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Concrete Representation of Martingales

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Abstract: Let (f_n) be a mean zero vector valued martingale sequence. Then there exist vector valued functions (d_n) from $[0,1]^n$ such that $\int_0^1 d_n(x_1,\dots,x_n)\,dx_n = 0$ for almost all $x_1,\dots,x_{n-1}$, and such that the law of (f_n) is the same as the law of $(\sum_{k=1}^n d_k(x_1,\dots,x_k))$. Similar results for tangent sequences and sequences satisfying condition (C.I.) are presented. We also present a weaker version of a result of McConnell that provides a Skorohod like representation for vector valued martingales.

Keywords: martingale, concrete representation, tangent sequence, condition (C.I.), UMD, Skorohod representation

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1. Introduction

In this paper, we seek to give a concrete representation of martingales. We will present theorems of the following form. Given a martingale sequence (f_n) (possibly vector valued), there is a martingale (g_n) on the probability space $[0,1]^{\mathbb{N}}$, with respect to the filtration $\mathcal{L}_n$, the minimal sigma field for which the first n coordinates of $[0,1]^{\mathbb{N}}$ are measurable, such that the sequence (f_n) has the same law as (g_n). Thus any martingale may be represented by a martingale

$$g_n((x_n)) = \sum_{k=0}^{n} d_k(x_1,\dots,x_k),$$

where $\int_0^1 d_n(x_1,\dots,x_n)\,dx_n = 0$ for $n \geq 1$. (Here, as in the rest of this paper, the notation (x_n) will refer to the sequence $(x_n)_{n\in\mathbb{N}}$, where $\mathbb{N}$ refers to the positive integers.)

The value of such a result is perhaps purely psychological. However, we will also present similar such results for tangent sequences, and also for sequences satisfying condition (C.I.). Tangent sequences and condition (C.I.), as defined in this paper, were introduced by Kwapień and Woyczyński [KW]. The abstract definition of tangent sequences can be, perhaps, a little hard to grasp. However, in this concrete setting, it is clear what their meaning is. To demonstrate the psychological advantage that this view gives, we will give a new proof of a result of McConnell [M2] that states that tangent sequence inequalities hold for the UMD spaces.

We will also give a weaker version of a result of McConnell [M1] that provides a Skorohod representation theorem for vector valued martingales, in that any martingale is a stopped continuous time stochastic process. This means that many martingale inequalities that are true for continuous time stochastic processes are automatically also true for general martingales.

2. Representations of sequences of random variables

In this section, we present the basic result upon which all our other results will depend.

First let us motivate this result by considering just one random variable, that is a measurable function f from the underlying probability space to a separable measurable space $(S, \mathcal{S})$. (A measurable space is said to be *separable* if its sigma field is generated by a countable collection of sets.) We seek to find a measurable function $g : [0,1] \to S$ that has the same law as f, that is, given any measurable set A, we have that $\Pr(f \in A) = \lambda(g \in A)$, or equivalently, given any measurable bounded function $F : S \to \mathbb{R}$, we have that $E(F(f)) = \int F(g)\, d\lambda$. Here λ refers to the Lebesgue measure on $[0,1]$.

First, it may be shown without loss of generality that $(S, \mathcal{S})$ is $\mathbb{R}$ with the Borel sets. (The argument for this may be found in Chapter 1 of [DM], and is presented below.) Then the idea is to let g be the so called increasing rearrangement of f, that is,

$$g(x) = \sup\{t \in \mathbb{R} : \Pr(f < t) < x\}.$$

That g has the required properties is easy to show.

Next, suppose that f is nowhere constant, that is, we have that $\Pr(f = s) = 0$ for all $s \in S$. Then it may be seen that g is a strictly increasing function. In that case, it may be seen that the minimal complete sigma field for which g is measurable is the collection of Lebesgue measurable sets.

Now let us move on to state the main result. We will start by setting our notation. We will work on two probability spaces: a generic one $(\Omega, \mathcal{F}, \Pr)$, and $([0,1]^{\mathbb{N}}, \mathcal{L}_{\mathbb{N}}, \lambda)$. Here $\mathcal{L}_{\mathbb{N}}$ refers to the Lebesgue measurable sets on $[0,1]^{\mathbb{N}}$, and λ refers to Lebesgue measure on $[0,1]^{\mathbb{N}}$.

Let us make some notational abuses. The reason for this is to make some expressions less cumbersome, while hopefully not being too obscure. We will always identify $[0,1]^n$ with the natural projection of $[0,1]^{\mathbb{N}}$ onto the first n coordinates. Any function g on $[0,1]^n$ will be identified with its canonical lifting on $[0,1]^{\mathbb{N}}$. The notation $\mathcal{L}_n$ refers to the Lebesgue measurable sets on $[0,1]^n$, and also to their canonical lifting onto $[0,1]^{\mathbb{N}}$. We will let λ also refer to Lebesgue measure on $[0,1]^n$.

Given a random variable f, and a sigma field $\mathcal{G}$, we will say that f is *nowhere constant* with respect to $\mathcal{G}$ if $\Pr(f = g) = 0$ for every $\mathcal{G}$ measurable function g. Let us illustrate this notion with respect to a measurable function f on $[0,1]^2$. It is nowhere constant with respect to the trivial sigma field if and only if it is nowhere constant as defined above. Let $\mathcal{G}$ be the sigma field generated by the first coordinate. Then f is nowhere constant with respect to $\mathcal{G}$ if and only if for almost every $x \in [0,1]$ the function $y \mapsto f(x,y)$ is nowhere constant. Now let $\mathcal{G}$ be the set of all Lebesgue measurable sets on $[0,1]^2$. In this case, f can never be nowhere constant with respect to $\mathcal{G}$.

