p-Summing Operators on Injective Tensor Products of Spaces

by

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Abstract Let $X, Y$ and $Z$ be Banach spaces, and let $\prod_p(Y, Z)$ $(1 \leq p < \infty)$ denote the space of $p$-summing operators from $Y$ to $Z$. We show that, if $X$ is a $\mathcal{L}_\infty$-space, then a bounded linear operator $T : X \hat{\otimes} \varepsilon Y \rightarrow Z$ is 1-summing if and only if a naturally associated operator $T^\# : X \rightarrow \prod_1(Y, Z)$ is 1-summing. This result need not be true if $X$ is not a $\mathcal{L}_\infty$-space. For $p > 1$, several examples are given with $X = C[0, 1]$ to show that $T^\#$ can be $p$-summing without $T$ being $p$-summing. Indeed, there is an operator $T$ on $C[0, 1] \hat{\otimes} \varepsilon \ell_1$ whose associated operator $T^\#$ is 2-summing, but for all $N \in \mathbb{N}$, there exists an $N$-dimensional subspace $U$ of $C[0, 1] \hat{\otimes} \varepsilon \ell_1$ such that $T$ restricted to $U$ is equivalent to the identity operator on $\ell_\infty^N$. Finally, we show that there is a compact Hausdorff space $K$ and a bounded linear operator $T : C(K) \hat{\otimes} \varepsilon \ell_1 \rightarrow \ell_2$ for which $T^\# : C(K) \rightarrow \prod_1(\ell_1, \ell_2)$ is not 2-summing.

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Introduction Let $X$ and $Y$ be Banach spaces, and let $X \hat{\otimes} Y$ denote their injective tensor product. In this paper, we shall study the behavior of those operators on $X \hat{\otimes} Y$ that are $p$-summing.

If $X$, $Y$ and $Z$ are Banach spaces, then every $p$-summing operator $T : X \hat{\otimes} Y \to Z$ induces a $p$-summing linear operator $T^\# : X \to \prod_p(Y, Z)$. This raises the following question: given two Banach spaces $Y$ and $Z$, and $1 \leq p < \infty$, for what Banach spaces $X$ is it true that a bounded linear operator $T : X \hat{\otimes} Y \to Z$ is $p$-summing whenever $T^\# : X \to \prod_p(Y, Z)$ is $p$-summing?

In [11], it was shown that whenever $X = C(\Omega)$ is a space of all continuous functions on a compact Hausdorff space $\Omega$, then $T : C(\Omega) \hat{\otimes} Y \to Z$ is $1$-summing if and only if $T^\# : C(\Omega) \to \prod_1(Y, Z)$ is $1$-summing. We will extend this result by showing that this result still remains true if $X$ is any $\mathcal{L}_\infty$-space. We will also give an example to show that the result need not be true if $X$ is not a $\mathcal{L}_\infty$-space. For this, we shall exhibit a $2$-summing operator $T$ on $\ell_2 \hat{\otimes} \ell_2$ that is not $1$-summing, but such that the associated operator $T^\#$ is $1$-summing.

The case $p > 1$ turns out to be quite different. Here, the $\mathcal{L}_\infty$-spaces do not seem to play any important role. We show that for each $1 < p < \infty$, there exists a bounded linear operator $T : C[0, 1] \hat{\otimes} \ell_2 \to \ell_2$ such that $T^\# : C[0, 1] \to \prod_p(\ell_2, \ell_2)$ is $p$-summing, but such that $T$ is not $p$-summing. We will also give an example that shows that, in general, the condition on $T^\#$ to be $2$-summing is too weak to imply any good properties for the operator $T$ at all. To illustrate this, we shall exhibit a bounded linear operator $T$ on $C[0, 1] \hat{\otimes} \ell_1$ with values in a certain Banach space $Z$, such that $T^\# : C[0, 1] \to \prod_2(\ell_1, Z)$ is $2$-summing, but for any given $N \in \mathbb{N}$, there exists a subspace $U$ of $C[0, 1] \hat{\otimes} \ell_1$, with $\dim U = N$, such that $T$ restricted to $U$ is equivalent to the identity operator on $\ell_\infty^N$.

Finally, we show that there is a compact Hausdorff space $K$ and a bounded linear operator $T : C(K) \hat{\otimes} \ell_1 \to \ell_2$ for which $T^\# : C(K) \to \prod_1(\ell_1, \ell_2)$ is not $2$-summing.
I - Definitions and Preliminaries

Let $E$ and $F$ be Banach spaces, and let $1 \leq q \leq p < \infty$. An operator $T : E \to F$ is said to be $(p, q)$-summing if there exists a constant $C \geq 0$ such that for any finite sequence $e_1, e_2, \ldots, e_n$ in $E$, we have

$$\left( \sum_{i=1}^{n} \| T(e_i) \|^p \right)^{\frac{1}{p}} \leq C \sup \left\{ \left( \sum_{i=1}^{n} |e^*(e_i)|^q \right)^{\frac{1}{q}} : e^* \in E^*, \| e^* \| \leq 1 \right\}.$$ 

We let $\pi_{p, q}(T)$ denote the smallest constant $C$ such that the above inequality holds, and let $\prod_{p, q}(E, F)$ be the space of all $(p, q)$-summing operators from $E$ to $F$ with the norm $\pi_{p, q}$. It is easy to check that $\prod_{p, q}(E, F)$ is a Banach space. In the case $p = q$, we will simply write $\prod_{p}(E, F)$ and $\pi_{p}$. We will use the fact that $T \in \prod_{p, q}(E, F)$ if and only if $\sum_{n} \| Te_n \|^p < \infty$ for every infinite sequence $(e_n)$ in $E$ with $\sum_{n} |e^*(e_n)|^q < \infty$ for each $e^* \in E^*$. That is to say, $T$ is in $\prod_{p, q}(E, F)$ if and only if $T$ sends all weakly $\ell^q$-summable sequences into strongly $\ell^p$-summable sequences. In what follows we shall mainly be interested in the case where $p = q$ and $p = 1$ or 2.

