

# Comparison of Sums of Independent Identically Distributed Random Variables

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ABSTRACT: Let  $S_k$  be the  $k$ -th partial sum of Banach space valued independent identically distributed random variables. In this paper, we compare the tail distribution of  $\|S_k\|$  with that of  $\|S_j\|$ , and deduce some tail distribution maximal inequalities.

The main result of this paper was inspired by the inequality from [dP–M] that says that  $\Pr(\|X_1\| > t) \leq 5 \Pr(\|X_1 + X_2\| > t/2)$  whenever  $X_1$  and  $X_2$  are independent identically distributed. Such results for  $L_p$  ( $p \geq 1$ ) such as  $\|X_1\|_p \leq \|X_1 + X_2\|_p$  are straightforward, at least if  $X_2$  has zero expectation. This inequality is also obvious if either  $X_1$  is symmetric, or  $X_1$  is real valued positive. However, for arbitrary random variables, this result is somewhat surprising to the author. Note that the identically distributed assumption cannot be dropped, as one could take  $X_1 = 1$  and  $X_2 = -1$ .

In this paper, we prove a generalization to sums of arbitrarily many independent identically distributed random variables. Note that all results in this paper are true for Banach space valued random variables. The author would like to thank Victor de la Peña for helpful conversations.

THEOREM 1. *There exist universal constants  $c_1 = 3$  and  $c_2 = 10$  such that if  $X_1, X_2, \dots$  are independent identically distributed random variables, and if we set*

$$S_k = \sum_{i=1}^k X_i,$$

*then for  $1 \leq j \leq k$*

$$\Pr(\|S_j\| > t) \leq c_1 \Pr(\|S_k\| > t/c_2).$$

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This result cannot be asymptotically improved. Consider, for example, the case where  $X_1 = 1$  with very small probability, and is zero otherwise. This shows that for the inequality to be true for all  $X_1$ , it must be that  $c_2$  must be larger than some universal constant for all  $j$  and  $k$ . Also, it is easy to see that  $c_1$  must be larger than some universal constant because it is easy to select  $X_1$  and  $t$  so that  $\Pr(\|S_j\| > t)$  is close to 1.

However, Latafala [L] has been able to obtain the same theorem with  $c_1 = 4$  and  $c_2 = 5$ , or  $c_1 = 2$  and  $c_2 = 7$ . In the case  $j = 1$  and  $k = 2$ , he has shown that  $\Pr(\|X_1\| > t) \leq 2 \Pr(\|X_1 + X_2\| > 2t/3)$ , and these constants cannot be improved.

In order to show this result, we will use the following definition. We will say that  $x$  is a  $t$ -concentration point for a random variable  $X$  if  $\Pr(\|X - x\| \leq t) > 2/3$ .

LEMMA 2. *If  $x$  is a  $t$ -concentration point for  $X$ , and  $y$  is a  $t$ -concentration point for  $Y$ , and  $z$  is a  $t$ -concentration point for  $X + Y$ , then  $\|x + y - z\| \leq 3t$ .*

*Proof:*

$$\begin{aligned} \Pr(\|x + y - z\| > 3t) &\leq \Pr(\|X - x + Y - y - (X + Y - z)\| > 3t) \\ &\leq \Pr(\|X - x\| > t) + \Pr(\|Y - y\| > t) + \Pr(\|X + Y - z\| > t) \\ &< 1. \end{aligned}$$

Hence  $\Pr(\|x + y - z\| \leq 3t) > 0$ . Since  $x$ ,  $y$  and  $z$  are fixed vectors, the result follows.  $\square$

COROLLARY 3. *If  $X_1, X_2, \dots$  are independent identically distributed random variables, and if the partial sums  $S_j = \sum_{i=1}^j X_i$  have  $t$ -concentration points  $s_j$  for  $1 \leq j \leq k$ , then  $\|ks_j - js_k\| \leq 3(k + j)t$ .*