If $f_1, \dots, f_n$ are random variables taking values in a measurable space $(S, \mathcal{S})$, we will let $\sigma(f_1, \dots, f_n)$ denote the minimal sigma field which contains all sets of measure zero, and for which $f_1, \dots, f_n$ are measurable. Note that a simple argument shows that the $\sigma(f_1, \dots, f_n)$ measurable functions coincide precisely with the functions that are almost everywhere equal to $F(f_1, \dots, f_n)$, where F is some measurable function on S^n.

Throughout these proofs we will use the following idea many times. Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space, and let $(S, \mathcal{S})$ be a measurable space. Suppose that $X : \Omega \to S$ is a measurable function such that $\sigma(X)$ is the whole of $\mathcal{F}$. If Y is any random variable on Ω taking values in $\mathbb{R}$, then there exists a measurable map $\phi : S \to \mathbb{R}$ such that $Y = \phi \circ X$ almost surely. This is easily seen by approximating Y by simple functions.

Theorem 2.1. *Let (f_n) be a sequence of random variables taking values in a separable measurable space $(S, \mathcal{S})$. Then there exists a sequence of measurable functions $(g_n : [0,1]^n \to S)$ that has the same law as (f_n).*

If further we have that f_{n+1} is nowhere constant with respect to $\sigma(f_1, \ldots, f_n)$ for all $n \geq 0$, then we may suppose that $\sigma(g_1, \ldots, g_n) = \mathcal{L}_n$, for all $n \geq 1$.

If even further we have that $(S, \mathcal{S})$ is $\mathbb{R}$ with the Borel sets, then we may suppose g_n is Borel measurable (that is, the preimages of Borel sets are Borel sets), and that $g_n(x_1, \ldots, x_n)$ is a strictly increasing function of $x_n \in (0,1)$ for almost all $x_1, \ldots, x_{n-1}$.

Proof: We may suppose without loss of generality that $S = \mathbb{R}$, and $\mathcal{S}$ is the Borel subsets of $\mathbb{R}$. To see that we may do this, see first that we may suppose without loss of generality that $\mathcal{S}$ separates points in S, that is, if $s \neq t \in S$, then there exists $A \in \mathcal{S}$ such that $s \in A$ and $t \notin A$. Let $\{\mathcal{C}_n\}$ be a countable collection of sets in $\mathcal{S}$ that generate $\mathcal{S}$. Notice then that the sequence $\{\mathcal{C}_n\}$ also separates points in S, that is, if $s \neq t \in S$, then there exists a number n such that one and only one of s or t is in $\mathcal{C}_n$. Define a map $\varphi : S \to \mathbb{R}$:

$$\varphi(s) = \sum_{m=1}^{\infty} \frac{I_{(s \in C_m)}}{3^m}.$$

Clearly φ is injective. Further φ maps C_n to $D_n \cap \varphi(S)$, where

$$D_n = \left\{ \sum_{m=1}^{\infty} \frac{e_m}{3^m} : e_m = 0 \text{ or } 1,\ e_n = 0 \right\},$$

and thus φ maps any element of $\mathcal{S}$ to a Borel subset of $\mathbb{R}$ intersected with $\varphi(S)$. Conversely, the preimage of D_n under φ is C_n, and hence the preimage of any Borel set under φ is in $\mathcal{S}$. (This argument may be found in Chapter 1 of [DM]).

Now apply the theorem to $(\varphi \circ f_n)$, to obtain (g_n). Since the law of g_n is the same as $\varphi \circ f_n$, the range of g_n lies in the range of $\varphi(S)$ with probability one. Then the sequence $(\varphi^{-1} \circ g_n)$ will have the same law as (f_n).

So let us suppose as the induction hypothesis that we have obtained $(g_n : [0,1]^n \to \mathbb{R})_{1 \leq n \leq N}$ that has the same law as $(f_n)_{1 \leq n \leq N}$, and that g_n is Borel measurable, for all $n \leq N$, where N is a non-negative integer. (The induction is started with $N = 0$ in which case the hypothesis is vacuously true. The arguments that follow simplify greatly in this case.)

For each $t \in \mathbb{Q}$, let

$$p_t = \mathbb{E}(I_{(f_{N+1} < t)} | \sigma(f_1, \ldots, f_N)).$$

Since p_t is $\sigma(f_1, \ldots, f_N)$ measurable, we may write

$$p_t = q_t(f_1, \ldots, f_N) \quad \text{almost surely}$$

for some Borel measurable function $q_t : \mathbb{R}^N \to [0,1]$. Define the Borel measurable functions r_t by

$$r_t = q_t(g_1, \dots, g_N).$$

Since the sequence $(r_t)_{t\in\mathbb{Q}}$ has the same law as $(p_t)_{t\in\mathbb{Q}}$, we see that there is a Borel set $B \subset [0,1]^N$ of full measure such that if $(x_1,\dots,x_N) \in B$, then $r_t(x_1,\dots,x_N)$ is an increasing function of $t \in \mathbb{Q}$, that tends to 0 as $t \to -\infty$, and tends to 1 as $t \to \infty$.