Given two Banach spaces $E$ and $F$, we will let $E\hat{\otimes}_\epsilon F$ denote their injective tensor product, that is, the completion of the algebraic tensor product $E \otimes F$ under the cross norm $\| \cdot \|_\epsilon$ given by the following formula. If $\sum_{i=1}^{n} e_i \otimes x_i \in E \otimes F$, then

$$\| \sum_{i=1}^{n} e_i \otimes x_i \|_\epsilon = \sup \left\{ \left| \sum_{i=1}^{n} e^*(e_i) x^*(x_i) \right| : \| e^* \| \leq 1, \| x^* \| \leq 1, e^* \in E^*, x^* \in F^* \right\}.$$ 

We will say that a bounded linear operator $T$ between two Banach spaces $E$ and $F$ is called an integral operator if the bilinear form $\tau$ defines an element of $(E\hat{\otimes}_\epsilon F^*)^*$, where $\tau$ is induced by $T$ according to the formula $\tau(e, x^*) = x^*(Te) \ (e \in E, x^* \in F^*)$. We will define the integral norm of $T$, denoted by $\| T \|_{\text{int}}$, by

$$\| T \|_{\text{int}} = \sup \left\{ \left| \sum_{i=1}^{n} x^*_i(Te_i) \right| : \sum_{i=1}^{n} e_i \otimes x_i^* \|_\epsilon \leq 1 \right\}.$$
The space of all integral operators from a Banach space $E$ into a Banach space $F$ will be denoted by $I(E, F)$. We note that $I(E, F)$ is a Banach space under the integral norm $\| \|_{\text{int}}$.

We will say that a Banach space $X$ is a $\mathcal{L}_\infty$-space if, for some $\lambda > 1$, we have that for every finite dimensional subspace $B$ of $X$, there exists a finite dimensional subspace $E$ of $X$ containing $B$, and an invertible bounded linear operator $T : E \rightarrow \ell_\infty^{\dim E}$ such that $\| T \| \| T^{-1} \| \leq \lambda$.

It is well known that for any Banach spaces $E$ and $F$, if $T$ is in $I(E, F)$, then it is also in $\prod_1(E, F)$, with $\pi_1(T) \leq \| T \|_{\text{int}}$. But $I(E, F)$ is strictly included in $\prod_1(E, F)$. It was shown in [12, p. 477] that a Banach space $E$ is a $\mathcal{L}_\infty$-space if and only if for any Banach space $F$, we have that $I(E, F) = \prod_1(E, F)$. We will use this characterization of $\mathcal{L}_\infty$-spaces in the sequel.

Finally, we note the following characterization of 1-summing operators (called right semi-integral by Grothendieck in [5]), which will be used later.

**Proposition 1** Let $E$ and $F$ be Banach spaces. Then the following properties about a bounded linear operator $T$ from $E$ to $F$ are equivalent:

(i) $T$ is 1-summing;

(ii) There exists a Banach space $F_1$, and an isometric injection $\varphi : F \rightarrow F_1$, such that $\varphi \circ T : E \rightarrow F_1$ is an integral operator.

For all other undefined notions we shall refer the reader to either [3], [7] or [10].
II 1-Summing and Integral Operators

Let \( X \) and \( Y \) be Banach spaces with injective tensor product \( X \hat{\otimes} Y \). For a Banach space \( Z \), any bounded linear operator \( T : X \hat{\otimes} Y \to Z \) induces a linear operator \( T^# \) on \( X \) by

\[
T^#x(y) = T(x \otimes y) \quad (y \in Y).
\]

It is clear that the range of \( T^# \) is the space \( \mathcal{L}(Y, Z) \) of bounded linear operators from \( Y \) into \( Z \), and that \( T^# \) is a bounded linear operator.

In this section, we are going to investigate the 1-summing operators, and the integral operators, on \( X \hat{\otimes} Y \). We will use Proposition 1 to relate these two ideas together. First of all, we have the following result.

**Theorem 2** Let \( X, Y \) and \( Z \) be Banach spaces, and let \( T : X \hat{\otimes} Y \to Z \) be a bounded linear operator. Denote by \( i : Z \to Z^{**} \) the isometric embedding of \( Z \) into \( Z^{**} \). Then the following two properties are equivalent:

(i) \( T \in I(X \hat{\otimes} Y, Z) \);

(ii) \( \hat{i} \circ T \in I(X, I(Y, Z^{**})) \), where \( \hat{i} : I(Y, Z) \to I(Y, Z^{**}) \) is defined by \( \hat{i}(U) = i \circ U \) for each \( U \in I(Y, Z) \).

In particular, if \( T^# \in I(X, I(Y, Z)) \), then \( T \in I(X \hat{\otimes} Y, Z) \).

**Proof:** First, we show that \((X \otimes Y) \hat{\otimes} Z^* \) and \(X \hat{\otimes} (Y \otimes Z^*) \) are isometrically isomorphic to one another. To see this, note that the algebraic tensor product is an associative operation, that is, \((X \otimes Y) \otimes Z^* \) and \(X \otimes (Y \otimes Z^*) \) are algebraically isomorphic. Also, they are both generated by elements of the form \( \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^* \), where \( x_i \in X \), \( y_i \in Y \) and \( z_i^* \in Z^* \). Now, if we let \( B(X^*) \), \( B(Y^*) \) and \( B(Z^{**}) \) denote the dual unit balls of \( X^* \), \( Y^* \) and \( Z^{**} \) equipped with their respective weak* topologies, then the spaces \((X \otimes Y) \otimes Z^\ast\) and \(X \otimes (Y \otimes Z^*) \) embed isometrically into \( C(B(X^*) \times B(Y^*) \times B(Z^{**})) \) in a natural way, by

\[
\langle \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^*, (x^*, y^*, z^{**}) \rangle = \sum_{i=1}^{n} x^*(x_i)y^*(y_i)z^{**}(z_i^*),
\]
where \( \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^* \) is in \( (X \otimes \epsilon Y) \otimes \epsilon Z^* \) or \( X \otimes \epsilon \{ Y \otimes \epsilon Z^* \} \). Thus both spaces \((X \hat{\otimes}_\epsilon Y) \hat{\otimes}_\epsilon Z^* \) and \(X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z^*)\) can be thought of as the closure in \( C(B(X^*) \times B(Y^*) \times B(Z^{**}))\) of the algebraic tensor product of \(X, Y\) and \(Z^*\).

Now let us assume that \( T : X \hat{\otimes}_\epsilon Y \longrightarrow Z \) is an integral operator. Then the bilinear map \( \tau \) on \( X \hat{\otimes}_\epsilon Y \times Z^* \), given by \( \tau(u, z^*) = z^*(Tu) \) for \( u \in X \hat{\otimes}_\epsilon Y \) and \( z^* \in Z^* \), defines an element of \( (X \hat{\otimes}_\epsilon Y \hat{\otimes}_\epsilon Z^*)^* \), that is,

\[
(*) \quad \| T \|_{\text{int}} = \sup \left\{ \left\| \sum_{i=1}^{n} z_i^* (T(x_i \otimes y_i)) \right\| : \left\| \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^* \right\|_{\epsilon} \leq 1 \right\}.
\]

To show that for every \( x \) in \( X \) the operator \( T^#x \) is in \( I(Y, Z) \), with

\[
\| T^#x \|_{\text{int}} \leq \| x \| \| T \|_{\text{int}},
\]

is easy. This is because, for each \( x \in X \), the operator \( T^#x \) is the composition of \( T \) with the bounded linear operator from \( Y \) to \( X \hat{\otimes}_\epsilon Y \), which to each \( y \) in \( Y \) gives the element \( x \otimes y \).

