The Gaussian Cotype of Operators from $C(K)$

S.J. Montgomery-Smith

Abstract

We show that the canonical embedding $C(K) \to L_\Phi(\mu)$ has Gaussian cotype p, where μ is a Radon probabilty measure on K, and Φ is an Orlicz function equivalent to $t^p (\log t)^{\frac{p}{2}}$ for large t.

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In [6], I showed that the Gaussian cotype 2 constant of the canonical embedding $l_\infty^N \to L_{2,1}^N$ is bounded by $\log\log N$. Talagrand [9] showed that this embedding does not have uniformly bounded cotype 2 constant. In fact, a careful study of his proof yields that the cotype 2 constant is bounded below by $\sqrt{\log\log N}$. In this paper, we will show that this is the correct value for the Gaussian cotype 2 constant of this operator. However, we will show this via a different result, which we will give presently. First, let us define our terms.

We will write Φ_p for an Orlicz function such that $\Phi_p(t) \approx t^p (\log t)^{\frac{p}{2}}$ for large t.

For any bounded linear operator $T : X \to Y$, where X and Y are Banach spaces, and any $2 \le p < \infty$, we say that T has *Gaussian cotype* p if there is a number $C < \infty$ such that for all sequences $x_1,\, x_2, \ldots \in X$ we have

$$\mathbf{E} \left\| \sum_{s=1}^\infty \gamma_s x_s \right\| \ge C^{-1} \left(\sum_{s=1}^\infty \|T x_s\|^p \right)^{\frac{1}{p}} .$$

(Here, as elsewhere, $\gamma_1,\, \gamma_2, \ldots$ denote independent $N(0,1)$ Gaussian random variables.) We call the least value of C the *Gaussian cotype* p *constant* of T, and denote it by $\beta^{(p)}(T)$.

Throughout this paper, we shall use the letter c to denote a positive finite constant, whose value may change with each occurence. We shall write $A \approx B$ to mean $A \le c\, B$ and $B \le c\, A$.

Theorem 1. *Let μ be a Radon probability measure on a compact Hausdorff topological space K, and let $2 \le p < \infty$. Then the canonical embedding $C(K) \to L_{\Phi_p}(\mu)$ has Gaussian cotype p.*

Finding the Gaussian cotype p constant of an operator from $C(K)$ involves finding lower bounds for the quantity $\mathbf{E} \left\| \sum_{s=1}^\infty \gamma_s x_s \right\|_\infty$, where $x_1,\, x_2, \ldots \in C(K)$. In fact, since we really only need to consider finite sequences $x_1, x_2, \dots, x_S \in C(K)$, in order to prove Theorem 1, it is sufficient to show that the Gaussian cotype p constant of the canonical embedding $C(K) \to L_{\Phi_p}(\mu)$ is uniformly bounded over all *finite* K. Now we see that we are trying to find lower bounds for the supremum of the finite Gaussian process, $\sup_{\omega\in K} |\Gamma_\omega|$, where $\Gamma_\omega = \sum_{s=1}^\infty \gamma_s x_s(\omega)$. Hence we can apply the following result due to Talagrand [8].

Theorem 2. *Let* $(\Gamma_\omega : \omega \in K)$ *be a finite Gaussian process.*

i) Let

$$V_1 = \mathbf{E}\left(\sup_{\omega\in K} |\Gamma_\omega|\right).$$

ii) Let V_2 *be the infimum of*

$$\left(\sup_{t\ge 1} \sqrt{1+\log t}\left(\mathbf{E}\,|Y_t|^2\right)^{\frac{1}{2}}\right)\left(\sup_{\omega\in K}\sum_{t=1}^\infty |\alpha_t(\omega)|\right)$$

over all Gaussian processes $(Y_t)_{t=1}^\infty$ *and over all sequences* $(\alpha_t)_{t=1}^\infty$ *of functions on* K *such that* $\Gamma_\omega = \sum_{t=1}^\infty \alpha_t(\omega) Y_t$.

Then $V_1 \approx V_2$.