If $(x_1,\dots,x_N) \in B$ and $x_{N+1} \in (0,1)$, let

$$g_{N+1}(x_1,\dots,x_{N+1}) = \sup\{t \in \mathbb{Q} : r_t(x_1,\dots,x_N) < x_{N+1}\},$$

and let it be zero otherwise. We see that g_{N+1} is Borel measurable, since g_{N+1} on $B\times(0,1)$ is the supremum of countably many Borel measurable functions $(s_t)_{t\in\mathbb{Q}}$ where

$$s_t(x_1,\dots,x_{N+1}) = \begin{cases} t & \text{if } r_t(x_1,\dots,x_N) < x_{N+1} \\ -\infty & \text{otherwise.} \end{cases}$$

It is easy to see that g_{N+1} is always finite.

Let us now show that $(f_1,\dots,f_{N+1})$ has the same law as $(g_1,\dots,g_{N+1})$. It is sufficient to show that for any Borel set $A \subset \mathbb{R}^N$, and for $t \in \mathbb{Q}$, that

$$\Pr((f_1,\dots,f_N) \in A \text{ and } f_{N+1} < t) = \lambda((g_1,\dots,g_N) \in A \text{ and } g_{N+1} < t).$$

So let us begin.

$$\begin{aligned}
&\lambda((g_1,\dots,g_N) \in A \text{ and } g_{N+1} < t) \\
&= \lambda(\{(g_1,\dots,g_N) \in A \text{ and } g_{N+1} < t\} \cap B \times (0,1)) \\
&\le \lambda(\{(x_1,\dots,x_{N+1}) : (g_1(x_1),\dots,g_N(x_1,\dots,x_N)) \in A \text{ and } r_t(x_1,\dots,x_N) \ge x_{N+1}\} \\
&\qquad\qquad \cap B \times (0,1)) \\
&= \lambda(\{(x_1,\dots,x_{N+1}) : (g_1(x_1),\dots,g_N(x_1,\dots,x_N)) \in A \text{ and } r_t(x_1,\dots,x_N) \ge x_{N+1}\}) \\
&= \int_{[0,1]^N} \int_0^1 I_{(r_t \ge x)} I_{((g_1,\dots,g_N)\in A)} \, dx \, d\lambda \\
&= \int_{[0,1]^N} r_t I_{((g_1,\dots,g_N)\in A)} \, d\lambda \\
&= \mathbb{E}(p_t I_{((f_1,\dots,f_N)\in A)}) \\
&= \mathbb{E}(I_{(f_{N+1}<t)} I_{((f_1,\dots,f_N)\in A)}) \\
&= \Pr((f_1,\dots,f_N) \in A \text{ and } f_{N+1} < t).
\end{aligned}$$

Similarly

$$\lambda((g_1,\dots,g_N) \in A \text{ and } g_{N+1} \le t) \ge \Pr((f_1,\dots,f_N) \in A \text{ and } f_{N+1} < t),$$

and so by Lebesgue's monotone convergence theorem

$$\begin{aligned}\lambda((g_1,\dots,g_N)\in A \text{ and } g_{N+1}<t) &= \lim_{\substack{s\nearrow t\\ s\in\mathbb{Q}}} \lambda((g_1,\dots,g_N)\in A \text{ and } g_{N+1}\le s)\\ &\ge \lim_{\substack{s\nearrow t\\ s\in\mathbb{Q}}} \Pr((f_1,\dots,f_N)\in A \text{ and } f_{N+1}<s)\\ &= \Pr((f_1,\dots,f_N)\in A \text{ and } f_{N+1}<t).\end{aligned}$$

Thus we have shown that $(f_1,\dots,f_{N+1})$ has the same law as $(g_1,\dots,g_{N+1})$.

Now let us add the assumption that f_{n+1} is nowhere constant with respect to $\sigma(f_1,\dots,f_n)$ for all $n\ge 0$. Let us assume the inductive hypothesis that $\sigma(g_1,\dots,g_N)=\mathcal{L}_N$,

Then there exist Borel measurable functions $(\alpha_n:\mathbb{R}^n\to[0,1])_{1\le n\le N}$ such that for $1\le n\le N$ we have

$$\alpha_n(g_1(x_1),\dots,g_N(x_1,\dots,x_N)) = x_n \quad \text{almost everywhere.}$$

Let $\beta_n=\alpha_n(g_1,\dots,g_N)$, and let $\gamma_n=\alpha_n(f_1,\dots,f_N)$.

Notice that $g_{N+1}(x_1,\dots,x_{N+1})$ is an increasing function in $x_{N+1}\in(0,1)$ for fixed $(x_1,\dots,x_N)\in B$. Let us show that $g_{N+1}(x_1,\dots,x_{N+1})$ is a strictly increasing function in $x_{N+1}\in(0,1)$ for $(x_1,\dots,x_N)\in C$ where C is a subset of B of full measure. For suppose otherwise. Then the following set has positive measure: the set of $(x_1,\dots,x_{N+1})$ such that for some rational number $t\in[0,1]$ we have that $g_{N+1}(x_1,\dots,x_N,x_{N+1}) = g_{N+1}(x_1,\dots,x_N,t)$. But this is equal to the set

$$\bigcup_{t\in[0,1]\cup\mathbb{Q}} \{g_{N+1}=g_{N+1}(\beta_1,\dots,\beta_N,t)\}.$$

But the measure of this set is the same as the probability of the set

$$\bigcup_{t\in[0,1]\cup\mathbb{Q}} \{f_{N+1}=g_{N+1}(\gamma_1,\dots,\gamma_N,t)\}.$$

Since f_{N+1} is nowhere constant with respect to $\sigma(f_1,\dots,f_N)$, it follows that this last set has probability zero.

