \mathbb{R}^n$ *is also a* C^k *parametrization.*

Remark 4.3.6. *In the case in which* $n \geqslant 3$*, the rank condition* (4.78) *is further equivalent to*

$$\left\|\frac{\partial P}{\partial u_1} \times \frac{\partial P}{\partial u_2} \times \cdots \times \frac{\partial P}{\partial u_{n-1}}\right\| \neq 0, \quad \forall\, u = (u_1, ..., u_{n-1}) \in \mathcal{O}, \tag{4.86}$$

by using Lemma 4.2.8 and (4.85).

Proposition 4.3.7. *Assume that* $n \geqslant 2$ *and that* $\mathcal{O}$ *is an open subset of* $\mathbb{R}^{n-1}$*. Let*

$$P : \mathcal{O} \longrightarrow \mathbb{R}^n \tag{4.87}$$

be a C^k *parametrization, for some* $k \in \mathbb{N} \cup \{\infty\}$*. Furthermore, suppose that* $\mathcal{U}$ *is an open subset of* $\mathbb{R}^{n-1}$ *and* $\varphi : \mathcal{U} \to \mathcal{O}$ *is a* C^k *diffeomorphism. Then*

$$P \circ \varphi : \mathcal{U} \longrightarrow \mathbb{R}^n \tag{4.88}$$

is a also C^k *parametrization.*

Proof. Let us start this proof with a remark:

Remark 4.3.8. *Let* $A \in \mathcal{M}_{m \times n}$ *be a given matrix. Then the following hold.*

(i) The row-rank of A is equal to the column-rank of A. This integer will, henceforth, be referred to as the **rank** *of A and is going to be denoted by* $\operatorname{rank}(A)$*.*

(ii) $\operatorname{rank}(A) = \operatorname{rank}(A^\top)$.

(iii) If $B \in \mathcal{M}_{n\times k}$ and $\operatorname{rank}(B) = n$, *then* $\operatorname{rank}(AB) = \operatorname{rank}(A)$.

(iv) If $C \in \mathcal{M}_{k\times m}$ and $\operatorname{rank}(C) = m$, *then* $\operatorname{rank}(CA) = \operatorname{rank}(A)$.

(v) A matrix $A \in \mathcal{M}_{n\times n}$ is invertible if and only if $\operatorname{rank}(A) = n$.

(vi) $\operatorname{rank}(A) = \dim\{AX : X \in \mathbb{R}^n\} = \dim\{X \in \mathbb{R}^n : AX = 0\}$.

To check the rank condition, use the Chain Rule and Remark 4.3.8. Concretely, since $\operatorname{rank}[D\varphi(w)] = n-1$ for every $w \in \mathcal{U}$, given that φ is a C^1 diffeomorphism, we have

$$\begin{aligned} \operatorname{rank}[D(P\circ\varphi)(w)] &= \operatorname{rank}[DP(\varphi(w))\cdot D\varphi(w)] \\ &= \operatorname{rank}[DP(\varphi(w))] = n-1, \end{aligned} \tag{4.89}$$

as desired. □

Lemma 4.3.9. *Assume that $n \geqslant 2$ and that $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$. Also, suppose that*

$$P : \mathcal{O} \longrightarrow \mathbb{R}^n \tag{4.90}$$

is a C^k parametrization, for some $k \in \mathbb{N}\cup\{\infty\}$. Finally, assume that $U \subseteq \mathbb{R}^n$ is an open set containing $P(\mathcal{O})$, and $F : U \to \mathbb{R}^n$ is a C^k function with $\det(DF) \neq 0$ *on U. Then*

$$F\circ P : \mathcal{O} \longrightarrow \mathbb{R}^n \tag{4.91}$$

is a also C^k parametrization.

Proof. To check the rank condition, use the Chain Rule and Remark 4.3.8. More specifically, since $\operatorname{rank}[DF(X)] = n$ for every $X \in U$, by the Chain Rule, we have

$$\begin{aligned} \operatorname{rank}[D(F\circ P)(u)] &= \operatorname{rank}[(DF)(P(u))\cdot DP(u)] \\ &= \operatorname{rank}[DP(u)] = n-1. \end{aligned} \tag{4.92}$$

as desired. □

Convention. Assume that $n \geqslant 2$. Given a C^1 parametrization $P : \mathcal{O} \to \mathbb{R}^n$, the function $P : \mathcal{O} \to P(\mathcal{O})$ is both injective and surjective, hence bijective. Hereafter, we agree to denote by

$$P^{-1} : P(\mathcal{O}) \to \mathcal{O} \tag{4.93}$$

the inverse of this function.

Definition 4.3.10. *Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be two arbitrary sets and assume that $f : U \to V$ is a given function. Call f an* **open map** *(or* **open mapping***) provided $f(\mathcal{O})$ is a relatively open subset of V whenever $\mathcal{O}$ is a relatively open subset of U.*

Remark 4.3.11. *Suppose that $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ are arbitrary sets and $f : U \to V$ is an invertible function then f is an open map if and only if its inverse, f^{-1}, is continuous.*

Proposition 4.3.12. *If $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$ and $P : \mathcal{O} \to \mathbb{R}^n$ is a C^1 local parametrization, then $P : \mathcal{O} \to P(\mathcal{O})$ is an open map (cf. Definition 4.3.10). As a consequence (cf. Remark 4.3.11), $P^{-1} : P(\mathcal{O}) \to \mathcal{O}$ is a continuous function, so, in fact, $P : \mathcal{O} \to P(\mathcal{O})$ is a homeomorphism.*

Proof. Let us start wirh a remark:

Remark 4.3.13. *Open mappings enjoy the following properties:*

(i) *The compositions of two open mappings is an open mapping.*

(ii) *The restriction of an open mapping to a subset of its domain continues to be an open mapping.*

(iii) *Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be two arbitrary sets and let $\phi : U \to V$ be a given, arbitrary function. Denote by G_ϕ the graph of ϕ, i.e., $G_\phi = \{(X, \phi(X)) : X \in U\}$. Then $\Phi : U \to G_\phi$ defined by $\Phi(X) := (X, \phi(X))$ for every $X \in U$, is an open map.*

(iv) *Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be two arbitrary sets and assume that $f : U \to V$ is a given function. Then f is an open mapping if and only if f is a* **locally open mapping**

(i.e., for every $X \in U$ there exists O, a relatively open subset of U, which contains X and such that $f|_O : O \to V$ is an open mapping).

(v) Suppose $n \in \mathbb{N}$ and $N \subseteq \{1, ..., n\}$ then the coordinate projection map

$$\pi_N : \mathbb{R}^n \longrightarrow \mathbb{R}^{\# N}, \qquad \pi_N(x_1, ..., x_n) := (x_j)_{j \in N}, \tag{4.94}$$

is an open mapping.

Recall from Remark 4.3.13 that being an open mapping is a local property. Therefore, it suffices to show that if $u^* \in \mathcal{O}$ is arbitrary, then there exists an open subset $O \subset \mathbb{R}^{n-1}$ such that $u^* \in O \subseteq \mathcal{O}$ and $P|_O : O \to P(\mathcal{O})$ is an open mapping. We introduce the auxiliary function

$$\widetilde{P} : \mathcal{O} \times \mathbb{R} \to \mathbb{R}^n, \qquad \widetilde{P}(u, t) := P(u) + tv, \quad \forall\, u \in \mathcal{O},\, t \in \mathbb{R}, \tag{4.95}$$

where $v \in \mathbb{R}^n$ is to be chosen later. It is immediate that $\widetilde{P}$ is of class C^1 and a direct computation gives that

$$(D\widetilde{P})(u, t) = (DP(u)\, v) \in \mathcal{M}_{n \times n} \quad \forall\, u \in \mathcal{O},\, t \in \mathbb{R}. \tag{4.96}$$

Since P is a C^1 local parametrization, we have that $\operatorname{rank}[DP(u)] = n - 1$ for all $u \in \mathcal{O}$. Hence, there exists $v \in \mathbb{R}^n$ such that the vectors $\partial_1 P(u^*)$, ..., $\partial_{n-1} P(u^*)$, v, form a basis in $\mathbb{R}^n$. Fix such a vector v and returning with it to $\widetilde{P}$ we see that $(D\widetilde{P})(u^*, 0)$ is an invertible $n \times n$ matrix. By the Inverse Function Theorem, it follows that there exists U open neighborhood of $(u^*, 0)$, with $U \subseteq \mathcal{O} \times \mathbb{R}$ such that $V := \widetilde{P}(U)$ is open in $\mathbb{R}^n$ and $\widetilde{P} : U \to V$ is a C^1 diffeomorphism. Let $O \subseteq \mathbb{R}^{n-1}$ be the set such that

$$O \times \{0\} = U \cap (\mathcal{O} \times \{0\}). \tag{4.97}$$

Then $u^* \in O$ and O is open (note that the mapping $\iota : \mathcal{O}' \to \mathbb{R}^n$, $\iota(x) = (x, 0)$ is continuous and $O = \iota^{-1}(U)$). Also, let $D \subseteq O$ be an arbitrary open set. The goal is to show that $P(D)$ is a relatively open subset of $P(\mathcal{O})$. The set $A := U \cap (D \times \mathbb{R})$ is open in $\mathbb{R}^n$, $A \subseteq U$, and

$$A \cap (\mathbb{R}^{n-1} \times \{0\}) = A \cap (\mathcal{O} \times \{0\}) = D \times \{0\}. \tag{4.98}$$

Hence, the set $W := \widetilde{P}(A)$ is open in $\mathbb{R}^n$ and using (4.95) and (4.98) we can write

$$\begin{aligned} P(D) &= \widetilde{P}(D \times \{0\}) = \widetilde{P}\Big(A \cap (\mathbb{R}^{n-1} \times \{0\})\Big) \\ &= \widetilde{P}\Big(A \cap (\mathcal{O} \times \{0\})\Big) = W \cap \widetilde{P}(\mathcal{O} \times \{0\}) = W \cap P(\mathcal{O}). \end{aligned} \tag{4.99}$$

This proves that $P(D)$ is a relatively open subset of $P(\mathcal{O})$ and the proof of the proposition is complete. ☐

Proposition 4.3.14 (The Structure of Parametrizations). *Assume that $n \geqslant 2$, $\mathcal{O}$, $\mathcal{U}$ are open subsets of $\mathbb{R}^{n-1}$ and that*

$$P : \mathcal{O} \longrightarrow \mathbb{R}^n, \qquad Q : \mathcal{U} \longrightarrow \mathbb{R}^n \tag{4.100}$$

are two C^k local parametrizations, for some $k \in \mathbb{N} \cup \{\infty\}$, with the property that the sets $P(\mathcal{O})$ and $Q(\mathcal{U})$ coincide. Then there exists a C^k diffeomorphism $\varphi : \mathcal{U} \to \mathcal{O}$ with the property that

$$Q = P \circ \varphi. \tag{4.101}$$

Proof. Consider the function $\varphi := P^{-1} \circ Q : \mathcal{U} \to \mathcal{O}$, which is well-defined given that $P(\mathcal{O})$ and $Q(\mathcal{U})$ coincide. We claim that this function is of class C^k. To see this, fix an arbitrary point $X^* \in P(\mathcal{O}) = Q(\mathcal{U})$ and consider $u^* \in \mathcal{O}$ and $w^* \in \mathcal{U}$ such that $P(u^*) = Q(w^*) = X^*$. Also, assume that $j \in \{1, ..., n\}$ is such that

$$\det\left(\frac{D(P_1, \dots, P_{j-1}, P_{j+1}, \dots, P_n)}{D(u_1, \dots, u_{n-1})}\right)(u^*) \neq 0. \tag{4.102}$$

Let $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection mapping defined by

$$\pi'(X) := (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n), \qquad \forall\, X = (x_1, ..., x_n) \in \mathbb{R}^n, \tag{4.103}$$

and consider

$$P' := \pi' \circ P = (P_1, \dots, P_{j-1}, P_{j+1}, \dots, P_n) : \mathcal{O} \longrightarrow \mathbb{R}^{n-1}. \tag{4.104}$$

Given that we are assuming that P is of class C^k, it follows that P' is of class C^k as well. In fact, by virtue of the Inverse Function Theorem (recall that (4.102) holds), there exists an open neighborhood O of u^*, $O \subseteq \mathcal{O}$, with the property that $P'(O)$ is an open neighborhood of $\pi'(P(u^*))$ in $\mathbb{R}^{n-1}$ and $P' : O \to P'(O)$ is a C^k diffeomorphism. Hence,

$$(P')^{-1} : P'(O) \to O \quad \text{ is a function of class } C^k. \tag{4.105}$$

Then

$$P \circ (P')^{-1} : P'(O) \longrightarrow P(O) \ \text{ is well-defined and bijective.} \tag{4.106}$$

We claim that the inverse of this function is $\pi' : P(O) \to P'(O)$. To justify this, we note that $\pi' \circ P \circ (P')^{-1} = P' \circ (P')^{-1}$ is the identity on $P'(O)$, whereas if $X = P(u) \in P(O)$ for some $u \in O$, then

$$\begin{aligned} (P \circ (P')^{-1})(\pi'(X)) &= (P \circ (P')^{-1})(\pi' \circ P)(u) \\ &= (P \circ (P')^{-1} \circ P')(u) = P(u) = X, \end{aligned} \tag{4.107}$$

proving our claim. Since $P = (P \circ (P')^{-1}) \circ P'$ on O, we have

$$P^{-1} = (P')^{-1} \circ (P \circ (P')^{-1})^{-1} = (P')^{-1} \circ \pi' \ \text{ on } \ P(O). \tag{4.108}$$

Observe now that, by Proposition 4.3.12, $P(O)$ is a relatively open subset of $P(\mathcal{O}) = Q(\mathcal{U})$ containing $X^* = Q(w^*)$. Since by Proposition 4.3.12 $Q : \mathcal{U} \to Q(\mathcal{U})$ is continuous, it follows that $Q^{-1}(P(O))$ is a neighborhood of w^* which is mapped by Q into $P(O)$. As a consequence, if $Q_1, ..., Q_n$ are the components of Q then, in $Q^{-1}(P(O))$, we have

$$\begin{aligned} \varphi &= P^{-1} \circ Q = (P')^{-1} \circ \pi' \circ Q \\ &= (P')^{-1} \circ (Q_1, ..., Q_{j-1}, Q_{j+1}, ..., Q_n). \end{aligned} \tag{4.109}$$

Since $(P')^{-1} : P'(O) \to O$ is of class C^k and $Q' := (Q_1, ..., Q_{j-1}, Q_{j+1}, ..., Q_n)$, as function from $Q^{-1}(P(O))$ into $P'(O)$, is of class C^k it follows that φ is of class C^k in $Q^{-1}(P(O))$, a neighborhood of w^*. Given that w^* has been chosen arbitrarily in $\mathcal{U}$ and since the quality of being of class C^k is local in nature, we may therefore conclude that $\varphi : \mathcal{U} \to \mathcal{O}$ is of class C^k on $\mathcal{U}$.

In fact, φ is invertible and its inverse $\varphi^{-1} = Q^{-1} \circ P : \mathcal{O} \to \mathcal{U}$ is, for the same reasons as above, of class C^k. This proves that φ is a C^k diffeomorphism. Clearly, $P \circ \varphi = Q$ on $\mathcal{U}$ and this concludes the proof of the proposition. □

4.4 Surfaces in $\mathbb{R}^n$

Definition 4.4.1. *Given $n \geqslant 2$ and $k \in \mathbb{N} \cup \{\infty\}$, a C^k* **surface** *(or, surface of class C^k) in $\mathbb{R}^n$ is a subset Σ of $\mathbb{R}^n$ with the property that for every $X^* \in \Sigma$ there exists $r > 0$ such that*

$$\Sigma \cap B(X^*, r) = P(\mathcal{O}) \tag{4.110}$$

where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$ and

$$P : \mathcal{O} \longrightarrow \mathbb{R}^n \quad \text{is a } C^k \text{ parametrization.} \tag{4.111}$$

In the sequel, we shall refer to the function P in (4.111) *as a* **local parametrization near** *X^*, and call $\Sigma \cap B(X^*, r)$ a* **parametrizable patch**.

In the case when (4.110) *holds when we formally take $r = +\infty$, i.e., when $\Sigma = P(\mathcal{O})$, we shall call P a* **global parametrization** *of the surface Σ.*

Remark 4.4.2. *A nonempty compact set cannot be homeomorphic with an open set. As a consequence, we observe that a compact C^1 surface in $\mathbb{R}^n$, $n \geqslant 2$, cannot have a global C^1 parametrization.*

Remark 4.4.3. [`Surfaces as Graphs`] *Assume that $n \geqslant 2$ and that $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ is an open set. Also, suppose that $\phi : \mathcal{O} \to \mathbb{R}$ is a function of class C^k, for some $k \in \mathbb{N} \cup \{\infty\}$. Then that the graph of ϕ, i.e.,*

$$G_\phi := \{(x', \phi(x')) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, x' \in \mathcal{O}\} \tag{4.112}$$

is a C^k surface in $\mathbb{R}^n$. In fact, for every fixed $j \in \{1, ..., n\}$,

$$G_{\phi,j} = \{(x_1, ..., x_{j-1}, \phi(x'), x_{j+1}, ..., x_n) : x' = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) \in \mathcal{O}\} \tag{4.113}$$

is a C^k surface in $\mathbb{R}^n$. Conversely, suppose $\Sigma \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a surface of class C^k, $k \in \mathbb{N} \cup \{\infty\}$, and fix an arbitrary point $X^* \in \Sigma$. Then there exist a number $R > 0$, an index $j \in \{1, ..., n\}$, an open set $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ and a function of class C^k, $\phi : \mathcal{O} \to \mathbb{R}$, with the property that

$$\begin{aligned} &\Sigma \cap B(X^*, R) \qquad\qquad (4.114) \\ &\quad = \{(x_1, ..., x_{j-1}, \phi(x'), x_{j+1}, ..., x_n) : \, x' = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) \in \mathcal{O}\}. \end{aligned}$$

Remark 4.4.4. *Let $r > 0$ and $X^* \in \mathbb{R}^n$, $n \geqslant 2$, be given, then $\partial B(X^*, r)$ is a C^∞ surface in $\mathbb{R}^n$.*

Indeed, we may consider

$$\begin{aligned} &P_r : (0,\pi)^{n-2} \times (0, 2\pi) \longrightarrow \mathbb{R}^n \quad \text{given by} \\ &P_r(\varphi_1, \varphi_2, \dots, \varphi_{n-1}) := X^* + (x_1, x_2, \dots, x_n), \end{aligned} \tag{4.115}$$

where

$$\begin{aligned} x_1 &:= r \cos \varphi_1, \\ x_2 &:= r \sin \varphi_1 \cos \varphi_2, \\ x_3 &:= r \sin \varphi_1 \sin \varphi_2 \cos \varphi_3, \\ &\ \vdots \\ x_{n-1} &:= r \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \cos \varphi_{n-1}, \\ x_n &:= r \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \sin \varphi_{n-1}. \end{aligned} \tag{4.116}$$

Using Proposition 4.8.5 we can show that this function is a C^∞ parametrization of a good portion of $\partial B(X^*, r)$.

Definition 4.4.5. *Let Σ be a C^1 surface in $\mathbb{R}^n$, $n \geqslant 2$. The* **algebraic tangent plane** *to Σ at some point $X^* \in \Sigma$ is then defined as*

$$\mathit{\Pi}_{X^*}\Sigma := \text{ the linear span of } (\partial_1 P)(u^*), \dots, (\partial_{n-1} P)(u^*) \tag{4.117}$$

where $P : \mathcal{O} \longrightarrow \mathbb{R}^n$, is a C^1 parametrization near X^* (with $\mathcal{O}$ open subset of $\mathbb{R}^{n-1}$), and $u^* \in \mathcal{O}$ is such that $P(u^*) = X^*$.

Also, the **geometric tangent plane** *to Σ at some point $X^* \in \Sigma$ is taken to be*

$$T_{X^*}\Sigma := X^* + \mathit{\Pi}_{X^*}\Sigma. \tag{4.118}$$

A vector $v \in \mathbb{R}^n$ is called **tangent to the surface** *Σ at the point X^* provided $v \in T_{X^*}\Sigma$.*

Theorem 4.4.6. *Let Σ be a C^1 surface in $\mathbb{R}^n$, $n \geqslant 2$, and assume that $X^* \in \Sigma$. Then the algebraic (hence, also geometric) tangent plane to Σ at X^* is unambiguously defined. That is, the definition of $\mathit{\Pi}_{X^*}\Sigma$ in* (4.117)*, hence also the definition of $T_{X^*}\Sigma$ in* (4.118)*, is independent of the particular local parametrization of Σ near X^*. Furthermore, $\mathit{\Pi}_{X^*}\Sigma$*

is a linear subspace of $\mathbb{R}^n$ *of dimension* $n-1$.

Proof. Let $Q : \mathcal{U} \to \mathbb{R}^n$ be another local parametrization of Σ near X^*, and denote by w^* the (unique) point in $\mathcal{U}$ mapped by Q into X^*. By Proposition 4.3.14, there exists a C^k diffeomorphism φ mapping a small neighborhood of w^* onto a small neighborhood of u^* with the property that

$$\varphi(w^*) = u^* \quad \text{and} \quad Q = P \circ \varphi \quad \text{near} \;\; w^*. \tag{4.119}$$

From this and Chain Rule we then obtain

$$\partial_j Q(w^*) = \sum_{k=1}^{n-1} (\partial_j \varphi_k)(w^*)(\partial_k P)(u^*), \qquad \forall\, j \in \{1, ..., n\}, \tag{4.120}$$

which shows that the linear span of $(\partial_1 Q)(w^*), \dots, (\partial_{n-1} Q)(w^*)$ is included in the linear span of $(\partial_1 P)(u^*), \dots, (\partial_{n-1} P)(u^*)$. Changing the roles of P and Q gives the opposite inclusion, hence, the two linear subspaces of $\mathbb{R}^n$ spanned by these two sets of vectors actually coincide. Hence, the definition of $\mathit{\Pi}_{X^*}\Sigma$ is independent of the local parametrization of the surface near the point X^*.

As for the very last claim in the statement of the theorem, we note that, by definition, $(\partial_1 P)(u^*), \dots, (\partial_{n-1} P)(u^*)$ span $\mathit{\Pi}_{X^*}\Sigma$. Furthermore, by (4.85), these vectors are linearly independent so, in fact,

$$\text{the vectors } \;\partial_1 P(u^*), \dots, \partial_{n-1} P(u^*) \;\text{ form a basis for }\; \mathit{\Pi}_{X^*}\Sigma. \tag{4.121}$$

This shows that $\mathit{\Pi}_{X^*}\Sigma$ is a linear subspace of $\mathbb{R}^n$ of dimension $n-1$. $\square$

Definition 4.4.7. *Assume that* Σ *is a* C^1 *surface in* $\mathbb{R}^n$*,* $n \geqslant 2$*. The* **unit normals** *to* Σ *at some point* $X^* \in \Sigma$ *are the two unit vectors in* $\mathbb{R}^n$ *(which, in fact, are opposite to one another), uniquely determined by the requirement that they are perpendicular to all vectors from* $\mathit{\Pi}_{X^*}\Sigma$*. The common direction of these unit normals will be referred to as the* **normal direction** *to* Σ *at* X^**.*

Lemma 4.4.8. *Assume that* Σ *is a* C^1 *surface in* $\mathbb{R}^n$*, and fix a point* $X^* \in \Sigma$*. Also, let* $\nu = (\nu_1, ..., \nu_n) \in \mathbb{R}^n$ *be a unit normal to* Σ *at* X^**. Then the vectors*

$$t_{jk} := \nu_k \mathbf{e}_j - \nu_j \mathbf{e}_k = (0, ..., 0, -\nu_k, 0, ..., 0, \nu_j, 0, ...0) \in \mathbb{R}^n,\; j, k \in \{1, ..., n\}, \tag{4.122}$$

span $\Pi_{X^*}\Sigma$, the algebraic tangent plane to Σ at X^*.

Proof. Let us check that $t_{jk} \cdot \nu = 0$ for every $j, k \in \{1, ..., n\}$, then that the family (4.122) contains $n-1$ linearly independent vectors. To justify the latter claim, we have to choose $j \in \{1, ..., n\}$ with $\nu_j \neq 0$, and $\{t_{jk} : 1 \leqslant k \leqslant n,\ k \neq j\}$. $\square$

We now discuss how the unit normals to a surface can be expressed in terms of a local parametrization.

Theorem 4.4.9. *Assume that $\Sigma \subset \mathbb{R}^n$ is a C^1 surface in $\mathbb{R}^n$, $n \geqslant 2$. Fix $X^* \in \Sigma$ and pick a local C^1 parametrization $P = (P_1, ..., P_n) : \mathcal{O} \to \mathbb{R}^n$ (where, as always, $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$) of Σ near X^*. Set $u^* := P^{-1}(X^*) \in \mathcal{O}$. Then the two unit normals to the surface Σ at X^* are given by*

$$\nu(X^*) = \pm \frac{\partial_1 P(u^*) \times \partial_2 P(u^*) \times \ldots \times \partial_{n-1} P(u^*)}{\|\partial_1 P(u^*) \times \partial_2 P(u^*) \times \ldots \times \partial_{n-1} P(u^*)\|} \qquad \text{if } n \geqslant 3, \tag{4.123}$$

and

$$\nu(X^*) = \pm \frac{(P_2'(u^*), -P_1'(u^*))}{\|P'(u^*)\|} \qquad \text{if } n = 2. \tag{4.124}$$

Proof. In the case when $n \geqslant 3$, it suffices to observe that, by (ii) in Proposition 4.2.2, $\partial_1 P(u^*) \times \ldots \times \partial_{n-1} P(u^*)$ is perpendicular to any of the vectors $\partial_j P(u^*)$, $j = 1, ..., n-1$. Hence, the unit normals to Σ at $X^* = P(u^*)$ are, in this case, given by (4.123). When $n = 2$, it is clear that $(P_2', -P_1')$ is perpendicular on $P' = (P_1', P_2')$, so the desired conclusion follows. $\square$

Definition 4.4.10. *Assume that Σ is an oriented C^1 surface in $\mathbb{R}^n$, $n \geqslant 2$, and denote by ν the choice of a continuous unit normal to Σ. Call a local parametrization $P : \mathcal{O} \to \mathbb{R}^n$ of Σ near $X^* \in \Sigma$* **positive** *if* (4.123)*, for $n \geqslant 3$, and* (4.124) *for $n = 2$, hold for the choice "plus" of the sign, and call P* **negative** *otherwise.*

Proposition 4.4.11. *Let $\Sigma \subset \mathbb{R}^n$, $n \geqslant 2$, be a surface of class C^1. Assume that $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ is an open set and that $\varphi : \mathcal{O} \to \mathbb{R}$ is a C^1 function with the property that*

$P : \mathcal{O} \to \mathbb{R}^n$, $P(x') = (x', \varphi(x'))$ *for every* $x' \in \mathcal{O}$ *is a local parametrization of* Σ *near* $X^* = P(x'_*) \in \Sigma$, $x'_* \in \mathcal{O}$. *Then*

$$\nu(x', \varphi(x')) = \pm \frac{(-\nabla\varphi(x'),\, 1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}, \qquad \textit{for every } x' \in \mathcal{O}. \tag{4.125}$$

Proof. For every $x' \in \mathcal{O}$, we have that

$$\partial_j P(x') = (0, \dots, 1, 0, \dots, \partial_j\varphi(x')) = \mathbf{e}_j + \partial_j\varphi(x')\mathbf{e}_j \quad \text{for} \quad j = 1, \dots, n-1. \tag{4.126}$$

Thus, if $n \geqslant 3$, by definition,

$$\begin{aligned}
\partial_1 P(x') \times \cdots \times \partial_{n-1} P(x') &= \det \begin{pmatrix} 1 & 0 & \cdots & 0 & \partial_1\varphi(x') \\ 0 & 1 & \cdots & 0 & \partial_2\varphi(x') \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \partial_{n-1}\varphi(x') \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_{n-1} & \mathbf{e}_n \end{pmatrix} \\
&= (-1)^{n-1}(-1)^{n-2}\partial_1\varphi(x')\mathbf{e}_1 + \cdots + (-1)^{2n-1}\partial_{n-1}\varphi(x')\mathbf{e}_{n-1} + (-1)^{2n}\mathbf{e}_n \\
&= (-\nabla\varphi(x'), 1),
\end{aligned} \tag{4.127}$$

for every $x' \in \mathcal{O}$. In particular,

$$\|\partial_1 P(x') \times \cdots \times \partial_{n-1} P(x')\| = \sqrt{1 + \|\nabla\varphi(x')\|^2} \qquad \text{for every } x' \in \mathcal{O}', \tag{4.128}$$

Formula (4.125) now follows from this and Theorem 4.4.9. This completes the case when $n \geqslant 3$. When $n = 2$, (4.125) is a direct consequence of (4.124). □

We would now like to express the two unit normals to a C^1 surface Σ without recourse to the multi-dimensional cross product.

Theorem 4.4.12. *Assume that* $\Sigma \subset \mathbb{R}^n$, $n \geqslant 2$, *is a* C^1 *surface,* $X^* \in \Sigma$, *and suppose that* $P : \mathcal{O} \to \mathbb{R}^n$, *where* $\mathcal{O}$ *is an open subset of* $\mathbb{R}^{n-1}$, *is a local parametrization of* Σ *near* X^*. *Let* $u^* \in \mathcal{O}$ *be the unique point with the property that* $P(u^*) = X^*$. *Then the two unit normals to* Σ *at* X^* *can be described as*

$$\nu(X^*) = \pm \left(\frac{(-1)^{j+1} \det\left(A_j(u^*)\right)}{\left(\sum\limits_{k=1}^{n} \left[\det\left(\frac{\mathrm{D(P_1 \dots \widehat{P}_j \dots P_n)}}{\mathrm{D(u_1 \dots u_{n-1})}} \right)(\mathrm{u}^*) \right]^2 \right)^{\frac{1}{2}}} \right)_{1 \leqslant j \leqslant n}, \tag{4.129}$$

where

$$A_j(u^*) = \begin{pmatrix} \partial_1 P_1(u^*) & ... & \partial_1 P_{j-1}(u^*) & \partial_1 P_{j+1}(u^*) & ... & \partial_1 P_n(u^*) \\ \vdots & ... & \vdots & \vdots & ... & \vdots \\ \partial_{n-1} P_1(u^*) & ... & \partial_{n-1} P_{j-1}(u^*) & \partial_{n-1} P_{j+1}(u^*) & ... & \partial_{n-1} P_n(u^*) \end{pmatrix},$$

with the convention that a "hat" over P_j indicates that this function is omitted for the corresponding enumeration.

Proof. Observe that, when $n \geqslant 3$, (4.39) implies

$$\begin{aligned} \partial_1 P \times \ldots \times \partial_{n-1} P &= \det \begin{pmatrix} \partial_1 P_1 & \partial_1 P_2 & \ldots & \partial_1 P_n \\ \partial_2 P_1 & \partial_2 P_2 & \ldots & \partial_2 P_n \\ \vdots & \vdots & \ldots & \vdots \\ \partial_{n-1} P_1 & \partial_{n-1} P_2 & \ldots & \partial_{n-1} P_n \\ \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n \end{pmatrix} \\ &= \sum_{j=1}^{n} (-1)^{j+1} \det A_j \, \mathbf{e}_j, \end{aligned} \tag{4.130}$$

where, as before,

$$A_j := \begin{pmatrix} \partial_1 P_1 & \ldots & \partial_1 P_{j-1} & \partial_1 P_{j+1} & \ldots & \partial_1 P_n \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ \partial_{n-1} P_1 & \ldots & \partial_{n-1} P_{j-1} & \partial_{n-1} P_{j+1} & \ldots & \partial_{n-1} P_n \end{pmatrix}. \tag{4.131}$$

Consequently,

$$\partial_1 P \times \ldots \times \partial_{n-1} P = \Big((-1)^{j+1} \det A_j \Big)_{1 \leqslant j \leqslant n}, \tag{4.132}$$

and, hence,

$$\| \partial_1 P \times \ldots \times \partial_{n-1} P \| = \left(\sum_{j=1}^{n} \big(\det A_j \big)^2 \right)^{\frac{1}{2}}. \tag{4.133}$$

Given (4.131), the last expression above can also be expressed as

$$\begin{aligned} &\left(\left[det \begin{pmatrix} \partial_1 P_2 & ... & \partial_1 P_n \\ \vdots & ... & \vdots \\ \partial_{n-1} P_2 & ... & \partial_{n-1} P_n \end{pmatrix} \right]^2 + ... + \left[det \begin{pmatrix} \partial_1 P_1 & ... & \partial_1 P_{n-1} \\ \vdots & ... & \vdots \\ \partial_{n-1} P_1 & ... & \partial_{n-1} P_{n-1} \end{pmatrix} \right]^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{j=1}^{n} \left[\det \left(\frac{D(P_1 \ldots \widehat{P_j} \ldots P_n)}{D(u_1 \ldots u_{n-1})} \right) \right]^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{4.134}$$

Having establishes this last formula, we now use (4.123) with the numerator replaced by (4.132) and the denominator replaced by (4.134), when both are evaluated at the point $u^* \in \mathcal{O}$. This proves (4.129) when $n \geqslant 3$. Finally, when $n = 2$, then (4.129) follows straight from (4.124). $\square$

Definition 4.4.13. *A* $(n-1)$*-dimensional* **plane** π *in* $\mathbb{R}^n$ *is a subset of* $\mathbb{R}^n$ *of the form*

$$\pi := \{X \in \mathbb{R}^n : N \cdot (X - X^*) = 0\}, \tag{4.135}$$

for fixed $N \in \mathbb{R}^n \setminus \{0\}$ *and* $X^* \in \mathbb{R}^n$*. The vector* N *is said to be* **normal** *to* π*.*

Remark 4.4.14. *Let* $\Sigma \subset \mathbb{R}^n$*,* $n \geqslant 2$*, be a* C^1 *surface. We may conclude from Theorem 4.4.9 that the geometric tangent plane to* Σ *at the point* $X^* \in \Sigma$ *is a* $(n-1)$*-dimensional plane in the sense of Definition 4.4.13, which has the description*

$$T_{X^*}\Sigma = \{X \in \mathbb{R}^n : \nu(X^*) \cdot (X - X^*) = 0\}, \tag{4.136}$$

for either choice of the unit normal $\nu(X^*)$ *to* Σ *at* X^**.*

Proposition 4.4.15 (Level Surfaces)**.** *Let* $W \subseteq \mathbb{R}^n$*,* $n \geqslant 2$*, be an open set, and assume that* $F : W \to \mathbb{R}$ *is a function of class* C^k*, for some* $k \in \mathbb{N} \cup \{\infty\}$*. Assume that* $X^* \in W$ *is such that*

$$\|\nabla F(X^*)\| \neq 0, \tag{4.137}$$

and set $c := F(X^*) \in \mathbb{R}$*. Then there exists an open neighborhood* $\mathcal{N}$ *of* X^* *in* $\mathbb{R}^n$ *with the property that* $\Sigma := \mathcal{N} \cap F^{-1}(\{c\})$ *is a* C^k *surface, whose unit normals are given by*

$$\nu(X) = \pm \frac{\nabla F(X)}{\|\nabla F(X)\|}, \qquad \forall\, X \in \Sigma. \tag{4.138}$$

As a corollary, if

$$\|\nabla F(X)\| \neq 0 \quad \forall\, X \in F^{-1}(\{c\}), \tag{4.139}$$

then $\Sigma := F^{-1}(\{c\})$ *is a surface of class* C^k *in* $\mathbb{R}^n$*.*

Proof. Condition (4.137) ensures the existence of an index $j_o \in \{1, ..., n\}$ with the property that $\partial_{j_o} F(X^*) \neq 0$. Let $(x_1^*, ..., x_n^*)$ be the components of $X^* \in \mathbb{R}^n$ and select an open neighborhoods $U \subseteq \mathbb{R}^{n-1}$ and $V \subseteq \mathbb{R}$ of $(x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*)$ and $x_{j_o}^*$, respectively, such that $U \times V \subseteq W$.

Then, by the Implicit Function Theorem (applied to the function $F(X) - c$ near $(x_1^*, ..., x_n^*)$), there exist $\rho, \eta > 0$ such that

$$B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho) \subseteq U, \quad (x_{j_o}^* - \eta, x_{j_o}^* + \eta) \subseteq V, \tag{4.140}$$

along with a function of class C^k

$$\varphi : B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho) \longrightarrow (x_{j_o}^* - \eta, x_{j_o}^* + \eta) \tag{4.141}$$

with the property that

$$\{x \in \mathbb{R}^n :\ (x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) \in B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho),$$

$$x_{j_o} \in (x_{j_o}^* - \eta, x_{j_o}^* + \eta) \text{ and } F(x_1, ..., x_n) = c\}$$

$$= \{(x_1, ..., x_{j_o-1}, \varphi(x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n), x_{j_o+1}, ..., x_n) : \tag{4.142}$$

$$(x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) \in B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho)\}.$$

Then

$$\begin{aligned} & P : B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho) \longrightarrow \mathbb{R}^n \\ & P(x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) := \\ & \qquad (x_1, ..., x_{j_o-1}, \varphi(x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n), x_{j_o+1}, ..., x_n), \end{aligned} \tag{4.143}$$

is a function of class C^k, which is injective, and satisfies rank $DP = n-1$ at every point in the ball $B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho)$. Define

$$\mathcal{N} = \{x \in \mathbb{R}^n : (x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) \in B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho),$$

$$\text{and} \quad x_{j_o} \in (x_{j_o}^* - \eta, x_{j_o}^* + \eta)\} \tag{4.144}$$

and note that $\mathcal{N}$ is an open neighborhood of X^* in $\mathbb{R}^n$. Furthermore, with this piece of notation, (4.142) becomes

$$\{X \in \mathcal{N} :\ F(X) = c\} = P\big(B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho)\big), \tag{4.145}$$

i.e.,

$$\Sigma = \mathcal{N} \cap F^{-1}(\{c\}) = P\big(B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho)\big). \tag{4.146}$$

It follows from this and Proposition 4.3.12 that P is a global C^k parametrization of Σ and, hence, Σ is a C^1 surface.

There remains to establish (4.138). To this end, observe that, by design,

$$\forall\, (x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) \in B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho)$$

$$\Longrightarrow F(P(x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n)) = c. \tag{4.147}$$

Thus, for every $j \in \{1, ..., j_o - 1, j_0 + 1, ..., n\}$,

$$\sum_{k=1}^{n} [(\partial_k F) \circ P] \partial_j P_k = 0 \quad \text{on} \quad B((x_1^*, ..., x_{j_o-1}^*, x_{j_o+1}^*, ..., x_n^*), \rho). \tag{4.148}$$

That is, when evaluated at points in $B((x_1^*,...,x_{j_o-1}^*,x_{j_o+1}^*,...,x_n^*),\rho)$, the vector $(\nabla F)\circ P$ is perpendicular on each of the following vectors:

$$\partial_1 P,...,\partial_{j_o-1}P,\partial_{j_0+1}P,...,\partial_n P. \tag{4.149}$$

Since these vectors span precisely the algebraic tangent plane to Σ, it follows that, when evaluated at points in $B((x_1^*,...,x_{j_o-1}^*,x_{j_o+1}^*,...,x_n^*),\rho)$, the vector $(\nabla F)\circ P$ is parallel to the normal to Σ. From this, (4.138) easily follows. $\square$

Example 4.4.16. *Let $n\in\mathbb{N}$ be fixed. Then for each number $c>n$*

$$\Sigma := \Big\{X=(x_1,...,x_n)\in\mathbb{R}^n : \sum_{j=1}^n e^{x_j^2}=c\Big\} \tag{4.150}$$

is a compact C^∞ surface.

Remark 4.4.17. *Suppose that $\mathcal{O}\subseteq\mathbb{R}^{n-1}$, $n\geqslant 2$, is an open set and that $\varphi:\mathcal{O}\to\mathbb{R}$ is a function of class C^1. Using Proposition 4.4.15 we can give another proof (compare with Proposition 4.4.11) of the fact that that the two unit normals to the surface*

$$G_\varphi=\big\{(x',\varphi(x'))\in\mathbb{R}^{n-1}\times\mathbb{R} : x'\in\mathcal{O}\big\} \tag{4.151}$$

at the point $(x'\varphi(x'))$, where $x'\in\mathcal{O}$ is arbitrary, are given by

$$\nu(x',\varphi(x'))=\pm\frac{(\nabla\varphi(x'),-1)}{\sqrt{1+\|\nabla\varphi(x')\|^2}}. \tag{4.152}$$

Indeed, consider the open set $W:=\mathcal{O}\times\mathbb{R}$ in $\mathbb{R}^n$ and the C^1 function

$$F:W\to\mathbb{R},\qquad F(X)=\varphi(x')-x_n,\quad \forall\, X=(x',x_n)\in W, \tag{4.153}$$

and note that $G_\varphi=F^{-1}(\{0\})$ and $\nabla F(X)\neq 0$ for every $X\in W$. Then, according to Proposition 4.4.15, G_φ is a C^1 surface whose two unit normals are given by

$$\begin{aligned}\nu(x',\varphi(x')) &= \pm\frac{\nabla F(x,\varphi(x'))}{\|\nabla F(x,\varphi(x'))\|}=\pm\frac{(\partial_1\varphi(x'),\dots,\partial_{n-1}\varphi(x'),-1)}{\sqrt{1+\sum\limits_{j=1}^{n-1}(\partial_j\varphi(x'))^2}}\\ &= \pm\frac{(\nabla\varphi(x'),-1)}{\sqrt{1+\|\nabla\varphi(x')\|^2}}.\end{aligned} \tag{4.154}$$

Theorem 4.4.18. [`Invariance of Surfaces under Diffeomorphisms`] *Let Σ be a C^k surface in $\mathbb{R}^n$, where $n\geqslant 2$ and $k\in\mathbb{N}\cup\{\infty\}$, and assume that $F:U\to V$ is a C^k*

diffeomorphism, where $U, V \subseteq \mathbb{R}^n$ are two open sets, with $\overline{\Sigma}$ contained in U. Then $\widetilde{\Sigma} := F(\Sigma)$ is also a C^k surface in $\mathbb{R}^n$.

Proof. This is a direct consequence of Lemma 4.3.9 and definitions. $\square$

We next explore how the unit normals to a surface change as the surface Σ is mapped by a diffeomorphism into another surface $\widetilde{\Sigma}$.

Theorem 4.4.19 (Covariance of Unit Normals under Diffeomorphisms)**.** *Let $\Sigma \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a C^1 surface and let ν denote a choice of the unit normal for Σ. Also, suppose that $U, V \subseteq \mathbb{R}^n$ are open sets such that $\overline{\Sigma} \subseteq U$ and that $F : U \to V$ is a C^1 diffeomorphism. Then $\widetilde{\Sigma} := F(\Sigma)$ is a C^1 surface in $\mathbb{R}^n$ (cf. Theorem 4.4.18) whose unit normals are given by*

$$\widetilde{\nu} = \pm\frac{(DF^{-1})^\top(\nu\circ F^{-1})}{\|(DF^{-1})^\top(\nu\circ F^{-1})\|} = \pm\frac{(((DF)^{-1})^\top\circ F^{-1})(\nu\circ F^{-1})}{\|(((DF)^{-1})^\top\circ F^{-1})(\nu\circ F^{-1})\|} \quad \text{on } \widetilde{\Sigma}. \tag{4.155}$$

Proof. Assume first that $n \geqslant 3$. Given that ν and $\widetilde{\nu}$ are defined locally, there is no loss of generality in assuming that Σ has a global parametrization $P : \mathcal{O} \longrightarrow \mathbb{R}^n$ Assuming that this is the case (and viewing F as a $\mathbb{R}^n$-valued function), Lemma 4.3.9 also gives that

$$F\circ P : \mathcal{O} \longrightarrow \mathbb{R}^n \quad \text{is a global parametrization for } \widetilde{\Sigma}. \tag{4.156}$$

Set $A := (DF)\circ P$. Granted (4.156), formula (4.123) gives

$$\begin{aligned}
\pm\widetilde{\nu}\circ(F\circ P) &= \frac{\partial_1(F\circ P)\times\ldots\times\partial_{n-1}(F\circ P)}{\|\partial_1(F\circ P)\times\ldots\times\partial_{n-1}(F\circ P)\|} = \frac{A\,\partial_1 P\times\ldots\times A\,\partial_{n-1}P}{\|A\,\partial_1 P\times\ldots\times A\,\partial_{n-1}P\|} \\
&= \frac{\det A\,(A^{-1})^\top(\partial_1 P\times\ldots\times\partial_{n-1}P)}{|\det A|\,\|(A^{-1})^\top(\partial_1 P\times\ldots\times\partial_{n-1}P)\|} \qquad (4.157) \\
&= \operatorname{sign}(\det A)\,\frac{(A^{-1})^\top\left(\frac{\partial_1 P\times\ldots\times\partial_{n-1}P}{\|\partial_1 P\times\ldots\times\partial_{n-1}P\|}\right)}{\left\|(A^{-1})^\top\left(\frac{\partial_1 P\times\ldots\times\partial_{n-1}P}{\|\partial_1 P\times\ldots\times\partial_{n-1}P\|}\right)\right\|} \\
&= \operatorname{sign}(\det[(DF)\circ P])\,\frac{[((DF)^{-1})^\top\circ P](\nu\circ P)}{\|[((DF)^{-1})^\top\circ P](\nu\circ P)\|} \qquad \text{on } \mathcal{O}.
\end{aligned}$$

Consequently,

$$\widetilde{\nu}\circ F = \pm\frac{((DF)^{-1})^\top\nu}{\|((DF)^{-1})^\top\nu\|} \quad \text{on } \Sigma. \tag{4.158}$$

Composing with F^{-1} and recalling, that by the Inverse Function Theorem

$$(DF)^{-1} = (DF^{-1}) \circ F, \tag{4.159}$$

then (4.155) follows. Finally, the case $n = 2$ is dealt with analogously, except that, in this situation, one works with (4.124). This finishes the prof of the theorem. $\square$

Orientation

Definition 4.4.20. *Call a C^1 surface $\Sigma \subseteq \mathbb{R}^n$* **orientable** *provided there exists a continuous choice of the unit normal, that is, if there exists a continuous function $\nu : \Sigma \to S^{n-1}$ with the property that, for every $X \in \Sigma$, the direction of $\nu(X)$ is the normal direction to Σ at X.*

A continuous choice of the unit normal will be referred to as an **orientation** *for the surface Σ.*

Remark 4.4.21. *Orientability is a genuine demand only at the global level. Indeed, from Theorem 4.4.9, we know that a continuous choice of the unit normal can be made in the neighborhood of any point on a C^1 surface.*

Lemma 4.4.22. *Any given orientable, connected, C^1 surface $\Sigma \subseteq \mathbb{R}^n$ has precisely two orientations.*

Proof. Fix a continuous choice of the unit normal ν to Σ. If $\widetilde{\nu}$ is another continuous choice of the unit normal to Σ, introduce

$$\mathcal{O}_1 := \{X \in \Sigma : \nu(X) = \widetilde{\nu}(X)\}, \qquad \mathcal{O}_2 := \{X \in \Sigma : \nu(X) \neq \widetilde{\nu}(X)\}. \tag{4.160}$$

Remark 4.4.23. *Assume that $U \subseteq \mathbb{R}^n$ and that $f, g : U \to \mathbb{R}^m$ are two continuous functions. Then the set where they disagree, i.e.,*

$$D := \{X \in U : f(X) \neq g(X)\} \tag{4.161}$$

is relatively open in U, and that the set where they agree, i.e.,

$$A := \{X \in U : f(X) = g(X)\} \tag{4.162}$$

is relatively closed in U.

Clearly, $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$ and $\Sigma = \mathcal{O}_1 \cup \mathcal{O}_2$. To prove that both $\mathcal{O}_1$ and $\mathcal{O}_2$ are relatively open subsets of Σ. For $\mathcal{O}_2$, use a similar approach as in the proof of the fact that the set in (4.161) is relatively open. As regards $\mathcal{O}_1$, proceed as for (4.162) in order to show that $\Sigma \setminus \mathcal{O}_1 = \{X \in \Sigma : \nu(X) = -\widetilde{\nu}(X)\}$ is a relatively closed subset of Σ. Then recall that Σ is connected in order to conclude. $\square$

Lemma 4.4.24. *The* **Möbius strip** *is defined as* $\Sigma := P([0, 2\pi] \times (-1, 1)) \subseteq \mathbb{R}^3$*, where*

$$P : [0, 2\pi] \times (-1, 1) \to \mathbb{R}^3,$$
$$P(u, v) := \big(2\cos u + v\sin(u/2)\cos u, 2\sin u + v\sin(u/2)\sin u,\ v\cos(u/2)\big). \tag{4.163}$$

Then Σ is a C^∞ surface, which is not orientable.

Sketch of Proof: Picture a choice of the unit normal at some point $X \in \Sigma$, then to move it continuously all the way around the Möbius strip, until it arrives again at X. This time, however, this vector ends up having the opposite sense of the one initially considered! $\square$

Remark 4.4.25. *An immediate consequence of Proposition 4.4.11, is that a continuous choice of the unit normal to Σ is given by*

$$\nu(x', \varphi(x')) := \frac{(\nabla\varphi(x'), -1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}, \qquad \text{for every } x' \in \mathcal{O}. \tag{4.164}$$

With the assumption that $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ is an open set and that $\varphi : \mathcal{O} \to \mathbb{R}$ is a function of class C^1 then the graph of φ, i.e., $\Sigma := \{(x', \varphi(x')) : x' \in \mathcal{O}\}$ becomes an orientable C^1 surface.

Remark 4.4.26. *For every $X^* \in \mathbb{R}^n$, $n \geqslant 2$, and $r > 0$, $\Sigma := \partial B(X^*, r)$ is an orientable C^∞ surface, since a continuous choice of the unit normal is $\nu(X) := (X - X^*)/r$, $X \in \partial B(X^*, r)$.*

Theorem 4.4.27. [`Invariance of Orientability under Diffeomorphisms`]
Let Σ be an orientable C^1 surface in $\mathbb{R}^n$, $n \geqslant 2$, and assume that $F : U \to V$ is a C^1 diffeomorphism, where $U, V \subseteq \mathbb{R}^n$ are two open sets, with $\overline{\Sigma}$ contained in U. Then $\widetilde{\Sigma} := F(\Sigma)$ is also an orientable C^1 surface in $\mathbb{R}^n$.

Proof. The proof of this theorem follows easily from Theorem 4.4.27, and from Theorem 4.4.19 and definitions. ☐

Theorem 4.4.28. *Assume that Σ is a surface of class C^1 in $\mathbb{R}^n$. Then Σ is orientable surface if and only if the following holds. There exists a family of C^1 parametrizations $P_j : \mathcal{O}_j \to \mathbb{R}^n$, $j \in J$, with the property that*

$$\Sigma = \bigcup_{j \in J} P_j(\mathcal{O}_j) \tag{4.165}$$

and

$$\det D(P_j^{-1} \circ P_k) > 0 \quad \text{on} \quad P_k^{-1}(\mathcal{O}_j \cap \mathcal{O}_k) \quad \forall\, j, k \in J. \tag{4.166}$$

Proof. Suppose $n \geqslant 3$ (the case $n = 2$ is treated analogously). If there exists a family of C^1 parametrizations $P_j : \mathcal{O}_j \to \mathbb{R}^n$, $j \in J$, for which (4.165)-(4.166) hold, we consider the function $\nu : \Sigma \to S^{n-1}$ defined by the requirement that, if $j \in J$ then

$$\nu(P_j(u)) = \frac{\partial_1 P_j(u) \times \partial_2 P_j(u) \times \ldots \times \partial_{n-1} P_j(u)}{\|\partial_1 P_j(u) \times \partial_2 P_j(u) \times \ldots \times \partial_{n-1} P_j(u)\|} \quad \text{for } u \in \mathcal{O}_j. \tag{4.167}$$

Since, by Chain Rule,

$$\partial_\ell P_j = \partial_\ell [P_k \circ (P_k^{-1} \circ P_j)] = \sum_{i=1}^{n} (\partial_i P_k) \circ (P_k^{-1} \circ P_j) \partial_\ell (P_k^{-1} \circ P_j)_i \tag{4.168}$$

formula (4.46) gives that, for each $u \in P_j^{-1}(P_j(\mathcal{O}_j) \cap P_k(\mathcal{O}_k))$,

$$\begin{aligned} &\frac{\partial_1 P_j(u) \times \partial_2 P_j(u) \times \ldots \times \partial_{n-1} P_j(u)}{\|\partial_1 P_j(u) \times \partial_2 P_j(u) \times \ldots \times \partial_{n-1} P_j(u)\|} \\ &\qquad = \frac{\det A}{|\det A|} \frac{\partial_1 P_k(w) \times \partial_2 P_k(w) \times \ldots \times \partial_{n-1} P_k(w)}{\|\partial_1 P_k(w) \times \partial_2 P_k(w) \times \ldots \times \partial_{n-1} P_k(w)\|}, \end{aligned} \tag{4.169}$$

where we have set $w := (P_k^{-1} \circ P_j)u$, and $A := D(P_k^{-1} \circ P_j)(u)$. By (4.166), we know that $\det A / |\det A| = 1$ which, in light of (4.169), proves that $\nu(P_j(u)) = \nu(P_k(w))$. Hence, $\nu : \Sigma \to S^{n-1}$ is unambiguously defined, and is a continuous functions. Thus Σ has a choice of a continuous unit normal, i.e., is orientable.