If \( i : Z \longrightarrow Z^{**} \) denotes the isometric embedding of \( Z \) into \( Z^{**} \), it induces a bounded linear operator \( \hat{i} : I(Y, Z) \longrightarrow I(Y, Z^{**}) \) given by \( \hat{i}(U) = i \circ U \) for all \( U \in I(Y, Z) \). It is immediate that \( \hat{i} \) is an isometry. We will now show that the operator \( \hat{i} \circ T^# : X \longrightarrow I(Y, Z^{**}) \) is integral. It is well known (see [3, p. 237]) that the space \( I(Y, Z^{**}) \) is isometrically isomorphic to the dual space \( (Y \hat{\otimes}_\epsilon Z^*)^* \). Thus to show that \( \hat{i} \circ T^# : X \longrightarrow (Y \hat{\otimes}_\epsilon Z^*)^* \) is an integral operator, we need to show that it induces an element of \( (X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z^*))^* \). For this, it is enough to note that, by our discussion concerning the isometry of \((X \hat{\otimes}_\epsilon Y) \hat{\otimes}_\epsilon Z^* \) and \(X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z^*)\), that

\[
(**) \quad \| \hat{i} \circ T^# \|_{\text{int}} = \sup \left\{ \left\| \sum_{i=1}^{n} i \circ T^#x_i, y_i \otimes z_i^* \right\| : \left\| \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^* \right\|_{\epsilon} \leq 1 \right\}.
\]
But for each \( x \in X, \ y \in Y \) and \( z^* \in Z^* \), we have
\[
\langle \hat{i} \circ T^#x, y \otimes z^* \rangle = \langle T(x \otimes y), z^* \rangle.
\]
Hence, from (*) and (**), it follows that
\[
\| \hat{i} \circ T \|_{\text{int}} = \| T \|_{\text{int}}.
\]
Thus we have shown that (i) \( \Rightarrow \) (ii). The proof of (ii) \( \Rightarrow \) (i) follows in a similar way. If \( \hat{i} \circ T^# : X \to I(Y, Z^{**}) \) is an integral operator, then one can show that \( i \circ T : \hat{X} \otimes_\varepsilon Y \to Z^{**} \) is integral, which in turn implies that \( T \) itself is integral (see [3, p. 233]).

Finally, the last assertion follows easily, since if \( T^# : X \to I(Y, Z) \) is integral, then \( \hat{i} \circ T \) is integral (see [3, p. 232]).

Since the mapping \( \hat{i} : I(Y, Z) \to I(Y, Z^{**}) \) is an isometry, Proposition 1 coupled with Theorem 2 implies that, if \( T : \hat{X} \otimes_\varepsilon Y \to Z \) is an integral operator, then \( T^# : X \to I(Y, Z) \) is 1-summing. This result can be shown directly from the definitions. In what follows we shall present a sketch of that alternative approach.

**Theorem 3** Let \( X, Y \) and \( Z \) be Banach spaces, and let \( T : \hat{X} \otimes_\varepsilon Y \to Z \) be a bounded linear operator. If \( T \) is integral, then \( T^# : X \to I(Y, Z) \) is 1-summing. If in addition \( X \) is a \( \ell_\infty \)-space, then \( T : \hat{X} \otimes_\varepsilon Y \to Z \) is integral if and only if \( T^# : X \to I(Y, Z) \) is integral.

**Proof:** First, we will show that, if \( T : \hat{X} \otimes_\varepsilon Y \to Z \) is an integral operator, then \( T^# \) is in \( \prod_1 (X, I(Y, Z)) \) with \( \pi_i(T^#) \leq \| T \|_{\text{int}} \). Let \( x_1, x_2, \ldots, x_n \) be in \( X \), and fix \( \varepsilon > 0 \). For each \( i \leq n \), there exists \( n_i \in \mathbb{N} \), \( (y_{ij})_{j \leq n_i} \) in \( Y \), and \( (z^*_{ij})_{j \leq n_i} \) in \( Z^* \), such that
\[
\| \sum_{j=1}^{n_i} y_{ij} \otimes z^*_{ij} \|_\varepsilon \leq 1,
\]
and
\[
\| T^#x_i \|_{\text{int}} \leq \sum_{j=1}^{n_i} z^*_{ij} (T(x_i \otimes y_{ij})) + \frac{\varepsilon}{2^i}.
\]
Since $T$ is an integral operator, and
\[
\| \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \otimes y_{ij} \otimes z_{ij}^* \|_\varepsilon \leq \sup \left\{ \sum_{i=1}^{n} |x^*(x_i)| : \| x^* \| \leq 1, x^* \in X^* \right\},
\]

it follows that
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij}^* (T(x_i \otimes y_{ij})) \leq \| T \|_{\text{int}} \sup \left\{ \sum_{i=1}^{n} |x^*(x_i)| : \| x^* \| \leq 1, x^* \in X^* \right\}.
\]

Therefore
\[
\sum_{i=1}^{n} \| T^#x_i \|_{\text{int}} \leq \| T \|_{\text{int}} \sup \left\{ \sum_{i=1}^{n} |x^*(x_i)| : x^* \in X^*, \| x^* \| \leq 1 \right\} + \varepsilon.
\]

Now, if in addition $X$ is a $L_\infty$-space, then by [12, p. 477], the operator $T^#$ is indeed integral.

**Remark 4** If $X = C(\Omega)$ is a space of continuous functions defined on a compact Hausdorff space $\Omega$, one can deduce a similar result to Theorem 3 from the main result of [13].

Our next result extends a result of [16] to $L_\infty$-spaces, where it was shown that whenever $X = C(\Omega)$, a space of all continuous functions on a compact Hausdorff space $\Omega$, then a bounded linear operator $T : C(\Omega) \hat{\otimes}_{\varepsilon} Y \rightarrow Z$ is 1-summing if and only if $T^# : C(\Omega) \rightarrow \prod_1(Y, Z)$ is 1-summing. This also extends a result of [14] where similar conclusions were shown to be true for $X = A(K)$, a space of continuous affine functions on a Choquet simplex $K$ (see [2]).