We can rewrite this corollary in the following way. First, let us define the following spaces (here we are assuming K is finite).

$$\mathcal{G} = \left\{ \big(x_s \in C(K)\big)_{s=1}^\infty : \|(x_s)\|_{\mathcal{G}} = \left\|\sum_{s=1}^\infty \gamma_s x_s\right\|_\infty < \infty \right\},$$

$$C(K,l_1) = \left\{ \big(\alpha_t \in C(K)\big)_{t=1}^\infty : \|(\alpha_t)\|_{C(K,l_1)} = \left\|\sum_{t=1}^\infty |\alpha_t|\right\|_\infty < \infty \right\},$$

$$\mathcal{Y} = \left\{ \big(y_t \in l_2\big)_{t=1}^\infty : \|(y_t)\|_{\mathcal{Y}} = \sup_{t\ge 1}\sqrt{1+\log t}\,\|y_t\|_2 < \infty \right\}.$$

Let $m : C(K,l_1) \times \mathcal{Y} \to \mathcal{G}$ be the bilinear map $m\big((\alpha_t),(y_t)\big) = (x_s)$, where

$$x_s = \sum_{t=1}^\infty y_t(s)\alpha_t.$$

Corollary 3. *The map* m *has the following two properties:*

i) m *is bounded;*

ii) m is open, that is, if $\|(x_s)\|_{\mathcal{G}} \leq 1$, then there are $\|(\alpha_t)\|_{C(K,l_1)} \leq c$ and $\|(y_t)\|_{\mathcal{Y}} \leq c$ such that $m\big((\alpha_t),(y_t)\big) = (x_s)$.

Proof: This is just restating Theorem 2, setting $\Gamma_\omega = \sum_{s=1}^\infty \gamma_s x_s(\omega)$, and $Y_t = \sum_{s=1}^\infty \gamma_s y_t(s)$. □

From this we obtain the following corollary, for which we first give a definition.

Definition. *If $2 \leq p < \infty$, and $T : C(K) \to Y$ is a bounded linear map, where K is a finite Hausdorff space, and Y is a Banach space, then we set*

$$H^{(p)}(T) = \sup\left\{ \left(\sum_{s=1}^\infty \|Tx_s\|^p \right)^{\frac{1}{p}} \right\},$$

where the supremum is over all $x_s = \sum_{t=1}^\infty y_t(s)\alpha_t$, with $\alpha_1,\ \alpha_2, \dots$ pairwise disjoint elements of the unit ball of $C(K)$, and $\|(y_t)\|_2 \leq \frac{1}{\sqrt{1+\log t}}$ for each $t \geq 1$.

Corollary 4. *For any $2 \leq p < \infty$, and for any bounded linear operator $T : C(K) \to Y$, where K is a finite Hausdorff space, and Y is a Banach space, we have*

$$H^{(p)}(T) \approx \beta^{(p)}(T).$$

Proof: This follows straight away from Corollary 3 and the following lemma.

Lemma 5. *Let B be the set of $(\alpha_t) \in C(K, l_1)$ such that the α_t are pairwise disjoint elements of the unit ball of $C(K)$. Then the closed convex hull of B is the unit ball of $C(K, l_1)$.*

Proof: See [5], Lemma 4 or [3], Proposition 14.4. □

Now we are almost in a position to prove Theorem 1; we just need the following properties of $L_{\Phi_p}(\mu)$.

Lemma 6. *If μ is a Radon probability measure on a compact Hausdorff space K, then*

i) for any Borel subset I of K, we have $\|\chi_I\|_{\Phi_p} \approx \big(\mu(I)\big)^{\frac{1}{p}} \sqrt{\log \frac{1}{\mu(I)}}$;

ii) the space L_{Φ_p} satisfies an upper p estimate.

Proof of Theorem 1: We want to show that $H^{(p)}\big(C(K) \to L_{\Phi_p}(\mu)\big) \leq c$, where μ is a probability measure on a finite Hausdorff space K. So consider (x_s), (α_t) and (y_t) as given in the definition of $H^{(p)}(T)$. Then we need to show that

$$\sum_{s=1}^{\infty} \|x_s\|_{\Phi_p}^p \le c.$$

First note, by Lemma 6, that

$$\begin{aligned}\|x_s\|_{\Phi_p}^p &\le c \sum_{t=1}^{\infty} y_t(s)^p \, \|a_t\|_{\Phi_p}^p \\ &\le c \sum_{t=1}^{\infty} y_t(s)^p \mu(I_t) \left(\log \frac{1}{\mu(I_t)}\right)^{\frac{p}{2}},\end{aligned}$$