Let us now show that $\sigma(g_1,\dots,g_{N+1})=\mathcal{L}_{N+1}$. For any $a\in[0,1]$ we have that

$$\{g_{N+1}\le g_{N+1}(\beta_1,\dots,\beta_N,a)\} = [0,1]^N\times[0,a] \quad \text{up to a set of measure zero.}$$

Thus $\sigma(g_1,\dots,g_{N+1})$ contains all set of the form $[0,1]^N\times[0,a]$. Since it also contains $\sigma(g_1,\dots,g_N)=\mathcal{L}_N$, the result follows.

Q.E.D.

3. Representation of martingales — 1

Theorem 3.1. *Let (d_n) be a Bochner integrable martingale difference sequence taking values in a Banach space X. Then there exists a sequence of Bochner measurable functions $(e_n : [0,1]^n \to X)$ such that (d_n) has the same law as (e_n), and such that for almost every $x_1, \dots, x_n$ we have*

$$\int_0^1 e_n(x_1, \dots, x_n)\, dx_n = 0.$$

Proof: First we show that without loss of generality that we may assume that d_{n+1} is nowhere constant with respect to $\sigma(d_1, \dots, d_n)$ for all $n \geq 0$. To do this, replace Ω by $\Omega \times [0,1]^{\mathbb{N}}$, replace X by $X \times \mathbb{R}$ (where $\mathbb{R}$ is equipped with the sigma field of Borel sets), and replace d_n by $(\omega, (x_n)) \mapsto (d_n(\omega), x_n - \frac{1}{2})$. Apply the theorem to this sequence. After obtaining the resulting (e_n), compose these functions with the natural projection of $X \times \mathbb{R}$ to X.

Further, since (d_n) is Bochner integrable, we may suppose without loss of generality that X is separable.

Apply Theorem 2.1 to obtain the sequence (e_n). Suppose that $\phi : [0,1]^{n-1} \to \mathbb{R}$ is any bounded measurable function. Then there exists a bounded Borel measurable function $\theta : X^{n-1} \to \mathbb{R}$ such that $\phi(x_1, \dots, x_{n-1}) = \theta(e_1(x_1), \dots, e_{n-1}(x_1, \dots, x_{n-1}))$ for almost all $x_1, \dots, x_{n-1}$. Then

$$\int_{[0,1]^n} \phi e_n \, d\lambda = \mathbb{E}(\theta(d_1, \dots, d_{n-1}) d_n),$$

and this is zero because $\mathbb{E}(d_n | \sigma(d_1, \dots, d_{n-1})) = 0$. The result follows.

Q.E.D.

4. Representation of tangent sequences

The following definition may be found in [KW]. Let $(\mathcal{F}_n)_{n\geq 0}$ be an increasing sequence of sigma fields on $(\Omega, \mathcal{F}, \mathrm{Pr})$, where $\mathcal{F}_0$ is the trivial sigma field. Two adapted sequences (f_n) and (g_n) taking values in a measurable space $(S, \mathcal{S})$ are said to be *tangent* if for each $n \geq 1$ we have that the law of f_n conditionally on $\mathcal{F}_{n-1}$ is the same as the law of g_n conditionally on $\mathcal{F}_{n-1}$, that is, $\mathbb{E}(I_{(f_n \in A)} | \mathcal{F}_{n-1}) = \mathbb{E}(I_{(g_n \in A)} | \mathcal{F}_{n-1})$ for any $A \in \mathcal{S}$.

Notice then that if $F : S^{2n-1} \to \mathbb{R}$ is any measurable bounded function, that

$$\mathbb{E}(F(f_1, g_1, \ldots, f_{n-1}, g_{n-1}, f_n)) = \mathbb{E}(F(f_1, g_1, \ldots, f_{n-1}, g_{n-1}, g_n)).$$

This is easily seen by reducing it to the case where F is a finite linear combination of characteristic functions of sets of the form $A_1 \times \ldots \times A_{2n-1}$ where $A_1, \ldots, A_{2n-1} \in \mathcal{S}$.

Theorem 4.1. *Let $(S, \mathcal{S})$ be a separable measurable space. Let (f_n) and (g_n) be S-valued measurable sequences adapted to $(\mathcal{F}_n)_{n\geq 0}$ that are tangent. Then there exists a sequence $(h_n : [0,1]^n \to S)$, and a sequence of Borel measurable functions $(\phi_n : [0,1]^n \to [0,1])$ such that the map $\phi_n(x_1, \ldots, x_{n-1}, \cdot)$ is a measure preserving map on $[0,1]$ for almost every $x_1, \ldots, x_{n-1}$, such that the law of (f_n, g_n) is the same as the law of (h_n, k_n), where*

$$k_n(x_1, \ldots, x_n) = h_n(x_1, \ldots, x_{n-1}, \phi_n(x_1, \ldots, x_n)).$$

Further, if S is a separable Banach space, and (f_n) is a martingale difference sequence with respect to $\mathcal{F}_n$, then

$$\int_0^1 h_n(x_1, \ldots, x_n) \, dx_n = 0$$

for almost all $x_1, \ldots, x_{n-1}$.