We note that one implication follows with no restriction on $X$. If $X, Y$ and $Z$ are Banach spaces, and $T : X \hat{\otimes}_{\varepsilon} Y \rightarrow Z$ is a 1-summing operator, then $T^#$ takes its values in $\prod_1(Y, Z)$. This follows from the fact that for each $x \in X$, the operator $T^#x$ is the composition of $T$ with the bounded linear operator from $Y$ into $X \hat{\otimes}_{\varepsilon} Y$ which to each $y$ in $Y$ gives the element $x \otimes y$ in $X \hat{\otimes}_{\varepsilon} Y$, and hence
\[
\pi_1(T^#x) \leq \| x \| \pi_1(T).
\]
Moreover, one can proceed as in [16] to show that $T^\# : X \to \prod_1(Y, Z)$ is 1-summing.

**Theorem 5** If $X$ is a $\mathcal{L}_\infty$ space, then for any Banach spaces $Y$ and $Z$, a bounded linear operator $T : X \hat{\otimes}_e Y \to Z$ is 1-summing if and only if $T^\# : X \to \prod_1(Y, Z)$ is 1-summing.

**Proof:** Let $T : X \hat{\otimes}_e Y \to Z$ be such that $T^\# : X \to \prod_1(Y, Z)$ is 1-summing. Since $X$ is a $\mathcal{L}_\infty$-space, it follows from [14, p. 477] that $T^\# : X \to \prod_1(Y, Z)$ is an integral operator. Let $\varphi$ denote the isometric embedding of $Z$ into $C(B(Z^*))$, the space of all continuous scaler functions on the unit ball $B(Z^*)$ of $Z^*$ with its weak*-topology. This induces an isometry

$$\hat{\varphi} : \prod_1(Y, Z) \to \prod_1((Y, C(B(Z^*))),$$

$$\hat{\varphi}(U) = \varphi \circ U \quad \text{for all } U \in \prod_1(Y, Z).$$

Now, it follows from [15, p. 301], that $\prod_1(Y, C(B(Z^*)))$ is isometric to $I(Y, C(B(Z^*)))$. Hence we may assume that $\hat{\varphi} \circ T^\# : X \to I(Y, C(B(Z^*)))$ is an integral operator. Moreover, it is easy to check that $(\varphi \circ T)^\# = \hat{\varphi} \circ T^\#$. By Theorem 2 the operator $\varphi \circ T : X \hat{\otimes}_e Y \to C(B(Z^*))$ is an integral operator, and hence $T$ is in $\prod_1(X \hat{\otimes}_e Y, Z)$ by Proposition 1.

In the following section we shall, among other things, exhibit an example that illustrates that it is crucial for the space $X$ to be a $\mathcal{L}_\infty$-space if the conclusion of Theorem 5 is to be valid.

### III 2-summing Operators and some Counter-examples.

In this section we shall study the behavior of 2-summing operators on injective tensor product spaces. As we shall soon see, the behavior of such operators when $p = 2$ is quite different from when $p = 1$. For instance, unlike the case $p = 1$, the $\mathcal{L}_\infty$-spaces don’t seem to play any particular role. In fact, we shall exhibit operators $T$ on $C[0, 1] \hat{\otimes}_e \ell_2$ which are not 2-summing, yet their corresponding operators $T^\#$ are. We will also give other interesting examples that answer some other natural questions.
We will present the next theorem for $p = 2$, but the same result is true for any $1 \leq p < \infty$, with only minor changes.

**Theorem 6** Let $X, Y$ and $Z$ be Banach spaces. If $T : X \hat{\otimes}_c Y \to Z$ is a 2-summing operator, then $T^\# : X \to \prod_2(Y, Z)$ is a 2-summing operator.

**Proof:** If $T : X \hat{\otimes}_c Y \to Z$ is 2-summing, then using the same kind of arguments that we have given above, it can easily be shown that for each $x \in X$, that $T^\# x \in \prod_2(Y, Z)$, with $\pi_2(T^\# x) \leq \pi_2(T) \|x\|$.

Now we will show that $T^\# : X \to \prod_2(Y, Z)$ is 2-summing. Let $(x_n)$ be in $X$ such that $\sum_n |x^*(x_n)|^2 < \infty$ for each $x^*$ in $X^*$. Fix $\epsilon > 0$. For each $n \geq 1$, let $(y_{nm})$ be a sequence in $Y$ such that

$$\sup \left\{ \left( \sum_{m=1}^{\infty} |y^*(y_{nm})|^2 \right)^{1/2} : y^* \leq 1, y^* \in Y^* \right\} \leq 1,$$

and

$$\pi_2(T^\# x_n) \leq \left( \sum_{m=1}^{\infty} \| T(x_n \otimes y_{nm}) \|^2 \right)^{1/2} + \frac{\epsilon}{2^n}.$$

Then

$$\left[ \pi_2(T^\# x_n) \right]^2 \leq \sum_{m=1}^{\infty} \| T(x_n \otimes y_{nm}) \|^2 + \frac{\epsilon}{2^{n-1}} \left( \sum_{m=1}^{\infty} \| T(x_n \otimes y_{nm}) \|^2 \right)^{1/2} + \frac{\epsilon^2}{2^{2n}}.$$

Now, consider the sequence $(x_n \otimes y_{nm})$ in $X \hat{\otimes}_c Y$. For each $\xi \in (X \hat{\otimes}_c Y)^* \simeq I(X, Y^*)$ we have that

$$\sum_{n,m} |\xi(x_n)(y_{nm})|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\xi(x_n)(y_{nm})|^2$$

$$\leq \sum_{n=1}^{\infty} \| \xi(x_n) \|^2.$$

Since $\xi \in I(X, Y^*)$, it follows that $\xi \in \prod_2(X, Y^*)$, and so

$$\sum_{n=1}^{\infty} \| \xi(x_n) \|^2 < \infty.$$
Hence we have shown that for all $\xi \in (X \hat{\otimes} Y)^*$,

$$\sum_{m,n} |\xi(x_n)(y_{nm})|^2 < \infty.$$  

Since $T \in \prod_2 (X \hat{\otimes}, Y, Z)$, we have that

$$\sum_{m,n} \|T(x_n \otimes y_{nm})\|^2 < \infty,$$

and therefore

$$\sum_n \left[ \pi_2 (T^#x_n) \right]^2 < \infty.$$  

\[\square\]

**Remark 7** The above result extends a result of [1], where it was shown that if $T : X \hat{\otimes} Y \to Z$ is $p$-summing for $1 \leq p < \infty$, then $T^# : X \to \ell(Y, Z)$ is $p$-summing.

Now we shall give the example that we promised at the end of section II.

**Theorem 8** There exists a bounded linear operator $T : \ell_2 \hat{\otimes} \ell_2 \to \ell_2$ such that $T$ is not 1-summing, yet $T^# : \ell_2 \to \pi_1(\ell_2, \ell_2)$ is 1-summing.