*Proof:* We prove the result by induction. It is obvious if  $j = k$ . Otherwise,

$$\begin{aligned} \|js_k - ks_j\| &\leq \|js_{k-j} - (k-j)s_j\| + \|js_k - js_{k-j} - js_j\| \\ &\leq 3(k-j+j)t + 3jt = 3(k+j)t. \end{aligned}$$

(The observant reader will notice that we are, in fact, following the steps of Euclid's algorithm. The same proof could show  $\|ks_j - js_k\| \leq 3(j+k-2h)t$  where  $h$  is the highest common factor of  $j$  and  $k$ .)  $\square$

*Proof of Theorem 1:* We consider three cases. First suppose that  $\Pr(\|S_{k-j}\| > 9t/10) \leq 1/3$ . Note that  $S_k - S_j$  is independent of  $S_j$ , and identically distributed to  $S_{k-j}$ . Then

$$\Pr(\|S_j\| > t) \leq 3/2 \Pr(\|S_j\| > t \text{ and } \|S_k - S_j\| \leq 9t/10) \leq 3/2 \Pr(\|S_k\| > t/10).$$

For the second case, suppose that there is a  $1 \leq i \leq k$  such that  $S_i$  does not have any  $(t/10)$ -concentration point. Then

$$\Pr(\|S_i + X_{i+1} + \dots + X_k\| > t/10 \mid \sigma(X_{i+1}, \dots, X_k)) \geq 1/3,$$

and hence  $\Pr(\|S_k\| > t/10) \geq 1/3 \geq 1/3 \Pr(\|S_i\| > t)$ .

Finally, we are left with the third case where  $\Pr(\|S_{k-j}\| > 9t/10) > 1/3$ , and  $S_i$  has a  $(t/10)$ -concentration point  $s_i$  for all  $1 \leq i \leq k$ . Clearly  $\|s_{k-j}\| \geq 8t/10$ . Also, by Corollary 3,

$$\|s_k\| \geq \frac{k}{k-j} \|s_{k-j}\| - \frac{3(2k-j)t}{10(k-j)} \geq \frac{8kt}{10(k-j)} - \frac{6kt}{10(k-j)} \geq \frac{2t}{10}.$$

Therefore,  $\Pr(\|S_k\| \geq t/10) \geq \Pr(\|S_k - s_k\| \leq t/10) \geq 2/3 \geq 2/3 \Pr(\|S_j\| > t)$ , and we are done.  $\square$

One might be emboldened to conjecture the following. Suppose that  $X_1, X_2, \dots$  are independent identically distributed random variables, and that  $\alpha_i > 0$ . Let

$$S_k = \sum_{i=1}^k \alpha_i X_i.$$

Then one might conjecture that there is a universal constant such that for  $1 \leq j \leq k$

$$\Pr(\|S_j\| > t) \leq c \Pr(\|S_k\| > t/c).$$

As it turns out, this is not the case. Let  $Y_1, Y_2, \dots$  be real valued independent identically distributed random variables such that

$$\begin{aligned} \Pr(Y_i = N-1) &= 1/N \\ \Pr(Y_i = -1) &= (N-1)/N. \end{aligned}$$

Then by the central limit theorem, there exists  $M \geq N^3$  such that

$$\Pr\left(\left|\frac{1}{M^{2/3}} \sum_{i=1}^M Y_i\right| > \frac{1}{N}\right) \leq \frac{1}{N}.$$

Now let  $X_i = Y_i + 1/M^{1/3}$ , and let

$$S_M = \frac{1}{M^{2/3}} \sum_{i=1}^M X_i.$$

Then  $\Pr(|S_M| > 1/2) \geq 1 - 1/N$ , whereas  $\Pr(|S_M + X_{M+1}| > 3/N) \leq 2/N$ .

Theorem 1 has several corollaries.

COROLLARY 4. *There is a universal constant  $c$  such that if  $X_1, X_2, \dots$  are independent identically distributed random variables, and if we set*

$$S_k = \sum_{i=1}^k X_i,$$

then

$$\Pr\left(\sup_{1 \leq j \leq k} \|S_j\| > t\right) \leq c \Pr(\|S_k\| > t/c).$$

Latalo [L] has been able to obtain this result with  $c_1 = 4$  and  $c_2 = 6$ , or with  $c_1 = 2$  and  $c_2 = 8$ .