Conversely, if Σ is orientable then there exists a choice of a continuous unit normal $\nu : \Sigma \to S^{n-1}$. Fix a point $X^* \in \Sigma$ and assume that $P : \mathcal{O} \to \mathbb{R}^n$ is a C^1 local parametrization of Σ near X^*. By eventually replacing $\mathcal{O}$ with a smaller open neighborhood of $P^{-1}(X^*)$, it can be assumed that $\mathcal{O}$ is a connected set. In this scenario, it follows from Theorem 4.4.9 (and a pattern of reasoning outlined in the hint for Lemma 4.4.22) that either

$$\nu(P(u)) = \frac{\partial_1 P(u) \times \partial_2 P(u) \times \ldots \times \partial_{n-1} P(u)}{\|\partial_1 P(u) \times \partial_2 P(u) \times \ldots \times \partial_{n-1} P(u)\|} \qquad \forall\, u \in \mathcal{O}, \tag{4.170}$$

or

$$\nu(P(u)) = -\frac{\partial_1 P(u) \times \partial_2 P(u) \times \ldots \times \partial_{n-1} P(u)}{\|\partial_1 P(u) \times \partial_2 P(u) \times \ldots \times \partial_{n-1} P(u)\|} \qquad \forall\, u \in \mathcal{O}. \tag{4.171}$$

In the second case, we pick a linear mapping L of $\mathbb{R}^{n-1}$ into itself with $\det L = -1$ and replace P by the local C^1 parametrization $\widetilde{P} : \widetilde{\mathcal{O}} \to \mathbb{R}^n$ of Σ near X^* given by $\widetilde{P} := P \circ L$ and $\widetilde{\mathcal{O}} := L^{-1}(\mathcal{O})$. Thus, it can be assumed that (4.170) always holds. In this fashion, we create a collection of C^1 local parametrizations $P_j : \mathcal{O}_j \to \mathbb{R}^n$, $j \in J$, with the property that (4.165) and (4.167) hold. Since the latter condition entails (4.169), we deduce that $\det A > 0$, i.e., (4.166) holds. This finishes the proof of the theorem. $\square$

4.5 Integration on Surfaces

Definition 4.5.1. *Let $\Sigma \subset \mathbb{R}^n$, $n \geqslant 2$, be a C^1 surface and assume that $P : \mathcal{O} \to \mathbb{R}^n$, with $\mathcal{O}$ open subset of $\mathbb{R}^{n-1}$, is a local parametrization of Σ near some point $X^* \in \Sigma$. Also, suppose that $f : \Sigma \to \mathbb{R}$ is a continuous function on Σ which vanishes outside of a compact subset of $P(\mathcal{O})$. We then define*

$$\int_\Sigma f\, d\sigma := \int_{\mathcal{O}} (f \circ P)(u)\ \|\partial_1 P(u) \times \ldots \times \partial_{n-1} P(u)\|\ du_1 \ldots du_{n-1} \quad \text{if } n \geqslant 3, \tag{4.172}$$

and

$$\int_\Sigma f\, d\sigma := \int_{\mathcal{O}} (f \circ P)(u) \|P'(u)\|\, du \quad \text{if } \ n = 2. \tag{4.173}$$

In (4.172)*, $d\sigma$ stands for the* **surface measure** *(or, surface area element), whereas in* (4.173)*, $d\sigma$ stands for the* **arc-length measure**.

Proposition 4.5.2. *Definition 4.5.1 is unambiguous in the sense that the right-hand side of* (4.172) *does not depend on the particular parametrization used to write this expression.*

Proof. Retain the context of Definition 4.5.1. In addition, suppose $\mathcal{U} \subseteq \mathbb{R}^{n-1}$ is an open set and $Q : \mathcal{U} \to \mathbb{R}^n$ is another local parametrization of Σ near $X^* \in \Sigma$, with the property that Q agrees with P near the support of f. In the case $n \geqslant 3$, our goal is to show that

$$\begin{aligned} &\int_{\mathcal{O}} (f \circ P)(u)\ \|\partial_1 P(u) \times \ldots \times \partial_{n-1} P(u)\|\ du_1 \ldots du_{n-1} \\ &\qquad = \int_{\mathcal{U}} (f \circ Q)(w)\ \|\partial_1 Q(w) \times \ldots \times \partial_{n-1} Q(w)\|\ dw_1 \ldots dw_{n-1}. \end{aligned} \tag{4.174}$$

From Proposition 4.3.14, we know that

$$\exists\, \varphi = (\varphi_1, ..., \varphi_{n-1}) : \mathcal{U} \to \mathcal{O} \ \ C^1 \text{ diffeomorphism, such that } Q = P \circ \varphi. \tag{4.175}$$

We may then use the Chain Rule to compute

$$\partial_j Q(w) = \sum_{k=1}^{n-1} \partial_j \varphi_k(w) (\partial_k P)(\varphi(w)), \quad \forall\, j \in \{1, ..., n\},\ \ \forall\, w \in \mathcal{U}. \tag{4.176}$$

Consequently, by (vii) in Proposition 4.2.2, we have

$$\partial_1 Q(w) \times \ldots \times \partial_{n-1} Q(w)$$

$$= \det (D\varphi)(w)\, \partial_1 P(\varphi(w)) \times \ldots \times \partial_{n-1} P(\varphi(w)), \quad \forall\, w \in \mathcal{U}. \tag{4.177}$$

On account of this and the Change of Variable Theorem, we then compute

$$\int_{\mathcal{U}} (f \circ Q)(w) \ \|\partial_1 Q(w) \times \ldots \times \partial_{n-1} Q(w)\| \ dw_1 \ldots dw_{n-1}$$

$$= \int_{\mathcal{U}} (f \circ P)(\varphi(w)) \ |\det (D\varphi)(w)| \cdot$$

$$\cdot \|(\partial_1 P)(\varphi(w)) \times \ldots \times (\partial_{n-1} P)(\varphi(w))\| \, dw_1 \ldots dw_{n-1}$$

$$= \int_{\mathcal{O}} (f \circ P)(u) \ \|\partial_1 P(u) \times \ldots \times \partial_{n-1} P(u)\| \ du_1 \ldots du_{n-1}, \tag{4.178}$$

as desired. Finally, the simpler case $n = 2$ is dealt with similarly. □

For certain applications it is of interest to express the surface integral of a function in a way which avoids using the multi-dimensional cross product. The scope of the theorem below is to do just that.

Theorem 4.5.3. *Let $\Sigma \subset \mathbb{R}^n$, $n \geqslant 2$, be a C^1 surface and assume that $P : \mathcal{O} \to \mathbb{R}^n$, with $\mathcal{O}$ open subset of $\mathbb{R}^{n-1}$, is a local parametrization of Σ near some point $X^* \in \Sigma$. Also, suppose that $f : \Sigma \to \mathbb{R}$ is a continuous function on Σ which vanishes outside of a compact subset of $P(\mathcal{O})$. Then*

$$\int_{\Sigma} f \, d\sigma = \int_{\mathcal{O}} (f \circ P) \sqrt{\sum_{j=1}^{n} \left[\det \left(\frac{D(P_1 \ldots \widehat{P_j} \ldots P_n)}{D(u_1 \ldots u_{n-1})} \right) (u) \right]^2} \, du_1 \ldots du_{n-1}, \tag{4.179}$$

with the convention that a "hat" over P_j indicates that this function is omitted for the corresponding enumeration.

Proof. This is an immediate consequence of (4.172) and (4.133)-(4.134). □

Definition 4.5.4. *Let $\Sigma \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a C^1 surface and suppose that $f : \Sigma \to \mathbb{R}$ is a given continuous function which vanishes outside of a compact subset of Σ. Let $\{B(X_j, r_j)\}_{j \in J}$ be a finite collection of balls in $\mathbb{R}^n$ centered at points in Σ, with the property that, for each $j \in J$, there exists a parametrization*

$$P_j : \mathcal{O}_j \longrightarrow \mathbb{R}^n, \qquad \mathcal{O}_j \subseteq \mathbb{R}^{n-1} \ \textit{open set}, \tag{4.180}$$

of Σ near X_j, such that $P_j(\mathcal{O}_j) = B(X_j, r_j) \cap \Sigma$. Assume that

$$\operatorname{supp} f \subseteq \bigcup_{j\in J} P_j(\mathcal{O}_j), \tag{4.181}$$

and, in addition, suppose that $\{\psi_j\}_{j\in J}$ is a partition of unity subordinate to the open cover $\{B(X_j, r_j)\}_{j\in J}$ of the compact set $\operatorname{supp} f$*. We then define the integral of f on Σ as*

$$\int_\Sigma f\,d\sigma := \sum_{j\in J} \int_\Sigma \psi_j f\,d\sigma \tag{4.182}$$

where each integral in the right-hand side is interpreted in the sense of Definition 4.5.1. Finally, whenever convenient, we shall write $\int_\Sigma f(X)\,d\sigma(X)$ in place of $\int_\Sigma f\,d\sigma$ in order to stress the variable with respect to which the integration is performed.

Lemma 4.5.5. *The above definition of $\int_\Sigma f\,d\sigma$ is independent of the partition of unity use to define the right-hand side of* (4.182)*.*

Proof. Assume that $\{B(X_j, r_j)\}_{j\in J}$ and $\{B(X_k, r_k)\}_{k\in K}$ are finite open covers for $\operatorname{supp} f$ with the property that $\Sigma \cap B(X_j, r_j)$, $j \in J$, and $\Sigma \cap B(X_k, r_k)$, $k \in K$, are images of local C^1 parametrizations of the surface Σ. Also, assume that $\{\psi_j\}_{j\in J}$ is a partition of unity subordinate to the open cover $\{B(X_j, r_j)\}_{j\in J}$ of $\operatorname{supp} f$, and that $\{\eta_k\}_{k\in K}$ is a partition subordinate to the open cover $\{B(X_k, r_k)\}_{k\in K}$ of $\operatorname{supp} f$. We need to show that

$$\sum_{j\in J} \int_\Sigma \psi_j f\,d\sigma = \sum_{k\in K} \int_\Sigma \eta_k f\,d\sigma. \tag{4.183}$$

To justify this, we write

$$\begin{aligned}
\sum_{j\in J} \int_\Sigma \psi_j f\,d\sigma &= \sum_{j\in J} \int_\Sigma \Big(\sum_{k\in K} \eta_k\Big) \psi_j f\,d\sigma = \sum_{j\in J}\sum_{k\in K} \int_\Sigma \eta_k \psi_j f\,d\sigma \\
&= \sum_{k\in K} \int_\Sigma \eta_k \Big(\sum_{j\in J} \psi_j\Big) f\,d\sigma = \sum_{k\in K} \int_\Sigma \eta_k f\,d\sigma,
\end{aligned} \tag{4.184}$$

as wanted. $\square$

Remark 4.5.6. *An immediate consequence of definition* (4.172)*, Proposition 4.5.2 and Proposition 4.4.11 is the following. Suppose that $U \subseteq \mathbb{R}^{n-1}$, $n \geqslant 2$, is an open set and assume that $\phi : U \to \mathbb{R}$ a C^1 function. Then, as already discussed in Remark 4.4.3, the graph of this function, i.e., $\Sigma := \{(x', \phi(x')) : x' \in U\}$, is a C^1 surface in $\mathbb{R}^n$. Then if*

$f : \Sigma \to \mathbb{R}$ is a continuous function which vanishes outside of a compact subset of Σ, then

$$\int_\Sigma f\,d\sigma = \int_U f(x', \phi(x'))\sqrt{1 + \|\nabla\phi(x')\|^2}\,dx'. \tag{4.185}$$

Remark 4.5.7. *If Σ is flat, i.e. $\Sigma = \mathbb{R}^{n-1} \times \{0\}$, then $\phi = 0$ and (4.185) becomes the equality between $\int_\Sigma f\,d\sigma$ and $\int_{\mathbb{R}^{n-1}} f(x', 0)\,dx'$. This simple observation is in support of (4.172) being a natural definition for the concept of integral on a surface.*

Theorem 4.5.8 (Surface to Surface Change of Variables)**.** *Suppose Σ is a C^1 surface in $\mathbb{R}^n$, $n \geqslant 2$, and let ν stand for a choice of the unit normal to Σ. Also, assume that $F : U \to V$ is a C^1 diffeomorphism, where $U, V \subseteq \mathbb{R}^n$ are open sets, with $\overline{\Sigma}$ contained in U. It follows that $\widetilde{\Sigma} := F(\Sigma)$ is a C^1 surface (cf. Theorem 4.4.18), and we denote by $\widetilde{\sigma}$ its surface measure. Then, for every continuous function $f : \widetilde{\Sigma} \to \mathbb{R}$ which vanishes outside of a compact subset of $\widetilde{\Sigma}$, one has*

$$\begin{aligned} \int_{\widetilde{\Sigma}} f\,d\widetilde{\sigma} &= \int_\Sigma (f \circ F)\,|\det(DF)|\,\big\|((DF)^{-1})^\top \nu\big\|\,d\sigma \\ &= \int_\Sigma (f \circ F)\,|\det(DF)|\,\big\|[(DF^{-1})^\top \circ F]\nu\big\|\,d\sigma. \end{aligned} \tag{4.186}$$

Proof. We shall address in detail the case $n \geqslant 3$, and leave it as an exercise to check the validity of (4.186) when $n = 2$. To get started, note that formula (4.186) has local character, in the sense that if it holds when $\operatorname{supp} f$ is suitably small, then it also holds in general. Indeed, it is easy to see that one can use a partition of unity in order to "glue" together these local results into a global one. For this reason, there is no loss of generality in assuming that Σ has a global parametrization, i.e., $\Sigma = P(\mathcal{O})$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$ and $P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is a global C^1 parametrization of Σ.

Assuming that this is the case, we note that, by the Chain Rule,

$$\partial_j(F \circ P) = A\partial_j P, \qquad \forall j \in \{1, \ldots, n\}, \tag{4.187}$$

where A is the $\mathcal{M}*_n \times n$-valued function defined in $\mathcal{O}$ by

$$A := (DF) \circ P. \tag{4.188}$$

Recall next formula (4.41). Taking the norms of both sides gives

$$\|Av_1 \times \ldots \times Av_{n-1}\| = |\det A| \left\|(A^{-1})^\top (v_1 \times \ldots \times v_{n-1})\right\|. \tag{4.189}$$

Then we can write for a given continuous function $f : \widetilde{\Sigma} \to \mathbb{R}$ the following sequence of identities:

$$\begin{aligned}
\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} &= \int_{\mathcal{O}} f \circ (F \circ P) \, \|\partial_1 (F \circ P) \times \ldots \times \partial_{n-1}(F \circ P)\| \, du_1 \ldots du_{n-1} \\
&= \int_{\mathcal{O}} (f \circ F) \circ P \, \|A \, \partial_1 P \times A\partial_2 P \times \ldots \times A \, \partial_{n-1} P\| \, du_1 \ldots du_{n-1} \\
&= \int_{\mathcal{O}} (f \circ F) \circ P \, |\det A| \left\|(A^{-1})^\top (\partial P_1 \times \ldots \times \partial_{n-1} P)\right\| \, du_1 \ldots du_{n-1},
\end{aligned} \tag{4.190}$$

where A is as in (4.188). Replacing A with its actual formula then gives

$$\begin{aligned}
\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} &= \int_{\mathcal{O}} (f \circ F) \circ P \, |\det[(DF) \circ P]| \times \\
&\qquad \times \left\|[((DF)^{-1})^\top \circ P](\partial_1 P \times \ldots \times \partial_{n-1} P)\right\| \, du_1 \ldots du_{n-1}.
\end{aligned} \tag{4.191}$$

Next, write

$$\begin{aligned}
&[((DF)^{-1})^\top \circ P](\partial_1 P \times \ldots \times \partial_{n-1} P) \\
&\quad = [((DF)^{-1})^\top \circ P] \left(\frac{\partial_1 P \times \ldots \times \partial_{n-1} P}{\|\partial_1 P \times \ldots \times \partial_{n-1} P\|} \right) \|\partial_1 P \times \ldots \times \partial_{n-1} P\|,
\end{aligned} \tag{4.192}$$

and recall formula (4.123). Together, these give

$$\begin{aligned}
&[((DF)^{-1})^\top \circ P](\partial_1 P \times \ldots \times \partial_{n-1} P) \\
&\qquad = [((DF)^{-1})^\top \circ P](\nu \circ P) \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\|.
\end{aligned} \tag{4.193}$$

Returning with (4.193) back in (4.191) then gives

$$\begin{aligned}
\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} &= \int_{\mathcal{O}} (f \circ F) \circ P \, |[\det(DF)] \circ P| \left\|[((DF)^{-1})^\top \circ P](\nu \circ P)\right\| \cdot \\
&\qquad \cdot \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du_1 \ldots du_{n-1}.
\end{aligned} \tag{4.194}$$

At this stage, recalling (4.172), formula (4.194) can be re-written in the form of the first equality in (4.186). Finally, the second equality in (4.186) follows from (4.159). $\square$

Remark 4.5.9. *It is possible to prove a version of Theorem 4.5.8 in which it is assumed that $F : U \to V$ is a function of class C^1, where $U, V \subseteq \mathbb{R}^n$ are open sets, with $\overline{\Sigma}$ contained in U, and such that $F|_{\overline{\Sigma}}$ is injective and* $\det DF \neq 0$ *on $\overline{\Sigma}$.*

Definition 4.5.10. *Let $\Sigma \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a C^1 surface and suppose that $f : \Sigma \to \mathbb{R}$ is* *a continuous function. We say that $\int_\Sigma f\,d\sigma$ exists, provided the following holds. There exist an at most countable collection $\{B(X_j,r_j)\}_{j\in J}$ of balls in $\mathbb{R}^n$ centered at points in Σ, with the property that each $B(X_j,r_j)\cap\Sigma$ is a parametrizable patch, along with a partition of unity $\{\psi_j\}_{j\in J}$ subordinate to the open family $\{B(X_j,r_j)\}_{j\in J}$, such that*

$$\sum_{j\in J}\int_\Sigma \psi_j|f|\,d\sigma < +\infty. \tag{4.195}$$

In this case, the integral of f on Σ, defined as in (4.182)*, is a well-defined number.*

Definition 4.5.11. *Given a C^1 surface $\Sigma\subseteq\mathbb{R}^n$, $n\geqslant 2$, we define its* **area** *as*

$$\mathcal{A}(\Sigma) := \int_\Sigma 1\,d\sigma, \tag{4.196}$$

provided the integral exists (in which case we write $\mathcal{A}(\Sigma)<+\infty$). In the case $n=2$, we shall refer to the right-hand side of (4.196) *as the* **length** *of Σ.*

Remark 4.5.12. *If Σ is the graph of the function $\phi:(a,b)\to\mathbb{R}$, then the length of Σ is given by*

$$\textit{length}\,(\Sigma) = \int_a^b \sqrt{1+|\phi'(x)|^2}\,dx, \tag{4.197}$$

in agreement with (4.196) *when $n=2$, and $f\equiv 1$.*

Remark 4.5.13. *Let $a,b\in\mathbb{R}$, $a<b$, and suppose that $f:[a,b]\to\mathbb{R}$ is a nonnegative, function of class C^1. The the area of the surface Σ obtained by revolving the graph of f, i.e., the curve $\{(x,f(x):\,x\in(a,b)\}$, about the x-axis is given by*

$$\mathcal{A}(\Sigma) = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx. \tag{4.198}$$

Indeed, let $P:(a,b)\times(0,2\pi)\to\mathbb{R}^3$ be the C^1 global parametrization of the surface Σ given by $P(x,\theta):=(x,f(x)\cos\theta,f(x)\sin\theta)$. Then $\|\partial_x P(x,\theta)\times\partial_\theta P(x,\theta)\| = f(x)\sqrt{1+[f'(x)]^2}$ so that

$$\mathcal{A}(\Sigma) = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx. \tag{4.199}$$

Remark 4.5.14. *Suppose Σ is a C^1 surface in $\mathbb{R}^n$, $n\geqslant 2$, with the property that $\mathcal{A}(\Sigma)<+\infty$. Then any continuous and bounded function $f:\Sigma\to\mathbb{R}$ is integrable and*

$$\left|\int_\Sigma f\,d\sigma\right| \leqslant \mathcal{A}(\Sigma)\sup_{X\in\Sigma}|f(X)|. \tag{4.200}$$

Proposition 4.5.15. *Suppose Σ is a C^k surface in $\mathbb{R}^n$, for some $n \geqslant 2$ and $k \in \mathbb{N}\cup\{\infty\}$, and let $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ be a rotation about the origin. Then $\mathcal{R}(\Sigma)$ is a C^k surface in $\mathbb{R}^n$ and, for every continuous function $f : \Sigma \to \mathbb{R}$ which vanishes outside of a compact subset of Σ, one has*

$$\int_{\Sigma} f\, d\sigma = \int_{\mathcal{R}(\Sigma)} f \circ \mathcal{R}^{-1}\, d\sigma_{\mathcal{R}(\Sigma)}, \tag{4.201}$$

where $d\sigma_{\mathcal{R}(\Sigma)}$ represents the surface measure of $\mathcal{R}(\Sigma)$. In particular, the integration on a sphere centered at the origin in $\mathbb{R}^n$ is rotation invariant. That is, for every $r > 0$ and any continuous function $f : \partial B(0,r) \to \mathbb{R}$, one has , with $d\sigma$ denoting the surface measure on $\partial B(0,r)$,

$$\int_{\partial B(0,r)} f \circ \mathcal{R}\, d\sigma = \int_{\partial B(0,r)} f\, d\sigma, \tag{4.202}$$

for every rotation $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ about the origin.

Proof. Since any rotation is a C^∞ diffeomorphism of the Euclidean space $\mathbb{R}^n$, it follows from Theorem 4.4.18 that $\mathcal{R}(\Sigma)$ is a C^k surface whenever Σ is a C^k surface in $\mathbb{R}^n$. To establish (4.201), note that the problem is local in character and, as such, it suffices to prove the corresponding result when Σ has a global C^1 parametrization $P : \mathcal{O} \to \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$ (again, such local results can be glued together via a partition of unity). Assuming that this is the case, it follows that $\mathcal{R} \circ P : \mathcal{O} \to \mathbb{R}^n$ is a global C^1 parametrization of $\mathcal{R}(\Sigma)$.

Remark 4.5.16. *If $U \in \mathcal{M}_{n\times n}$ invertible is a unitary matrix then $|\det U| = 1$ and $\|UX\| = \|X\|$ for every $X \in \mathbb{R}^n$.*

Since $\mathcal{R}$ is linear, Remark 4.5.16 then gives $|\det D\mathcal{R}| = |\det \mathcal{R}| = 1$. Also, if ν is a choice of the unit normal for Σ then, by Remark 4.2.4, $\|\mathcal{R}\nu\| = \|\nu\| = 1$.
Finally, upon observing that $\big\|[(D\mathcal{R}^{-1})^\top \circ \mathcal{R}]\nu\big\| = \big\|\mathcal{R}^\top\nu\big\| = 1$, formula (4.201) eventually follows from the Surface to Surface Change of Variables Theorem (Theorem 4.5.8).

Recall that integration on the entire Euclidean space is translation invariant, in the sense that for every continuous, compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$ and any vector $X \in \mathbb{R}^n$ one has

$$\int_{\mathbb{R}^n} f(X+Y)\, dY = \int_{\mathbb{R}^n} f(Y)\, dY. \tag{4.203}$$

as desired. $\square$

Our next lemma explores the case in which a general surface is used in place of $\mathbb{R}^n$.

Lemma 4.5.17. *Suppose Σ is a C^k surface in $\mathbb{R}^n$, for some $n \geqslant 2$ and $k \in \mathbb{N} \cup \{\infty\}$, and fix $X \in \mathbb{R}^n$. Then $X + \Sigma := \{X + Y : Y \in \Sigma\}$ is a C^k surface in $\mathbb{R}^n$ and, for every continuous function $f : \Sigma \to \mathbb{R}$ which vanishes outside of a compact subset of Σ, one has*

$$\int_{\Sigma} f\, d\sigma = \int_{X+\Sigma} f(X+Y)\, d\sigma_{X+\Sigma}(Y), \tag{4.204}$$

where $d\sigma_{X+\Sigma}$ denotes the surface measure of $X + \Sigma$.

Proof. Proceed as in the suggested solution for Proposition 4.5.15, taking $F : \mathbb{R}^n \to \mathbb{R}^n$ to be the C^∞ diffeomorphism given by $F(Y) := X + Y$ for every $Y \in \mathbb{R}^n$. Note that, in this case, $DF = I_{n\times n}$.

A rigid transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is obviously a homeomorphism of $\mathbb{R}^n$, so its action commutes with the operation of taking the closure, complement, interior, and boundary for sets. $\square$

An immediate consequence of Proposition 4.5.15 and Lemma 4.5.17 is formulated in' the next remark.

Remark 4.5.18. *Suppose Σ is a C^k surface in $\mathbb{R}^n$, for some $n \geqslant 2$ and $k \in \mathbb{N} \cup \{\infty\}$, and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a rigid transformation. Then $T(\Sigma)$ is a C^k surface in $\mathbb{R}^n$ and, for every continuous function $f : \Sigma \to \mathbb{R}$ which vanishes outside of a compact subset of Σ, one has*

$$\int_{\Sigma} f\, d\sigma = \int_{T(\Sigma)} f \circ T^{-1}\, d\sigma_{T(\Sigma)}, \tag{4.205}$$

where $d\sigma_{T(\Sigma)}$ denotes the surface measure of $T(\Sigma)$.

4.6 Domains of Class C^k

Definition 4.6.1. *We say that a nonempty, bounded open set $\Omega \subseteq \mathbb{R}^n$, where $n \geqslant 2$, is a C^k* **domain** *(or, a domain of class C^k), for some $k \in \mathbb{N} \cup \{\infty\}$, provided the following holds. For every point $X^* \in \partial\Omega$ there exist $R > 0$, an open interval $I \subset \mathbb{R}$ with $0 \in I$, a rigid transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(X^*) = 0$, along with a function φ of class C^k which maps $B(0, R) \subseteq \mathbb{R}^{n-1}$ into I with the property that $\varphi(0) = 0$ and, if $\mathcal{C}$ denotes the (open) cylinder $B(0, R) \times I \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$, then*

$$\mathcal{C} \cap T(\Omega) = \{X = (x', x_n) \in \mathcal{C} : \ x_n > \varphi(x')\}, \tag{4.206}$$

$$\mathcal{C} \cap \partial T(\Omega) = \{X = (x', x_n) \in \mathcal{C} : \ x_n = \varphi(x')\}, \tag{4.207}$$

$$\mathcal{C} \cap (\overline{T(\Omega)})^c = \{X = (x', x_n) \in \mathcal{C} : \ x_n < \varphi(x')\}. \tag{4.208}$$

Lemma 4.6.2. *Any C^1 domain $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, lies on only one side of its boundary, i.e.,*

$$\partial\overline{\Omega} = \partial\Omega. \tag{4.209}$$

Proof. Since for a rigid transformation T, the condition $\partial\overline{T(\Omega)} = \partial T(\Omega)$ implies $\partial\overline{\Omega} = \partial\Omega$, we can assume that T in conditions (4.207)-(4.208) is equal to the identity. Then, if $X^* = (x^*, \varphi(x^*)) \in \partial\Omega$, conditions (4.207)-(4.208) show that there exists a sequence of points $\{X_j\}_{j\in\mathbb{N}}$ in $(\bar{\Omega})^c$ (for example $X_j = (x^*, \varphi(x^*) - \frac{1}{j})$) with the property that $\lim_{j\to\infty} X_j = X^*$. Thus, $X^* \in \overline{(\bar{\Omega})^c}$ which ultimately proves that $\partial\Omega \subseteq \overline{(\bar{\Omega})^c}$.

Definition 4.6.3. *Given $E \subseteq \mathbb{R}^n$, define the* **boundary** *of E to be the set $\partial E \subseteq \mathbb{R}^n$ consisting of all points $X \in \mathbb{R}^n$ with the property that any neighborhood of X overlaps both with E and with E^c. Equivalently,*

$$\partial E = \{X \in \mathbb{R}^n : B(X, \varepsilon) \cap E \neq \varnothing \text{ and } B(X, \varepsilon) \cap E^c \neq \varnothing \text{ for every } \varepsilon > 0\}. \tag{4.210}$$

Remark 4.6.4. *Suppose that $E \subseteq \mathbb{R}^n$. Then the following are true.*

(1) $\partial E = \overline{E} \cap \overline{(E^c)}$.

(2) $\partial E = \overline{E} \setminus E^\circ$. *Hence,* $\partial E \subseteq \overline{E}$ *and* $E \setminus \partial E = E^\circ$.

(3) ∂E *is a closed subset of* $\overline{E}$.

(4) $\overline{E} = E \cup \partial E = E^\circ \cup \partial E$.

(5) $\partial E = \partial(E^c)$.

(6) $\partial(\overline{E}) \subseteq \partial E$.

(7) $\partial(E^\circ) \subseteq \partial E$.

(8) $(E^\circ)^c = \partial E \cup (\overline{E})^c$.

(9) $\mathbb{R}^n = E^\circ \cup \partial E \cup (\overline{E})^c$ *and the three sets appearing in the right-hand side are mutually disjoint.*

Hence, $\partial\Omega \subseteq \overline{(\bar{\Omega})} \cap \overline{(\bar{\Omega})^c} = \partial\overline{\Omega}$, by (1) in Remark 4.6.4. Since the opposite inclusion is always true (cf. (6) in Remark 4.6.4), (4.209) follows. □

Remark 4.6.5. *Conditions* (4.206)-(4.208) *are not independent. Specifically,*

$$(4.206) \quad and \quad (4.207) \Longrightarrow (4.208), \tag{4.211}$$

and, up to reversing the sense on the vertical axis in $\mathbb{R}^{n-1} \times \mathbb{R}$,

$$(4.207) \quad and \quad (4.209) \Longrightarrow (4.206), \ (4.208). \tag{4.212}$$

Remark 4.6.6. *Let* $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$, $n \geqslant 2$, *be two* C^k *domains, for some* $k \in \mathbb{N} \cup \{\infty\}$, *with the property that* $\overline{\Omega_1} \subseteq \Omega_2$. *Then* $\Omega := \Omega_2 \setminus \overline{\Omega_1}$ *is also a* C^k *domain. Moreover, for every point* $X^* \in \mathbb{R}^n$, $n \geqslant 2$, *and any numbers* $0 < r < R$, *the annulus* $\Omega := B(X^*, R) \setminus \overline{B(X^*, r)}$ *is a* C^∞ *domain in* $\mathbb{R}^n$.

Definition 4.6.7. *The (open) upper and lower half-spaces of the Euclidean space* $\mathbb{R}^n$ *are, respectively,*

$$\begin{aligned} \mathbb{R}^n_+ &:= \{X = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, x_n > 0\}, \\ \mathbb{R}^n_- &:= \{X = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, x_n < 0\}. \end{aligned} \tag{4.213}$$

Theorem 4.6.8. *Let* Ω *be a nonempty, open, bounded subset of* $\mathbb{R}^n$, $n \geqslant 2$, *and assume that* $k \in \mathbb{N} \cup \{\infty\}$. *Then* Ω *is a* C^k *domain if and only if for every point* $X^* \in \partial\Omega$ *there exist an open neighborhood* U *of* X^* *in* $\mathbb{R}^n$, $r > 0$, *and a* C^k *diffeomorphism*

$$\psi = (\psi_1, ..., \psi_n) : U \longrightarrow B(0, r) \tag{4.214}$$

for which $\psi(X^*) = 0$ *and which satisfies*

$$\begin{aligned} \psi(\Omega \cap U) &= B(0,r) \cap \mathbb{R}^n_+, \\ \psi((\overline{\Omega})^c \cap U) &= B(0,r) \cap \mathbb{R}^n_-, \\ \psi(\partial\Omega \cap U) &= B(0,r) \cap \partial\mathbb{R}^n_+. \end{aligned} \tag{4.215}$$

Proof. Assume first that Ω is a C^k domain in the sense of Definition 4.6.1. In this case, given $X^* \in \partial\Omega$ there exist an open cylinder $\mathcal{C} = B(0,R) \times I \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$ with $0 \in \mathcal{C}$, and a function $\varphi : B(0,R) \to I$ of class C^k satisfying $\varphi(0) = 0$ and such that conditions (4.206)-(4.208) hold. Fix $r > 0$ sufficiently small so that $B(0,r) \subseteq \mathcal{C}$. Then $U := T^{-1}(B(0,r))) \subseteq \mathbb{R}^n$ is an open neighborhood of X^* and we define $\psi : U \to B(0,r)$ by the requirement that

$$\psi(T^{-1}(X)) := (x', x_n - \varphi(x')) \quad \text{if} \quad X = (x', x_n) \in B(0,r) \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}. \tag{4.216}$$

It is then clear that ψ is a C^k diffeomorphism and $\psi(X^*) = 0$. Moreover, conditions (4.206)-(4.208) translate precisely into (4.215) Then ψ does the required job.

Conversely, suppose that Ω is a nonempty, open, bounded subset of $\mathbb{R}^n$, with the property that for every point $X^* \in \partial\Omega$ there exist an open neighborhood U of X^* in $\mathbb{R}^n$, $r > 0$, and a C^k diffeomorphism ψ as in (4.214) which satisfies $\psi(X^*) = 0$ as well as the conditions listed in (4.215). Our goal is to shows that Ω is a C^k domain in the sense of Definition 4.6.1. To get started, choose $j \in \{1, ..., n\}$ such that

$$(\partial_j \psi_n)(X^*) \neq 0. \tag{4.217}$$

This is possible since otherwise $\nabla\psi_n(X^*) = 0$, which would imply that the Jacobian matrix $D\psi(X^*)$ has a zero line. In turn, this would entail that $D\psi(X^*)$ is not invertible, a contradiction given that ψ is a C^1 diffeomorphism near X^*. To simplify the explanation assume that $j = n$, i.e., assume that

$$(\partial_n \psi_n)(X^*) \neq 0. \tag{4.218}$$

Using the Implicit Function theorem, we can then solve the equation $\psi_n(x', x_n) = 0$ for x_n in terms of x', near X^*. To be specific, assume that $(x_1^*, ..., x_n^*)$ are the components of $X^* \in \mathbb{R}^n$. Then it is possible to find a $(n-1)$-dimensional ball B' of radius $R > 0$, centered at $(x_1^*, ..., x_{n-1}^*)$, and an open interval J with midpoint x_n^* such that $B' \times J \subseteq U$, and for which the following holds. There exists a C^k function

$$\phi : B' \longrightarrow J \tag{4.219}$$

which satisfies $\phi(x_1^*, ..., x_{n-1}^*) = x_n^*$ and for which

$$\Big\{(x', x_n) \in B' \times J : \psi_n(x', x_n) = 0\Big\} = \Big\{(x', \phi(x')) : x' \in B'\Big\}, \tag{4.220}$$

$$\Big\{(x', x_n) \in B' \times J : \psi_n(x', x_n) > 0\Big\} = \Big\{(x', x_n) \in B' \times J : \phi(x') < x_n\Big\}, \tag{4.221}$$

$$\Big\{(x', x_n) \in B' \times J : \psi_n(x', x_n) < 0\Big\} = \Big\{(x', x_n) \in B' \times J : \phi(x') > x_n\Big\}. \tag{4.222}$$

In light of (4.215), the above equalities can be re-written in the form

$$(B' \times J) \cap \Omega = \Big\{(x', \phi(x')) : x' \in B'\Big\}, \tag{4.223}$$

$$(B' \times J) \cap \partial\Omega = \Big\{(x', x_n) \in B' \times J : \phi(x') < x_n\Big\}, \tag{4.224}$$

$$(B' \times J) \cap (\overline{\Omega})^c = \Big\{(x', x_n) \in B' \times J : \phi(x') > x_n\Big\}. \tag{4.225}$$

Define the rigid motion

$$T : \mathbb{R}^n \to \mathbb{R}^n, \qquad T(X) := X - X^* \ \text{ for every } \ X \in \mathbb{R}^n, \tag{4.226}$$

and note that the open interval $I := J - x_n^*$ contains 0. Also, consider the (open) cylinder $\mathcal{C} := B(0, R) \times I$ in $\mathbb{R}^n$, so that $T(B' \times J) = \mathcal{C}$. Finally, define the function $\varphi : B(0, R) \to I$ by setting

$$\varphi(x') := \phi(x' + (x_1^*, ..., x_{n-1}^*)) - x_n^* \ \text{ for every } \ x' \in B(0, R). \tag{4.227}$$

It follows that φ is of class C^k and $\varphi(0) = 0$. Granted (4.223)-(4.225), these choices then ensures that conditions (4.206)-(4.208) in Definition 4.6.1 are satisfied. Since $X^* \in \partial\Omega$ was arbitrary, we may then conclude that Ω is a C^k domain. $\square$

The next theorem shows that the class of C^k domains in $\mathbb{R}^n$ is invariant under C^k diffeomorphisms.

Theorem 4.6.9 (Invariance of Domains under Diffeomorphisms)**.** *Assume that Ω is a C^k domain in $\mathbb{R}^n$ for some $n \geqslant 2$ and $k \in \mathbb{N} \cup \{\infty\}$. Also, suppose that $V, W \subseteq \mathbb{R}^n$ are open sets with $\overline{\Omega} \subseteq V$ and that $F : V \to W$ is a C^k diffeomorphism. Then $F(\Omega)$ is also a C^k domain in $\mathbb{R}^n$.*

Proof. Since $\overline{\Omega}$ is a compact subset of V, it follows that $F(\overline{\Omega})$ is a compact subset of W. Hence, $F(\Omega)$ is a nonempty, open, bounded subset of $\mathbb{R}^n$. Since F is a homeomorphism, $\partial F(\Omega) = F(\partial\Omega)$. Consider an arbitrary point $Y^* \in \partial F(\Omega)$. Then there exists $X^* \in \partial\Omega$ such that $Y^* = F(X^*)$. Also, from Theorem 4.6.8, there exist an open neighborhood U of X^* in $\mathbb{R}^n$, $r > 0$, and a C^k diffeomorphism $\psi : U \longrightarrow B(0,r)$ for which $\psi(X^*) = 0$ and which satisfies the conditions listed in (4.215).

Since $X^* \in \partial\Omega \subseteq \overline{\Omega} \subseteq V$ and since V is open, there is no loss of generality we can assume that $U \subseteq V$ (this can always be arranged by decreasing $r > 0$ and taking a $U := \psi^{-1}(B(0,r)))$. Then $F(U)$ is an open neighborhood of Y^* and, if we consider the C^k diffeomorphism

$$\widetilde{\psi} := \psi \circ F^{-1} : F(U) \longrightarrow B(0,r), \tag{4.228}$$

it follows that

$$\begin{aligned} &\widetilde{\psi}(F(\Omega) \cap F(U)) = B(0,r) \cap \mathbb{R}^n_+, \\ &\widetilde{\psi}((\overline{F(\Omega)})^c \cap F(U)) = B(0,r) \cap \mathbb{R}^n_-, \\ &\widetilde{\psi}(\partial F(\Omega) \cap F(U)) = B(0,r) \cap \partial\mathbb{R}^n_+. \end{aligned} \tag{4.229}$$

On account of these and Theorem 4.6.8, we may therefore conclude that $F(\Omega)$ is a C^k domain in $\mathbb{R}^n$. ☐

Proposition 4.6.10. *Assume that Ω is a C^k domain in $\mathbb{R}^n$ for some $n \geqslant 2$ and $k \in \mathbb{N} \cup \{\infty\}$. Then $\Sigma := \partial\Omega$ is a compact C^k surface in $\mathbb{R}^n$.*

Proof. Since Ω is bounded, it follows that $\partial\Omega$ is compact. Fix an arbitrary point $X^* \in \partial\Omega$ and observe that, in the context of Definition 4.6.1, $T^{-1}(\mathcal{C})$ is an open neighborhood of X^*. Thus, there exists $r > 0$ with the property that $B(X^*, r) \subseteq T^{-1}(\mathcal{C})$. Condition (4.207) then ensures that there exists an open subset $\mathcal{O}$ of $B(0,R) \subseteq \mathbb{R}^{n-1}$ for which

$$B(X^*, r) \cap \partial\Omega = \{T^{-1}(x', \varphi(x')) : x' \in \mathcal{O}\}. \tag{4.230}$$

This shows that the function $P : \mathcal{O} \to \mathbb{R}^n$ defined by $P(x') := T^{-1}(x', \varphi(x'))$ for every $x' \in \mathcal{O}$ is a local C^k parametrization of $\partial\Omega$ near X^*. Hence, one can have $\Sigma := \partial\Omega$ is a compact C^k surface in $\mathbb{R}^n$. ☐

Remark 4.6.11. *Let Ω is a C^k domain in $\mathbb{R}^n$, $n \geqslant 2$, for some $k \in \mathbb{N} \cup \{\infty\}$. Then any relatively open subset Σ of $\partial\Omega$ is a surface of class C^k in $\mathbb{R}^n$.*

Lemma 4.6.12. *Let $E \subseteq \mathbb{R}^n$ be a bounded set and suppose that $f : E \to \mathbb{R}^m$ is a uniformly continuous function. Then the graph of f, i.e., $G_f := \{(X, f(X)) : X \in E\}$ is a set of content zero in $\mathbb{R}^{n+m}$.*

Proof. Fix $\varepsilon > 0$. Since f is uniformly continuous on E, there exists $\delta \in (0,1)$ with the property that $\|f(X) - f(Y)\| < \varepsilon$ whenever $X, Y \in E$ are such that $\|X - Y\| < \delta$. Also, since E is bounded, there exists a cube $Q \subseteq \mathbb{R}^n$ which contains $\{X \in \mathbb{R}^n : \operatorname{dist}(X, E) < 1\}$. Consider next the standard n-dimensional grid decomposition of $\mathbb{R}^n$ into cubes with side-length $\delta/\sqrt{n}$, and let $Q_1, ..., Q_N$ be the collection of all cubes from the grid which intersect E. Finally, pick $X_j \in Q_j \cap E$ for each $j = 1, ..., N$. Then

$$G_f \subseteq \bigcup_{1 \leqslant j \leqslant N} B_\infty(X_j, \delta) \times B_\infty(f(X_j), \varepsilon) \tag{4.231}$$

and notice that $R_j := B_\infty(X_j, \delta) \times B_\infty(f(X_j), \varepsilon)$ are (open) rectangles in $\mathbb{R}^{n+m}$ with

$$\begin{aligned} \sum_{j=1}^N v(R_j) &= \sum_{j=1}^N v(B_\infty(X_j, \delta))(2\varepsilon)^m = (2\varepsilon)^m \sum_{j=1}^N (2\sqrt{n})^n v(Q_j) \\ &\leqslant 2^{m+n} n^{n/2} \varepsilon^m v(Q), \end{aligned} \tag{4.232}$$

which this will finish the proof of Lemma 4.6.12. □

Remark 4.6.13. *Using Lemma 4.6.12 one can show that if Ω is a C^1 domain in $\mathbb{R}^n$, $n \geqslant 2$, then $\partial\Omega$ is a set of content zero in $\mathbb{R}^n$.*

Definition 4.6.14. *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a C^k domain, for some $k \in \mathbb{N} \cup \{\infty\}$. Then, given a point $X^* \in \partial\Omega$, the* **outward unit normal** *to Ω at X^* is the choice $\nu(X^*)$ of one of the two unit normals to the C^k surface $\partial\Omega$ at X^* which points towards the outside of Ω. Also, we call $-\nu(X^*)$ the* **inner unit normal** *to Ω at the point $X^* \in \partial\Omega$.*

Remark 4.6.15. *A clarification as to what "pointing towards the outside of Ω" means is in order. Fix a point $X^* \in \partial\Omega$ and let*

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad T(X) = \mathcal{R}(X - X^*), \quad \forall\, X \in \mathbb{R}^n \tag{4.233}$$

be the rigid transformation associated with this point as in Definition 4.6.1 which, given that $T(X^*) = 0$, *necessarily has this form for some rotation about the origin* $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$. *Then the outward unit normal to* Ω *at* X^* *is the choice of that unit normal to* $\partial\Omega$ *at* X^* *whose dot product with the vector* $\mathcal{R}^{-1}(\mathbf{e}_n)$ *is* < 0.

As a consequence of this convention, Proposition 4.4.11 and Theorem 4.4.19, it follows that, in the context of Definition 4.6.1, the outward unit normal to Ω *at* $X^* \in \partial\Omega$ *is given by*

$$\nu(X^*) = \frac{\mathcal{R}^{-1}(\nabla\varphi(0), -1)}{\sqrt{1 + \|\nabla\varphi(0)\|^2}}. \tag{4.234}$$

More generally, the outward unit normal to Ω *at the point* $T^{-1}(x', \varphi(x')) \in \partial\Omega$ *is, for every* $x' \in B(0, R) \subseteq \mathbb{R}^{n-1}$, *given by*

$$\nu(T^{-1}(x', \varphi(x'))) = \frac{\mathcal{R}^{-1}(\nabla\varphi(x'), -1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}. \tag{4.235}$$

The reason for making the choice (4.6.15) for the outward direction of the normal becomes more apparent from the result presented in the lemma below.

Lemma 4.6.16. *Assume that* $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, *is a* C^1 *domain, and denote by* ν *its outward unit normal. Then for every* $X^* \in \partial\Omega$ *there exists* $\varepsilon > 0$ *with the property that*

$$X^* + t\nu(X^*) \in (\overline{\Omega})^c \qquad \forall\, t \in (0, \varepsilon), \tag{4.236}$$

$$X^* - t\nu(X^*) \in \Omega \qquad \forall\, t \in (0, \varepsilon). \tag{4.237}$$

Proof. Since the problem is local, it suffices to consider the situation when $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ is an open set, $x^* \in \mathcal{O}$ is a fixed point, and $\varphi : \mathcal{O} \to \mathbb{R}$ is a function of class C^1. In this scenario, the role of X^* is played by $(x^*, \varphi(x^*))$ and $\nu(X^*)$ is a positive multiple of $(\nabla\varphi(x^*), -1)$. Since $(x^*, \varphi(x^*)) \pm t(\nabla\varphi(x^*), -1) = (x^* \pm t\nabla\varphi(x^*), \varphi(x^*) \mp t)$, for a small t, then

$$\varphi(x^* + t\nabla\varphi(x^*)) > \varphi(x^*) - t \text{ and } \varphi(x^* - t\nabla\varphi(x^*)) < \varphi(x^*) + t \text{ for } t > 0. \tag{4.238}$$

However, if for $|t|$ small we set $f(t) := \varphi(x^* + t\nabla\varphi(x^*)) - \varphi(x^*) + t$ then f is of class C^1 and $f'(0) = \|\nabla\varphi(x^*)\|^2 + 1 > 0$. Thus, $f(t) > f(0) = 0$ for $t > 0$ small, proving the first inequality in (4.238). The case of second inequality in (4.238) is similar. $\square$

Proposition 4.6.17. *Assume that Ω is a C^1 domain in $\mathbb{R}^n$, $n \geqslant 2$. Then $\Sigma := \partial\Omega$ is an orientable C^1 surface in $\mathbb{R}^n$.*

Proof. From the discussion in Remark 4.6.15 (cf. (4.235)) it follows that if $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a C^1 domain then

$$\nu : \partial\Omega \longrightarrow \mathbb{R}^n \tag{4.239}$$

is a continuous function, which satisfies $\|\nu(X)\| = 1$ for every $X \in \partial\Omega$. $\square$

Lemma 4.6.18. *Suppose that Ω is a C^k domain in $\mathbb{R}^n$, $n \geqslant 2$, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geqslant 2$. Then the outward unit normal ν in* (4.239) *is a function of class C^{k-1}. That is, there exist an open set $U \subseteq \mathbb{R}^n$, which contains $\partial\Omega$, along with a function $N : U \to \mathbb{R}^n$ of class C^{k-1} in U with the property that $N|_{\partial\Omega} = \nu$.*

Proof. Working locally, there exist an extension of ν to a neighborhood of $\Sigma := \partial\Omega$ which is of class C^{k-1}. Concretely, in the context of Remark 4.6.15, near each $X^* \in \partial\Omega$, we define $\widetilde{\nu}$ such that

$$(\widetilde{\nu} \circ T^{-1})(X) = \frac{\mathcal{R}^{-1}(\nabla\varphi(x'), -1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}, \qquad X = (x', x_n) \ \text{ near } \ T(X^*) = 0. \tag{4.240}$$

Since the right-hand side of (4.240) is a function of class C^{k-1} near the origin, it follows that $\widetilde{\nu}$ is a function of class C^{k-1} in some neighborhood $\mathcal{O}$ of X^*. Finally, we will cover $\partial\Omega$ with a finite family of open sets $(\mathcal{O}_j)_{j\in J}$ in each of which ν can be extended to a C^{k-1} function $\widetilde{\nu}_j$ (in the manner described above), then glue together these local extensions via a partition of unity $(\phi_j)_{j\in J}$ which is subordinate to $(\mathcal{O}_j)_{j\in J}$. That is, take $U := \bigcup\limits_{j\in J} \mathcal{O}_j$ and $N := \sum\limits_{j\in J} \phi_j \widetilde{\nu}_j$. $\square$

Proposition 4.6.19. *Assume that $U \subseteq \mathbb{R}^n$, $n \geqslant 2$, is an open set which contains $S^{n-1} = \partial B(0,1)$, the unit sphere centered at the origin in $\mathbb{R}^n$. Also, for a fix function*

$$\varphi : U \longrightarrow (0, \infty) \tag{4.241}$$

of class C^k, for some $k \in \mathbb{N} \cup \{\infty\}$, consider the subset of $\mathbb{R}^n$ described in polar coordinate as

$$\Omega := \{r\omega : \omega \in S^{n-1},\ 0 \leqslant r < \varphi(\omega)\}. \tag{4.242}$$

Then the following are true.

(i) Ω *is a* C^k *domain in* $\mathbb{R}^n$ *whose boundary is given by*

$$\partial\Omega = \{\varphi(\omega)\omega : \omega \in S^{n-1}\}. \tag{4.243}$$

(ii) The outward unit normal ν *to* Ω *is given by*

$$\nu(\varphi(\omega)\omega) = \frac{\varphi(\omega)\omega - (\nabla_{\tan}\varphi)(\omega)}{\sqrt{\|(\nabla_{\tan}\varphi)(\omega)\|^2 + \varphi(\omega)^2}}, \qquad \textit{for every } \omega \in S^{n-1}, \tag{4.244}$$

where

$$\nabla_{\tan}\varphi(\omega) := \nabla\varphi(\omega) - (\omega \cdot \nabla\varphi(\omega))\omega, \qquad \forall\, \omega \in S^{n-1}. \tag{4.245}$$

(iii) If $d\sigma$ *is the surface measure* $\partial\Omega$ *then, for every continuous function* $f : \partial\Omega \to \mathbb{R}$,

$$\int_{\partial\Omega} f\, d\sigma = \int_{S^{n-1}} f(\varphi(\omega)\omega)\, \varphi(\omega)^{n-2} \sqrt{\|(\nabla_{\tan}\varphi)(\omega)\|^2 + \varphi(\omega)^2}\, d\sigma_{S^{n-1}}(\omega), \tag{4.246}$$

where $d\sigma_{S^{n-1}}$ *is the surface measure on* S^{n-1}.

Proof. The function

$$F_\varphi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n, \qquad F_\varphi(X) := \varphi\left(\frac{X}{\|X\|}\right) X, \quad \forall\, X \in \mathbb{R}^n \setminus \{0\}, \tag{4.247}$$

is a C^k diffeomorphism. For a fix number $0 < \varepsilon < \inf\{\varphi(\omega) : \omega \in S^{n-1}\}$, we have that

$$F_\varphi(B(0,1) \setminus \overline{B(0,\varepsilon)}) = \{r\omega : \omega \in S^{n-1},\ \varepsilon < r < \varphi(\omega)\} \tag{4.248}$$

is, by Theorem 4.6.9 and Remark 4.6.6, a C^k domain as well. In turn, this readily implies that Ω in (4.242) is a C^k domain whose boundary, $\partial\Omega = F_\varphi(S^{n-1})$, is given by (4.243). Since the unit normals to S^{n-1} are $\nu_{S^{n-1}}(\omega) = \pm\omega$ at every point $\omega \in S^{n-1}$, Theorem 4.4.19 gives that $\nu(\varphi(\omega)\omega)$ is one of the two vectors

$$\pm \frac{(DF_\varphi^{-1}(\varphi(\omega)\omega))^\top(\nu_{S^{n-1}}(F_\varphi^{-1}(\varphi(\omega)\omega))}{\|(DF_\varphi^{-1}(\varphi(\omega)\omega))^\top(\nu_{S^{n-1}}(F_\varphi^{-1}(\varphi(\omega)\omega))\|} \tag{4.249}$$

Definition 4.6.20. *Given,* $a = (a_1, ..., a_n) \in \mathbb{R}^n$, $b = (b_1, ..., b_n) \in \mathbb{R}^n$, *define* $a \otimes b$ *to be the matrix*

$$a \otimes b := \big(a_j b_k\big)_{1 \leqslant j,k \leqslant n} \in \mathcal{M}_{n\times n}. \tag{4.250}$$

Remark 4.6.21. *Recall that* $S^{n-1} = \{X \in \mathbb{R}^n : \|X\| = 1\}$, *and let* $U \subseteq \mathbb{R}^n$ *be an open set containing* S^{n-1}. *Assume that* $\varphi : U \to (0,\infty)$ *is a differentiable function, and define*

$F_\varphi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_\varphi(X) := \begin{cases} \varphi\left(\frac{X}{\|X\|}\right) X & \text{if} \quad X \neq 0, \\ 0 & \text{if} \quad X = 0. \end{cases} \tag{4.251}$$

then the following are true

(i) The function F_φ is continuous in $\mathbb{R}^n$, differentiable in $\mathbb{R}^n \setminus \{0\}$, and (by Definition 4.6.20) that

$$DF_\varphi(X) = \varphi\left(\frac{X}{\|X\|}\right) I_{n\times n} + \frac{X}{\|X\|} \otimes (\nabla_{\tan}\varphi)\left(\frac{X}{\|X\|}\right) \tag{4.252}$$

for all $X \in \mathbb{R}^n \setminus \{0\}$, where we have set

$$(\nabla_{\tan}\varphi)(X) := \nabla\varphi(X) - \big(\nabla\varphi(X) \cdot X\big)X, \qquad \forall\, X \in S^{n-1}. \tag{4.253}$$

Use (4.252) *to conclude that*

$$\det\left((DF_\varphi)(X)\right) = \varphi\left(\frac{X}{\|X\|}\right)^n, \qquad \forall\, X \in \mathbb{R}^n \setminus \{0\}. \tag{4.254}$$

(ii) The function F_φ is bijective, its inverse $(F_\varphi)^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ continuous in $\mathbb{R}^n$ and differentiable in $\mathbb{R}^n \setminus \{0\}$.