**Proof:** First, we note the well known fact that $\ell_2 \hat{\otimes} \ell_2 = K(\ell_2, \ell_2)$, the space of all compact operators from $\ell_2$ to $\ell_2$. Now we define $T$ as the composition of two operators.

Let $P : K(\ell_2, \ell_2) \to c_0$ be the operator defined so that for each $K \in K(\ell_2, \ell_2)$,

$$P(K) = (K(e_n)(e_n)),$$

where $(e_n)$ is the standard basis of $\ell_2$. It is well known [10, p.145] that the sequence $(e_n \otimes e_n)$ in $\ell_2 \hat{\otimes} \ell_2$ is equivalent to the $c_0$-basis, and that the operator $P$ defines a bounded linear projection of $K(\ell_2, \ell_2)$ onto $c_0$.

Let $S : c_0 \to \ell_2$ be the bounded linear operator such that for each $(\alpha_n) \in c_0$

$$S(\alpha_n) = \left( \frac{\alpha_n}{n} \right).$$

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It is easily checked [7, p. 39] that $S$ is a 2-summing operator that is not 1-summing.

Now we define $T : \mathcal{K}(\ell_2, \ell_2) \to \ell_2$ to be $T = S \circ P$. Thus $T$ is 2-summing but not 1-summing. It follows from Theorem 6 that the induced operator $T^\# : \ell_2 \to \prod_2(\ell_2, \ell_2)$ is 2-summing. Since $\ell_2$ is of cotype 2, it follows from [10, p. 62], that for any Banach space $E$, we have $\prod_2(\ell_2, E) = \prod_1(\ell_2, E)$, and that there exists a constant $C > 0$ such that for all $U \in \prod_2(\ell_2, E)$ we have

$$\pi_1(U) \leq C \pi_2(U).$$

This implies that $T^\#$ is 1-summing as an operator taking its values in $\prod_1(\ell_2, \ell_2)$.

**Remark 9** We do not need to use Theorem 6 to show that $T^\#$ is 1-summing in the example above. Instead, we can use the following argument. First note that $T^\#$ factors as follows:

$$\begin{array}{ccc}
\ell_2 & \xrightarrow{T^\#} & \pi_1(\ell_2, \ell_2) \\
\downarrow & & \downarrow \\
\ell_2 & \xrightarrow{A} & B
\end{array}$$

Here $A : \ell_2 \to \ell_2$ is the 1-summing operator defined by

$$A(\alpha_n) = \left(\frac{\alpha_n}{n}\right),$$

for each $(\alpha_n) \in \ell_2$, and $B : \ell_2 \to \pi_1(\ell_2, \ell_2)$ is the natural embedding of $\ell_2$ into the space $\pi_1(\ell_2, \ell_2)$ defined by

$$B(\beta_n)(\gamma_n) = (\beta_n \gamma_n)$$

for each $(\beta_n), (\gamma_n) \in \ell_2$.

Now we will give two examples concerning the case when $p > 1$. We will show that we do not have a converse to Theorem 8, even when the underlying space $X$ is a $\mathbb{L}_\infty$-space.
First, let us fix some notation. In what follows we shall denote the space $\ell_p(\mathbb{Z})$ by $\ell_p$, and call its standard basis $\{e_n : n \in \mathbb{Z}\}$. Thus if $x = (x(n)) \in \ell_p$, then $x(n) = \langle x, e_n \rangle$, and

$$\|x\|_{\ell_p} = \left(\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^p \right)^{\frac{1}{p}}.$$ 

For $f \in L_p[0, 1]$, we let

$$\|f\|_{L_p} = \left(\int_{0}^{1} |f(t)|^p dt \right)^{\frac{1}{p}}.$$ 

If $\Omega$ is a compact Hausdorff space, and $Y$ is a Banach space, then $C(\Omega, Y) = C(\Omega) \hat{\otimes} Y$ will denote the Banach space of continuous $Y$-valued functions on $\Omega$ under the supremum norm.

We recall that since $\ell_2$ is of cotype 2, we have that $\prod_2(\ell_2, \ell_2) = \prod_1(\ell_2, \ell_2)$. We also recall that, if $u = \sum_{n=1}^{\infty} \alpha_n e_n \otimes e_n$ is a diagonal operator in $\prod_2(\ell_2, \ell_2)$, then

$$\pi_2(u) = \left(\sum_{n=1}^{\infty} |\alpha_n|^2 \right)^{\frac{1}{2}} = \text{the Hilbert-Schmidt norm of } u.$$

**Theorem 10** For each $1 < p < \infty$, there is a bounded linear operator $T : C([0, 1], \ell_2) \to \ell_2$ that is not $p$-summing, but such that $T^\#: C[0, 1] \to \Pi_p(\ell_2, \ell_2)$ is $p$-summing.

**Proof:** We present the proof for $p \leq 2$. The case where $p > 2$ follows by the same argument. For each $n \in \mathbb{Z}$, let $\epsilon_n(t) : [0, 1] \to \mathbb{C}$, $\epsilon_n(t) = e^{2\pi i n t}$ denote the standard trigonometric basis of $L_2[0, 1]$. If $f \in L_1[0, 1]$, let $\hat{f}(n) = \int_{0}^{1} f(t)\epsilon_n(t) dt$ denote the usual Fourier coefficient of $f$. For each $\lambda = (\lambda_n)$, where $|\lambda_n| \leq 1$ for all $n \in \mathbb{Z}$, define the operator

$$T_\lambda : C([0, 1], \ell_2) \to \ell_2$$

such that for $\varphi \in C([0, 1], \ell_2)$ we have

$$T_\lambda \varphi = (\lambda_n \langle \varphi(n), e_n \rangle).$$
Here $\hat{\varphi}(n) = \text{Bochner} - \int_0^1 \varphi(t) e_n(t) dt$.