where I_t is the support of α_t. Hence

$$\begin{aligned}\sum_{s=1}^{\infty} \|x_s\|_{\Phi_p}^p &\le c \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} y_t(s)^p \mu(I_t) \left(\log \frac{1}{\mu(I_t)}\right)^{\frac{p}{2}} \\ &\le c \sum_{t=1}^{\infty} \frac{1}{(1+\log t)^{\frac{p}{2}}} \mu(I_t) \left(\log \frac{1}{\mu(I_t)}\right)^{\frac{p}{2}},\end{aligned}$$

since $\|y_t\|_p \le \|y_t\|_2 \le \frac{1}{\sqrt{1+\log t}}$. But now, splitting the sum into the two cases $\mu(I_t) \ge \frac{1}{t^2}$ or $\mu(I_t) < \frac{1}{t^2}$, we deduce that this sum is bounded by some universal constant. □

Concluding Remarks

We first note that there is a nice way to calculate the Orlicz norms $\|\cdot\|_{\Phi_p}$ provided by the following result of Bennett and Rudnick.

Theorem 7. *If $1 \le p < \infty$ and $a \in \mathbf{R}$, then the Orlicz probability norm given by the function $\Theta(t) \approx t^p (\log t)^a$ (t large) is equivalent to the norm*

$$\|x\| = \left(\int_0^1 (1+\log \tfrac{1}{t})^a x^*(t)^p \, dt\right)^{\frac{1}{p}},$$

where x^* is the non-increasing rearrangement of $|x|$.

Proof: See [1], Theorem D. □

Thus we can now deduce the following result.

Theorem 8. *The Gaussian cotype 2 constant of the canonical embedding $l_\infty^N \to L_{2,1}^N$ is bounded by $\sqrt{\log\log N}$.*

Proof: Let $K = \{1, 2, \dots, N\}$, and let μ be the measure $\mu(A) = \frac{|A|}{N}$. Now notice that if $x \in l_\infty^N = C(K)$, then $x^*(t)$ is constant over $0 \le t \le \frac{1}{N}$, and hence

$$\begin{aligned}
\|x\|_{L_{2,1}^N} &= \frac{1}{2}\int_0^1 \frac{x^*(t)}{\sqrt{t}}\,dt \\
&= \frac{x^*(1/N)}{\sqrt{N}} + \frac{1}{2}\int_{\frac{1}{N}}^1 \frac{x^*(t)}{\sqrt{t}}\,dt \\
&\le \left(\int_0^{\frac{1}{N}} (1+\log\tfrac{1}{t}) x^*(t)^2\,dt\right)^{\frac{1}{2}} + \frac{1}{2}\left(\int_{\frac{1}{N}}^1 \frac{1}{t(1+\log\frac{1}{t})}\,dt\right)^{\frac{1}{2}} \left(\int_{\frac{1}{N}}^1 (1+\log\tfrac{1}{t}) x^*(t)^2\,dt\right)^{\frac{1}{2}} \\
&\le c\sqrt{\log\log N}\;\|x\|_{\Phi_2}.
\end{aligned}$$

This is sufficient to prove the result. □

An obvious question is the following.

Problem 9. *Is there a rearrangement invariant norm $\|\cdot\|_X$ on $[0,1]$ which is strictly larger than $\|\cdot\|_{\Phi_p}$, but for which the canonical embedding $C(K) \to X(\mu)$ has Gaussian cotype p?*

For $p > 2$, the answer is yes. The embedding $C(K) \to L_{p,1}(\mu)$ has cotype p (this follows from results in [2]). Hence $X = L_{\Phi_p} \cap L_{p,1}$ equipped with the norm $\|x\| = \max\{\|x\|_{\Phi_p}, \|x\|_{p,1}\}$ provides the counterexample.

For $p = 2$, the answer is no. Talagrand [10] has recently shown that if $C[0,1] \to X$ has Gaussian cotype 2, then $\|\cdot\|_X$ is bounded by a constant times $\|\cdot\|_{\Phi_2}$.

Another problem is also suggested by Theorem 1.

Problem 10. *If $T : C(K) \to X$ is a linear map with Gaussian cotpye 2, does it follow that there is a Radon probability measure μ on K such that $\|Tx\| \le c\,\|x\|_{L_{\Phi_2}(\mu)}$ for $x \in C(K)$?*

Talagrand [10] has recently shown that this not the case.

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S.J. Montgomery-Smith,
Department of Mathematics,
University of Missouri at Columbia,
Columbia, Missouri 65211,
U.S.A.