(iii) The function F_φ is differentiable at 0 if and only if φ is constant.

Given that (cf. Proposition 4.6.21) the inverse of (4.247) is given by $F_\varphi^{-1} = F_{1/\varphi}$, the above vectors can be re-written as

$$\pm\frac{(DF_{1/\varphi}(\varphi(\omega)\omega))^\top\omega}{\|(DF_{1/\varphi}(\varphi(\omega)\omega))^\top\omega\|}. \tag{4.255}$$

On the other hand, making use of Proposition 4.6.21, we have

$$\begin{aligned}(DF_{1/\varphi}(\varphi(\omega)\omega))^\top\omega &= \Big[(1/\varphi(\omega))I_{n\times n} + \omega \otimes \nabla_{\tan}(1/\varphi)(\omega)\Big]^\top\omega \\ &= \Big[(1/\varphi(\omega))I_{n\times n} + (\nabla_{\tan}(1/\varphi)(\omega)) \otimes \omega\Big]\omega \\ &= (1/\varphi(\omega))\omega + \nabla_{\tan}(1/\varphi)(\omega). \end{aligned}\tag{4.256}$$

Since $\nabla(1/\varphi) = -(\nabla\varphi)/\varphi^2$, it follows that $\nabla_{\tan}(1/\varphi) = -(\nabla_{\tan}\varphi)/\varphi^2$. Thus,

$$(DF_{1/\varphi}(\varphi(\omega)\omega))^\top\omega = \omega/\varphi(\omega) - (\nabla_{\tan}\varphi)(\omega)/\varphi(\omega)^2 \tag{4.257}$$

and, hence,

$$\|(DF_{1/\varphi}(\varphi(\omega)\omega))^\top\omega\| = \varphi(\omega)^{-2}\sqrt{\varphi(\omega)^2 + \|(\nabla_{\tan}\varphi)(\omega)\|^2}, \tag{4.258}$$

since $\omega \cdot (\nabla_{\tan}\varphi)(\omega) = 0$ for every $\omega \in S^{n-1}$. Hence, altogether, at every $\omega \in S^{n-1}$ the outward unit normal $\nu(\varphi(\omega)\omega)$ to $\partial\Omega$ is one of the two vectors

$$\pm\frac{\varphi(\omega)\omega - (\nabla_{\tan}\varphi)(\omega)}{\sqrt{\varphi(\omega)^2 + \|(\nabla_{\tan}\varphi)(\omega)\|^2}}. \tag{4.259}$$

A moment's reflection shows that the correct choice of the sign is "plus", and this finishes the proof of (4.244). Given that, by (4.254),

$$\det\left((DF_\varphi)(\varphi(\omega)\omega)\right) = \varphi(\omega)^n, \qquad \forall\,\omega \in S^{n-1}, \tag{4.260}$$

the surface-to-surface change of variable formula (4.186) finally gives (4.246), on account of (4.258). $\square$

Theorem 4.6.22. *Assume that $n \geqslant 2$ and $F : \mathbb{R}^n \to \mathbb{R}$ is function of class C^k, for some $k \in \mathbb{N} \cup \{\infty\}$. Suppose that $c \in \mathbb{R}$ is a number for which $F^{-1}((c, +\infty))$ is a bounded set and*

$$\|\nabla F(X)\| \neq 0 \quad \textit{for any} \;\; X \in F^{-1}(\{c\}). \tag{4.261}$$

Then

$$\Omega := \{X \in \mathbb{R}^n : \, F(X) > c\} \tag{4.262}$$

is a domain of class C^k in $\mathbb{R}^n$ with the boundary $\partial\Omega = \{X \in \mathbb{R}^n : \, F(X) = c\}$.

Proof. Clearly, Ω is an open and bounded subset of $\mathbb{R}^n$.

Remark 4.6.23. *Assume that $F : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, $c \in \mathbb{R}$, and suppose that F does not have any local minima or local maxima on $F^{-1}(\{c\})$. Then*

$$\Omega := \{X \in \mathbb{R}^n : \, F(X) > c\} \tag{4.263}$$

is an open subset of $\mathbb{R}^n$ whose boundary is given by

$$\partial\Omega = \{X \in \mathbb{R}^n : \, F(X) = c\}. \tag{4.264}$$

From the remark above, we know that $\partial\Omega = \{X \in \mathbb{R}^n : \, F(X) = c\}$. The goal is to show that the conditions in Definition 4.6.1 are satisfied near each $X^* \in \partial\Omega$. On this purpose we use ideas first employed in the proof of Proposition 4.4.15. More specifically, we apply the Implicit Function Theorem to

$$\widetilde{F}(X) := F(X + X^*) - F(X^*) = F(X + X^*) - c, \qquad X \in \mathbb{R}^n, \tag{4.265}$$

in order to obtain $\rho, \eta > 0$ and a

$$\text{a function } \varphi \text{ of class } C^k, \text{ mapping } B(0,\rho) \subseteq \mathbb{R}^{n-1} \text{ into } (-\eta,\eta) \tag{4.266}$$

which has the following properties. Consider the open neighborhood of the origin in $\mathbb{R}^n$ given by

$$\mathcal{N} = \{(x_1, ..., x_n) \in \mathbb{R}^n : (x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) \in B(0,\rho),\ x_{j_o} \in (-\eta,\eta)\}. \tag{4.267}$$

Then

$$\begin{aligned} \{X \in \mathcal{N} : \widetilde{F}(X) = 0\} = \{(x_1, ..., x_{j_o-1}, \varphi(x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n), x_{j_o+1}, ..., x_n) : \\ (x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) \in B(0,\rho)\}, \end{aligned} \tag{4.268}$$

and, up to changing $\widetilde{F}$ into $-\widetilde{F}$,

$$\begin{aligned} \{X \in \mathcal{N} : \widetilde{F}(X) > 0\} = \{(x_1, ..., x_n) \in \mathbb{R}^n : (x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) \in B(0,\rho), \\ x_{j_o} \in (-\eta,\eta) \text{ and } \varphi(x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) < x_{j_o}\}, \end{aligned} \tag{4.269}$$

and

$$\begin{aligned} \{X \in \mathcal{N} : \widetilde{F}(X) < 0\} = \{(x_1, ..., x_n) \in \mathbb{R}^n : (x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) \in B(0,\rho), \\ x_{j_o} \in (-\eta,\eta) \text{ and } \varphi(x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n) > x_{j_o}\}. \end{aligned} \tag{4.270}$$

To proceed, introduce the rotation about the origin in $\mathbb{R}^n$ given by

$$\mathcal{R}(X) := (x_1, ..., x_{j_o-1}, x_{j_o+1}, ..., x_n, x_{j_o}), \qquad \forall\, X = (x_1, ..., x_n) \in \mathbb{R}^n, \tag{4.271}$$

as well as the rigid transformation

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad T(X) := \mathcal{R}(X - X^*), \qquad \forall\, X \in \mathbb{R}^n. \tag{4.272}$$

Upon observing that

$$\begin{aligned} \{X \in \mathcal{N} :\ \widetilde{F}(X) = 0\} &= \mathcal{N} \cap (\partial\Omega - X^*), \\ \{X \in \mathcal{N} :\ \widetilde{F}(X) > 0\} &= \mathcal{N} \cap (\Omega - X^*), \\ \{X \in \mathcal{N} :\ \widetilde{F}(X) < 0\} &= \mathcal{N} \cap (\overline{\Omega} - X^*)^c, \end{aligned} \tag{4.273}$$

and that $\mathcal{R}$ maps $\mathcal{N}$ into the cylinder $\mathcal{C} := \{X = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} :\ \|x'\| < \rho,\ |x_n| < \eta\}$, it follows that

$$\begin{aligned} \mathcal{C} \cap T(\partial\Omega) &= \{X = (x', x_n) \in \mathcal{C} :\ \varphi(x') = x_n\}, \\ \mathcal{C} \cap T(\Omega) &= \{X = (x', x_n) \in \mathcal{C} :\ \varphi(x') < x_n\}, \\ \mathcal{C} \cap (\overline{T(\Omega)})^c &= \{X = (x', x_n) \in \mathcal{C} :\ \varphi(x') > x_n\}. \end{aligned} \tag{4.274}$$

This proves that the conditions in (4.206)-(4.208) are satisfied in this case. Finally, in the case when one is forced to change $\widetilde{F}$ into $-\widetilde{F}$ in (4.269)-(4.270), the idea is to replace φ by $\widetilde{\varphi} := -\varphi$ and to further compose T with the reflection $(x', x_n) \mapsto (x', -x_n)$. $\square$

Lemma 4.6.24. *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a domain of class C^k, for some $k \in \{1, 2\}$, and denote by ν the outward unit normal to Ω. Then for every $X \in \partial\Omega$,*

$$\nu(X) \cdot (X - Y) = \begin{cases} O(\|X - Y\|^2) & \text{as } \partial\Omega \ni Y \to X, \text{ if } k = 2, \\ o(\|X - Y\|) & \text{as } \partial\Omega \ni Y \to X, \text{ if } k = 1. \end{cases} \tag{4.275}$$

Proof. The problem is local in character, so there is no loss of generality in assuming that $\partial\Omega$ is the graph of a C^k function $\varphi : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O}$ is an open set in $\mathbb{R}^{n-1}$, $X = (x', \varphi(x'))$ and $Y = (y', \varphi(y'))$ with $x', y' \in \mathcal{O}$, and

$$\nu(X) = \frac{(\nabla\varphi(x'), -1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}} \in \mathbb{R}^n. \tag{4.276}$$

Then

$$\nu(X) \cdot (X - Y) = \frac{\varphi(y') - \varphi(x') - (y' - x') \cdot \nabla\varphi(x')}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}. \tag{4.277}$$

Since, by the Taylor Expansion Theorem we have

$$\varphi(y') - \varphi(x') - (y' - x') \cdot \nabla\varphi(x') = \begin{cases} O(\|x' - y'\|^2) \text{ as } y' \to x', \text{ if } k = 2, \\ o(\|x' - y'\|) \text{ as } y' \to x', \text{ if } k = 1, \end{cases} \tag{4.278}$$

and since $\|x' - y'\| \leqslant \|X - Y\|$, the desired conclusion follows. $\square$

Lemma 4.6.25. *Assume that Σ is a C^k surface in $\mathbb{R}^n$, $n \geqslant 2$, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geqslant 2$. Fix $X^* \in \Sigma$. Then there exist a C^k domain Ω in $\mathbb{R}^n$ and $r > 0$ with the property that $X^* \in \partial\Omega$ and*

$$\Sigma \cap B(X^*, r) = \partial\Omega \cap B(X^*, r). \tag{4.279}$$

Proof. Without loss of generality, it can be assumed that Σ is the graph of a function $\varphi : \mathcal{O} \to \mathbb{R}$ of class C^k, where $\mathcal{O}$ is an open neighborhood of the origin in $\mathbb{R}^{n-1}$, that $X^* = \mathbf{e}_n$ and $\varphi(0) = 1$. We can pick a cutoff function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ with small compact support, which is identically 1 near 0 and such that $0 \leqslant \psi \leqslant 1$. Then, if we take Ω to be the under-graph of the function $\psi\varphi$ lying in the open-half space augmented with a suitably chosen domain lying in the lower-half space, which has a sufficiently large flat boundary the desired result will be achived. $\square$

4.7 Integration by Parts in $\mathbb{R}^n$

As a preamble, we first establish the following useful result.

Proposition 4.7.1. *Assume that $O \subseteq \mathbb{R}^n$ is an open set and suppose that $f : O \to \mathbb{R}$ is a function of class C^1 which vanishes outside of a compact subset K of O. Then for every $j \in \{1, ..., n\}$, we have*

$$\int_O (\partial_j f)(X)\, dX = 0. \tag{4.280}$$

Proof. Let $(R_k)_{1 \leqslant k \leqslant N}$ be a finite open cover of K with (open) rectangles whose closures are contained in O. Also, denote by $\{\psi_k\}_{1 \leqslant k \leqslant N}$ a partition of unity subordinate to this cover. Since $\sum\limits_{k=1}^{N} \psi_k = 1$ in an open set containing K, we have that $\sum\limits_{k=1}^{N} \partial_j \psi_k = 0$ on the support of f. Consequently, if

$$R_k = \prod_{i=1}^{n} (a_i^{(k)}, b_i^{(k)}), \qquad 1 \leqslant k \leqslant N, \tag{4.281}$$

then each function $\psi_k f$ vanishes outside of the rectangle R_k so we may write

$$\begin{aligned} \int_O (\partial_j f)(X)\, dX &= \sum_{k=1}^{N} \int_O \partial_j(\psi_k f)(X)\, dX = \sum_{k=1}^{N} \int_{R_k} \partial_j(\psi_k f)(X)\, dX \qquad (4.282) \\ &= \sum_{k=1}^{N} \int_{a_1^{(k)}}^{b_1^{(k)}} \Big(\int_{a_2^{(k)}}^{b_2^{(k)}} \cdots \Big(\int_{a_j^{(k)}}^{b_j^{(k)}} \partial_j(\psi_k f)(x_1, ... x_n)\, dx_j \Big) \cdots dx_2 \Big) dx_n. \end{aligned}$$

However, this is zero since, for every $k \in \{1, ..., N\}$, the innermost integral above vanishes, as can be seen by integrating by parts in the variable x_j and using tha fact that $\psi_k f$ is zero near ∂R_k. This finishes the proof of the proposition. $\square$

Proposition 4.7.2. *Assume that $R > 0$ and that φ is a function of class C^1 which maps the ball $B(0, R) \subseteq \mathbb{R}^{n-1}$, $n \geqslant 2$, into an open, bounded interval $I = (a, b) \subset \mathbb{R}$. For this, consider its upper-graph domain, i.e.,*

$$\Omega := \left\{ X = (x', x_n) \in B(0, R) \times I \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n : x_n > \varphi(x') \right\}, \tag{4.283}$$

and define

$$\Sigma := \{(x', \varphi(x')) : x' \in B(0, R)\} \subseteq \partial\Omega, \tag{4.284}$$

which is a surface of class C^1. Finally, let $f : \overline{\Omega} \to \mathbb{R}$ be a function of class C^1 which *vanishes outside a relatively closed subset of* Ω*. Then, for each* $j \in \{1, ..., n\}$*, we have*

$$\int_\Sigma f\nu_j \, d\sigma = \int_\Omega (\partial_j f)(X) \, dX, \tag{4.285}$$

where $d\sigma$ *is the surface measure on* Σ *and* ν_j *is the* j*-th component of the canonical unit normal* ν *to the surface* Σ *(chosen such that* $\nu_n < 0$ *on* Σ*).*

Proof. We shall split the proof into two parts, depending on whether $j \neq n$ or $j = n$. We begin with:

`Case I.` Assume that $j \in \{1, \ldots, n-1\}$. Then the left-hand side of (4.285) becomes

$$\begin{aligned} \int_\Sigma f\nu_j \, d\sigma &= \int_{B(0,R)} (f\nu_j)(x', \varphi(x'))\sqrt{1 + \|\nabla\varphi(x')\|^2} \, dx' \\ &= \int_{B(0,R)} f(x', \varphi(x'))(\partial_j\varphi)(x') \, dx'. \end{aligned} \tag{4.286}$$

Consider next the right-hand side of (4.285). In this regard, introduce the function which flattens the lower bottom of Ω into the subregion of the upper-half space

$$D := \{Y = (y', y_n) \in \mathbb{R}^n_+ : y' \in B(0, R), \ 0 < y_n < b - \varphi(y')\}, \tag{4.287}$$

e.i.

$$\Phi : \Omega \to D, \, \Phi(X) := (x', x_n - \varphi(x')), \, \forall \, X = (x', x_n) \in \Omega \subseteq \mathbb{R}^{n-1} \times \mathbb{R}. \tag{4.288}$$

This is a clearly a bijection, whose inverse is given by

$$\Phi^{-1} : D \to \Omega, \, \Phi^{-1}(Y) := (y', \varphi(y') + y_n), \, \forall \, Y = (y', y_n) \in D \subseteq \mathbb{R}^{n-1} \times \mathbb{R}. \tag{4.289}$$

Since $\Phi^{-1}(y_1, ..., y_n) = (y_1, \ldots, y_{n-1}, \varphi(y') + y_n)$, we may then compute

$$D(\Phi^{-1})(Y) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \partial_1\varphi(y') & \partial_2\varphi(y') & \cdots & \partial_{n-1}\varphi(y') & 1 \end{bmatrix}, \tag{4.290}$$

for every $Y = (y_1, ..., y_n) \in D$. Hence, Φ^{-1} in (4.289) is a C^1 diffeomorphism with $\det D(\Phi^{-1})(Y) = 1$ for every $Y \in D$. Thus, by the Change of Variable Theorem,

$$\begin{aligned} \int_\Omega (\partial_j f)(X) \, dX &= \int_D (\partial_j f)(\Phi^{-1}(Y)) |\det D(\Phi^{-1})(Y)| \, dY \\ &= \int_D (\partial_j f)(\Phi^{-1}(Y)) \, dY. \end{aligned} \tag{4.291}$$

To continue, it is useful to recall the following.

Theorem 4.7.3 (Interchanging the Order of Integration)**.** *Let* $A \subseteq \mathbb{R}^n$ *and* $B \subseteq \mathbb{R}^m$ *be* *closed rectangles and let $f : A \times B \to \mathbb{R}$ be a Riemann integrable function on $A \times B$ with the property that*

for every $Y \in B$, the function $f(\cdot, Y) : A \to \mathbb{R}$ is Riemann integrable on A,

for every $X \in A$, the function $f(X, \cdot) : B \to \mathbb{R}$ is Riemann integrable on B.

Then the assignments

$$A \ni X \mapsto \int_B f(X,Y)\,dY \in \mathbb{R}, \qquad B \ni Y \mapsto \int_A f(X,Y)\,dX \in \mathbb{R}, \tag{4.292}$$

are Riemann integrable functions on A and on B, respectively, and the following formula, dealing with changing the order of integration, holds:

$$\int_{A\times B} f = \int_A \left(\int_B f(X,Y)\,dY \right) dX = \int_B \left(\int_A f(X,Y)\,dX \right) dY. \tag{4.293}$$

Returning to the proof of Proposition 4.7.2, by the previous theorem, we have

$$\begin{aligned} \int_\Omega (\partial_j f)(X)\,dX &= \int_D (\partial_j f)(\Phi^{-1}(Y))\,dY = \int_D (\partial_j f)(y', \varphi(y') + y_n)\,dY \\ &= \int_{B(0,R)} \left(\int_0^{b-\varphi(y')} (\partial_j f)(y', \varphi(y') + t)\,dt \right) dy'. \end{aligned} \tag{4.294}$$

On the other hand, given that f vanishes outside of a relatively closed subset of Ω, for every $y' \in B(0,R)$ we may write

$$\begin{aligned} f(y', \varphi(y'))(\partial_j \varphi)(y') &= -\left[f(y', \varphi(y') + t)(\partial_j \varphi)(y') \right] \Big|_{t=0}^{t=b-\varphi(y')} \\ &= -\int_0^{b-\varphi(y')} \frac{d}{dt} \left[f(y', \varphi(y') + t)(\partial_j \varphi)(y') \right] dt. \end{aligned} \tag{4.295}$$

As a result,

$$\begin{aligned} \int_\Sigma f \nu_j\,d\sigma &= \int_{B(0,R)} f(y', \varphi(y'))(\partial_j \varphi)(y')\,dy' \\ &= -\int_{B(0,R)} \left(\int_0^{b-\varphi(y')} \frac{d}{dt} \left[f(y', \varphi(y') + t)(\partial_j \varphi)(y') \right] dt \right) dy' \\ &= -\int_{B(0,R)} \left(\int_0^{b-\varphi(y')} (\partial_n f)(y', \varphi(y') + t)(\partial_j \varphi)(y')\,dt \right) dy' \\ &= \int_{B(0,R)} \Big(\int_0^{b-\varphi(y')} (\partial_j f)(y', \varphi(y') + t)\,dt \Big)\,dy' \\ &\quad - \int_{B(0,R)} \Big(\int_0^{b-\varphi(y')} \partial_j \left[f(y', \varphi(y') + t) \right] dt \Big)\,dy' \end{aligned} \tag{4.296}$$

where the last equality requires observing that for every $y' \in B(0,R)$ and $0 < t <$

$b-\varphi(y')$ we have

$$\partial_j\left[f(y',\varphi(y')+t)\right]=(\partial_j f)(y',\varphi(y')+t)+(\partial_n f)(y',\varphi(y')+t)(\partial_j\varphi)(y'), \qquad (4.297)$$

from which we deduce that

$$(\partial_n f)(y',\varphi(y')+t)(\partial_j\varphi)(y')=\partial_j\left[f(y',\varphi(y')+t)\right]-(\partial_j f)(y',\varphi(y')+t). \qquad (4.298)$$

Note, however, that if for every $t\in(0,b-a)$ we set $O_t:=\{y'\in B(0,R):\ \varphi(y')<b-t\}$, which is an open set, then Fubini's theorem allows us to write

$$\begin{aligned}&\int_{B(0,R)}\Big(\int_0^{b-\varphi(y')}\partial_j\left[f(y',\varphi(y')+t)\right]\,dt\Big)\,dy'\\ &\qquad=\int_0^{b-a}\Big(\int_{O_t}\partial_j\left[f(y',\varphi(y')+t)\right]\,dy'\Big)\,dt=0,\end{aligned} \qquad (4.299)$$

since, by virtue of Proposition 4.7.1, the innermost integral is zero, given that that for every $t\in(0,b-a)$ fixed the C^1 function $O_t\ni y'\mapsto f(y',\varphi(y')+t)\in\mathbb{R}$ vanishes outside of a compact subset of O_t. Thus, the very last integral in (4.296) is zero, whereas the penultimate one matches the right-most side of (4.294). Altogether, (4.285) is proved in this case.

`Case II.` Suppose now that $j=n$. Then the left-hand side of (4.285) becomes

$$\begin{aligned}\int_\Sigma f\nu_n\,d\sigma &= -\int_{B(0,R)}f(y',\varphi(y'))\,dy'\\ &= \int_{B(0,R)}\Big(\int_0^{b-\varphi(y')}\frac{d}{dt}\left[f(y',\varphi(y')+t)\right]\,dt\Big)\,dy'\\ &= \int_{B(0,R)}\Big(\int_0^{b-\varphi(y')}(\partial_n f)(y',\varphi(y')+t)\,dt\Big)\,dy'.\end{aligned} \qquad (4.300)$$

On the other hand, much as before, we can transform the right-hand side of (4.285) by making a change of variables into

$$\begin{aligned}\int_\Omega(\partial_n f)(X)\,dX &= \int_D(\partial_n f)(y',\varphi(y')+y_n)\,dY\\ &= \int_{B(0,R)}\Big(\int_0^{b-\varphi(y')}(\partial_n f)(y',\varphi(y')+t)\,dt\Big)\,dy'.\end{aligned} \qquad (4.301)$$

This matches the expression in (4.300), so formula (4.285) holds in this case. $\square$

Theorem 4.7.4. *Let Ω be a C^1 domain in $\mathbb{R}^n$, $n\geqslant 2$. We denote by $d\sigma$ the surface measure on the boundary $\partial\Omega$, and by $\nu=(\nu_1,...,\nu_n)$ the outward unit normal to Ω. If*

$f : \overline{\Omega} \to \mathbb{R}^n$ *is a function of class* C^1*, then for every* $j \in \{1, ..., n\}$*, one has*

$$\int_{\Omega} (\partial_j f)(X)\, dX = \int_{\partial\Omega} f\nu_j\, d\sigma. \tag{4.302}$$

Proof. Since Ω be a C^1 domain, for every point $X^* \in \partial\Omega$ there exist:

(1) a number $R > 0$ and an open interval $I \subset \mathbb{R}$ with $0 \in I$;

(2) a rigid transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ which has the form $T(X) = \mathcal{R}(X - X^*)$ for every $X \in \mathbb{R}^n$, where $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ is a rotation about the origin,

(3) a function φ of class C^k which maps $B(0, R) \subseteq \mathbb{R}^{n-1}$ into I with the property that $\varphi(0) = 0$ and such that if $\mathcal{C}$ denotes the cylinder $B(0, R) \times I \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$, then formulas (4.206)-(4.208) hold.

We first prove (4.302) under additional assumption that, for some $X^* \in \partial\Omega$,

$$f \text{ vanishes in } \Omega \text{ outside a compact subset of } \overline{\Omega} \cap T^{-1}\mathcal{C}. \tag{4.303}$$

In this scenario, we write

$$\begin{aligned}
\int_{\Omega} (\partial_j f)(X)\, dX &= \int_{\Omega \cap T^{-1}(\mathcal{C})} (\partial_j f)(X)\, dX \qquad (4.304)\\
&= \int_{T(\Omega)\cap\mathcal{C}} (\partial_j f)(T^{-1}(X)) |\det D(T^{-1})(X)|\, dX \\
&= \int_{T(\Omega)\cap\mathcal{C}} (\partial_j f)(T^{-1}(X)) |\det \mathcal{R}^{-1}|\, dX = \int_{T(\Omega)\cap\mathcal{C}} (\partial_j f)(T^{-1}(X))\, dX.
\end{aligned}$$

To continue, observe that $\partial T(\Omega) = T(\partial\Omega)$ and that (cf. Theorem 4.4.19) the outward unit normal to the C^1 domain $T(\Omega)$ is $\widetilde{\nu} = \mathcal{R}^{-1}(\nu \circ T^{-1})$. Also, if $(a_{jk})_{1 \leqslant j,k \leqslant n}$ is the unitary matrix associated with the rotation $\mathcal{R}$, and $(a^{jk})_{1 \leqslant j,k \leqslant n}$ is its inverse, then by Chain Rule

$$\partial_j (f \circ T^{-1})(X) = \sum_{k=1}^{n} (\partial_k f)(T^{-1}(X)) a^{jk} \tag{4.305}$$

$$(\partial_j f)(T^{-1}(X)) = \sum_{k=1}^{n} \partial_k (f \circ T^{-1})(X) a_{jk}. \tag{4.306}$$

Then, granted (4.206) and (4.207), we can then invoke Proposition 4.7.2 and condition

(4.303) in order to write

$$\begin{aligned}
\int_{T(\Omega)\cap\mathcal{C}} (\partial_j f)(T^{-1}(X))\,dX &= \sum_{k=1}^{n} a_{jk} \int_{T(\Omega)\cap\mathcal{C}} \partial_k(f\circ T^{-1})(X)\,dX \qquad (4.307)\\
&= \sum_{k=1}^{n} a_{jk} \int_{T(\partial\Omega)\cap\mathcal{C}} f\circ T^{-1}\,(\mathcal{R}^{-1}(\nu\circ T^{-1}))_k\,d\sigma_{T(\partial\Omega)}\\
&= \sum_{k=1}^{n} a_{jk} \int_{T(\partial\Omega)} f\circ T^{-1}\,(\mathcal{R}^{-1}(\nu\circ T^{-1}))_k\,d\sigma_{T(\partial\Omega)}
\end{aligned}$$

where $d\sigma_{T(\partial\Omega)}$ is the surface measure on the surface $T(\partial\Omega)$. Upon recalling (4.205), the last integral above is further equal to

$$\begin{aligned}
\sum_{k=1}^{n} a_{jk} \int_{\partial\Omega} f\,(\mathcal{R}^{-1}(\nu))_k\,d\sigma &= \int_{\partial\Omega} f \sum_{k=1}^{n} a_{jk}(\mathcal{R}^{-1}(\nu))_k\,d\sigma\\
&= \int_{\partial\Omega} f\,(\mathcal{R}\mathcal{R}^{-1}(\nu))_j\,d\sigma = \int_{\partial\Omega} f\,\nu_j\,d\sigma. \qquad (4.308)
\end{aligned}$$

Together, (4.304), (4.307) and (4.308) prove (4.302), granted the supplementary condition (4.303).

To treat the general case (i.e., when no assumption is made on the support of f), select a finite open cover $(O_k)_{1\leqslant k\leqslant N}$ of $\partial\Omega$ with sets which are rigid transformations of standard cylinders $\mathcal{C}_k$ in $\mathbb{R}^{n-1}\times\mathbb{R}$ which have the property that (4.302) holds for every function satisfying (4.303) with $\mathcal{C} := \mathcal{C}_k$. Supplement this with an open set O_0, with $\overline{O_0}\subseteq\Omega$, such that $O_0, O_1, ..., O_N$ is an open cover of $\overline{\Omega}$. Also, fix a partition of unity $\{\psi_k\}_{0\leqslant k\leqslant N}$ subordinate to this cover. Since $\sum\limits_{k=0}^{N}\psi_k = 1$ near $\overline{\Omega}$ and, hence, $\sum\limits_{k=0}^{N}\partial_j\psi_k = 0$ near $\overline{\Omega}$, we may then use the partial result we have proved up to this point in order to write

$$\int_{\Omega}(\partial_j f)(X)\,dX = \sum_{k=0}^{N}\int_{\Omega}\partial_j(\psi_k f)(X)\,dX = \sum_{k=1}^{N}\int_{\Omega}\partial_j(\psi_k f)(X)\,dX$$

$$= \sum_{k=1}^{N}\int_{\partial\Omega}\psi_k f\,\nu_j\,d\sigma = \int_{\partial\Omega}\Big(\sum_{k=1}^{N}\psi_k\Big) f\,\nu_j\,d\sigma = \int_{\partial\Omega} f\,\nu_j\,d\sigma, \qquad (4.309)$$

where, in second step, Proposition 4.7.1 is used to conclude $\int_{\Omega}\partial_j(\psi_0 f)(X)\,dX = 0$. This finishes the proof of the theorem. □

Theorem 4.7.5 (Integration by Parts in $\mathbb{R}^n$)**.** *Let Ω be a C^1 domain in $\mathbb{R}^n$, $n\geqslant 2$. We denote by $d\sigma$ the surface measure on the boundary $\partial\Omega$, and by $\nu=(\nu_1,...,\nu_n)$ the outward unit normal to Ω. If $f,g:\overline{\Omega}\to\mathbb{R}^n$ are two functions of class C^1, then for every*

$j \in \{1, ..., n\}$, one has

$$\int_\Omega (\partial_j f)(X) g(X)\, dX = -\int_\Omega f(X)(\partial_j g)(X)\, dX + \int_{\partial\Omega} f g\, \nu_j\, d\sigma. \tag{4.310}$$

Proof. This is a direct consequence of (4.302) written for fg in place of f and the fact that $\partial_j(fg) = (\partial_j f)g + f(\partial_j g)$. □

Theorem 4.7.6 (The Divergence Theorem)**.** *Let Ω be a C^1 domain in $\mathbb{R}^n$, $n \geqslant 2$. We denote by $d\sigma$ the surface measure on the boundary $\partial\Omega$, and by ν the outward unit normal to Ω. If $F : \overline{\Omega} \to \mathbb{R}^n$ is a function of class C^1, then*

$$\int_\Omega (\operatorname{div} F)(X)\, dX = \int_{\partial\Omega} F \cdot \nu\, d\sigma. \tag{4.311}$$

Proof. Assume that $(f_1, ..., f_n)$ are the components of F. Also, let $(\nu_1, ..., \nu_n)$ be the components of the outward unit normal ν. Then, making use of Theorem 4.7.4, we may write

$$\begin{aligned} \int_{\partial\Omega} F \cdot \nu\, d\sigma &= \int_{\partial\Omega} \sum_{j=1}^n f_j \nu_j\, d\sigma = \sum_{j=1}^n \int_{\partial\Omega} f_j \nu_j\, d\sigma \qquad (4.312) \\ &= \sum_{j=1}^n \int_\Omega (\partial_j f_j)(X)\, dX = \int_\Omega \sum_{j=1}^n (\partial_j f_j)(X)\, dX = \int_\Omega (\operatorname{div} F)(X)\, dX, \end{aligned}$$

as desired. □

Definition 4.7.7. *Let Ω be a domain of class C^1 in $\mathbb{R}^n$, $n \geqslant 2$, and denote by ν its outward unit normal. Given a function $u : \overline{\Omega} \to \mathbb{R}$ of class C^1, we define the* **normal derivative** *of u as the directional derivative of u along the outward unit normal to Ω, i.e.,*

$$\partial_\nu u := \nu \cdot (\nabla u) \qquad \text{on} \quad \partial\Omega. \tag{4.313}$$

Proposition 4.7.8 (Green's Formulas)**.** *Suppose that Ω is a domain of class C^1 in $\mathbb{R}^n$, $n \geqslant 2$, and denote by ν its outward unit normal and by $d\sigma$ the surface measure of its boundary. Also, assume that $u, v : \overline{\Omega} \to \mathbb{R}$ are two arbitrary functions of class C^2. Then*

$$\int_\Omega \Delta u v\, dX = \int_{\partial\Omega} v\, \partial_\nu u\, d\sigma - \int_\Omega \nabla u \cdot \nabla v\, dX, \tag{4.314}$$

$$\int_\Omega (v\Delta u - u\Delta v)\, dX = \int_{\partial\Omega} \{v\, \partial_\nu u - u\, \partial_\nu v\}\, d\sigma. \tag{4.315}$$

As a corollary, if $u : \overline{\Omega} \to \mathbb{R}$ is a function of class C^2, then

$$\int_{\Omega} \Delta u \, dX = \int_{\partial\Omega} \partial_\nu u \, d\sigma. \tag{4.316}$$

In particular, if $u : \overline{\Omega} \to \mathbb{R}$ is a function of class C^2, then

$$u \text{ is harmonic in } \Omega \Longrightarrow \int_{\partial\Omega} \partial_\nu u \, d\sigma = 0. \tag{4.317}$$

Proof. By virtue of the Divergence theorem we may write

$$\int_{\Omega} (\Delta u v + \nabla u \cdot \nabla v) \, dX = \int_{\Omega} \operatorname{div}(v \nabla u) \, dX = \int_{\partial\Omega} (v \nabla u \cdot \nu) \, d\sigma, \tag{4.318}$$

from which (4.314) follows. Since, obviously, the second Green formula follows from the first, the proof is finished. □

Lemma 4.7.9. *The Integration by Parts Theorem can be deduced from the Divergence Theorem (and, hence, it is equivalent to it).*

Proof. The goal is to show that if f, g are scalar-valued functions, then for every index $j \in \{1, ..., n\}$,

$$\int_{\Omega} f(\partial_j g) \, dX = \int_{\partial\Omega} f g \nu_j \, d\sigma - \int_{\Omega} g(\partial_j f) \, dX. \tag{4.319}$$

However, using the Divergence Theorem, we have

$$\int_{\Omega} f(\partial_j g) \, dX = \int_{\Omega} (\operatorname{div}(fg \, \mathbf{e}_j) - g \partial_j f) \, dX = \int_{\partial\Omega} f g \nu_j \, d\sigma - \int_{\Omega} g(\partial_j f) \, dX. \tag{4.320}$$

as desired. □

Lemma 4.7.10. *Suppose that Ω is a domain of class C^1 in $\mathbb{R}^n$, $n \geqslant 2$, and denote by ν its outward unit normal and by $d\sigma$ the surface measure of its boundary. Also, assume that $f : \overline{\Omega} \to \mathbb{R}$ and $G = (G_1, \ldots, G_n) : \overline{\Omega} \to \mathbb{R}^n$ are of class C^1. Then*

$$\int_{\Omega} (\nabla f) \cdot G \, dX = \int_{\partial\Omega} f \, G \cdot \nu \, d\sigma - \int_{\Omega} f \operatorname{div} G \, dX. \tag{4.321}$$

Proof. Observe that, by the Integration by Parts formula,

$$\begin{aligned}
\int_\Omega \nabla f \cdot G\, dX &= \int_\Omega \Big(\sum_{j=1}^n G_j \partial_j f\Big)\, dX = \sum_{j=1}^n \int_\Omega G_j (\partial_j f)\, dx \\
&= \sum_{j=1}^n \Big(\int_{\partial\Omega} f G_j \nu_j \, d\sigma - \int_\Omega f(\partial_j G_j)\, dX\Big) \\
&= \int_{\partial\Omega} f \Big(\sum_{j=1}^n G_j \nu_j\Big)\, d\sigma - \int_\Omega f \Big(\sum_{j=1}^n \partial_j G_j\Big)\, dX \\
&= \int_{\partial\Omega} f G \cdot \nu \, d\sigma - \int_\Omega f \operatorname{div} G\, dX, \qquad (4.322)
\end{aligned}$$

as desired. □

Remark 4.7.11. *Suppose that Ω is a domain of class C^1 in $\mathbb{R}^3$ and denote by ν its outward unit normal and by $d\sigma$ the surface measure of its boundary. Also, assume that $f, g : \overline{\Omega} \to \mathbb{R}^3$ two functions of class C^1. One can have*

$$\int_\Omega (\operatorname{curl} f) \cdot g\, dX = \int_\Omega f \cdot (\operatorname{curl} g)\, dX + \int_{\partial\Omega} (\nu \times f) \cdot g\, d\sigma. \qquad (4.323)$$

Indeed, after we expand $(\operatorname{curl} f) \cdot g$ and use the Integration by Parts formula for each term the result will follow.

Lemma 4.7.12. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a C^1 domain and assume $u : \overline{\Omega} \to \mathbb{R}$ is a function of class C^2. If $\Delta u = 0$ in Ω and $u|_{\partial\Omega} = 0$, then $u = 0$ in Ω.*

As a consequence of this, one can deduce the following uniqueness result for harmonic functions: If $v, w : \overline{\Omega} \to \mathbb{R}$ are two functions of class C^2 with the property that $\Delta u = \Delta v = 0$ in Ω and $v = w$ on $\partial\Omega$, then $v = w$ on Ω.

Proof. Notice that, by Green's formula and since $\Delta u = 0$ in Ω and $u|_{\partial\Omega} = 0$

$$\begin{aligned}
\int_\Omega \|\nabla u\|^2\, dX &= \int_\Omega \nabla u \cdot \nabla u\, dX = \int_{\partial\Omega} u \nabla u \cdot \nu\, d\sigma - \int_\Omega u \Delta u\, dX \\
&= \int_{\partial\Omega} u\, \partial_\nu u\, d\sigma = 0. \qquad (4.324)
\end{aligned}$$

Since $\|\nabla u\|^2 \geqslant 0$, it follows that $\|\nabla u\|^2 = 0$ a.e. in Ω so that, ultimately, $\|\nabla u\| = 0$ everywhere since this is a continuous function. Thus, $\nabla u = 0$ in Ω. The Mean Value Theorem then forces u to be locally constant in Ω. Hence, necessarily, $u = 0$ in Ω, since $u|_{\partial\Omega} = 0$. For the second part one can use the first part with $u := v - w$. □

Lemma 4.7.13. [`Rellich's Identity`] *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a C^1 domain and denote by ν its outward unit normal and by $d\sigma$ the surface measure of its boundary. Suppose that $u : \overline{\Omega} \to \mathbb{R}$ is a function of class C^2 which satisfies $\Delta u = 0$ in Ω. to prove that, for every fixed vector $h \in \mathbb{R}^n$, one has*

$$\int_{\partial\Omega} \|\nabla u\|^2 \, h \cdot \nu \, d\sigma = 2 \int_{\partial\Omega} (\nabla u \cdot \nu)(h \cdot \nabla u) \, d\sigma. \tag{4.325}$$

Proof. After one will check that $\operatorname{div}\left(\|\nabla u\|^2 h\right) = 2 \operatorname{div}\big((h \cdot \nabla u)\nabla u\big)$ and use the Divergence Theorem, the resul will follow. □

Proposition 4.7.14. [`Integration by Parts on the Boundary`] *Let $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a C^1 domain and suppose that $f, g : \overline{\Omega} \to \mathbb{R}$ are functions of class C^2. For $j, k \in \{1, \ldots, n\}$, define*

$$\frac{\partial f}{\partial \tau_{jk}} := (\nu_j \partial_k - \nu_k \partial_j) f \qquad \text{on } \ \partial\Omega, \tag{4.326}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outward unit normal to Ω. Then

$$\int_{\partial\Omega} \frac{\partial f}{\partial \tau_{jk}} g \, d\sigma = - \int_{\partial\Omega} f \frac{\partial g}{\partial \tau_{jk}} \, d\sigma. \tag{4.327}$$

Proof. Using Theorem 4.7.5 and the fact that $\partial_j \partial_k f = \partial_k \partial_j f$, we may write

$$\begin{aligned}
\int_{\partial\Omega} \frac{\partial f}{\partial \tau_{jk}} g \, d\sigma &= \int_{\partial\Omega} g (\nu_j \partial_k - \nu_k \partial_j) f \, d\sigma \\
&= \int_{\partial\Omega} g \nu_j \partial_k f \, d\sigma - \int_{\partial\Omega} g \nu_k \partial_j f \, d\sigma = \int_\Omega \partial_j (g \partial_k f) - \int_\Omega \partial_k (g \partial_j f) \\
&= \int_\Omega \Big\{ (\partial_j g \partial_k f) + g(\partial_j \partial_k f) - (\partial_k g \partial_j f + g(\partial_k \partial_j f)) \Big\} \\
&= \int_\Omega \Big\{ (\partial_j g \partial_k f) - (\partial_k g \partial_j f) \Big\}.
\end{aligned} \tag{4.328}$$

At this stage, observe that the last expression is antisymmetric in f and g (i.e., by switching the roles of f and g yields the opposite expression). As a consequence, the original expression must be antisymmetric in f and g. □

Remark 4.7.15. *Let $\theta : \mathbb{R}^n$ be a nonnegative C^∞ function with the property that* $\operatorname{supp} \theta \subseteq B(0,1)$ *and $\int_{\mathbb{R}^n} \theta = 1$. For each $\varepsilon > 0$, set*

$$\theta_\varepsilon(X) := \varepsilon^{-n} \theta(X/\varepsilon), \qquad X \in \mathbb{R}^n. \tag{4.329}$$

If $f:\mathbb{R}^n\to\mathbb{R}$ is an arbitrary continuous function, define for each $\varepsilon>0$,

$$f^{\varepsilon}(X):=\int_{\mathbb{R}^n}\theta_{\varepsilon}(X-Y)f(Y)\,dY,\qquad X\in\mathbb{R}^n. \tag{4.330}$$

Then f^{ε} is a function of class C^{∞} which satisfies

$$\sup_{X\in K}|f(X)-f^{\varepsilon}(X)|\to 0\quad \textit{as}\quad \varepsilon\to 0^{+}, \tag{4.331}$$

for every compact set K in $\mathbb{R}^n$. If, in addition, f has compact support then so does f^{ε} and (4.331) holds for $K=\mathbb{R}^n$.

Remark 4.7.16. *Using Remark 4.7.15 one can have that formula (4.327) remains valid if the functions f,g are of class C^1 only.*

Proposition 4.7.17. *Let $\Omega\subseteq\mathbb{R}^n$, $n\geqslant 2$, be a C^1 domain and suppose that $f:\mathbb{R}^n\to\mathbb{R}$ is a continuous function with compact support. Then the function*

$$F:\mathbb{R}^n\to\mathbb{R},\qquad F(X):=\int_{\Omega}f(X+Y)\,dY,\quad \forall\,X\in\mathbb{R}^n. \tag{4.332}$$

is of class C^1 in $\mathbb{R}$ and, for every $j\in\{1,\dots,n\}$, we have

$$\partial_j F(X)=\int_{\partial\Omega}f(X+Y)\nu_j(Y)\,d\sigma(Y),\qquad X\in\mathbb{R}^n, \tag{4.333}$$

where ν_j is the j-th component of the outward unit normal to Ω and $d\sigma$ is the surface measure on $\partial\Omega$.

Lemma 4.7.18. *Let $\Omega\subset\mathbb{R}^n$ be an open set and assume that $u:\Omega\to\mathbb{R}$ is a harmonic function. Fix $X^*\in\Omega$ along with $R>0$ such that $\overline{B(X^*,R)}\subset\Omega$. If $r,\rho\in(0,R]$ are such that $\rho^2=rR$, then*

$$\int_{S^{n-1}}u(X^*+r\omega)\,u(X^*+R\omega)\,d\sigma(\omega)=\int_{S^{n-1}}u^2(X^*+\rho\omega)\,d\sigma(\omega). \tag{4.334}$$

Proof. Consider the function

$$\phi(t):=\int_{S^{n-1}}u(X^*+t\omega)\,u(X^*+\rho^2t^{-1}\omega)\,d\sigma(\omega),\quad t\in[r,R], \tag{4.335}$$

and note that,

$$\begin{aligned}
\phi'(t) &= \int_{S^{n-1}} \Big\{ \omega\cdot(\nabla u)(X^*+t\omega)\,u(X^*+b^2t^{-1}\omega)\,d\sigma(\omega) \\
&\qquad\qquad -\rho^2t^{-2}u(X^*+t\omega)\,\omega\cdot(\nabla u)(X^*+\rho^2t^{-1}\omega)\Big\}\,d\sigma(\omega) \\
&= t^{-1}\int_{S^{n-1}} \Big\{ \partial_\nu[u(X^*+t\omega)]u(X^*+\rho^2t^{-1}\omega) \\
&\qquad\qquad -u(X^*+t\omega)\partial_\nu[u(X^*+\rho^2t^{-1}\omega)]\Big\}\,d\sigma(\omega) \\
&= t^{-1}\int_{B(0,1)} \Big\{ \Delta[u(X^*+tX)]\,u(X^*+\rho^2t^{-1}X) \\
&\qquad\qquad -u(X^*+tX)\,\Delta[u(X^*+\rho^2t^{-1}X)]\Big\}\,dX = 0,
\end{aligned} \tag{4.336}$$

by Green's formula and the harmonicity of u. Hence, ϕ is a constant function on $[r,R]$. As such, $\phi(r)=\phi(R)$, proving (4.334). $\square$

Theorem 4.7.19. [`Unique Continuation Property for Harmonic Functions`] *Let $\Omega\subset\mathbb{R}^n$ be an open, connected set and assume that $u,v:\Omega\to\mathbb{R}$ are two harmonic functions. Using Lemma 4.7.18 we conclude that if $u=v$ in the neighborhood of a point $X^*\in\Omega$ then $u=v$ everywhere in Ω.*

Proof. By considering $u-v$, which continues to be a harmonic function, in place of u, it can be assumed that $v=0$. Assuming that this is the case, fix $r>0$ such that $\overline{B(X^*,r)}\subset\Omega$ and $u(X)=0$ for every $X\in B(X^*,r)$, hence for every $X\in\partial B(X^*,r)$. Pick any number R with $r<R<\operatorname{dist}(X^*,\Omega^c)$. Therefore, $\overline{B(X^*,R)}\subset\Omega$ and, if we define $\rho:=\sqrt{rR}$, then (4.334) forces

$$\int_{S^{n-1}} u^2(X^*+\rho\omega)\,d\sigma(\omega)=0. \tag{4.337}$$

Since u^2 is continuous and $\geqslant 0$, it follows from (4.337) that $u=0$ on $\partial B(X^*,\rho)$. Repeated applications of this pattern of reasoning shows that $u=0$ on $B(X^*,R)$. To summarize, we have proved that

$$\text{if } X^*\in\Omega \text{ and } u=0 \text{ near } X^* \implies u=0 \text{ on } B(X^*,\operatorname{dist}(X^*,\Omega^c)). \tag{4.338}$$

To continue, introduce

$$U:=\{X\in\Omega:\ u=0 \text{ near } X\}, \tag{4.339}$$

is a nonempty, open subset of Ω. We claim that U is a relatively closed subset of Ω. To prove this claim, consider a sequence $\{X_j\}_{j\in\mathbb{N}}$ of points in U which convergence to some point $Y \in \Omega$. Set $r := \text{dist}\,(Y, \Omega^c)$ and pick $j \in \mathbb{N}$ large enough so that $X_j \in B(Y, r/2)$. In particular, $\text{dist}(X_j, \Omega^c) > r/2$. Then u vanishes identically on $B(X_j, \text{dist}(X_j, \Omega^c))$, hence on $B(X_j, r/2)$ which is a neighborhood of Y. Then $Y \in U$, proving the claim. Since Ω is connected, the fact that the nonempty set U is simultaneously relatively closed and open in Ω forces $U = \Omega$ and $u = 0$ in Ω. $\square$

4.8 More on Integration on Spheres

Remark 4.8.1. *If $X^* \in \mathbb{R}^n$, $n \geqslant 2$, and $r > 0$, then for any continuous scalar-valued function f, depending on n variables, we have*

$$\int_{\partial B(X^*,r)} f\, d\sigma = \int_{\partial B(0,r)} f(X+X^*)\, d\sigma(X), \tag{4.340}$$

$$\int_{\partial B(0,r)} f\, d\sigma = r^{n-1}\int_{\partial B(0,1)} f(rX)\, d\sigma(X), \tag{4.341}$$

$$\int_{\partial B(0,r)} f(-X)\, d\sigma(X) = \int_{\partial B(0,r)} f(X)\, d\sigma(X). \tag{4.342}$$

Also, for each $j \in \{1,2,...,n\}$,

$$\int_{S^{n-1}} f(x_1,...,x_{j-1},-x_j,x_{j+1},...,x_n)\, d\sigma(X) = \int_{S^{n-1}} f\, d\sigma, \tag{4.343}$$

and for $1 \leqslant j,k \leqslant n$,

$$\int_{S^{n-1}} f(x_1,...,x_{j-1},x_k,x_{j+1},...,x_{k-1},x_j,x_{k+1},...,x_n)\, d\sigma(X) = \int_{S^{n-1}} f\, d\sigma. \tag{4.344}$$

Remark 4.8.2. *Let f be a continuous, real-valued function, of one variable, with the property that there exists m such that $f(\lambda t) = \lambda^m f(t)$ for any $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$. Then one will have, for any $0 \neq \eta \in \mathbb{R}^n$, where $n \geqslant 2$,*

$$\int_{S^{n-1}} f(\eta\cdot\xi)\, d\sigma(\xi) = |\eta|^m \int_{S^{n-1}} f(\xi_1)\, d\sigma(\xi), \tag{4.345}$$

where ξ_1 is the first component of ξ.

Indeed, with the invariance of the integral on S^{n-1} under rotations the result above will follow. Assume $n \geqslant 3$, let $R > 0$ be fixed and, for $r \in (0,R)$, $\varphi_1 \in (0,\pi)$, $\varphi_2 \in (0,\pi)$, $\ldots$, $\varphi_{n-2} \in (0,\pi)$, and $\varphi_{n-1} \in (0,2\pi)$ set x_1, x_2, ... x_n as in (4.116).

Definition 4.8.3. *Assume that $X^* \in \mathbb{R}^n$, $n \geqslant 3$, is a fixed point, and $R > 0$ is arbitrary. The* **standard parametrization of the ball** *$B(X^*,R)$ is defined as*

$$\begin{aligned} &\mathcal{P} : (0,\pi)^{n-2} \times (0,2\pi) \times (0,R) \longrightarrow \mathbb{R}^n, \\ &\mathcal{P}(\varphi_1,\varphi_2,\ldots,\varphi_{n-1},r) := X^* + (x_1,x_2,\ldots,x_n). \end{aligned} \tag{4.346}$$

Remark 4.8.4. *One can have that $\mathcal{P}$, defined above, is an injective function of class C^∞, takes values in $B(X^*,R)$, its image differs from $B(X^*,R)$ by a subset of $\mathbb{R}^n$ having content zero, and*

$$\det (D\mathcal{P})(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, r) = r^{n-1}(\sin\, \varphi_1)^{n-2}(\sin\, \varphi_2)^{n-3} \ldots (\sin\, \varphi_{n-2}), \quad (4.347)$$

at every point in its domain.