The operator $T_\lambda$ is a bounded linear operator, with $\| T_\lambda \varphi \|_{\ell_2} \leq \| \varphi \|$. To see this, note that for $\varphi \in C([0, 1], \ell_2)$ we have

$$\| T_\lambda \varphi \|_{\ell_2}^2 = \sum_n |\lambda_n|^2 |\langle \hat{\varphi}(n), e_n \rangle|^2 \leq \sum_n |\langle \hat{\varphi}(n), e_n \rangle|^2 \leq \sum_n \int_0^1 |\langle \varphi(t), e_n \rangle|^2 dt = \int_0^1 \| \varphi(t) \|_{\ell_2}^2 dt \leq \sup_t \| \varphi(t) \|_{\ell_2}^2.$$

Now, note that if $f \in C([0, 1])$, and $x \in \ell_2$, then

$$T_\lambda(f \otimes x) = \left( \lambda_n \hat{f}(n) \langle x, e_n \rangle \right),$$

and hence the operator $T_\lambda^\#: C[0, 1] \to \mathcal{L}(\ell_2, \ell_2)$ is such that

$$T_\lambda^\# f(x) = \left( \lambda_n \hat{f}(n) \langle x, e_n \rangle \right).$$

Thus

$$\pi_2(T_\lambda^\# f) = \left( \sum_n |\lambda_n|^2 |\hat{f}(n)|^2 \right)^{\frac{1}{2}}.$$

Hence, by Hölder’s inequality,

$$\pi_2(T_\lambda^\# f) \leq \| \lambda_n \|_{\ell_r} \| \hat{f}(n) \|_{\ell_q},$$

where $\frac{1}{r} + \frac{1}{q} = \frac{1}{2}$. By the Hausdorff-Young inequality, we have that

$$\| \hat{f}(n) \|_{\ell_q} \leq \| f \|_{L_p},$$

where $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Thus

$$\pi_2(T_\lambda^\# f) \leq \| \lambda_n \|_{\ell_r} \| f \|_{L_p},$$

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for $1 \leq p \leq 2$, $2 \leq r \leq \infty$ and $\frac{1}{p} = \frac{1}{r} + \frac{1}{2}$. This shows that if $\parallel (\lambda_n) \parallel_r < \infty$, then

1) $T^\#_\lambda (C[0,1]) \subseteq \pi_2(\ell_2,\ell_2) = \pi_p(\ell_2,\ell_2)$;

2) $T^\#_\lambda : C[0,1] \longrightarrow \pi_p(\ell_2,\ell_2)$ is $p$-summing.

Now, let $U \subset C([0,1],\ell_2)$ be the closed linear span of $\{\epsilon_i \otimes e_i, a_i \in \mathbb{Z}\}$. Then $U$ is isometrically isomorphic to $\ell_2$. This is because

$$\| \sum_i \mu_i \epsilon_i \otimes e_i \| = \sup_{t \in [0,1]} \| (\mu_n \epsilon_n(t)) \|_{\ell_2} = \| (\mu_i \epsilon_i(t_0)) \|_{\ell_2},$$

for some $t_0 \in [0,1]$, and hence

$$\| \sum_i \mu_i \epsilon_i \otimes e_i \| = \left( \sum_i |\mu_i|^2 \right)^{\frac{1}{2}}.$$

Moreover

$T_\lambda(\epsilon_i \otimes e_i) = \lambda_i e_i$ for all $i \in \mathbb{Z}$.

Therefore, we have the following commuting diagram

$$\begin{array}{ccc}
U & \xrightarrow{T_\lambda U} & \ell_2 \\
\downarrow Q & & \downarrow S_\lambda \\
\ell_2 & & \\
\end{array}$$

where $Q : U \rightarrow \ell_2$ is the isomorphism from $U$ onto $\ell_2$ such that $Q(\epsilon_n \otimes e_n) = e_n$ for all $n \in \mathbb{Z}$, and $S_\lambda : \ell_2 \longrightarrow \ell_2$ is the operator given by $S_\lambda(\epsilon_n) = \lambda_n e_n$. So to show that $T_\lambda$ is not $p$-summing, it is sufficient to show that one can pick $\lambda = (\lambda_n)$ such that $S_\lambda$ is not $p$-summing. To do this, we consider two cases. If $p = 2$, we take $\lambda_n = 1$ for all $n \in \mathbb{Z}$. Then the map $S_\lambda$ induced on $\ell_2$ is the identity map which is not $s$-summing for any $s < \infty$. If $1 < p < 2$, let $\lambda_n = \frac{1}{n + 1 \frac{1}{2} \log |n + 1|}$, so that $\| (\lambda_n) \|_{\ell_s} < \infty$. Then the map $S_\lambda : \ell_2 \longrightarrow \ell_2$ is not $s$-summing for any $s < r$. To show this, we may assume, without loss of generality, that $s \geq 2$. Let $x_n = e_n$ for all $n \geq 1$, and note that

$$\sup_{x^* \in B(\ell_2)} \left( \sum_n |x^*(x_n)|^s \right)^{\frac{1}{s}} \leq x^* \|_{\ell_2} \leq 1,$$
whilst
\[
\left( \sum_n \| \lambda_n x_n \|^s \right)^{\frac{1}{s}} = \infty.
\]

While the operators \( T_\lambda \) in the previous example failed to be \( p \)-summing, they were all \((2,1)\)-summing. This suggests the following question: suppose \( T : C([0,1], Y) \longrightarrow Z \) is a bounded linear operator such that \( T^* : C[0,1] \longrightarrow \prod_2 (Y, Z) \) is 2-summing. What can we say about \( T \)? Is \( T \) \((2,1)\)-summing? The following example shows that \( T \) can be very bad.

**Theorem 11** There exists a Banach space \( Z \), and a bounded linear operator \( T : C([0,1], \ell_1) \rightarrow Z \) such that \( T^* : C[0,1] \rightarrow \prod_2 (\ell_1, Z) \) is 2-summing, with the property that, for any \( N \in \mathbb{N} \), there exists a subspace \( U \) of \( C([0,1], \ell_1) \) with \( \dim U = N \), such that \( T \) restricted to \( U \) behaves like the identity operator on \( \ell_\infty^N \). In particular \( T \) is not \((2,1)\)-summing.

**Proof:** If \( X \) and \( Y \) are Banach spaces, we denote by \( X \hat{\otimes}_\pi Y \) the projective tensor product, that is, the completion of the algebraic tensor product of \( X \) and \( Y \) under the norm
\[
\| u \|_\pi = \inf \left\{ \sum_{i=1}^n \| x_i \| \| y_i \| : u = \sum_{i=1}^n x_i \otimes y_i \right\}.
\]
It is well known that \( (X \hat{\otimes}_\pi Y)^* \) is isometrically isomorphic to the space \( L(X,Y^*) \) of all bounded linear operators from \( X \) to \( Y^* \).

Let \( Z = C([0,1], \ell_1) + L_2[0,1] \hat{\otimes}_\pi \ell_2 \) be the Banach space with the norm
\[
\| x \|_Z = \inf \{ \| x' \|_\epsilon + \| x'' \|_\pi : x = x' + x'' \},
\]
where \( \| \cdot \|_\epsilon \) denotes the sup norm in \( C([0,1], \ell_1) \), and \( \| \cdot \|_\pi \) denotes the norm of the projective tensor product \( L_2[0,1] \hat{\otimes}_\pi \ell_2 \). Let
\[
T : C([0,1], \ell_1) \longrightarrow Z
\]
be the identity operator.