*Proof:* This follows from Proposition 1.1.1 of [K–W], that states that if  $X_1, X_2, \dots$  are independent (not necessarily identically distributed), and if  $S_k = \sum_{i=1}^k X_i$ , then

$$\Pr\left(\sup_{1 \leq j \leq k} \|S_j\| > t\right) \leq 3 \sup_{1 \leq j \leq k} \Pr(\|S_j\| > t/3).$$

It is also possible to prove this result directly using the techniques of the proof of Theorem 1. The third case only requires that  $\Pr(\|S_{k-j}\| > 9t/10) > 1/3$  for one  $j = 1, 2, \dots, k$ . Hence, for the first case we may assume that  $\Pr(\|S_k - S_j\| > 9t/10) \leq 1/3$  for all  $1 \leq j \leq k$ . Let  $A_j$  be the event  $\{\|S_i\| \leq t \text{ for all } i < j \text{ and } \|S_j\| > t\}$ . Then

$$\Pr(A_j) \leq 3/2 \Pr(A_j \text{ and } \|S_k - S_j\| \leq 9t/10) \leq 3/2 \Pr(A_j \text{ and } \|S_j\| > t/10).$$

Summing over  $j$ , the result follows. □

COROLLARY 5. *There is a universal constant  $c$  such that if  $X_1, X_2, \dots$  are independent identically distributed random variables, and if  $|\alpha_i| \leq 1$ , then*

$$\Pr\left(\left\|\sum_{i=1}^k \alpha_i X_i\right\| > t\right) \leq c \Pr\left(\left\|\sum_{i=1}^k X_i\right\| > t/c\right).$$

*Proof:* The technique used in this proof is well known (see for example [KW], Proposition 1.2.1), but is included for completeness.

By taking real and imaginary parts of  $\alpha_i$ , we may suppose that the  $\alpha_i$  are real. Without loss of generality,  $1 \geq \alpha_1 \geq \dots \geq \alpha_k \geq -1$ . Then we may write  $\alpha_j = -1 + \sum_{i=j}^k \sigma_i$ , where

$\sigma_i \geq 0$ . Thus  $\sum_{i=1}^k |\sigma_i| \leq 2$ , and hence

$$\begin{aligned} \left\| \sum_{j=1}^k \alpha_j X_j \right\| &= \left\| \sum_{j=1}^k \left( -1 + \sum_{i=j}^k \sigma_i \right) X_j \right\| \\ &= \left\| - \left( \sum_{i=1}^k X_i \right) + \left( \sum_{j=1}^k \sigma_j \sum_{i=1}^j X_i \right) \right\| \\ &\leq \left\| \sum_{i=1}^k X_i \right\| + \left( \sum_{j=1}^k |\sigma_j| \right) \sup_{1 \leq j \leq k} \left\| \sum_{i=1}^j X_i \right\|. \end{aligned}$$

Applying Corollary 4, the result follows.  $\square$

**COROLLARY 6.** *There are universal constants  $c_1$  and  $c_2$  such that if  $X_1, X_2, \dots$  are independent identically distributed random variables, and if we set*

$$S_k = \sum_{i=1}^k X_i,$$

then for  $1 \leq k \leq j$

$$\Pr(\|S_j\| > t) \leq c_1 j/k \Pr(\|S_k\| > kt/c_2 j).$$

*Proof:* Let  $m$  be the least integer such that  $mk \geq j$ . By Theorem 1, it follows that  $\Pr(\|S_j\| > t) \leq c \Pr(\|S_{mk}\| > t/c)$ . That  $\Pr(\|S_{mk}\| > t) \leq m \Pr(\|S_k\| > t/m)$  is straightforward.  $\square$

The example where  $X_1$  is constant shows that  $c_2$  cannot be made smaller than some universal constant. The example where  $X_1 = 1$  with very small probability and is zero otherwise shows the same is true for  $c_1$ .

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