Proposition 4.8.5. *For $r > 0$ and $X^* \in \mathbb{R}^n$, $n \geqslant 3$, given, recall the function P_r from* (4.115)-(4.116). *Then* (4.347) *holds, i.e.,*

$$\left\| \frac{\partial P_r}{\partial \varphi_1} \times \frac{\partial P_r}{\partial \varphi_2} \times \cdots \times \frac{\partial P_r}{\partial \varphi_{n-1}} \right\| = r^{n-1}(\sin\, \varphi_1)^{n-2}(\sin\, \varphi_2)^{n-3} \ldots (\sin \varphi_{n-2}), \quad (4.348)$$

where, as usual, $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^n$. In particular, this shows that condition (4.86) *holds and, hence, P_r is a local parametrization of $\partial B(X^*, r)$.*

Proof. We shall use formulas (4.48)-(4.49) with $v_j := \partial P_r / \partial \varphi_j$, $1 \leqslant j \leqslant n-1$. Based on properties of determinants, we can factor out r from each of the $n-1$ rows of the matrices A_j, which accounts for the factor r^{n-1} in the right-hand side of (4.347). Hence, without any loss of generality, we can assume that $r = 1$ to begin with. To continue, recall the x_j's defined in (4.116) and, for each $j \in \{1, \ldots, n\}$, define the $(n-1) \times 1$ matrix

$$\nabla' x_j := \left(\frac{\partial x_j}{\partial \varphi_k} \right)_{1 \leqslant k \leqslant n-1} \quad (4.349)$$

viewed as a column, where ∇' is the gradient in $\mathbb{R}^{n-1}$. Then

$$\frac{\partial P_r}{\partial \varphi_1} \times \frac{\partial P_r}{\partial \varphi_2} \times \cdots \times \frac{\partial P_r}{\partial \varphi_{n-1}} = \sum_{j=1}^{n} (-1)^{j+n} \,(\det A_j)\, \mathbf{e}_j, \quad (4.350)$$

where, for each $j \in \{1, \ldots, n\}$, A_j is the $(n-1) \times (n-1)$ matrix given by

$$A_j := (\nabla' x_1, \ldots, \nabla' x_{j-1}, \nabla' x_{j+1}, \ldots, \nabla' x_n). \quad (4.351)$$

Consequently,

$$\left\| \frac{\partial P}{\partial \varphi_1} \times \frac{\partial P}{\partial \varphi_2} \times \cdots \times \frac{\partial P}{\partial \varphi_{n-1}} \right\| = \sqrt{\sum_{j=1}^{n} (\det A_j)^2}. \quad (4.352)$$

In order to facilitate the computation of the determinant of each A_j, observe that

$$\nabla' x_1 = x_1 \begin{pmatrix} -\tan\varphi_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \nabla' x_2 = x_2 \begin{pmatrix} \cot\varphi_1 \\ -\tan\varphi_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad ...$$

$$..., \quad \nabla' x_{n-1} = x_{n-1} \begin{pmatrix} \cot\varphi_1 \\ \vdots \\ \cot\varphi_{n-2} \\ -\tan\varphi_{n-1} \end{pmatrix}, \quad \nabla' x_n = x_n \begin{pmatrix} \cot\varphi_1 \\ \vdots \\ \cot\varphi_{n-1} \end{pmatrix}. \tag{4.353}$$

Using this, and also factoring out cot φ_1 from row$_1$ A_j, cot φ_2 from row$_2$ A_j, $\ldots$, cot φ_{n-1} from row$_{n-1}$ A_j, it follows that

$$\det A_j = \frac{\prod_{k=1}^{n} x_k}{x_j} \left(\prod_{k=1}^{n-1} \cot\varphi_k \right) \det B_j, \qquad \forall\, j \in \{1, \ldots, n\}, \tag{4.354}$$

where B_j is the $(n-1) \times (n-1)$ matrix obtained by eliminating column j from the $(n-1) \times n$ matrix

$$B := \begin{pmatrix} -\tan^2\varphi_1 & 1 & 1 & \ldots & 1 & 1 \\ 0 & -\tan^2\varphi_2 & 1 & \ldots & 1 & 1 \\ 0 & 0 & -\tan^2\varphi_3 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & -\tan^2\varphi_{n-1} & 1 \end{pmatrix}. \tag{4.355}$$

Since B_n is an upper triangular matrix, $\det B_n$ is just the product of the elements on its main diagonal. That is,

$$\det B_n = \prod_{k=1}^{n-1} \tan^2\varphi_k. \tag{4.356}$$

Similarly,

$$\det B_{n-1} = \prod_{k=1}^{n-2} \tan^2\varphi_k. \tag{4.357}$$

The computation of $\det B_{n-2}$ is somewhat more involved since this particular matrix is no longer upper-triangular. This time, after subtracting the last column of B_{n-2} from the second-to-last column of B_{n-2}, we arrive at

$$\det B_{n-2} = \det \begin{pmatrix} -\tan^2\varphi_1 & 1 & 1 & \dots & 0 & 1 \\ 0 & -\tan^2\varphi_2 & 1 & \dots & 0 & 1 \\ 0 & 0 & -\tan^2\varphi_3 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -1-\tan^2\varphi_{n-1} & 1 \end{pmatrix}$$

$$= (1+\tan^2\varphi_{n-1}) \det \begin{pmatrix} -\tan^2\varphi_1 & 1 & 1 & \dots & 1 & 1 \\ 0 & -\tan^2\varphi_2 & 1 & \dots & 1 & 1 \\ 0 & 0 & -\tan^2\varphi_3 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\tan^2\varphi_{n-3} & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$= \pm \left(\prod_{k=1}^{n-3} \tan^2\varphi_k \right) (1+\tan^2\varphi_{n-1}), \tag{4.358}$$

where the second equality follows by expanding the determinant according to the next-to-last column. The actual sign in front of the product is not important since we are going to eventually work with the square of this expression.

When computing $\det B_{n-3}$, we first subtract the last column from the second to the last column, then expend with respect to column $(n-2)$, then again subtract the last column from the second to the last column and expend over the second to the last column. The resulting determinant will be that of an upper-triangular matrix. As such, we obtain

$$\det B_{n-3} = \pm \left(\prod_{k=1}^{n-4} \tan^2\varphi_k \right) (1+\tan^2\varphi_{n-2})(1+\tan^2\varphi_{n-1}). \tag{4.359}$$

Inductively, one can prove that for $j \in \{i, \dots, n\}$,

$$\det B_j = \pm \prod_{k=1}^{j-1} \tan^2\varphi_k \prod_{k=j+1}^{n-1} (1+\tan^2\varphi_k) = \pm \prod_{k=1}^{j-1} \tan^2\varphi_k \prod_{k=j+1}^{n-1} \frac{1}{\cos^2\varphi_k}, \tag{4.360}$$

with the convention that when $j = 1$ the first product is not written, and when $j = n-1$, the second product is not written. Combining all these we arrive at the conclusion that

$$\det A_n = \frac{\prod\limits_{k=1}^{n} x_k}{x_n} \left(\prod_{k=1}^{n-1} \cot\varphi_k \right) \prod_{k=1}^{n-1} \tan^2\varphi_k = \left(\prod_{k=1}^{n} x_k \right) \frac{1}{x_n} \prod_{k=1}^{n-1} \tan\varphi_k$$

$$= \prod_{k=1}^{n} x_k \cdot \frac{1}{\prod\limits_{k=1}^{n-1} \cos\varphi_k}, \tag{4.361}$$

and, if $j \in \{1, \dots, n-1\}$,

$$\det A_j = \pm \left(\prod_{k=1}^{n} x_k \right) \frac{1}{x_j} \prod_{k=1}^{n-1} \cot\, \varphi_k \prod_{k=1}^{j-1} \tan^2 \varphi_k \prod_{k=j+1}^{n-1} \frac{1}{\cos^2 \varphi_k}$$

$$= \pm \prod_{k=1}^{n} x_k \cdot \frac{1}{\prod\limits_{k=1}^{n-1} \cos\, \varphi_k} \cdot \frac{\cos \varphi_j}{\prod\limits_{k=j}^{n-1} \sin\, \varphi_j}. \tag{4.362}$$

Thus,

$$\left\| \frac{\partial P_r}{\partial \varphi_1} \times \cdots \times \frac{\partial P_r}{\partial \varphi_{n-1}} \right\|^2 = \prod_{k=1}^{n} x_k^2 \cdot \frac{1}{\prod\limits_{k=1}^{n-1} \cos^2 \varphi_k} \left[\sum_{j=1}^{n-1} \frac{\cos^2 \varphi_j}{\prod\limits_{k=j}^{n-1} \sin^2 \varphi_j} + 1 \right]$$

$$= \frac{\prod\limits_{k=1}^{n} x_k^2}{\prod\limits_{k=1}^{n-1} \cos^2 \varphi_k} \cdot \frac{1}{\prod\limits_{k=1}^{n-1} \sin^2 \varphi_k} \Big[\cos^2 \varphi_1 + \sin^2 \varphi_1 \cos^2 \varphi_2 + \sin^2 \varphi_1 \sin^2 \varphi_2 \cos^2 \varphi_3$$

$$+ \cdots + \sin^2 \varphi_1 \sin^2 \varphi_2 \dots \sin^2 \varphi_{n-2} \cos^2 \varphi_{n-1} + \sin^2 \varphi_1 \dots \sin^2 \varphi_{n-1} \Big]$$

$$= \frac{\prod\limits_{k=1}^{n} x_k^2}{\prod\limits_{k=1}^{n-1} \cos^2 \varphi_k} \cdot \frac{1}{\prod\limits_{k=1}^{n-1} \sin^2 \varphi_k} = (\sin \varphi_1)^{2(n-1)} (\sin\, \varphi_2)^{2(n-3)} \dots (\sin\, \varphi_{n-2})^2, \tag{4.363}$$

as desired. This finishes the proof of the proposition. □

Theorem 4.8.6 (Spherical Fubini and Polar Coordinates)**.** *Let f be a scalar-valued, continuous function, depending on n variables. Then for each $X^* \in \mathbb{R}^n$, $n \geqslant 2$, and each $R > 0$ the following formulas hold:*

$$\int_{B(X^*,R)} f(X)\, dX = \int_0^R \Big(\int_{\partial B(X^*,r)} f\, d\sigma \Big) dr, \qquad (\textit{spherical Fubini}) \tag{4.364}$$

$$\int_{B(X^*,R)} f(X)\, dX = \int_0^R \int_{\partial B(0,1)} f(X^* + r\omega) r^{n-1}\, d\sigma(\omega)\, dr \tag{4.365}$$

$$= \int_0^R \int_{\partial B(X^*,1)} f(r\omega) r^{n-1}\, d\sigma(\omega)\, dr \quad (\textit{polar coordinates}).$$

In particular, when X^* is the origin of $\mathbb{R}^n$,

$$\int_{B(0,R)} f = \int_0^R \Big(\int_{\partial B(0,r)} f\, d\sigma \Big) dr, \qquad (\textit{spherical Fubini}) \tag{4.366}$$

$$\int_{B(0,R)} f = \int_0^R \int_{\partial B(0,1)} f(r\omega) r^{n-1}\, d\sigma(\omega)\, dr \qquad (\textit{polar coordinates}). \tag{4.367}$$

Proof. We shall treat in detail the case when $n \geqslant 3$, and leave the simpler case $n = 2$ as an exercise. Let $X^* \in \mathbb{R}^n$, $R > 0$ be fixed. Then, as in (4.116), for $\varphi_1 \in (0, \pi)$, $\varphi_2 \in (0, \pi)$, ..., $\varphi_{n-2} \in (0, \pi)$, $\varphi_{n-1} \in (0, 2\pi)$ and $r \in (0, R)$, recall the setting from (4.116). These are used (cf. Definition 4.8.3) to parametrize the ball $B(X^*, R)$ by

$$\begin{aligned} &\mathcal{P} : (0, \pi)^{n-2} \times (0, 2\pi) \times (0, R) \longrightarrow \mathbb{R}^n \\ &\mathcal{P}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, r) := X^* + (x_1, x_2, \ldots, x_n). \end{aligned} \tag{4.368}$$

Note that, for each fixed $r \in (0, R)$,

$$\mathcal{P}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, r) = P_r(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}), \tag{4.369}$$

where P_r, defined as in (4.346), is the standard parametrization of $\partial B(X^*, r)$. Then, using this parametrization the Change of Variable Theorem gives

$$\begin{aligned} &\int_{B(X^*,R)} f(X)\, dX \\ &\qquad = \int_0^R \int_{(0,\pi)^{n-2}\times(0,2\pi)} f(\mathcal{P}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, r)) \cdot \\ &\qquad\qquad \cdot |\det \mathcal{J}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, r)|\, d\varphi_1 \ldots d\varphi_{n-1}\, dr, \end{aligned} \tag{4.370}$$

where we have denoted by $\mathcal{J}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, r)$ the Jacobian matrix corresponding to the parametrization $\mathcal{P}$ above. In other words, $\mathcal{J}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, r)$ is the $n \times n$ matrix whose columns are given by

$$\begin{aligned} &\nabla' x_1 = x_1 \begin{pmatrix} -\tan\varphi_1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \nabla' x_2 = x_2 \begin{pmatrix} \cot\varphi_1 \\ -\tan\varphi_2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \ldots \\ &\ldots, \quad \nabla' x_{n-1} = x_{n-1} \begin{pmatrix} \cot\varphi_1 \\ \vdots \\ \cot\varphi_{n-2} \\ -\tan\varphi_{n-1} \\ 1 \end{pmatrix}, \quad \nabla' x_n = x_n \begin{pmatrix} \cot\varphi_1 \\ \vdots \\ \cot\varphi_{n-1} \\ 1 \end{pmatrix}, \end{aligned} \tag{4.371}$$

where, as before, ∇' is the gradient in $\mathbb{R}^{n-1}$. Much as in the proof of Proposition 4.8.5, after factoring out $\cot\varphi_1$ from the first row, $\cot\varphi_2$ from the second row, ..., $\cot\varphi_{n-1}$ from the $(n-1)$-th row of $\mathcal{J}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, r)$ we arrive at the conclusion that

$$\det\, \mathcal{J}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, r) = \prod_{k=1}^{n} x_k \left(\prod_{k=1}^{n-1} \cot\,\varphi_k \right) \det \mathcal{B}, \tag{4.372}$$

where $\mathcal{B}$ is the $n \times n$ matrix

$$\mathcal{B} := \begin{pmatrix} -\tan^2\varphi_1 & 1 & 1 & \dots & 1 & 1 \\ 0 & -\tan^2\varphi_2 & 1 & \dots & 1 & 1 \\ 0 & 0 & -\tan^2\varphi_3 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & -\tan^2\varphi_{n-1} & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}. \tag{4.373}$$

Compare this with the $(n-1) \times n$ matrix B from (4.355), used in the proof of Proposition 4.8.5. It is immediate that, after subtracting the last column of $\mathcal{B}$ from each of the other $n-1$ columns of $\mathcal{B}$, we obtain a triangular matrix whose determinant is equal to $(-1)^{n-1}\prod_{k=1}^{n-1}(1+\tan^2\varphi_k)$. Utilizing this back in (4.372) proves that

$$|\det \mathcal{J}(\varphi_1, \varphi_2, \dots, \varphi_{n-1}, r)| = r^{n-1}(\sin\varphi_1)^{n-2}(\sin\varphi_2)^{n-3}\dots(\sin\varphi_{n-2}). \tag{4.374}$$

In concert with Proposition 4.8.5, this shows that

$$|\det \mathcal{J}(\varphi_1, \varphi_2, \dots, \varphi_{n-1}, r)| = \left\| \frac{\partial P_r}{\partial \varphi_1} \times \dots \times \frac{\partial P_r}{\partial \varphi_{n-1}} \right\|, \tag{4.375}$$

the Jacobian of the standard parametrization P_r of the sphere $\partial B(0,r)$ in $\mathbb{R}^n$. Consequently, Definition 4.5.1 gives

$$\begin{aligned} &\int\limits_{(0,\pi)^{n-2}\times(0,2\pi)} f(\mathcal{P}(\varphi_1, \varphi_2, \dots, \varphi_{n-1}, r))|\det \mathcal{J}(\varphi_1, \varphi_2, \dots, \varphi_{n-1}, r)|\, d\varphi_1 \dots d\varphi_{n-1} \\ &\qquad = \int\limits_{\partial B(X^*,r)} f(Y)\, d\sigma(Y) \end{aligned} \tag{4.376}$$

which, when combined with (4.370), completes the proof of (4.364). Going further, observe that

$$\mathcal{P}(\varphi_1, \varphi_2, \dots, \varphi_{n-1}, r) = X^* + r\,P_1(\varphi_1, \varphi_2, \dots, \varphi_{n-1}), \qquad \forall\, r > 0, \tag{4.377}$$

where $P_1(\varphi_1, \varphi_2, \dots, \varphi_{n-1})$ denotes the standard parametrization of the unit sphere centered at the origin in $\mathbb{R}^n$ (here, as usual, $\varphi_1 \in (0,\pi)$, $\varphi_2 \in (0,\pi)$, $\dots$, $\varphi_{n-2} \in (0,\pi)$ and $\varphi_{n-1} \in (0, 2\pi)$). From this, (4.370) and (4.348), we then have

$$\begin{aligned} \int\limits_{B(X^*,R)} f(X)\, dX &= \int_0^R \int\limits_{(0,\pi)^{n-2}\times(0,2\pi)} f(X^* + rP_1(\varphi_1, \varphi_2, \dots, \varphi_{n-1})) \times \\ &\qquad \times r^{n-1}(\sin\varphi_1)^{n-2}(\sin\varphi_2)^{n-3}\dots(\sin\varphi_{n-2})\, d\varphi_1 \dots d\varphi_{n-1}\, dr \\ &= \int_0^R \int\limits_{\partial B(0,1)} f(X^* + r\omega) r^{n-1}\, d\sigma(\omega)\, dr, \end{aligned} \tag{4.378}$$

and (4.366) is proved. $\square$

Lemma 4.8.7. *For a fixed $X^* \in \mathbb{R}^n$, $n \geqslant 2$, and f a continuous, real-valued function, on n variables, then*

$$\frac{d}{dR}\Big(\int_{B(X^*,R)} f(X)\,dX\Big) = \int_{\partial B(X^*,R)} f\,d\sigma. \tag{4.379}$$

Proof. This is a consequence of (4.364) and the Fundamental Theorem of Calculus, as soon as we show that the assignment

$$(0,\infty) \ni r \mapsto \int_{\partial B(X^*,r)} f\,d\sigma \in \mathbb{R} \tag{4.380}$$

is a continuous function. This, however, can be justified using (4.376). $\square$

Proposition 4.8.8. *Let f be a real-valued, continuous function of n variables, $n \geqslant 2$, and fix $R > 0$. Denote by $B_n(0,R)$ the n-dimensional ball of radius R centered at the origin. Then*

$$\int_{\mathbb{R}^n_\pm \cap \partial B_n(0,R)} f\,d\sigma = \int_{B_{n-1}(0,R)} f(x', \pm\sqrt{R^2-|x'|^2})\frac{R}{\sqrt{R^2-|x'|^2}}\,dx'. \tag{4.381}$$

Proof. Consider the parametrization of $\mathbb{R}^n_\pm \cap \partial B_n(0,R)$ given by

$$P_\pm : B_{n-1}(0,R) \to \mathbb{R}^n_\pm \cap \partial B_n(0,R), \qquad P_\pm(x') = (x', \pm\sqrt{R^2-|x'|^2}). \tag{4.382}$$

The key aspect is that

$$\begin{aligned}\left\|\frac{\partial P_\pm}{\partial x_1}\times\cdots\times\frac{\partial P_\pm}{\partial x_{n-1}}\right\| &= \left\|\left(\frac{\pm x_1}{\sqrt{R^2-|x'|^2}},\dots,\frac{\pm x_{n-1}}{\sqrt{R^2-|x'|^2}},1\right)\right\| \\ &= \frac{R}{\sqrt{R^2-|x'|^2}}.\end{aligned} \tag{4.383}$$

Indeed,

$$\frac{\partial P_\pm}{\partial x_1}\times\cdots\times\frac{\partial P_\pm}{\partial x_{n-1}} := \begin{pmatrix} 1 & 0 & 0 & & 0 & \mp\frac{x_1}{\sqrt{R^2-|x'|^2}} \\ 0 & 1 & 0 & & 0 & \mp\frac{x_2}{\sqrt{R^2-|x'|^2}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & \mp\frac{x_{n-1}}{\sqrt{R^2-|x'|^2}} \\ e_1 & e_2 & e_3 & & e_{n-1} & e_n \end{pmatrix}. \tag{4.384}$$

The j-th component ($1 \leqslant j \leqslant n$) of this vector is equal to $(-1)^{j+n}\det A_j^\pm$ where $A_j^\pm$ is the $(n-1)\times(n-1)$-matrix obtained from the matrix above by eliminating the j-th column along with the n-th row. It follows that

$$\det A_j^\pm = \begin{cases} 1 & \text{if } j = n, \\ (-1)^{j+n-1}\frac{\mp x_j}{\sqrt{R^2-|x'|^2}} & \text{if } j < n. \end{cases} \tag{4.385}$$

This is clear when $j = n$, since $A_n^{\pm} = I_{(n-1)\times(n-1)}$. When $j < n$, each of the first $n-2$ columns in the matrix $A_j^{\pm}$ contains just one nonzero entry (which is equal to 1). Furthermore, when regarded as vectors in $\mathbb{R}^{n-1}$, these are linearly independent. More precisely, they can be identified with $\mathbf{e}'_1, ..., \mathbf{e}'_{j-1}, \mathbf{e}'_{j+1}, ..., \mathbf{e}'_{n-1}$, where $\{\mathbf{e}'_k\}_{1\leqslant k\leqslant n-1}$ is the standard orthonormal basis in $\mathbb{R}^{n-1}$. As such, these can be used to obtain zeroes on all but the j-th entry in the last column, which is $\frac{\mp x_j}{\sqrt{R^2-|x'|^2}}$. This justifies (4.385). Once this has been established, (4.383) readily follows. In turn, this and Definition 4.5.1 show that (4.381) holds. $\square$

Proposition 4.8.9. *Retain our earlier convention to the effect that $B_n(0,R)$ denotes the n-dimensional ball. Then, for every continuous, real-valued function f of n variables, $n \geqslant 2$*

$$\int_{\partial B_n(0,R)} f\,d\sigma = \int_{-R}^{R}\Big(\int_{\partial B_{n-1}(0,\sqrt{R^2-s^2})} f(s,\theta)\,d\sigma(\theta)\Big)\frac{R}{\sqrt{R^2-s^2}}\,ds. \tag{4.386}$$

Proof. The proof is a consequence of Proposition 4.8.8 (applied to each hemisphere!), the spherical Fubini formula (cf. (4.366)), and the change of variables $s := \sqrt{R^2-r^2}$ $\square$

Definition 4.8.10. *Recall that S^{n-1} denotes $\partial B(0,1)$, the canonical unit sphere in $\mathbb{R}^n$, $n \geqslant 2$. Define the* **stereographic projection**

$$\Pi : S^{n-1} \setminus \{\mathbf{e}_n\} \longrightarrow \mathbb{R}^{n-1} \tag{4.387}$$

by setting

$$\begin{aligned}\Pi(z) \;:=\;& \text{the intersection point between the } (n-1)\text{-dimensional}\\ & \text{plane } \mathbb{R}^{n-1}\times\{0\} \text{ with the line joining } \mathbf{e_n} \text{ with } z.\end{aligned} \tag{4.388}$$

Remark 4.8.11. *The stereographic projection Π in (4.387)-(4.388) is a bijection whose inverse, $\Pi^{-1} : \mathbb{R}^{n-1} \to S^{n-1}\setminus\{\mathbf{e}_n\} \subset \mathbb{R}^n$, $n \geqslant 2$, is given by*

$$\Pi^{-1}(z') = \left(\frac{2z'}{1+|z'|^2}, \frac{|z'|^2-1}{|z'|^2+1}\right), \qquad \forall\, z' \in \mathbb{R}^{n-1}. \tag{4.389}$$

Proposition 4.8.12. *With the above notation and conventions,*

$$\left\|\frac{\partial \Pi^{-1}}{\partial z_1}\times\frac{\partial \Pi^{-1}}{\partial z_2}\times\cdots\times\frac{\partial \Pi^{-1}}{\partial z_{n-1}}\right\| = \left(\frac{2}{1+|z'|^2}\right)^{n-1} \quad \forall\, z' = (z_1,...,z_{n-1}) \in \mathbb{R}^{n-1}. \tag{4.390}$$

Consequently, Π^{-1} is a global parametrization of $S^{n-1}\setminus\{\mathbf{e}_n\}$ and, for every real-valued, continuous function f of n variables, $n\geqslant 2$,

$$\int\limits_{S^{n-1}} f\,d\sigma=\int\limits_{\mathbb{R}^{n-1}} f\Big(\frac{2z_1}{1+|z'|^2},\frac{2z_2}{1+|z'|^2},\dots\frac{2z_{n-1}}{1+|z'|^2},\frac{|z'|^2-1}{|z'|^2+1}\Big)\Big(\frac{2}{1+|z'|^2}\Big)^{n-1}dz'.\tag{4.391}$$

Proof. If we write

$$x_j:=\frac{2z_j}{1+|z'|^2}\ \text{if}\ 1\leqslant j\leqslant n-1,\quad\text{and}\ \ x_n=\frac{1-|z'|^2}{1+|z'|^2},\tag{4.392}$$

then, much as in (4.352),

$$\Big\|\frac{\partial\Pi^{-1}}{\partial z_1}\times\frac{\partial\Pi^{-1}}{\partial z_2}\times\cdots\times\frac{\partial\Pi^{-1}}{\partial x_n}\Big\|=\sqrt{\sum_{j=1}^{n}(\det A_j)^2},\tag{4.393}$$

where the matrices A_j are as in (4.351) (with x_j as in (4.392)). Note that (with ∇' denoting the gradient in $\mathbb{R}^{n-1}$), we have

$$\nabla' x_j=\begin{cases}\frac{2}{1+|z'|^2}\,\mathbf{e}'_j-\frac{4z_j}{(1+|z'|^2)^2}z' & \text{if } 1\leqslant j\leqslant n-1,\\ \frac{4}{(1+|z'|^2)^2}z' & \text{if } j=n,\end{cases}\tag{4.394}$$

where, for $1\leqslant j\leqslant n-1$, $\mathbf{e}'_j$ is the j-th vector in the standard orthonormal basis in $\mathbb{R}^{n-1}$.

When $1\leqslant j\leqslant n-1$, relocate the last column to be the j-th one and add suitable multiples of this column to the other $n-2$ columns in A_j so that any column (except the j-th) has at most one nonzero entry, located on the main diagonal. Note that these operations preserve $\det A_j$ up to a sign which in turn, is the product of the entries on the main diagonal in the resulting matrix. These are $\frac{2}{1+|z'|^2}$, showing up $n-2$ times, and $\frac{4z_j}{(1+|z'|^2)^2}$, appearing once. Hence,

$$|\det A_j|=z_j\Big(\frac{2}{1+|z'|^2}\Big)^n,\qquad\text{if}\ \ 1\leqslant j\leqslant n-1.\tag{4.395}$$

As for $\det A_n$, factoring out $-\frac{4z_j}{(1+|z'|^2)^2}$ from the j-th column, $1\leqslant j\leqslant n-1$, and then z_j from the j-th line, $1\leqslant j\leqslant n-1$, allows us to bring it in the form

$$\frac{(-4z_1^2)(-4z_2^2)\cdot(-4z_{n-1}^2)}{(1+|z'|^2)^{2(n-1)}}\cdot\det\Big(1-\delta_{jk}(1+|z'|^2)/(2z_j^2)\Big)_{1\leqslant j,k\leqslant n-1}.\tag{4.396}$$

Since, generally speaking,

$$\det\Big(1+b_j\delta_{jk}\Big)_{1\leqslant j,k\leqslant n-1}=\Big(\prod_{j=1}^{n-1}b_j\Big)\Big(1-\sum_{j=1}^{n-1}\frac{1}{b_j}\Big),\tag{4.397}$$

for any $b_1,\dots,b_{n-1}\in\mathbb{R}$, we may finally conclude from this and (4.396) that

$$\det A_n=2^{n-1}\cdot\frac{1-|z'|^2}{(1+|z'|^2)^n}.\tag{4.398}$$

All together, (4.395) and (4.398) imply

$$\sqrt{\sum_{j=1}^{n}(\det A_j)^2} = \left(\frac{2}{1+|z'|^2}\right)^{n-1} \tag{4.399}$$

which, in concert with (4.393), proves (4.390). $\square$

Chapter 5

Geometric Analysis

5.1 Domains Satisfying Uniform Ball and Cone Conditions

Definition 5.1.1. *A function $\varphi : \mathcal{O} \to \mathbb{R}$ (where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$) is of said to be of* **class** $C^{k,1}$, *for some $k \in \mathbb{N}$, if it is of class C^k and, for each multi-index $\alpha \in \mathbb{N}_0^n$ of length k, $\partial^\alpha \varphi$ is a Lipschitz function (that is, there exists a number $M > 0$ with the property that $|(\partial^\alpha \varphi)(x) - (\partial^\alpha \varphi)(y)| \leqslant M\|x - y\|$ for every $x, y \in \mathcal{O}$).*

Definition 5.1.2. *(i) Given $k \in \mathbb{N}$, a C^k domain $\Omega \subseteq \mathbb{R}^n$ is called a* **domain of class** $C^{k,1}$ *provided the function φ appearing in Definition 4.6.1 is of class $C^{k,1}$.*

(ii) Corresponding to the case $k = 0$, a set $\Omega \subseteq \mathbb{R}^n$ is called a **domain of class** $C^{0,1}$ *provided it satisfies all conditions in Definition 4.6.1, the only alteration being that the function φ appearing in this definition is actually Lipschitz.*

Lemma 5.1.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain of class C^1, and denote by ν its outward unit normal. Then Ω is a domain of class $C^{1,1}$ if and only if $\nu : \partial\Omega \to \mathbb{R}^n$ is a Lipschitz function.*

Proof. In one direction, if Ω is a domain of class $C^{1,1}$, then the outward unit normal is locally Lipschitz, hence Lipschitz since $\partial\Omega$ is a compact set (cf. Remark 5.1.4 below).

Remark 5.1.4. *Let $K \subseteq \mathbb{R}^n$ be a compact set and assume that $f : K \to \mathbb{R}^m$ is a* **locally Lipschitz function**, *i.e., f has the property that for every $X^* \in K$ there exists $r > 0$ such that $f|_{K \cap B(X^*, r)}$ is Lipschitz. Then f is Lipschitz.*

Indeed, a reason by contradiction, and making use of local Lipschitzianity property and the Bolzano-Weierstrass theorem will give the desired result.

Conversely, assume that $\Omega \subseteq \mathbb{R}^n$ is a domain of class C^1 and that $\nu : \partial\Omega \to \mathbb{R}^n$ is Lipschitz. Let $\varphi : \mathcal{O} \to \mathbb{R}$, with $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ open, be a function which locally describes the boundary of $\partial\Omega$ (after a suitable rigid transformation of the Euclidean space). Then the functions $g, f_j : \mathcal{O} \to \mathbb{R}$ given by

$$g(x') := \frac{1}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}, \qquad f_j(x') := \frac{\partial_j\varphi(x')}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}, \qquad 1 \leqslant j \leqslant n-1, \tag{5.1}$$

are Lipschitz. By eventually shrinking $\mathcal{O}$ it can be assumed that there exists $c > 0$ with the property that $c \leqslant g(x') \leqslant 1$ for every $x' \in \mathcal{O}$. Using this we obtain that $1/g$ is Lipschitz and, further, that each $\partial_j\varphi = (1/g)f_j$ is Lipschitz. Hence, ultimately, φ is of class $C^{1,1}$, proving that Ω is a domain of class $C^{1,1}$. $\square$

Definition 5.1.5. *Fix a proper, non-empty, arbitrary subset D of $\mathbb{R}^n$.*

(1) We say that D satisfies a **uniform exterior ball condition** *(UEBC) if there exists a number $r > 0$ such that for any $X^* \in \partial D$ there exists a point $X \in \mathbb{R}^n$ for which $B(X,r) \cap D = \varnothing$ and $X^* \in \partial B(X,r)$. We shall call the supremum of all such numbers r the* **UEBC constant** *of D.*

(2) We say that D satisfies a **uniform interior ball condition** *(UIBC) if $\mathbb{R}^n \setminus D$ satisfies a uniform exterior ball condition. The UEBC constant of $\mathbb{R}^n \setminus D$ will be referred to as the* **UIBC constant** *of D.*

(3) We say that $D \subseteq \mathbb{R}^n$ satisfies a **uniform two-sided ball condition** *if D satisfies both a UEBC and a UIBC. We shall refer to the minimum between the UEBC UIBC constants of D as the* **uniform two-sided ball condition constant** *of D.*

Remark 5.1.6. *(i) It is clear that $D \subseteq \mathbb{R}^n$ satisfies a uniform two-sided ball condition if and only if there exists a number $r > 0$ such that for any $X^* \in \partial D = \partial(\mathbb{R}^n \setminus D)$ there exist $X_\pm \in \mathbb{R}^n$ for which*

$$\begin{gathered} B(X_-,r) \cap D = \varnothing, \quad B(X_+,r) \cap (\mathbb{R}^n \setminus D) = \varnothing, \\ \textit{and} \;\; X^* \in \partial B(X_+,r) \cap \partial B(X_-,r). \end{gathered} \tag{5.2}$$

(ii) Informally speaking, a UEBC (respectively, UIBC) for a set Ω is equivalent with the requirement that one can roll a ball of fixed radius along the boundary of Ω, on the outside (respectively, inside) of Ω.

Lemma 5.1.7. *If $\Omega \subseteq \mathbb{R}^n$ satisfies a Uniform Exterior Ball Condition, then $\partial\Omega = \partial(\overline{\Omega})$.*

Proof. The UEBC implies that $\partial\Omega \subseteq \overline{(\Omega^c)^\circ}$, or $\partial\Omega \subseteq \overline{\overline{(\Omega)}} \bigcap \overline{(\overline{\Omega})^c} = \partial(\overline{\Omega})$. Hence, $\partial\Omega \subseteq \partial(\overline{\Omega})$. Since the opposite inclusion is always true (cf. (6) in Theorem 4.6.4), we ultimately have $\partial\Omega = \partial(\overline{\Omega})$. $\square$

Proposition 5.1.8. *Assume that $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ is an open neighborhood of the origin and $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a function satisfying $\varphi(0) = 0$, which is differentiable, and whose derivative is continuous at $0 \in \mathbb{R}^{n-1}$. Let $\mathcal{R}$ be a rotation about the origin in $\mathbb{R}^n$ with the property that*

$$\mathcal{R} \text{ maps the vector } \frac{(\nabla\varphi(0), -1)}{\sqrt{1 + \|\nabla\varphi(0)\|^2}} \text{ into } -\mathbf{e}_n \in \mathbb{R}^n. \tag{5.3}$$

Then there exists a continuous, real-valued function ψ defined in a small neighborhood of $0 \in \mathbb{R}^{n-1}$ with the property that $\psi(0) = 0$ and whose graph coincides, in a small neighborhood of $0 \in \mathbb{R}^n$, with the graph of φ rotated by $\mathcal{R}$.

Furthermore, if φ is of class C^k, for some $k \in \mathbb{N} \cup \{\infty\}$, then so is ψ.

Proof. Matching the graph of φ, after being rotated by $\mathcal{R}$, by that of a function φ comes down to ensuring that ψ is such that $\mathcal{R}(x', \varphi(x')) = (y', \psi(y'))$ can be solved both for x' in terms y', as well as for y' in terms x', near the origin in $\mathbb{R}^{n-1}$ in each instance. Let $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection map of $\mathbb{R}^n$ onto the first $n-1$ coordinates, and $\pi_n : \mathbb{R}^n \to \mathbb{R}$ be the coordinate projection map of $\mathbb{R}^n$ onto the last coordinate. Then,

$$\begin{aligned}(y', y_n) = \mathcal{R}(x', \varphi(x')) \;&\Leftrightarrow\; \mathcal{R}^{-1}(y', y_n) = (x', \varphi(x')) \\ &\Leftrightarrow\; \pi'\mathcal{R}^{-1}(y', y_n) = x' \text{ and } \pi_n\mathcal{R}^{-1}(y', y_n) = \varphi(x') \\ &\Leftrightarrow\; F(y', y_n) = 0 \text{ and } x' = \pi'\mathcal{R}^{-1}(y', y_n), \end{aligned} \tag{5.4}$$

where F is the real-valued function defined in a neighborhood of origin in $\mathbb{R}^n$ by

$$F(y', y_n) := \varphi(\pi'\mathcal{R}^{-1}(y', y_n)) - \pi_n\mathcal{R}^{-1}(y', y_n). \tag{5.5}$$

Then a direct calculation shows that $F(0,0)=0$ and

$$\partial_n F(y',y_n) = \sum_{j=1}^{n-1}(\partial_j\varphi)(\pi'\mathcal{R}^{-1}(y',y_n))(\mathcal{R}^{-1}\mathbf{e}_n)\cdot\mathbf{e}_j - (\mathcal{R}^{-1}\mathbf{e}_n)\cdot\mathbf{e}_n \tag{5.6}$$

$$= (\mathcal{R}^{-1}\mathbf{e}_n)\cdot((\nabla\varphi)(\pi'\mathcal{R}^{-1}(y',y_n))\,,\,-1) = \mathbf{e}_n\cdot\mathcal{R}((\nabla\varphi)(\pi'\mathcal{R}^{-1}(y',y_n))\,,\,-1).$$

In particular, by (5.3),

$$\partial_n F(0,0) = -\sqrt{1+\|\nabla\varphi(0)\|^2} \neq 0. \tag{5.7}$$

Thus, by the Implicit Function Theorem, there exists a continuous real-valued function ψ defined in a neighborhood of $0\in\mathbb{R}^{n-1}$ such that $\psi(0)=0$ and for which

$$F(y',y_n)=0 \Longleftrightarrow y_n=\psi(y') \quad \text{whenever } (y',y_n) \text{ is near } 0. \tag{5.8}$$

From this and (5.4) the desired conclusions follow. $\square$

Theorem 5.1.9. *Any domain Ω of class $C^{1,1}$ in $\mathbb{R}^n$ satisfies a uniform two-sided ball condition.*

Proof. Fix $X^*\in\partial\Omega$ and assume that the rigid transformation T, the function φ, the $(n-1)$-dimensional ball $B(0,R)$, the open interval I, and the open cylinder $\mathcal{C}$ retain their meanings from Definition 4.6.1. Write $I=(-h,h)$ and re-denote $T(\Omega)$ by Ω. Thus:

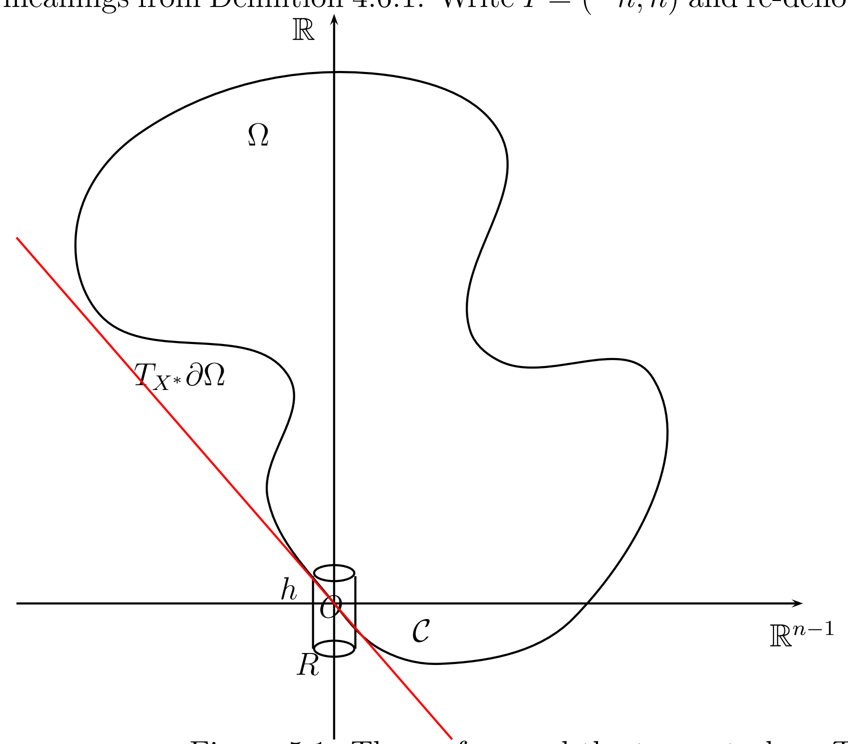

Figure 5.1: The surface and the tangent plane T.

Since the domain Ω is bounded, its boundary, $\partial\Omega$, is a compact set. This ensures that we can choose h and R independent of the point X^*. In fact, the same applies to the Lipschitz constant M of the function $\nabla\varphi : B(0,R) \to \mathbb{R}^{n-1}$.

To continue, pick a rotation $\mathcal{R}$ about the origin in $\mathbb{R}^n$ which maps the geometric tangent plane $T_{X^*}\partial\Omega$ to $\partial\Omega$ at X^* into $\mathbb{R}^{n-1} \times \{0\}$. Making use of Proposition 5.1.8, we can then assume that, in addition to the properties listed in Definition 4.6.1, the function φ also satisfies $\nabla\varphi(0) = 0$. A careful inspection of the way the Implicit Function Theorem works in the context of Proposition 5.1.8 shows that that we can still retain our condition of uniformity for the constants R, h and M. Assuming that this is the case, we once again retain the notation Ω for the domain $\mathcal{R}(\Omega)$. A picture of the current case is as follows:

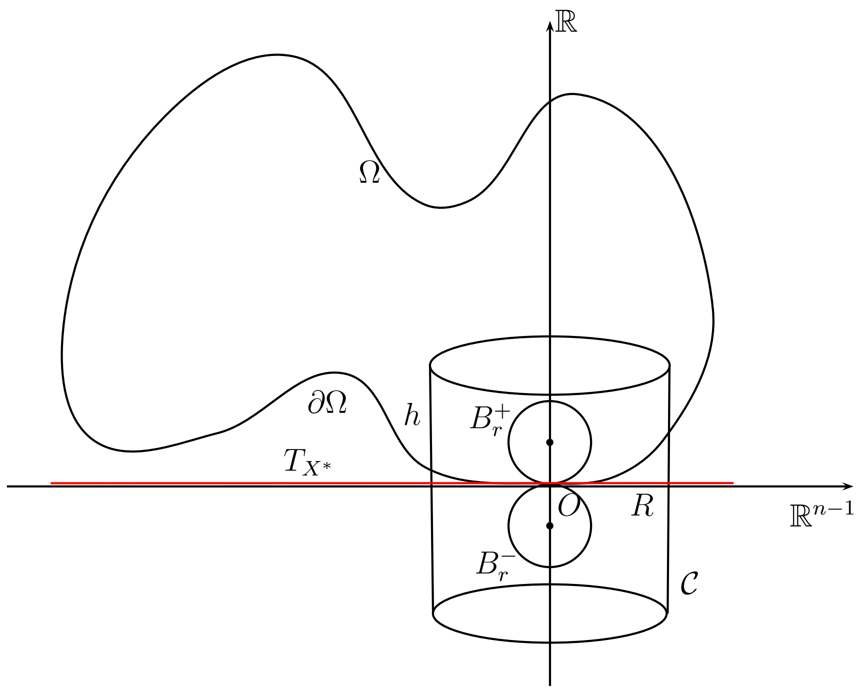

Figure 5.2: The surface, the horizontal tangent plane, and UIBC/UEBC.

To summarize, we are in a situation when the function $\varphi : B(0,R) \to (-h,h)$ is of class C^1, $\nabla\varphi$ Lipschitz with constant $\leqslant M$, $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$, where R, M and h depend only on the domain Ω. We want to find $r > 0$ such that if $B_r^{\pm} := B(\pm r\mathbf{e}_n, r)$ then $B_r^+ \subseteq \Omega$, $B_r^- \subseteq \mathbb{R}^n \setminus \Omega$ and $0 \in \partial B_r^{\pm}$. The latter condition is automatically satisfied by construction. If $0 < r < \min\{h/2, R\}$ then it is clear that the inner/outer balls $B_r^{\pm}$ are contained in the cylinder $\mathcal{C}$.

To finish the proof, we need to show that the inner ball lies above of the graph of φ, while the outer ball lies below the graph of φ (since this will imply that the inner ball is in Ω, while the outer ball is contained in the complement of Ω). As a preamble, we shall first establish the following.

`Claim 1.` *After eventually narrowing the cylinder $\mathcal{C}$, as well as decreasing the radius r (in a controlled fashion), the following property holds: any point $X \in \mathcal{C}$ lying below or on the graph of φ does not belong to the inner ball B_r^+.*

Proof of Claim 1. Pick a point $X \in \mathcal{C}$ below or on the graph of φ. This means that $X = (x', x_n)$ with $x_n \leqslant \varphi(x')$, $\|x'\| < R$, and $-h < x_n < h$. Clearly, $X \notin B_r^+$ if and only if $\|X - (0, ...0, r)\| > r$. Upon noticing that

$$\|X - (0, ...0, r)\|^2 = \|(x', x_n - r)\|^2 = \|x'\|^2 + x_n^2 - 2x_n r + r^2, \tag{5.9}$$

it follows that of Claim 1 is proved as soon as we show that

$$x_n \leqslant \varphi(x') \ \text{ and } \ \|x'\| < R \Longrightarrow \|x'\|^2 + x_n^2 - 2rx_n > 0. \tag{5.10}$$

In general, for any fixed $r > 0$, the function $f(t) := t^2 - 2rt$ is decreasing if $t < r$. Consequently, $f(x_n) \geqslant f(\varphi(x'))$ if $x_n \leqslant \varphi(x') \leqslant r$. Thus, $x_n^2 - 2rx_n \geqslant \varphi(x')^2 - 2r\varphi(x')$ and, further,

$$\|x'\|^2 + x_n^2 - 2rx_n \geqslant \|x'\|^2 + \varphi(x')^2 - 2r\varphi(x') \geqslant \|x'\|^2 - 2r\varphi(x'), \tag{5.11}$$

provided $x_n \leqslant \varphi(x') \leqslant r$.

We now seek to prove that

$$r > 0 \text{ small} \Longrightarrow \|x'\|^2 - 2r\varphi(x') > 0. \tag{5.12}$$

Since $\nabla\varphi$ is a Lipschitz function with constant $\leqslant M$, it follows that $\|\nabla\varphi(0) - \nabla\varphi(x')\| \leqslant M\|0 - x'\|$ for any $x' \in B(0, R)$. Since $\nabla\varphi(0) = 0$ this implies $\|\nabla\varphi(x')\| \leqslant M\|x'\|$ if $\|x'\| < R$. Using this and relying on the Mean Value Theorem we can then write

$$|\varphi(0) - \varphi(x')| \leqslant \|0 - x'\| \sup_{y' \in [0, x']} \|\nabla\varphi(y')\| \leqslant M\|x'\|^2. \tag{5.13}$$

Hence, $|\varphi(x')| \leqslant M\|x'\|^2$ if $\|x'\| < R$. This implies that $\|x'\|^2 - 2r\varphi(x') \geqslant \|x'\|^2(1 - 2rM)$ if $\|x'\| < R$. Hence, we have $\|x'\|^2 - 2r\varphi(x') > 0$ whenever $\|x'\| < R$, granted that

$0 < r < \frac{1}{2M}$. Concretely, we shall pick $r := \min\left\{\frac{1}{4M}, \frac{h}{2}, R\right\}$, in which case we need to guarantee that $\varphi(x') \leqslant \min\left\{\frac{1}{4M}, \frac{h}{2}, R\right\}$. Since for any x' we know that $|\varphi(x')| \leqslant M\|x'\|^2$ it is enough to have $M\|x'\|^2 \leqslant \min\left\{\frac{1}{4M}, \frac{h}{2}, R\right\}$, or $\|x'\| \leqslant \min\left\{\frac{1}{2M}, \frac{h^{1/2}}{(2M)^{1/2}}, \frac{R^{1/2}}{M^{1/2}}\right\}$. To ensure that this happens as a result of the fact that $\|x'\| < R$, we eventually decrease R so that $0 < R \leqslant \min\left\{\frac{1}{2M}, \frac{h^{1/2}}{(2M)^{1/2}}\right\}$. The choice $R := \frac{1}{2}\min\left\{\frac{1}{2M}, \frac{h^{1/2}}{(2M)^{1/2}}\right\}$ then leads to a value of r which depends only on Ω itself (and not on the particular boundary point near which the current considerations have been developed). This concludes the proof of Claim 1.

In a similar fashion, via an argument which closely parallels the proof of Claim 1, we can also establish:

`Claim 2.` *After eventually narrowing the cylinder $\mathcal{C}$ and decreasing the radius r (in a controlled fashion), the following property holds: any point in the cylinder $\mathcal{C}$ lying above or on the graph of φ does not belong to B_r^-.*

Given that, as pointed out already, the (common) radius, r, of the inner and outer balls depends only on general geometric characteristics of the domain Ω, the uniformity feature in Definition 5.1.5 is satisfied. This concludes the proof of Theorem 5.1.9. $\square$

Remark 5.1.10. *It is possible to work out a version of the proof of Theorem 5.1.9 in which one avoids making a rotation which transforms the geometric tangent plane $T_{X^*}\partial\Omega$ into a horizontal $(n-1)$-dimensional plane $\mathbb{R}^{n-1} \times \{0\}$ (i.e., work directly with the situation depicted in Figure 2, than the one in Figure 3).*

Indeed, in the context of Figure 2, one can take $B_r^\pm := B(\mp r\nu(0), r)$. In this scenario, in place of the last inequality in (5.10) this time we have

$$\|x'\|^2 + x_n^2 + \frac{2r}{\sqrt{1+\|\nabla\varphi(0)\|^2}}\Big(x'\cdot\nabla\varphi(0) - x_n\Big) > 0, \tag{5.14}$$

and note that, if $x_n \leqslant \varphi(x') \leqslant r/\sqrt{1+\|\nabla\varphi(0)\|^2}$, then

$$\begin{aligned}\|x'\|^2 + x_n^2 &+ \frac{2r}{\sqrt{1+\|\nabla\varphi(0)\|^2}}\Big(x'\cdot\nabla\varphi(0) - x_n\Big)\\ &\geqslant \|x'\|^2 + \frac{2r}{\sqrt{1+\|\nabla\varphi(0)\|^2}}\Big(x'\cdot\nabla\varphi(0) - \varphi(x')\Big)\end{aligned} \tag{5.15}$$

Thus, in place of (5.12), it is enough to ensure that

$$r > 0 \text{ small} \implies \|x'\|^2 + \frac{2r}{\sqrt{1+\|\nabla\varphi(0)\|^2}}\Big(x'\cdot\nabla\varphi(0) - \varphi(x')\Big) > 0. \tag{5.16}$$

At this stage, one can use the fact that

$$|x'\cdot\nabla\varphi(0) - \varphi(x')| \leqslant M\|x'\|^2 \quad \text{if} \quad \|x'\| < R, \tag{5.17}$$

in order to arrive at the desired conclusion. Here is the Lemma that finish the proof.

Lemma 5.1.11. *Assume that $S \subseteq \mathbb{R}^n$ is an open convex set and that $f : S \to \mathbb{R}^m$ is a function which is differentiable at every point in S. Then for every $X, Y \in S$,*

$$\|f(X) - f(Y) - Df(Y)(X-Y)\| \leqslant \|X-Y\| \sup_{Z\in(X,Y)} \|Df(Z) - Df(Y)\|, \tag{5.18}$$

provided $\{\|Df(Z) - Df(Y)\| : Z \in (X,Y)\}$ is bounded from above.

Proof. Fix a point $Y \in S$ and apply Theorem 5.1.12 below to the function $g : S \to \mathbb{R}^m$ given by $g(X) := f(X) - (Df(Y))X$. $\square$

Theorem 5.1.12 (The Mean-Value Theorem in Inequality Form)**.** *Assume that $S \subseteq \mathbb{R}^n$ is an open convex set and that $f : S \to \mathbb{R}^m$ is differentiable at every point in S. Then for any two pints $X, Y \in S$ we have*

$$\|f(X) - f(Y)\| \leqslant \|X-Y\| \sup_{Z\in(X,Y)} \|Df(Z)\|, \tag{5.19}$$

provided $\{\|Df(Z)\| : Z \in (X,Y)\}$ is bounded from above.

The philosophy behind the concept of UEBC of a set D is that ∂D does not exhibit any inward pointing singularities. For example, UEBC rules out the presence of interior cusps and re-entrant corners. Likewise, D satisfies UIBC then, heuristically, ∂D does not exhibit any outward pointing singularities. Consequently, when considered in concert, UEBC and UIBC are expected to ensure that the boundary of D is fairly regular. This is the point of the theorem below (which is the converse of Theorem 5.1.9).

Theorem 5.1.13. *If Ω is an open, bounded, nonempty subset of $\mathbb{R}^n$ which satisfies both a uniform exterior ball condition (UEBC) and a uniform interior ball condition (UIBC), then then Ω is a domain of class $C^{1,1}$.*

Proof. We shall show that the conditions stipulated in Definition 5.1.2 hold. This is accomplished in a series of steps, starting with:

`Step I.` *Definition of the function φ.* Fix $X^* \in \partial\Omega$ arbitrary and denote by $B_R^{\pm}$ the interior and exterior balls at X^*, where $R > 0$ is the common vales of the radii of these balls. These balls have a common $(n-1)$-dimensional tangent plane H passing through X^* and we denote by N the unit normal to H (pointing towards the interior ball). This situation is depicted in the following picture:

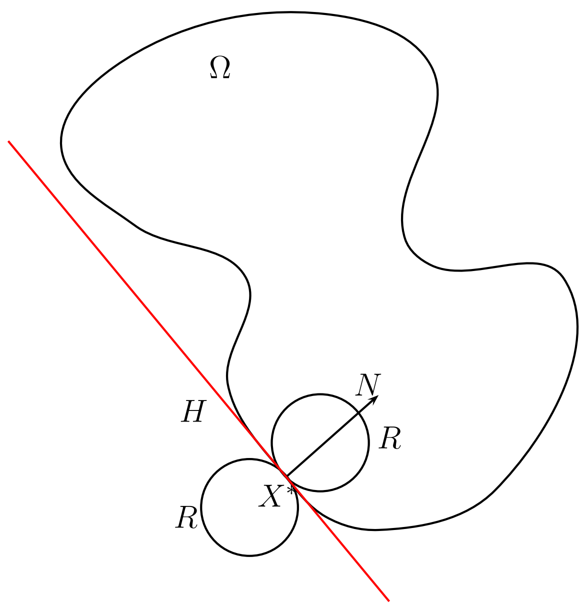

Figure 5.3: The surface, the tangent plane, and UIBC/UEBC.

Make a translation and a rotation such that X^* becomes the origin in $\mathbb{R}^n$ and the plane H becomes the horizontal $(n-1)$-dimensional plane $\mathbb{R}^{n-1} \times \{0\}$). Below we make the convention that B_{n-1} will denote an $(n-1)$-dimensional ball in the plane $H \equiv \mathbb{R}^{n-1}$. Fix a small number $\lambda \in (0,1)$ to be specified later. Our goal is to find a $C^{1,1}$ function $\varphi : B_{n-1}(0', \lambda R) \to \mathbb{R}$ (where $0'$ is the origin in $\mathbb{R}^{n-1}$) which, among other things (see Definition 5.1.2) has the property that its graph coincide with $\partial\Omega$ in a neighborhood of $X^* = 0$.