We first see that for each \( f \in C[0, 1] \), the operator \( T# f : \ell_1 \to Z \) is 2-summing with

\[
\pi_2(T# f) \leq \pi_2(I) \| T# f \|_{\mathcal{L}(\ell_2, Z)},
\]

where \( I : \ell_1 \to \ell_2 \) is the natural mapping. This is because, for each \( f \in C[0, 1] \), and each \( x \in \ell_1 \), we have that

\[
\| T(f \otimes x) \|_2 \leq \| f \otimes x \|_{L_2(\ell_2, \ell_2)} \leq \| f \|_{L_2} \| x \|_{\ell_2}.
\]

To see that \( T# : C[0, 1] \to \prod_2(\ell_1, X) \) is 2-summing, note that \( \| T# f \|_{\mathcal{L}(\ell_2, Z)} \leq \| f \|_{L_2} \), and hence if \( f_1, \ldots, f_n \in C[0, 1] \), then

\[
\left( \sum_{k=1}^n \left[ \pi_2(T# f_k) \right]^2 \right)^{\frac{1}{2}} \leq \pi_2(I) \left( \sum_{k=1}^n \| f_k \|_{L_2}^2 \right)^{\frac{1}{2}} \leq \pi_2(I) \pi_2(J) \sup_{t \in [0, 1]} \left\| \left( \sum_{K=1}^n |f_k(t)|^2 \right)^{\frac{1}{2}} \right\|.
\]

Here \( J : C[0, 1] \to L_2[0, 1] \) denotes the natural mapping.

Now we define the space \( U \), a closed linear subspace of \( C([0, 1], \ell_1) \). Let \( \{ f_{ij} : 1 \leq i, j \leq N \} \) be disjoint functions in \( C[0, 1] \), for which \( 0 \leq f_{ij} \leq 1, \| f_{ij} \| = 1 \), each \( f_{ij} \) is supported in an interval of length \( \frac{1}{N^2} \), and

\[
\int_0^1 f_{ij} dt = \frac{1}{2N^2} \text{ and } \int_0^1 f_{ij}^2 dt = \frac{1}{3N^2}.
\]

Let \( \{ e_{ij} : 1 \leq i, j \leq N \} \) be distinct unit vectors in \( \ell_1 \). We let \( U = \{ \sum_{i,j} \lambda_{i,j} f_{ij} \otimes e_{ij}, \lambda_i \in \mathbb{R} \} \).

Now we consider \( T \) restricted to \( U \). If \( \sum_{i,j} \lambda_{i,j} f_{ij} \otimes e_{ij} \in U \), then

\[
\| \sum_{i,j} \lambda_{i,j} f_{ij} \otimes e_{ij} \|_r \leq \sup_i |\lambda_i|,
\]

and hence

\[
\| \sum_{i,j} \lambda_{i,j} f_{ij} \otimes e_{ij} \|_Z \leq \sup_i |\lambda_i|.
\]
Let $y^*_i = N \sum j f_{ij} \otimes e_{ij}$, and set $x = \sum_{i,j} \lambda_{i} f_{ij} \otimes e_{ij}$. Then whenever $x = x' + x''$, with $x' \in C([0, 1], \ell_1)$ and $x'' \in L_2[0, 1] \hat{\otimes} \ell_2$, we know that

$$|y^*_i(x)| \leq |y^*_i(x')| + |y^*_i(x'')|.$$ 

Hence

$$|y^*_i(x)| \leq \|y^*_i\|_{C([0, 1], \ell_1)^*} \|x'\|_e + \|y^*_i\|_{(L_2([0, 1] \hat{\otimes} \ell_2)^*)} \|x''\|_\pi.$$

But

$$\|y^*_i\|_{C([0, 1], \ell_1)^*} = N \sum_{i,j} \int_{\operatorname{supp} f_{ij}} |f_{ij}| dt = N \cdot \frac{N}{2N^2} = \frac{1}{2},$$

and, since $(L_2[0, 1] \hat{\otimes} \ell_2)^*$ is isometric to $\mathcal{L}(L_2[0, 1], \ell_2)$,

$$\|y^*_i\|_{(L_2([0, 1] \hat{\otimes} \ell_2)^*)} = \sup \left\{ N \sum_{j=1}^N \left( N \int_0^1 f_{ij}^2 dt \right)^{\frac{1}{2}} : g \|_{L_2} \leq 1 \right\} \leq \sup \left\{ N \left( \int_0^1 f_{ij}^2 dt \right)^{\frac{1}{2}} : g \|_{L_2} \leq 1 \right\} \leq N \sqrt{3} \sup \|g\|_2 \leq 1 \right\} \leq \frac{1}{\sqrt{3}}.$$

Therefore

$$|y^*_i(x)| \leq \frac{1}{2} \|x'\|_e + \frac{1}{\sqrt{3}} \|x''\|_\pi \leq \frac{1}{\sqrt{3}} \|x\|.$$

However,

$$y^*_i(x) = N \sum_{j=1}^N \lambda_{i} \int_0^1 f_{ij}^2 dt = N^2 \lambda_{i} \frac{1}{3N^2} = \frac{\lambda_{i}}{3}.$$ 

Therefore

$$\|\sum_{i,j} \lambda_{i} f_{ij} \otimes e_{ij}\| \geq \sqrt{3} \sup_i |y^*_i(x)| \geq \frac{1}{\sqrt{3}} \sup_i |\lambda_{i}|.$$
Thus the space $U$ is isomorphic to $\ell^N_\infty$, and we have the commuting diagram

$$
\begin{array}{ccc}
U & \xrightarrow{T|_U} & T(U) \\
\downarrow A & & \uparrow A^{-1} \\
\ell^N_\infty & \xrightarrow{id} & \ell^N_\infty
\end{array}
$$

where $A : U \to \ell^N_\infty$ is the isomorphism between $U$ and $\ell^N_\infty$.

\[ \square \]

IV Operators that factor through a Hilbert space

It is well known that $\mathcal{L}(X, \ell_2) = \prod_2(X, \ell_2)$ whenever $X$ is $C(K)$ or $\ell_1$. One might ask whether this is true when $X = C(K, \ell_1)$. Indeed one could ask the weaker question: if $T : C(K, \ell_1) \to \ell_2$ is bounded, does it follow that the induced operator $T^\#$ is 2-summing? We answer this question in the negative.

**Theorem 12** There is a compact Hausdorff space $K$ and a bounded linear operator $T : C(K, \ell_1) \to \ell_2$ for which $T^\# : C(K) \to \prod_1(\ell_1, \ell_2)$ is not 2-summing.