Lemma 4.2. *Let $(S, \mathcal{S}, \mu)$, $(T, \mathcal{T}, \nu)$ and $(U, \mathcal{U}, \tau)$ be measure spaces such that $\mathcal{U}$ is separable. Let $\phi : S \times T \to U$ be a measurable map such that for $A \in \mathcal{S}$ and $C \in \mathcal{U}$ we have that*

$$\mu \otimes \nu((s,t) : s \in A \text{ and } \phi(s,t) \in C) = \mu(A)\tau(C).$$

Then $\phi(s, \cdot)$ is a measure preserving map $T \to U$ for μ almost every s.

Proof: The hypothesis of the lemma can be recast as:

$$\int_{s\in A} \nu(t : \phi(s,t) \in C) \, d\mu(s) = \int_A \tau(C) \, d\mu,$$

from which we conclude that for every $C \in \mathcal{U}$, we have that $\nu(t : \phi(s,t) \in C) = \tau(C)$ for μ almost every s.

Let $\mathcal{C}$ be a countable subcollection of $\mathcal{U}$ that generates $\mathcal{U}$. Then there exists a set $A \in \mathcal{S}$ of full measure such that $\nu(t : \phi(s,t) \in C) = \tau(C)$ for $C \in \mathcal{C}$ and $s \in A$. Let

$$\mathcal{M} = \{C \in \mathcal{U} : \nu(t : \phi(s,t) \in C) = \tau(C) \text{ for } s \in A\}.$$

Notice that $\mathcal{M}$ is a sigma field that contains $\mathcal{C}$, and hence $\mathcal{M} = \mathcal{U}$.

Q.E.D.

Proof of Theorem 4.1: Without loss of generality (at least until we prove the last part), let us suppose that $S = \mathbb{R}$ with the Borel sets. To do this, we use the same argument that we gave at the beginning of the proof of Theorem 2.1.

Consider the sequence $(f_1, g_1, f_2, g_2, \ldots)$. By the kind of argument we presented at the beginning of Theorem 3.1, we may suppose that each element of this sequence is nowhere constant with respect to the sigma field generated by those elements in the sequence that precede it.

Applying Theorem 2.1, we create a sequence of Borel measurable functions $(\tilde{h}_1, \tilde{k}_1, \tilde{h}_2, \tilde{k}_2, \ldots)$ with the same law, where $\tilde{h}_n : [0,1]^{2n-1} \to \mathbb{R}$, $\tilde{k}_n : [0,1]^{2n} \to \mathbb{R}$, and $\tilde{h}_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n)$ is a strictly increasing function of $x_n \in (0,1)$ for almost every $x_1, y_1, \ldots, x_{n-1}, y_{n-1}$.

Define the function

$$r_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, s) = \sup\{t \in \mathbb{Q} \cap (0,1) : \tilde{h}_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, t) < s\},$$

where we will set the first supremum equal to zero if the set is empty. Notice that r_n is the supremum of countably many Borel measurable functions, and hence is Borel measurable. Also, since $\tilde{h}_n(x_1, y_1, \ldots, y_{n-1}, x_{n-1}, x_n)$ is a strictly increasing function of $x_n \in (0,1)$ for almost every $x_1, y_1, \ldots, x_{n-1}, y_{n-1}$, we see that

$$r_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, \tilde{h}_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n)) = x_n$$

for almost every $x_1, y_1, \ldots, x_n$. Define

$$\tilde{\phi}_n(x_1, y_1, \ldots, x_n, y_n) = r_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, \tilde{k}_n(x_1, y_1, \ldots, x_n, y_n)).$$

Now let α_n, β_n be functions such that

$$\begin{aligned} x_n &= \alpha_n(\tilde{h}_1(x_1), \tilde{k}_1(x_1, y_1), \ldots, \tilde{h}_n(x_1, y_1, \ldots, x_n)) \quad \text{almost everywhere} \\ y_n &= \beta_n(\tilde{h}_1(x_1), \tilde{k}_1(x_1, y_1), \ldots, \tilde{k}_n(x_1, y_1, \ldots, x_n, y_n)) \quad \text{almost everywhere,} \end{aligned}$$

and let γ_n, δ_n be random variables defined by

$$\begin{aligned} \gamma_n &= \alpha_n(f_1, g_1, \ldots, f_n) \\ \delta_n &= \beta_n(f_1, g_1, \ldots, f_n, g_n). \end{aligned}$$

If A is any Borel subset of $[0,1]^{2n-2}$, and C is a Borel subset of $[0,1]$, then since f_n and g_n have the same law with respect to $\mathcal{F}_{n-1}$,

$$\begin{aligned} &\lambda((x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in A \text{ and } \tilde{\phi}_n(x_1, y_1, \ldots, x_n, y_n) \in C) \\ &= \Pr((\gamma_1, \delta_1, \ldots, \gamma_{n-1}, \delta_{n-1}) \in A \text{ and } r_n(\gamma_1, \delta_1, \ldots, \gamma_{n-1}, \delta_{n-1}, g_n) \in C) \\ &= \Pr((\gamma_1, \delta_1, \ldots, \gamma_{n-1}, \delta_{n-1}) \in A \text{ and } r_n(\gamma_1, \delta_1, \ldots, \gamma_{n-1}, \delta_{n-1}, f_n) \in C) \\ &= \lambda((x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in A \text{ and } r_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, \tilde{h}_n(x_1, y_1, \ldots, x_n)) \in C) \\ &= \lambda((x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in A \text{ and } x_n \in C) \\ &= \lambda(A)\lambda(C). \end{aligned}$$