In the context of the above picture, we shall do the natural thing and define the function $\varphi : B_{n-1}(0', \lambda R) \longrightarrow \mathbb{R}$ such that the graph of φ coincides with the boundary of Ω inside

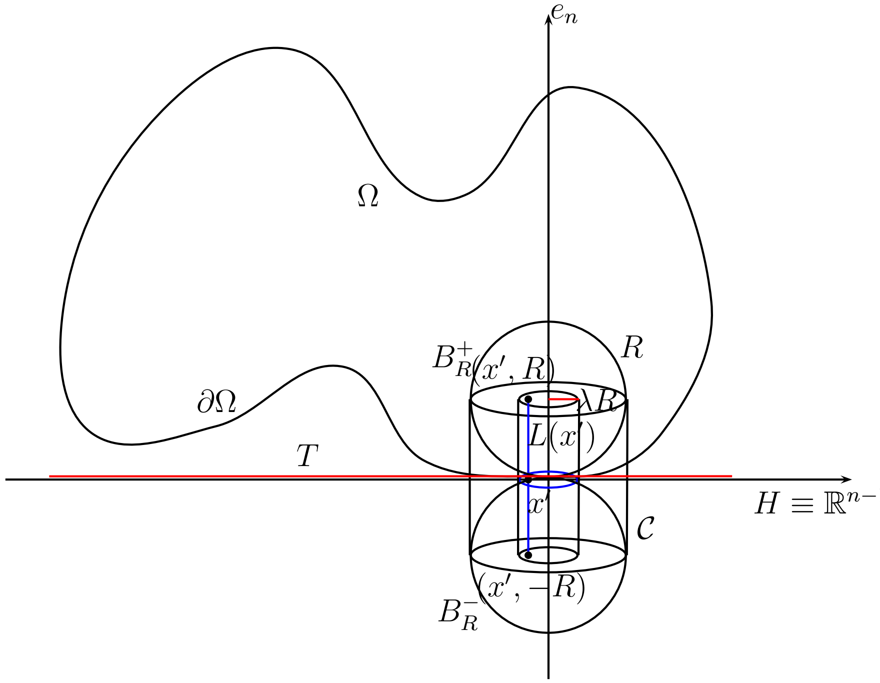

Figure 5.4: The horizontal tangent plane, and the segment $L(x)$.

the cylinder $\mathcal{C}$. Concretely, for each $x' \in B_{n-1}(0', \lambda R)$, we take $\varphi(x')$ to be the height of the vertical segment emerging from x' until it intersects the boundary of Ω. In particular, this implies that $\varphi(0') = 0$. Of course, we need to make sure that the function defined in the manner described above is well-defined (i.e., we need to show that the Vertical Line Test is not violated).

To prove this, for each $x' \in B_{n-1}(0', \lambda R)$ define $L(x') = [(x', R),\ (x', -R)]$. In other words, $L(x')$ is the line segment with end-points (x', R) and $(x', -R)$. At this stage, we make the following the claim.

$$\lambda > 0 \text{ small} \implies \#(L(x') \cap \partial\Omega) = 1, \qquad \forall\, x' \in B_{n-1}(0', \lambda R). \tag{5.20}$$

To prove this,we first note the fact that $L(x')$ is aconnected set and property (9) in Remark 4.6.4 ensure that

$$L(x') \cap \partial\Omega \neq \varnothing, \qquad \forall\, x' \in B_{n-1}(0', \lambda R). \tag{5.21}$$

Granted this, we only need to show that $L(x')$ intersects $\partial\Omega$ only once. To justify this, fix a point $x' \in B_{n-1}(0', \lambda R)$, and consider the vertical line segment $L(x')$ as before. From what we have proved so far we know that exists a point $u \in L(x')$ such that $u \in \partial\Omega$ (since the segment intersects the boundary at least once). Consider the interior and the exterior balls $B^{\pm}(u)$ as in Uniform Interior Ball Condition/Uniform Exterior Ball Condition for this new point $u \in \partial\Omega$. If we can show that the portion of $L(x')$ lying above u is contained in the interior, or the exterior, ball (which, in turn, are included in Ω and Ω^c, respectively), and that the portion of $L(x')$ lying below u is contained in the exterior, or interior, ball (which, in turn, are included in Ω^c and in Ω, respectively), then there can be no other points from the boundary of Ω on $L(x')$ other than u. Any other possible combination will fit into the pattern above.

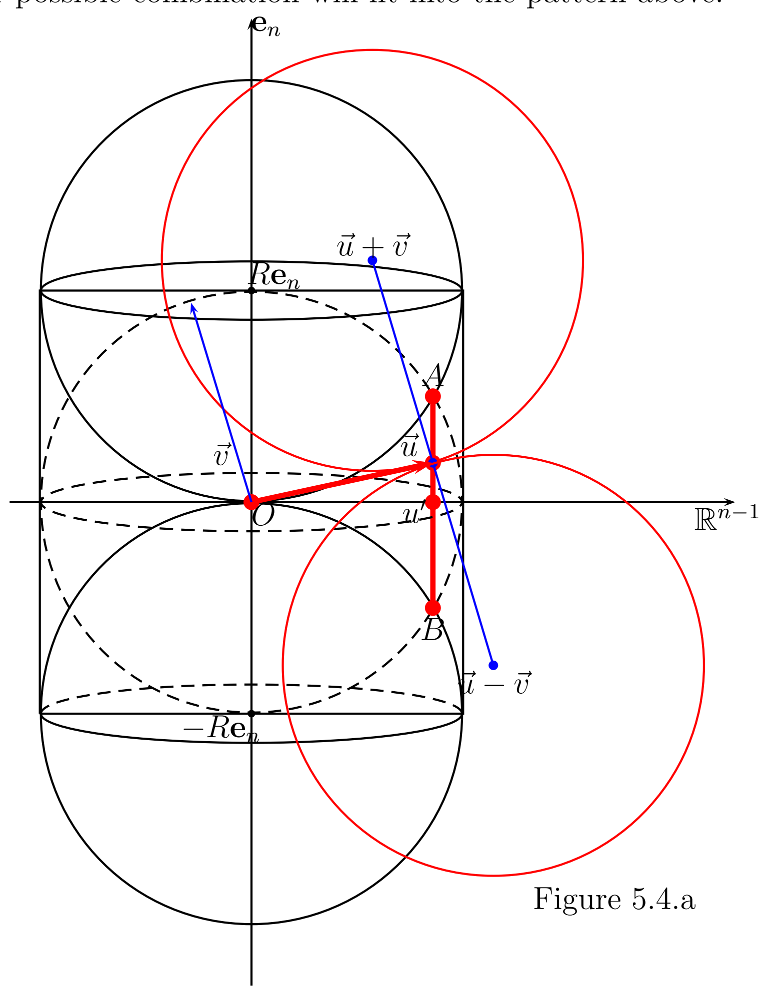

Figure 5.4.a

In fact, we only need to check that the aforementioned inclusion properties apply only to the portion of $L(x')$ not already contained in the "original" interior/exterior balls (i.e., the interior and exterior balls touching at the origin). With this goal in mind, we write $u = (u', u_n)$ and observe that $u' = x' \in B(0, \lambda R)$ (since $u \in L(x')$). For further reference, let us explicitly point out that this implies

$$\|u'\| < \lambda R. \tag{5.22}$$

Let $L[u']$ denote that the portion of $L(u')$ not contained in the interior/exterior balls touching at origin. In Figure 6, this is the segment $[A, B]$. Our goal then becomes showing that the points A, B belong to the interior/exterior balls touching at u. In the picture above v is the direction of the line joining the centers of the "new" balls (the interior/exterior balls touching at u). We normalize v such that $\|v\| = R$, and write $v = (v', v_n)$, with $\|v'\| < R$ and $\|v'\|^2 + v_n^2 = R^2$.

Let us now describe the coordinates of the points A, B in Figure 6. To fix ideas, assume that A is a point on the boundary of the original interior ball. Generally speaking, a point $x = (x', x_n)$ belongs to this ball if $0 < x_n < R$ and $\|x'\|^2 + (x_n - R)^2 = R^2$. Note that this forces $x_n^2 - 2Rx_n + \|x'\|^2 = 0$, so $x_n = R \pm \sqrt{R^2 - \|x'\|^2}$. Hence, the coordinates of the point A are $\left(u',\ R - \sqrt{R^2 - \|u'\|^2}\right)$. Likewise, the coordinates of the point B are $\left(u',\ -R + \sqrt{R^2 - \|u'\|^2}\right)$. If we now write $u = \theta A + (1-\theta)B$ with $\theta \in (0,1)$, it follows that $u_n = t\left(R - \sqrt{R^2 - \|u'\|^2}\right)$, where $t = 2\theta - 1 \in (-1, 1)$. This implies that

$$-R + \sqrt{R^2 - \|u'\|^2} < u_n < R - \sqrt{R^2 - \|u'\|^2} \quad \text{or} \quad |u_n| < R - \sqrt{R^2 - \|u'\|^2}. \tag{5.23}$$

After this preamble, we now come to the central question we wish to answer, which is:

$$\text{does the point } A \text{ belong to } B(u+v, R) \cup B(u-v, R)\,? \tag{5.24}$$

Of course, $A \in B(u+v, R)$ if and only if $\|A - (u+v)\|^2 < R^2$. The last statement holds if and only if

$$\left\| \left(-v', \left(R - \sqrt{R^2 - \|u'\|^2} - u_n\right) - v_n\right) \right\|^2 < R^2, \tag{5.25}$$

or $\|-v'\|^2 + v_n^2 + \left(R - \sqrt{R^2 - \|u'\|^2} - u_n\right)^2 - 2v_n\left(R - \sqrt{R^2 - \|u'\|^2} - u_n\right) < R^2$. Knowing that $\|-v'\|^2 + v_n^2 = R^2$, the above reduces to

$$\left(R - \sqrt{R^2 - \|u'\|^2} - u_n\right)\left(R - \sqrt{R^2 - \|u'\|^2} - u_n - 2v_n\right) < 0. \tag{5.26}$$

Since the first factor above is positive, we need to check that $R - \sqrt{R^2 - \|u'\|^2} - 2v_n < u_n$. Upon recalling (5.23), this holds whenever $R - \sqrt{R^2 - \|u'\|^2} < v_n$. Likewise, it can be shown that $A \in B(u-v,R)$ whenever $R-\sqrt{R^2-\|u'\|^2}<-v_n$. Combining these two results proves that

$$R-\sqrt{R^2-\|u'\|^2}<|v_n| \Longrightarrow A \in B(u+v,R)\cup B(u-v,R). \tag{5.27}$$

To proceed, we shall derive some estimates for u_n, $\|u\|$, and v_n. First, combining (5.23) and the fact that $\|u'\|<\lambda R$ (cf. (5.22)) we obtain

$$|u_n|<R\left(1-\sqrt{1-\lambda^2}\right), \tag{5.28}$$

so that

$$\|u'\|^2+u_n^2<R^2\left(\lambda^2+1+(1-\lambda^2)-2\sqrt{1-\lambda^2}\right). \tag{5.29}$$

Thus, we can write $\|u\|^2<2R^2(1-\sqrt{1-\lambda^2})$ and, further,

$$\|u\|^2<2R^2(1-\sqrt{1-\lambda^2})=\frac{2R^2\lambda^2}{1+\sqrt{1-\lambda^2}}<2R^2\lambda^2. \tag{5.30}$$

In particular,

$$\|u\|<\sqrt{2}R\lambda. \tag{5.31}$$

On the other hand, from

$$B(u+v,R)\subseteq\mathbb{R}^n\setminus\Omega \quad\text{and}\quad B(-R\mathbf{e}_n,R)\subseteq\Omega \tag{5.32}$$

we deduce $B(u+v,R)\cap B(-R\mathbf{e}_n,R)=\varnothing$ which further gives

$$\|(u+v)-(-R\mathbf{e}_n)\|>2R. \tag{5.33}$$

In a similar fashion, based on

$$B(u-v,R)\subseteq\Omega \quad\text{and}\quad B(R\mathbf{e}_n,R)\subseteq\mathbb{R}^n\setminus\Omega \tag{5.34}$$

we can write $B(u-v,R)\cap B(R\mathbf{e}_n,R)=\varnothing$ and, hence,

$$\|(u-v)-R\mathbf{e}_n\|>2R. \tag{5.35}$$

Combining (5.33) and (5.35), using the triangle inequality and (5.31), we can write

$$2R<\|u+(v+R\mathbf{e}_n)\|\leqslant\|u\|+\|v+R\mathbf{e}_n\|\leq\sqrt{2}\lambda R+\|v+R\mathbf{e}_n\|. \tag{5.36}$$

From this we therefore obtain $R(2-\sqrt{2}\lambda)\leqslant\|v+R\mathbf{e}_n\|$. After squaring both sides and using $\|v\|=R$ we arrive at $R^2(2-\sqrt{2}\lambda)^2\leqslant 2R^2+2Rv_n$ which forces

$$v_n>R[(\sqrt{2}-\lambda)^2-1]. \tag{5.37}$$

Note that (5.22) implies that $R-\sqrt{R^2-\|u'\|^2}<R\left(1-\sqrt{1-\lambda^2}\right)$. Thanks to this and (5.37), it follows that the inequality in the left-hand side of (5.27) is true provided we

have $1-\sqrt{1-\lambda^2} < (\sqrt{2}-\lambda)^2 - 1$. However, for $\lambda > 0$ small the latter inequality is true since the left-hand side tends to 0 as $\lambda \to 0$, whereas the right-hand side tends to 1 as $\lambda \to 0$. Thus, for $\lambda > 0$ small, the implication in (5.27) is true and this shows that the question in (5.24) has a positive answer if $\lambda > 0$ is small. In fact, a very similar argument shows that

$$\lambda > 0 \text{ small} \Longrightarrow \text{the point } B \text{ belongs to } (u+v,R) \cup B(u-v,R). \tag{5.38}$$

To summarize, the above reasoning proves (5.20). In turn, this shows that the function φ is well-defined.

`Step II.` *The function φ is continuous.* Fix an arbitrary point $u' \in B_{n-1}(0', \lambda R)$. Next, for some $\delta \in (0,R)$ to be specified later, pick a point

$$(x', x_n) \in B_{n-1}(u', \delta) \times (-R, R) \text{ such that } \|(x',x_n) - (u \pm v)\| \geqslant R. \tag{5.39}$$

Then the last condition in (5.39) implies

$$\begin{aligned} R^2 &\leqslant \|(x',x_n) - (u \pm v)\|^2 \\ &= \|x'-u'\|^2 + (x_n - u_n)^2 \pm 2v \cdot (x'-u', x_n - u_n) + R^2 \end{aligned} \tag{5.40}$$

so that

$$2|v' \cdot (x'-u') + v_n(x_n - u_n)| \leqslant \|x'-u'\|^2 + (x_n-u_n)^2 \leqslant \delta^2 + (x_n - u_n)^2 \tag{5.41}$$

From (5.28) and $x_n \in (-R,R)$ we also have $|x_n - u_n| \leqslant |x_n| + |u_n| < \big(2 - \sqrt{1-\lambda^2}\big)R$. Using this and (5.41) we may therefore estimate

$$\begin{aligned} 2|v_n(x_n-u_n)| &\leqslant 2|v' \cdot (x'-u')| + \delta^2 + (x_n-u_n)^2 \\ &\leqslant 2\delta\|v'\| + \delta^2 + (x_n - u_n)^2 \\ &\leqslant 2\delta R + \delta^2 + \big(2-\sqrt{1-\lambda^2}\big)R|x_n - u_n|. \end{aligned} \tag{5.42}$$

From (5.42) and (5.37) we therefore obtain (recall that $0 < \delta < R$)

$$2R[(\sqrt{2}-\lambda)^2 - 1]|x_n - u_n| \leqslant 3\delta R + \big(2 - \sqrt{1-\lambda^2}\big)R|x_n - u_n|. \tag{5.43}$$

In summary, this shows that if $\lambda \in (0,1)$ is small then there exists a constant $C > 0$ such that if $\delta \in (0,R)$ then

$$\|x'-u'\| < \delta,\ |x_n| < R \text{ and } \|(x',x_n) - (u \pm v)\| \geqslant R \Longrightarrow |x_n - u_n| \leqslant C\delta. \tag{5.44}$$

Having established (5.44), it is easy now to complete the proof of the continuity of φ at an arbitrary point $u' \in B_{n-1}(0', \lambda R)$. Concretely, let $\varepsilon > 0$ be an arbitrary threshold and assume that $x' \in B_{n-1}(0', \lambda R)$ is such that $\|x' - u'\| < \delta$. Then $\varphi(x') \in (-R, R)$ and $(x', \varphi(x'))$ lies outside of the balls $B_n(u \pm v, R)$, where $u = (u', \varphi(u'))$. According to (5.44), this forces $|\varphi(x') - \varphi(u')| \leqslant C\delta$ so that, if $0 < \delta < \min\{\varepsilon/(3C), R\}$, we obtain $|\varphi(x') - \varphi(u')| \leqslant \varepsilon$. Thus, φ is continuous at u'.

`Step III.` *The function φ is differentiable.* Granted the uniform two-sided ball condition satisfied by Ω as well as the specific way in which φ has been defined, it is clear that the Point of Impact Differentiability Criterion from Theorem 5.1.14 below readily gives that the function $\varphi : B_{n-1}(0', \lambda R) \to \mathbb{R}$ is differentiable at every point in its domain (here we make use of the result proved in Step II). In fact, the very last part in the statement of Theorem 5.1.14 also gives that

$$\frac{v}{R} = \frac{(-\nabla\varphi(u'), 1)}{\sqrt{1 + \|\nabla\varphi(u')\|^2}}. \tag{5.45}$$

This is going to play an important role shortly.

Theorem 5.1.14 (The "Point of Impact" Differentiability Criterion)**.** *Assume that $U \subseteq \mathbb{R}^n$ is a an arbitrary set, and that $X^* \in U^\circ$. Given a function $f : U \to \mathbb{R}$, we denote by G_f the graph of f, i.e., $G_f := \{(X, f(X)) : X \in U\} \subseteq \mathbb{R}^{n+1}$.*

Then f is differentiable at the point X^* if and only if f is continuous at X^* and there exists a non-horizontal vector $N \in \mathbb{R}^{n+1}$ (i.e., satisfying $N \cdot \mathbf{e}_{n+1} \neq 0$) with the following significance. For every angle $\theta \in (0, \pi/2)$ there exists $\delta > 0$ with the property that $G_f \cap B_{n+1}((X^*, f(X^*)), \delta)$ lies between the cones $\mathcal{C}_{n+1}((X^*, f(X^*)), N, \theta)$ and $\mathcal{C}_{n+1}((X^*, f(X^*)), -N, \theta)$, i.e.,

$$\begin{aligned} &G_f \cap B_{n+1}((X^*, f(X^*)), \delta) \\ &\qquad \subseteq \mathbb{R}^{n+1} \setminus \Big[\mathcal{C}_{n+1}((X^*, f(X^*)), N, \theta) \cup \mathcal{C}_{n+1}((X^*, f(X^*)), -N, \theta)\Big]. \end{aligned} \tag{5.46}$$

If this happens, then necessarily N is a scalar multiple of $(\nabla f(X^*), -1) \in \mathbb{R}^{n+1}$.

Proof. " $\Longrightarrow$ " Assume that f is differentiable at X^*. Then, by Theorem 5.1.15 below,

f is continuous at X^*. To proceed, take

$$N := \frac{(\nabla f(X^*), -1)}{\sqrt{1 + \|\nabla f(X^*)\|^2}} \in \mathbb{R}^{n+1}. \tag{5.47}$$

Clearly, $\|N\| = 1$ and $N \cdot \mathbf{e}_{n+1} = -(1 + \|\nabla f(X^*)\|^2)^{-1/2} \neq 0$, so N is non-horizontal. Then, given $\theta \in (0, \pi/2)$, the fact that f is differentiable at X^* implies that there exists $\delta > 0$ such that for all $X \in B(X^*, \delta) \cap U$ we obtain

$$|f(X) - f(X^*) - (\nabla f(X^*)) \cdot (X - X^*)| < (\cos\theta)\|X - X^*\|. \tag{5.48}$$

For any $X \in B(X^*, \delta) \cap U$ we may then estimate

$$\begin{aligned} \Big|((X, f(X)) - (X^*, f(X^*))) \cdot N\Big| &= \frac{|(\nabla f(X^*)) \cdot (X - X^*) - f(X) + f(X^*)|}{\sqrt{1 + \|\nabla f(X^*)\|^2}} \\ &\leqslant |(\nabla f(X^*)) \cdot (X - X^*) - f(X) + f(X^*)| < (\cos\theta)\|X - X^*\| \\ &< (\cos\theta)\|(X, f(X)) - (X^*, f(X^*))\|, \end{aligned} \tag{5.49}$$

which (recall that $\|N\| = 1$) shows that

$$X \in B(X^*, \delta) \cap U \Longrightarrow (X, f(X)) \notin \mathcal{C}_{n+1}((X^*, f(X^*)), \pm N, \theta). \tag{5.50}$$

Upon observing that any point in $G_f \cap B_{n+1}((X^*, f(X^*)), \delta)$ is of the form $(X, f(X))$ for some $X \in B(X^*, \delta) \cap U$, based on (5.50) we may conclude that (5.46) holds.

" $\Longleftarrow$ " For the converse implication, suppose that f is continuous at X^* and assume that there exists a non-horizontal vector $N \in \mathbb{R}^{n+1}$ with the property that for every angle $\theta \in (0, \pi/2)$ there exists $\delta > 0$ such that (5.46) holds. By dividing N by the non-zero number $-N \cdot \mathbf{e}_{n+1}$, we may assume that the $(n+1)$-th component of N is -1 to begin with. That is, $N = (N', -1)$ for some $N' \in \mathbb{R}^n$.

Fix an arbitrary number $\varepsilon \in (0, 1/2)$ and pick an angle $\theta \in (0, \pi/2)$ sufficiently close to $\pi/2$ so that $0 < \cos\theta < \varepsilon/\sqrt{1 + \|N'\|^2}$. Then, by assumption, there exists $\delta_0 > 0$ with the property that if $X \in U$ is such that $\|(X, f(X)) - (X^*, f(X^*))\| < \delta_0$ then $(X, f(X)) \notin \mathcal{C}_{n+1}((X^*, f(X^*)), \pm N, \theta)$, i.e.,

$$\begin{aligned}
&\Big|((X, f(X)) - (X^*, f(X^*))) \cdot (N', -1)\Big| \\
&\qquad\qquad \leqslant (\cos\theta)\|(N', -1)\|\|(X, f(X)) - (X^*, f(X^*))\| \\
&\qquad\qquad \leqslant \varepsilon\sqrt{\|X - X^*\|^2 + (f(X) - f(X^*))^2} \\
&\qquad\qquad \leqslant \varepsilon\Big[\|X - X^*\| + |f(X) - f(X^*)|\Big].
\end{aligned} \tag{5.51}$$

In turn, this forces (recall that $0 < \varepsilon < \frac{1}{2}$)

$$\begin{aligned}
|f(X) - f(X^*)| &\leqslant \Big|((X, f(X)) - (X^*, f(X^*))) \cdot (N', -1)\Big| + |(X - X^*) \cdot N'| \\
&\leqslant \varepsilon\Big[\|X - X^*\| + |f(X) - f(X^*)|\Big] + \|X - X^*\|\|N'\| \\
&\leqslant (\tfrac{1}{2} + \|N'\|)\|X - X^*\| + \tfrac{1}{2}|f(X) - f(X^*)|.
\end{aligned} \tag{5.52}$$

Absorbing the last term above in the left-most side of (5.52) yields

$$\tfrac{1}{2}|f(X) - f(X^*)| \leqslant (\tfrac{1}{2} + \|N'\|)\|X - X^*\|. \tag{5.53}$$

We have therefore proved that there exists $\delta_0 > 0$ for which

$$\begin{aligned}
&X \in U \ \text{ and } \ \|(X, f(X)) - (X^*, f(X^*))\| < \delta_0 \\
&\qquad \Longrightarrow |f(X) - f(X^*)| \leqslant (1 + 2\|N'\|)\|X - X^*\|.
\end{aligned} \tag{5.54}$$

Returning with this back in (5.51) then yields

$$\begin{aligned}
&X \in U \ \text{ and } \ \|(X, f(X)) - (X^*, f(X^*))\| < \delta_0 \Longrightarrow \\
&\qquad \Big|((X, f(X)) - (X^*, f(X^*))) \cdot (N', -1)\Big| \leqslant \varepsilon(2 + 2\|N'\|)\|X - X^*\|.
\end{aligned} \tag{5.55}$$

Since we are assuming that f is continuous at the point X^*, it follows that there exists $\delta_1 > 0$ with the property that

$$X \in U \ \text{ and } \ \|X - X^*\| < \delta_1 \Longrightarrow |f(X) - f(X^*)| < \frac{\delta_0}{\sqrt{2}}. \tag{5.56}$$

Introducing

$$\delta := \min\Big\{\delta_1\,,\ \frac{\delta_0}{\sqrt{2}}\Big\}, \tag{5.57}$$

implication (5.56) therefore guarantees that

$$X \in U \ \text{ and } \ \|X - X^*\| < \delta \Longrightarrow \|(X, f(X)) - (X^*, f(X^*))\| < \delta_0. \tag{5.58}$$

Consequently, from this and (5.55) we deduce that $X \in B(X^*, \delta) \cap U$ will imply

$$\left|\frac{((X, f(X)) - (X^*, f(X^*))) \cdot (N', -1)}{\|X - X^*\|}\right| \leqslant \varepsilon(2 + 2\|N'\|). \tag{5.59}$$

Since $\varepsilon \in (0, 1/2)$ was arbitrary, this translates into saying that

$$\lim_{X \to X^*,\, X \in U} \frac{f(X) - f(X^*) - N' \cdot (X - X^*)}{\|X - X^*\|} = 0. \tag{5.60}$$

This proves that f is differentiable at X^* and, in fact, $\nabla f(X^*) = N'$ (hence, in particular, N is a scalar multiple of $(N', -1) = (\nabla f(X^*), -1)$). The proof of the theorem is therefore finished. $\square$

Theorem 5.1.15 (Global Differentiability Implies Continuity). *Let $U \subseteq \mathbb{R}^n$ be a set and fix $X^* \in U^\circ$. If $f : U \to \mathbb{R}^m$ is a function which is differentiable at X^*, then f is continuous at X^*.*

Proof. Let us first state the following Proposition.

Proposition 5.1.16. *Let $U \subseteq \mathbb{R}^n$ and fix some $X \in U^\circ$. Then for a given, arbitrary, function $f : U \to \mathbb{R}^m$ the following are equivalent:*

(i) f is differentiable at X;

(ii) there exist a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$, a number $0 < r < \operatorname{dist}(X, U^c)$, and a function $E_X : B(0, r) \to \mathbb{R}^m$ such that $\lim_{h \to 0,\, h \in \mathbb{R}^n} E_X(h) = 0$ and

$$f(X + h) - f(X) - Lh = \|h\|\, E_X(h) \quad \textit{for all} \quad h \in B(0, r) \subseteq \mathbb{R}^n. \tag{5.61}$$

Returnig to our proof, suppose that $L := (Df)(X^*)$ exists. By Proposition 5.1.16, one can find a number $0 < r < \operatorname{dist}(X, U^c)$ and a mapping $E_{X^*} : B(0, r) \to \mathbb{R}^m$ such that

$$\lim_{h \to 0,\, h \in \mathbb{R}^n} E_{X^*}(h) = 0 \tag{5.62}$$

and

$$f(X) - f(X^*) - L(X - X^*) = \|X - X^*\|\, E_{X^*}(X - X^*), \ \ \forall\, X \in B(X^*, r). \tag{5.63}$$

Thus, for any $X \in B(X^*, r)$ we can estimate

$$\begin{aligned}
\|f(X) - f(X^*)\| &\leqslant \|f(X) - f(X^*) - L(X - X^*)\| + \|L(X - X^*)\| \\
&\leqslant \|E_{X^*}(X - X^*)\|\, \|X - X^*\| + \|L\| \|X - X^*\| \\
&\leqslant \|X - X^*\| \Big(\|E_{X^*}(X - X^*)\| + \|L\| \Big).
\end{aligned} \tag{5.64}$$

From (5.62) and (5.64) is then clear that $\lim_{X\to X^*,\, X\in U} f(X) = f(X^*)$, so f is continuous at X^*. $\square$

Step IV. *The function $\nabla\varphi$ is Lipschitz.* Equating the n-th components of the vectors in (5.45) gives $v_n/R = 1/\sqrt{1+\|\nabla\varphi(u')\|^2}$. Since $v_n/R > (\sqrt{2}-\lambda)^2 - 1$, this shows that there exists $M > 0$ with the property that $\|\nabla\varphi(u')\| \leqslant M$ for any $\|u'\| < \lambda R$. In other words, $\nabla\varphi$ is bounded so that φ is Lipschitz.

Next we concentrate on proving that $\nabla\varphi$ is Lipschitz (near $0'$). The strategy we adopt for this purpose is to show that the assignment

$$\partial\Omega \ni u \mapsto v \in \mathbb{R}^n \tag{5.65}$$

is Lipschitz. Let us assume for a moment that the mapping (5.65) is Lipschitz and explain how this leads to the conclusion that $\nabla\varphi$ is a Lipschitz function near $0'$. Concretely, since φ is Lipschitz, it follows that the mapping $B_{n-1}(0',\lambda R) \ni u' \mapsto u = (u', \varphi(u')) \in \partial\Omega$ is Lipschitz. Granted this, the Lipschitzianity of (5.65) and formula (5.45), via a reasoning similar to the hint given for Remark 5.1.10, we may then conclude that each $\partial_j\varphi$ is Lipschitz in $B_{n-1}(0',\lambda R)$ (after possibly decreasing λ). In fact, it follows from (5.45) that the Lipschitzianity of (5.65) near a boundary point where $\partial\Omega$ is given as the graph of a function $\varphi : B_{n-1}(0',\lambda R) \to \mathbb{R}$ as before, is equivalent to having $\nabla\varphi$ Lipschitz near $0'$.

Therefore, it remains to show that the mapping (5.65) is Lipschitz. Write $v(u)$ in place of v to stress the dependence on u. By Remark 5.1.4, it suffices to show that $u \mapsto v(u)$ is locally Lipschitz. That is, we need to show that for every $u^* \in \partial\Omega$ there exist $r > 0$ and $C > 0$ with the property that

$$\|v(u) - v(u_0)\| \leqslant C\|u - u_0\| \quad \text{whenever} \quad u, u_0 \in B(u^*, r) \cap \partial\Omega. \tag{5.66}$$

Choose $r > 0$ small enough so that $B(u^*, r) \cap \partial\Omega$ becomes, after a rigid transformation of the Euclidean space, the graph of a Lipschitz function $\varphi : B_{n-1}(0',\lambda R) \to \mathbb{R}$ as in the first part of the proof. Also, without loss of generality, we may assume that u_0 is the origin in $\mathbb{R}^n$ and $\varphi(0') = 0$. Once this choice has been made, we find it convenient to revert from the task of proving (5.66) to to the (equivalent) task of proving that there exists $c > 0$ for which

$$\|\nabla\varphi(0') - \nabla\varphi(u')\| \leqslant c\|0' - u'\|. \tag{5.67}$$

With this goal in mind, pick another arbitrary point $u_0' \in B_{n-1}(0, \lambda R)$ and set $u_0 := (u_0', \varphi(u_0')) \in \partial\Omega$. Then u_0 does not belong to the interior/exterior balls $B(u \pm v, R)$. The latter condition forces $\|(u - u_0) \pm v\|^2 > R^2$, or $\|u - u_0\|^2 + \|v\|^2 \pm 2(u - u_0) \cdot v > R^2$. Since $\|v\|^2 = R^2$, we obtain $\|u - u_0\|^2 > \pm 2(u - u_0) \cdot v$, or $|(u - u_0) \cdot v| < \|u - u_0\|^2/2$. Together with (5.45) and the fact that φ is Lipschitz with Lipschitz constant $\leqslant M$, this gives

$$\begin{aligned}
&\Big|\Big((u_0', \varphi(u_0')) - (u', \varphi(u'))\Big) \cdot (-\nabla\varphi(u'), 1)\Big| \\
&\quad = \sqrt{1 + \|\nabla\varphi(u')\|^2}\,\Big|(u_0 - u) \cdot \frac{v}{R}\Big| < \frac{\sqrt{1 + M^2}}{2R}\,\|u - u_0\|^2 \\
&\quad = \frac{\sqrt{1 + M^2}}{2R}\,\|(u_0', \varphi(u_0')) - (u', \varphi(u'))\|^2 \leqslant \frac{(1 + M^2)^{3/2}}{2R}\,\|u_0' - u'\|^2.
\end{aligned} \tag{5.68}$$

In summary, this shows that there exists $C > 0$ with the property that for every points $u_0', u' \in B_{n-1}(0', \lambda R)$ we have

$$\Big|\Big((u_0', \varphi(u_0')) - (u', \varphi(u'))\Big) \cdot (-\nabla\varphi(u'), 1)\Big| \leqslant C\|u_0' - u'\|^2. \tag{5.69}$$

Fix now $u' \in B_{n-1}(0', \lambda R/2)$. If $\nabla\varphi(0') = 0'$ then (5.67) is trivially true. If, on the other hand, $\nabla\varphi(0') \neq 0'$, we introduce $u_0' := u' + h$ where

$$h := -\|u'\|\frac{\nabla\varphi(u')}{\|\nabla\varphi(u')\|} \in B_{n-1}(0', \lambda R/2) \subseteq \mathbb{R}^{n-1}. \tag{5.70}$$

It follows that $u_0', u' \in B_{n-1}(0', \lambda R)$ and if we plug in (5.69) these choices we therefore obtain

$$\Big|\Big((u' + h, \varphi(u' + h)) - (u', \varphi(u'))\Big) \cdot (\nabla\varphi(u'), -1)\Big| \leqslant C\|h\|^2, \tag{5.71}$$

hence

$$|h \cdot \nabla\varphi(u') - (\varphi(u' + h) - \varphi(u'))| \leqslant C\|u'\|^2. \tag{5.72}$$

Since, near the origin, the graph of φ is contained in between the graphs of

$$y = R - \sqrt{R^2 - \|u'\|^2} = \frac{\|u'\|^2}{R + \sqrt{R^2 - \|u'\|^2}}, \qquad \|u'\| \leqslant R, \tag{5.73}$$

and

$$y = -R + \sqrt{R^2 - \|u'\|^2} = -\frac{\|u'\|^2}{R + \sqrt{R^2 - \|u'\|^2}}, \qquad \|u'\| \leqslant R, \tag{5.74}$$

it follows that there exists $c > 0$ with the property that $|\varphi(u')| \leqslant c\|u'\|^2$, for u' is near $0'$. Based on this, we obtain $\nabla\varphi(0') = 0'$ and $|\varphi(u' + h)| \leqslant c\|u' + h\|^2 \leqslant 2c\|u'\|^2$. After replacing h by its expression from (5.70), we deduce from these estimates and (5.72) that $\|u'\|\|\nabla\varphi(u')\| \leqslant c\|u'\|^2$. Thus, since $\nabla\varphi(0') = 0'$, (5.67) follows.

The bottom line is that, near the origin $0 = (0', \varphi(0')) \in \partial\Omega$, the boundary of Ω coincides with the graph of φ, where $\varphi : B_{n-1}(0', \lambda R) \to \mathbb{R}$ is of class $C^{1,1}$. Since the Uniform Exterior Ball Condition implies (cf. Lemma 5.1.7) that $\partial\Omega = \partial(\overline{\Omega})$, we may conclude from this and (4.212) that Ω is a domain of class $C^{1,1}$. This concludes the proof of Theorem 5.1.13. $\square$

Remark 5.1.17. *Let Ω be a nonempty convex set in $\mathbb{R}^n$. Then for each number $\varepsilon > 0$, the set $\Omega_\varepsilon := \{X \in \mathbb{R}^n : \operatorname{dist}(X, \Omega) < \varepsilon\}$ is also convex.*

Theorem 5.1.18. *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, open, bounded, convex set and, for some $\varepsilon > 0$, consider*

$$\Omega_\varepsilon := \{X \in \mathbb{R}^n : \operatorname{dist}(X, \Omega) < \varepsilon\}. \tag{5.75}$$

Then Ω_ε is a convex domain of class $C^{1,1}$.

Proof. Clearly, Ω_ε is an open, bounded set which contains Ω (and, hence, it is nonempty as well). Also, from Remark 5.1.17, we know that Ω_ε is a convex set and, from (5.80) below,

$$\partial\Omega_\varepsilon = \{X \in \mathbb{R}^n : \operatorname{dist}(X, \Omega) = \varepsilon\}. \tag{5.76}$$

In order to show that Ω_ε is a domain of class $C^{1,1}$, we will employ Theorem 5.1.13. A key observation in this respect is that if $X^* \in \partial\Omega_\varepsilon$ and $Y^* \in \partial\Omega$ are such that $\|X^* - Y^*\| = \varepsilon$ then

$$B\big(\tfrac{1}{2}(X^* + Y^*)\,,\, \tfrac{\varepsilon}{2}\big) \subseteq \Omega_\varepsilon. \tag{5.77}$$

Indeed, if $X \in B\big(\frac{1}{2}(X^* + Y^*)\,,\, \frac{\varepsilon}{2}\big)$ then the distance from X to Y^* is strictly less than the diameter of the ball, i.e., $\|X - Y^*\| < \varepsilon$. This shows that $\operatorname{dist}(X, \Omega) = \operatorname{dist}(X, \overline{\Omega}) \leqslant$

$\|X - Y^*\| < \varepsilon$, so that $X \in \Omega_\varepsilon$. This finishes the justification of (5.77). In turn, since $X^* \in \partial\Omega_\varepsilon$ was arbitrary and $X^* \in \partial B\big(\frac{1}{2}(X^*+Y^*)\,,\,\frac{\varepsilon}{2}\big)$, we may conclude that Ω_ε satisfies a UIBC. Next, we propose to show that

$$B\big(\tfrac{3}{2}X^* - \tfrac{1}{2}Y^*\,,\,\tfrac{\varepsilon}{2}\big) \subseteq \mathbb{R}^n \setminus \Omega_\varepsilon. \tag{5.78}$$

Reasoning by contradiction, assume that (5.78) does not hold, i.e., there exists a point $X \in B\big(\frac{3}{2}X^* - \frac{1}{2}Y^*\,,\,\frac{\varepsilon}{2}\big) \cap \Omega_\varepsilon$. Granted this, (5.77) and the fact that Ω_ε is convex, we may then conclude that

$$O := \{Z \in \mathbb{R}^n : \exists\, Y \in B\big(\tfrac{1}{2}(X^* + Y^*)\,,\,\tfrac{\varepsilon}{2}\big) \text{ such that } Z \in (X, Y)\} \tag{5.79}$$

is a subset of Ω_ε. However, it is not difficult to see that O is a neighborhood of X^*. This would then imply that $X^* \in \Omega_\varepsilon$ which, given that Ω_ε is open, contradicts the assumption that $X^* \in \partial\Omega_\varepsilon$. Thus, (5.78) holds. Since the point $X^* \in \partial\Omega_\varepsilon$ has been arbitrarily chosen, and $X^* \in \partial B\big(\frac{3}{2}X^* - \frac{1}{2}Y^*\,,\,\frac{\varepsilon}{2}\big)$, the above reasoning shows that Ω_ε satisfies a UEBC as well. Hence Theorem 5.1.13 applies and gives that Ω_ε is a $C^{1,1}$ domain. □

Lemma 5.1.19. *Let $E \subseteq \mathbb{R}^n$ be an arbitrary set and, for each number $\varepsilon > 0$, define $E_\varepsilon := \{X \in \mathbb{R}^n : \operatorname{dist}(X, E) < \varepsilon\}$. Then*

$$\partial E_\varepsilon = \{X \in \mathbb{R}^n : \operatorname{dist}(X, E) = \varepsilon\}. \tag{5.80}$$

Proof. Observe that if $X^* \in \partial E_\varepsilon$ then there exist a sequence $\{Y_j^*\}_{j\in\mathbb{N}}$ of points in E_ε along with a sequence $\{Z_j^*\}_{j\in\mathbb{N}}$ of points in $\mathbb{R}^n \setminus E_\varepsilon$ with the property that $\lim\limits_{j\to\infty} Y_j^* = X^* = \lim\limits_{j\to\infty} Z_j^*$. Hence, by the generalized triangle inequality,

$$\operatorname{dist}(X^*, E) \leqslant \|X^* - Y_j\| + \operatorname{dist}(Y_j^*, E) < \|X^* - Y_j\| + \varepsilon \to \varepsilon \text{ as } j \to \infty \tag{5.81}$$

and

$$\varepsilon \leqslant \operatorname{dist}(Z_j^*, E) \leqslant \|Z_j^* - X^*\| + \operatorname{dist}(X^*, E) \to \operatorname{dist}(X^*, E) \text{ as } j \to \infty. \tag{5.82}$$

On account of these, we may then conclude that $\operatorname{dist}(X^*, E) = \varepsilon$. This proves the left-to-right inclusion in (5.80). Conversely, consider a point $X^* \in \mathbb{R}^n$ such that $\operatorname{dist}(X^*, E) = \varepsilon$. Then, from the definition of E_ε, we have $X^* \notin E_\varepsilon$. Thus, in order to conclude that $X^* \in \partial E_\varepsilon$ it suffices to show that $X^* \in \overline{E_\varepsilon}$. Seeking a contradiction, assume that there exists $r > 0$ with the property that $B(X^*, r) \cap E_\varepsilon = \varnothing$. This implies that $\operatorname{dist}(X, E) \geqslant \varepsilon$ for every $X \in B(X^*, r)$ hence, ultimately, $B(X^*, r+\varepsilon) \cap E = \varnothing$ by the Remark 5.1.20 below. In turn, this forces $\varepsilon = \operatorname{dist}(X^*, E) \geqslant r + \varepsilon$, a contradiction. $\square$

Remark 5.1.20. *If $X^* \in \mathbb{R}^n$ and $r_1, r_2 > 0$, then one can have*

$$\bigcup_{X \in B(X^*, r_1)} B(X, r_2) = B(X^*, r_1 + r_2). \tag{5.83}$$

Remark 5.1.21. *The conclusion in Theorem 5.1.18 is optimal. Indeed, if we take $n := 2$, $\Omega := [-1, 0] \times [1, 2]$ and $\varepsilon := 1$, then $0 \in \partial\Omega_1$ and the boundary of Ω_1 coincides near 0 with the graph of the function*

$$f : (-1, 1) \to \mathbb{R}, \qquad f(x) := \begin{cases} 1 - \sqrt{1 - x^2} & \text{if } 0 < x < 1, \\ 0 & \text{if } -1 < x \leqslant 0. \end{cases} \tag{5.84}$$

Note that while f is of class C^1, its derivative

$$f' : (-1, 1) \to \mathbb{R}, \qquad f'(x) = \begin{cases} \frac{x}{\sqrt{1-x^2}} & \text{if } 0 < x < 1, \\ 0 & \text{if } -1 < x \leqslant 0, \end{cases} \tag{5.85}$$

is a Lipschitz function which fails to be differentiable at 0.

Theorem 5.1.22. *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a C^1 convex domain, and denote by ν its outward unit normal. Then*

$$\nu(X) \cdot (X - Y) \geqslant 0, \qquad \forall\, X \in \partial\Omega, \quad \forall\, Y \in \overline{\Omega}. \tag{5.86}$$

Proof. Fix $X^* \in \partial\Omega$. Without loss of generality, we can assume that there exists an open set $\mathcal{O} \subseteq \mathbb{R}^{n-1}$, a point $x^* \in \mathcal{O}$, an open interval $I \subseteq \mathbb{R}$ and a C^1 function $\varphi : \mathcal{O} \to I$ with the property that $X^* = (x^*, \varphi(x^*))$ and, if $\mathcal{C} := \mathcal{O} \times I$,

$$\mathcal{C} \cap \Omega = \{X = (x', x_n) \in \mathcal{C} :\, x_n > \varphi(x')\}, \tag{5.87}$$

$$\mathcal{C} \cap \partial\Omega = \{X = (x', x_n) \in \mathcal{C} :\, x_n = \varphi(x')\}, \tag{5.88}$$

$$\mathcal{C} \cap (\overline{\Omega})^c = \{X = (x', x_n) \in \mathcal{C} :\, x_n < \varphi(x')\}. \tag{5.89}$$

Denote by G_φ the graph of φ, i.e., $G_\varphi := \{(x', \varphi(x')) :\, x' \in \mathcal{O}\} \subseteq \mathbb{R}^n$. From the "Point of Impact" Differentiability Criterion (cf. Theorem 5.1.14), we know that for every angle $\theta \in (0, \pi/2)$ there exists $\delta > 0$ with the property that $G_\varphi \cap B(X^*, \delta)$ lies in between the cones $\mathcal{C}_n(X^*, \nu(X^*), \theta)$ and $\mathcal{C}_n(X^*, -\nu(X^*), \theta)$ i.e.,

$$G_\varphi \cap B(X^*, \delta) \subseteq \mathbb{R}^n \setminus \Big[\mathcal{C}_n(X^*, \nu(X^*), \theta) \cup \mathcal{C}_n(X^*, -\nu(X^*), \theta)\Big]. \tag{5.90}$$

In particular, if $\delta > 0$ is sufficiently small, $\mathcal{C}_n(X^*, \nu(X^*), \theta) \cap B(X^*, \delta)$ is a connected set, contained in the cylinder $\mathcal{C} = \mathcal{O} \times I$, which is disjoint from G_φ, and which (cf. Lemma 4.6.16) contains points from $(\overline{\Omega})^c$. It follows that

$$\forall\, \theta \in (0, \pi/2),\ \exists\, \delta > 0 \ \text{ such that } \ \mathcal{C}_n(X^*, \nu(X^*), \theta) \cap B(X^*, \delta) \subseteq (\overline{\Omega})^c. \tag{5.91}$$

We now claim that, in fact,

$$\mathcal{C}_n(X^*, \nu(X^*), \theta) \subseteq (\overline{\Omega})^c \qquad \forall\, \theta \in (0, \pi/2). \tag{5.92}$$

Indeed, if $\theta \in (0, \pi/2)$ is such that there exists $X \in \mathcal{C}_n(X^*, \nu(X^*), \theta) \cap \overline{\Omega}$ then $[X^*, X] \subseteq \overline{\Omega}$ by (iii) in Remark 5.1.23 below. However, this would force

$$[X^*, X] \cap B(X^*, \delta) \subseteq \mathcal{C}_n(X^*, \nu(X^*), \theta) \cap B(X^*, \delta) \cap \overline{\Omega} = \varnothing, \tag{5.93}$$

a contradiction. Thus, (5.92) is proved. In turn, this implies

$$\{X \in \mathbb{R}^n : \nu(X^*) \cdot (X - X^*) > 0\} = \bigcup_{\theta \in (0, \pi/2)} \mathcal{C}_n(X^*, \nu(X^*), \theta) \subseteq (\overline{\Omega})^c \tag{5.94}$$

so that

$$\begin{aligned} \overline{\Omega} &\subseteq \{X \in \mathbb{R}^n : \nu(X^*) \cdot (X - X^*) > 0\}^c \\ &= \{X \in \mathbb{R}^n : \nu(X^*) \cdot (X - X^*) \leqslant 0\} \\ &= \{X \in \mathbb{R}^n : \nu(X^*) \cdot (X^* - X) \geqslant 0\}, \end{aligned} \tag{5.95}$$

as desired. $\square$

Remark 5.1.23.

(i) Any convex set is path-wise connected, hence also connected.

(ii) The intersection of any family of convex sets is a convex set.

(iii) The closure of any convex set is a convex set.

(iv) The interior of any convex set is a convex set.

(v) If $S_1, S_2 \subseteq \mathbb{R}^n$ are convex and $\lambda_1, \lambda_2 \in \mathbb{R}$, then $\lambda_1 S_1 + \lambda_2 S_2$ is also convex.

Remark 5.1.24. *Assume that $\varnothing \neq E \subset \mathbb{R}^n$ is an arbitrary set and fix an arbitrary point $X \in \mathbb{R}^n \setminus E^\circ$. Then*

$$\operatorname{dist}(X, E) = \operatorname{dist}(X, \partial E). \tag{5.96}$$

Indeed, observe that in one direction, by (5.97) below, the fact that ∂E is a subset of $\overline{E}$

and the definition of distance, $\operatorname{dist}(X,E)=\operatorname{dist}(X,\overline{E})\leqslant\operatorname{dist}(X,\partial E)$. For the opposite inequality, use (5.104) to find a point $Y\in\overline{E}$ such that $\operatorname{dist}(X,E)=\operatorname{dist}(X,\overline{E})=\|X-Y\|$. This precludes Y from belonging to E° (explain!) so, necessarily, $Y\in\overline{E}\setminus E^\circ=\partial E$. In turn, this forces $\operatorname{dist}(X,E)=\|X-Y\|\geqslant\operatorname{dist}(X,\partial E)$.

Remark 5.1.25. *Given a point $X\in\mathbb{R}^n$ and a nonempty set $E\subseteq\mathbb{R}^n$, then*

$$\operatorname{dist}(X,E)=\operatorname{dist}(X,\overline{E}). \tag{5.97}$$

Remark 5.1.26. *Let $E\subseteq\mathbb{R}^n$ be an arbitrary set and, for each number $\varepsilon>0$, define $E_\varepsilon:=\{X\in\mathbb{R}^n:\operatorname{dist}(X,E)<\varepsilon\}$. Then for every point $X\in\partial(\overline{E})$ we have*

$$\operatorname{dist}(X,\partial E_\varepsilon)\to 0 \quad \textit{as} \quad \varepsilon\to 0^+. \tag{5.98}$$

Otherwise there exist $r>0$ and a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ of positive numbers which converges to zero such that $\operatorname{dist}(X,\partial E_{\varepsilon_j})\geqslant r$ for every $j\in\mathbb{N}$. Since $X\in\partial(\overline{E})\subseteq\overline{E}\subseteq E_{\varepsilon_j}$ for every $j\in\mathbb{N}$, it follows that $X\notin((E_{\varepsilon_j})^c)^\circ$ for every $j\in\mathbb{N}$. Thus, by Remark 5.1.24, $\operatorname{dist}(X,(E_{\varepsilon_j})^c)=\operatorname{dist}(X,\partial E_{\varepsilon_j})\geqslant r$ for every $j\in\mathbb{N}$. This forces $B(X,r)\subseteq E_{\varepsilon_j}$ for every $j\in\mathbb{N}$ hence, ultimately, $B(X,r)\subseteq\bigcap\limits_{j\in\mathbb{N}}E_{\varepsilon_j}=\overline{E}$, by (5.101) below. This would, however, place X in $(\overline{E})^\circ$, in contradiction with the fact that $X\in\partial(\overline{E})=\overline{E}\setminus(\overline{E})^\circ$.

Remark 5.1.27. *Let $E\subseteq\mathbb{R}^n$ be an arbitrary set. Then for each number $\varepsilon>0$, the set*

$$E_\varepsilon:=\{X\in\mathbb{R}^n:\operatorname{dist}(X,E)<\varepsilon\} \tag{5.99}$$

is open, has the property that

$$E_\varepsilon=E+B(0,\varepsilon), \tag{5.100}$$

$\overline{E_\varepsilon}=\{X\in\mathbb{R}^n:\operatorname{dist}(X,E)\leqslant\varepsilon\}$, *and*

$$\overline{E}=\bigcap_{\varepsilon>0}E_\varepsilon. \tag{5.101}$$

Remark 5.1.28. *Suppose $A\subseteq\mathbb{R}^n$ is compact, $B\subseteq\mathbb{R}^n$ is closed and $A\cap B=\varnothing$. Then there exist two points $X^*\in A$ and $Y^*\in B$ with the property that*

$$\operatorname{dist}(A,B)=\|X^*-Y^*\|, \tag{5.102}$$

and this will imply that

$$\operatorname{dist}(A,B)>0. \tag{5.103}$$

Moreover, the distance from a point to a closed set is always attained, i.e.,

$$\forall\, X^* \in \mathbb{R}^n,\ \forall\, E \subseteq \mathbb{R}^n \text{ closed} \Longrightarrow \exists Y^* \in E \text{ so that } \operatorname{dist}(X^*, E) = \|X^* - Y^*\|. \tag{5.104}$$

Indeed, if we consider the function $f : A \times B \to \mathbb{R}$ given by $f(X,Y) := \|X - Y\|$. Then f is continuous and $\operatorname{dist}(A,B)$ can be viewed as the infimum of f on $A \times B$. One can have that this infimum is attained. If B is bounded then B is compact, hence, so is $A \times B$. Hence, in this case, the desired conclusion follows from the Extreme Value Theorem. For example, if $A = \mathbb{N}$ and $B = \{j + \frac{1}{j+1} : j \in \mathbb{N}\}$, then A and B are closed, and $A \cap B = \varnothing$. However, if we let $x_j = j \in A$ and $y_j = j + \frac{1}{j+1} \in B$, then

$$\operatorname{dist}(A,B) \leqslant \|x_j - y_j\| = \frac{1}{j+1}, \quad \forall j \in \mathbb{N}, \tag{5.105}$$

which implies that $\operatorname{dist}(A,B) = 0$.