**Proof:** First, we show that there is a compact Hausdorff space $K$, and an operator $R : C(K) \to \ell_\infty$ that is (2,1)-summing but not 2-summing. To see this, let $K = [0,1]$, and consider the natural embedding $C[0,1] \to L_{2,1}[0,1]$, where $L_{2,1}[0,1]$ is the Lorentz space on $[0,1]$ with the Lebesgue measure (see [6]). By [11], it follows that this map is (2,1)-summing. To show that this map is not 2-summing, we argue in a similar fashion to [8]. For $n \in \mathbb{N}$, consider the functions $e_i(t) = f(t + \frac{1}{i} \mod 1)$ ($1 \leq i \leq n$), where $f(t) = \frac{1}{\sqrt{t}}$ if $t \geq \frac{1}{n}$ and $\sqrt{n}$ otherwise. Then it is an easy matter to verify that for some constant $C > 0$,

$$
\left( \sum_{i=1}^{n} |e^*(e_i)|^2 \right)^{\frac{1}{2}} \leq C \sqrt{\log n}
$$

for every $e^*$ in the unit ball of $C[0,1]^*$, whereas

$$
\left( \sum_{i=1}^{n} \|e_i\|_{L_{2,1}[0,1]}^2 \right)^{\frac{1}{2}} \geq C^{-1} \log n.
$$
Finally, since $L_{2,1}[0,1]$ is separable, it embeds isometrically into $\ell_{\infty}$.

Define $T : C(K, \ell_1) \rightarrow \ell_2$ as follows: for $\varphi = (f_n) \in C(K, \ell_1)$, let

$$T(f_n) = \sum_n Rf_n(n)e_n.$$  

Then $T$ is bounded, for

$$\|T(f_n)\|_2 = \left(\sum_n |Rf_n(n)|^2\right)^{\frac{1}{2}} \leq \left(\sum_n \|RF_n\|_{\ell_\infty}^2\right)^{\frac{1}{2}} \leq \pi_{2,1}(R) \sup_{t \in K} \sum_n |f_n(t)|.$$  

Thus

$$\|T\| \leq \pi_{2,1}(R).$$  

But $T^# : C(K) \rightarrow \ell(\ell_1, \ell_2)$ is not 2-summing, because for each $f \in C(K)$, the operator $T^# f : \ell_1 \rightarrow \ell_2$ is the diagonal operator $\sum_n Rf_n(n)e_n \otimes e_n$. Hence the strong operator norm of $T^# f$ is

$$\|T^# f\| = \sup_n |Rf_n(n)| = \|RF\|_{\ell_\infty}.$$  

Thus $T^# : C(K) \rightarrow \ell(\ell_1, \ell_2)$ is not 2-summing, because $R : C(K) \rightarrow \ell_\infty$ is not 2-summing.  

**Discussions and concluding remarks**

**Remark 13** Theorem 12 shows that if $X$ and $Y$ are Banach spaces such that $\ell(X, \ell_2) = \prod_2(X, \ell_2)$ and $\ell(Y, \ell_2) = \prod_2(X, \ell_2)$, then $X \hat{\otimes} Y$ need not share this property. This observation could also be deduced from arguments presented in [4] (use Example 3.5 and the proof of Proposition 3.6 to show that there is a bounded operator $T : (\ell_1 \oplus \ell_1 \oplus \ldots \oplus \ell_1)_{\ell_\infty} \rightarrow \ell_2$ that is not $p$-summing for any $p < \infty$).
Remark 14 In the proof of Theorem 2 we showed that the injective tensor product is an associative operation, that is, if \(X, Y\) and \(Z\) are Banach spaces, then \((X \hat{\otimes} Y) \hat{\otimes} Z\) is isometrically isomorphic to \(X \hat{\otimes} (Y \hat{\otimes} Z)\). It is not hard to see that the same is true for the projective tensor product. However, we can conclude from Theorem 12 that what is known as the \(\gamma^*_2\)-tensor product is not an associative operation.

If \(E\) and \(F\) are Banach spaces, and \(T : E \rightarrow F\) is a bounded linear operator, following [10], we say that \(T\) factors through a Hilbert space if there is a Hilbert space \(H\), and operators \(B : E \rightarrow H\) and \(A : H \rightarrow F\) such that \(T = A \circ B\). We let \(\gamma_2(T) = \inf\{\| A \| \| B \|\}\), where the infimum runs over all possible factorization of \(T\), and denote the space of all operators \(T : E \rightarrow F\) that factor through a Hilbert space by \(\Gamma_2(E,F)\). It is not hard to check that \(\gamma_2\) defines a norm on \(\Gamma_2(E,F)\), making \(\Gamma_2(E,F)\) a Banach space. We define the \(\gamma^*_2\)-norm \(\| \|_*\) on \(E \otimes F\) (see [9] or [10]) in which the dual of \(E \otimes F\) is identified with \(\Gamma_2(E,F^*)\), and let \(E \hat{\otimes} \gamma^*_2 F\) denote the completion of \((E \otimes F, \| \|_*)\).

The operator \(T : C(K) \hat{\otimes} \gamma^*_2 \ell_1 \rightarrow \ell_2\) exhibited in Theorem 12, induces a bounded linear functional on \([[(C(K) \hat{\otimes} \gamma_2^* \ell_1) \hat{\otimes} \gamma_2^* \ell_2]^*\]. Now we see that if \((C(K) \hat{\otimes} \gamma_2^* \ell_1) \hat{\otimes} \gamma_2^* \ell_2\) were isometrically isomorphic to \((C(K) \hat{\otimes} \gamma_2^* \ell_1) \hat{\otimes} \gamma_2^* \ell_2\), then the operator \(T^* : C(K) \rightarrow \mathcal{L}(\ell_1, \ell_2)\) would induce a bounded linear functional on \([((C(K) \hat{\otimes} \gamma_2^* \ell_1) \hat{\otimes} \gamma_2^* \ell_2)]^*\), showing that \(T^* \in \Gamma_2(C(K), \mathcal{L}(\ell_1, \ell_2))\), implying that \(T^*\) would be 2-summing [10, p. 62]. This contradiction shows that \((C(K) \hat{\otimes} \gamma_2^* \ell_1) \hat{\otimes} \gamma_2^* \ell_2\) and \((C(K) \hat{\otimes} \gamma_2^* \ell_1) \hat{\otimes} \gamma_2^* \ell_2\) cannot be isometrically isomorphic.

Another example showing that the \(\gamma^*_2\)-tensor product is not associative was given by Pisier (private communication).
Bibliography


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