Thus, by Lemma 4.2, we have that $\tilde{\phi}(x_1, y_1, \dots, x_{n-1}, y_{n-1}, \cdot, \cdot)$ is a measure preserving map from $[0,1]^2$ to $[0,1]$ for almost every $x_1, y_1, \dots, x_{n-1}, y_{n-1}$.

Now let $x \mapsto (\theta_1(x), \theta_2(x))$ be any Borel measurable measure preserving map from $[0,1]$ to $[0,1]^2$. Let

$$\begin{aligned} h_n(x_1, \dots, x_n) &= \tilde{h}_n(\theta_1(x_1), \theta_2(x_1), \dots, \theta_1(x_n)) \\ k_n(x_1, \dots, x_n) &= \tilde{k}_n(\theta_1(x_1), \theta_2(x_1), \dots, \theta_1(x_n), \theta_2(x_n)). \end{aligned}$$

Notice that

$$k_n(x_1, \dots, x_n) = h_n(x_1, \dots, x_{n-1}, \phi_n(x_1, \dots, x_n)),$$

where

$$\phi_n(x_1, \dots, x_n) = \tilde{\phi}_n(\theta_1(x_1), \theta_2(x_1), \dots, \theta_1(x_n), \theta_2(x_n)).$$

The last part of the theorem follows by exactly the same argument as for Theorem 3.1.

Q.E.D.

5. Representation of sequences satisfying condition (C.I.)

The following definition may be found in [KW]. Let $(\mathcal{F}_n)_{n \ge 0}$ be an increasing sequence of sigma fields on $(\Omega, \mathcal{F}, \Pr)$, where $\mathcal{F}_0$ is the trivial sigma field. An adapted sequence (f_n) taking values in a measurable space $(S, \mathcal{S})$ is said to satisfy *condition (C.I.)* if there exists a sigma field $\mathcal{G} \subset \mathcal{F}$ such that the law of f_n conditionally on $\mathcal{F}_{n-1}$ is the same as the law of f_n conditionally on $\mathcal{G}$, that is, $\mathbb{E}(I_{(f_n \in A)} | \mathcal{F}_{n-1}) = \mathbb{E}(I_{(f_n \in A)} | \mathcal{G})$ for any $A \in \mathcal{S}$, and if the sequence (f_n) is conditionally independent with respect to $\mathcal{G}$, that is, for any sequence of sets $A_n \in \mathcal{S}$ we have

$$\mathbb{E}(I_{(f_1 \in A_1)} \cdots I_{(f_n \in A_n)} | \mathcal{G}) = \mathbb{E}(I_{(f_1 \in A_1)} | \mathcal{G}) \cdots \mathbb{E}(I_{(f_n \in A_n)} | \mathcal{G}).$$

It is shown in [KW] that given any sequence (f_n) adapted to some filtration, that after possibly enlarging the underlying probability space, that there exists a sequence $(\tilde{f}_n)$ that is tangent to (f_n) and that satisfies condition (C.I.).

Indeed, using Theorem 2.1, we can show a technically weaker but essentially identical result as follows. Let (g_n) be the sequence constructed by Theorem 2.1. Enlarge the probability space $[0,1]^{\mathbb{N}}$ to $[0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$, and define the sequences

$$\begin{aligned} u_n((x_n), (y_n)) &= g_n(x_1, \dots, x_{n-1}, x_n) \\ v_n((x_n), (y_n)) &= g_n(x_1, \dots, x_{n-1}, y_n). \end{aligned}$$

Then (u_n) has the same law as (f_n), and (v_n) is tangent to (u_n) with respect to the filtration $(\mathcal{L}_n \otimes \mathcal{L}_0)$, and (v_n) satisfies condition (C.I.). (Here $\mathcal{L}_n$ is the minimal complete sigma field on $[0,1]^{\mathbb{N}}$ for which the first n coordinate functions are measurable. In particular, $\mathcal{L}_0$ denotes the trivial complete measure sigma field on $[0,1]^{\mathbb{N}}$.)

This is the motivation behind the next result, which shows that this previous construction is the canonical representation.

Theorem 5.1. *Let $(S, \mathcal{S})$ be a separable measurable space. Let (f_n) be an S-valued measurable sequence adapted to $(\mathcal{F}_n)_{n\geq 0}$ that satisfies condition (C.I.). Then there exists a sequence $(h_n : [0,1]^n \to S)$ such that if we define the functions u_n on $[0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$ by*

$$u_n((x_n),(y_n)) = h_n(x_1,\ldots,x_{n-1},y_n),$$

then the laws of (f_n), (h_n) and (u_n) are the same.

Further, if S is a separable Banach space, and (f_n) is a martingale difference sequence with respect to $\mathcal{F}_n$, then

$$\int_0^1 h_n(x_1,\ldots,x_n)\,dx_n = 0$$

for almost all $x_1,\ldots,x_n$.