Theorem 5.1.29. *Any open convex set $D \subset \mathbb{R}^n$ satisfies a UEBC with constant $+\infty$.*

Proof. Working with $D \cap B(0,R)$ (with $R > 0$ arbitrary), there is no loss of generality in assuming that D is a bounded set to begin with. Next, fix $X^* \in \partial D$ and, for each number $\varepsilon > 0$ define $\Omega_\varepsilon := \{X \in \mathbb{R}^n : \operatorname{dist}(X, D) < \varepsilon\}$. Let $X_\varepsilon \in \partial\Omega_\varepsilon$ be a point with the property that $\|X^* - X_\varepsilon\| = \operatorname{dist}(X^*, \partial\Omega_\varepsilon)$ (cf. (5.104)). Then, by Remark 5.1.26 and Remark 5.1.30 below, $\lim\limits_{\varepsilon \to 0^+} \operatorname{dist}(X^*, \partial\Omega_\varepsilon) = 0$. Thus,

$$\lim_{\varepsilon \to 0^+} \|X^* - X_\varepsilon\| = 0. \tag{5.106}$$

By Theorem 5.1.18, Ω_ε is a $C^{1,1}$ convex domain, and if we denote by ν_ε its outward unit normal, then Theorem 5.1.22 gives that

$$\nu_\varepsilon(X_\varepsilon) \cdot (X_\varepsilon - Y) \geqslant 0, \qquad \forall\, Y \in \overline{\Omega_\varepsilon}. \tag{5.107}$$

The sequence $\{\nu_\varepsilon(X_\varepsilon)\}_{\varepsilon > 0}$ is bounded so, by the Bolzano-Weierstrass theorem, there exist a vector $v \in S^{n-1}$ and a sequence of positive numbers $\{\varepsilon_j\}_{j \in \mathbb{N}}$ with $\lim\limits_{j \to \infty} \varepsilon_j = 0$ and $\lim\limits_{j \to \infty} \nu_{\varepsilon_j}(X_{\varepsilon_j}) = v$. Now, if $Y \in D$, it follows that $Y \in \overline{\Omega_{\varepsilon_j}}$ for every $j \in \mathbb{N}$ so, $\nu_{\varepsilon_j}(X_{\varepsilon_j}) \cdot (X_{\varepsilon_j} - Y) \geqslant 0$ for every $j \in \mathbb{N}$. Passing to limit $j \to \infty$ then yields, on account of (5.106)

$$v \cdot (X^* - Y) \geqslant 0, \qquad \forall\, Y \in D. \tag{5.108}$$

Since for every $r > 0$ we have (compare with Remark 5.1.31)

$$Y \in B(X^* + rv, r) \Longrightarrow v \cdot (X^* - Y) < 0, \tag{5.109}$$

we deduce from (5.108) that $B(X^* + rv, r) \cap D = \varnothing$ for every $r > 0$. Given that we have $X^* \in \partial B(X^* + rv, r)$ and that $X^* \in \partial D$ has been arbitrarily chosen, it follows that D satisfies a UEBC with constant $+\infty$. $\square$

Remark 5.1.30. *One can have the following implications*

$$\Omega \subseteq \mathbb{R}^n \text{ convex } \Longrightarrow \Omega^\circ = (\overline{\Omega})^\circ. \tag{5.110}$$

$$\Omega \subseteq \mathbb{R}^n \text{ convex } \Longrightarrow \partial\Omega = \partial(\overline{\Omega}). \tag{5.111}$$

Remark 5.1.31. *If $X \in \mathbb{R}^n$ is a unit vector, then $X \cdot Y > 0$ for every $Y \in \mathbb{R}^n$ with $\|X - Y\| < 1$.*

Indeed, $1 > \|X - Y\|^2 = \|X\|^2 - 2X \cdot Y + \|Y\|^2 = 1 - 2X \cdot Y + \|Y\|^2$ forces $2X \cdot Y > \|Y\|^2 \geqslant 0$.

Remark 5.1.32. *Let $D \subseteq \mathbb{R}^n$ be given. Call a $(n-1)$-dimensional plane π in $\mathbb{R}^n$ a* **support hyperplane for** *D if $\pi \cap \overline{D} \neq \varnothing$ and one of the two components of $\mathbb{R}^n \setminus \pi$ is disjoint from D (hence, $\overline{D}$ lies on only one side of π). Then through every boundary point of a convex proper subset of $\mathbb{R}^n$ there passes at least one support hyperplane.*

Indeed, recall (5.108) to obtain the existence of a support hyperplane π through a boundary point of D.

Theorem 5.1.33. *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, open, bounded, convex set. Then there exist $\varepsilon_o > 0$ and two families of convex $C^{1,1}$ domains $\{\Omega_\varepsilon^\pm\}_{\varepsilon_o > \varepsilon > 0}$ satisfying*

$$\overline{\Omega_\varepsilon^+} \subset \Omega, \qquad \overline{\Omega} \subset \Omega_\varepsilon^-, \qquad \forall\, \varepsilon \in (0, \varepsilon_o), \tag{5.112}$$

and

$$\Omega = \bigcup_{\varepsilon_o > \varepsilon > 0} \Omega_\varepsilon^+ \quad \text{and} \quad 0 < \varepsilon' < \varepsilon'' < \varepsilon_o \Longrightarrow \overline{\Omega_{\varepsilon''}^+} \subset \Omega_{\varepsilon'}^+, \tag{5.113}$$

$$\overline{\Omega} = \bigcap_{\varepsilon_o > \varepsilon > 0} \Omega_\varepsilon^- \quad \text{and} \quad 0 < \varepsilon' < \varepsilon'' < \varepsilon_o \Longrightarrow \overline{\Omega_{\varepsilon'}^-} \subset \Omega_{\varepsilon''}^-. \tag{5.114}$$

Furthermore, with Dist $[\cdot,\cdot]$ *denoting the Hausdorff distance function, we have*

$$\operatorname{Dist}[\partial\Omega, \partial\Omega_\varepsilon^\pm] = \varepsilon \quad \text{for every} \quad \varepsilon \in (0, \varepsilon_o). \tag{5.115}$$

Proof. Fix $\varepsilon_o > 0$ sufficiently small (e.g., $0 < \varepsilon_o < \frac{1}{4}\,\mathrm{diam}\,(\Omega)$ will do). For each number $\varepsilon \in (0, \varepsilon_o)$ then define

$$\Omega_\varepsilon^- := \{X \in \mathbb{R}^n :\, \mathrm{dist}\,(X, \Omega) < \varepsilon\}. \tag{5.116}$$

Then, by Theorem 5.1.18, $\{\Omega_\varepsilon^-\}_{\varepsilon_o > \varepsilon > 0}$ is a nested family of convex $C^{1,1}$ domains exhausting Ω. In addition, Lemma 5.1.19 gives that for every $X \in \partial\Omega_\varepsilon^-$ there exists $X_\varepsilon \in \partial\Omega$ such that $\|X - X_\varepsilon\| = \varepsilon$. To prove that $\mathrm{Dist}\,[\partial\Omega, \partial\Omega_\varepsilon^-] = \varepsilon$ we need to show that for every $X \in \partial\Omega$ there exists $X_\varepsilon \in \partial\Omega_\varepsilon^-$ such that $\|X - X_\varepsilon\| = \varepsilon$. Fix $X \in \partial\Omega$ and let π be a support hyperplane for Ω through X. Let v be the unit vector in $\mathbb{R}^n$ such that $X + v$ is pointing into the connected component of $\mathbb{R}^n \setminus \pi$ which is disjoint from $\overline{\Omega}$. Then the distance from $X_\varepsilon := X + \varepsilon v$ to π is ε, hence, $\mathrm{dist}\,(X_\varepsilon, \partial\Omega) = \varepsilon$. Accordingly, $X_\varepsilon \in \partial\Omega_\varepsilon^-$ by Lemma 5.1.19 and $\|X - X_\varepsilon\| = \varepsilon$. Altogether, this reasoning shows that $\mathrm{Dist}\,[\partial\Omega, \partial\Omega_\varepsilon^-] = \varepsilon$.

To construct $\{\Omega_\varepsilon^+\}_{\varepsilon_o > \varepsilon > 0}$, consider first $\Omega^\varepsilon := \{X \in \mathbb{R}^n :\, \mathrm{dist}\,(X, \Omega^c) > \varepsilon\}$ for every $\varepsilon > 0$. Then, by Remark 5.1.34 below, this is a convex domain, so if we take

$$\Omega_\varepsilon^+ := \big(\Omega^{2\varepsilon}\big)_\varepsilon := \{X \in \mathbb{R}^n :\, \mathrm{dist}\,(X, \Omega^{2\varepsilon}) < \varepsilon\}, \qquad \forall\, \varepsilon \in (0, \varepsilon_o), \tag{5.117}$$

then the same type of argument shows that $\{\Omega_\varepsilon^+\}_{\varepsilon_o > \varepsilon > 0}$ has all the desired properties as well. $\square$

Remark 5.1.34. *Assume that $\Omega \subseteq \mathbb{R}^n$ is an open convex set and that $\varepsilon > 0$. Then the set*

$$\Omega^\varepsilon := \{X \in \mathbb{R}^n :\, \mathrm{dist}\,(X, \Omega^c) > \varepsilon\} \tag{5.118}$$

is convex and satisfies

$$\partial\Omega^\varepsilon = \{X \in \mathbb{R}^n :\, \mathrm{dist}\,(X, \Omega^c) = \varepsilon\}, \qquad \mathrm{dist}\,(\partial\Omega^\varepsilon, \partial\Omega) = \varepsilon. \tag{5.119}$$

Indeed, if one will pick $X, Y \in \Omega^\varepsilon$. Then there exists $r > \varepsilon$ with the property that $B(X, r), B(Y, r) \subseteq \Omega$. Consequently, if we define

$$O := \bigcup_{X^* \in B(X,r),\, Y^* \in B(Y,r)} [X^*, Y^*] \tag{5.120}$$

then, given that Ω is convex, we have $O \subseteq \Omega$. Thus, $\Omega^c \subseteq O^c$. Now, if $Z \in [X, Y]$, it follows that $\mathrm{dist}\,(Z, \Omega^c) \geqslant \mathrm{dist}\,(Z, O^c) = r > \varepsilon$, so that $Z \in \Omega^\varepsilon$. Hence, $[X, Y] \subseteq \Omega^\varepsilon$

which proves that Ω^ε is convex. Next, the first formula in (5.119) is proved much as (5.80). Finally, the second formula in (5.119) can be justified from (5.121) below.

Remark 5.1.35. *Let $\varnothing \neq E \subset \mathbb{R}^n$ be an arbitrary set and, for each number $\varepsilon > 0$, define $E_\varepsilon := \{X \in \mathbb{R}^n \,:\, \mathrm{dist}\,(X, E) < \varepsilon\}$.*

$$\mathrm{dist}\,(\partial E_\varepsilon, \partial E) = \varepsilon. \tag{5.121}$$

Remark 5.1.36. *Let $v \in \mathbb{R}^n$ be a vector, and assume that $\theta \in (0, \pi)$. Denote by $\mathcal{C}_n(X^*, v, \theta)$ the open, solid circular* **cone** *in $\mathbb{R}^n$ with vertex at $X^* \in \mathbb{R}^n$, whose axis is along v, and has aperture θ. That is, $\mathcal{C}_n(X^*, v, \theta)$ consists of those vectors $X \in \mathbb{R}^n$ with the property that the angle made by $X - X^*$ with v is $< \theta$. Then $\mathcal{C}_n(X^*, v, \theta)$ can be described as*

$$\mathcal{C}_n(X^*, v, \theta) = \{X \in \mathbb{R}^n : \, (X - X^*) \cdot v > (\cos\,\theta)\|X - X^*\|\|v\|\}. \tag{5.122}$$

Definition 5.1.37. *A set $\Omega \subseteq \mathbb{R}^n$ is said to satisfy a* **uniform cone property** *provided for every $X^* \in \partial\Omega$ there exist $R, r > 0$, $\theta \in (0, \pi/2)$ and a unit vector $v \in \mathbb{R}^n$ with the property that (cf. (5.122))*

$$\mathcal{C}_n(X, v, \theta) \cap B(X, R) \subseteq \Omega \quad \text{ and } \quad \mathcal{C}_n(X, -v, \theta) \cap B(X, R) \subseteq \mathbb{R}^n \setminus \Omega \tag{5.123}$$

for every $X \in B(X^*, r) \cap \partial\Omega$.

Remark 5.1.38. *Let $\zeta, \eta \in \mathbb{R}^n$, $n \geqslant 2$, be two linearly independent unit vectors, and consider the mapping*

$$\mathcal{R}(\xi) := \xi - \Big[\frac{\xi \cdot (\eta + \zeta)}{1 + \eta \cdot \zeta}\Big]\zeta + \Big[\frac{\xi \cdot [(1 + 2\eta \cdot \zeta)\zeta - \eta]}{1 + \eta \cdot \zeta}\Big]\eta, \qquad \forall\, \xi \in \mathbb{R}^n, \tag{5.124}$$

regarded as an application $R : \mathbb{R}^n \to \mathbb{R}^n$. Then this is a rotation about the origin in $\mathbb{R}^n$ which satisfies

$$\mathcal{R}(\zeta) = \eta \quad \text{and} \quad \mathcal{R}^\top(\eta) = \zeta. \tag{5.125}$$

Indeed, via a direct calculation, one can have that

$$\mathcal{R}^\top(w) = w - \Big[\frac{w \cdot [\zeta - (1 + 2\eta \cdot \zeta)\eta]}{1 + \eta \cdot \zeta}\Big]\zeta - \Big[\frac{w \cdot (\zeta + \eta)}{1 + \eta \cdot \zeta}\Big]\eta, \qquad \forall\, w \in \mathbb{R}^n, \tag{5.126}$$

then use this to obtain that $\mathcal{R}\mathcal{R}^\top = I_{n \times n}$. This guarantees that $\mathcal{R}$ is a rotation in $\mathbb{R}^n$. Then (5.125) is seen straight from (5.124).

Recall (ii) in Definition 5.1.2, and use the Remark 5.1.38 above to conclude the next Teorem.

Theorem 5.1.39. *A nonempty, bounded, open set $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, satisfies a uniform cone property if and only if Ω is a domain of class $C^{0,1}$.*

Proof. Assume first that $\Omega \subseteq \mathbb{R}^n$ is a nonempty, bounded, open set which satisfies a uniform cone property. Fix an arbitrary point $X^* \in \partial\Omega$. Then there exist $R, r > 0$, $\theta \in (0, \pi/2)$ and a unit vector $v \in \mathbb{R}^n$ with the property that (5.123) holds for every $X \in B(X^*, r) \cap \partial\Omega$. Since for any rotation $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\begin{aligned} &\mathcal{R}(\mathcal{C}_n(X, \pm v, \theta)) = \mathcal{C}_n(\mathcal{R}(X), \pm\mathcal{R}(v), \theta)), \\ &\|\mathcal{R}(v)\| = 1, \quad \mathcal{R}(B(X,R)) = B(\mathcal{R}(X), R), \end{aligned} \tag{5.127}$$

there is no loss of generality (cf. Remark 5.1.38) in assuming that $v = \mathbf{e}_n$. Also, performing a suitable translation, we can assume that $X^* = 0 \in \mathbb{R}^n$. Granted these, fix some small positive number h, say,

$$0 < h < \min\Big\{R\cos\theta\,,\, \frac{r}{\sqrt{1+\cos^2\theta}}\Big\}, \tag{5.128}$$

and consider the cylinder

$$\mathcal{C} := B_{n-1}(0', h\cos\theta) \times (-h, h) \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n. \tag{5.129}$$

Then the top and bottom lids of $\mathcal{C}$ are contained in $\mathcal{C}_n(X, v, \theta) \cap B(X,R) \subseteq \Omega$ and $\mathcal{C}_n(X, -v, \theta) \cap B(X,R) \subseteq \mathbb{R}^n \setminus \Omega$, respectively. In particular, the bottom lid is contained in $\mathcal{C}_n(X, -v, \theta) \cap B(X,R) \subseteq (\Omega^c)^\circ = (\overline{\Omega})^c$. It follows from this and Proposition 5.1.40 below that, for every $x' \in B_{n-1}(0', h\cos\theta)$, the line segment

$$L(x') := [(x', h), (x', -h)] \tag{5.130}$$

intersects $\partial\Omega$ at least once, and we claim that, in fact,

$$\#(L(x') \cap \partial\Omega) = 1. \tag{5.131}$$

To justify this, let $X = (x', x_n) \in L(x') \cap \partial\Omega$. Then

$$\|X - X^*\| = \|X\| = \sqrt{\|x'\|^2 + x_n^2} \leqslant \sqrt{h^2\cos^2\theta + h^2} = h\sqrt{\cos^2\theta + 1} < r, \tag{5.132}$$

so $X \in B(X^*, r) \cap \partial\Omega$. Consequently, from (5.123) and conventions,

$$\mathcal{C}_n(X, \mathbf{e}_n, \theta) \cap B(X,R) \subseteq \Omega \quad\text{and}\quad \mathcal{C}_n(X, -\mathbf{e}_n, \theta) \cap B(X,R) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ. \tag{5.133}$$

This forces

$$(X, X + R\,\mathbf{e}_n) \subseteq \Omega \quad \text{and} \quad (X, X - R\,\mathbf{e}_n) \subseteq (\overline{\Omega})^c \tag{5.134}$$

and, hence, $(X - R\,\mathbf{e}_n, X + R\,\mathbf{e}_n) \cap \partial\Omega = \{X\}$. Then (5.131) follows from this, after noticing that $L(x') \subseteq (X - R\,\mathbf{e}_n, X + R\,\mathbf{e}_n)$ (since $h < R\cos\theta < R$).

Having established (5.131), it is then possible to define a function

$$\varphi : B_{n-1}(0', h\cos\,\theta) \longrightarrow (-h, h) \tag{5.135}$$

in an unambiguous fashion by setting, for every $x' \in B_{n-1}(0', h\cos\,\theta)$,

$$\varphi(x') := x_n \quad \text{if} \quad (x', x_n) \in L(x') \cap \partial\Omega. \tag{5.136}$$

Then, by design (recall (5.129)), we have

$$\mathcal{C} \cap \partial\Omega = \{X = (x', x_n) \in \mathcal{C} :\, x_n = \varphi(x')\}, \tag{5.137}$$

and we now proceed to show that φ defined in (5.135)-(5.136) is a Lipschitz function. Concretely, if $x', y' \in B_{n-1}(0', h\cos\,\theta)$ are two arbitrary points, then $(y', \varphi(y')) \in \partial\Omega$ therefore $(y', \varphi(y')) \notin \mathcal{C}_n((x', \varphi(x')), \pm\mathbf{e}_n, \theta)$ which implies

$$\begin{aligned} \pm((y', \varphi(y')) - (x', \varphi(x'))) \cdot \mathbf{e}_n \;&\leqslant\; (\cos\,\theta)\|(y', \varphi(y')) - (x', \varphi(x'))\| \\ &\leqslant\; (\cos\,\theta)\|y' - x'\|. \end{aligned} \tag{5.138}$$

Thus, ultimately, $|\varphi(y') - \varphi(x')| \leqslant \cos\,\theta\|y' - x'\|$, which shows that φ is a Lipschitz function, with Lipschitz constant $\leqslant \cos\,\theta$. Going further, since the uniform cone condition also entails that points in $\partial\Omega$ are limits points for $(\mathbb{R}^n \setminus \Omega)^\circ$, we have $\partial\Omega \subseteq \overline{(\bar{\Omega})^c}$. Accordingly,

$$\partial\Omega \subseteq \overline{\Omega} \cap \overline{(\overline{\Omega})^c} = \overline{\Omega} \setminus (\overline{\Omega})^\circ = \partial(\overline{\Omega}). \tag{5.139}$$

Since the opposite inclusion is always true (cf. (6) in Theorem 4.6.4), we ultimately have $\partial\Omega = \partial(\overline{\Omega})$. With this and (5.137) in hand, we may then invoke Remark 4.6.5 in order to conclude that Ω is a $C^{0,1}$ domain.

Conversely, assume next that Ω is a $C^{0,1}$ domain. Then for every point $X^* \in \partial\Omega$ there exist $\rho > 0$, an open interval $I \subset \mathbb{R}$ with $0 \in I$, a rigid transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(X^*) = 0$, along with a Lipschitz function $\varphi : B_{n-1}(0', \rho) \to I$ the property that

$\varphi(0')=0$ and, if $\mathcal{C}$ denotes the (open) cylinder $B_{n-1}(0',\rho)\times I\subseteq\mathbb{R}^{n-1}\times\mathbb{R}=\mathbb{R}^n$, then

$$\mathcal{C}\cap T(\Omega)=\{X=(x',x_n)\in\mathcal{C}:\ x_n>\varphi(x')\}, \tag{5.140}$$

$$\mathcal{C}\cap\partial T(\Omega)=\{X=(x',x_n)\in\mathcal{C}:\ x_n=\varphi(x')\}, \tag{5.141}$$

$$\mathcal{C}\cap(\overline{T(\Omega)})^c=\{X=(x',x_n)\in\mathcal{C}:\ x_n<\varphi(x')\}. \tag{5.142}$$

The fact that Ω satisfies a uniform cone condition will follow from (5.140)-(5.142) as soon as we show that there exist $R,r>0$ and $\theta\in(0,\pi/2)$ with the property that

$$\begin{gathered}X\in\partial\Omega\cap B(0,r)\text{ and }Y=(y',y_n)\in\mathcal{C}_n(X,\mathbf{e}_n,\theta)\cap B(X,R)\\ \Longrightarrow y'\in B_{n-1}(0',\rho)\ \text{ and }\ y_n>\varphi(y'),\end{gathered} \tag{5.143}$$

and

$$\begin{gathered}X\in\partial\Omega\cap B(0,r)\text{ and }Y=(y',y_n)\in\mathcal{C}_n(X,-\mathbf{e}_n,\theta)\cap B(X,R)\\ \Longrightarrow y'\in B_{n-1}(0',\rho)\ \text{ and }\ y_n<\varphi(y').\end{gathered} \tag{5.144}$$

In fact, we shall only deal with (5.143), as the treatment of (5.144) is similar. Taking $r\in(0,\rho/2)$ guarantees that if $X=(x',x_n)\in B(0,r)$ then $x'\in B_{n-1}(0',\rho)$ and $x_n=\varphi(x')$. If also take $R\in(0,\rho/2)$ then, with the angle $\theta\in(0,\pi/2)$ to be selected later, for every point $Y=(y',y_n)\in\mathcal{C}_n(X,\mathbf{e}_n,\theta)\cap B(X,R)$ we have $\|y'\|\leqslant\|y'-x'\|+\|x'\|<\rho/2+\rho/2=\rho$, i.e., $y'\in B_{n-1}(0',\rho)$. Also, the fact that the angle between $Y-X$ and $\mathbf{e}_n$ is $<\theta$ implies

$$\begin{aligned}\frac{(Y-X)\cdot\mathbf{e}_n}{\|Y-X\|}>\cos\theta\ &\Longrightarrow\ y_n-x_n>(\cos\theta)\|Y-X\|\\ &\Longrightarrow\ (y_n-x_n)^2>(\cos\theta)^2[\,\|y'-x'\|^2+(y_n-x_n)^2]\\ &\Longrightarrow\ y_n-x_n>(\cot\theta)\|y'-x'\|.\end{aligned} \tag{5.145}$$

Pick a number $M>0$ larger than the Lipschitz constant of φ. Based on (5.145) we may then estimate

$$\begin{aligned}\varphi(y')&=((\varphi(y')-\varphi(x'))+\varphi(x')<M\|y'-x'\|+\varphi(x')\\ &=M\|y'-x'\|+x_n\leqslant M(\tan\theta)(y_n-x_n)+x_n\\ &<(y_n-x_n)+x_n=y_n,\end{aligned} \tag{5.146}$$

where the last inequality holds if we choose

$$\theta\in(0,\pi/2)\quad\text{ such that }\quad\theta<\arctan(M^{-1}). \tag{5.147}$$

This justifies (5.143) and finishes the proof of the theorem. □

Proposition 5.1.40. *Assume that $E \subset \mathbb{R}^n$ is a given set, and that two points $X \in E^\circ$ and $Y \in (\overline{E})^c$ have been fixed. Then $(X, Y) \cap \partial E \neq \varnothing$.*

Remark 5.1.41. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a nonempty, bounded, open set with the property that for every $X^* \in \partial\Omega$ there exist $R, r > 0$, $\theta \in (0, \pi/2)$ and a unit vector $v \in \mathbb{R}^n$ with the property that*

$$\mathcal{C}_n(X, v, \theta) \cap B(X, R) \subseteq \Omega \quad \textit{for every} \quad X \in B(X^*, r) \cap \Omega. \tag{5.148}$$

Then Ω is a $C^{0,1}$ domain.

Indeed, one can have that Ω satisfies a uniform cone property, then invoke Theorem 5.1.39. More specifically, if Ω satisfies (5.148), then also

$$X \in B(X^*, r/2) \cap \partial\Omega \Longrightarrow \mathcal{C}_n(X, -v, \theta) \cap B(X, R) \subseteq \Omega^c, \tag{5.149}$$

at least if the number R is sufficiently small relative to r. Indeed, the existence of a point $Y \in \mathcal{C}_n(X, -v, \theta) \cap B(X, R) \cap \Omega$ would entail $X \in \mathcal{C}_n(Y, v, \theta)$. Since $Y \in \Omega$ is also close to X^* (assuming that R is small, relative to r), (5.148) further implies that X belongs to the open set Ω, in contradiction with $X \in \partial\Omega$.

Recall that S^{n-1} is the unit sphere in $\mathbb{R}^n$ centered at the origin.

Theorem 5.1.42. *Assume that*

$$\varphi : S^{n-1} \longrightarrow (0, \infty) \quad \textit{is Lipschitz} \tag{5.150}$$

and consider the set $\Omega \subseteq \mathbb{R}^n$ which, in polar coordinates, has the description

$$\Omega := \left\{ r\omega : \omega \in S^{n-1}, \quad 0 \leqslant r < \varphi(\omega) \right\}. \tag{5.151}$$

Then Ω is a $C^{0,1}$ domain.

Proof. Fix a point

$$X^* \in \partial\Omega = \left\{ r\omega : \omega \in S^{n-1}, \quad r = \varphi(\omega) \right\}. \tag{5.152}$$

We wish to show that, after a rigid transformation, $\partial\Omega$ can be made to coincide, near X^*, with the graph of a Lipschitz function. After possibly rotating Ω, there is no loss of generality in assuming that X^* lies at the intersection of the vertical axis in $\mathbb{R}^n$ with $\partial\Omega$. That is,

$$X^* = (0', -\lambda), \quad \text{where} \quad \lambda := \varphi(-\mathbf{e}_n) > 0. \tag{5.153}$$

In this setting, we will establish the existence of a cylinder

$$\mathcal{C} := \left\{ X = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, \|x'\| < r, \quad |x_n| < h \right\}, \tag{5.154}$$

where $r > 0$, $h > \lambda$, and a function

$$\psi : B_{n-1}(0', r) \to (-h, h) \;\text{ Lipschitz} \tag{5.155}$$

with the property that

$$X^* = (0', \psi(0')) \in \mathcal{C} \tag{5.156}$$

and for which

$$\partial\Omega \cap \mathcal{C} = \{(x', \psi(x') : \, x' \in B_{n-1}(0', r)\} \cap \mathcal{C}. \tag{5.157}$$

Condition (5.157) requires that if $x_n = \psi(x')$

$$\varphi\left(\frac{(x', x_n)}{\|(x', x_n)\|}\right) = \|(x', x_n)\| \tag{5.158}$$

and the idea is to find the Lipschitz function ψ by solving (5.158) for x_n as a function of x'. To this end, it is convenient to rephrase (5.158) as

$$F(x', x_n) = 0, \qquad (x', x_n) \;\text{ near }\; (0', -\lambda), \tag{5.159}$$

where

$$F(x', x_n) := \varphi^2\left(\frac{(x', x_n)}{\|(x', x_n)\|}\right) - \|x'\|^2 - x_n^2, \qquad (x', x_n) \;\text{ near }\; (0', -\lambda). \tag{5.160}$$

Note that

$$F(0', -\lambda) = \varphi^2(-\mathbf{e}_n) - \lambda^2 = 0. \tag{5.161}$$

We want to use the Implicit Function Theorem for Lipschitz Functions in order to solve (5.159) for x_n in terms of x' such that

$$x_n(0') = -\lambda. \tag{5.162}$$

At this stage, we need to check the hypothesis of the Implicit Function Theorem for F given in (5.160). First, let us check that F is a Lipschitz function (in x' and x_n). To this end, we make use of (5.160) and the fact that φ is Lipschitz in order to write

$$\begin{aligned} |F(x', x_n) - F(y', y_n)| \;&\leqslant\; \left|\varphi^2\left(\frac{(x', x_n)}{\|(x', x_n)\|}\right) - \varphi^2\left(\frac{(y', y_n)}{\|(y', y_n)\|}\right)\right| \\ &\quad + \left|\|x'\|^2 - \|y'\|^2\right| + \left|x_n^2 - y_n^2\right| \\ &=: \; I + II. \end{aligned} \tag{5.163}$$

Note that

$$\begin{aligned} I &= \left|\varphi\left(\frac{(x',x_n)}{\|(x',x_n)\|}\right)-\varphi\left(\frac{(y',y_n)}{\|(y',y_n)\|}\right)\right|\cdot\left|\varphi\left(\frac{(x',x_n)}{\|(x',x_n)\|}\right)+\varphi\left(\frac{(y',y_n)}{\|(y',y_n)\|}\right)\right| \\ &\leqslant 2\big(\sup_{\omega\in S^{n-1}}\varphi(\omega)\big)\mathrm{Lip}\,(\varphi;S^{n-1})\left|\frac{(x',x_n)}{\|(x',x_n)\|}-\frac{(y',y_n)}{\|(y',y_n)\|}\right|. \end{aligned} \tag{5.164}$$

To continue, let us bring in a general identity

Remark 5.1.43. *If $\lambda_1,\lambda_2\geqslant 0$ and $X_1,X_2\in\mathbb{R}^n$ are unit vectors, then*

$$\|\lambda_1X_1-\lambda_2X_2\|^2=|\lambda_1-\lambda_2|^2+\lambda_1\lambda_2\|X_1-X_2\|^2. \tag{5.165}$$

Rewrite it as

$$|\omega_1-\omega_2|^2=r_1^{-1}r_2^{-1}|r_1\omega_1-r_2\omega_2|^2-r_1^{-1}r_2^{-1}|r_1-r_2|^2, \tag{5.166}$$

valid for any $\omega_1,\omega_2\in S^{n-1}$, $r_1,r_2>0$. Making use of (5.166) with

$$\omega_1:=\frac{(x',x_n)}{\|(x',x_n)\|},\quad r_1:=\|(x',x_n)\|,\qquad \omega_2:=\frac{(y',y_n)}{\|(y',y_n)\|}\quad r_2:=\|(y',y_n)\|, \tag{5.167}$$

gives

$$\begin{aligned} &\left|\frac{(x',x_n)}{\|(x',x_n)\|}-\frac{(y',y_n)}{\|(y',y_n)\|}\right|^2 \\ &=\|(x',x_n)\|^{-1}\|(y',y_n)\|^{-1}\left\{\|(x',x_n)-(y',y_n)\|^2-\Big|\|(x',x_n)\|-\|(y',y_n)\|\Big|^2\right\}. \end{aligned} \tag{5.168}$$

Using the triangle inequality, it is then easy to bound

$$\begin{gathered} \left|\frac{(x',x_n)}{\|(x',x_n)\|}-\frac{(y',y_n)}{\|(y',y_n)\|}\right|\leqslant C\|(x',x_n)-(y',y_n)\| \\ \text{for } (x',x_n),\ (y',y_n) \text{ near } (0',-\lambda), \end{gathered} \tag{5.169}$$

where $C>0$ is a fixed constant (which depends on λ). Plugging this back in (5.164) then gives

$$I\leqslant C\|(x',x_n)-(y',y_n)\|\quad\text{for } (x',x_n),\ (y',y_n) \text{ near } (0',-\lambda). \tag{5.170}$$

In a more direct fashion (using just the triangle inequality), we also obtain

$$II\leqslant C\|(x',x_n)-(y',y_n)\|\quad\text{for } (x',x_n),\ (y',y_n) \text{ near } (0',-\lambda). \tag{5.171}$$

All together, (5.163) and (5.171)-(5.172) give that

$$\begin{gathered} |F(x',x_n)-F(y',y_n)|\leqslant C\|(x',x_n)-(y',y_n)\| \\ \text{for } (x',x_n),\ (y',y_n) \text{ near } (0',-\lambda). \end{gathered} \tag{5.172}$$

This proves that F is Lipschitz in a neighborhood of $(0',-\lambda)$. Thus, for Theorem 4.1.4 to apply we need to check that F is bi-Lipschitz in the second variable. That is, we seek an estimate of the form

$$C\,|y_1 - y_2| \leqslant |F(x', y_1) - F(x', y_2)| \quad \text{for } x' \text{ near } 0' \text{ and } y_1, y_2 \text{ near } -\lambda. \tag{5.173}$$

To justify this, we first write

$$\begin{aligned} F(x', x_n) - F(y', y_n) &= \varphi^2\left(\frac{(x', y_1)}{\|(x', y_1)\|}\right) - \varphi^2\left(\frac{(x', y_2)}{\|(x', y_2)\|}\right) \\ &\quad + y_2^2 - y_1^2 \\ &=: III + IV, \end{aligned} \tag{5.174}$$

and note that

$$|IV| = |y_1 - y_2||y_1 + y_2| \geqslant \lambda |y_1 - y_2| \quad \text{for } y_1, y_2 \text{ sufficiently close to } -\lambda. \tag{5.175}$$

As for III in (5.174), we may recycle (5.164) in the form

$$|III| \leqslant 2\big(\sup_{\omega \in S^{n-1}} \varphi(\omega)\big) \mathrm{Lip}\,(\varphi; S^{n-1}) \left| \frac{(x', y_1)}{\|(x', y_1)\|} - \frac{(x', y_2)}{\|(x', y_2)\|} \right|. \tag{5.176}$$

Using the Mean Value Theorem we then estimate

$$\left| \frac{(x', y_1)}{\|(x', y_1)\|} - \frac{(x', y_2)}{\|(x', y_2)\|} \right| \leqslant |y_1 - y_2| \left(\max_{1 \leqslant j \leqslant n} \sup_{x_n \in [y_1, y_2]} \left| \partial_n \left(\frac{(x', x_n)}{\|(x', x_n)\|} \right)_j \right| \right). \tag{5.177}$$

Now, observe that for each $j \in \{1, ..., n\}$,

$$\partial_n \left(\frac{(x', x_n)}{\|(x', x_n)\|} \right)_j = \frac{\delta_{jn}}{\|(x', x_n)\|} - \frac{x_j x_n}{\|(x', x_n)\|^3}, \tag{5.178}$$

and that, since $x_n \in [y_1, y_2]$ and x' is assumed to be near $0'$,

$$\begin{aligned} y_1, y_2 \text{ close to } -\lambda &\implies x_n \text{ close to } -\lambda \implies \\ (x', x_n) \text{ close to } (0', -\lambda) &\implies \frac{\delta_{jn}}{\|(x', x_n)\|} - \frac{x_j x_n}{\|(x', x_n)\|^3} \text{ close to } 0. \end{aligned} \tag{5.179}$$

Consequently,

$$|III| \leqslant \frac{\lambda}{2}\,|y_1 - y_2| \quad \text{if } (x', y_1),\ (x', y_2) \text{ sufficiently close to } (0', -\lambda). \tag{5.180}$$

Relying on this and recalling (5.175), we may now deduce from (5.174) that

$$\begin{aligned} |F(x', y_1) - F(x', y_2)| &= |III + IV| \geqslant |IV| - |III| \\ &\geqslant \lambda |y_1 - y_2| - \frac{\lambda}{2}\,|y_1 - y_2| = \frac{\lambda}{2}\,|y_1 - y_2|, \end{aligned} \tag{5.181}$$

whenever $(x' y_1)$ and (x', y_2) are sufficiently close to $(0', -\lambda)$. This finishes the justification of 5.173.

At this point, Theorem 4.1.4 applies and gives the existence of a neighborhood V of $0'$

in $\mathbb{R}^{n-1}$ along with a Lipschitz function

$$\psi : V \longrightarrow \mathbb{R}, \qquad \text{with} \;\; \psi(0') = -\lambda, \tag{5.182}$$

and such that

$$F(x', \psi(x')) = 0 \quad \text{for every} \;\; x' \in V. \tag{5.183}$$

Given the definition of F in (5.160), this entails

$$\varphi\left(\frac{(x, \psi(x'))}{\|(x', \psi(x'))\|}\right) = \|(x', \psi(x'))\| \quad \text{for every} \;\; x' \in V. \tag{5.184}$$

With this in mind and observing that

$$\partial\Omega = \{\varphi(\omega)\omega : \, \omega \in S^{n-1}\}, \tag{5.185}$$

it follows (by specializing $\omega := \frac{(x', \psi(x'))}{\|(x', \psi(x'))\|} \in S^{n-1}$) that

$$(x', \psi(x')) = \varphi\left(\frac{(x', \psi(x'))}{\|(x', \psi(x'))\|}\right)\frac{(x', \psi(x'))}{\|(x', \psi(x'))\|} \in \partial\Omega \quad \text{for every} \;\; x' \in V. \tag{5.186}$$

Thus,

$$\text{the graph of } \psi \text{ is included in } \partial\Omega. \tag{5.187}$$

On the other hand, if $X = (x', x_n) \in \partial\Omega$ is near $X^* = -\lambda\mathbf{e}_n = (0', -\lambda)$, then there exists $\omega \in S^{n-1}$ near $-\mathbf{e}_n$ such that

$$\varphi(\omega)\omega = X \tag{5.188}$$

which forces

$$\|X\| = \varphi(\omega) \quad \text{and} \quad \omega = \frac{X}{\|X\|}. \tag{5.189}$$

Thus,

$$F(X) = \varphi^2\left(\frac{X}{\|X\|}\right) - \|X\|^2 = \|X\|^2 - \|X\|^2 = 0 \tag{5.190}$$

and since X is near $-\lambda\mathbf{e}_n$, (4.6) tells us that X belongs to the graph of ψ. Consequently,

$$\partial\Omega \text{ near } X^* = -\lambda\mathbf{e}_n \text{ is contained in the graph of } \psi. \tag{5.191}$$

From (5.187) and (5.191)

$$\partial\Omega \text{ and the graph of } \psi \text{ coincide near } X^*. \tag{5.192}$$

Consequently, there exists an open, upright, doubly truncated circular cylinder $\mathcal{C}$ in $\mathbb{R}^{n-1} \times \mathbb{R}$ such that

$$\mathcal{C} \cap \partial\Omega = \{(x', x_n) \in \mathcal{C} : \, x_n = \psi(x')\}, \tag{5.193}$$

in agreement with (4.207).

Upon observing that

$$\overline{\Omega} = \{r\omega : 0 \leqslant r \leqslant \varphi(\omega),\, \omega \in S^{n-1}\} \tag{5.194}$$

we may conclude (using also (5.185)) that

$$\partial\overline{\Omega} = \{\varphi(\omega)\omega : \omega \in S^{n-1}\} = \partial\Omega. \tag{5.195}$$

Hence, $\partial\overline{\Omega} = \partial\Omega$ and, having already established (4.207), conditions (4.206), (4.208) are now guaranteed by the Remark 5.1.44 below. This proves that Ω is a $C^{0,1}$ domain. $\square$

Remark 5.1.44. *Assume that $\Omega \subset \mathbb{R}^n$ is an open set, and fix $X_0 \in \partial\Omega$. Also, assume that $B' \subset \mathbb{R}^{n-1}$ is an open ball, $I \subseteq \mathbb{R}$ is an open interval, and*

$$\varphi : B' \longrightarrow I \quad \textit{is continuous.} \tag{5.196}$$

Denote by $\Sigma := \{(x', \varphi(x')) : x' \in B'\}$ the graph of φ. Assume that the open cylinder

$$\mathcal{C} := B' \times I \subset \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n \tag{5.197}$$

contains X_0 and satisfies

$$\Sigma = \mathcal{C} \cap \partial\Omega. \tag{5.198}$$

Finally, set

$$D^+ := \{(x', x_n) \in \mathcal{C} : \varphi(x') < x_n\}, \qquad D^- := \{(x', x_n) \in \mathcal{C} : \varphi(x') > x_n\}. \tag{5.199}$$

Then one of the following three alternatives holds:

$$\Omega \cap \mathcal{C} = D^+ \quad \textit{and} \quad (\overline{\Omega})^c \cap \mathcal{C} = D^-, \tag{5.200}$$

or

$$\Omega \cap \mathcal{C} = D^- \quad \textit{and} \quad (\overline{\Omega})^c \cap \mathcal{C} = D^+, \tag{5.201}$$

or

$$X_0 \in (\overline{\Omega})^\circ. \tag{5.202}$$

Definition 5.1.45.

1. *Call an open set Ω starlike with respect to $x_0 \in \Omega$ if $\mathcal{I}(x, x_0) \subseteq \Omega$ for all $x \in \Omega$, where $\mathcal{I}(x, x_0) :=$ the open segment with endpoint x and x_0.*

2. *Call an open set Ω starlike with respect to a ball $B \subseteq \Omega$ if $\mathcal{I}(x, x_0) \subseteq \Omega$ for all $x \in \Omega$ and $y \in B$. That is, Ω is starlike with respect to any point in B.*

Theorem 5.1.46. *Let Ω be a bounded, open, non-empty set which is starlike with respect to some ball. Then Ω is a Lipschitz domain.*

Proof. Without loss of generality, assume that Ω is starlike with respect to $B := B(0, \rho)$, for some $\rho > 0$.

Step I. Assume that for all $x \in \overline{\Omega}$, $x_0 \in B \Longrightarrow \mathcal{I}(x, x_0) \subseteq \Omega$.

Proof of Step I. Since $x \in \overline{\Omega}$, there exists $x_j \in \Omega$ such that $x_j \to x$. Fix $y \in \mathcal{I}(x, x_o)$, then there exists $t \in (0, 1)$ such that $y = x_0 + t(x - x_0)$ and consider $y_j := x_0 + t(x_j - x_0) \in \mathcal{I}(x, x_0) \subseteq \Omega$. Then

$$\lim_{j\to\infty} y_j = \lim_{j\to\infty} [x_0 + t(x_j - x_0)] = x_0 + t(x - x_0) = y \tag{5.203}$$

Since $y_j \in \Omega$ for every j, this entails $y \in \overline{\Omega}$. Since y was an arbitrary point in $\mathcal{I}(x, x_0)$, this implies that

$$\mathcal{I}(x, x_0) \subseteq \overline{\Omega}. \tag{5.204}$$

Thus so far, we have that

$$\mathcal{I}(x, x_0) \subseteq \overline{\Omega}, \text{ for all } x \in \overline{\Omega} \text{ and } x_0 \in B. \tag{5.205}$$

We now claim that

$$\mathcal{I}(x, x_0) \cap \partial\Omega = \emptyset. \tag{5.206}$$

Reasoning by contradiction, assume that there exists $z \in \mathcal{I}(x, x_0)$ such that $z \in \partial\Omega$. Define the cone-like region

$$\mathcal{C} := \bigcup_{y \in B} \mathcal{I}(z, y). \tag{5.207}$$

Note that $\mathcal{C}$ is open. By (5.205) we know that $\mathcal{C} \subseteq \overline{\Omega}$. Also, $\mathcal{I}(x, z) \subseteq \mathcal{I}(x, x_0) \subseteq \overline{\Omega}$ and $\mathcal{I}(x, z) \cap \Omega = \emptyset$. Thus, if $w \in \mathcal{I}(x, z) \cap \Omega$ then $z \in \mathcal{I}(w, x_0) \subseteq \Omega$ which is a contradiction. Hence

$$\mathcal{I}(x, z) \subseteq \partial\Omega. \tag{5.208}$$

Let $z_0 \in \mathcal{I}(x, z)$ then there exists B_0 ball centered at z_0 such that $B_0 \subseteq \mathcal{C} \subseteq \overline{\Omega}$. Then there exists three collinear points a, b, c (mutually different) with $a \in B, b \in \mathcal{I}(x, z)$, and $c \in B$. This means that $a \notin \Omega$, otherwise $\mathcal{I}(a, c) \subseteq \Omega$, which implies that $b \in \Omega$. However, $b \in \mathcal{I}(x, z) \subseteq \partial\Omega$. Since, nonetheless, $a \in B_0 \subseteq \mathcal{C} \subseteq \overline{\Omega}$ implies $a \in \partial\Omega$.

Perturbing a, implies there exists $r > 0$ such that $B(a,r) \subseteq B_0$ and such that for all $a \in B(a,r) \subseteq \partial\Omega$ implies that $a' \in \partial\Omega$ (i.e. $B(a,r) \subseteq \partial\Omega$. However, $a \in \partial\Omega$ implies $B(a,r) \cap \Omega \neq \emptyset$, in contradiction with

$$B(a,r) \cap \Omega \subseteq \partial\Omega \cap \Omega = \emptyset. \tag{5.209}$$

Hence this proves (5.207). From (5.205) and (5.207), the conclusion in Step I follows.

Step II. Let us assume that for all $\omega \in S^{n-1}$, the open half line

$$L_w := \{r\omega : r > 0\} \tag{5.210}$$

must intersect $\partial\Omega$.

Proof of Step II. Note that $L_w \subseteq \mathbb{R}^n = \Omega \cup \partial\Omega \cup (\overline{\Omega})^c$, disjoint union. Thus, if $L_w \cap \partial\Omega = \emptyset$ then this implies that Ω and $(\overline{\Omega})^c$ are and open, disjoint, cover of the connected set L_w. Consequently, either

(i) $L_w \cap \Omega = \emptyset$, or

(ii) $L_w \cap (\overline{\Omega})^c = \emptyset$.

Since ω is open and $\mathcal{O} \in \Omega$ then this implies (i) cannot occur. Now, (ii) implies that $L_w \subseteq \overline{\Omega}$ which implies is impossible since Ω is bounded. Thus this contradiction proves Step II.

Step III. Assume that if $\omega \in S^{n-1}$ then $L_w \cap \partial\Omega$ contains just one point.

Proof of Step III. By applying the results of Step II we know that $L_w \cap \partial\Omega \neq \emptyset$. If $x_1 \neq x_2 \in L_w \cap \partial\Omega$ then either,

(i) $x_2 \in \mathcal{I}(x_1, 0)$ or

(ii) $x_1 \in \mathcal{I}(x_2, 0)$.

By Step I, if (i) happens then, since $x_1 \in \partial\Omega$ and $\partial\Omega \ni x_2 \in \mathcal{I}(x_1, 0) \subseteq \Omega$, which is a contradiction. By Step I, if (ii) occurs, then (since $x_2 \in \partial\Omega$) $\partial\Omega \ni x_1 \in \mathcal{I}(x_2, 0) \subseteq \Omega$, which is a contradiction. Hence, this finishes the proof of Step III.

Step IV. If for all $\omega \in S^{n-1}$ we set $\varphi(\omega) :=$ the distance from the point in $L_w \cap \partial\Omega$ to the origin then

$$\varphi : S^{n-1} \longrightarrow (0, \infty) \tag{5.211}$$

is well-defined and

$$\partial\Omega = \left\{ \varphi(\omega)\omega : \ \ \omega \in S^{n-1} \right\}. \tag{5.212}$$

Proof of Step IV. This follows directly from the fact that $\mathcal{O} \in \Omega$ and Step III.

Step V. Assume that there exists a $C > 0$ such that for all

$$\omega_1, \omega_2 \in S^{n-1} \ \text{ and } \ |\omega_1 - \omega_2| < 1 \tag{5.213}$$

then

$$|\varphi(\omega_1) - \varphi(\omega_2)| \leq C\, |\omega_1 - \omega_2|\,. \tag{5.214}$$

Proof of Step V. Let $x_j := \varphi(\omega_j)\omega_j \in \partial\Omega$ where, by Step IV, $\varphi(\omega_j)\omega_j \in \partial\Omega$ for $j = 1, 2$. This implies that

$$|\varphi(\omega_1) - \varphi(\omega_2)| \leq |x_1 - x_2|\,. \tag{5.215}$$

Let us also note that if $\theta := \sphericalangle(\omega_1, \omega_2)$ then

$$1 > |\omega_1 - \omega_2| = 2sin\left(\frac{\theta}{2}\right) \tag{5.216}$$

then

$$\theta \in \left(0, \frac{\pi}{3}\right). \tag{5.217}$$

Recall $B = B(0, \rho)$.

Claim 1. The line L_{x_1,x_2} passing through x_1, x_2 does not intersect $B(0, \frac{\rho}{2})$.

Proof of Claim 1. Otherwise, there exists $z \in B(0, \frac{\rho}{2}) \cap L_{x_1,x_2}$. Since $z \in B$, then by Step I, $\mathcal{I}(x_j, z) \subseteq \Omega$, $j = 1, 2$. Since $x_j \in \partial\Omega$, this implies that

$$z \in \mathcal{I}(x_1, x_2). \tag{5.218}$$

Now $x_1, x_2 \in \partial\Omega$ and $B(0, \rho) \subseteq \Omega$ implies that $|x_1|, |x_2| \geq \rho$. Thus the following implications occur

$$|x_j - z| \geq |x_j| - |z| \geq \rho - \frac{\rho}{2} = \frac{\rho}{2}, \tag{5.219}$$

$|z|$ is the shortest side in $\Delta\mathcal{O}zx_j$, α_j is the smallest angle in $\Delta\mathcal{O}zx_j$, and

$$\alpha \leq \frac{\pi}{3}, \quad j = 1, 2. \tag{5.220}$$

Recall that $\theta < \frac{\pi}{3}$. This makes it impossible to have $\alpha_1 + \alpha_2 + \theta = \pi$. Thus this proves Claim 1 made above.

Claim 2. It holds that

$$\text{distance}\,(0, L_{x_1,x_2}) = \frac{|x_1|\ |x_2|\ sin\theta}{|x_1 - x_2|}. \tag{5.221}$$

This is just elementary geometry

$$\frac{h \cdot c}{s} \quad = \quad \text{area}\,\Delta \quad = \quad \frac{b \cdot a \cdot sin\theta}{2}. \tag{5.222}$$

For us,

$$\begin{aligned} h &= \text{dist}(0, L_{x_1,x_2}) \\ a &:= |x_2|\,,\ \ b = |x_1|\,,\ \text{and} \\ c &= |x_1 - x_2|\,. \end{aligned} \tag{5.223}$$

From (5.222), the desired conclusion readily follows.

Next by Claim 1 and Claim 2,

$$\frac{|x_1|\ |x_2|\ \sin\theta}{|x_1 - x_2|} \geq \frac{\rho}{2} \tag{5.224}$$

and

$$\begin{aligned} |x_1 - x_2| &\leq 2\rho^{-1}\,|x_1|\ |x_2|\ sin\theta \\ &= 4\rho^{-1}\,|x_1|\ |x_2|\ \sin\left(\frac{\theta}{2}\right)\ \cos\left(\frac{\theta}{2}\right) \\ &\leq 4\rho^{-1}(diam(\Omega))^2\ |\omega_1 - \omega_2|\,. \end{aligned} \tag{5.225}$$

Hence, all in all,

$$|\varphi(\omega_1) - \varphi(\omega_2)| \leq |x_1 - x_2| \leq \mathrm{C}\ |\omega_1 - \omega_2|\,, \tag{5.226}$$

proving Step V.

Step VI. Show that

$$\Omega\ \text{ is a Lipschitz domain.} \tag{5.227}$$

Proof of Step VI. By applying our results from Step V we know that φ is locally Lipschitz. Since S^{n-1} is compact, then φ is Lipschitz. Thus,

$$\Omega = \left\{r\omega :\ 0 \leq r < \varphi(\omega),\ \omega \in S^{n-1}\right\} \tag{5.228}$$

with

$$\varphi : S^{n-1} \longrightarrow (0, \infty) \text{ Lipschitz,} \tag{5.229}$$

and it is known that this implies that Ω is a Lipschitz domain. $\square$

5.2 The Differentiability of the Distance Function

Proposition 5.2.1. *Let $A \subseteq \mathbb{R}^n$ be a nonempty, compact, and convex set. Then, for any $X \in \mathbb{R}^n$ there exists a unique point $\mathcal{P}(X) \in A$ with* $\operatorname{dist}(X, A) = \|X - \mathcal{P}(X)\|$. *Furthermore, the induced function $\mathcal{P} : \mathbb{R}^n \to A$ is Lipschitz, with Lipschitz constant $\leqslant 1$.*

Proof. Pick an arbitrary vector $X \in \mathbb{R}^n$ and assume that $Y_1, Y_2 \in A$ are two points with the property that

$$\operatorname{dist}(X, A) = \|X - Y_1\| = \|X - Y_2\|. \tag{5.230}$$

In particular, Remark 5.2.2 below, gives that $2X - Y_1 - Y_2$ is perpendicular on $Y_1 - Y_2$, so

$$\begin{aligned} \|2X - Y_1 - Y_2\|^2 + \|Y_1 - Y_2\|^2 &= \|(2X - Y_1 - Y_2) - (Y_1 + Y_2)\|^2 \\ &= \|2(X - Y_1)\|^2, \end{aligned} \tag{5.231}$$

by the Pythagorean Theorem. However, since A is a convex set, we can write $Y^* := \frac{1}{2}(Y_1 + Y_2) \in A$ so (5.230) and (5.231) give

$$\begin{aligned} \operatorname{dist}(X, A)^2 &\leqslant \|X - Y^*\|^2 \leqslant \|X - Y^*\|^2 + \tfrac{1}{4}\|Y_1 - Y_2\|^2 \\ &= \|X - Y_1\|^2 = \operatorname{dist}(X, A)^2. \end{aligned} \tag{5.232}$$

In turn, this forces $\|Y_1 - Y_2\| = 0$, i.e., $Y_1 = Y_2$. Thus, there exists a unique nearest point to X in A, which we shall denote by $\mathcal{P}(X)$. There remains to prove that

$$\|\mathcal{P}(X) - \mathcal{P}(Y)\| \leqslant \|X - Y\|, \qquad \forall\, X, Y \in \mathbb{R}^n. \tag{5.233}$$

To get started, we claim that

$$(X - \mathcal{P}(X)) \cdot (Z - \mathcal{P}(X)) \leqslant 0, \qquad \forall\, Z \in A. \tag{5.234}$$

Indeed, if $Z \in A$ and $\lambda \in (0, 1)$ then $\lambda Z + (1 - \lambda)\mathcal{P}(X) \in A$, since $\mathcal{P}(X) \in A$ and A is convex. Hence

$$\|X - \mathcal{P}(X)\|^2 \leqslant \|X - (\lambda Z + (1 - \lambda)\mathcal{P}(X))\|^2 \tag{5.235}$$

$$= \|(X - \mathcal{P}(X)) - \lambda(Z - \mathcal{P}(X))\|^2$$

$$= \|(X - \mathcal{P}(X))\|^2 + \lambda^2\|Z - \mathcal{P}(X)\|^2 - 2\lambda(X - \mathcal{P}(X)) \cdot (Z - \mathcal{P}(X)),$$

which implies (after simple algebra)

$$(X - \mathcal{P}(X)) \cdot (Z - \mathcal{P}(X)) \leqslant \frac{\lambda}{2} \|Z - \mathcal{P}(X)\|^2. \tag{5.236}$$

Passing to the limit $\lambda \to 0^+$ then yields (5.234). Hence, the claim holds.

Next, given $X, Y \in \mathbb{R}^n$ arbitrary, use estimate (5.234) for the choice $Z := \mathcal{P}(Y)$. This gives $(\mathcal{P}(Y) - \mathcal{P}(X)) \cdot (X - \mathcal{P}(X)) \leqslant 0$, hence $(\mathcal{P}(X) - \mathcal{P}(Y)) \cdot (Y - \mathcal{P}(Y)) \leqslant 0$ after interchanging X and Y. Adding these two inequalities we therefore obtain

$$\begin{aligned} 0 &\geqslant (\mathcal{P}(X) - \mathcal{P}(Y)) \cdot (Y - \mathcal{P}(Y) + \mathcal{P}(X) - X) \\ &= \|\mathcal{P}(X) - \mathcal{P}(Y)\|^2 + (\mathcal{P}(X) - \mathcal{P}(Y)) \cdot (Y - X), \end{aligned} \tag{5.237}$$

which, by Cauchy-Schwarz's inequality, further implies

$$\|\mathcal{P}(X) - \mathcal{P}(Y)\| \|X - Y\| \geqslant (\mathcal{P}(X) - \mathcal{P}(Y)) \cdot (X - Y) \geqslant \|\mathcal{P}(X) - \mathcal{P}(Y)\|^2. \tag{5.238}$$

This readily gives (5.233), and finishes the proof of the proposition. ☐

Remark 5.2.2. *For any $X, Y \in \mathbb{R}^n$, one can have*

$$\|X\| = \|Y\| \Longleftrightarrow (X - Y) \perp (X + Y). \tag{5.239}$$

We are interested in studying the differentiability properties of the function $X \mapsto \mathcal{P}(X)$. Informally speaking, our next result shows that the shortest path to the boundary of a C^1 domain is along the unit normal.

Proposition 5.2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a C^1 domain, with outward unit normal ν, and fix an arbitrary point $X^* \in \mathbb{R}^n \setminus \partial\Omega$. Then for every point $Y^* \in \partial\Omega$ with the property that*

$$\operatorname{dist}(X^*, \partial\Omega) = \|X^* - Y^*\| \tag{5.240}$$

there exists $\lambda \in \mathbb{R}$ (in fact, either $\lambda = \|X^* - Y^*\|$ or $\lambda = -\|X^* - Y^*\|$) such that

$$X^* - Y^* = \lambda\, \nu(Y^*). \tag{5.241}$$

Proof. To begin with, we observe that the existence of at least one point $Y^* \in \partial\Omega$ for which (5.240) holds is guaranteed by Remark 5.1.28. Fix such a point. From Definition 4.6.1 and the fact that rigid transformations preserve distances, there is no loss of generality in assuming that there exists a C^1 function $\varphi : B_{n-1}(0', R) \to I$, where $B_{n-1}(0', R) \subset \mathbb{R}^{n-1}$ and $I \subset \mathbb{R}$ is an open interval, such that

$$\mathcal{C} \cap \partial\Omega = \{X = (x', x_n) \in \mathcal{C} : \, x_n = \varphi(x')\} \quad \text{if } \mathcal{C} := B_{n-1}(0', R) \times I \subseteq \mathbb{R}^n, \tag{5.242}$$

and

$$(0', \varphi(0')) = Y^*, \qquad \nu(Y^*) := \frac{(\nabla\varphi(0'),\, -1)}{\sqrt{1 + \|\nabla\varphi(0')\|^2}}. \tag{5.243}$$

To proceed, introduce the C^1 function

$$F : B_{n-1}(0', R) \to \mathbb{R}, \qquad F(x') := \|X^* - (x', \varphi(x'))\|^2, \qquad \forall\, x' \in B_{n-1}(0', R), \tag{5.244}$$

so that, by (5.240), $0' \in \mathbb{R}^{n-1}$ is a local minimum for F. Consequently, $(\nabla F)(0') = 0'$ by Proposition 5.2.4 below. Thus, if $X^* = (x'_*, x_{*n})$ and $Y^* = (y'_*, y_{*n})$, we obtain

$$x'_* - y'_* + (x_{*n} - y_{*n})(\nabla\varphi)(0') = 0. \tag{5.245}$$

With this in hand, equation 5.241 now holds for the choice

$$\lambda := -(x_{*n} - \varphi(0'))\sqrt{1 + \|\nabla\varphi(0')\|^2}. \tag{5.246}$$

This finishes the proof of the proposition. $\square$

Proposition 5.2.4 (Fermat's Criterion: Multivariabe Version)**.** *Let $U \subseteq \mathbb{R}^n$ be an arbitrary set, and fix a point $X^* \in U^\circ$. Assume that $f : U \to \mathbb{R}$ is a function with the property that X^* is a local extremum, and all partial derivatives of f at X^* exist. Then X^* is a critical point for the function f.*

Proof. Since $X^* \in U^\circ$, there exists $r > 0$ with the property that $B(X^*, r) \subseteq U$. For each $j \in \{1, ..., n\}$, consider $g_j : (-r, r) \to \mathbb{R}$ given by $g_j(t) := f(X^* + te_j)$ for $|t| < r$. The fact that f has a local extremum at X^* implies that each g_j has a local extremum at $t = 0$. Also, the existence of $\partial_j f(X^*)$ entails, by Proposition 5.2.5 below, that g_j is differentiable at $t = 0$. In addition, $g'_j(0) = \partial_j f(X^*)$. Consequently, the one-dimensional version of Fermat's Theorem implies that $g'_j(0) = 0$ and, hence, $\partial_j f(X^*) = 0$ for $1 \leqslant j \leqslant n$. Thus $(\nabla f)(X^*) = 0$, so X^* is a critical point for f. $\square$

Proposition 5.2.5. *Suppose that $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$. Pick $X \in U^\circ$ and $r > 0$ such that $B(X, r) \subseteq U$. Also, fix an index $j \in \{1, ..., n\}$ and define the function $g : (-r, r) \to \mathbb{R}$ by setting $g(t) := f(X + t\mathbf{e}_j)$ for $-r < t < r$. Then $\partial_j f(X)$ exists if and only if g is differentiable at 0. Furthermore, whenever $\partial_j f(X)$ exists, $\partial_j f(X) = g'(0)$.*

Proof. Note that $\lim_{t\to 0} \frac{g(t)-g(0)}{t} = \lim_{t\to 0} \frac{f(X+t\mathbf{e}_j)-f(X)}{t}$. Therefore, $g'(0)$ exists if and only if $\partial_j f(X)$ exists. Clearly, $g'(0) = \partial_j f(X)$ whenever they exist. $\square$

Remark 5.2.6. *One can use the Lagrange Multiplier Method to give an alternative proof of Proposition 5.2.3.*

Proposition 5.2.7. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a C^1 domain. Assume that $X^* \in \partial\Omega$, $X \in \mathbb{R}^n$ and $R > 0$ are such that $X^* \in \partial B(X, R)$ and either $B(X, R) \subseteq \Omega$ or $B(X, R) \subseteq \mathbb{R}^n \setminus \overline{\Omega}$. Then the geometric tangent plane $T_{X^*}\partial\Omega$ to the C^1 surface $\partial\Omega$ at X^* is also tangent to the ball $B(X, R)$.*

Proof. Under the current hypotheses, it is clear that we have $\operatorname{dist}(X, \partial\Omega) = R = \|X - X^*\|$. Then Proposition 5.2.3 gives that either $X - X^* = R\nu(X^*)$ or $X - X^* = -R\nu(X^*)$. Since $\nu(X^*)$ is orthogonal to $\mathit{\Pi}_{X^*}\partial\Omega$, this proves that $T_{X^*}\partial\Omega$ tangent to the ball $B(X, R)$. $\square$

Theorem 5.2.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a C^1 domain satisfying a UIBC (with radius r). Then for every $X \in \Omega$ such that the dist$(X, \partial\Omega) < r$, there exists a unique point $X^* \in \partial\Omega$ with the property that*

$$\operatorname{dist}(X, \partial\Omega) = \|X - X^*\|. \tag{5.247}$$

Furthermore, an analogous result holds for the case of C^1 domains in $\mathbb{R}^n$ satisfying a UEBC granted that, this time, $X \in (\overline{\Omega})^c$.

Proof. Let $X \in \Omega$ be such that $\operatorname{dist}(X, \partial\Omega) < r$. Then, by Remark 5.1.28, there exists some $X^* \in \partial\Omega$ such that equation 5.247 holds. Also, by the UIBC, there exists $Y \in \Omega$ for which

$$B(Y, r) \subseteq \Omega \quad \text{and} \quad X^* \in \partial B(Y, r). \tag{5.248}$$

The previous equation implies that

$$\operatorname{dist}(Y, \partial\Omega) = \|Y - X^*\|, \tag{5.249}$$

so that, by Proposition 5.2.3,

$$Y - X^* = \lambda_0\, \nu(X^*), \qquad \text{for some} \quad \lambda_0 \in \mathbb{R}. \tag{5.250}$$

In the same vein, (5.247) and Proposition 5.2.3 imply

$$X - X^* = \lambda_1\, \nu(X^*), \qquad \text{for some} \quad \lambda_1 \in \mathbb{R}. \tag{5.251}$$

Thus X^*, X, and Y are collinear and $\rho := \|X - X^*\| = \operatorname{dist}(X, \partial\Omega) < r = \|X^* - Y\|$. This fact is illustrated in the picture below:

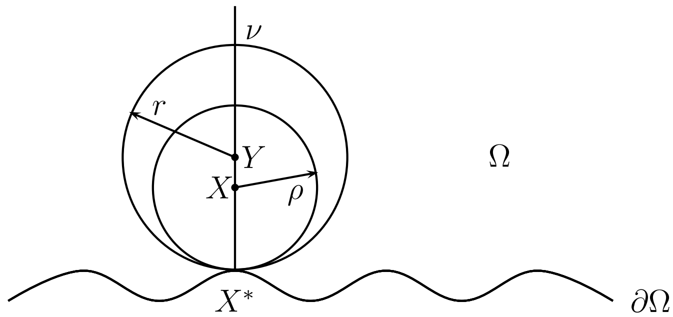

Figure 5.5: The boundary of Ω with the normal direction ν to the boundary.

By recalling $\rho = \|X - X^*\|$, it follows that

$$B(X, \rho) \subseteq B(Y, r) \subseteq \Omega \quad \text{and} \quad X^* \in \partial\Omega \cap \partial B(X, \rho). \tag{5.252}$$

This implies that

$$\operatorname{dist}(X, \partial\Omega \setminus \{X^*\}) > \rho \quad \text{and} \quad \|X - X^*\| = \rho, \tag{5.253}$$

which, in turn, shows that $X^* \in \partial\Omega$ is the unique point satisfying (5.247). Finally, the last claim in the theorem is dealt with analogously. □

Proposition 5.2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$ domain. Then there exists an open set $U \subseteq \mathbb{R}^n$ such that $\partial\Omega \subseteq U$, along with a function $\mathcal{P} : U \longrightarrow \partial\Omega$ with the property that*

$$\mathcal{P}(X) := \textit{the unique nearest point on } \partial\Omega \textit{ to } X \in U, \quad \forall\, X \in U. \tag{5.254}$$

Proof. This is an immediate corollary of Theorem 5.2.8. □

Definition 5.2.10. *Let $\Omega \subseteq \mathbb{R}^n$ be a $C^{1,1}$ domain. Then the* **nearest boundary point function** *is the mapping $\mathcal{P} : U \longrightarrow \partial\Omega$, where $U \subseteq \mathbb{R}^n$ is a suitable open set containing $\partial\Omega$, defined in* (5.254).

Proposition 5.2.11. *Let $\Omega \subseteq \mathbb{R}^n$ be a $C^{1,1}$ domain. Then the nearest boundary point function (cf. Definition 5.2.10) is continuous.*

Proof. Assume $\mathcal{P}$ is not continuous at $X^* \in U$. Then there exists $\varepsilon_0 > 0$ and a sequence $\{X_j\}_{j\in\mathbb{N}}$ of points in U with the property that

$$\lim_{j\to\infty} X_j = X^* \quad \text{and} \quad \|\mathcal{P}(X^*) - \mathcal{P}(X_j)\| \geqslant \varepsilon_0, \quad \text{for all } \ j \in \mathbb{N}. \tag{5.255}$$

Using the fact that, by Lemma 5.2.12 below, $|\operatorname{dist}(X_j, \partial\Omega) - \operatorname{dist}(X^*, \partial\Omega)| \leqslant \|X_j - X^*\|$, we then obtain

$$\begin{aligned} \|\mathcal{P}(X_j) - X^*\| &\leqslant \|\mathcal{P}(X_j) - X_j\| + \|X_j - X^*\| = \operatorname{dist}(X_j, \partial\Omega) + \|X_j - X^*\| \\ &\leqslant |\operatorname{dist}(X_j, \partial\Omega) - \operatorname{dist}(X^*, \partial\Omega)| + \operatorname{dist}(X^*, \partial\Omega) + \|X_j - X^*\| \\ &\leqslant \operatorname{dist}(X^*, \partial\Omega) + 2\|X_j - X^*\|, \quad \forall\, j \in \mathbb{N}. \end{aligned} \tag{5.256}$$

Hence, the sequence $\{\mathcal{P}(X_j)\}_{j\in\mathbb{N}}$ is bounded. Thus, by eventually passing to a subsequence (cf. The Bolzano-Weierstrass Theorem in $\mathbb{R}^n$), it can be assumed that

$$\mathcal{P}(X_j) \ \to \ Z \in \partial\Omega \quad \text{ as } \ j \to \infty. \tag{5.257}$$

Now, since $X^* \in U$, and

$$\operatorname{dist}(X^*, \partial\Omega) = \lim_{j\to\infty} \operatorname{dist}(X_j, \partial\Omega) = \lim_{j\to\infty} \|\mathcal{P}(X_j) - X_j\| = \|Z - X^*\|, \tag{5.258}$$

and since $Z \in \partial\Omega$, we may conclude that

$$\mathcal{P}(X^*) = Z. \tag{5.259}$$

Formulas 5.255, (5.257) and (5.259) lead to an obvious contradiction, so the assumption that $\mathcal{P}$ is not continuous at $X^* \in U$ must be false. In conclusion, $\mathcal{P}$ is continuous at each point $X^* \in U$. □

Lemma 5.2.12. *Consider an arbitrary nonempty set $E \subseteq \mathbb{R}^n$. Then the function $f(X) := \operatorname{dist}(X, E)$, $X \in \mathbb{R}^n$ is Lipschitz with Lipschitz constant $\leqslant 1$.*

Proof. Note that if $X, Z \in \mathbb{R}^n$ and $Y \in E$ are arbitrary, we have $f(X) \leqslant \|X - Y\| \leqslant \|Z - Y\| + \|X - Z\|$. Taking the infimum over $Y \in E$, the latter implies that $f(X) - f(Z) \leqslant \|X - Z\|$. By either proceeding similarly, or simply changing the roles of X and Z, we also obtain that $f(Z) - f(X) \leqslant \|X - Z\|$. Hence, in fact, $|f(X) - f(Z)| \leqslant \|X - Z\|$ for every $X, Z \in \mathbb{R}^n$, from which the desired conclusions follow. □

Proposition 5.2.13. *If $\Omega \subseteq \mathbb{R}^n$ is a domain of class $C^{1,1}$ with outward unit normal ν, then there exists $\varepsilon > 0$ such that*

$$\operatorname{dist}(X^* \mp t\,\nu(X^*)) = t \qquad \textit{whenever } \ 0 < t < \varepsilon \ \ \textit{and } X^* \in \partial\Omega, \tag{5.260}$$

and

$$\mathcal{P}(X^* \mp t\,\nu(X^*)) = X^* \qquad \textit{whenever } \ 0 < t < \varepsilon \ \ \textit{and } X^* \in \partial\Omega. \tag{5.261}$$

Proof. Here is a picture illustrating the situation described in the statement of the proposition:

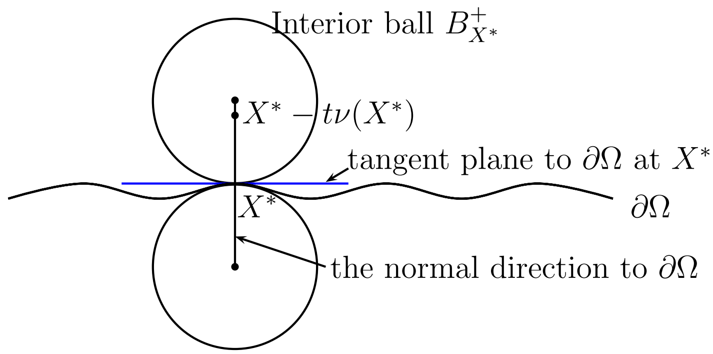

Figure 5.6: The boundary of Ω with the tangent plane to the boundary.

Turning in earnest to the proof of the claim made in the proposition, fix an arbitrary point $X^* \in \partial\Omega$. Since Ω is a domain of class $C^{1,1}$, it follows from Theorem 5.1.9 that Ω satisfies both a UIBC as well as UEBC with some radius $r > 0$. Denote by $B^{\pm}_{X^*}$ the interior and exterior balls at X^*. If we take $\varepsilon := r$ then, assuming that $0 < t < \varepsilon$, it follows that $B(X^* - t\nu(X^*), t) \subseteq B^{+}_{X^*}$ and, since $X^* \in \partial\Omega$, we therefore obtain $\operatorname{dist}(X^* - t\nu(x^*)) = \|(X^* - t\nu(X^*)) - X^*\| = t$, as desired. The fact that $\operatorname{dist}(X^* + t\nu(X^*)) = -t$ for every $0 < t < \varepsilon$ is proved analogously. Finally, the fact that

$$\overline{B(X^* \mp t\nu(X^*))} \subset B^{\pm}_{X^*} \cup \{X^*\} \qquad \text{whenever} \quad 0 < t < \varepsilon, \tag{5.262}$$

readily gives (5.261). $\square$

Proposition 5.2.14. *Assume that $\Omega \subset \mathbb{R}^n$ is a $C^{1,1}$ domain, and recall the nearest boundary point function $\mathcal{P} : U \to \partial\Omega$ introduced in Definition 5.2.10. Set*

$$U_+ := \Omega \cap U \quad \textit{and} \quad U_- := U \setminus \overline{\Omega}. \tag{5.263}$$

Then, (cf. figure below),

$$\pm\,\nu\,(\mathcal{P}(X)) = \frac{\mathcal{P}(X) - X}{\|\mathcal{P}(X) - X\|}, \quad \textit{for all} \quad X \in U_{\pm}. \tag{5.264}$$

Proof. A picture illustrating the phenomenon described in (5.264) is as follows:

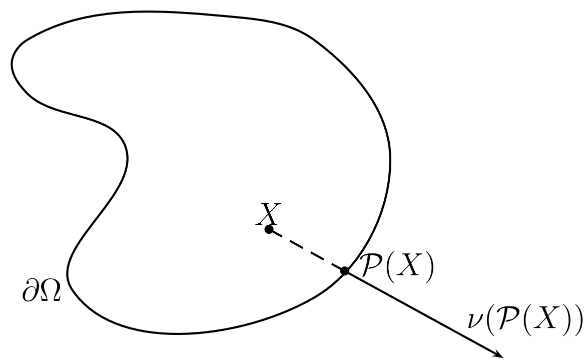

Figure 5.7: The boundary of Ω with the normal direction ν.

As far as formula (5.264) is concerned, this is a direct consequence of Proposition 5.2.9 and Proposition 5.2.3. □

Remark 5.2.15. *Assume that $\Omega \subset \mathbb{R}^n$ is a $C^{1,1}$ domain, and recall the nearest boundary point function $\mathcal{P} : U \to \partial\Omega$ introduced in Definition 5.2.10. One can prove that*

$$\lim_{X \to X^*,\, X \in U_{\pm}} \frac{\mathcal{P}(X) - X}{\|\mathcal{P}(X) - X\|} = \pm\,\nu(X^*), \quad \textit{for all } X^* \in \partial\Omega. \tag{5.265}$$

Hint: Use (5.264) and Proposition 5.2.11.

Theorem 5.2.16 (Differentiability of the Nearest Boundary Point Function)**.** *Assume that Ω is a domain of class C^k in $\mathbb{R}^n$, $n \geqslant 2$, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geqslant 2$. Then the nearest boundary point function $\mathcal{P} : U \to \partial\Omega$ from Definition 5.2.10 is of class C^{k-1} in U (after possibly shrinking the original set U).*

Proof. Fix a small number $r > 0$ with the property that, if

$$U := \{X \in \mathbb{R}^n : \operatorname{dist}(X, \partial\Omega) < r\} \tag{5.266}$$

then the nearest boundary point function $\mathcal{P}$ is well-defined and continuous in U. Next, pick an arbitrary point $X_0 = (x_0', \, x_{0n}) \in U$ and set $Y_0 := \mathcal{P}(X_0) \in \partial\Omega$. Hence, in particular,

$$\operatorname{dist}(X_0, \partial\Omega) = \|X_0 - Y_0\| < r. \tag{5.267}$$

Without loss of generality, we can assume that there exist a number $M > 0$ which depends only on Ω alone, an open set $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ and a function $\varphi : \mathcal{O} \to \mathbb{R}$ of class C^k such that $Y_0 = (y_0',\ \varphi(y_0'))$ for some $y_0' \in \mathcal{O}$, the boundary of Ω near Y_0 coincides with the graph of φ near y_0', and

$$\sum_{|\alpha|\leqslant 2} \left(\sup_{y'\in\mathcal{O}} |\partial^\alpha \varphi(y')| \right) \leqslant M. \tag{5.268}$$

The idea will be to choose $r > 0$ small, relative to this parameter M.

Moving on, consider the function

$$\begin{gathered} F : \mathbb{R}^n \times \mathcal{O} \to \mathbb{R}^{n-1}, \qquad F(X, y') := x' - y' + (x_n - \varphi(y'))\nabla\varphi(y'), \\ \text{for every } \ X = (x',\ x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n \ \text{ and every } \ y' \in \mathcal{O}. \end{gathered} \tag{5.269}$$

Since φ is of class C^k, the function F is of class C^{k-1} on its domain. Also, $F(X_0, y_0') = 0$, by (5.267) and an argument similar to the one leading up to (5.245), while a direct calculation shows that the Jacobian matrix of F with respect to y' is

$$(D_{y'}F)(X, y') = \Big(-\delta_{jk} - \partial_j\varphi(y')\partial_k\varphi(y') + (x_n - \varphi(y'))(\partial_j\partial_k\varphi)(y') \Big)_{1\leqslant j,k\leqslant n-1}, \tag{5.270}$$

where δ_{jk} is the Kronecker symbol. For the purpose of using the Implicit Function Theorem we need to verify that

$$\det\left[(D_{y'}F)(X_0, y_0') \right] \neq 0. \tag{5.271}$$

To this end, introduce the $(n-1) \times (n-1)$ matrices

$$M_0 := \Big(-\delta_{jk} - (\partial_j\varphi)(y_0')(\partial_k\varphi)(y_0') \Big)_{1\leqslant j,\, k\leqslant n-1} \tag{5.272}$$

and

$$M_1 := (x_{0n} - \varphi(y_0'))H_\varphi(y_0'), \tag{5.273}$$

where $H_\varphi(y_0') := \Big((\partial_j\partial_k\varphi)(y_0') \Big)_{1\leqslant j,\, k\leqslant n-1}$ is the Hessian matrix of φ at y_0'. Then

$$\|M_1\| \leqslant M\,|x_{0n} - \varphi(y_0')| \leqslant M\|X_0 - Y_0\| \leqslant Mr. \tag{5.274}$$

On the other hand, $M_0 = -(I_{(n-1)\times(n-1)} + \nabla\varphi(y_0') \otimes \nabla\varphi(y_0'))$, hence, by (4) in Remark 5.2.17 below, we have

$$\det(-M_0) = 1 + \|\nabla\varphi(y_0')\|^2 \geqslant 1. \tag{5.275}$$

Now, (5.271) follows from (5.274), (5.275), and the continuity of the determinant function, granted that $r > 0$ is small enough (relative to M).

For this choice of $r > 0$, the Implicit Function Theorem then implies that there exists a function $\psi : A_0 \longrightarrow B_0$ of class C^{k-1}, where $A_0 \subseteq \mathbb{R}^n$ is an open neighborhood of X_0 and $B_0 \subseteq \mathcal{O}$ is an open neighborhood of y_0', with the property that

$$\psi(X_0) = y_0', \tag{5.276}$$

and

$$\{(X, y') \in A_0 \times B_0 : \; F(X, y') = 0\} = \{(X, \; \psi(X)) : \; X \in A_0\}. \tag{5.277}$$

In particular, $F(X, \psi(X)) = 0$ for every $X \in A_0$. This shows that, for each fixed point $X \in A_0$, the function G_X defined as

$$G_X : B_0 \longrightarrow \mathbb{R}, \qquad G_X(y') := \|X - (y', \varphi(y'))\|^2 \qquad \forall\, y' \in B_0, \tag{5.278}$$

has a unique critical point, namely $y' = \psi(X)$ (since $F(X, y') = \frac{1}{2}\,(DG_X)(y')$). Recall from Proposition 5.2.11 that $\mathcal{P} : U \to \partial\Omega$ is continuous, and that $\mathcal{P}(X_0) = Y_0 = (y_0', \varphi(y_0'))$ with $y_0' \in B_0$. By further shrinking A_0 if necessary, it can be therefore assumed that

$$A_0 \subseteq U \quad \text{ and } \quad X \in A_0 \Longrightarrow \mathcal{P}(X) = (z', \; \varphi(z')) \qquad \text{for some } \; z' \in B_0. \tag{5.279}$$

It follows that z' is a local minimum for the function G_X. Thus, z' is a critical point for G_X and this implies that $z' = \psi(X)$ by the uniqueness of such a critical point for this function. In turn, this can be re-interpreted as saying that

$$\mathcal{P}(X) = (\psi(X), \; \varphi(\psi(X))), \qquad \forall\, X \in A_0. \tag{5.280}$$

From (5.276) and (5.280) we now conclude that $\mathcal{P}$ is of class C^{k-1} in a neighborhood of X_0. In turn, since membership to the class C^{k-1} is a local property, it follows that $\mathcal{P}$ is of class C^{k-1} in U, as desired. $\square$

Remark 5.2.17. *The following are true:*

(1) $(a \otimes b)^\top = b \otimes a$ *for all* $a, b \in \mathbb{R}^n$*;*

(2) $\mathrm{Tr}\,(a \otimes b) = a \cdot b$ *for all* $a, b \in \mathbb{R}^n$*;*

(3) $(a \otimes b)c = (b \cdot c)a$ *for all* $a, b, c \in \mathbb{R}^n$*;*

(4) $\det(I_{n\times n} + a\otimes b) = 1 + a\cdot b$ *for all* $a, b \in \mathbb{R}^n$*;*

(5) $(a\otimes b)(c\otimes d) = (b\cdot c)\, a\otimes d$ *for all* $a, b, c, d \in \mathbb{R}^n$*;*

(6) for each $a \in \mathbb{R}^n$*, the matrix* $I_{n\times n} + a\otimes a \in \mathcal{M}_{n\times n}$ *is positive definite;*

(7) for every $a \in \mathbb{R}^n$ *and every numbers* $\mu, \lambda \in \mathbb{R}$ *with* $\mu \neq 0$ *and* $\mu \neq -\lambda\|a\|^2$*, the matrix* $\mu I_{n\times n} + \lambda\, a\otimes a$ *is invertible and*

$$\big(\mu I_{n\times n} + \lambda\, a\otimes a\big)^{-1} = \frac{1}{\mu}\Big[I_{n\times n} - \Big(\frac{\lambda}{\mu + \lambda\|a\|^2}\Big) a\otimes a\Big]; \tag{5.281}$$

(8) $\|\mu I_{n\times n} + \lambda\, a\otimes b\| = \sqrt{n\mu^2 + 2\mu\lambda\, a\cdot b + \lambda^2\|a\|^2\|b\|^2}$ *for every* $a, b \in \mathbb{R}^n$ *and every* $\mu, \lambda \in \mathbb{R}$*.*

Proposition 5.2.18. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a domain of class* C^2*, and recall the nearest boundary point function* $\mathcal{P} : U \to \partial\Omega$ *from Definition 5.2.10. Then*

$$(D\,\mathcal{P}(X))^{\top}(\mathcal{P}(X) - X) = 0, \quad \text{for all} \quad X \in U. \tag{5.282}$$

Proof. Fix an arbitrary vector $v \in \mathbb{R}^n$ and, with X fixed in U, set

$$f(t) := \|X - \mathcal{P}(X + tv)\|^2, \quad \text{for } |t| \text{ small}. \tag{5.283}$$

Then

$$\|X - \mathcal{P}(X + tv)\|^2 \geqslant \|X - \mathcal{P}(X)\|^2 \tag{5.284}$$

so that f has a global minimum at zero. Since f is of class C^1, this forces $f'(0) = 0$. To proceed, recall the directional derivative operator D_v along v. In turn, this allows us to write

$$\begin{aligned} 0 &= f'(0) = \frac{d}{dt}\Big[(X - \mathcal{P}(X + tv))\cdot(X - \mathcal{P}(X + tv))\Big]\Big|_{t=0} \\ &= (-2)(\mathcal{P}(X) - X)\cdot(D_v\mathcal{P})(X) = (-2)(\mathcal{P}(X) - X)\cdot(D\mathcal{P}(X)v) \\ &= (-2)[(D\mathcal{P}(X))^{\top}(\mathcal{P}(X) - X)]\cdot v. \end{aligned} \tag{5.285}$$

Given that the vector $v \in \mathbb{R}^n$ was arbitrary, this proves (cf. (4.56)) that (5.282) holds. $\square$

Theorem 5.2.19. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a domain of class C^k, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geqslant 2$, and define the* **distance to the boundary function** *by*

$$\delta(X) := \operatorname{dist}(X, \partial\Omega), \qquad \forall\, X \in \mathbb{R}^n. \tag{5.286}$$

Then there exists an open neighborhood U of $\partial\Omega$ with the property that δ is of class C^k in $U \setminus \partial\Omega = U_+ \cup U_-$.

Proof. With U as in Theorem 5.2.16, we have $\delta(X) = \|X - \mathcal{P}(X)\|$ for every $X \in U$. Since, by Theorem 5.2.16, the function $\mathcal{P}$ is of class C^{k-1} in U (and $X - \mathcal{P}(X)$ vanishes precisely when $X \in \partial\Omega$), it follows that δ is of class C^{k-1} in $U \setminus \partial\Omega$. This and the fact that $k \geqslant 2$ allow us to compute

$$\nabla\delta(X) = \frac{(I_{n\times n} - (D\mathcal{P})(X))^{\top}(X - \mathcal{P}(X))}{\|X - \mathcal{P}(X)\|}, \qquad \forall\, X \in U \setminus \partial\Omega. \tag{5.287}$$

Taking into account Proposition 5.2.18, we further obtain from (5.287) that

$$\nabla\delta(X) = \frac{X - \mathcal{P}(X)}{\|X - \mathcal{P}(X)\|}, \qquad \text{for all} \quad X \in U \setminus \partial\Omega. \tag{5.288}$$

Upon once again recalling that $\mathcal{P}$ is of class C^{k-1} in U, this finally shows that $\nabla\delta$ is of class C^{k-1} in $U \setminus \partial\Omega$. Hence, ultimately, the function δ is of class C^k in $U \setminus \partial\Omega$ as desired. $\square$

Remark 5.2.20. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a domain of class C^2. Then its unit normal satisfies*

$$\nu(\mathcal{P}(X)) = \mp\nabla\delta(X), \quad \textit{for all} \quad X \in U_{\pm}, \tag{5.289}$$

where, as before, $U_+ := U \cap \Omega$, $U_- := U \setminus \overline{\Omega}$, and $\mathcal{P} : U \to \partial\Omega$ is the nearest boundary point function.

Indeed, if one will make use of (5.288) and (5.264), the desired conclusion can be reached.

Example 5.2.21. *Let $\Omega := B(0,1)$ be the unit ball in $\mathbb{R}^n$ centered at the origin, and consider $U := \mathbb{R}^n \setminus \{0\}$. Then the distance to the boundary function is given by*

$$\delta(X) := \begin{cases} 1 - \|X\| & \textit{if} \quad \|X\| \leqslant 1, \\ \|X\| - 1 & \textit{if} \quad \|X\| \geqslant 1. \end{cases} \tag{5.290}$$

Hence, via a direct calculation,

$$\nabla\delta(X) = \begin{cases} -\frac{X}{\|X\|} & \textit{if} \quad 0 < \|X\| < 1, \\ \frac{X}{\|X\|} & \textit{if} \quad \|X\| > 1. \end{cases} \tag{5.291}$$

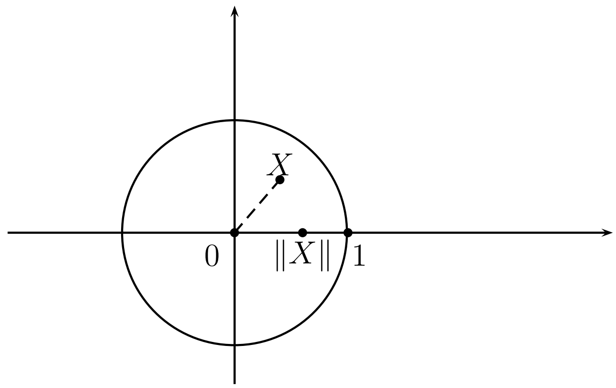

Figure 5.8: The unit ball is the domain Ω.

Note that, in this situation, $\mathcal{P}(X) = \frac{X}{\|X\|}$ *for every* $X \in U$ *and* $\nu(X) = X$ *whenever* $\|X\| = 1$*. Thus, as already predicted,* $\nabla\delta(X) = \mp\nu(\mathcal{P}(X))$ *for every point* $X \in U_\pm$*, where we have set* $U_+ := \{X \in \mathbb{R}^n : 0 < \|X\| < 1\}$ *and* $U_- := \{X \in \mathbb{R}^n : \|X\| > 1\}$*. Also,*

$$\begin{aligned}
\frac{\mathcal{P}(X) - X}{\|\mathcal{P}(X) - X\|} &= \frac{\frac{X}{\|X\|} - X}{\left\|\frac{X}{\|X\|} - X\right\|} = \frac{\left(\frac{1}{\|X\|} - 1\right) X}{\left|\frac{1}{\|X\|} - 1\right| \|X\|} \\
&= \operatorname{sign}(1 - \|X\|) \frac{X}{\|X\|} = \pm\,\nu(\mathcal{P}(X)) \qquad \forall\, X \in U_\pm, \quad (5.292)
\end{aligned}$$

in agreement with (5.264).

Definition 5.2.22. *Given an arbitrary set* $\Omega \subseteq \mathbb{R}^n$*, the* **signed distance** *(to its boundary) is the function* $d : \mathbb{R}^n \longrightarrow \mathbb{R}$*, defined by*

$$d(X) := \begin{cases} +\operatorname{dist}(X, \partial\Omega) & \text{if } X \in \Omega, \\ -\operatorname{dist}(X, \partial\Omega) & \text{if } X \in \Omega^c. \end{cases} \qquad (5.293)$$

Remark 5.2.23. *If* $\Omega \subseteq \mathbb{R}^n$*,* $n \geqslant 2$*, is a domain of class* C^k *for some* $k \in \mathbb{N} \cup \{\infty\}$*,* $k \geqslant 2$*, then* $d > 0$ *in* U_+*,* $d < 0$ *in* U_-*,* $d = 0$ *on* $\partial\Omega$ *and, from Theorem 5.2.19, we have that* d *is of class* C^k *in* $U_\pm$*. In the theorem below we shall prove the stronger claim that actually* d *is of class* C^k *in* U*.*

Theorem 5.2.24. *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a domain of class C^k, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geqslant 2$. Then there exists an open set $U \subseteq \mathbb{R}^n$, containing $\partial\Omega$, with the property that*

$$\textit{the signed distance function } d \textit{ is of class } C^k \textit{ in } U. \tag{5.294}$$

Proof. To begin with, we note that

$$d(X) = \nu(\mathcal{P}(X)) \cdot (\mathcal{P}(X) - X), \qquad \forall\, X \in U. \tag{5.295}$$

Indeed, both sides of (5.295) vanish when $X \in \partial\Omega$, whereas for $X \in U \setminus \partial\Omega = U_+ \cup U_-$, the desired conclusion follows from (5.264). Having established (5.295) and upon recalling from Lemma 4.6.18 that ν is extendible to a function of class C^{k-1} in a neighborhood of $\partial\Omega$, we may deduce, with the help of Theorem 5.2.16, that the signed distance function d is of class C^{k-1} in some neighborhood U of $\partial\Omega$. With this in hand and observing that, as readily seen from (5.289),

$$(\nabla d)(X) = \nu(\mathcal{P}(X)), \qquad \text{for every } \ X \in U_\pm, \tag{5.296}$$

we may conclude that (5.296) actually holds for every $X \in U$ (here, $k \geqslant 2$ is used). Since, as already remarked, both ν and $\mathcal{P}$ are of class C^{k-1} it follows from (5.296) that ∇d is of class C^{k-1} in U. Hence, (5.294) follows. $\square$

Remark 5.2.25. *Generally speaking, the signed distance function only exhibits good differentiability properties in a small neighborhood of the boundary of a C^k domain. For example, in the context of Example 5.2.21, $d(X) = 1 - \|X\|$ is not differentiable at $0 \in \mathbb{R}^n$.*

We continue by discussing a version of Theorem 5.2.16 for domains of class $C^{1,1}$. It is worth pointing out that, for this larger class of domains, the classical Implicit Function Theorem does not apply. We nonetheless have the following result.

Theorem 5.2.26. *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a domain of class $C^{1,1}$. Then there exists a neighborhood U of $\partial\Omega$ with the property that the nearest boundary point function $\mathcal{P} : U \to \partial\Omega$ is Lipschitz.*

Proof. We follow the broad outline of the proof of Theorem 5.2.16 (and also retain most of the notation introduced on that occasion), the most notable difference being that a suitable version of the Implicit Function Theorem is used in this setting. More specifically, we will implement Theorem 4.1.4 for the function F defined in (5.269). To get started, since φ is of class $C^{1,1}$, it follows that the function F is Lipschitz.

Next, we note that if $X=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ and $y_1',y_2'\in\mathcal{O}$ then

$$F(X,y_1')-F(X,y_2') = y_2'-y_1'+(x_n-\varphi(y_1'))\nabla\varphi(y_1')-(x_n-\varphi(y_2'))\nabla\varphi(y_2') \tag{5.297}$$

$$= y_2'-y_1'+(\varphi(y_2')-\varphi(y_1'))\nabla\varphi(y_2')+(x_n-\varphi(y_1'))(\nabla\varphi(y_1')-\nabla\varphi(y_2')).$$

Compute the difference

$$\begin{aligned}\varphi(y_2')-\varphi(y_1') &= \int_0^1 \frac{d}{dt}\left[\varphi(y_1'+t(y_2'-y_1'))\right]dt\\ &= (y_2'-y_1')\cdot\left(\int_0^1 (\nabla\varphi)\,(y_1'+t(y_2'-y_1'))\;dt\right),\end{aligned} \tag{5.298}$$

then express

$$\int_0^1 (\nabla\varphi)\,(y_1'+t(y_2'-y_1'))\;dt$$

$$=\nabla\varphi(y_2')+\int_0^1\left[(\nabla\varphi)(y_1'+t(y_2'-y_1'))-(\nabla\varphi)(y_2')\right]dt \tag{5.299}$$

and, finally, using the fact that $\nabla\varphi$ is Lipschitz with Lipschitz constant M, estimate

$$\left|\int_0^1\left[(\nabla\varphi)(y_1'+t(y_2'-y_1'))-(\nabla\varphi)(y_2')\right]dt\right|$$

$$\leqslant \int_0^1 \left|(\nabla\varphi)(y_1'+t(y_2'-y_1'))-(\nabla\varphi)(y_2')\right|dt$$

$$\leqslant M\|y_1'-y_2'\|\left(\int_0^1(1-t)\,dt\right)=\frac{M}{2}\|y_1'-y_2'\|. \tag{5.300}$$

The bottom line is that if $y_j'\in\mathcal{O}$, $j=1,2$, and $X\in\mathbb{R}^n$, then

$$\begin{aligned}F(X,y_1')-F(X,y_2') &= y_2'-y_1'+\Big((y_2'-y_1')\cdot\nabla\varphi(y_2')\Big)(\nabla\varphi)(y_2')\\ &\quad +O(\|y_1'-y_2'\|^2)+O(|x_n-\varphi(y_1')|\|y_1'-y_2'\|).\end{aligned} \tag{5.301}$$

Let us now restrict X to a small neighborhood of X_0 and restrict y_1', y_2' to a small neighborhood of y_0', say, $X \in B(X_0, \rho)$ and $y_1', y_2' \in B(y_0', \varepsilon)$ with $\rho, \varepsilon > 0$ small. Assuming that $\varepsilon > 0$ is small enough, this entails

$$O(\|y_1' - y_2'\|^2) \leqslant \tfrac{1}{4\sqrt{n-1}}\|y_1' - y_2'\|, \qquad \forall\, y_1', y_2' \in B(y_0', \varepsilon). \tag{5.302}$$

Also, since

$$\begin{aligned} |x_n - \varphi(y_1')| &\leqslant \|X - (y_1', \varphi(y_1'))\| \leqslant \|X - X_0\| + \|X_0 - Y_0\| + \|Y_0 - (y_1', \varphi(y_1'))\| \\ &\leqslant \rho + r + \varepsilon\sqrt{1 + M^2} \end{aligned} \tag{5.303}$$

we obtain

$$\begin{gathered} O(|x_n - \varphi(y_1')|\|y_1' - y_2'\|) \leqslant \tfrac{1}{4\sqrt{n-1}}\|y_1' - y_2'\|, \\ \forall\, y_1', y_2' \in B(y_0', \varepsilon), \qquad \forall\, X \in B(X_0, \rho), \end{gathered} \tag{5.304}$$

provided $\varepsilon, r, \rho > 0$ are chosen small enough.

Turning to the task of estimating the term $y_2' - y_1' + \Big((y_2' - y_1') \cdot \nabla\varphi(y_2')\Big)(\nabla\varphi)(y_2')$ from (5.301), we abbreviate $v := y_2' - y_1' \in \mathbb{R}^{n-1}$ and $a := \nabla\varphi(y_2') \in \mathbb{R}^{n-1}$. Then, with $I := I_{(n-1)\times(n-1)}$, we have

$$y_2' - y_1' + \Big((y_2' - y_1') \cdot \nabla\varphi(y_2')\Big)(\nabla\varphi)(y_2') = (I + a \otimes a)v. \tag{5.305}$$

On the other hand,

$$\|v\| = \|(I + a\otimes a)^{-1}(I + a \otimes a)v\| \leqslant \|(I + a \otimes a)^{-1}\|\|(I + a \otimes a)v\| \tag{5.306}$$

so we may estimate

$$\|(I + a \otimes a)v\| \geqslant \|(I + a \otimes a)^{-1}\|^{-1}\|v\|. \tag{5.307}$$

In concert with the fact that (cf. Remark 5.2.17)

$$\|(I + a \otimes a)^{-1}\| = \Big\|I - \Big(\frac{1}{1 + \|a\|^2}\Big)a \otimes a\Big\| \leqslant \sqrt{n-1}, \tag{5.308}$$

the estimate (5.307) further yields

$$\Big\|y_2' - y_1' + \Big((y_2' - y_1') \cdot \nabla\varphi(y_2')\Big)(\nabla\varphi)(y_2')\Big\| \geqslant \frac{1}{\sqrt{n-1}}\|y_1' - y_2'\|, \tag{5.309}$$

for every $y_1', y_2' \in \mathcal{O}$.

Altogether, if $\rho, \varepsilon > 0$ are small enough,

$$\begin{gathered} \|F(X, y_1') - F(X, y_2')\| \geqslant \tfrac{1}{2\sqrt{n-1}}\|y_1' - y_2'\|, \\ \forall\, X \in B(X_0, \rho), \qquad \forall\, y_1', y_2' \in B(y_0', \varepsilon). \end{gathered} \tag{5.310}$$

This completes the verification of the hypotheses of Theorem 4.1.4 in the current case. Hence, this version of the Implicit Function Theorem applies and gives that there exist that there exist a Lipschitz function $\psi : A_0 \longrightarrow B_0$ where $A_0 \subseteq \mathbb{R}^n$ is an open neighborhood of X_0 and $B_0 \subseteq \mathcal{O}$ is an open neighborhood of y_0', with the property that (5.276)-(5.277) hold. With this in hand, the same reasoning as in the proof of Theorem 5.2.16 gives 5.280 which shows that $\mathcal{P} : U \to \partial\Omega$ is a Lipschitz function. $\square$

Theorem 5.2.27 (The Collar Neighborhood Theorem)**.** *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a domain of class C^k, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geqslant 2$, and denote by ν its outward unit normal. Then there exist $\varepsilon > 0$ and an open set $U \subseteq \mathbb{R}^n$ which contains $\partial\Omega$, such that the function*

$$F : \partial\Omega \times (-\varepsilon, \varepsilon) \longrightarrow U, \quad F(X,t) := X - t\nu(X), \quad \forall\, (X,t) \in \partial\Omega \times (-\varepsilon, \varepsilon), \tag{5.311}$$

is a homeomorphism. Moreover, F is of class C^{k-1}, the function

$$G : U \longrightarrow \partial\Omega \times (-\varepsilon, \varepsilon), \quad G(X) := (\mathcal{P}(X), d(X)) \quad \forall\, X \in U, \tag{5.312}$$

is well-defined, of class C^{k-1}, and is an inverse for F in (5.311)*. That is, one has*

$$X = \mathcal{P}(X) - d(X)\nu(\mathcal{P}(X)), \qquad \forall\, X \in U, \tag{5.313}$$

as well as

$$d(X - t\nu(X)) = t \quad \textit{and} \quad \mathcal{P}(X - t\nu(X)) = X, \qquad \forall\, (X,t) \in \partial\Omega \times (-\varepsilon, \varepsilon). \tag{5.314}$$

Finally, in the case when Ω is a $C^{1,1}$ domain in $\mathbb{R}^n$, $n \geqslant 2$, the above results continue to hold if "being of class C^{k-1}" is, in this situation, interpreted as "being Lipschitz."

Proof. Formula (5.313) is a consequence of (5.264) and (5.293). Formulas (5.314) are seen from (5.260) and (5.261). The differentiability properties of F, G are consequence of Lemma 4.6.18, Theorem 5.2.16 and Theorem 5.2.24.

The final part of the statement is a consequence of Theorem 5.2.26 and (5.295), which according to which $\mathcal{P}$ and d are Lipschitz functions in this case. $\square$

Remark 5.2.28. *Assume that Ω is a $C^{1,1}$ domain in $\mathbb{R}^n$. Then the signed distance function is of class $C^{1,1}$ in a neighborhood of $\partial\Omega$.*

Theorem 5.2.29. *Let Ω be a C^2 domain in $\mathbb{R}^n$, and let ρ be a* **defining function** *for Ω (that is, ρ is a function of class C^1 which is positive in Ω, negative in $(\overline{\Omega})^c$, and $\nabla\rho \neq 0$*

on $\partial\Omega$). Then there exists $\varepsilon > 0$ such that for every function $f \in C_0^\infty(U)$

$$\int_U f(X)\,dX = \int_{-\varepsilon}^{\varepsilon}\Big(\int_{\Sigma_{\phi,\,t}} f\,\|\nabla\rho\|^{-1}\,d\sigma_t\Big)\,dt \tag{5.315}$$

where $\Sigma_{\phi,t}$ is the surface $\{\rho = t\}$ and σ_t is the surface measure on $\Sigma_{\phi,t}$.

Proof. Let ρ be a defining function for Ω. Augment ρ to create a new function, say $F = (f_1, ..., f_{n-1}, \rho) : U \longrightarrow O \times (-\varepsilon, \varepsilon)$, which is a C^1 diffeomorphism, where $O \subseteq \mathbb{R}^{n-1}$ open, and $\varepsilon > 0$ is a small number (this step uses the Implicit Function Theorem). Then $\{d = t\} = F_t(\partial\Omega)$ where $F_t(X) := X + t\nu(X)$, which implies that $\Sigma_{\phi,t} = F_t(\partial\Omega)$ a C^k surface. Indeed, the fact that $F^{-1} : O \times (-\varepsilon,\, \varepsilon) \longrightarrow U$ is a diffeomorphism for any $t \in (-\varepsilon,\, \varepsilon)$ implies that

$$F^{-1}(\cdot,\, t) \,:\, O \longrightarrow U \quad \text{is a parametrization for } \Sigma_{\phi,\,t} := \{\rho = t\}. \tag{5.316}$$

The strategy is to make use of our earlier Surface-to-Surface change of variables theorem. The key claim, which grately facilitates the application of this result, is as follows:

$$\|\nabla\rho\|^{-1}\|\partial_1 F^{-1}(y',\, t) \times ... \times \partial_{n-1}F^{-1}(y',\, t)\| = |\det(DF^{-1})(y',\, t)|. \tag{5.317}$$

To justify (5.317) we first note from the definition of cross-product that

$$\begin{aligned}\det(DF^{-1})(y',\, t) \;&=\; \|\partial_1 F^{-1}(y',\, t) \times ... \times \partial_{n-1}F^{-1}(y',\, t)\| \\ &\quad \cdot \Big\langle \frac{\partial_1 F^{-1}(y',\, t) \times ... \times \partial_{n-1}F^{-1}(y',\, t)}{\|\partial_1 F^{-1}(y',\, t) \times ... \times \partial_{n-1}F^{-1}(y',\, t)\|},\, \partial_t F^{-1}(y',\, t)\Big\rangle .\end{aligned} \tag{5.318}$$

Denote by ν_t the unit normal to surface $\Sigma_{\phi,t}$. Relying on (5.316) we may then write

$$\pm\frac{\nabla\rho}{\|\nabla\rho\|} = \nu_t = \pm\frac{\partial_1 F^{-1}(y',\, t) \times ... \times \partial_{n-1}F^{-1}(y',\, t)}{\|\partial_1 F^{-1}(y',\, t) \times ... \times \partial_{n-1}F^{-1}(y',\, t)\|}. \tag{5.319}$$

Also, since $F(F^{-1}(y',\, t)) = (y',\, t)$ we obtain $\rho(F^{-1}(y'\, t)) = t$ which further implies

$$\nabla\rho \cdot \partial_t F^{-1}(y',\, t) = 1. \tag{5.320}$$

By virtue of (5.318), (5.319) and (5.320) we see that claim (5.317) holds.

Having established (5.317), we make use of the change of variable formula between surfaces in order to write

$$\begin{aligned}\int_U f(X)\,dX &= \int_{O\times(-\varepsilon,\,\varepsilon)} f(F^{-1}(y',\,t))|\det DF^{-1}(y',\,t)|\,dy'\,dt \\ &= \int_{\varepsilon}^{-\varepsilon}\Big(\int_O f(F^{-1}(y',\,t))|\det DF^{-1}(y',\,t)|\,dy'\Big)\,dt \\ &= \int_{\varepsilon}^{-\varepsilon}\Big(\int_{\Sigma_{\phi,\,t}} f\,\|\nabla\rho\|^{-1}\,d\sigma_t\Big)\,dt, \qquad (5.321)\end{aligned}$$

where we have used the fact that $F^{-1}(y',\,t))$ is $F^{-1}(\cdot,\,t)$ is a parametrization of $\Sigma_{\phi,\,t}$, and that $|\det DF^{-1}(y',\,t)| = \|\nabla\rho\|^{-1}\|\partial_1 F^{-1}(y',\,t)\times ...\times\partial_{n-1}F^{-1}(y',\,t)\|$. This finishes the proof of the theorem. $\square$

5.3 The Unit Normal and Mean Curvature

Theorem 5.3.1 (A Distinguished Extension of the Outward Unit Normal of a Domain)**.** *Assume that* $\Omega \subseteq \mathbb{R}^n$*,* $n \geqslant 2$*, is a domain of class* C^k*, for some* $k \in \mathbb{N} \cup \{\infty\}$*,* $k \geqslant 2$*. Then, with* d *denoting the signed distance from* (5.293)*, there exists* U *open neighborhood of* $\partial\Omega$ *and a vector field*

$$N = (N_1, ..., N_n) : U \longrightarrow \mathbb{R}^n, \qquad N(X) := (\nabla d)(X), \qquad \forall\, X \in U, \tag{5.322}$$

is a vector-valued function of class C^{k-1} *in* U *which has the following properties:*

(1) $\|N(X)\| = 1$ *for every* $X \in U$*;*

(2) $N\Big|_{\partial\Omega} = \nu$*, the outward unit normal to* Ω*;*

(3) $\partial_j N_k = \partial_k N_j$ *in* U*, for all* j*,* $k \in \{1, ..., n\}$*;*

(4) for every $j \in \{1, ..., n\}$*, the directional derivative* $D_N N_j$ *vanishes in* U*.*

Proof. The fact that N in (5.322) is of class C^{k-1} in U follows from the Definition 5.2.22 (cf. (5.294)). Also,

$$\|N(X)\| = \|(\nabla d)(X)\| = \|\nu(\mathcal{P}(X))\| = 1, \qquad \forall\, X \in U, \tag{5.323}$$

proving (1). Next, since for every point $X \in \partial\Omega$ we have $\mathcal{P}(X) = X$, we deduce that $N(X) = (\nabla d)(X) = \nu(\mathcal{P}(X)) = \nu(X)$, for every $X \in \partial\Omega$. This proves (2).