Proof: As usual, we suppose without loss of generality that f_n is nowhere constant with respect to $\sigma(f_1,\ldots,f_{n-1})$. Let (h_n) be the sequence generated by Theorem 2.1. Then for any $A_1,\ldots,A_n \in \mathcal{S}$ we have that

$$\begin{aligned}
&\Pr((f_1,\ldots,f_n) \in A_1 \times \cdots \times A_n) \\
&= \mathbb{E}(\mathbb{E}(I_{(f_1\in A_1)} \cdots I_{(f_n\in A_n)}|\mathcal{G})) \\
&= \mathbb{E}(\mathbb{E}(I_{(f_1\in A_1)}|\mathcal{G}) \cdots \mathbb{E}(I_{(f_n\in A_n)}|\mathcal{G})) \\
&= \mathbb{E}(\mathbb{E}(I_{(f_1\in A_1)}|\mathcal{F}_0) \cdots \mathbb{E}(I_{(f_n\in A_n)}|\mathcal{F}_{n-1})).
\end{aligned}$$

Now for each $k = 1,\ldots,n$, let $F_k : S^{k-1} \to [0,1]$ be the measurable function such that

$$\mathbb{E}(I_{(f_k\in A_k)}|\mathcal{F}_{k-1}) = F_k(f_1,\ldots,f_{k-1}) \quad \text{almost surely.}$$

Notice that if $G : S^{n-1} \to \mathbb{R}$ is bounded measurable, then

$$\mathbb{E}(I_{(f_k\in A_k)}G(f_1,\ldots,f_{k-1})) = \mathbb{E}(F_k(f_1,\ldots,f_{k-1})G(f_1,\ldots,f_{k-1})),$$

and hence

$$\mathbb{E}(I_{(h_k\in A_k)}G(h_1,\ldots,h_{k-1})) = \mathbb{E}(F_k(h_1,\ldots,h_{k-1})G(h_1,\ldots,h_{k-1})).$$

Since $\sigma(h_1,\ldots,h_{k-1}) = \mathcal{L}_{k-1}$, we see that

$$\mathbb{E}(I_{(h_k\in A_k)}|\mathcal{L}_{k-1}) = F_k(h_1,\ldots,h_{k-1}) \quad \text{almost everywhere.}$$

Thus the sequence

$$(\mathbb{E}(I_{(f_k\in A_k)}|\mathcal{F}_{k-1}))_{1\leq k\leq n}$$

has the same law as the sequence

$$(\mathbb{E}(I_{(h_k\in A_k)}|\mathcal{L}_{k-1}))_{1\leq k\leq n},$$

and this last sequence is very easily seen to have the same law as the sequence

$$(\mathbb{E}(I_{(u_k \in A_k)} | \mathcal{L}_{k-1} \otimes \mathcal{L}_0))_{1 \le k \le n}.$$

Hence

$$\begin{aligned}
&\mathbb{E}(\mathbb{E}(I_{(f_1 \in A_1)} | \mathcal{F}_0) \cdots \mathbb{E}(I_{(f_n \in A_n)} | \mathcal{F}_{n-1})) \\
&= \int (\mathbb{E}(I_{(u_1 \in A_1)} | \mathcal{L}_0 \otimes \mathcal{L}_0) \cdots \mathbb{E} I_{(u_n \in A_n)} | \mathcal{L}_{n-1} \otimes \mathcal{L}_0) \, d\lambda \otimes \lambda,
\end{aligned}$$

and this last quantity is easily computed to be

$$\lambda \otimes \lambda((u_1, \ldots, u_n) \in A_1 \times \cdots \times A_n).$$

Finally, the last part of the theorem follows exactly as in Theorem 3.1.

Q.E.D.

6. Tangent sequences in UMD spaces

The concept of UMD spaces was introduced by Aldous [A], and extensively explored by Burkholder [Bu]. It is from this second reference that the following definition of UMD (along with many other equivalent definitions) may be found.

A Banach space X is said to be *UMD* if for some $1 < p < \infty$ (equivalently all $1 < p < \infty$), there exists a positive constant c depending only upon X and p such that if (d_n) is a Bochner integrable X-valued martingale difference sequence, then for any signs $\epsilon_n = \pm 1$, we have that

$$\left\| \sum_{n=1}^{N} \epsilon_n d_n \right\|_{L_p(X)} \le c \left\| \sum_{n=1}^{N} d_n \right\|_{L_p(X)}.$$

This section is devoted to providing a new proof of the following result. This is essentially the same as the first part of Theorem 2.2 from the paper by McConnell [M2]. While the following proof can be rewritten so as to avoid needing the representation theorems, to do so would be to remove the motivation behind this proof.

Theorem 6.1. *Let X be a UMD space. Then given $1 < p < \infty$, there is a positive constant C, depending only upon X and p, such that given a filtration $(\mathcal{F}_n)_{n \ge 0}$, and two martingales (f_n) and (g_n) adapted to this filtration that are X-valued Bochner integrable and tangent to each other, we have that*

$$\left\| \sum_{n=1}^{N} g_n \right\|_{L_p(X)} \le C \left\| \sum_{n=1}^{N} f_n \right\|_{L_p(X)}.$$

Proof: It is sufficient to prove the inequality in the case when one of the sequences satisfies condition (C.I.), since the inequality may be then obtained by comparing both sides with another sequence tangent to them both that satisfies condition (C.I.). We will prove it in the case when the sequence (g_n) satisfies condition (C.I.), but it will be evident that the proof also works when the sequence (f_n) satisfies condition (C.I.).