Moving on, we note that $\partial_j N_k = \partial_j(\partial_k d) = \partial_k(\partial_j d) = \partial_k N$ in U, which proves (3). Finally, for every $j \in \{1, ..., n\}$, we have

$$D_N N_j = N \cdot \nabla N_j = \nabla d \cdot \nabla(\partial_j d) = \tfrac{1}{2}\partial_j \|\nabla d\|^2 = 0, \tag{5.324}$$

since $\|\nabla d(X)\| = \|\nu(\mathcal{P}(X))\| = 1$ for each $X \in U$. This justifies (4) and completes the proof of the theorem. ∎

Next, we discuss the Gauss mean curvature of a C^2 surface $\Sigma \subseteq \mathbb{R}^n$, which is a local concept. To define this properly, requires some preliminary analysis. Let $X^* \in \Sigma$ be an arbitrary point and let ν be a fixed continuous (hence, C^1) choice of a unit normal to Σ near X^*. Then, for any local C^2 parametrization $P : \mathcal{O} \to \mathbb{R}^n$ of Σ near X^*, we have

$$\nu(P(x')) \cdot \nu(P(x')) = \|\nu(P(x'))\|^2 = 1 \qquad \forall\, x' \in \mathcal{O}. \tag{5.325}$$

Via partial differentiation, this implies that, for all $j = 1, \dots, n-1$,

$$\frac{\partial}{\partial x_j}[\nu(P(x'))] \cdot \nu(P(x')) = 0 \qquad \forall\, x' \in \mathcal{O}. \tag{5.326}$$

Fix $x' \in \mathcal{O}$. As a consequence of (5.326), we then have that for every $1 \leqslant j \leqslant n-1$ the vector $\frac{\partial}{\partial x_j}[\nu(P(x'))]$ is tangent to Σ at the point $P(x')$. Given that the family of vectors $\{\partial_{x_k} P(x') : \, k = 1, \dots, n-1\}$ is a basis for the algebraic tangent plane to Σ at $P(x')$, it follows that there exist real numbers b_{jk}, for $1 \leqslant j, k \leqslant n-1$, such that

$$\frac{\partial}{\partial x_j}[\nu(P(x'))] = \sum_{k=1}^{n-1} b_{jk}[\partial_{x_k} P(x')] \quad \text{for all} \quad 1 \leqslant j \leqslant n-1. \tag{5.327}$$

For further considerations it is useful to organize the numbers b_{jk}'s as the $(n-1) \times (n-1)$ matrix

$$B := (b_{jk})_{1 \leqslant j,k \leqslant n-1} = \left(\begin{array}{cccc} b_{11} & b_{12} & \cdots & b_{1\,n-1} \\ b_{21} & b_{22} & \cdots & b_{2\,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n-1\,1} & \cdots & \cdots & b_{n-1\,n-1} \end{array} \right). \tag{5.328}$$

Definition 5.3.2. *Retaining the above setting and conventions, the Gauss mean curvature $\mathcal{G}$ of the C^2 surface Σ (relative to the continuous choice of a unit normal to Σ) at the point $P(x')$ (where $x' \in \mathcal{O}$) is then defined as*

$$\mathcal{G}(P(x')) := \operatorname{Tr} B = \sum_{j=1}^{n-1} b_{jj}. \tag{5.329}$$

In the case when Σ is the boundary of a C^2 domain Ω, we make the convention that the unit normal to Σ with respect to which the Gauss mean curvature is computed is the outward unit normal to Ω.

Theorem 5.3.3. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a domain of class C^2 and denote by $\mathcal{G}$ the Gauss mean curvature of the surface $\Sigma := \partial\Omega$. Then the extension N of the outward unit normal to Ω defined in Theorem 5.3.1 also satisfies*

$$\operatorname{div} N = \mathcal{G} \qquad \textit{at every point on} \;\; \partial\Omega. \tag{5.330}$$

Proof. Fix an arbitrary point $X^* \in \partial\Omega$ and let $P : \mathcal{O} \to \mathbb{R}^n$ be a local C^2 parametrization of the C^2 surface $\Sigma := \partial\Omega$ near X^*. Granted that $N|_\Sigma$ is the outward unit normal to Ω, formula (5.327) then reads in matrix form

$$D(N \circ P) = DP\, B^\top \quad \text{ or } \quad ((DN) \circ P)DP = DP\, B^\top. \tag{5.331}$$

To continue, let us augment the matrix DP by including $N \circ P$ as the n-th column, i.e., introduce the $n \times n$ matrix given in terms of its columns as

$$A := (\partial_1 P, ..., \partial_{n-1} P, N \circ P). \tag{5.332}$$

Let us also augment B by zeroes to a $n \times n$ matrix, i.e., consider

$$\widetilde{B} := \left(\begin{array}{c|c} B & 0_{n-1\times 1} \\ \hline 0_{1\times n-1} & 0 \end{array}\right) \in \mathcal{M}_{n\times n}. \tag{5.333}$$

Then, upon noticing that, by (4) in Theorem 5.3.1, for every $j \in \{1, ..., n\}$ we have

$$\mathrm{row}_j((DN) \circ P) \cdot \mathrm{col}_n A = ((\nabla N_j) \circ P) \cdot N \circ P = (D_N N_j) \circ P = 0, \tag{5.334}$$

we may re-write the last formula in (5.331) as

$$((DN) \circ P)A = A\,\widetilde{B}^\top. \tag{5.335}$$

To continue, observe that the columns of the square-matrix A in (5.332) are linearly independent (cf. (4.85), (4.121) and Definition 4.4.7). On account of this, (5.338) and (5.341) below, we may then write

$$\begin{aligned} (\mathrm{div}\, N) \circ P &= \mathrm{Tr}\,((DN) \circ P) = \mathrm{Tr}\,[A\,\widetilde{B}^\top A^{-1}] = \mathrm{Tr}\,\widetilde{B}^\top = \mathrm{Tr}\,[B^\top] \\ &= \mathrm{Tr}\, B = \mathcal{G} \circ P, \end{aligned} \tag{5.336}$$

from which we deduce that $\mathrm{div}\, N = \mathcal{G}$ for points on $\partial\Omega$ near X^*. Since $X^* \in \partial\Omega$ was arbitrary, (5.330) follows. □

Remark 5.3.4. *Assume that $U \subseteq \mathbb{R}^n$ is an open set and that $f : U \to \mathbb{R}$ is a function of class C^2 on U. Then the trace of the Hessian of f is the Laplacian of f, i.e.,*

$$\mathrm{Tr}\,(H_f(X)) = \Delta f(X) \qquad \textit{for every} \quad X \in U. \tag{5.337}$$

Furthermore, if $U \subseteq \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}^n$ is a function of class C^1 on U, then

$$\mathrm{div}\, f = \mathrm{Tr}\,(Df). \tag{5.338}$$

Remark 5.3.5. *The following statements are true*

$$(A^\top)^\top = A \quad \textit{for every} \quad A \in \mathcal{M}_{m\times n}, \tag{5.339}$$

$$(AB)^\top = B^\top A^\top \quad \textit{for every} \quad A \in \mathcal{M}_{m\times n},\ B \in \mathcal{M}_{n\times k}, \tag{5.340}$$

$$\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA) \quad \textit{for every} \quad A \in \mathcal{M}_{m\times n},\ B \in \mathcal{M}_{n\times m}, \tag{5.341}$$

$$\mathrm{Tr}\,(A^\top) = \mathrm{Tr}\,A \quad \textit{for every} \quad A \in \mathcal{M}_{n\times n}. \tag{5.342}$$

Lemma 5.3.6. *Recall the concept of directional derivative from Definition 5.3.8. Let $U \subseteq \mathbb{R}^n$ be an open set and assume that $f : U \to \mathbb{R}^n$ is a function of class C^1 on U. Also, assume that $u_1, ..., u_n$ is an orthonormal basis for $\mathbb{R}^n$. Then*

$$\mathrm{div}\, f = \sum_{j=1}^n u_j \cdot D_{u_j} f \qquad \textit{in} \quad U. \tag{5.343}$$

Proof. To set the stage, consider the matrix $U \in \mathcal{M}^*_{n\times n}$ uniquely determined by the requirement that $U\mathbf{e}_j = u_j$, $1 \leqslant j \leqslant n$ (i.e., $\mathrm{col}_j U = u_j$ for every j). Then U is unitary and

$$\begin{aligned}
\sum_{j=1}^n u_j \cdot D_{u_j} f &= \sum_{j=1}^n u_j \cdot (Df)u_j = \sum_{j=1}^n (U\mathbf{e}_j)\cdot[(Df)(U\mathbf{e}_j)] \\
&= \sum_{j=1}^n \mathbf{e}_j \cdot [(U^{-1}(Df)U)\mathbf{e}_j] = \mathrm{Tr}\,[U^{-1}(Df)U] = \mathrm{Tr}\,(Df) \\
&= \mathrm{div}\, f,
\end{aligned} \tag{5.344}$$

by (4.52), (5.341) and (5.338). ☐

Proposition 5.3.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain of class C^2, with outward unit normal denoted by ν. Suppose that $U \subseteq \mathbb{R}^n$ in an open set which contains $\partial\Omega$, and that $\widetilde{\nu} : U \to \mathbb{R}^n$ is a function which has the properties*

$$\widetilde{\nu}\Big|_{\partial\Omega} = \nu, \quad \widetilde{\nu} \textit{ is of class } C^1 \textit{ on } U, \quad \|\widetilde{\nu}\| = 1 \textit{ in } U. \tag{5.345}$$

Then the restriction of $\mathrm{div}\,\widetilde{\nu}$ to $\partial\Omega$ is independent of $\widetilde{\nu}$. In fact,

$$(\mathrm{div}\,\widetilde{\nu})\Big|_{\partial\Omega} = \mathcal{G}, \tag{5.346}$$

the Gauss mean curvature of $\partial\Omega$.

Proof. Assume that $T_1, ..., T_{n-1} : \partial\Omega \to \mathbb{R}^n$ are such that

$$T_1, ..., T_{n-1},\ \nu \quad \text{is an orthonormal basis for } \mathbb{R}^n \tag{5.347}$$

at each point on $\partial\Omega$. Next, set $T_n := \nu$ and recall from Lemma 5.3.6 that, at each point on $\partial\Omega$,

$$\operatorname{div} \widetilde{\nu} = \sum_{j=1}^{n} T_j \cdot D_{T_j} \widetilde{\nu} = \sum_{j=1}^{n-1} T_j \cdot D_{T_j} \widetilde{\nu}, \tag{5.348}$$

where the last equality is a consequence of the fact that $\|\widetilde{\nu}\| = 1$ in U and

$$\nu \cdot D_\nu \widetilde{\nu} = \widetilde{\nu} \cdot D_\nu \widetilde{\nu} = \tfrac{1}{2} D_\nu \|\widetilde{\nu}\|^2 = 0 \text{ on } \partial\Omega. \tag{5.349}$$

To proceed, note that by (5.347), the vectors T_j, $1 \leqslant j \leqslant n-1$, are tangent to $\partial\Omega$. Hence, by Lemma 5.3.9 below, as a function on $\partial\Omega$, each $D_{T_j}\widetilde{\nu}$ does not depend on the particular C^1 extension $\widetilde{\nu}$ of ν. Consequently, the restriction of $\sum_{j=1}^{n-1} T_j \cdot D_{T_j}\widetilde{\nu}$ to $\partial\Omega$ is independent of $\widetilde{\nu}$. This justifies the claim made in the first part of the statement. Finally, (5.346) follows from what we have proved so far and Theorem 5.3.3. □

Definition 5.3.8. *Let $U \subseteq \mathbb{R}^n$ be an arbitrary set, and assume that $f : U \to \mathbb{R}^m$ is a given function. Given a vector $u \in \mathbb{R}^n$ and a point $X^* \in U^\circ$, the* **directional derivative** *of f along u at X^* is defined as*

$$D_u f(X^*) := \lim_{t \to 0} \frac{f(X^* + tu) - f(X^*)}{t}, \tag{5.350}$$

whenever this limit exists in $\mathbb{R}^m$.

Lemma 5.3.9. *Recall the concept of directional derivative from Definition 5.3.8. Let $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ be an open set, consider a differentiable function $P : \mathcal{O} \to \mathbb{R}^n$ and set $\mathcal{S} = \{P(x') : x' \in \mathcal{O}\}$. Assume that $\mathcal{S} \subseteq U$ and that there exists $x'_* \in \mathcal{O}$ such that $P(x'_*) = X^*$. Finally, fix a vector $\tau \in \mathbb{R}^n$ which belongs to the linear span of the vectors*

$$\left\{ \frac{\partial P}{\partial x_j}(x'_*) : j = 1, ..., n-1 \right\}. \tag{5.351}$$

Then the restriction of a real-valued function f, originally defined in an open neighborhood of $\mathcal{S}$, to the set $\mathcal{S}$ is a constant function then $D_\tau f(X^*) = 0$.

Proof. Observe that since f is constant on $\mathcal{S}$, the function $f \circ P$ is constant on the open set $\mathcal{O}$. Given that f is also differentiable at X^*, we obtain $D(f \circ P)(x'_*) = 0$ and, hence, by the Chain Rule Formula,

$$0 = (Df)(X^*)DP(x'_*). \tag{5.352}$$

Then, since τ belongs to the linear span of the vectors in (5.351), we deduce that there exists $\xi = (\xi_1, ..., \xi_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$\tau = \sum_{j=1}^{n-1} \xi_j \frac{\partial P}{\partial x_j}(x'_*) = (DP(x'_*))\xi. \tag{5.353}$$

Then combining (5.353) and (5.352),

$$D_\tau f(X^*) = (\nabla f)(X^*) \cdot \tau = (Df)(X^*)DP(x'_*)\xi = 0, \tag{5.354}$$

as wanted. $\square$

Remark 5.3.10. *It is also possible to compute the Gauss mean curvature of a sphere $\partial B(X^*, R)$ in $\mathbb{R}^n$, by considering* (5.346) *with $\widetilde{\nu}(X) := (X - X^*)/R$, for $X \in \mathbb{R}^n$.*

Lemma 5.3.11. *Let $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ be an open set, suppose $\varphi : \mathcal{O} \to \mathbb{R}$ is a given C^2 function, and denote by Σ the graph of φ. Then for every $x' \in \mathcal{O}$, the Gauss mean curvature $\mathcal{G}$ of Σ at the point $(x', \varphi(x'))$ is*

$$\begin{aligned} \mathcal{G}(x', \varphi(x')) &= \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j}\left[\frac{\partial_j \varphi(x')}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}\right] \\ &= \frac{1}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}\left[\operatorname{Tr} H_\varphi(x') - \frac{\nabla\varphi(x') \cdot \big(H_\varphi(x')\nabla\varphi(x')\big)}{1 + \|\nabla\varphi(x')\|^2}\right], \end{aligned} \tag{5.355}$$

where $H_\varphi(x')$ is the Hessian matrix of φ at x'.

Proof. The problem is local hence, without loss of generality, it can be assumed (cf. Lemma 4.6.25) that $\Sigma = \partial\Omega$, where Ω is a C^2 domain in $\mathbb{R}^n$. In this case, one can use (5.346) for the following extension of the outward unit normal to Σ:

$$\widetilde{\nu}(X) := \frac{(\nabla\varphi(x'),\, -1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}, \quad \text{for all} \quad X = (x',\, x_n) \in \mathcal{O} \times \mathbb{R}. \tag{5.356}$$

$\square$

Definition 5.3.12. *Assume that $\Sigma \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a surface of class C^2, $X^* \in \Sigma$ is an arbitrary point, and ν is a continuous choice of the unit normal ν for Σ near X^*. Let $\widetilde{\nu}$ be an extension of class C^1 of ν to a neighborhood of X^* in $\mathbb{R}^n$. Then*

$$\mathbb{I}_{X^*}(v, w) := (D\widetilde{\nu}(X^*)v) \cdot w, \qquad v, w \in \mathit{\Pi}_{X^*}\Sigma, \tag{5.357}$$

is called the **second fundamental form** *of* Σ *at* X^*.

Proposition 5.3.13. *Let* $\Sigma \subseteq \mathbb{R}^n$, $n \geqslant 2$, *be a surface of class* C^2. *Then the definition of the second fundamental form* $\mathrm{I\!I}_{X^*}$ *of* Σ *at* $X^* \in \Sigma$ *is independent of the particular* C^1 *extension of the chosen continuous unit normal near* X^*, *and* $\mathrm{I\!I}_{X^*}$ *is a bilinear and symmetric.*

Proof. Since the second fundamental form is a local concept, there is no loss of generality in assuming that $\Sigma = \partial\Omega$ where $\Omega \subseteq \mathbb{R}^n$ is a domain of class C^2. In addition, we can also assume that N, the distinguished extension of the outward unit normal to Ω described in Theorem 5.3.1, coincides with ν near X^* on Σ.

Next, let $\widetilde{\nu} : B(X^*, r) \to \mathbb{R}^n$, for some small $r > 0$, be a function of class C^1 with the property that $\widetilde{\nu}|_\Sigma$ coincides with ν near X^* on Σ. Then, if $v, w \in \mathit{\Pi}_{X^*}\Sigma$, we have

$$\begin{aligned} \mathrm{I\!I}_{X^*}(v,w) &= (D\widetilde{\nu}(X^*)v) \cdot w \\ &= w \cdot D_v\widetilde{\nu}(X^*) = w \cdot D_v N(X^*) = (DN(X^*)v) \cdot w, \end{aligned} \tag{5.358}$$

since the directional derivative along the tangent vector v to Σ at X^* depends only on the restriction of $\widetilde{\nu}$ to Σ near X^* (cf. Lemma 5.3.9). This proves that $\mathrm{I\!I}_{X^*}$ does not depend on the particular extension of the unit normal. Also, the fact that $\mathrm{I\!I}_{X^*}$ is bilinear is clear. As for symmetry, we note that for any $v, w \in \mathit{\Pi}_{X^*}\Sigma$

$$\begin{aligned} \mathrm{I\!I}_{X^*}(v,w) &= (DN(X^*)v) \cdot w = v \cdot (DN(X^*)^\top w) \\ &= v \cdot (DN(X^*)w) = \mathrm{I\!I}_{X^*}(w,v) \end{aligned} \tag{5.359}$$

since, by Theorem 5.3.1, the Jacobian of N at points in Σ is a symmetric matrix. □

Remark 5.3.14. *Assume that* $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, *is a domain of class* C^2 *with outward unit normal* ν. *Denote by* N *the distinguished extension of* ν *to an open neighborhood* U *of* $\partial\Omega$ *described in Theorem 5.3.1. Then the following are true*

(1) The Jacobian matrix DN *is symmetric at all points in* U, *and its restriction to* $\partial\Omega$ *depends only on the domain* Ω *itself.*

(2) *For every $X \in U$ $(DN(X))N(X) = 0$. In particular, $(DN(X))\nu(X) = 0$ for every $X \in \partial\Omega$.*

(3) *One has $(DN(X)v) \cdot w = \mathrm{II}_X(v, w)$, the second fundamental form of $\partial\Omega$, for every point $X \in \partial\Omega$ and any vectors $v, w \in \Pi_X \partial\Omega$.*

(4) *For every $X \in \partial\Omega$ one has $\mathrm{Tr}\,(DN(X)) = \mathcal{G}(X)$, the Gauss mean curvature of $\partial\Omega$ at X.*

(5) *If $X \in \partial\Omega$ and $\lambda_1(X), ..., \lambda_n(X)$ are the eigenvalues of the symmetric matrix $DN(X)$, then $\mathcal{G}(X) = \lambda_1(X) + \cdots + \lambda_n(X)$.*

Proposition 5.3.15. *Let $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ be an open set, K a compact subset of $\mathcal{O}$, $\varphi : \mathcal{O} \to \mathbb{R}$ a function of class C^2, and denote by Σ the graph of φ, i.e., $\Sigma := \{(x', \varphi(x')) :\, x' \in \mathcal{O}\}$. Then the following assertions are equivalent.*

(i) *There exists $R > 0$ with the property that*
$$B\Big(R\frac{(\nabla\varphi(x'), -1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}} + (x', \varphi(x'))\,,\, R\Big) \cap \Sigma = \varnothing, \qquad \forall\, x' \in K. \tag{5.360}$$

(ii) *There exist $\varepsilon > 0$ small and $C_0 > 0$ large such that for every $x' \in K$ and for every $h \in \mathbb{R}^{n-1}$ with $\|h\| \leqslant \varepsilon$, there holds*
$$2\varphi(x') - \varphi(x' + h) - \varphi(x' - h) \leqslant C_0\|h\|^2. \tag{5.361}$$

(iii) *There exists $C_1 > 0$ with the property that for every $x' \in K$ one has*
$$-(H_\varphi(x')h) \cdot h \leqslant C_1\|h\|^2, \qquad \forall\, h \in \mathbb{R}^{n-1}, \tag{5.362}$$
where $H_\varphi(x') := \Big(\frac{\partial^2\varphi}{\partial x_j \partial x_k}(x')\Big)_{1 \leqslant j,k \leqslant n-1}$ is the Hessian of φ at x'.

(iv) *There exists $C_1 > 0$ with the property that for every $x' \in K$ one has*
$$\min_{1 \leqslant j \leqslant n-1} \lambda_j(x') \geqslant -C_1, \tag{5.363}$$
where $\lambda_1(x'), ...\lambda_{n-1}(x')$ are the eigenvalues of the symmetric matrix $H_\varphi(x')$.

Proof. *(i)* $\Rightarrow$ *(ii)*. Fix $0 < \varepsilon < \mathrm{dist}\,(K, \partial\mathcal{O})$. Then, for each $x' \in K$ and each $h \in \mathbb{R}^{n-1}$ with $\|h\| \leqslant \varepsilon$, condition 5.360 implies that
$$\Big\|(x' + h, \varphi(x' + h)) - R\frac{(\nabla\varphi(x'), -1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}} - (x', \varphi(x'))\Big\|^2 \geqslant R^2, \tag{5.364}$$

i.e.,

$$\|h\|^2 + (\varphi(x'+h) - \varphi(x'))^2$$
$$\geqslant \frac{2R}{\sqrt{1+\|\nabla\varphi(x')\|^2}}(\varphi(x') + h\cdot\nabla\varphi(x') - \varphi(x'+h)). \tag{5.365}$$

Set

$$M := \sup\{\|\nabla\varphi(x')\| : \, x' \in K\}. \tag{5.366}$$

Then (5.365) entails

$$(2R)^{-1}\sqrt{1+M^2}\,\|h\|^2 \geqslant \varphi(x') + h\cdot\nabla\varphi(x') - \varphi(x'+h). \tag{5.367}$$

By writing (5.367) for $-h$ in place of h, then adding it to (5.367) we therefore obtain

$$(2R)^{-1}\sqrt{1+M^2}\,\|h\|^2 \geqslant 2\varphi(x') - \varphi(x'+h) - \varphi(x'+h), \tag{5.368}$$

which shows that (5.361) holds for

$$C_0 := (2R)^{-1}\sqrt{1+M^2}. \tag{5.369}$$

(ii) $\Rightarrow$ *(iii)*. Fix an arbitrary point $x' \in K$. By expanding φ in a Taylor series at x', the left-hand side of (5.361) becomes

$$-(H_\varphi(x')h)\cdot h + o(\|h\|^2) \quad \text{as} \quad \|h\| \to 0. \tag{5.370}$$

Since, by (5.361), the expression in (5.370) is $\leqslant C_0\|h\|^2$ for $\|h\| \leqslant \varepsilon$, it follows that $-(H_\varphi(x')h)\cdot h \leqslant 2C_0\|h\|^2$ if $\|h\|$ is sufficiently small. Hence, by homogeneity, (5.362) holds with for every $h \in \mathbb{R}^{n-1}$ if we take

$$C_1 := 2C_0. \tag{5.371}$$

(iii) $\Rightarrow$ *(i)*. Fix a number r with $0 < r < \operatorname{dist}(K, \partial\mathcal{O})$. We claim that if r is small enough then it is possible to choose some $R > 0$ with the property that (5.364) holds for every point $x' \in K$ and any vector $h \in \mathbb{R}^{n-1}$ with $\|h\| < r$. Note that this claim is equivalent to asserting that, for every $x' \in K$,

$$B\Big(R\frac{(\nabla\varphi(x'), -1)}{\sqrt{1+\|\nabla\varphi(x')\|^2}} + (x', \varphi(x'))\,,\, R\Big) \cap \Sigma \cap \big(B(x', r)\times\mathbb{R}\big) = \varnothing. \tag{5.372}$$

By eventually decreasing R to ensure that this number is $\leqslant r/2$, it follows that

$$B\Big(R\frac{(\nabla\varphi(x'), -1)}{\sqrt{1+\|\nabla\varphi(x')\|^2}} + (x', \varphi(x'))\,,\, R\Big) \subseteq B(x', r)\times\mathbb{R} \tag{5.373}$$

in which scenario (5.372) reduces to (5.360).

Thus, there remains to prove the validity of the claim made at the beginning of the previous paragraph. By (5.365), this comes down to verifying that if $r > 0$ is small then $\exists\, R > 0$ such that for all $x' \in K$ and for all $h \in \mathbb{R}^{n-1}$ with $\|h\| < r$

$$\Longrightarrow \|h\|^2 + (\varphi(x'+h) - \varphi(x'))^2 \geqslant 2R\,\frac{\varphi(x') + h\cdot\nabla\varphi(x') - \varphi(x'+h)}{\sqrt{1+\|\nabla\varphi(x')\|^2}}. \tag{5.374}$$

By expanding φ in a Taylor series at x' we see that, for $h \in \mathbb{R}^{n-1}$ satisfying $\|h\| < r$ with r sufficiently small, the numerator of the fraction in (5.374) is $\leqslant -\frac{1}{2}(H_\varphi(x')h)\cdot h + \frac{1}{2}\|h\|^2$ hence $\leqslant \frac{1}{2}C_1\|h\|^2 + \frac{1}{2}\|h\|^2$ by (5.362) Choosing $0 < R < (1+C_1)^{-1}$ therefore makes the entire right-hand side of (5.374) $\leqslant \|h\|^2 \leqslant \|h\|^2 + (\varphi(x'+h) - \varphi(x'))^2$. This proves that (5.374) holds and finishes the proof of (i).

$(iii) \Rightarrow (iv)$. From Remark 1 in the proof of Theorem 5.3.16 below, we know that for each $x' \in K$ there exists a unitary matrix $U \in \mathcal{M}_{(n-1)\times(n-1)}$ invertible, with the property that

$$UH_\varphi(x')U^{-1} = D, \qquad \text{where} \quad D := \begin{pmatrix} \lambda_1(x') & 0 & ... & 0 \\ 0 & \lambda_2(x') & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_{n-1}(x') \end{pmatrix}. \tag{5.375}$$

Since for every $h \in \mathbb{R}^{n-1}$ we have $(H_\varphi(x')h)\cdot h = (U^{-1}DUh)\cdot h = (Dv)\cdot v$ where $v := Uh$, and $\|v\| = \|h\|$, condition (5.362) becomes equivalent to $-(Dv)\cdot v \leqslant C_1$. Specializing this to the case when v belongs to the standard orthonormal basis in $\mathbb{R}^{n-1}$ then yields (5.363).

$(iv) \Rightarrow (iii)$. Note that if (5.363) holds then $\sum\limits_{j=1}^{n-1} \lambda_j(x')v_j^2 \geqslant -C_1\|v\|^2$ for every $x' \in K$ and every $v = (v_1, ..., v_{n-1}) \in \mathbb{R}^{n-1}$. As such, (5.363) is equivalent to the condition that $-(Dv)\cdot v \leqslant C_1$ for all $v \in \mathbb{R}^{n-1}$ (where D is as in (5.375)). By reasoning as above, it then follows that (5.362) holds. $\square$

Theorem 5.3.16. *For a matrix $A \in \mathcal{M}_{n\times n}$ the following statements are equivalent:*

(i) the matrix A is symmetric and semi-positive definite;

(ii) there exist a unitary matrix U and real numbers $\lambda_i \geqslant 0$ for $i = 1, 2, ..., n$ such that

$$UAU^{-1} = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix}. \tag{5.376}$$

If (i) (hence also (ii)) is satisfied then the eigenvalues of A are precisely the numbers $\lambda_1, \lambda_2, ..., \lambda_n$, and

$$\det A = \prod_{i=1}^{n} \lambda_i. \tag{5.377}$$

Furthermore, for a symmetric matrix $A \in \mathcal{M}_{n\times n}$, one has the following sequence of equivalences

$$\begin{aligned} A \textit{ is positive definite} &\iff A \textit{ is semi-positive definite with positive eigenvalues} \\ &\iff A \textit{ is semi-positive definite and } \det A \neq 0. \end{aligned} \tag{5.378}$$

We are going to use Lemma 5.3.11 and Proposition 5.3.15 to prove the next lemma.

Lemma 5.3.17. *If $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a C^2 domain, hence satisfying an uniform exterior ball condition with some constant $R > 0$, then the Gauss mean curvature of the C^2 surface $\partial\Omega$ is bounded from below by $-C/R$ where $C > 0$ depends only on the first-order derivatives of the C^2 functions whose graphs locally describe $\partial\Omega$.*

Proof. The problem is local so there is no loss of generality to assume that $\partial\Omega$ is actually the graph of a C^2 function $\varphi : \mathcal{O} \to \mathbb{R}$ and that we seek to estimate the Gauss mean curvature $\mathcal{G}(x', \varphi(x'))$ of the graph of φ for $x' \in K$, with K a compact subset of $\mathcal{O}$. In this setting, use Proposition 5.2.7 and Proposition 5.3.15 to deduce that, if $\lambda_1(x'), ...\lambda_{n-1}(x')$ are the eigenvalues of the symmetric matrix $H_\varphi(x')$, then (5.363) holds for some C_1 of the form C/R where C depends only on $\sup\{\|\nabla\varphi(x')\| : x' \in K\}$ (cf. (5.366), (5.369) and (5.371)). In the context of (5.355), we need to estimate $\mathrm{Tr}\, H_\varphi(x') - v\cdot(H_\varphi(x')v)/(1+\|v\|^2)$ where $v := \nabla\varphi(x') \in \mathbb{R}^{n-1}$. This, however, can be written as (cf. (5.375))

$$(1 + \|v\|^2)^{-1}\Big[\sum_{j=1}^{n-1} \lambda_j(x')(1 + \|v\|^2) - \sum_{j=1}^{n-1} \lambda_j(x')(Uv)_j^2\Big] \tag{5.379}$$

where $U \in \mathcal{M}_{(n-1)\times(n-1)}$ invertible is a unitary matrix and $(Uv)_j$ is the j-th component of Uv. Therefore, a lower bound for (5.379) is given by

$$\begin{aligned} &(1 + \|v\|^2)^{-1}\Big[\sum_{j=1}^{n-1} \lambda_j(x')(1 + \|Uv\|^2) - \sum_{j=1}^{n-1} \lambda_j(x')(Uv)_j^2\Big] \\ &\qquad \geqslant (1 + \|v\|^2)^{-1} \sum_{j=1}^{n-1} \lambda_j(x') \geqslant -C(1 + \|v\|^2)^{-1}/R. \end{aligned} \tag{5.380}$$

Since $(1 + \|v\|^2)^{-1}$ also depends only on the first-order derivative of φ, the desired conclusion finally follows from this and (5.355). $\square$

Remark 5.3.18. *Retain the notation and background assumptions made in Proposition 5.3.15. One can have that the following statements are equivalent.*

(i) There exists $R > 0$ with the property that

$$B\Big(R\frac{(-\nabla\varphi(x'),1)}{\sqrt{1+\|\nabla\varphi(x')\|^2}} + (x', \varphi(x'))\,,\, R\Big) \cap \Sigma = \varnothing, \qquad \forall\, x' \in K. \tag{5.381}$$

(ii) There exist $\varepsilon > 0$ small and $C_0 > 0$ large such that for every $x' \in K$ and for every $h \in \mathbb{R}^{n-1}$ with $\|h\| \leqslant \varepsilon$, there holds

$$-2\varphi(x') + \varphi(x'+h) + \varphi(x'-h) \leqslant C_0\|h\|^2. \tag{5.382}$$

(iii) There exists $C_1 > 0$ with the property that for every $x' \in K$ one has

$$(H_\varphi(x')h)\cdot h \leqslant C_1\|h\|^2, \qquad \forall\, h \in \mathbb{R}^{n-1}. \tag{5.383}$$

(iv) There exists $C_1 > 0$ with the property that for every $x' \in K$ one has

$$\min_{1\leqslant j\leqslant n-1} \lambda_j(x') \leqslant C_1, \tag{5.384}$$

where $\lambda_1(x'), ...\lambda_{n-1}(x')$ are the eigenvalues of the symmetric matrix $H_\varphi(x')$.

Remark 5.3.19. *One can use Remark 5.3.18 and Proposition 5.3.15 to obtain that if $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a C^2 domain, hence satisfying an uniform interior ball condition with some constant $R > 0$, then the Gauss mean curvature of the C^2 surface $\partial\Omega$ is bounded from above by C/R where $C > 0$ depends only on the first-order derivatives of the C^2 functions whose graphs locally describe $\partial\Omega$.*

Indeed, a similar procedure as in the proof of Lemma 5.3.17 will come to the desired conclusion.

5.4 First and Second Variations of Area

Let Σ be an oriented C^2 surface in $\mathbb{R}^n$, with canonical unit normal ν, and surface measure σ. Fix $\phi : \mathbb{R}^n \to \mathbb{R}$ a C^∞ function with compact support and such that

$$\Sigma \cap \operatorname{supp}\phi \text{ is a compact subset of } \Sigma. \tag{5.385}$$

Associated with these, consider the one-parameter variation of the original surface:

$$\Sigma_{\phi,t} := \{X + t\phi(X)\nu(X) : X \in \Sigma\} \quad \text{for} \quad t > 0 \quad \text{fixed.} \tag{5.386}$$

Theorem 5.4.1 (The First Variation Formula of Area)**.** *With the above notation and assumptions, $\Sigma_{\phi,t}$ continues to be a C^2 surface for $|t|$ small and*

$$\frac{d}{dt}\left[\operatorname{Area}(\Sigma_{\phi,t})\right]\Big|_{t=0} = \int_\Sigma \phi\,\mathcal{G}\,d\sigma, \tag{5.387}$$

where $\mathcal{G}$ is the Gauss mean curvature of Σ.

Proof. Let $N = (N_1, ..., N_n)$ be the extension of the canonical unit normal described in Theorem 5.3.1 in a neighborhood of $\Sigma \cap \operatorname{supp}\phi$. In particular, this satisfies

$$[\operatorname{div} N]\Big|_\Sigma = \mathcal{G} \quad \text{and, near } \Sigma, \quad \|N\| = 1, \quad \partial_N N_j = 0 \text{ for every } j \in \{1, ..., n\}, \tag{5.388}$$

where $\mathcal{G}$ is the Gauss mean curvature of Σ, and $\sigma_{\phi,t}$ is the surface area on $\Sigma_{\phi,t}$. Consider

$$F_{\phi,t}(X) := X + t\,\phi(X)N(X), \quad \text{for } X \text{ near } \Sigma, \tag{5.389}$$

and note $[DF_{\phi,t}(X)]\Big|_{t=0} = I_{n\times n}$, so $F_{\phi,t}$ is locally a C^2-diffeomorphism, if $|t|$ is small, and X is near Σ, by the Inverse Function Theorem. Consequently, $\Sigma_{\phi,t} = F_{\phi,t}(\Sigma)$ is a C^2 surface for $|t|$ small. In order to prove (5.387), we use the Surface-to-Surface Changes of Variables Theorem with $\widetilde{\Sigma} := \Sigma_{\phi,t}$, and $F = F_{\phi,t}$. Using Theorem 4.5.8 we compute

$$\operatorname{Area}(\Sigma_{\phi,t}) = \int_{\Sigma_{\phi,t}} 1\,d\sigma_{\phi,t} = \int_\Sigma |\det(DF_{\phi,t})|\,\|((DF_{\phi,t})^{-1})^\top N\|\,d\sigma. \tag{5.390}$$

Keeping in mind (5.388) we obtain

$$(DF_{\phi,t})(X) = I_{n\times n} + t\phi(X)(DN)(X) + tN(X) \otimes \nabla\phi(X), \quad \text{for } X \text{ near } \Sigma, \tag{5.391}$$

which implies that

$$\begin{aligned} \frac{d}{dt}\left[\operatorname{Area}(\Sigma_{\phi,t})\right]\Big|_{t=0} &= \int_\Sigma \frac{d}{dt}\left[\det(DF_{\phi,t})\right]\Big|_{t=0} \|((DF_{\phi,t})^{-1})^\top \nu\|\Big|_{t=0}\,d\sigma \\ &+ \int_\Sigma |\det(DF_{\phi,t})|\Big|_{t=0} \frac{d}{dt}\|((DF_{\phi,t})^{-1})^\top \nu\|\Big|_{t=0}\,d\sigma. \end{aligned} \tag{5.392}$$

From 5.391 it follows that

$$\frac{d}{dt}[\det(DF_{\phi,t})]\Big|_{t=0} = \mathrm{Tr}\,[\phi\,DN + \nu\otimes\nabla\phi] = \phi\,\mathrm{Tr}\,(DN) + \partial_\nu\phi. \tag{5.393}$$

Upon recalling that $\mathcal{G} = [\mathrm{Tr}(DN)]\big|_\Sigma$ represents the Gauss mean curvature, and that $(DF_{\phi,t})\Big|_{t=0} = I_{n\times n}$, we obtain that $\|((DF_{\phi,t})^{-1})^\top\nu\| = \|\nu\| = 1$ on Σ. Hence,

$$\int\limits_\Sigma \frac{d}{dt}[\det(DF_{\phi,t})]\Big|_{t=0}\|((DF_{\phi,t})^{-1})^\top\nu\|\Big|_{t=0}\,d\sigma = \int\limits_\Sigma \phi\,\mathcal{G}\,d\sigma + \int\limits_\Sigma \partial_\nu\phi\,d\sigma. \tag{5.394}$$

Also, from (5.391) we have $|\det(DF_{\phi,t})|\Big|_{t=0} = \det I_{n\times n} = 1$. Note that by taking the transpose in 5.391 we also obtain

$$(DF_{\phi,t})^\top = I_{n\times n} + tB, \quad \text{where} \quad B := I_{n\times n} + t\phi\,(DN)^\top + t\nu\otimes\nabla\phi, \tag{5.395}$$

so if we set $v(t) := (I+tB)^{-1}\nu$ we obtain

$$\frac{d}{dt}\left[|(I+tB)^{-1}\nu|\right]\Big|_{t=0} = \frac{d}{dt}[|v(t)|]\Big|_{t=0} = \left(\frac{v'(t)\cdot v(t)}{\|v(t)\|}\right)\Big|_{t=0}. \tag{5.396}$$

For $t=0$ we have $v(0)=\nu$ which entails $\|v(0)\|=1$, hence

$$\frac{d}{dt}\left[|(I+tB)^{-1}\nu|\right]\Big|_{t=0} = v'(0)\cdot\nu. \tag{5.397}$$

Recall that, generally speaking, $\frac{d}{dt}[A(t)w] = A'(t)w$, so

$$v'(0) = \frac{d}{dt}\left[(I+tB)^{-1}\nu\right]\Big|_{t=0} = \left(\frac{d}{dt}[(I+tB)^{-1}]\Big|_{t=0}\right)\nu. \tag{5.398}$$

Also,

$$\frac{d}{dt}\left[(I+tB)^{-1}\right] = -(I+tB)^{-1}\frac{d}{dt}[(I+tB)](I+tB)^{-1}. \tag{5.399}$$

Since $\frac{d}{dt}[(I+tB)] = B$ we have that $\frac{d}{dt}\left[(I+tB)^{-1}\right]\Big|_{t=0} = -B$, and we can write that $v'(0)\cdot\nu$ as $-(B\nu)\cdot\nu$. With this observations, the (5.397) can be continued as

$$v'(0)\cdot\nu = -\phi\,((DN)^\top\nu)\cdot\nu - ((\nabla\phi\otimes\nu)\nu)\cdot\nu = -\phi\,\nu\cdot((DN)\nu) - \partial_\nu\phi. \tag{5.400}$$

Note that $\nu\cdot((DN)\nu) = \sum_{j,k=1}^n N_kN_j\partial_jN_k = N\cdot\partial_N N = 0$ and since $\partial_N N = 0$ near Σ, we obtain

$$\int\limits_\Sigma |\det(DF_{\phi,t})|\Big|_{t=0}\frac{d}{dt}\|((DF_{\phi,t})^{-1})^\top\nu\|\Big|_{t=0}\,d\sigma = -\int\limits_\Sigma \partial_\nu\phi\,d\sigma. \tag{5.401}$$

In summary,

$$\frac{d}{dt}[\mathrm{Area}\,(\Sigma_{\phi,t})]\Big|_{t=0} = \int\limits_\Sigma \phi\,\mathcal{G}\,d\sigma + \int\limits_\Sigma \partial_\nu\phi\,d\sigma - \int\limits_\Sigma \partial_\nu\phi\,d\sigma = \int\limits_\Sigma \phi\,\mathcal{G}\,d\sigma, \tag{5.402}$$

finishing the proof of (5.387). □

We next consider the second order variation of the area of a C^2 surface in $\mathbb{R}^n$.

Theorem 5.4.2 (The Second Variation Formula of Area)**.** *In the context of Theorem 5.4.1 we have*

$$\frac{d^2}{dt^2}[\mathrm{Area}\,(\Sigma_{\phi,t})]\Big|_{t=0} = \int\limits_{\Sigma}\Big[2\phi^2\Big(\sum_{1\leqslant j<k\leqslant n-1}\lambda_j\lambda_k\Big) + \|\nabla_{tan}\phi\|^2\Big]\,d\sigma, \tag{5.403}$$

where $\lambda_1, \ldots, \lambda_{n-1}$ are the principal curvatures of the surface Σ, and $\nabla_{tan}\phi := \nabla\phi - (\partial_\nu\phi)\nu$ is the tangential gradient of ϕ on Σ.

Proof. Assume that the matrix $A = (a_{jk})_{j,k} \in \mathcal{M}_{n\times n}$ is given by $A = \phi DN + \nu \otimes \nabla\phi$, i.e., $a_{jk} = \phi\,\partial_k\nu_j + \partial_k\phi\,\nu_j$, for every $j,k \in \{1,\ldots,n\}$. Then

$$\mathrm{Tr}\,(A^2) = \sum_{j,k=1}^{n} a_{jk}\,a_{kj} = \sum_{j,k=1}^{n}(\phi\,\partial_k N_j + \partial_k\phi\,\nu_j)(\phi\,\partial_j N_k + \partial_j\phi\,\nu_k) \tag{5.404}$$

$$= \sum_{j,k=1}^{n}\phi^2\partial_k N_j\,\partial_j N_k + \sum_{j,k=1}^{n}\phi\,\partial_j\phi\,\nu_k\,\partial_k N_j + \sum_{j,k=1}^{n}\phi\,\partial_k\phi\,\nu_j\partial_j N_k + \sum_{j,k=1}^{n}\partial_j\phi\,\partial_k\phi\,\nu_j\nu_k. \tag{5.405}$$

Since $\partial_j N_k = \partial_k N_j$, we have that $\sum_{k=1}^{n}\nu_k\partial_k N_j = \partial_\nu N_j = 0$, $\sum_{j=1}^{n}\nu_j\partial_j N_k = \partial_\nu N_k = 0$, and $\sum_{j,k=1}^{n}\partial_j\phi\,\partial_k\phi\,\nu_j\nu_k = (\partial_\nu\phi)^2$, so the previous equation becomes

$$\mathrm{Tr}\,(A^2) = \phi^2\|DN\|^2 + (\partial_\nu\phi)^2 \quad \text{if} \quad A = \phi DN + \nu\otimes\nabla\phi. \tag{5.406}$$

Also, from an earlier calculations, recall that $\mathrm{Tr}\,(A) = \phi\,\mathcal{G} + \partial_\nu\phi$. If we set $v(t) := A(t)\nu$ then $v'(t) = A'(t)\nu$ and $v''(t) = A''(t)\nu$. Based on this we can then write

$$\frac{d^2}{dt^2}[\|A(t)\nu\|] = \frac{A''(t)\nu\cdot A(t)\nu}{\|A(t)\nu\|} + \frac{\|A'(t)\nu\|^2}{\|A(t)\nu\|} - \frac{[A'(t)\nu\cdot A(t)\nu]^2}{\|A(t)\nu\|^3}. \tag{5.407}$$

We continue by computing second order derivatives for the area variation:

$$\frac{d^2}{dt^2}[\mathrm{Area}\,(\Sigma_{\phi,t})]\Big|_{t=0} = I + II + III + IV \tag{5.408}$$

where

$$I = \int\limits_{\Sigma}\frac{d^2}{dt^2}[\det(DF_{\phi,t})]\Big|_{t=0}\|((DF_{\phi,t})^{-1})^{\top}\nu\|\Big|_{t=0}\,d\sigma, \tag{5.409}$$

$$II = III := \int\limits_{\Sigma}\frac{d}{dt}[\det(DF_{\phi,t})]\Big|_{t=0}\frac{d}{dt}\|((DF_{\phi,t})^{-1})^{\top}\nu\|\Big|_{t=0}\,d\sigma, \tag{5.410}$$

$$IV := \int\limits_{\Sigma}\det(DF_{\phi,t})\Big|_{t=0}\frac{d^2}{dt^2}\|((DF_{\phi,t})^{-1})^{\top}\nu\|\Big|_{t=0}\,d\sigma. \tag{5.411}$$

Note that, from earlier calculations,

$$II = III = \int_{\Sigma} (\phi\,\mathcal{G} + \partial_\nu \phi)(-\partial_\nu \phi)\,d\sigma. \tag{5.412}$$

Let us now consider the first term in the sum in the right-hand side of (5.408), namely I. By calculations made before, we obtain

$$\begin{aligned} \frac{d^2}{dt^2}[\det(DF_{\phi,t})]\Big|_{t=0} &= (\phi\,\mathcal{G} + \partial_\nu\phi)^2 - \phi^2\|DN\|^2 - (\partial_\nu\phi)^2 \\ &= \phi^2[\mathcal{G}^2 - \|DN\|^2] + 2\mathcal{G}\,\phi\,\partial_\nu\phi, \end{aligned} \tag{5.413}$$

hence,

$$I = \int_{\Sigma} \phi^2\left(\mathcal{G}^2 - \|DN\|^2\right)\,d\sigma + 2\int_{\Sigma} \mathcal{G}\,\phi\,\partial_\nu\phi\,d\sigma. \tag{5.414}$$

Since $(DF_{\phi,t})^\top = I + tB$, for $A(t) = (I+tB)^{-1}$ and $B := \phi(DN)^\top + \nabla\phi\otimes\nu$, we have

$$\begin{aligned} \frac{d^2}{dt^2}\|((DF_{\phi,t})^\top)^{-1}\nu\|\Big|_{t=0} &= \frac{d^2}{dt^2}[\|A(t)\nu\|]\Big|_{t=0} \\ &= 2(B^2\nu)\cdot\nu + \|B\nu\|^2 - ((B\nu)\cdot\nu)^2, \end{aligned} \tag{5.415}$$

and

$$B\nu = \phi(DN)^\top\nu + (\nabla\phi\otimes\nu)\nu = \sum_{j,k=1}^{n} \phi(\partial_j N_k)N_k + \nabla\phi. \tag{5.416}$$

Since for every $j \in \{1,...,n\}$ we have

$$\sum_{k=1}^{n}(\partial_j N_k)N_k = \frac{1}{2}\partial_j\|N\|^2 = 0 \quad \text{near} \quad \Sigma, \tag{5.417}$$

we obtain that $B\nu = \nabla\phi$. Hence, we can write $\|B\nu\|^2 = \|\nabla\phi\|^2$ and $((B\nu)\cdot\nu)^2 = (\partial_\nu\phi)^2$. Together, these two terms amount to $\|\nabla_{tan}\phi\|^2$. Next, since $(B^2\nu)\cdot\nu = (B\nu)\cdot(B^\top\nu)$ and $B^\top = \phi DN + \nu\otimes\nabla\phi$, we may conclude that

$$B^\top\nu = \sum_{j,k=1}^{n}\Big(\phi(\partial_k N_j)N_k + (\partial_k\phi)\nu_j\nu_k\Big). \tag{5.418}$$

Since for every $j \in \{1,...,n\}$ we have

$$\sum_{k=1}^{n}(\partial_k N_j)N_k = \partial_\nu N_j = 0, \tag{5.419}$$

we obtain $B^\top\nu = (\partial_\nu\phi)\nu$. Consequently,

$$2(B^2\nu)\cdot\nu = 2(\nabla\phi)\cdot((\partial_\nu\phi)\nu) = 2(\partial_\nu\phi)^2. \tag{5.420}$$

The bottom line is that

$$IV = 2\int_{\Sigma}(\partial_\nu\phi)^2\,d\sigma + \int_{\Sigma}\|\nabla_{tan}\phi\|^2\,d\sigma. \tag{5.421}$$

All in all,

$$\begin{aligned}\frac{d^2}{dt^2}[\text{Area}\,(\Sigma_{\phi,\,t})]\Big|_{t=0} &= \int\limits_{\Sigma} \phi^2(\mathcal{G}^2 - \|DN\|^2)\,d\sigma + 2\int\limits_{\Sigma} \mathcal{G}\phi\,\partial_\nu\phi\,d\sigma - 2\int\limits_{\Sigma} \mathcal{G}\phi\,\partial_\nu\phi\,d\sigma \\ &\quad -2\int\limits_{\Sigma}(\partial_\nu\phi)^2\,d\sigma + 2\int\limits_{\Sigma}(\partial_\nu\phi)^2\,d\sigma + \int\limits_{\Sigma}\|\nabla_{tan}\phi\|^2\,d\sigma \\ &= \int\limits_{\Sigma}[\phi^2(\mathcal{G}^2 - \|DN\|^2) + \|\nabla_{tan}\phi\|^2]\,d\sigma. \qquad (5.422)\end{aligned}$$

Thus, (5.403) is proved as soon as we show that

$$\mathcal{G}^2 - \|DN\|^2 = 2\sum_{1\leqslant j<k\leqslant n-1}\lambda_j\lambda_k \qquad (5.423)$$

where $\lambda_1, ..., \lambda_{n-1}$ are the principal curvatures of the surface Σ. Set $\lambda_n := 0$ and recall that $\lambda_1, ..., \lambda_n$ are the eigenvalues of the symmetric matrix DN at points on Σ. Consequently, there exists a unitary matrix $U \in \mathcal{M}_{n\times n}$ such that

$$DN = U\mathcal{D}U^{-1}, \qquad \text{where } \mathcal{D} := \begin{pmatrix}\lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n\end{pmatrix}. \qquad (5.424)$$

We may then compute

$$\begin{aligned}\|DN\|^2 &= \text{Tr}\,((DN)^\top(DN)) = \text{Tr}\,((DN)^2) = \text{Tr}\,((U\mathcal{D}U^{-1})^2) \\ &= \text{Tr}\,(U\mathcal{D}^2U^{-1}) = \text{Tr}\,(\mathcal{D}^2) = \sum_{j=1}^{n}\lambda_j^2, \qquad (5.425)\end{aligned}$$

so that

$$\mathcal{G}^2 - \|DN\|^2 = \Big(\sum_{j=1}^{n}\lambda_j\Big)^2 - \sum_{j=1}^{n}\lambda_j^2 = 2\sum_{1\leqslant j<k\leqslant n}\lambda_j\lambda_k, = 2\sum_{1\leqslant j<k\leqslant n-1}\lambda_j\lambda_k, \qquad (5.426)$$

as desired. □

Remark 5.4.3. *(i) For a C^2 surface Σ in $\mathbb{R}^n$ $\int\limits_{\Sigma}\|\nabla_{tan}\phi\|^2\,d\sigma = -\int\limits_{\Sigma}(\Delta_\Sigma\phi)\phi\,d\sigma$ where Δ_Σ is the Laplacian on Σ, granted that* (5.385) *holds. As a result, we may write*

$$\frac{d^2}{dt^2}[\text{Area}\,(\Sigma_{\phi,\,t})]\Big|_{t=0} = \int\limits_{\Sigma}\Big[2\phi^2\Big(\sum_{1\leqslant j<k\leqslant n-1}\lambda_j\lambda_k\Big) - (\Delta_\Sigma\phi)\phi\Big]\,d\sigma. \qquad (5.427)$$

(ii) If $\mathcal{G} = 0$, equation (5.427) *becomes*

$$\frac{d^2}{dt^2}[\text{Area}\,(\Sigma_{\phi,\,t})]\Big|_{t=0} = -\int\limits_{\Sigma}\Big[\phi^2\Big(\sum_{j=1}^{n-1}\lambda_j^2\Big) + \phi(\Delta_\Sigma\phi)\Big]\,d\sigma. \qquad (5.428)$$

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VITA

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