So applying Theorems 4.1 and 5.1, we see that there exists a sequence $h_n : [0,1]^n \to X$ of X-valued Bochner integrable functions such that

$$\int_0^1 h_n(x_1, \ldots, x_n)\, dx_n = 0$$

for almost all $x_1, \ldots, x_{n-1}$, and such that if we define the the functions $d_n, e_n : [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} \to X$ by

$$\begin{aligned} d_n((x_n),(y_n)) &= h_n(x_1, \ldots, x_{n-1}, x_n) \\ e_n((x_n),(y_n)) &= h_n(x_1, \ldots, x_{n-1}, y_n), \end{aligned}$$

then the sequence $(f_1, g_1, f_2, g_2, \ldots)$ has the same law as the sequence $(d_1, e_1, d_2, e_2, \ldots)$.

Now we define a martingale difference sequence r_n on $[0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$ by $r_{2n-1} = d_n + e_n$ and $r_{2n} = d_n - e_n$, where the filtration is $\mathcal{G}_{2n} = \mathcal{L}_n \otimes \mathcal{L}_n$, and $\mathcal{G}_{2n-1}$ is the collection of those sets in $\mathcal{G}_{2n}$ generated by sets like

$$\{((x_n),(y_n)) : (x_1, \ldots, x_n) \in A \times C,\ (y_1, \ldots, y_n) \in B \times C\},$$

where $A, B \in \mathcal{L}_{n-1}$, and $C \in \mathcal{L}_1$.

Then we see that $\sum_{n=1}^N d_n = \frac{1}{2} \sum_{n=1}^{2N} r_n$ and $\sum_{n=1}^N e_n = \frac{1}{2} \sum_{n=1}^{2N} (-1)^{n+1} r_n$, and thus the result follows immediately from the fact that X is a UMD space.

Q.E.D.

7. Representations of martingales — 2

In this section, we present a version of a result of McConnell [M1]. This result provides a Skorohod like representation for vector valued martingales. Our result is weaker than that of McConnell in that his process is adapted to a filtration generated by a one-dimensional Brownian motion. The process presented here is adapted to a two dimensional Brownian motion.

Theorem 7.1. *Let (d_n) be a Bochner integrable martingale difference sequence taking values in a Banach space X. Then there exists a continuous time X-valued stochastic process $(F_t)_{t\ge 0}$ with continuous sample paths such that (d_n) has the same law as $(F_n - F_{n-1})$. Furthermore, the process (F_t) is adapted to a two dimensional brownian motion.*

Proof: Let $D = \{z \in \mathbb{C} : |z| < 1\}$, $\bar{D} = \{z \in \mathbb{C} : |z| \le 1\}$ and $\partial D = \{z \in \mathbb{C} : |z| = 1\}$, and give ∂D normalized Lebesgue measure $d|z|/2\pi$. Since ∂D is measure equivalent to $[0,1]$, we may apply Theorem 3.1 to obtain a sequence $(e_n : \partial D^n \to X)$ that has the same law as (f_n), and such that

$$\int_{\partial D} e_n(z_1,\dots,z_n)\,\frac{d|z_n|}{2\pi} = 0$$

for almost every $z_1,\dots,z_{n-1}$.

For each $z_1,\dots,z_{n-1}$, extend the function $z_n \mapsto e_n(z_1,\dots,z_n)$ to its harmonic extension on $\bar{D}$. We will identify e_n with this extension.

Now let $(b_t^{(n)})_{n\in\mathbb{N}}$ be a sequence of independent Brownian motions into $\mathbb{C}$ with origin at zero. Let τ_n be the stopping time $\tau_n = \inf\{t : |b_t^{(n)}| \ge 1\}$. Notice that τ_n is finite almost surely.

By the Itô calculus, we have that

$$e_n(b_{\tau_1}^{(1)},\dots,b_{\min\{t,\tau_n\}}^{(n)}) = \int_0^{\min\{t,\tau_n\}} \nabla_n e_n(b_{\tau_1}^{(1)},\dots,b_{\tau_{n-1}}^{(n-1)},b_s^{(n)})\cdot db_s^{(n)},$$

where ∇_n refers to taking the gradient with respect to the nth coordinate. Define a process

$$G_{n,t} = \left(\sum_{m=1}^{n-1} e_m(b_{\tau_1}^{(1)},\dots,b_{\tau_m}^{(m)})\right) + e_n(b_{\tau_1}^{(1)},\dots,b_{\min\{t,\tau_n\}}^{(n)}).$$

Let $\phi : [0,1) \to [0,\infty)$ be defined by $\phi(t) = t/(1-t)$, and let

$$F_t = G_{[t+1],\phi(t-[t])},$$

where $[t]$ represents the largest integer less than t.

To see that (F_t) may be adapted to a two-dimensional brownian motion, let $\tilde{b}_t$ be a brownian motion taking values in $\mathbb{C}$, and set

$$b_t^{(n)} = \int_{n-1}^{n-1+\phi^{-1}(t)} \sqrt{\phi'(s)}\,d\tilde{b}_s.$$

Q.E.D.

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